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# On the codimension-two cohomology of $\mathrm{SL}_n(\mathbb{Z})$

Benjamin Brück [a,1], Jeremy Miller [b,*,2], Peter Patzt [c,3], Robin J. Sroka [d,4,5], Jennifer C.H. Wilson [e,6]

[a] *Institut für Mathematische Logik und Grundlagenforschung, Universität Münster, Germany*
[b] *Department of Mathematics, Purdue University, USA*
[c] *Department of Mathematics, University of Oklahoma, USA*
[d] *Department of Mathematics & Statistics, McMaster University, Canada*
[e] *Department of Mathematics, University of Michigan, USA*

* Corresponding author.
*E-mail addresses:* benjamin.brueck@uni-muenster.de (B. Brück), jeremykmiller@purdue.edu (J. Miller), ppatzt@ou.edu (P. Patzt), robinjsroka@uni-muenster.de (R.J. Sroka), jchw@umich.edu (J.C.H. Wilson).

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[5] *Current address*: Mathematisches Institut, Universität Münster, Germany.
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A B S T R A C T

Borel–Serre proved that $\mathrm{SL}_n(\mathbb{Z})$ is a virtual duality group of dimension $\binom{n}{2}$ and the Steinberg module $\mathrm{St}_n(\mathbb{Q})$ is its dualizing module. This module is the top-dimensional homology group of the Tits building associated to $\mathrm{SL}_n(\mathbb{Q})$. We determine the "relations among the relations" of this Steinberg module. That is, we construct an explicit partial resolution of length two of the $\mathrm{SL}_n(\mathbb{Z})$-module $\mathrm{St}_n(\mathbb{Q})$. We use this partial resolution to show the codimension-2 rational cohomology group $H^{\binom{n}{2}-2}(\mathrm{SL}_n(\mathbb{Z});\mathbb{Q})$ of $\mathrm{SL}_n(\mathbb{Z})$ vanishes for $n \geq 3$. This resolves a case of a conjecture of Church–Farb–Putman. We also produce lower bounds for the codimension-1 cohomology of certain congruence subgroups of $\mathrm{SL}_n(\mathbb{Z})$.



**Contents**

## 1. Introduction

### 1.1. Steinberg modules and Borel–Serre duality

Although the Steinberg module was initially introduced as an object of study in representation theory, the work of Borel–Serre [3] showed its importance to the study of cohomology of arithmetic groups. In this paper, we are interested in the arithmetic group $\mathrm{SL}_n(\mathbb{Z})$ and its congruence subgroups. We use their relationship to the Steinberg module for $\mathrm{SL}_n(\mathbb{Q})$ to obtain new insights about the high-dimensional cohomology of these groups.

We begin by recalling the relevant definitions. Let $\mathbb{F}$ be a field. The *Tits building* associated to $\mathrm{SL}_n(\mathbb{F})$, denoted $\mathcal{T}_n(\mathbb{F})$, is the geometric realisation of the poset of proper nonzero subspaces of $\mathbb{F}^n$. It is $(n-2)$-spherical by the Solomon–Tits Theorem [21] and its one potentially nonvanishing reduced homology group is called the Steinberg module

$$\mathrm{St}_n(\mathbb{F}) := \widetilde{H}_{n-2}(\mathcal{T}_n(\mathbb{F})).$$

The group $\mathrm{SL}_n(\mathbb{F})$ acts on the Tits building and hence the Steinberg module is a representation of $\mathrm{SL}_n(\mathbb{F})$. The results of Borel–Serre [3] show that $\mathrm{SL}_n(\mathbb{Z})$ is a virtual duality group of dimension $\binom{n}{2}$ and that the Steinberg module $\mathrm{St}_n(\mathbb{Q})$ is the virtual dualizing module. Thus, for any finite index subgroup $\Gamma \subseteq \mathrm{SL}_n(\mathbb{Z})$, we have $H^k(\Gamma;\mathbb{Q}) = 0$ for $k > \binom{n}{2}$ and

$$H^{\binom{n}{2}-i}(\Gamma;\mathbb{Q}) \cong H_i(\Gamma;\mathrm{St}_n(\mathbb{Q}) \otimes \mathbb{Q}). \tag{1}$$

If $\Gamma$ is torsion-free, then $H^{\binom{n}{2}-i}(\Gamma) \cong H_i(\Gamma;\mathrm{St}_n(\mathbb{Q}))$. We call the cohomology group $H^{\binom{n}{2}-i}(\Gamma)$ the *codimension-i* cohomology of $\Gamma$.

### 1.2. Resolutions of Steinberg modules

Borel–Serre duality is useful because it translates questions about the high-degree cohomology of $\mathrm{SL}_n(\mathbb{Z})$ and its finite index subgroups to questions about their low-degree homology, at the cost of working with twisted coefficients. One can compute this group homology with twisted coefficients by constructing a projective resolution of the coefficient module. The main achievement of this work is the construction of a partial resolution of $\mathrm{St}_n(\mathbb{Q})$,

$$\mathcal{M}_2 \longrightarrow \mathcal{M}_1 \longrightarrow \mathcal{M}_0 \longrightarrow \mathrm{St}_n(\mathbb{Q}) \longrightarrow 0,$$

where the $\mathrm{SL}_n(\mathbb{Z})$-modules $\mathcal{M}_i$ for $i = 0, 1, 2$ have generating sets that allow for an easy description of the $\mathrm{SL}_n(\mathbb{Z})$-action (see below).

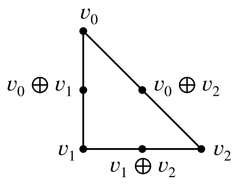


**Fig. 1.** The apartment $A_{\{\vec{v}_0,\vec{v}_1,\vec{v}_2\}}$ in $\mathcal{T}_3(\mathbb{Q})$.

This extends work of Solomon–Tits [21], Ash–Rudolph [1], and Bykovskiĭ [4]: Given a basis $\beta = \{\vec{v}_0, \dots, \vec{v}_{n-1}\}$ of $\mathbb{Q}^n$, let $A_\beta$ be the full subcomplex of $\mathcal{T}_n(\mathbb{Q})$ of all subspaces that are spans of nonempty proper subsets of $\{\vec{v}_0, \dots, \vec{v}_{n-1}\}$. This subcomplex is called an *apartment*. It is homeomorphic to $S^{n-2}$ and this sphere has a canonical fundamental class $[A_\beta]$ (well-defined up to sign). See Fig. 1. By the Solomon–Tits Theorem, the images of all these homology classes form a generating set for the Steinberg module $\mathrm{St}_n(\mathbb{Q}) = \widetilde{H}_{n-2}(\mathcal{T}_n(\mathbb{Q}))$. Ash–Rudolph [1] showed that in fact, a generating set is given by the *integral apartment classes* (also known as *modular symbols*), i.e. the images of $[A_\beta]$, where $\beta = \{\vec{v}_0, \dots, \vec{v}_{n-1}\}$ is a basis of $\mathbb{Z}^n$. Bykovskiĭ [4] extended this to a presentation. Now our resolution computes the two-syzygies (the relations among the relations) of $\mathrm{St}_n(\mathbb{Q})$.

Our partial resolution admits the following "combinatorial" description: We define the groups $\mathcal{M}_i$ as quotients of free abelian groups, generated by formal symbols $[[\vec{v}_0, \dots, \vec{v}_k]]$, where $\vec{v}_0, \dots, \vec{v}_k$ are certain sets of vectors in $\mathbb{Z}^n$. The action of $\mathrm{SL}_n(\mathbb{Z})$ on $\mathbb{Z}^n$ induces an action on the sets of these formal symbols, given by $\phi \cdot [[\vec{v}_0, \dots, \vec{v}_k]] = [[\phi(\vec{v}_0), \dots, \phi(\vec{v}_k)]]$.
**Generators:** Let $\mathcal{M}_0$ be the quotient of the free abelian group

$$\langle [[\vec{v}_0, \dots, \vec{v}_{n-1}]] \mid \ \vec{v}_0, \dots, \vec{v}_{n-1} \text{ a basis of } \mathbb{Z}^n \rangle_{\mathbb{Z}}$$

by the relations:

i) $[[\vec{v}_0, \dots, \vec{v}_{n-1}]] = \mathsf{sgn}(\sigma)[[\vec{v}_{\sigma(0)}, \dots, \vec{v}_{\sigma(n-1)}]]$ for all permutations $\sigma \in \mathrm{Sym}(n)$,
ii) $[[\vec{v}_0, \dots, \vec{v}_{n-1}]] = [[\pm\vec{v}_0, \dots, \pm\vec{v}_{n-1}]]$, (with the $n$ signs each chosen independently).

**Relations:** Let $\mathcal{M}_1$ be the quotient of the free abelian group

$$\left\langle [[\vec{v}_0, \dots, \vec{v}_n]] \;\middle|\; \begin{array}{l} \text{there exist indices } i, j, k \text{ with} \\ \bullet\ \vec{v}_0, \dots, \vec{v}_{i-1}, \vec{v}_{i+1}, \dots, \vec{v}_n \text{ is a basis of } \mathbb{Z}^n, \\ \bullet\ \vec{v}_i = \pm\vec{v}_j \pm \vec{v}_k \text{ or } \vec{v}_i = \pm\vec{v}_j \pm \vec{v}_k \pm \vec{v}_l \text{ for } l \neq i, j, k \end{array} \right\rangle_{\mathbb{Z}}$$

by the relations

i) $[[\vec{v}_0, \dots, \vec{v}_n]] = \mathsf{sgn}(\sigma)[[\vec{v}_{\sigma(0)}, \dots, \vec{v}_{\sigma(n)}]]$ for all permutations $\sigma \in \mathrm{Sym}(n+1)$,
ii) $[[\vec{v}_0, \dots, \vec{v}_n]] = [[\pm\vec{v}_0, \dots, \pm\vec{v}_n]]$ (signs chosen independently).

**Relations among the relations:** Let $\mathcal{M}_2$ be the quotient of the free abelian group

$$\left\langle [[\vec{v}_0,\dots,\vec{v}_{n+1}]] \;\middle|\; \begin{array}{l} \text{there exist distinct indices } i,j,k,l,m \text{ with} \\ \bullet\ \vec{v}_0,\dots,\vec{v}_{i-1},\vec{v}_{i+1},\dots,\vec{v}_{j-1},\vec{v}_{j+1},,\dots,\vec{v}_n \text{ is a basis of } \mathbb{Z}^n, \\ \bullet\ \vec{v}_i = \pm\vec{v}_k \pm \vec{v}_l \\ \bullet\ \vec{v}_j = \pm\vec{v}_m \pm \vec{v}_l \text{ or } \vec{v}_j = \pm\vec{v}_m \pm \vec{v}_p \text{ for } p \neq i,j,k,l,m \end{array} \right\rangle_{\mathbb{Z}}$$

by the relations

i) $[[\vec{v}_0,\dots,\vec{v}_{n+1}]] = \mathsf{sgn}(\sigma)[[\vec{v}_{\sigma(0)},\dots,\vec{v}_{\sigma(n+1)}]]$ for all permutations $\sigma \in \mathrm{Sym}(n+2)$,
ii) $[[\vec{v}_0,\dots,\vec{v}_{n+1}]] = [[\pm\vec{v}_0,\dots,\pm\vec{v}_{n+1}]]$ (signs chosen independently).

**Maps in the resolution:** Let $\delta_1 \colon \mathcal{M}_1 \to \mathcal{M}_0$ and $\delta_2 \colon \mathcal{M}_2 \to \mathcal{M}_1$ be the maps

$$\delta_1 \colon [[\vec{v}_0,\dots,\vec{v}_n]] \longmapsto \sum_i (-1)^i\ [[\vec{v}_0,\dots,\vec{v}_{i-1},\vec{v}_{i+1},\dots,\vec{v}_n]].$$

$$\delta_2 \colon [[\vec{v}_0,\dots,\vec{v}_{n+1}]] \longmapsto \sum_i (-1)^{i+1}\ [[\vec{v}_0,\dots,\vec{v}_{i-1},\vec{v}_{i+1},\dots,\vec{v}_{n+1}]].$$

For these maps, we define the symbols $[[\vec{v}_0,\dots,\vec{v}_{i-1},\vec{v}_{i+1},\dots,\vec{v}_n]]$ and $[[\vec{v}_0,\dots,\vec{v}_{i-1},\vec{v}_{i+1},\dots,\vec{v}_{n+1}]]$ to be zero if the vectors do not span $\mathbb{Z}^n$.

The map $\epsilon \colon \mathcal{M}_0 \to \mathrm{St}_n(\mathbb{Q})$ is the "integral apartment class map" mentioned above. More precisely, it is defined as follows. If $[[\vec{v}_0,\dots,\vec{v}_{n-1}]]$ is a generator of $\mathcal{M}_0$, then $\beta = \{\vec{v}_0,\dots,\vec{v}_{n-1}\}$ is a basis of $\mathbb{Z}^n$ that comes with an order that is well-defined up to the action of the alternating group. This order determines a sign of the corresponding apartment class $[A_\beta]$. Define $\epsilon$ via the formula:

$$\begin{aligned} \epsilon \colon \mathcal{M}_0 &\longrightarrow \mathrm{St}_n(\mathbb{Q}) \\ [[\vec{v}_0,\dots,\vec{v}_{n-1}]] &\longmapsto [A_\beta]. \end{aligned}$$

**Theorem A.** *The sequence*

$$\mathcal{M}_2 \xrightarrow{\delta_2} \mathcal{M}_1 \xrightarrow{\delta_1} \mathcal{M}_0 \xrightarrow{\epsilon} \mathrm{St}_n(\mathbb{Q}) \longrightarrow 0$$

*is exact.*

Exactness of

$$\mathcal{M}_0 \xrightarrow{\epsilon} \mathrm{St}_n(\mathbb{Q}) \longrightarrow 0$$

is due to Ash–Rudolph [1] and exactness of

$$\mathcal{M}_1 \xrightarrow{\delta_1} \mathcal{M}_0 \xrightarrow{\epsilon} \mathrm{St}_n(\mathbb{Q}) \longrightarrow 0$$

follows from Bykovskiĭ [4]. See Church–Putman [6] for an alternate proof.

*1.3. Applications to the cohomology of* $\mathrm{SL}_n(\mathbb{Z})$

Using Theorem A, we show that the codimension-2 rational homology of $\mathrm{SL}_n(\mathbb{Z})$ vanishes for large $n$.

**Theorem B.** *For* $n \geq 3$, $H^{\binom{n}{2}-2}(\mathrm{SL}_n(\mathbb{Z}); \mathbb{Q}) \cong 0$.

A standard transfer argument implies that $H^i(\mathrm{GL}_n(\mathbb{Z}); \mathbb{Q})$ is a summand of $H^i(\mathrm{SL}_n(\mathbb{Z}); \mathbb{Q})$. Thus $H^{\binom{n}{2}-2}(\mathrm{GL}_n(\mathbb{Z}); \mathbb{Q}) \cong 0$ for $n \geq 3$. Theorem B resolves the codimension-2 case of a conjecture of Church–Farb–Putman [7, Conjecture 2].

**Conjecture 1.1** *(Church–Farb–Putman). For* $n \geq i+2$, $H^{\binom{n}{2}-i}(\mathrm{SL}_n(\mathbb{Z}); \mathbb{Q}) \cong 0$.

For codimension $i = 0$, this conjecture is true and due to Lee–Szczarba [11]. Vanishing in codimension-0 follows easily from Ash–Rudolph's [1] generating set for $\mathrm{St}_n(\mathbb{Q})$. For codimension-1, the conjecture was established by Church–Putman [6] and follows from the Bykovskiĭ presentation [4] of $\mathrm{St}_n(\mathbb{Q})$. Similarly, Theorem B follows readily from our result determining the relations among the relations in the Steinberg module, Theorem A.

The rational cohomology of $\mathrm{SL}_n(\mathbb{Z})$ has been completely computed for $n \leq 7$ (Soulé [22] for $n = 3$, Lee–Szczarba [12] for $n = 4$, and Elbaz-Vincent–Gangl–Soulé [9] for $n = 5$, 6, and 7). These calculations verify Conjecture 1.1 for $n \leq 7$ and also show that the vanishing range predicted by Conjecture 1.1 is not sharp for $n = 3$, 5, or 7. This failure of sharpness is reflected in the fact that Theorem B implies that the codimension-2 rational cohomology vanishes for $n \geq 3$ while the codimension-2 case of Conjecture 1.1 only concerns vanishing for $n \geq 4$.

*1.4. Applications to the cohomology of congruence subgroups*

The principal level $p$-congruence subgroup of $\mathrm{SL}_n(\mathbb{Z})$, denoted $\Gamma_n(p)$, is defined to be the kernel of the mod-$p$ reduction map

$$\mathrm{SL}_n(\mathbb{Z}) \longrightarrow \mathrm{SL}_n(\mathbb{Z}/p).$$

Using Theorem A, we obtain a combinatorial chain complex computing $H_1(\Gamma_n(p); \mathrm{St}_n(\mathbb{Q})) \cong H^{\binom{n}{2}-1}(\Gamma_n(p))$ (see Proposition 9.2). In the case $p = 3$ or 5, we use this to obtain the following numerical estimate on the size of the codimension-1 homology. For a field $\mathbb{F}$ let $Gr_k^m(\mathbb{F})$ denote the Grassmannian of $k$-planes in $\mathbb{F}^m$.

**Theorem C.** *For* $p = 3$ *or* 5, $\dim_{\mathbb{Q}} H^{\binom{n}{2}-1}(\Gamma_n(p); \mathbb{Q}) \geq p^{\binom{n-2}{2}} |Gr_2^n(\mathbb{F}_p)| \left(\frac{p-1}{2}\right)^{n-2}$.

See [16, Corollary 1.2] for an upper bound of a similar flavour in the case $p = 3$.

### 1.5. Proof structure and paper outline

Following Lee–Szczarba [11], Church–Farb–Putman [8], and Church–Putman [6], we will construct our resolution of $\mathrm{St}_n(\mathbb{Q})$ by proving that certain simplicial complexes are highly-connected. The complexes relevant to our paper are called $\mathrm{BAA}_n$. These complexes are related to Maazen's complex of partial bases [13,14] with added augmentations in the sense of Church–Putman [6]. The augmentations are inspired by the Voronoi tessellation of the symmetric spaces associated to the groups $\mathrm{SL}_n(\mathbb{Z})$. In Section 2, we define $\mathrm{BAA}_n$ and some variants. In the following sections, we adapt an argument of Church–Putman [6] to prove $\mathrm{BAA}_n$ is highly-connected, and in fact Cohen–Macaulay of dimension $n+1$. Because of the added complexity needed to study the relations among the relations, we use computer calculations for one step in the proof. In Section 3, we construct a retraction map that is used in the connectivity argument and is based on the Euclidean algorithm. The last step of the construction of this retraction uses that certain finite subcomplexes of $\mathrm{BAA}_4$ are highly-connected. This is proved in Section 4 using computer calculations. In Section 5 and Section 6, we complete the proof that $\mathrm{BAA}_n$ is highly-connected. In Section 7, we recall a spectral sequence due to Quillen [20] concerning maps of posets. We use this spectral sequence in Section 8 to study the codimension-2 cohomology of $\mathrm{SL}_n(\mathbb{Z})$ and in Section 9 to study the codimension-1 cohomology of congruence subgroups.

### 1.6. Code for the computer calculations

The code that was used to perform the computer calculations described in Section 4 is publicly available under https://github.com/benjaminbrueck/codim2_cohomology_SLnZ. Comments on runtime and verifiability of the results can be found in Section 4.4.

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## 2. Definitions

Following Church–Putman [6] (building on ideas of Church–Farb–Putman [8] and Lee–Szczarba [11]), we will construct our partial resolution of Steinberg modules using highly-connected complexes. In this section, we define the relevant complexes.

*2.1. Definition of* $\mathrm{B}_n$ *and* $\mathrm{BA}_n$

We begin by recalling a variant of Maazen's complex of partial bases $\mathrm{B}_n$ [13,14]. Church–Farb–Putman [8] observed that high connectivity of this complex can be used to construct generators for Steinberg modules. We will then recall the definition of a large complex of augmented partial bases, denoted $\mathrm{BA}_n$, that was introduced by Church–Putman to study relations in Steinberg modules.

**Definition 2.1.** Let $\Lambda$ be a PID. A vector $\vec{v} \in \Lambda^n$ is called *primitive* if it spans a summand.

Recall that a vector $\vec{v} \in \Lambda^n$ is primitive if and only if the greatest common divisor of its entries is a unit. If $\Lambda$ is a field, every $\vec{v} \in \Lambda^n \setminus \{0\}$ is primitive.

**Convention 2.2.** Throughout this text, we take $\Lambda$ to be either $\mathbb{Z}$ or $\mathbb{F}_p$. Given a primitive vector $\vec{v}$, the equivalence class $\pm\vec{v}$ is denoted by $v$. Given an equivalence class $v$, we let $\vec{v}$ denote an (arbitrary) choice of representative of $v$. We refer to equivalence classes $v$ as $\pm$-*vectors*. If $\Lambda = \mathbb{Z}$, we also call $v$ a *line*, since in this case there is a bijection between rank-1 summands (lines) in $\mathbb{Z}^n$ and equivalence classes of primitive vectors.

For $\vec{v}_0, \dots, \vec{v}_k \in \Lambda^n$, we write $\langle \vec{v}_0, \dots, \vec{v}_k \rangle_\Lambda$ for the $\Lambda$-span of $\vec{v}_0, \dots, \vec{v}_k$. If $\Lambda = \mathbb{Z}$, we shorten this notation to $\langle \vec{v}_0, \dots, \vec{v}_k \rangle \coloneqq \langle \vec{v}_0, \dots, \vec{v}_k \rangle_\mathbb{Z}$.

**Definition 2.3.** Let $\Lambda$ be $\mathbb{Z}$ or $\mathbb{F}_p$. Let $V_n^\pm(\Lambda)$ be the set

$$V_n^\pm(\Lambda) := \{v \mid \vec{v} \in \Lambda^n \text{ is primitive}\}.$$

A subset

$$\sigma = \{v_0, \dots, v_k\} \subset V_n^\pm(\Lambda)$$

of $(k+1)$ $\pm$-vectors is called

i) a *standard* simplex, if $\langle \vec{v}_0, \dots, \vec{v}_k \rangle_\Lambda$ is a rank-$(k+1)$ summand of $\Lambda^n$ and if $k = n-1$, the determinant of $[\vec{v}_0 \cdots \vec{v}_{n-1}]$ is $\pm 1$;
ii) a *2-additive* simplex, if (possibly after re-indexing)

$$\vec{v}_0 = \pm\vec{v}_1 \pm \vec{v}_2$$

for some choice of signs and $\sigma \setminus \{v_0\}$ is a standard simplex.

Note that the condition in Definition 2.3 Part i) that the determinant of $[\vec{v}_0 \cdots \vec{v}_{n-1}]$ be $\pm 1$ is always true in the case $\Lambda = \mathbb{Z}$ and is only an extra condition in the case $\Lambda = \mathbb{F}_p$.

**Definition 2.4.** Let $\Lambda$ be $\mathbb{Z}$ or $\mathbb{F}_p$ and $n \in \mathbb{N}_0$. The simplicial complexes $\mathrm{B}_n^\pm(\Lambda)$ and $\mathrm{BA}_n^\pm(\Lambda)$ have $V_n^\pm(\Lambda)$ as their vertex set, and

i) the simplices of $\mathrm{B}_n^{\pm}(\Lambda)$ are all standard simplices;
ii) the simplices of $\mathrm{BA}_n^{\pm}(\Lambda)$ are all either standard simplices or 2-additive simplices.

### *2.2. Definition of* $\mathrm{BAA}_n$

We now introduce a larger complex denoted $\mathrm{BAA}_n$. This complex captures relations among the relations in Steinberg modules. The second "A" indicates that we add even more augmentations.

**Definition 2.5.** Let $\Lambda$ be $\mathbb{Z}$ or $\mathbb{F}_p$. A subset

$$\sigma = \{v_0, \dots, v_k\} \subset V_n^{\pm}(\Lambda)$$

of $(k+1)$ $\pm$-vectors is called

i) a *3-additive* simplex, if (possibly after re-indexing)

$$\vec{v}_0 = \pm\vec{v}_1 \pm \vec{v}_2 \pm \vec{v}_3,$$

for some choice of signs and $\sigma \setminus \{v_0\}$ is a standard simplex;

ii) a *double-triple* simplex, if (possibly after re-indexing)

$$\vec{v}_0 = \pm\vec{v}_2 \pm \vec{v}_3, \qquad \vec{v}_1 = \pm\vec{v}_2 \pm \vec{v}_4,$$

for some choice of signs and $\sigma \setminus \{v_0, v_1\}$ is a standard simplex;

iii) a *double-double* simplex, if (possibly after re-indexing)

$$\vec{v}_0 = \pm\vec{v}_2 \pm \vec{v}_3, \qquad \vec{v}_1 = \pm\vec{v}_4 \pm \vec{v}_5,$$

for some choice of signs and $\sigma \setminus \{v_0, v_1\}$ is a standard simplex.

We remark that the name "double-triple" reflects that, after performing a change of basis and re-indexing, a double-triple simplex is represented by vectors of the form

$$\vec{v}_0 = \pm\vec{v}_2 \pm \vec{v}_3, \quad \vec{v}_1 = \pm\vec{v}_2 \pm \vec{v}_3 \pm \vec{v}_4, \quad \vec{v}_2, \quad \vec{v}_3, \quad \dots, \quad \vec{v}_k$$

for some choice of signs and a partial basis $\vec{v}_2, \vec{v}_3, \dots, \vec{v}_k$ of $\Lambda^n$. See also Observation 4.11.

**Definition 2.6.** Let $\Lambda$ be $\mathbb{Z}$ or $\mathbb{F}_p$ and $n \in \mathbb{N}_0$. The simplicial complex $\mathrm{BAA}_n^{\pm}(\Lambda)$ has $V_n^{\pm}(\Lambda)$ as its vertex set. The simplices of $\mathrm{BAA}_n$ are precisely the ones introduced in Definition 2.3 and Definition 2.5.

**Convention 2.7.** When $\Lambda = \mathbb{Z}$, we also write $\mathrm{B}_n$, $\mathrm{BA}_n$ and $\mathrm{BAA}_n$ for $\mathrm{B}_n^{\pm}(\mathbb{Z})$, $\mathrm{BA}_n^{\pm}(\mathbb{Z})$ and $\mathrm{BAA}_n^{\pm}(\mathbb{Z})$, respectively.

### 2.3. Definition of $\widehat{\mathsf{Link}}$, $\mathrm{B}_n^m$, $\mathrm{BA}_n^m$, $\mathrm{BAA}_n^m$ and $\widehat{\mathsf{Link}}^<$

In this subsection, we specialise to the case $\Lambda = \mathbb{Z}$. We will define some subcomplexes of links of simplices. Throughout this section, let $\vec{e}_1, \dots, \vec{e}_k$ denote the standard basis elements of $\mathbb{Z}^k$ and $e_1, \dots, e_k$ the corresponding lines.

**Definition 2.8.** Let $n \in \mathbb{N}_0$ and let $X_n$ denote the complex $\mathrm{B}_n$, $\mathrm{BA}_n$ or $\mathrm{BAA}_n$. Consider a simplex $\sigma = \{w_0, \dots, w_k\}$ of $X$. Then $\widehat{\mathsf{Link}}_{X_n}(\sigma)$ is defined to be the full subcomplex of $\mathsf{Link}_{X_n}(\sigma)$ on the vertex set

$$\{v \in \mathsf{Link}_{X_n}(\sigma) \mid \vec{v} \notin \langle \vec{w}_0, \dots, \vec{w}_k \rangle\}.$$

**Definition 2.9.** Let $m, n \in \mathbb{N}_0$ and let $X_{m+n}$ denote the complex $\mathrm{B}_{m+n}$, $\mathrm{BA}_{m+n}$ or $\mathrm{BAA}_{m+n}$. Consider the standard simplex $\Delta^m = \{e_1, \dots, e_m\}$ contained in $X_{m+n}$. We set

$$X_n^m \coloneqq \widehat{\mathsf{Link}}_{X_{m+n}}(\Delta^m).$$

When $X_{m+n}$ is $\mathrm{B}_{m+n}$, $\mathrm{BA}_{m+n}$ or $\mathrm{BAA}_{m+n}$, respectively, we write $\mathrm{B}_n^m$, $\mathrm{BA}_n^m$ or $\mathrm{BAA}_n^m$, respectively, for $X_n^m$.

The majority of the paper will be devoted to proving the following theorem. It is our main technical tool and the main theorems follow fairly quickly from it.

**Theorem 2.10.** *Let $n \geq 1$. Then $\mathrm{BAA}_n^m$ is $n$-connected.*

For the cases where $n + m \leq 2$, this immediately follows from results of Church–Putman: The complex $\mathrm{BAA}_1^0 = \mathrm{B}_1$ is a single point given by the unique line spanning $\mathbb{Z}$; the complex $\mathrm{BAA}_1^1 = \mathrm{BA}_1^1$ is isomorphic to the Cayley graph of $\mathbb{Z}$ with respect to the generating set $\{e_1\}$, so it is a line [6, Proof of Theorem C', base case]; the complex $\mathrm{BAA}_2^0 = \mathrm{BA}_2$ is contractible as well by [6, Remark 1.4].

In the present article, we prove that Theorem 2.10 also holds if $n + m > 2$. In this case, the following stronger statement is true.

**Theorem 2.11.** *Let $n \geq 1$ and $m + n \geq 3$. Then $\mathrm{BAA}_n^m$ is Cohen–Macaulay of dimension $n + 1$.*

Recall that a simplicial complex is called Cohen–Macaulay of dimension $d$ if it is $d$-dimensional, $(d-1)$-connected, and links of $p$-simplices are $(d-1-p)$-connected. In fact, to deduce the main theorems, it will be sufficient to prove the connectivity result Theorem 2.10 for the case $m = 0$. The complexes $\mathrm{BAA}_n^m$ are "relative versions" of this complex that naturally show up in our inductive proof. The Cohen–Macaulay property is not directly needed for this induction or the main theorems; it however follows rather easily from the steps of our proof.

We need to consider the following subcomplex of links.

**Definition 2.12.** Let $m, n \in \mathbb{N}_0$ and let $X_n^m$ denote the complex $\mathrm{B}_n^m$, $\mathrm{BA}_n^m$ or $\mathrm{BAA}_n^m$. Consider a simplex $\sigma = \{w_0, \dots, w_k\}$ of $X_n^m$. Then $\widehat{\mathsf{Link}}_{X_n^m}(\sigma)$ is defined to be the full subcomplex of $\mathsf{Link}_{X_n^m}(\sigma)$ on the vertex set

$$\{v \in \mathsf{Link}_{X_n^m}(\sigma) \mid \vec{v} \notin \langle \vec{e}_1, \dots, \vec{e}_m, \vec{w}_0, \dots, \vec{w}_k \rangle\}.$$

**Definition 2.13.** Let $R \in \mathbb{Z}_{\geq 1}$, let $X_n^m$ denote the complex $\mathrm{B}_n^m$, $\mathrm{BA}_n^m$ or $\mathrm{BAA}_n^m$ and consider a simplex $\sigma = \{w_0, \dots, w_k\}$ of $X_n^m$. We write $\widehat{\mathsf{Link}}^{<R}_{X_n^m}(\sigma)$ for the full subcomplex of $\widehat{\mathsf{Link}}_{X_n^m}(\sigma)$ on the vertex set

$$\{v \in \widehat{\mathsf{Link}}_{X_n^m}(\sigma) \mid \vec{v} = c_1\vec{e}_1 + \cdots + c_{m+n}\vec{e}_{m+n} \text{ with } |c_{m+n}| < R\}.$$

We will use the notation $\widehat{\mathsf{Link}}^{<}_{X_n^m}(\sigma) = \widehat{\mathsf{Link}}^{<R}_{X_n^m}(\sigma)$ with $R$ equal to the absolute value of the maximum nonzero last coordinate of the vectors in $\sigma$.

## 3. Constructing the retraction

In this section, we present the main technical result that enables us to show that $\mathrm{BAA}_n$ is spherical of dimension $n+1$. To prove it, we build on ideas of Church–Putman [6] and Maazen [13].

**Theorem 3.1.** *Let $n \geq 2$, $m \geq 0$ and $w = \langle \vec{w} \rangle \in \mathrm{BAA}_n^m$ be a vertex. Assume the last coordinate of $\vec{w} \in \mathbb{Z}^{m+n}$ is nonzero. Then, the inclusion*

$$i\colon \widehat{\mathsf{Link}}^{<}_{\mathrm{BAA}_n^m}(w) \hookrightarrow \widehat{\mathsf{Link}}_{\mathrm{BAA}_n^m}(w)$$

*admits a* topological *retraction*

$$r\colon \widehat{\mathsf{Link}}_{\mathrm{BAA}_n^m}(w) \twoheadrightarrow \widehat{\mathsf{Link}}^{<}_{\mathrm{BAA}_n^m}(w).$$

The definition of the retraction map occurring in Theorem 3.1 is inspired by work of Church–Putman [6, Section 4] and Maazen [13, Chapter III]. On vertices, the retraction is given by using the Euclidean algorithm to "reduce" the last coordinate of vertices in the domain "modulo $R$", where $R > 0$ is the last coordinate of a fixed vector $\vec{w} \in \mathbb{Z}^{m+n}$ (compare with Definition 3.9). Church–Putman [6, Section 4.1] demonstrated that this map can be used to prove that the complex of partial frames $\mathrm{B}_n$ is spherical (compare with Proposition 3.14). However, the method does not directly apply to the complex of augmented partial frames $\mathrm{BA}_n$. To show that $\mathrm{BA}_n$ is spherical, Church–Putman [6] need to modify the definition of the retraction. The reason for this additional difficulty in the paper of Church–Putman comes from the following algebraic fact. For an integer $z$, let us denote by $(z \mod R) \in \{0, \dots, R-1\}$ the remainder of division of $z$ by $R > 0$. Let $R > 0$ and $a, b \geq 0$ be nonnegative integers. Then

$$(a \mod R) + (b \mod R) = \begin{cases} (a+b \mod R) \text{ or} \\ (a+b \mod R) + R. \end{cases}$$

It is a consequence of this fact that the *simplicial* retraction maps defined for $\mathrm{B}_n$ do not extend to *simplicial* retraction maps for $\mathrm{BA}_n$. The problem is that, because of the second case in the equation above, the image of a 2-additive simplex might not always span a simplex [6, p. 1020]. To circumvent this problem, Church–Putman subdivide all problematic 2-additive simplices, which they call *carrying* simplices, in the domain of the retraction. They do this by adding a single vertex at the barycentre of every carrying simplex and extending this subdivision to the whole complex. Then, they specify the value that their "modified" retraction takes at these newly introduced vertices and prove that the resulting map is a *topological* retraction (compare with Proposition 3.20). In our construction of the retraction map for $\mathrm{BAA}_n$, i.e. Theorem 3.1, we need to deal additionally with 3-additive simplices. For these simplices, the following algebraic fact is the main source of trouble. Let $R > 0$ and $a, b, c \geq 0$ be nonnegative integers. Consider the integer $a + b + c$ or $a + b - c$. Then

i) $(a \mod R) + (b \mod R) + (c \mod R) = \begin{cases} (a+b+c \mod R), \\ (a+b+c \mod R) + R, \text{ or} \\ (a+b+c \mod R) + 2R \end{cases}$ for the sum $a + b + c$ and

ii) $(a \mod R) + (b \mod R) - (c \mod R) = \begin{cases} (a+b-c \mod R), \\ (a+b-c \mod R) + R, \text{ or} \\ (a+b-c \mod R) - R \end{cases}$ for the sum $a + b - c$.

Similarly to the difficulty for $\mathrm{BA}_n$, the problem is that, because of the second and third case in both item i) and ii), the image of a 3-additive simplex might not always span a simplex. To circumvent this, we subdivide these problematic *carrying* 3-additive simplices by adding a new vertex at their barycentre and, in analogy with Church–Putman, construct a *topological* retraction map for $\mathrm{BAA}_n$. However, since double-double and double-triple simplices might contain multiple problematic 2-additive and 3-additive facets (codimension-1 faces) we face novel difficulties. We not only need to explain how 2-additive and 3-additive simplices are subdivided but also need to describe how higher dimensional simplices can be subdivided in a *compatible* fashion. For the most complicated case, we use computer calculations to show the existence of a retraction and do not make the corresponding subdivisions explicit (see Lemma 3.52 et seq. and Section 4).

We now start working towards the proof of Theorem 3.1 by introducing and fixing some notation. In the next subsection, we discuss the results that Church–Putman obtained for $\mathrm{BA}_n$ in greater detail. In each of the following subsections, we explain how the retraction maps can be defined on and extended over double-double, 3-additive, and double-triple simplices, respectively.

**Convention 3.2.** Throughout this section, we work in the setting of Theorem 3.1. We fix the natural numbers $n \geq 2$, $m \geq 0$. For each line $v$ in $\mathbb{Z}^{m+n}$, we let $\bar{v}$ denote a choice of primitive vector in $v$ with nonnegative last coordinate. Note that the vector $\bar{v}$ is uniquely defined unless its last coordinate is zero. The line $w$ occurring in the statement of Theorem 3.1 is fixed throughout this section and $R$ always denotes the last coordinate of $\bar{w}$, which by assumption satisfies $R > 0$.

The following notions will be frequently used for $\widehat{\mathsf{Link}}_{\mathrm{BAA}_n^m}(w)$ in this section, and for $\mathrm{BAA}_n^m$ in the subsequent section.

**Definition 3.3.** Let $\sigma$ be a simplex of $\widehat{\mathsf{Link}}_{\mathrm{BAA}_n^m}(w)$ or $\mathrm{BAA}_n^m$. We say that $\sigma$ is a *standard*, *2-additive*, *3-additive*, *double-double* or *double-triple* simplex of $\widehat{\mathsf{Link}}_{\mathrm{BAA}_n^m}(w)$ or $\mathrm{BAA}_n^m$ if the *underlying simplex*

$$\sigma * \{e_1, \dots, e_m, w\} \text{ or } \sigma * \{e_1, \dots, e_m\}$$

in $\mathrm{BAA}_{m+n}$ is a simplex of this type.

**Example 3.4.** Let $\vec{v}_1, \vec{v}_2 \in \mathbb{Z}^{m+n}$ such that $\{\vec{v}_1, \vec{v}_2, \vec{e}_1, \dots, \vec{e}_m, \vec{w}\}$ is a partial basis. Then $\{v_1, \langle \vec{v}_1 + \vec{w} \rangle\}$ is a standard simplex in $\mathrm{BAA}_{m+n}$ and $\mathrm{BAA}_n^m$, but it is a 2-additive simplex in $\widehat{\mathsf{Link}}_{\mathrm{BAA}_n^m}(w)$. Similarly, $\{v_1, v_2, \langle \vec{v}_1 + \vec{v}_2 + \vec{e}_1 \rangle\}$ is a 3-additive simplex in $\mathrm{BAA}_n^m$ (and in $\widehat{\mathsf{Link}}_{\mathrm{BAA}_n^m}(w)$) and $\{v_1, \langle \vec{v}_1 + \vec{e}_1 \rangle, \langle \vec{v}_1 + \vec{w} \rangle\}$ is a double-triple simplex in $\widehat{\mathsf{Link}}_{\mathrm{BAA}_n^m}(w)$.

Any simplex $\sigma$ that is not a standard simplex contains a unique minimal face that determines its type. This is the content of the next definition.

**Definition 3.5.** Let $\sigma$ be a simplex of $\widehat{\mathsf{Link}}_{\mathrm{BAA}_n^m}(w)$ or $\mathrm{BAA}_n^m$.

i) The simplex $\sigma$ is called a *minimal* simplex of 2-additive, 3-additive, double-double, or double-triple type (in $\widehat{\mathsf{Link}}_{\mathrm{BAA}_n^m}(w)$ or $\mathrm{BAA}_n^m$) if $\sigma$ is of this type and $\sigma$ does not contain a proper face also of this type.

ii) The *additive core* of a nonstandard simplex $\sigma$ is the unique minimal face of $\sigma$ with the same type as $\sigma$.

**Example 3.6.** Let again $\vec{v}_1, \vec{v}_2 \in \mathbb{Z}^{m+n}$ such that $\{\vec{v}_1, \vec{v}_2, \vec{e}_1, \dots, \vec{e}_m, \vec{w}\}$ is a partial basis. The simplices $\{v_1, v_2, \langle \vec{v}_1 + \vec{v}_2 \rangle\}$ and $\{v_1, \langle \vec{v}_1 + \vec{w} \rangle\}$ are minimal in $\widehat{\mathsf{Link}}_{\mathrm{BAA}_n^m}(w)$. In particular, these simplices form their own additive cores. The simplex $\{v_1, v_2, \langle \vec{v}_1 + \vec{w} \rangle\}$ is not minimal in $\widehat{\mathsf{Link}}_{\mathrm{BAA}_n^m}(w)$. Its additive core is $\{v_1, \langle \vec{v}_1 + \vec{w} \rangle\}$.

The next definition is parallel to [6, Definition 4.9].

**Definition 3.7.** Let $\sigma$ be a simplex of $\widehat{\mathsf{Link}}_{\mathrm{BAA}_n^m}(w)$ or $\mathrm{BAA}_n^m$.

i) We say that $\sigma$ is *external* if the additive core of the underlying simplex in $\mathrm{BAA}_{m+n}$ contains $e_i$ for some $1 \le i \le m$.
ii) We say that $\sigma$ is *$w$-related* if $\sigma$ is a simplex in $\widehat{\mathsf{Link}}_{\mathrm{BAA}_n^m}(w)$ and the additive core of the underlying simplex in $\mathrm{BAA}_{m+n}$ contains $w$.
iii) We say that $\sigma$ is *internal* if the additive core of the underlying simplex in $\mathrm{BAA}_{m+n}$ is contained in $\sigma$.

Note that an internal simplex is neither external nor $w$-related.

**Example 3.8.** Among the simplices in Example 3.4 and Example 3.6, in $\widehat{\mathsf{Link}}_{\mathrm{BAA}_n^m}(w)$,

- $\{v_1, \langle \vec{v}_1 + \vec{w} \rangle\}$ is $w$-related,
- $\{v_1, v_2, \langle \vec{v}_1 + \vec{v}_2 + \vec{e}_1 \rangle\}$ is external,
- $\{v_1, \langle \vec{v}_1 + \vec{e}_1 \rangle, \langle \vec{v}_1 + \vec{w} \rangle\}$ is external and $w$-related,
- $\{v_1, v_2, \langle \vec{v}_1 + \vec{v}_2 \rangle\}$ is internal.

*3.1. Definition on vertices, standard and 2-additive simplices*

We start by defining the retraction maps on the set of vertices $\mathsf{Vert}(\widehat{\mathsf{Link}}_{\mathrm{BAA}_n^m}(w))$ of the simplicial complex $\widehat{\mathsf{Link}}_{\mathrm{BAA}_n^m}(w)$.

**Definition 3.9.** Let $\mathsf{Vert}(X)$ denote the vertex set of the simplicial complex $X$. Then, we define

$$\begin{aligned} r\colon \mathsf{Vert}(\widehat{\mathsf{Link}}_{\mathrm{BAA}_n^m}(w)) &\longrightarrow \mathsf{Vert}(\widehat{\mathsf{Link}}^{<}_{\mathrm{BAA}_n^m}(w)) \\ v &\longmapsto \langle \bar{v} - a\bar{w} \rangle \end{aligned}$$

where $a \in \mathbb{Z}$ is chosen so that $\bar{v} - a\bar{w}$ has last coordinate in the interval $[0, R)$.

The constant $a \in \mathbb{Z}$ in Definition 3.9 is determined by the Euclidean algorithm. We note that, although the vector $\bar{v}$ is not uniquely determined by $v$ if its last coordinate is zero, the line $r(v)$ is still uniquely determined because $r(v) = v$. More generally, we observe that $r(v) = v$ if the last coordinate of $\bar{v}$ is contained in $[0, R)$.

**Convention 3.10.** Consider $v \in \mathsf{Vert}(\widehat{\mathsf{Link}}_{\mathrm{BAA}_n^m}(w))$, then the line $r(v) \in \mathsf{Vert}(\widehat{\mathsf{Link}}^{<}_{\mathrm{BAA}_n^m}(w))$ is spanned by a vector $\overline{r(v)}$. Recall that this vector is *not well-defined* if the last coordinate of $\overline{r(v)}$ is zero (see Convention 3.2). We use the following notational convention for $\overline{r(v)}$. Let $\bar{v}$ be a vector representing $v$ that has nonnegative last coordinate $aR + b$ where $a \in \mathbb{Z}$ and $b \in [0, R)$. In this situation, $\overline{r(v)}$ always denotes the vector $\bar{v} - a\bar{w}$.

Before we discuss the effect of this map on standard and 2-additive simplices, we record some facts that will help us to calculate the value of $r$ on certain vertices. The following observation is elementary but useful.

**Observation 3.11.** Let $\{\bar{v}_0, \bar{v}_1, \bar{v}_2, \dots, \bar{v}_p, \bar{w}\}$ be a partial basis for $\mathbb{Z}^{m+n}$. Then, if we replace any element $\bar{v}_i$ by $\bar{v}_i + a\bar{w}$ for any $a \in \mathbb{Z}$, the result is still a partial basis for $\mathbb{Z}^{m+n}$ and spans the same summand. In particular, $\bar{v}_i + a\bar{w}$ is necessarily primitive.

The next lemma describes some properties of the map $r$. Its proof is easy and left to the reader.

**Lemma 3.12.** *Let $v \in \mathsf{Vert}(\widehat{\mathsf{Link}}_{\mathrm{BAA}_n^m}(w))$ and let $\epsilon_1, \epsilon_2 \in \{-1, +1\}$ be two signs. Then, the map $r$ introduced in Definition 3.9 has the following properties.*

*i) If $u \in \mathsf{Vert}(\widehat{\mathsf{Link}}_{\mathrm{BAA}_n^m}(w)) \sqcup \{e_1, \dots, e_m\}$ is a line such that the last coordinate of $\bar{u}$ is zero, then*

$$r(\langle \epsilon_1 \bar{v} + \epsilon_2 \bar{u} \rangle) = \langle \epsilon_1 \overline{r(v)} + \epsilon_2 \bar{u} \rangle.$$

*ii) It holds that*

$$r(\langle \epsilon_1 \bar{v} + \epsilon_2 \bar{w} \rangle) = r(v), \text{ if } \epsilon_1 = \epsilon_2,$$

*and, if $\epsilon_1 \neq \epsilon_2$, then*

$$r(\langle \epsilon_1 \bar{v} + \epsilon_2 \bar{w} \rangle) = \begin{cases} r(v), & \text{if the last coordinate of } \bar{v} \text{ is in } \{0\} \sqcup [R, \infty), \\ \langle \epsilon_1 \overline{r(v)} + \epsilon_2 \bar{w} \rangle = \langle \bar{w} - \overline{r(v)} \rangle = \langle \bar{w} - \bar{v} \rangle, & \text{if the last coordinate of } \\ \quad \bar{v} \text{ is in } (0, R). \end{cases}$$

*iii) Given two vertices $v_1, v_2 \in \mathsf{Vert}(\widehat{\mathsf{Link}}_{\mathrm{BAA}_n^m}(w))$. Let $a_i R + b_i$ for $a_i \geq 0$ and $b_i \in [0, R)$ denote the last coordinate of $\bar{v}_i$. Then,*

$$r(\langle \bar{v}_1 + \bar{v}_2 \rangle) = \begin{cases} \langle \overline{r(v_1)} + \overline{r(v_2)} \rangle, & \text{if } b_1 + b_2 \in [0, R), \\ \langle \overline{r(v_1)} + \overline{r(v_2)} - \bar{w} \rangle, & \text{if } b_1 + b_2 \in [R, 2R). \end{cases}$$

We now discuss the effect of the retraction on standard simplices. This has been studied by Church–Putman in Section 4.1 of [6] as part of their proof that $\mathrm{B}_n^m$ is a Cohen–Macaulay complex of dimension $n - 1$ [6, Theorem 4.2].

**Definition 3.13.** Let $\widehat{\mathsf{Link}}_{\mathrm{B}_n^m}(w)$ and $\widehat{\mathsf{Link}}^{<}_{\mathrm{B}_n^m}(w)$ denote the subcomplexes of $\widehat{\mathsf{Link}}_{\mathrm{BAA}_n^m}(w)$ and $\widehat{\mathsf{Link}}^{<}_{\mathrm{BAA}_n^m}(w)$, respectively, that consists of all standard simplices in the sense of Definition 3.3.

Note that the vertex sets of $\widehat{\mathsf{Link}}_{\mathrm{B}_n^m}(w)$ and $\widehat{\mathsf{Link}}_{\mathrm{BAA}_n^m}(w)$ are equal. The following result is a case of Church–Putman [6, Lemma 4.5].

**Proposition 3.14** *([6, Lemma 4.5]). Let $n, m \geq 0$. Then, Definition 3.9 induces a simplicial map*

$$r\colon \widehat{\mathsf{Link}}_{\mathrm{B}_n^m}(w) \twoheadrightarrow \widehat{\mathsf{Link}}^{<}_{\mathrm{B}_n^m}(w) \hookrightarrow \widehat{\mathsf{Link}}^{<}_{\mathrm{BAA}_n^m}(w)$$

*that restricts to the inclusion on the subcomplex $\widehat{\mathsf{Link}}^{<}_{\mathrm{B}_n^m}(w)$ of $\widehat{\mathsf{Link}}_{\mathrm{B}_n^m}(w)$. In particular, $\widehat{\mathsf{Link}}^{<}_{\mathrm{B}_n^m}(w)$ is a simplicial retract of $\widehat{\mathsf{Link}}_{\mathrm{B}_n^m}(w)$.*

We now explain how Church–Putman extended the *simplicial* retraction of $\widehat{\mathsf{Link}}_{\mathrm{B}_n^m}(w)$ onto $\widehat{\mathsf{Link}}^{<}_{\mathrm{B}_n^m}(w)$ over 2-additive simplices to a *topological* retraction between the following two simplicial complexes.

**Definition 3.15.** Let $\widehat{\mathsf{Link}}_{\mathrm{BA}_n^m}(w)$ and $\widehat{\mathsf{Link}}^{<}_{\mathrm{BA}_n^m}(w)$ denote the subcomplexes of $\widehat{\mathsf{Link}}_{\mathrm{BAA}_n^m}(w)$ and $\widehat{\mathsf{Link}}^{<}_{\mathrm{BAA}_n^m}(w)$ respectively that consist of all standard and 2-additive simplices in the sense of Definition 3.3.

The following definition captures the reason why Definition 3.9 does not induce a *simplicial* retraction $r\colon \widehat{\mathsf{Link}}_{\mathrm{BA}_n^m}(w) \twoheadrightarrow \widehat{\mathsf{Link}}^{<}_{\mathrm{BA}_n^m}(w)$ as one might initially hope.

**Definition 3.16.** Let $\sigma = \tau_1 * \tau_2$ be a 2-additive simplex in $\widehat{\mathsf{Link}}_{\mathrm{BAA}_n^m}(w)$, where $\tau_1$ is a minimal 2-additive simplex and $\tau_2$ is a standard simplex. $\sigma$ is called *carrying* if one of the following equivalent conditions holds

i) The set $r(\tau_1)$ does not span a simplex in $\widehat{\mathsf{Link}}^{<}_{\mathrm{BA}_n^m}(w)$. (Note the $\mathrm{BA}_n^m$ subscript.)
ii) $\tau_1 = \{v_0, v_1, v_2 = \langle \bar{v}_0 + \bar{v}_1 \rangle\}$ is internally 2-additive with $b_0 + b_1 \in [R, 2R)$, where $a_i R + b_i$ with $a_i \geq 0$ and $b_i \in [0, R)$ is the last coordinate of $\bar{v}_i$.

We remark that in Condition ii), $v_2$ is the unique vertex in $\tau_1$ with last coordinate of $\bar{v}_i$ maximal and $r(\tau_1) = \{r(v_0), r(v_1), r(v_2) = \langle \overline{r(v_0)} + \overline{r(v_1)} - \bar{w} \rangle\}$ by Part iii) of Lemma 3.12. The equivalence of i) and ii) in Definition 3.16 follows from [6, §4.4. Claim 1-4 and the discussion on p. 1022].

**Remark 3.17.** In the first condition of Definition 3.16, we highlighted the $\mathrm{BA}_n^m$ subscript because $r(\tau_1)$ does form a $w$-related 3-additive simplex in $\widehat{\mathsf{Link}}^{<}_{\mathrm{BAA}_n^m}(w)$, as visible in the second condition.

**Example 3.18.** Let $\bar{w} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 10 \end{bmatrix}$, $\bar{v}_0 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 9 \end{bmatrix}$, and $\bar{v}_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 6 \end{bmatrix}$. Then $\{v_0, v_1, v_2 = \langle \bar{v}_0 + \bar{v}_1 \rangle\}$ is a 2-additive simplex in $\widehat{\mathsf{Link}}_{\mathrm{BAA}_3^1}(w)$. However,

$$r(\{v_0, v_1, v_2\}) = \{v_0, v_1, r(v_2) = \langle \bar{v}_1 + \bar{v}_2 - \bar{w} \rangle\}$$

is not a simplex of $\widehat{\mathsf{Link}}^{<}_{\mathrm{BA}_3^1}(w)$ and therefore $\{v_0, v_1, v_2\}$ is an example of a carrying 2-additive simplex.

To circumvent this problem and, instead, construct a *topological* retraction

$$r\colon \widehat{\mathsf{Link}}_{\mathrm{BA}_n^m}(w) \twoheadrightarrow \widehat{\mathsf{Link}}^{<}_{\mathrm{BA}_n^m}(w),$$

Church–Putman modify the definition of $r$ on all carrying 2-additive simplices. To do this, they pass to the following subdivision of $\widehat{\mathsf{Link}}_{\mathrm{BA}_n^m}(w)$.

**Definition 3.19.** Let $\mathrm{sd}(\widehat{\mathsf{Link}}_{\mathrm{BA}_n^m}(w))$ denote the coarsest subdivision of $\widehat{\mathsf{Link}}_{\mathrm{BA}_n^m}(w)$, where every carrying minimal 2-additive simplex $\tau_1 = \{v_0, v_1, v_2 = \langle \bar{v}_0 + \bar{v}_1 \rangle\}$ is subdivided by inserting a single vertex $t(\tau_1)$ at the barycentre of $\tau_1$.

Concretely, $\mathrm{sd}(\widehat{\mathsf{Link}}_{\mathrm{BA}_n^m}(w))$ in Definition 3.19 is constructed as follows: Let $\sigma = \tau_1 * \tau_2$ be a 2-additive simplex of $\widehat{\mathsf{Link}}_{\mathrm{BA}_n^m}(w)$, where $\tau_1 = \{v_0, v_1, v_2 = \langle \bar{v}_0 + \bar{v}_1 \rangle\}$ is a carrying minimal 2-additive simplex and $\tau_2$ is standard. Then, when passing from $\widehat{\mathsf{Link}}_{\mathrm{BA}_n^m}(w)$ to $\mathrm{sd}(\widehat{\mathsf{Link}}_{\mathrm{BA}_n^m}(w))$, each such simplex $\sigma$ is replaced by $\mathrm{sd}(\sigma)$, its subdivision into the three simplices $\{v_0, \dots, \hat{v}_i, \dots, v_2, t(\tau_1)\} * \tau_2$ for $i = 0, 1, 2$. Here the notation $\hat{v}_i$ means $v_i$ is omitted. Note that $\widehat{\mathsf{Link}}_{\mathrm{B}_n^m}(w)$ and $\widehat{\mathsf{Link}}^{<}_{\mathrm{BA}_n^m}(w)$ are subcomplexes of $\mathrm{sd}(\widehat{\mathsf{Link}}_{\mathrm{BA}_n^m}(w))$.

The following is the main technical result of Church–Putman [6], and the key input for their proof that $\mathrm{BA}_n^m$ is a Cohen–Macaulay complex of dimension $n$ [6, Theorem C'].

**Proposition 3.20** *([6, Proposition 4.17.]). Let $n \geq 2$ and $m \geq 0$. Then, the simplicial map constructed in Proposition 3.14*

$$r\colon \widehat{\mathsf{Link}}_{\mathrm{B}_n^m}(w) \twoheadrightarrow \widehat{\mathsf{Link}}^{<}_{\mathrm{B}_n^m}(w) \hookrightarrow \widehat{\mathsf{Link}}^{<}_{\mathrm{BAA}_n^m}(w)$$

*extends to a simplicial map*

$$r\colon\ \mathrm{sd}(\widehat{\mathsf{Link}}_{\mathrm{BA}_n^m}(w)) \twoheadrightarrow \widehat{\mathsf{Link}}^{<}_{\mathrm{BA}_n^m}(w) \hookrightarrow \widehat{\mathsf{Link}}^{<}_{\mathrm{BAA}_n^m}(w)$$

*that restricts to the inclusion on the subcomplex* $\widehat{\mathsf{Link}}^{<}_{\mathrm{BA}_n^m}(w)$ *of* $\mathrm{sd}(\widehat{\mathsf{Link}}_{\mathrm{BA}_n^m}(w))$*. The value of* $r$ *on the barycentre* $t(\tau_1)$ *of a carrying minimal* 2*-additive simplex* $\tau_1 = \{v_0, v_1, v_2 = \langle \bar{v}_0 + \bar{v}_1 \rangle\}$ *is defined by the formula*

$$r(t(\tau_1)) = \langle \overline{r(v_l)} - \bar{w} \rangle$$

*where* $l \in \{0,1\}$ *is arbitrarily chosen, i.e.* $v_l$ *is one of the two lines in* $\tau_1$ *with the property that the last coordinate of* $\bar{v}_l$ *is not maximal. In particular, it follows that* $\widehat{\mathsf{Link}}^{<}_{\mathrm{BA}_n^m}(w)$ *is a* topological *retract of* $\widehat{\mathsf{Link}}_{\mathrm{BA}_n^m}(w) \cong \mathrm{sd}(\widehat{\mathsf{Link}}_{\mathrm{BA}_n^m}(w))$*.*

This completes our discussion of the definition of $r$ on vertices, and standard and 2-additive simplices. We close this subsection by presenting a proof of the following lemma. It will frequently be used to reduce the question of whether the map $r$ extends over a simplex $\sigma = \tau_1 * \tau_2$ with additive core $\tau_1$ to the question whether $r$ extends over the additive core $\tau_1$. To shorten notation, we write $\langle \nu \rangle := \langle \vec{v} \mid \langle \vec{v} \rangle \in \nu \rangle$ for the $\mathbb{Z}$-linear span of a set of lines $\nu$ in $\mathbb{Z}^{m+n}$.

**Lemma 3.21.** *Let* $\sigma = \tau_1 * \tau_2$ *be a simplex of* $\widehat{\mathsf{Link}}_{\mathrm{BAA}_n^m}(w)$ *such that the additive core of* $\sigma$ *is contained in* $\tau_1$ *and* $\tau_2$ *is a standard simplex. Let* $\nu$ *be a set of lines in* $\mathbb{Z}^{m+n}$ *such that* $\langle \nu \rangle \subseteq \langle \tau_1 \cup \{e_1, \dots, e_m, w\} \rangle$*. If* $\nu$ *spans a simplex in* $\widehat{\mathsf{Link}}^{<}_{\mathrm{BAA}_n^m}(w)$*, then* $\nu * r(\tau_2)$ *spans a simplex of the same type.*

**Proof.** Since $\nu$ is a simplex in $\widehat{\mathsf{Link}}^{<}_{\mathrm{BAA}_n^m}(w)$, there exists a (not necessarily unique) maximal standard simplex $\nu' \subseteq \nu$ that is contained in $\nu$. Observe that $\langle \nu' \rangle \oplus \langle \{e_1, \dots, e_m, w\} \rangle = \langle \nu \cup \{e_1, \dots, e_m, w\} \rangle$ is a direct summand of $\mathbb{Z}^{m+n}$. Since $\tau_1 * \tau_2$ is a simplex in $\widehat{\mathsf{Link}}_{\mathrm{BAA}_n^m}(w)$ and the additive core of it is contained in $\tau_1$, it follows that $\langle \tau_1 \cup \{e_1, \dots, e_m, w\} \rangle \oplus \langle \tau_2 \rangle$ is a direct summand of $\mathbb{Z}^{m+n}$. The assumption that $\langle \nu \rangle \subseteq \langle \tau_1 \cup \{e_1, \dots, e_m, w\} \rangle$ implies that $\langle \nu' \rangle \oplus \langle \{e_1, \dots, e_m, w\} \rangle \subseteq \langle \tau_1 \cup \{e_1, \dots, e_m, w\} \rangle$. We conclude that $\langle \nu' \rangle \oplus \langle \{e_1, \dots, e_m, w\} \rangle \oplus \langle \tau_2 \rangle$ is a direct summand of $\mathbb{Z}^{m+n}$ as well, using e.g. [6, Lemma 2.6]. Proposition 3.14 implies that $r(\tau_2)$ is a standard simplex and Observation 3.11 yields

$$\langle \{e_1, \dots, e_m, w\} \rangle \oplus \langle \tau_2 \rangle = \langle \{e_1, \dots, e_m, w\} \rangle \oplus \langle r(\tau_2) \rangle.$$

Hence, $\langle \nu' \rangle \oplus \langle \{e_1, \dots, e_m, w\} \rangle \oplus \langle r(\tau_2) \rangle$ is a direct summand of $\mathbb{Z}^{m+n}$. It follows that $\nu' * r(\tau_2)$ is a standard simplex in $\widehat{\mathsf{Link}}^{<}_{\mathrm{BAA}_n^m}(w)$. The fact that $\nu$ is a simplex in $\widehat{\mathsf{Link}}^{<}_{\mathrm{BAA}_n^m}(w)$ means that the vertices in $\nu \setminus \nu'$ can in an appropriate way be written as sums of the vectors spanning the lines $\nu' \cup \{e_1, \dots, e_m, w\}$. Therefore, $\nu * r(\tau_2)$ spans a simplex of the same type as $\nu$ in $\widehat{\mathsf{Link}}^{<}_{\mathrm{BAA}_n^m}(w)$. □

*3.2. Extending over double-double simplices*

The goal of this subsection is to extend the map

$$r\colon \operatorname{sd}(\widehat{\mathsf{Link}}_{\mathrm{BA}_n^m}(w)) \to \widehat{\mathsf{Link}}^{<}_{\mathrm{BAA}_n^m}(w)$$

defined in Proposition 3.20 over all double-double simplices. For this, we need to study minimal double-double simplices in the sense of Definition 3.5.

**Observation 3.22.** A minimal double-double simplex $\tau_1$ in $\widehat{\mathsf{Link}}_{\mathrm{BAA}_n^m}(w)$ is the join $\tau_{1,1} * \tau_{1,2}$ of two minimal 2-additive simplices in $\widehat{\mathsf{Link}}_{\mathrm{BAA}_n^m}(w)$. In particular, any facet of $\tau_1$ is 2-additive.

If one of the two minimal 2-additive simplices in a double-double simplex $\sigma$ in $\widehat{\mathsf{Link}}_{\mathrm{BAA}_n^m}(w)$ is carrying, then the set $r(\sigma)$ might or might not span a simplex in $\widehat{\mathsf{Link}}_{\mathrm{BAA}_n^m}(w)$. This is illustrated in the next example.

**Example 3.23.** Consider a minimal double-double simplex $\tau_1$ in $\widehat{\mathsf{Link}}_{\mathrm{BAA}_n^m}(w)$ of the form

$$\tau_1 = \{v_0, v_1, v_2 = \langle \bar{v}_0 + \bar{v}_1 \rangle, v_3, \langle \bar{v}_3 + \epsilon \cdot \bar{w} \rangle\}$$

for $\epsilon \in \{+1, -1\}$. Assume that $\{v_0, v_1, v_2 = \langle \bar{v}_0 + \bar{v}_1 \rangle\}$ is carrying. If $\epsilon = +1$, then Lemma 3.12 implies that $r(\tau_1) = \{r(v_0), r(v_1), \langle \overline{r(v_0)} + \overline{r(v_1)} - \bar{w} \rangle, r(v_3)\}$, which spans a $w$-related 3-additive simplex in $\widehat{\mathsf{Link}}_{\mathrm{BAA}_n^m}(w)$. If $\epsilon = -1$ and the last coordinate of $\bar{v}_3$ is contained in $(0, R)$, then Lemma 3.12 implies that $r(\tau_1) = \{r(v_0), r(v_1), \langle \overline{r(v_0)} + \overline{r(v_1)} - \bar{w} \rangle, r(v_3), \langle \bar{w} - \overline{r(v_3)} \rangle\}$, which does not define a simplex in $\widehat{\mathsf{Link}}_{\mathrm{BAA}_n^m}(w)$.

Because we decided to construct the retraction maps $r$ for $\mathrm{BAA}_n^m$ occurring in Theorem 3.1 as extensions of the retraction maps that Church–Putman defined for $\mathrm{BA}_n^m$ (compare with Proposition 3.20), we nevertheless subdivide *every* minimal double-double simplex that contains a carrying 2-additive face. This leads us to the following definition.

**Definition 3.24.** Let $\sigma = \tau_1 * \tau_2$ be a double-double simplex of $\widehat{\mathsf{Link}}_{\mathrm{BAA}_n^m}(w)$, where $\tau_1 = \tau_{1,1} * \tau_{1,2}$ is a minimal double-double simplex and $\tau_2$ is a standard simplex. Then $\sigma$ is called *carrying* if one of the following equivalent conditions holds.

i) $\tau_1$ has a carrying facet.
ii) One of the two 2-additive simplices $\tau_{1,1}$ or $\tau_{1,2}$ is carrying in the sense of Definition 3.16.

Since any carrying 2-additive simplex has been subdivided in $\operatorname{sd}(\widehat{\mathsf{Link}}_{\mathrm{BA}_n^m}(w))$, we need to subdivide every carrying double-double simplex in a compatible fashion. This is done in the next definition.

**Definition 3.25.** Let $\widehat{\mathsf{Link}}_{\mathrm{DD}_n^m}(w)$ and $\widehat{\mathsf{Link}}^{<}_{\mathrm{DD}_n^m}(w)$ denote the subcomplexes of $\widehat{\mathsf{Link}}_{\mathrm{BAA}_n^m}(w)$ and $\widehat{\mathsf{Link}}^{<}_{\mathrm{BAA}_n^m}(w)$, respectively, consisting of all simplices that are standard, 2-additive, or of type double-double in the sense of Definition 3.3. Let $\mathrm{sd}(\widehat{\mathsf{Link}}_{\mathrm{DD}_n^m}(w))$ denote the coarsest subdivision of $\widehat{\mathsf{Link}}_{\mathrm{DD}_n^m}(w)$ that contains $\mathrm{sd}(\widehat{\mathsf{Link}}_{\mathrm{BA}_n^m}(w))$ as a subcomplex.

Concretely, $\mathrm{sd}(\widehat{\mathsf{Link}}_{\mathrm{DD}_n^m}(w))$ in Definition 3.25 is constructed as follows: Let $\sigma = \tau_1 * \tau_2$ be a double-double simplex of $\widehat{\mathsf{Link}}_{\mathrm{DD}_n^m}(w)$, where $\tau_1 = \tau_{1,1} * \tau_{1,2}$ is a carrying minimal double-double simplex and $\tau_2$ is standard. Then, when passing from $\widehat{\mathsf{Link}}_{\mathrm{DD}_n^m}(w)$ to $\mathrm{sd}(\widehat{\mathsf{Link}}_{\mathrm{DD}_n^m}(w))$, each such simplex $\sigma$ is replaced by the simplicial join

$$\mathrm{sd}(\sigma) = \mathrm{sd}(\tau_{1,1}) * \mathrm{sd}(\tau_{1,2}) * \tau_2,$$

where $\mathrm{sd}(\tau_{1,i})$ for $i \in \{1,2\}$ denotes the subdivision of the 2-additive simplex $\tau_{1,i}$ (see Definition 3.19) if it is carrying, and $\mathrm{sd}(\tau_{1,i}) = \tau_{1,i}$ if it is not carrying. Note that $\widehat{\mathsf{Link}}^{<}_{\mathrm{DD}_n^m}(w)$ and $\mathrm{sd}(\widehat{\mathsf{Link}}_{\mathrm{BA}_n^m}(w))$ are subcomplexes of $\mathrm{sd}(\widehat{\mathsf{Link}}_{\mathrm{DD}_n^m}(w))$.

The main result of this subsection is the following proposition.

**Proposition 3.26.** *The simplicial map constructed in Proposition 3.20*

$$r\colon\ \mathrm{sd}(\widehat{\mathsf{Link}}_{\mathrm{BA}_n^m}(w)) \to \widehat{\mathsf{Link}}^{<}_{\mathrm{BAA}_n^m}(w)$$

*extends to a simplicial map*

$$r\colon\ \mathrm{sd}(\widehat{\mathsf{Link}}_{\mathrm{DD}_n^m}(w)) \to \widehat{\mathsf{Link}}^{<}_{\mathrm{BAA}_n^m}(w)$$

*that restricts to the inclusion*

$$\widehat{\mathsf{Link}}^{<}_{\mathrm{DD}_n^m}(w) \hookrightarrow \widehat{\mathsf{Link}}^{<}_{\mathrm{BAA}_n^m}(w)$$

*on the subcomplex* $\widehat{\mathsf{Link}}^{<}_{\mathrm{DD}_n^m}(w)$ *of* $\mathrm{sd}(\widehat{\mathsf{Link}}_{\mathrm{DD}_n^m}(w))$.

**Proof.** Our goal is to check that $r$ is simplicial on all (possibly subdivided) double-double simplices. Let $\sigma = \tau_1 * \tau_2$ be a double-double simplex of $\widehat{\mathsf{Link}}_{\mathrm{DD}_n^m}(w)$, where $\tau_1 = \tau_{1,1} * \tau_{1,2}$ is a minimal double-double simplex and $\tau_2$ is standard. We need to argue that $r$ extends over its subdivision $\mathrm{sd}(\sigma) = \mathrm{sd}(\tau_1) * \tau_2$. We will show that if $\alpha \subseteq \mathrm{sd}(\tau_1) = \mathrm{sd}(\tau_{1,1}) * \mathrm{sd}(\tau_{1,2})$ is a simplex, then $r(\alpha)$ is a simplex in $\widehat{\mathsf{Link}}^{<}_{\mathrm{BAA}_n^m}(w)$. An application of Lemma 3.21 for $\sigma = \tau_1 * \tau_2$ and $\nu = r(\alpha)$ then yields that $r(\alpha * \tau_2) = r(\alpha) * r(\tau_2)$ is a simplex of $\widehat{\mathsf{Link}}^{<}_{\mathrm{BAA}_n^m}(w)$ as well and the claim follows. The use of Lemma 3.21 is justified because the definition of $r$ on carrying 2-additive simplices (compare with Proposition 3.20) implies that $\nu = r(\alpha) \subset \langle \tau_1, e_1, \ldots, e_m, w \rangle$.

Firstly, assume that $\tau_1$ is not carrying. Then, $\mathrm{sd}(\tau_{1,1}) = \tau_{1,1}$ and $\mathrm{sd}(\tau_{1,2}) = \tau_{1,2}$, i.e. neither of the two 2-additive simplices $\tau_{1,1}$ and $\tau_{1,2}$ is subdivided. We will show that $r$ extends over $\tau_1 = \alpha$. Because $\tau_1$ is a double-double simplex, it is impossible that $\tau_{1,1}$ and $\tau_{1,2}$ are both $w$-related or that $\tau_{1,1}$ and $\tau_{1,2}$ are both externally 2-additive involving the same $e_j$. The following is verified in the proof of [6, Section 4.4, Claim 2-4]. If $\tau_{1,i}$ is . . .

- $w$-related 2-additive, then $r(\tau_{1,i})$ is a $w$-related 2-additive or standard simplex,
- externally 2-additive involving $e_j$, then $r(\tau_{1,i})$ is externally 2-additive involving $e_j$,
- internally 2-additive, then $r(\tau_{1,i})$ is internally 2-additive.

This implies that it also is impossible that the simplices $r(\tau_{1,1})$ and $r(\tau_{1,2})$ are both $w$-related 2-additive or that $r(\tau_{1,1})$ and $r(\tau_{1,2})$ are both externally 2-additive involving the same $e_j$. We now compute and compare the two summands $\langle r(\tau_{1,i})\rangle$ of $\mathbb{Z}^{m+n}$ obtained for $i \in \{1,2\}$. Let $\eta_{1,i} \subset \tau_{1,i}$ be a maximal standard simplex for $i \in \{1,2\}$. Then $\eta_1 = \eta_{1,1} * \eta_{1,2}$ is a maximal standard simplex in $\tau_1$. By Observation 3.11, it holds that $r(\eta_1) = r(\eta_{1,1}) * r(\eta_{1,2})$ is a standard simplex in $\widehat{\mathsf{Link}}^{<}_{\mathrm{BAA}^m_n}(w)$ of the same dimension as $\eta_1$. In particular, $\langle r(\eta_{1,1})\rangle \oplus \langle r(\eta_{1,2})\rangle \oplus w \oplus \langle \bar{e}_1, \dots, \bar{e}_m\rangle$ is a direct summand of $\mathbb{Z}^{m+n}$. The summand $\langle r(\tau_{1,i})\rangle$ is equal to . . .

- $\langle r(\eta_{1,i})\rangle \oplus w$ if $r(\tau_{1,i})$ is $w$-related 2-additive,
- $\langle r(\eta_{1,i})\rangle \oplus e_j$ if $r(\tau_{1,i})$ is externally additive involving $e_j$, and
- $\langle r(\eta_{1,i})\rangle$ if $r(\tau_{1,i})$ is standard or internally 2-additive.

Hence, the previous conclusion implies that the two summands $\langle r(\tau_{1,1})\rangle$ and $\langle r(\tau_{1,2})\rangle$ of $\mathbb{Z}^{m+n}$ intersect trivially. Since at least one of the two simplices $r(\tau_{1,1})$ and $r(\tau_{1,2})$ is 2-additive and the other one is either a standard simplex or 2-additive as well, we conclude that $r(\tau_1) = r(\tau_{1,1}) * r(\tau_{1,2})$ spans a 2-additive or double-double simplex.

Secondly, assume that $\tau_1$ is carrying such that $\tau_{1,1}$ is a carrying 2-additive simplex and $\tau_{1,2}$ is carrying or not. Then, $\tau_{1,1} = \{v_0, v_1, v_2 = \langle \bar{v}_0 + \bar{v}_1\rangle\}$ and $r(\mathrm{sd}(\tau_{1,1}))$ consists of the following three simplices where we write $\{l, l'\} = \{0,1\}$,

- $\{r(v_l), r(v_{l'}), r(t(\tau_{1,1})) = \langle \overline{r(v_l)} - \bar{w}\rangle\}$, which is $w$-related 2-additive,
- $\{r(v_l), r(t(\tau_{1,1})) = \langle \overline{r(v_l)} - \bar{w}\rangle, r(v_2) = \langle \overline{r(v_l)} + \overline{r(v_{l'})} - \bar{w}\rangle\}$, which is $w$-related 2-additive,
- $\{r(t(\tau_{1,1})) = \langle \overline{r(v_l)} - \bar{w}\rangle, r(v_{l'}), r(v_2) = \langle \overline{r(v_l)} + \overline{r(v_{l'})} - \bar{w}\rangle\}$, which is internally 2-additive.

Let $\alpha_1 \subset \mathrm{sd}(\tau_{1,1})$ and $\alpha_2 \subset \mathrm{sd}(\tau_{1,2})$ be simplices of maximal dimension. We show that $r$ extends over $\alpha = \alpha_1 * \alpha_2$. If it is not the case that both $r(\alpha_1)$ and $r(\alpha_2)$ are $w$-related 2-additive, we can argue as in the first part to see that the two summands $\langle r(\alpha_1)\rangle$ and $\langle r(\alpha_2)\rangle$ of $\mathbb{Z}^{m+n}$ intersect trivially and conclude that $r(\alpha) = r(\alpha_1) * r(\alpha_2)$ spans a 2-

additive or double-double simplex. If both $r(\alpha_1)$ and $r(\alpha_2)$ are $w$-related 2-additive, then they are of the form $\{v, \langle \bar{v} \pm \bar{w}\rangle, v'\}$ and $\{u, \langle \bar{u} \pm \bar{w}\rangle, u'\}$ where $\{v, v', u, u'\}$ is a standard simplex in $\widehat{\mathsf{Link}}^{<}_{\mathrm{BAA}_n^m}(w)$ or of the form $\{v, \langle \bar{v}\pm\bar{w}\rangle, v'\}$ and $\{u, \langle \bar{u}\pm\bar{w}\rangle\}$ where $\{v, v', u\}$ is a standard simplex in $\widehat{\mathsf{Link}}^{<}_{\mathrm{BAA}_n^m}(w)$. In both cases it follows that $r(\alpha_1 * \alpha_2) = r(\alpha_1) * r(\alpha_2)$ is a $w$-related double-triple simplex. □

### 3.3. Extending over 3-additive simplices

The goal of this subsection is to extend the map

$$r\colon \operatorname{sd}(\widehat{\mathsf{Link}}_{\mathrm{DD}_n^m}(w)) \to \widehat{\mathsf{Link}}^{<}_{\mathrm{BAA}_n^m}(w)$$

defined in the previous subsection over all 3-additive simplices. For this, we need to study minimal 3-additive simplices in the sense of Definition 3.5.

**Observation 3.27.** A 3-additive simplex of $\widehat{\mathsf{Link}}_{\mathrm{BAA}_n^m}(w)$ is minimal if all of its facets are standard.

As in the 2-additive case, studied by Church–Putman [6], the difficulty is to extend $r$ over carrying simplices; that is 3-additive simplices in $\widehat{\mathsf{Link}}_{\mathrm{BAA}_n^m}(w)$ whose image under $r$ is not a simplex in the target $\widehat{\mathsf{Link}}^{<}_{\mathrm{BAA}_n^m}(w)$.

**Definition 3.28.** Let $\sigma = \tau_1 * \tau_2$ be a 3-additive simplex in $\widehat{\mathsf{Link}}_{\mathrm{BAA}_n^m}(w)$, where $\tau_1$ is a minimal 3-additive simplex and $\tau_2$ is a standard simplex. $\sigma$ is called *carrying* if the set $r(\tau_1)$ does not span a simplex of $\widehat{\mathsf{Link}}^{<}_{\mathrm{BAA}_n^m}(w)$.

As part of our discussion in this subsection, we will find the following characterisation of carrying 3-additive simplices.

**Lemma 3.29.** *Let $\sigma = \tau_1 * \tau_2$ be a 3-additive simplex of $\widehat{\mathsf{Link}}_{\mathrm{BAA}_n^m}(w)$ such that $\tau_1$ is minimal 3-additive and $\tau_2$ is a standard simplex. For any vertex $v_i = \langle \bar{v}_i \rangle$, write the last coordinate of $\bar{v}_i$ as $a_i R + b_i$ with $a_i \geq 0$ and $0 \leq b_i < R$. Then $\sigma$ is carrying if and only if $\tau_1$ is of one of the following two types for some $\epsilon \in \{-1, +1\}$:*

*i) $\tau_1 = \{v_0, v_1, v_2 = \langle \bar{v}_0 + \bar{v}_1 + \epsilon \bar{e}_i \rangle\}$ is minimal externally 3-additive and $b_0 + b_1 \notin [0, R)$.*
*ii) $\tau_1 = \{v_0, v_1, v_2, v_3 = \langle \bar{v}_0 + \bar{v}_1 + \epsilon \bar{v}_2 \rangle\}$ is minimal internally 3-additive and $b_0 + b_1 + \epsilon b_2 \notin [0, R)$.*

This lemma follows from Lemma 3.32, Lemma 3.33 and Lemma 3.34, which are proved below.

To extend the map $r$ over these carrying 3-additive simplices, we need to subdivide them. This leads us to the definition of the complex $\mathrm{sd}(\widehat{\mathsf{Link}}_{\mathrm{TA}_n^m}(w))$ that will serve as the new domain of the map $r$ when extending over 3-additive simplices.

**Definition 3.30.** Let $\widehat{\mathsf{Link}}_{\mathrm{TA}_n^m}(w)$ and $\widehat{\mathsf{Link}}^{<}_{\mathrm{TA}_n^m}(w)$ denote the subcomplexes of $\widehat{\mathsf{Link}}_{\mathrm{BAA}_n^m}(w)$ and $\widehat{\mathsf{Link}}^{<}_{\mathrm{BAA}_n^m}(w)$, respectively, consisting of all simplices that are standard, 2-additive, double-double, or 3-additive. Let $\mathrm{sd}(\widehat{\mathsf{Link}}_{\mathrm{TA}_n^m}(w))$ denote the coarsest subdivision of $\widehat{\mathsf{Link}}_{\mathrm{TA}_n^m}(w)$ that contains $\mathrm{sd}(\widehat{\mathsf{Link}}_{\mathrm{DD}_n^m}(w))$ as a subcomplex and that subdivides every carrying minimal 3-additive simplex $\tau_1$ by inserting a single vertex $t(\tau_1)$ at its barycentre.

Using Lemma 3.29, this means that $\mathrm{sd}(\widehat{\mathsf{Link}}_{\mathrm{DD}_n^m}(w))$ in Definition 3.30 is constructed as follows: Let $\sigma = \tau_1 * \tau_2$ be a 3-additive simplex of $\widehat{\mathsf{Link}}_{\mathrm{TA}_n^m}(w)$, where $\tau_1$ is a carrying minimal 3-additive simplex and $\tau_2$ is standard. Then, when passing from $\widehat{\mathsf{Link}}_{\mathrm{TA}_n^m}(w)$ to $\mathrm{sd}(\widehat{\mathsf{Link}}_{\mathrm{TA}_n^m}(w))$, each such simplex $\sigma$ is replaced as follows.

- If $\tau_1 = \{v_0, v_1, v_2\}$ is a carrying minimal *externally* 3-additive simplex, we replace $\sigma$ by $\mathrm{sd}(\sigma) = \mathrm{sd}(\tau_1) * \tau_2$, the subdivision of $\sigma$ into the three simplices $\{v_0, \dots, \hat{v}_i, \dots, v_2, t(\tau_1)\} * \tau_2$ for $i = 0, 1, 2$.
- If $\tau_1 = \{v_0, v_1, v_2, v_3\}$ is a carrying minimal *internally* 3-additive simplex, we replace $\sigma$ by $\mathrm{sd}(\sigma) = \mathrm{sd}(\tau_1) * \tau_2$, the subdivision of $\sigma$ into the four simplices $\{v_0, \dots, \hat{v}_i, \dots, v_3, t(\tau_1)\} * \tau_2$ for $i = 0, 1, 2, 3$.

In addition, the subdivisions described in Definition 3.25 are performed on the subcomplex $\widehat{\mathsf{Link}}_{\mathrm{DD}_n^m}(w)$ of $\widehat{\mathsf{Link}}_{\mathrm{TA}_n^m}(w)$. Note that $\mathrm{sd}(\widehat{\mathsf{Link}}_{\mathrm{DD}_n^m}(w))$ and $\widehat{\mathsf{Link}}^{<}_{\mathrm{TA}_n^m}(w)$ are subcomplexes of $\mathrm{sd}(\widehat{\mathsf{Link}}_{\mathrm{TA}_n^m}(w))$.

The main result of this subsection is the following proposition.

**Proposition 3.31.** *The simplicial map constructed in Proposition 3.26*

$$r\colon \mathrm{sd}(\widehat{\mathsf{Link}}_{\mathrm{DD}_n^m}(w)) \to \widehat{\mathsf{Link}}^{<}_{\mathrm{BAA}_n^m}(w)$$

*extends to a simplicial map*

$$r\colon \mathrm{sd}(\widehat{\mathsf{Link}}_{\mathrm{TA}_n^m}(w)) \to \widehat{\mathsf{Link}}^{<}_{\mathrm{BAA}_n^m}(w)$$

*that restricts to the inclusion*

$$\widehat{\mathsf{Link}}^{<}_{\mathrm{TA}_n^m}(w) \hookrightarrow \widehat{\mathsf{Link}}^{<}_{\mathrm{BAA}_n^m}(w)$$

*on the subcomplex* $\widehat{\mathsf{Link}}^{<}_{\mathrm{TA}_n^m}(w)$ *of* $\mathrm{sd}(\widehat{\mathsf{Link}}_{\mathrm{TA}_n^m}(w))$.

The proof of this proposition and the definition of the extension of $r$ is split into several lemmas, which we present below. We start by proving that $r$ extends over all 3-additive simplices that cannot be possibly carrying (compare Lemma 3.29).

**Lemma 3.32.** *The map $r$ in Proposition 3.31 extends over all 3-additive simplices $\sigma = \tau_1 * \tau_2$ of $\widehat{\mathsf{Link}}_{\mathrm{BAA}_n^m}(w)$, where $\tau_2$ is a standard simplex and $\tau_1$ is a minimal 3-additive simplex that is not internally 3-additive or externally 3-additive of dimension two. In these cases, the set $r(\sigma) = r(\tau_1) * r(\tau_2)$ is a simplex of $\widehat{\mathsf{Link}}^{<}_{\mathrm{BAA}_n^m}(w)$, so in particular, $\sigma$ is not carrying.*

**Proof.** Any minimal 3-additive simplex $\tau_1 = \{v_0, \dots, v_{\dim(\tau_1)}\}$ satisfies $1 \leq \dim(\tau_1) \leq 3$. The underlying simplex of $\tau_1$ in $\mathrm{BAA}_{m+n}$ is a subset $\{v_0, v_1, v_2, v_3\} \subseteq \{e_1, \dots, e_m, w, v_0, v_1, \dots, v_{\dim(\tau_1)}\}$, where $\{e_1, \dots, e_m, w, v_1, \dots, v_{\dim(\tau_1)}\}$ is a standard simplex and $v_0 = \langle \bar{v}_1 + \epsilon_2 \bar{v}_2 + \epsilon_3 \bar{v}_3 \rangle$ for some choice of signs $\epsilon_2, \epsilon_3 \in \{-1, +1\}$. We consider the possible minimal 3-additive simplices $\tau_1$, one after the other, to prove this lemma.

Firstly, assume that $\dim(\tau_1) = 1$ and write $\tau_1 = \{v_0, v_1\}$. Then there are two cases.

*Case (a)*: If $v_2 = e_i, v_3 = e_j$ for some $1 \leq i \neq j \leq m$, then $v_0 = \langle \bar{v}_1 + \epsilon_2 \bar{e}_i + \epsilon_3 \bar{e}_j \rangle$. Hence, $r(v_0) = \langle \overline{r(v_1)} + \epsilon_2 \bar{e}_i + \epsilon_3 \bar{e}_j \rangle$ by Lemma 3.12 and it follows that $r(\tau_1)$ is a 3-additive edge in $\widehat{\mathsf{Link}}^{<}_{\mathrm{BAA}_n^m}(w)$ as well. Hence by Lemma 3.21, $r(\sigma) = r(\tau_1) * r(\tau_2)$ is a 3-additive simplex.

*Case (b)*: If $v_2 = w, v_3 = e_i$ for some $1 \leq i \leq m$, then $v_0 = \langle \bar{v}_1 + \epsilon_2 \bar{w} + \epsilon_3 \bar{e}_i \rangle$ and, by Lemma 3.12, it holds that $r(v_0) = \langle \epsilon \overline{r(\langle \bar{v}_1 + \epsilon_2 \bar{w} \rangle)} + \epsilon_3 \bar{e}_i \rangle$ where $\epsilon = -1$ if the last coordinate of $\bar{v}_1 + \epsilon_2 \bar{w}$ is negative, and $\epsilon = +1$ otherwise. By Lemma 3.12 we furthermore have that $r(\langle \bar{v}_1 + \epsilon_2 \bar{w} \rangle) \in \{r(v_1), \langle \bar{w} - \overline{r(v_1)} \rangle\}$. Note that $r(\langle \bar{v}_1 + \epsilon_2 \bar{w} \rangle) = \langle \bar{w} - \overline{r(v_1)} \rangle$ requires that $\epsilon_2 = -1$ and that $\epsilon = -1$. Resolving the signs, it follows that $r(\tau_1) = \{r(v_0), r(v_1)\}$ is either an externally 2-additive edge $\{\langle \overline{r(v_1)} + \epsilon_3 \bar{e}_i \rangle, r(v_1)\}$ or an externally $w$-related 3-additive edge $\{\langle \overline{r(v_1)} - \bar{w} + \epsilon_3 \bar{e}_i \rangle, r(v_1)\}$. Hence by Lemma 3.21, $r(\sigma) = r(\tau_1) * r(\tau_2)$ is a 2-additive or 3-additive simplex in $\widehat{\mathsf{Link}}^{<}_{\mathrm{BAA}_n^m}(w)$.

Secondly, assume that $\dim(\tau_1) = 2$ and write $\tau_1 = \{v_0, v_1, v_2\}$. Assume further that $\tau_1$ is $w$-related, i.e. $v_3 = w$. Then,

$$r(v_0) = r(\langle \bar{v}_1 + \epsilon_2 \bar{v}_2 + \epsilon_3 \bar{w} \rangle) \in \{r(\langle \bar{v}_1 + \epsilon_2 \bar{v}_2 \rangle), \langle \bar{w} - \overline{r(\langle \bar{v}_1 + \epsilon_2 \bar{v}_2 \rangle)} \rangle\}$$

by Lemma 3.12. Note that the value of $r(v_0)$ depends on the last coordinate of $\bar{v}_1 + \epsilon_2 \bar{v}_2$, which might be negative, and the sign $\epsilon_3$ (compare Lemma 3.12). There are different cases that can occur, depending on how $\overline{r(\langle \bar{v}_1 + \epsilon_2 \bar{v}_2 \rangle)}$ compares to $r(v_1)$ and $r(v_2)$. We use the internally 2-additive simplex $\{v_1, v_2, v_0' = \langle \bar{v}_1 + \epsilon_2 \bar{v}_2 \rangle\}$ to list these cases.

*Case (a)*: If $\{v_1, v_2, v_0' = \langle \bar{v}_1 + \epsilon_2 \bar{v}_2 \rangle\}$ is not carrying, then its image under $r$ is internally 2-additive [6, Section 4.4, Claim 4]. It follows that $r(\tau_1)$ is internally 2-additive if $r(v_0) = r(v_0')$, and $w$-related 3-additive if $r(v_0) = \langle \bar{w} - \overline{r(v_0')} \rangle$. Hence by Lemma 3.21, $r(\sigma) = r(\tau_1) * r(\tau_2)$ is a 2-additive or 3-additive simplex.

*Case (b)*: If $\{v_1, v_2, v_0' = \langle \bar{v}_1 + \epsilon_2 \bar{v}_2 \rangle\}$ is carrying, then it contains a unique vertex whose coordinate is maximal in absolute value (see Definition 3.16 et seq.). We consider two subcases.

*Case (b.1)*: If the absolute value of the last coordinate of $v_1$ (or similarly $v_2$) is maximal among $\{v_1, v_2, v_0'\}$, then $\overline{r(v_1)} = \overline{r(v_2)} + \overline{r(v_0')} - \bar{w}$ (see Definition 3.16 et seq.). In the case where $r(v_0) = r(v_0')$, it follows that $f(\tau_1)$ is $w$-related 3-additive. In the case $r(v_0) = \langle \bar{w} - \overline{r(v_0')} \rangle$, it follows that $f(\tau_1)$ is 2-additive. Hence by Lemma 3.21, $r(\sigma) = r(\tau_1) * r(\tau_2)$ is a 3-additive or 2-additive simplex.

*Case (b.2)*: If on the other hand, the absolute value of the last coordinate of $v_0'$ is maximal among $\{v_1, v_2, v_0'\}$, then we must have $\epsilon_2 = +1$ and $\overline{r(v_0')} = \overline{r(v_1)} + \overline{r(v_2)} - \bar{w}$. In the case where $r(v_0) = r(v_0')$, it follows that $f(\tau_1)$ is $w$-related 3-additive. Hence by Lemma 3.21, $r(\sigma) = r(\tau_1) * r(\tau_2)$ is a 3-additive simplex. The case where $r(v_0) = \langle \bar{w} - \overline{r(v_0')} \rangle$ cannot occur, because this only happens if the last coordinate of $\overline{v_0'} = \bar{v}_1 + \bar{v}_2$ in $(0, R)$ (see Lemma 3.12), which is impossible under the assumption that the last coordinate of $\overline{v_0'}$ is the maximum of the carrying simplex $\{v_1, v_2, v_0'\}$.

Thirdly and lastly, the remaining two possibilities are those where $\tau_1$ has dimension two and is externally 3-additive and the one where $\tau_1$ has dimension three, which is equivalently to it being internally 3-additive. These are the cases we excluded in this lemma. □

We now deal with minimal externally 3-additive simplices of type $\tau_1 = \{v_0 = \langle \bar{v}_1 \pm \bar{v}_2 \pm \bar{e}_i \rangle, v_1, v_2\}$.

**Lemma 3.33.** *The map $r$ in Proposition 3.31 extends over all externally 3-additive simplices $\sigma = \tau_1 * \tau_2$ of $\widehat{\mathsf{Link}}_{\mathrm{BAA}_n^m}(w)$, where $\tau_2$ is a standard simplex and $\tau_1$ is minimal externally 3-additive of dimension two.*

More precisely, in the proof of Lemma 3.33 we check that the map

$$r\colon \operatorname{sd}(\widehat{\mathsf{Link}}_{\mathrm{DD}_n^m}(w)) \to \widehat{\mathsf{Link}}^{<}_{\mathrm{BAA}_n^m}(w)$$

in Proposition 3.31 extends over the simplex $\sigma = \tau_1 * \tau_2$ if it is not carrying, and over the subdivision $\operatorname{sd}(\sigma)$ described in Definition 3.30 if it is carrying. The carrying case occurs if and only if $\tau_1$ is as case $i$) of Lemma 3.29; we then define $r(t(\tau_1)) = r(\langle \bar{v}_1 + \bar{v}_2 \rangle) = \langle \overline{r(v_1)} + \overline{r(v_2)} - \bar{w} \rangle$, where $v_1, v_2 \in \tau_1$ are the two unique vertices whose last coordinate is not maximal in absolute value.

**Proof.** There is an ordering of the vertices of $\tau_1$ such that $\tau_1 = \{v_0, v_1, v_2\}$, where

$$\bar{v}_0 = \bar{v}_1 + \bar{v}_2 \pm \bar{e}_i$$

for some $1 \leq i \leq m$, an appropriate choice of sign and where $v_0$ is a (possibly not unique) vertex of $\tau_1$ whose last coordinate is maximal in absolute value.

Let $R_i$ denote the last coordinate of $\bar{v}_i$ and write $R_i = Ra_i + b_i$ with $b_i \in [0, R)$. Note that $R_0 = R_1 + R_2$. There are two cases: either $b_0 = b_1 + b_2 \in [0, R)$ or $b_0 + R = b_1 + b_2 \notin [0, R)$. In the following we use that $r(v_i) = \langle \bar{v}_i - a_i\bar{w}\rangle$ and $r(\langle \bar{u} \pm \bar{e}_i\rangle) = \langle \overline{r(u)} \pm \bar{e}_i\rangle$ for all lines $u$.

Firstly, assume that $b_0 = b_1 + b_2 \in [0, R)$. Then

$$r(\tau_1) = \{\langle \overline{r(v_1)} + \overline{r(v_2)} \pm \bar{e}_i\rangle, r(v_1), r(v_2)\}$$

forms an externally 3-additive simplex. Hence it follows from Lemma 3.21 that $r(\sigma) = r(\tau_1) * r(\tau_2)$ is a 3-additive simplex in $\widehat{\mathsf{Link}}^{<}_{\mathrm{BAA}^m_n}(w)$.

Secondly, assume that $b_0 + R = b_1 + b_2 \notin [0, R)$. Then

$$\overline{r(v_0)} = \langle (\bar{v}_1 - a_1\bar{w}) + (\bar{v}_2 - a_2\bar{w}) \pm \bar{e}_i - \bar{w}\rangle = \langle \overline{r(v_1)} + \overline{r(v_2)} \pm e_i - \bar{w}\rangle,$$

and $r(\tau_1)$ does not form a simplex in $\widehat{\mathsf{Link}}^{<}_{\mathrm{BAA}^m_n}(w)$. Hence, $\tau_1$ is a carrying minimal externally 3-additive simplex (compare Lemma 3.29) and, in $\mathrm{sd}(\widehat{\mathsf{Link}}_{\mathrm{TA}^m_n}(w))$, the simplex $\sigma = \tau_1 * \tau_2$ has been subdivided as $\mathrm{sd}(\sigma) = \mathrm{sd}(\tau_1) * \tau_2$ into three simplices (compare Definition 3.30)

$$\alpha_i * \tau_2 = \{v_0, \ldots, \hat{v}_i, \ldots, v_2, t(\tau_1)\} * \tau_2 \quad \text{for } i = 0, 1, 2.$$

Observe that $b_0 + R = b_1 + b_2 \in [R, 2R)$ implies that all $\bar{v}_i$ have nonzero last coordinate and hence that $v_0$ is the unique vertex in $\tau_1$ whose last coordinate is maximal in absolute value. To see that $r$ extends over $\mathrm{sd}(\sigma)$ by defining $r(t(\tau_1)) = r(\langle \bar{v}_1 + \bar{v}_2\rangle) = \langle \overline{r(v_1)} + \overline{r(v_2)} - \bar{w}\rangle$, we first observe that the three sets $r(\alpha_i)$ span 2-simplices in $\widehat{\mathsf{Link}}^{<}_{\mathrm{BAA}^m_n}(w)$. Indeed,

- $r(\alpha_2) = \{\langle \overline{r(v_1)} + \overline{r(v_2)} \pm \bar{e}_i - \bar{w}\rangle, r(v_1), \langle \overline{r(v_1)} + \overline{r(v_2)} - \bar{w}\rangle\}$ is externally 2-additive,[7]
- $r(\alpha_1) = \{\langle \overline{r(v_1)} + \overline{r(v_2)} \pm \bar{e}_i - \bar{w}\rangle, r(v_2), \langle \overline{r(v_1)} + \overline{r(v_2)} - \bar{w}\rangle\}$ is externally 2-additive, and
- $r(\alpha_0) = \{r(v_1), r(v_2), \langle \overline{r(v_1)} + \overline{r(v_2)} - \bar{w}\rangle\}$ is a $w$-related 3-additive in $\widehat{\mathsf{Link}}^{<}_{\mathrm{BAA}^m_n}(w)$.

Then, we invoke Lemma 3.21 for $\sigma = \tau_1 * \tau_2$ to conclude that $r(\alpha_i * \tau_2) = r(\alpha_i) * r(\tau_2)$ spans a simplex of the same type. □

We are left with proving that we can extend over internally 3-additive simplices. This is done in the next lemma, whose proof also yields a description of the possible values that the vertices of a carrying internally 3-additive simplex can take under $r$.

[7] For better readability, we highlight the vertices that are contained in the additive core of the simplex in light blue (for interpretation of the colours in the text, the reader is referred to the web version of this article).

**Lemma 3.34.** *The map $r$ in Proposition 3.31 extends over all internally 3-additive simplices $\sigma$ of $\widehat{\mathsf{Link}}_{\mathrm{BAA}_n^m}(w)$.*

More precisely, in Lemma 3.34 we consider an internally 3-additive simplex $\sigma = \tau_1 * \tau_2$ of $\widehat{\mathsf{Link}}_{\mathrm{BAA}_n^m}(w)$ such that $\tau_1 = \{v_0, v_1, v_2, v_3\}$ is a minimal internally 3-additive simplex and $\tau_2$ is a standard simplex. We may assume that the last coordinate of $\bar{v}_3$ is maximal (perhaps not uniquely). Then, the proof of Lemma 3.34 establishes the following sequence of claims: Possibly after reordering, we have that

$$\bar{v}_3 = \bar{v}_0 + \bar{v}_1 + \bar{v}_2 \text{ or } \bar{v}_3 = \bar{v}_0 + \bar{v}_1 - \bar{v}_2.$$

Letting $R_i = a_i R + b_i$ for $a_i \geq 0$ and $b_i \in [0, R)$ denote the last coordinate of $\bar{v}_i$, one of the following is true:

- i) $\overline{r(v_3)} = \overline{r(v_0)} + \overline{r(v_1)} + \overline{r(v_2)} - 2\bar{w}$ and $b_0 + b_1 \in [R, 2R)$,
- ii) $\overline{r(v_3)} = \overline{r(v_0)} + \overline{r(v_1)} + \overline{r(v_2)} - \bar{w}$,
- iii) $\overline{r(v_3)} = \overline{r(v_0)} + \overline{r(v_1)} + \overline{r(v_2)}$,
- iv) $\overline{r(v_3)} = \overline{r(v_0)} + \overline{r(v_1)} - \overline{r(v_2)} - \bar{w}$,
- v) $\overline{r(v_3)} = \overline{r(v_0)} + \overline{r(v_1)} - \overline{r(v_2)}$, or
- vi) $\overline{r(v_3)} = \overline{r(v_0)} + \overline{r(v_1)} - \overline{r(v_2)} + \bar{w}$ and $b_0 + b_1 \in [0, R)$.

In case *iii*) and case *v*), it holds that $b_0 + b_1 \pm b_2 \in [0, R)$,[8] that the set $r(\tau_1)$ forms a simplex in $\widehat{\mathsf{Link}}^{<}_{\mathrm{BAA}_n^m}(w)$ and that $r$ extends over the simplex $\sigma = \tau_1 * \tau_2$. In all other cases, it holds that $b_0 + b_1 \pm b_2 \notin [0, R)$, that $r(\tau_1)$ is not a simplex (i.e. $\sigma$ is carrying) and that $r$ extends over the subdivision $\mathrm{sd}(\sigma)$ by defining $r(t(\tau_1)) = r(\langle \bar{v}_0 + \bar{v}_1 \rangle)$. Here, $r(t(\tau_1))$ is equal to $\langle \overline{r(v_0)} + \overline{r(v_1)} - \bar{w} \rangle$ if $b_0 + b_1 \in [R, 2R)$ or $\langle \overline{r(v_0)} + \overline{r(v_1)} \rangle$ if $b_0 + b_1 \in [0, R)$. In particular, the definition of $r(t(\tau_1))$ depends on a choice of $v_0$ and $v_1$ as above.

**Proof.** Firstly, assume that $v_i$ has last coordinate zero for some $0 \leq i \leq 3$. Possibly after reordering we may assume that $\bar{v}_2$ has last coordinate zero and that $\bar{v}_3$ has maximal last coordinate (perhaps not uniquely). It follows that $\bar{v}_3 = \bar{v}_0 + \bar{v}_1 \pm \bar{v}_2$ and Lemma 3.12 implies that $\overline{r(v_3)} = \langle \overline{r(\bar{v}_0 + \bar{v}_1 \rangle)} \pm \bar{v}_2 \rangle$. There are two subcases.

(1) If $b_0 + b_1 = b_0 + b_1 \pm b_2 \in [0, R)$, then $\overline{r(v_3)} = \overline{r(v_0)} + \overline{r(v_1)} \pm \overline{r(v_2)}$. It follows that the set $r(\tau_1)$ is an internally 3-additive simplex in $\widehat{\mathsf{Link}}^{<}_{\mathrm{BAA}_n^m}(w)$ and hence by Lemma 3.21 that $r(\sigma) = r(\tau_1) * r(\tau_2)$ is a 3-additive simplex as well.

(2) If $b_0 + b_1 = b_0 + b_1 \pm b_2 \in [R, 2R)$, then $\overline{r(v_3)} = \overline{r(v_0)} + \overline{r(v_1)} \pm \overline{r(v_2)} - \overline{w}$ and $r(\tau_1)$ is not a simplex in $\widehat{\mathsf{Link}}^{<}_{\mathrm{BAA}_n^m}(w)$. At the end of this proof we will discuss how $r$ can be extended over $\mathrm{sd}(\sigma) = \mathrm{sd}(\tau_1) * \tau_2$ in this case.

Secondly, assume that the last coordinate of all $\bar{v}_i$ for $0 \leq i \leq 3$ is nonzero. Let us assume that $\bar{v}_3$ has maximal last coordinate (possibly after reordering and perhaps not

[8] Here and in the following sentence, "$\pm$" is to be understood as "the same sign as the one in front of $\overline{r(v_2)}$".

uniquely). By the definition of 3-additive, it follows that $\bar{v}_3 = \pm\bar{v}_0 \pm \bar{v}_1 \pm \bar{v}_2$ for some choice of signs. Then, there are three cases up to reordering

a) $\bar{v}_3 = \bar{v}_0 + \bar{v}_1 + \bar{v}_2$ (no minus signs)
b) $\bar{v}_3 = \bar{v}_0 + \bar{v}_1 - \bar{v}_2$ (one minus sign) or
c) $\bar{v}_3 = \bar{v}_0 - \bar{v}_1 - \bar{v}_2$ (two minus signs).

The last case cannot occur because then the last coordinate of $\bar{v}_0$ is bigger than the last coordinate of $\bar{v}_3$ (violating the assumption that the last coordinate of $\bar{v}_3$ is maximal). It follows that either

$$\bar{v}_3 = \bar{v}_0 + \bar{v}_1 + \bar{v}_2 \text{ or } \bar{v}_3 = \bar{v}_0 + \bar{v}_1 - \bar{v}_2.$$

Observe that $b_0 + b_1 + b_2 \in [0, 3R)$ and $b_0 + b_1 - b_2 \in (-R, 2R)$.

(3) In case a) and if $b_0+b_1+b_2 \in [2R, 3R)$, it follows that $\overline{r(v_3)} = \overline{r(v_0)}+\overline{r(v_1)}+\overline{r(v_2)}-2w$ and that $r(\tau_1)$ is not a simplex in $\widehat{\mathsf{Link}}^{<}_{\mathrm{BAA}_n^m}(w)$. Observe that we must have $b_0+b_1 \in [R, 2R)$ in this case.
(4) In case a) and if $b_0+b_1+b_2 \in [R, 2R)$, it follows that $\overline{r(v_3)} = \overline{r(v_0)}+\overline{r(v_1)}+\overline{r(v_2)}-w$ and that $r(\tau_1)$ is not a simplex in $\widehat{\mathsf{Link}}^{<}_{\mathrm{BAA}_n^m}(w)$.
(5) In case a) and if $b_0 + b_1 + b_2 \in [0, R)$, it follows that $\overline{r(v_3)} = \overline{r(v_0)} + \overline{r(v_1)} + \overline{r(v_2)}$, that $r(\tau_1)$ is an internally 3-additive simplex and hence by Lemma 3.21 that $r(\sigma) = r(\tau_1) * r(\tau_2)$ is a 3-additive simplex in $\widehat{\mathsf{Link}}^{<}_{\mathrm{BAA}_n^m}(w)$ as well.
(6) In case b) and if $b_0+b_1-b_2 \in [R, 2R)$, it follows that $\overline{r(v_3)} = \overline{r(v_0)}+\overline{r(v_1)}-\overline{r(v_2)}-w$ and that $r(\tau_1)$ is not a simplex in $\widehat{\mathsf{Link}}^{<}_{\mathrm{BAA}_n^m}(w)$.
(7) In case b) and if $b_0 + b_1 - b_2 \in [0, R)$, it follows that $\overline{r(v_3)} = \overline{r(v_0)} + \overline{r(v_1)} - \overline{r(v_2)}$, that $r(\tau_1)$ is an internally 3-additive simplex and hence by Lemma 3.21 that $r(\sigma) = r(\tau_1) * r(\tau_2)$ is a 3-additive simplex in $\widehat{\mathsf{Link}}^{<}_{\mathrm{BAA}_n^m}(w)$ as well.
(8) In case b) and if $b_0+b_1-b_2 \in (-R, 0)$, it follows that $\overline{r(v_3)} = \overline{r(v_0)}+\overline{r(v_1)}-\overline{r(v_2)}+w$ and that $r(\tau_1)$ is not a simplex in $\widehat{\mathsf{Link}}^{<}_{\mathrm{BAA}_n^m}(w)$. Observe that we must have $b_0+b_1 \in [0, R)$ in this case.

This establishes the first three claims in the paragraph after Lemma 3.34. To finish, we are left with proving that the map extends over $\mathrm{sd}(\sigma)$ whenever $\sigma = \tau_1 * \tau_2$ is carrying, i.e. in the situations (2), (3), (4), (6) and (8). In $\mathrm{sd}(\widehat{\mathsf{Link}}_{\mathrm{TA}_n^m}(w))$ the simplex $\sigma$ has been subdivided as $\mathrm{sd}(\sigma) = \mathrm{sd}(\tau_1) * \tau_2$ into four simplices

$$\alpha_i * \tau_2 = \{v_0, \dots, \hat{v}_i, \dots, v_3, t(\tau_1)\} * \tau_2 \quad \text{for } i = 0, 1, 2, 3.$$

To see that $r$ extends over $\alpha_i * \tau_2$ by defining $r(t(\tau_1)) = r(\langle \bar{v}_0 + \bar{v}_1 \rangle)$, we first note that $r(\langle \bar{v}_0 + \bar{v}_1 \rangle) \in \widehat{\mathsf{Link}}^{<}_{\mathrm{BAA}_n^m}(w)$ by definition, and that hence all elements in the set $r(\alpha_i * \tau_2)$

are vertices of $\widehat{\mathsf{Link}}^{<}_{\mathrm{BAA}_n^m}(w)$. We only need to check that they from simplices. For this, we distinguish two cases depending on whether $b_0 + b_1 \in [0, R)$ or $b_0 + b_1 \in [R, 2R)$.

*Assume that* $b_0+b_1 \in [R, 2R)$. Observe that this is always true in the situation (2) and (3), might happen in situation (4) and (6), and is impossible in situation (8) described above. By Lemma 3.12 we have that $r(\langle \overline{v}_0 + \overline{v}_1 \rangle) = \langle \overline{r(v_0)} + \overline{r(v_1)} - \bar{w} \rangle$. With the value of $r(v_3)$ calculated above, it follows that in $\widehat{\mathsf{Link}}^{<}_{\mathrm{BAA}_n^m}(w)$,

- $r(\alpha_3) = \{r(v_0), r(v_1), r(v_2), \langle \overline{r(v_0)} + \overline{r(v_1)} - \bar{w} \rangle\}$ is $w$-related 3-additive,
- $r(\alpha_2) = \{r(v_0), r(v_1), r(v_3), \langle \overline{r(v_0)} + \overline{r(v_1)} - \bar{w} \rangle\}$ is $w$-related 3-additive,
- $r(\alpha_1) = \{r(v_0), r(v_2), r(v_3), \langle \overline{r(v_0)} + \overline{r(v_1)} - \bar{w} \rangle\}$ is $w$-related 3-additive in situation (3) or 2-additive in situation (2), (4) and (6), and
- $r(\alpha_0) = \{r(v_1), r(v_2), r(v_3), \langle \overline{r(v_0)} + \overline{r(v_1)} - \bar{w} \rangle\}$ is $w$-related 3-additive in situation (3) or 2-additive in situation (2), (4) and (6).

Invoking Lemma 3.21 for $\sigma = \tau_1 * \tau_2$, we conclude that $r(\alpha_i * \tau_2) = r(\alpha_i) * r(\tau_2)$ spans a simplex of the same type. Hence, we can extend over $\mathrm{sd}(\sigma)$ in this case.

*Assume that* $b_0 + b_1 \in [0, R)$. Observe that this is always true in the situation (8), might happen in situation (4) and (6), and is impossible in situation (2) and (3) described above. By Lemma 3.12 we have that $r(\langle \overline{v}_0 + \overline{v}_1 \rangle) = \langle \overline{r(v_0)} + \overline{r(v_1)} \rangle$. With the value of $r(v_3)$ calculated above, it follows that in $\widehat{\mathsf{Link}}^{<}_{\mathrm{BAA}_n^m}(w)$,

- $r(\alpha_3) = \{r(v_0), r(v_1), r(v_2), \langle \overline{r(v_0)} + \overline{r(v_1)} \rangle\}$ is 2-additive,
- $r(\alpha_2) = \{r(v_0), r(v_1), r(v_3), \langle \overline{r(v_0)} + \overline{r(v_1)} \rangle\}$ is 2-additive,
- $r(\alpha_1) = \{r(v_0), r(v_2), r(v_3), \langle \overline{r(v_0)} + \overline{r(v_1)} \rangle\}$ is $w$-related 3-additive, and
- $r(\alpha_0) = \{r(v_1), r(v_2), r(v_3), \langle \overline{r(v_0)} + \overline{r(v_1)} \rangle\}$ is $w$-related 3-additive.

Invoking Lemma 3.21 for $\sigma = \tau_1 * \tau_2$, we conclude that $r(\alpha_i * \tau_2) = r(\alpha_i) * r(\tau_2)$ spans a simplex of the same type. Hence, we can extend over $\mathrm{sd}(\sigma)$ in this case as well. □

Lemma 3.32, Lemma 3.33 and Lemma 3.34 imply Proposition 3.31 and Lemma 3.29, so this concludes our discussion of 3-additive simplices.

### *3.4. Extending over double-triple simplices*

The goal of this subsection is to extend the map

$$r\colon \mathrm{sd}(\widehat{\mathsf{Link}}_{\mathrm{TA}_n^m}(w)) \to \widehat{\mathsf{Link}}^{<}_{\mathrm{BAA}_n^m}(w)$$

defined in the previous subsection over all double-triple simplices. For this, we need to study minimal double-triple simplices in the sense of Definition 3.5.

**Observation 3.35.** A double-triple simplex of $\mathrm{BAA}_n^m$ is *minimal* if all of its facets are 2-additive or 3-additive.

The difficulty is to extend $r$ over carrying double-triple simplices, i.e. double-triple simplices that have a carrying facet.

**Definition 3.36.** Let $\sigma = \tau_1 * \tau_2$ be a double-triple simplex in $\widehat{\mathsf{Link}}_{\mathrm{BAA}_n^m}(w)$, where $\tau_1$ is a minimal double-triple simplex and $\tau_2$ is a standard simplex. The simplex $\sigma$ is called *carrying* if $\tau_1$ has a carrying facet.

We use the following characterisation of carrying double-triple simplices.

**Lemma 3.37.** *Let $\sigma = \tau_1 * \tau_2$ be a double-triple simplex of $\widehat{\mathsf{Link}}_{\mathrm{BAA}_n^m}(w)$ such that $\tau_1$ is a minimal double-triple and $\tau_2$ is a standard simplex. For any vertex $v_i = \langle \bar{v}_i \rangle$, write the last coordinate of $\bar{v}_i$ as $a_i R + b_i$ with $a_i \geq 0$ and $0 \leq b_i < R$. Then $\sigma$ is carrying if and only if $\tau_1$ is of one of the following types for some $\epsilon \in \{-1, +1\}$:*

- $\tau_1 = \{v_0, v_1, \langle \bar{v}_0 + \bar{v}_1 + \epsilon \bar{w} \rangle, \langle \bar{v}_0 + \bar{v}_1 \rangle\}$ *and* $b_0 + b_1 \notin [0, R)$,
- $\tau_1 = \{v_0, v_1, \langle \bar{v}_0 + \bar{v}_1 + \epsilon \bar{e}_i \rangle, \langle \bar{v}_0 + \epsilon \bar{e}_i \rangle\}$ *for some* $i \leq m$, $b_0 + b_1 \notin [0, R)$,
- $\tau_1 = \{v_0, v_1, \langle \bar{v}_0 + \bar{v}_1 + \epsilon \bar{e}_i \rangle, \langle \bar{v}_0 + \bar{v}_1 \rangle\}$ *for some* $i \leq m$ *and* $b_0 + b_1 \notin [0, R)$, *or*
- $\tau_1 = \{v_0, v_1, v_2, \langle \bar{v}_0 + \bar{v}_1 + \epsilon \bar{v}_2 \rangle, \langle \bar{v}_0 + \bar{v}_1 \rangle\}$ *and* $b_0 + b_1 \notin [0, R)$ *or* $b_0 + b_1 + \epsilon b_2 \notin [0, R)$.

This follows from Lemma 3.43, Lemma 3.45, Lemma 3.48 and Lemma 3.52, which are proved below.

Since all carrying 2-additive and 3-additive simplices have been subdivided in $\mathrm{sd}(\widehat{\mathsf{Link}}_{\mathrm{TA}_n^m}(w))$, we will need to subdivide every double-triple simplex in a compatible fashion. The general type of subdivision of $\widehat{\mathsf{Link}}_{\mathrm{BAA}_n^m}(w)$ that we will be considering is described in the next definition. The construction of such a subdivision will be part of the proof of the main result of this subsection.

**Definition 3.38.** Assume that for every carrying minimal double-triple simplex $\tau_1$ in $\widehat{\mathsf{Link}}_{\mathrm{BAA}_n^m}(w)$, we are given a simplicial disc $\mathrm{sd}(\tau_1)$ whose boundary sphere is exactly the subcomplex $\mathrm{sd}(\partial\tau_1)$ of $\mathrm{sd}(\widehat{\mathsf{Link}}_{\mathrm{TA}_n^m}(w))$. Let $\mathrm{sd}(\widehat{\mathsf{Link}}_{\mathrm{BAA}_n^m}(w))$ denote the coarsest subdivision of $\widehat{\mathsf{Link}}_{\mathrm{BAA}_n^m}(w)$ that contains $\mathrm{sd}(\widehat{\mathsf{Link}}_{\mathrm{TA}_n^m}(w))$ as a subcomplex and that subdivides every carrying minimal double-triple simplex $\tau_1$ according to $\mathrm{sd}(\tau_1)$.

Concretely, $\mathrm{sd}(\widehat{\mathsf{Link}}_{\mathrm{BAA}_n^m}(w))$ in Definition 3.38 is constructed as follows: In addition to the subdivisions described in Definition 3.30 on the subcomplex $\widehat{\mathsf{Link}}_{\mathrm{TA}_n^m}(w)$ of $\widehat{\mathsf{Link}}_{\mathrm{BAA}_n^m}(w)$, we subdivide carrying double-triple simplices of $\widehat{\mathsf{Link}}_{\mathrm{BAA}_n^m}(w)$ in the following fashion. Let $\sigma = \tau_1 * \tau_2$ be a double-triple simplex of $\widehat{\mathsf{Link}}_{\mathrm{BAA}_n^m}(w)$, where $\tau_1$ is a carrying minimal double-triple simplex and $\tau_2$ is standard. Then, when passing from $\widehat{\mathsf{Link}}_{\mathrm{BAA}_n^m}(w)$ to $\mathrm{sd}(\widehat{\mathsf{Link}}_{\mathrm{BAA}_n^m}(w))$, each such simplex is replaced by the simplicial join

$$\mathrm{sd}(\sigma) = \mathrm{sd}(\tau_1) * \tau_2$$

where $\mathrm{sd}(\tau_1)$ is the simplicial disc associated to $\tau_1$ that we fixed before. Note that $\mathrm{sd}(\widehat{\mathsf{Link}}_{\mathrm{TA}_n^m}(w))$ and $\widehat{\mathsf{Link}}^{<}_{\mathrm{BAA}_n^m}(w)$ are subcomplexes of $\mathrm{sd}(\widehat{\mathsf{Link}}_{\mathrm{BAA}_n^m}(w))$.

The main result of this subsection is the following proposition, which implies Theorem 3.1.

**Proposition 3.39.** *There exists a subdivision* $\mathrm{sd}(\widehat{\mathsf{Link}}_{\mathrm{BAA}_n^m}(w))$ *of* $\widehat{\mathsf{Link}}_{\mathrm{BAA}_n^m}(w)$ *as in Definition 3.38 such that the simplicial map constructed in Proposition 3.31*

$$r\colon\ \mathrm{sd}(\widehat{\mathsf{Link}}_{\mathrm{TA}_n^m}(w)) \to \widehat{\mathsf{Link}}^{<}_{\mathrm{BAA}_n^m}(w)$$

*extends to a simplicial map*

$$r\colon\ \mathrm{sd}(\widehat{\mathsf{Link}}_{\mathrm{BAA}_n^m}(w)) \to \widehat{\mathsf{Link}}^{<}_{\mathrm{BAA}_n^m}(w)$$

*that restricts to the identity map on the subcomplex* $\widehat{\mathsf{Link}}^{<}_{\mathrm{BAA}_n^m}(w)$ *of* $\mathrm{sd}(\widehat{\mathsf{Link}}_{\mathrm{BAA}_n^m}(w))$.

The proof of this proposition, the precise definition of $\mathrm{sd}(\widehat{\mathsf{Link}}_{\mathrm{BAA}_n^m}(w))$, and the definition of the extension of $r$ is split into several lemmas, which we present below.

Recall that the extension of the simplicial map $r$ over carrying 2-additive and 3-additive simplices involved a subdivision as well as a choice of vertices. This is the main source of difficulty in this section. The following discussion shows that for carrying 2-additive simplices the two possible extensions of $r$ are "homotopic".

### *3.4.1. Different extensions of $r$ over 2-additive simplices are "homotopic"*

Let $\beta \in \widehat{\mathsf{Link}}_{\mathrm{BAA}_n^m}(w)$ be a minimal carrying 2-additive simplex. Then, Definition 3.16 implies that

$$\beta = \{v_0, v_1, v_2 = \langle \bar{v}_0 + \bar{v}_1 \rangle\}.$$

The definition of the map $r$ in Proposition 3.20 on the subdivision $\mathrm{sd}(\beta)$ of $\beta$ involves a choice $v_l \in \{v_0, v_1\}$. This choice allowed Church–Putman [6] to specify $r$ on the new vertex $t(\beta) \in \mathrm{sd}(\beta)$, the barycentre of $\beta$, by the formula

$$r(t(\beta)) = \langle \overline{r(v_l)} - \bar{w} \rangle.$$

Let us write $r_0\colon\ \mathrm{sd}(\beta) \to \widehat{\mathsf{Link}}_{\mathrm{BAA}_n^m}(w)$ for the map defined using $l = 0$ and $r_1\colon\ \mathrm{sd}(\beta) \to \widehat{\mathsf{Link}}_{\mathrm{BAA}_n^m}(w)$ for the map defined using $l = 1$. The next lemma shows that these two maps are homotopic relative to the boundary $\partial\beta$ in $\widehat{\mathsf{Link}}^{<}_{\mathrm{BAA}_n^m}(w)$.

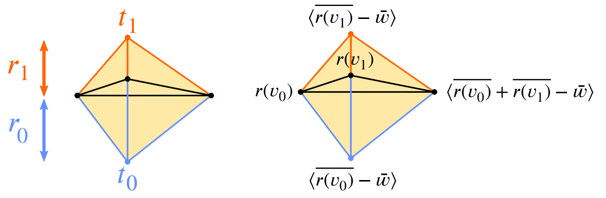


**Fig. 2.** The simplices $\Delta^2 * \{t_0(\beta), t_1(\beta)\}$ and their image under $h$.

**Lemma 3.40.** *Let* $\beta = \{v_0, v_1, v_2 = \langle \bar{v}_0 + \bar{v}_1\rangle\} \in \widehat{\mathsf{Link}}_{\mathrm{BAA}_n^m}(w)$ *be a carrying* 2*-additive simplex. Then, the two maps* $r_0$ *and* $r_1$*, where* $r_l$ *for* $l = 0, 1$ *is as above, are homotopic relative to* $\partial\beta$ *via the simplicial map*

$$h\colon \Delta^2 * \{t_0(\beta), t_1(\beta)\} \to \widehat{\mathsf{Link}}^{<}_{\mathrm{BAA}_n^m}(w),$$

*where* $h(t_0(\beta)) = r_0(t(\beta))$ *and* $h(t_1(\beta)) = r_1(t(\beta))$ *(see Fig. 2).*

**Proof.** Observe that the set of vertices in

$$r_l(\mathrm{sd}(\beta)) = \{r(v_0), r(v_1), \langle\overline{r(v_0)} + \overline{r(v_1)} - \bar{w}\rangle, \langle\overline{r(v_l)} - \bar{w}\rangle\}$$

spans a double-triple simplex for each of the two choices, $l = 0$ and $l = 1$, and that the two simplices share their 3-additive facet $\{r(v_0), r(v_1), \langle\overline{r(v_0)} + \overline{r(v_1)} - \bar{w}\rangle\}$. □

This observation allows us to perform the following construction, which we will use later.

**Corollary 3.41.** *Let* $\tau_1$ *be a carrying minimal double-triple simplex of* $\widehat{\mathsf{Link}}_{\mathrm{BAA}_n^m}(w)$ *that contains a* 2*-dimensional carrying* facet $\beta$ *that is* 2*-additive. Then,*

$$r|_{\partial\tau_1} \cup h\colon \mathrm{sd}(\partial\tau_1) \cup_{\mathrm{sd}(\beta)} D(\beta) \to \widehat{\mathsf{Link}}^{<}_{\mathrm{BAA}_n^m}(w)$$

*defines a simplicial homotopy between the two possible definitions of the map* $r$ *on* $\mathrm{sd}(\partial\tau_1)$*, one obtained from* $r_0$ *and one obtained from* $r_1$ *as discussed above. Here,* $D(\beta) = \Delta^2 * \{t_0(\beta), t_1(\beta)\}$ *is the domain of the homotopy defined in Lemma 3.40.*

The homotopy in Corollary 3.41 is illustrated in Fig. 3 for $\tau_1$ a 3-dimensional double-triple simplex with a carrying 2-additive face.

We now start proving the main results of this subsection.

### *3.4.2. Proof of Proposition 3.39 and Lemma 3.37*

Note that any minimal double-triple simplex $\tau_1$ in $\widehat{\mathsf{Link}}_{\mathrm{BAA}_n^m}(w)$ satisfies $2 \leq \dim(\tau_1) \leq 4$, that any minimal double-triple simplex has a unique 3-additive face and all other faces are 2-additive (see Observation 4.11).

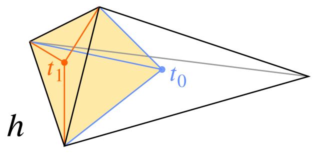


**Fig. 3.** The homotopy $r|_{\partial\tau_1} \cup h$.

**Convention 3.42.** Let $\tau_1$ be a minimal double-triple simplex $\tau_1$ in $\widehat{\mathsf{Link}}_{\mathrm{BAA}_n^m}(w)$. In the remainder of this section, the unique 3-additive face of $\tau_1$ will be denoted by $\gamma$. The simplex $\tau_1$ can then be written as a join $\gamma * z$, where $z \in \tau_1$ is a vertex. We remark that the vertex $z$ has the property that it is contained in the additive core of every 2-additive facet of $\tau_1$.

The next lemma shows that the map $r$ can be extended over minimal double-triple simplices of dimension two without any subdivisions.

**Lemma 3.43.** *The map* $r\colon \widehat{\mathsf{Link}}_{\mathrm{TA}_n^m}(w) \to \widehat{\mathsf{Link}}^{<}_{\mathrm{BAA}_n^m}(w)$ *in Proposition 3.31 extends simplicially over all double-triple simplices* $\sigma = \tau_1 * \tau_2$ *in* $\widehat{\mathsf{Link}}_{\mathrm{BAA}_n^m}(w)$*, where* $\tau_1$ *is a minimal double-triple simplex of dimension two and* $\tau_2$ *is a standard simplex. I.e. for any such simplex it holds that* $r(\sigma) = r(\tau_1) * r(\tau_2)$ *forms a simplex in* $\widehat{\mathsf{Link}}^{<}_{\mathrm{BAA}_n^m}(w)$*, so* $\sigma$ *is not carrying.*

**Proof.** Let $\tau_1 = \gamma * z$, where $\gamma = \{v_0, v_1\}$ is the unique 3-additive facet of $\tau_1$. In this proof, we will use exactly the same strategy as in the proof of Lemma 3.32 and consider the possible minimal 3-additive simplices $\gamma$, one after the other. The underlying simplex of $\gamma$ in $\mathrm{BAA}_{m+n}$ is a subset $\{v_0, v_1, v_2, v_3\} \subseteq \{e_1, \ldots, e_m, w, v_0, v_1\}$, where $\{e_1, \ldots, e_m, w, v_1\}$ is a standard simplex and $v_0 = \langle \bar{v}_1 + \epsilon_2 \bar{v}_2 + \epsilon_3 \bar{v}_3 \rangle$ for some choice of signs $\epsilon_2, \epsilon_3 \in \{-1, +1\}$. As in the first part of the proof of Lemma 3.32, we need to consider two cases.

*Case (a)*: If $v_2 = e_i$ and $v_3 = e_j$ for some $0 \leq i \neq j \leq m$, then $v_0 = \langle \bar{v}_1 + \epsilon_2 \bar{e}_i + \epsilon_3 \bar{e}_j \rangle$ and $z = \langle \bar{v}_1 + \epsilon_2 \bar{e}_i \rangle$ or $z = \langle \bar{v}_1 + \epsilon_3 \bar{e}_j \rangle$. Hence, $r(v_0) = \langle \overline{r(v_1)} + \epsilon_2 \bar{e}_i + \epsilon_3 \bar{e}_j \rangle$ and $r(z) = \langle \overline{r(v_1)} + \epsilon_2 \bar{e}_i \rangle$ or $r(z) = \langle \overline{r(v_1)} + \epsilon_3 \bar{e}_j \rangle$ by Lemma 3.12. It follows that $r(\tau_1)$ is a double-triple simplex in $\widehat{\mathsf{Link}}^{<}_{\mathrm{BAA}_n^m}(w)$. An application of Lemma 3.21 implies that $r(\sigma) = r(\tau_1) * r(\tau_2)$ is a double-triple simplex as well.

*Case (b)*: If $v_2 = w, v_3 = e_i$ for some $1 \leq i \leq m$, then $v_0 = \langle \bar{v}_1 + \epsilon_2 \bar{w} + \epsilon_3 \bar{e}_i \rangle$ and $z = \langle \bar{v}_1 + \epsilon_2 \bar{w} \rangle$ or $z = \langle \bar{v}_1 + \epsilon_3 \bar{e}_i \rangle$. By Lemma 3.12, $r(v_0) = \langle \epsilon \overline{r(\langle \bar{v}_1 + \epsilon_2 \bar{w} \rangle)} + \epsilon_3 \bar{e}_i \rangle$ where $\epsilon = -1$ if the last coordinate of $\bar{v}_1 + \epsilon_2 \bar{w}$ is negative, and $\epsilon = +1$ otherwise. Lemma 3.12 also implies that $r(\langle \bar{v}_1 + \epsilon_3 \bar{e}_i \rangle) = \langle \overline{r(v_1)} + \epsilon_3 \bar{e}_i \rangle$ and that $r(\langle \bar{v}_1 + \epsilon_2 \bar{w} \rangle) \in \{r(v_1), \langle \bar{w} - \overline{r(v_1)} \rangle\}$. Note that $r(\langle \bar{v}_1 + \epsilon_2 \bar{w} \rangle) = \langle \bar{w} - \overline{r(v_1)} \rangle$ requires that $\epsilon_2 = -1$ and that $\epsilon = -1$ (compare with Lemma 3.12). Resolving the signs, it follows that if $z = \langle \bar{v}_1 + \epsilon_3 \bar{e}_i \rangle$, then

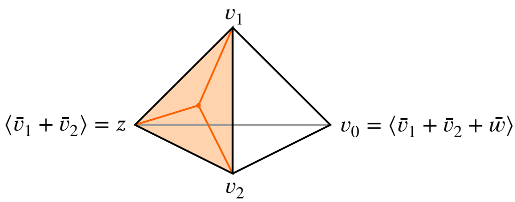


**Fig. 4.** The subdivision of $\partial\tau_1$ in the carrying case.

$$r(\tau_1) = \{r(v_0), r(v_1), r(z)\} = \left.\begin{matrix} \langle \overline{r(v_1)} + \epsilon_3 \bar{e}_i \rangle \\ \langle \overline{r(v_1)} - \bar{w} + \epsilon_3 \bar{e}_i \rangle \end{matrix}\right\} * r(v_1) * \langle \overline{r(v_1)} + \epsilon_3 \bar{e}_i \rangle$$

is a 2-additive or double-triple simplex. If $z = \langle \bar{v}_1 + \epsilon_2 \bar{w} \rangle$, then $r(\tau_1) = \{r(v_0), r(v_1), r(z)\}$ is equal to either

$$\{\langle \overline{r(v_1)} + \epsilon_3 \bar{e}_i \rangle, r(v_1), r(v_1)\},$$

which is externally 2-additive, or

$$\{\langle \overline{r(v_1)} - \bar{w} + \epsilon_3 \bar{e}_i \rangle, r(v_1), \langle \bar{w} - \overline{r(v_1)} \rangle\},$$

which is a double-triple simplex. Hence $r(\tau_1)$ forms a simplex in each case. By Lemma 3.21 we therefore conclude that $f(\sigma) = f(\tau_1) * f(\tau_2)$ is a simplex of the same type in $\widehat{\mathsf{Link}}^{<}_{\mathrm{BAA}_n^m}(w)$. □

We now work towards extending the retraction over minimal double-triple simplices that are 3-dimensional. The unique 3-additive facet of such simplices is either $w$-related or externally 3-additive. We start by considering 3-dimensional double-triple simplices whose unique 3-additive facet is $w$-related. The next observation explains why such minimal double-triple simplices can only have one carrying facet.

**Observation 3.44.** Let $\tau_1 = \gamma * z$ be a minimal double-triple simplex of dimension 3 whose unique 3-additive facet $\gamma = \{v_0 = \langle \bar{v}_1 + \epsilon_2 \bar{v}_2 + \epsilon_3 \bar{w} \rangle, v_1, v_2\}$ is $w$-related, where $\epsilon_2, \epsilon_3 \in \{-1, +1\}$. If $z = \langle \epsilon_k \bar{v}_k + \epsilon_3 \bar{w} \rangle$ for $k \in \{1, 2\}$ and $\epsilon_1 := +1$, then $\tau_1$ cannot have any carrying facet. If $z = \langle \bar{v}_1 + \epsilon_2 \bar{v}_2 \rangle$, then there exists a unique 2-additive facet $\{v_1, v_2, z\}$ that might be carrying, as in Fig. 4.

**Proof.** Recall that $z$ is contained in the additive core of any 2-additive facet of $\tau_1$. If $z = \langle \epsilon_k \bar{v}_k + \epsilon_3 \bar{w} \rangle$ for $k \in \{1, 2\}$, then any facet of $\tau_1$ is $w$-related and it follows from Definition 3.16 and Lemma 3.29 that no such simplex can be carrying. If $z = \langle \bar{v}_1 + \epsilon_2 \bar{v}_2 \rangle$, then all facets but the 2-additive facet $\{v_1, v_2, z\}$ are $w$-related. Hence, it follows from Definition 3.16 and Lemma 3.29 that $\{v_1, v_2, z\}$ is the unique possibly carrying facet. □

We now extend the retraction over the first type of minimal double-triple simplex of dimension 3.

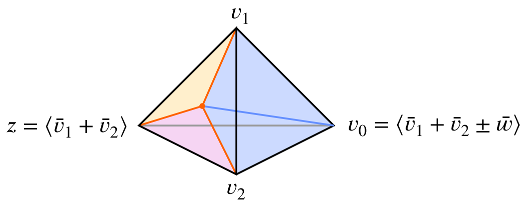


**Fig. 5.** The subdivision of $\tau_1 = \gamma * z$ in the carrying case.

**Lemma 3.45.** *The map $r$ introduced in Proposition 3.31 extends over all double-triple simplices $\sigma = \tau_1 * \tau_2$ in $\widehat{\mathsf{Link}}_{\mathrm{BAA}_n^m}(w)$, where $\tau_1 = \gamma * z$ is a minimal double-triple simplex of dimension 3 with $w$-related 3-additive facet $\gamma$ and $\tau_2$ is a standard simplex.*

More precisely, in the proof of Lemma 3.45 we check that the map

$$r\colon \operatorname{sd}(\widehat{\mathsf{Link}}_{\mathrm{TA}_n^m}(w)) \to \widehat{\mathsf{Link}}^{<}_{\mathrm{BAA}_n^m}(w)$$

in Proposition 3.31 extends over the simplex $\sigma = \tau_1 * \tau_2$ if the simplex is not carrying, and over a subdivision $\operatorname{sd}(\sigma) = \operatorname{sd}(\tau_1) * \tau_2$ if the simplex is carrying. Here, $\operatorname{sd}(\tau_1) = \operatorname{sd}(\eta) * v_0$ is the coarsest subdivision of $\tau_1$ that is compatible with the subdivision of its unique carrying 2-additive facet $\eta = \{v_1, v_2, z = \langle \bar{v}_1 + \bar{v}_2 \rangle\}$ described in Definition 3.19. The carrying case (illustrated in Fig. 5) occurs if and only if $\tau_1 = \{v_0 = \langle \bar{v}_1 + \bar{v}_2 \pm \bar{w} \rangle, v_1, v_2, z = \langle \bar{v}_1 + \bar{v}_2 \rangle\}$ contains a unique carrying 2-additive facet $\{v_1, v_2, z = \langle \bar{v}_1 + \bar{v}_2 \rangle\}$.

**Proof.** Let $\tau_1 = \gamma * z$ and $\gamma = \{v_0, v_1, v_2\}$ with $v_0 = \langle \bar{v}_1 + \epsilon_2 \bar{v}_2 + \epsilon_3 \bar{w} \rangle$ for two signs $\epsilon_2, \epsilon_3 \in \{-1, +1\}$. Then, $z \in \{\langle \bar{v}_1 + \epsilon_2 \bar{v}_2 \rangle, \langle \bar{v}_1 + \epsilon_3 \bar{w} \rangle, \langle \epsilon_2 \bar{v}_2 + \epsilon_3 \bar{w} \rangle\}$. We will sometimes use the convention that $\epsilon_1 := +1$. In this proof, we use exactly the same strategy as in the proof of Lemma 3.32. The underlying simplex of $\gamma$ in $\mathrm{BAA}_{m+n}$ is the set $\{v_0, v_1, v_2, v_3 = w\}$. We need to consider three cases, which are similar to the cases in the second part of the proof of Lemma 3.32: We again set $v_0' = \langle \bar{v}_1 + \epsilon_2 \bar{v}_2 \rangle$ and recall that

$$r(v_0) = r(\langle \bar{v}_1 + \epsilon_2 \bar{v}_2 + \epsilon_3 \bar{w} \rangle) \in \{r(v_0'), \langle \bar{w} - \overline{r(v_0')} \rangle\}$$

by Lemma 3.12. Furthermore, we note that the value of $r(v_0)$ depends on the last coordinate of $\bar{v}_1 + \epsilon_2 \bar{v}_2$, which might be negative, and the sign $\epsilon_3$ (compare with Lemma 3.12). There are three subcases.

*Case (a):* Assume that $\{v_1, v_2, v_0' = \langle \bar{v}_1 + \epsilon_2 \bar{v}_2 \rangle\}$ is not carrying and hence $r(v_0') = \langle \overline{r(v_1)} + \epsilon_2 \overline{r(v_2)} \rangle$. Using Lemma 3.12 to calculate $r(v_0)$ and $r(z)$, the possible values of $r(\tau_1) = \{r(v_0), r(v_1), r(v_2), r(z)\}$ are of the following form. If $z = \langle \bar{v}_1 + \epsilon_2 \bar{v}_2 \rangle$, then $r(\tau_1)$ is

$$\left.\begin{matrix} \langle \overline{r(v_1)} + \epsilon_2 \overline{r(v_2)} \rangle \\ \langle \overline{r(v_1)} + \epsilon_2 \overline{r(v_2)} + \epsilon_3 \bar{w} \rangle \end{matrix}\right\}$$

$$* \, r(v_1) * r(v_2) * \langle \overline{r(v_1)} + \epsilon_2 \overline{r(v_2)} \rangle \text{ which is 2-additive or double-triple.}$$

Let $k \in \{1,2\}$ and $\epsilon_1 := +1$. If $z = \langle \epsilon_k \bar{v}_k + \epsilon_3 \bar{w} \rangle$ and $r(z) = r(v_k)$, then $r(\tau_1)$ is

$$\left.\begin{matrix} \langle \overline{r(v_1)} + \epsilon_2 \overline{r(v_2)} \rangle \\ \langle \overline{r(v_1)} + \epsilon_2 \overline{r(v_2)} + \epsilon_3 \bar{w} \rangle \end{matrix}\right\} * r(v_1) * r(v_2) * r(v_k) \text{ which is 2-additive or 3-additive.}$$

If $z = \langle \epsilon_k \bar{v}_k + \epsilon_3 \bar{w} \rangle$ and $r(z) = \langle \epsilon_k \overline{r(v_k)} + \epsilon_3 \bar{w} \rangle$, then $r(\tau_1)$ is

$$\left.\begin{matrix} \langle \overline{r(v_1)} + \epsilon_2 \overline{r(v_2)} \rangle \\ \langle \overline{r(v_1)} + \epsilon_2 \overline{r(v_2)} + \epsilon_3 \bar{w} \rangle \end{matrix}\right\} * r(v_1) * r(v_2) * \langle \epsilon_k \overline{r(v_k)} + \epsilon_3 \bar{w} \rangle \text{ which is double-triple.}$$

It follows that $r(\tau_1)$ is a simplex in $\widehat{\mathsf{Link}}^{<}_{\mathrm{BAA}_n^m}(w)$. Then, an application of Lemma 3.21 implies that $r(\sigma) = r(\tau_1) * r(\tau_2)$ is a simplex in $\widehat{\mathsf{Link}}^{<}_{\mathrm{BAA}_n^m}(w)$ as claimed.

*Case (b)*: Assume that $\{v_1, v_2, v_0' = \langle \bar{v}_1 + \epsilon_2 \bar{v}_2 \rangle\}$ is carrying and that $z \in \{\langle \bar{v}_1 + \epsilon_3 \bar{w} \rangle, \langle \epsilon_2 \bar{v}_2 + \epsilon_3 \bar{w} \rangle\}$. We start by recording two observations.

- If the absolute value of the last coordinate of $v_k$ is maximal among $\{v_1, v_2, v_0'\}$ for $k \in \{1,2\} = \{k, k'\}$, then one obtains the relation $\overline{r(v_k)} = \overline{r(v_{k'})} + \overline{r(v_0')} - \bar{w}$. Note that in this case, the last coordinate of $\bar{v}_k$ is contained in $[R, \infty)$ and hence that, if $\epsilon_k \neq \epsilon_3$, it is impossible that $r(\langle \epsilon_k \bar{v}_k + \epsilon_3 \bar{w} \rangle) = \langle \bar{w} - \overline{r(v_k)} \rangle$ (compare with Lemma 3.12). Therefore, this case will not be considered below.
- If the absolute value of the last coordinate of $v_0'$ is maximal among $\{v_1, v_2, v_0'\}$, then one obtains the relation $\overline{r(v_0')} = \overline{r(v_1)} + \overline{r(v_2)} - \bar{w}$. As observed in case (b.2) in the proof of Lemma 3.32, it is impossible that $v_0'$ is maximal and $r(v_0) = \langle \bar{w} - \overline{r(v_0')} \rangle$. Hence, this case will not be considered below.

Using Lemma 3.12 to calculate $r(v_0)$ and $r(z)$, the possible values of $r(\tau_1) = \{r(v_0), r(v_1), r(v_2), r(z)\}$ are of the following form. Let $l \in \{1,2\}$. If $r(v_0) = r(v_0')$, then

$$\left.\begin{matrix} \langle \overline{r(v_k)} - \overline{r(v_{k'})} + \bar{w} \rangle \\ \langle \overline{r(v_1)} + \overline{r(v_2)} - \bar{w} \rangle \end{matrix}\right\} * r(v_1) * r(v_2) * r(v_l) \text{ is 3-additive or 2-additive,}$$

$$\{\langle \overline{r(v_k)} - \overline{r(v_{k'})} + \bar{w} \rangle, r(v_1), r(v_2), \langle \bar{w} - \overline{r(v_{k'})} \rangle\} \text{ is double-triple, or}$$

$$\{\langle \overline{r(v_1)} + \overline{r(v_2)} - \bar{w} \rangle, r(v_1), r(v_2), \langle \bar{w} - \overline{r(v_l)} \}\rangle\} \text{ is double-triple.}$$

If $r(v_0) = \langle \bar{w} - \overline{r(v_0')} \rangle$, then

$$\langle \overline{r(v_k)} - \overline{r(v_{k'})} \rangle * r(v_1) * r(v_2) * \begin{cases} r(v_l) \text{ is 2-additive, or} \\ \langle \bar{w} - r(v_l) \rangle \text{ is double-triple.} \end{cases}$$

It follows that $r(\tau_1)$ is a simplex in $\widehat{\mathsf{Link}}^{<}_{\mathrm{BAA}_n^m}(w)$. Hence Lemma 3.21 implies that $r(\sigma) = r(\tau_1) * r(\tau_2)$ is a simplex in $\widehat{\mathsf{Link}}^{<}_{\mathrm{BAA}_n^m}(w)$ as claimed.

*Case (c)*: Assume that $\eta = \{v_1, v_2, v_0' = \langle \bar{v}_1 + \epsilon_2 \bar{v}_2 \rangle\}$ is carrying and that $z = \langle \bar{v}_1 + \epsilon_2 \bar{v}_2 \rangle$. Then, the carrying facet $\eta = \{v_1, v_2, z\}$ of $\tau_1$ is subdivided into three simplices in $\mathrm{sd}(\widehat{\mathsf{Link}}_{\mathrm{TA}_n^m}(w))$ and we extend this subdivision to $\tau_1$ by replacing $\tau_1$ with the simplicial join $\mathrm{sd}(\tau_1) = \mathrm{sd}(\eta) * v_0$. The resulting subdivision $\mathrm{sd}(\tau_1)$ of $\tau_1$ consists of the following three 3-simplices

$$\{v_0, t(\eta)\} * \begin{cases} \{v_1, v_2\}, \\ \{v_1, \langle \bar{v}_1 + \epsilon_2 \bar{v}_2 \rangle\} \text{ and} \\ \{v_2, \langle \bar{v}_1 + \epsilon_2 \bar{v}_2 \rangle\}. \end{cases}$$

Recall that the barycentre $t(\eta)$ is mapped to $r(t(\eta)) = \langle \overline{r(v_l)} - \bar{w} \rangle$ for some choice $l \in \{1, 2\}$. Note that we must have $r(v_0) = r(v_0')$ if $\eta$ is carrying, i.e. we can't have $r(v_0) = \langle \bar{w} - \overline{r(v_0)} \rangle$ (compare with Lemma 3.12). The images of these simplices under $r$ are therefore given by the following.

$$\{\langle \overline{r(v_1)} + \overline{r(v_2)} - \bar{w} \rangle, \langle \overline{r(v_l)} - \bar{w} \rangle\} * \begin{cases} \{r(v_1), r(v_2)\} \text{ is double-triple,} \\ \{r(v_1), \langle \overline{r(v_1)} + \overline{r(v_2)} - \bar{w} \rangle\} \text{ is 2-additive and} \\ \{r(v_2), \langle \overline{r(v_1)} + \overline{r(v_2)} - \bar{w} \rangle\} \text{ is 2-additive.} \end{cases}$$

It follows that $r$ extends over $\mathrm{sd}(\tau_1)$. Hence, Lemma 3.21 implies that $r$ extends over any simplex in $\mathrm{sd}(\sigma) = \mathrm{sd}(\tau_1) * \tau_2 = \mathrm{sd}(\eta) * v_0 * \tau_2$. □

In the next step, we extend the retraction over all minimal double-triple simplices of dimension 3 whose unique 3-additive facet is externally 3-additive. The next observation records that if such a simplex is carrying, then it has exactly two carrying facets.

**Observation 3.46.** Let $\tau_1 = \gamma * z$ be a minimal double-triple simplex of dimension 3 whose unique 3-additive facet $\gamma$ is externally 3-additive. Assuming that $\bar{v}_0$ has maximal last coordinate, we get that $\gamma = \{v_0 = \langle \bar{v}_1 + \epsilon_2 \bar{v}_2 + \epsilon_3 \bar{e}_i \rangle, v_1, v_2\}$. If $\tau_1$ is carrying, then $\tau_1$ has exactly two carrying facets and it holds that $v_0 = \langle \bar{v}_1 + \bar{v}_2 + \epsilon_3 \bar{e}_i \rangle$, i.e. $\epsilon_2 = +1$. See an illustration in Fig. 6. In this case, the 3-additive facet $\gamma$ has to be carrying and the second carrying facet $\beta$ is the unique internally 2-additive facet of $\tau_1$, which is one of the following

$$\beta = \{v_0, v_{k'}, z = \langle \bar{v}_k + \epsilon_3 \bar{e}_i \rangle\} \text{ for } \{k, k'\} = \{1, 2\} \text{ or } \beta = \{v_1, v_2, z = \langle \bar{v}_1 + \bar{v}_2 \rangle\}.$$

**Proof.** If $\gamma$ is carrying, then it follows from Lemma 3.29 that $\gamma = \{v_0 = \langle \bar{v}_1 + \bar{v}_2 + \epsilon_3 \bar{e}_i \rangle, v_1, v_2\}$ with $b_1 + b_2 \in [R, 2R)$. It follows that $z \in \{\langle \bar{v}_l + \epsilon_3 \bar{e}_i \rangle, \langle \bar{v}_1 + \bar{v}_2 \rangle\}$ for

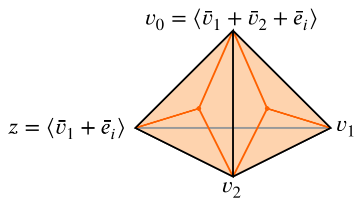


**Fig. 6.** The subdivision of $\partial\tau_1$ in the carrying case.

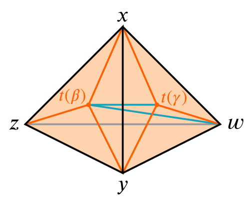


**Fig. 7.** The subdivision $\mathrm{sd}(\Delta^3)$.

$l \in \{1, 2\}$. Recall from Definition 3.16 that the only carrying 2-additive simplices are internally 2-additive. $\tau_1$ contains a unique internally 2-additive facet $\beta$ spanned by $\beta = \{v_0, v_{k'}, z = \langle \bar{v}_k + \epsilon_3 \bar{e}_i \rangle\}$ for $\{k, k'\} = \{1, 2\}$ or $\beta = \{v_1, v_2, z = \langle \bar{v}_1 + \bar{v}_2 \rangle\}$. Observe that $\beta$ has to be carrying, because $b_1 + b_2 \in [R, 2R)$. If $\gamma$ is not carrying, then the unique internally 2-additive facet $\beta$ cannot be carrying since adding or subtracting $\bar{e}_i$ does not change the last coordinate (compare with Lemma 3.12). □

We now finish our discussion on how to extend the retraction over minimal double-triple simplices of dimension 3. The following simplicial 3-disc and Corollary 3.41 will be used to describe the subdivision $\mathrm{sd}(\tau_1)$ of a carrying minimal double-triple simplex with externally 3-additive facet.

**Definition 3.47.** Let $\mathrm{sd}(\partial\Delta^3)$ be subdivision of the standard simplicial 2-sphere $\partial\Delta^3$ on the vertex set $\{w, x, y, z\}$ obtained by subdividing the facet $\gamma = \{w, x, y\}$ by placing the vertex $t(\gamma)$ at its barycentre and the facet $\beta = \{x, y, z\}$ by placing the vertex $t(\beta)$ at its barycentre. Let $\mathrm{sd}(\Delta^3)$ be the simplicial 3-disc that is obtained by extending the subdivision of $\mathrm{sd}(\partial\Delta^3)$ to a subdivision of the 3-simplex $\Delta^3$ using the following five 3-simplices (shown in Fig. 7),

$$\{t(\gamma), t(\beta), x, y\}, \quad \{t(\gamma), t(\beta), w, x\}, \quad \{t(\gamma), t(\beta), w, y\}, \quad \{t(\beta), w, x, z\}, \quad \text{and} \quad \{t(\beta), w, y, z\}.$$

**Lemma 3.48.** *The map $r$ in Proposition 3.31 extends over all double-triple simplices $\sigma = \tau_1 * \tau_2$ in $\widehat{\mathsf{Link}}_{\mathrm{BAA}_n^m}(w)$, where $\tau_1 = \gamma * z$ is a minimal double-triple simplex of dimension 3 with externally 3-additive facet $\gamma$ and $\tau_2$ is a standard simplex.*

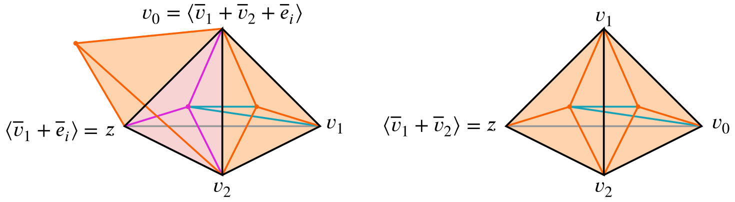


**Fig. 8.** The carrying cases of Lemma 3.48.

More precisely, in the proof of Lemma 3.48 we check that the map

$$r\colon \operatorname{sd}(\widehat{\mathsf{Link}}_{\mathrm{TA}_n^m}(w)) \to \widehat{\mathsf{Link}}^{<}_{\mathrm{BAA}_n^m}(w)$$

in Proposition 3.31 extends over the simplex $\sigma = \tau_1 * \tau_2$ if the simplex is not carrying, and over a subdivision $\operatorname{sd}(\sigma) = \operatorname{sd}(\tau_1) * \tau_2$ of $\sigma$ if the simplex is carrying. Here, the subdivision $\operatorname{sd}(\tau_1)$ is of the form $\operatorname{sd}(\Delta^3)$ described in Definition 3.47 or of the form $\operatorname{sd}(\Delta^3) \cup_{\operatorname{sd}(\beta)} D(\beta)$ using Definition 3.47 and applying Corollary 3.41 once to the internally 2-additive carrying facet $\beta$ of $\tau_1$. The carrying case occurs if and only if $\tau_1 = \{v_1, v_2, \langle \bar{v}_1 + \bar{v}_2 + \epsilon_3 \bar{e}_i \rangle, \langle \bar{v}_1 + \epsilon_3 \bar{e}_i \rangle\}$ for some $i \leq m$ and $\epsilon_3 \in \{+1, -1\}$ or $\tau_1 = \{v_1, v_2, \langle \bar{v}_1 + \bar{v}_2 + \epsilon_3 \bar{e}_i \rangle, \langle \bar{v}_1 + \bar{v}_2 \rangle\}$ for some $i \leq m$ and $\epsilon_3 \in \{+1, -1\}$. These cases are illustrated in Fig. 8.

**Proof.** Let $\tau_1 = \gamma * z$ and $\gamma = \{v_0, v_1, v_2\}$ with $v_0 = \langle \bar{v}_1 + \epsilon_2 \bar{v}_2 + \epsilon_3 \bar{e}_i \rangle$ for $\epsilon_2, \epsilon_3 \in \{+1, -1\}$. Then, $z \in \{\langle \bar{v}_1 + \epsilon_2 \bar{v}_2 \rangle, \langle \bar{v}_1 + \epsilon_3 \bar{e}_i \rangle, \langle \epsilon_2 \bar{v}_2 + \epsilon_3 \bar{e}_i \rangle\}$.

Firstly, assume that $\tau_1$ is not carrying. Then, it holds that $r(v_0) = \langle \overline{r(v_1)} + \epsilon_2 \overline{r(v_2)} + \epsilon_3 \bar{e}_i \rangle$ and $r(z) \in \{\langle \overline{r(v_1)} + \epsilon_2 \overline{r(v_2)} \rangle, \langle \overline{r(v_1)} + \epsilon_3 \bar{e}_i \rangle, \langle \epsilon_2 \overline{r(v_2)} + \epsilon_3 \bar{e}_i \rangle\}$ using Lemma 3.12. It follows that

$$\{\langle \overline{r(v_1)} + \epsilon_2 \overline{r(v_2)} + \epsilon_3 \bar{e}_i \rangle, r(v_1), r(v_2)\} * \begin{cases} \langle \overline{r(v_1)} + \epsilon_2 \overline{r(v_2)} \rangle \text{ is double-triple,} \\ \langle \overline{r(v_1)} + \epsilon_3 \bar{e}_i \rangle \text{ is double-triple, and} \\ \langle \epsilon_2 \overline{r(v_2)} + \epsilon_3 \bar{e}_i \rangle \text{ is double-triple.} \end{cases}$$

Hence, $r(\tau_1)$ spans a simplex in $\widehat{\mathsf{Link}}^{<}_{\mathrm{BAA}_n^m}(w)$ and, by Lemma 3.21, it therefore follows that $r(\sigma) = r(\tau_1) * r(\tau_2)$ is a simplex in $\widehat{\mathsf{Link}}^{<}_{\mathrm{BAA}_n^m}(w)$ as claimed.

Secondly, assume that $\tau_1$ is carrying. Then, it holds by Observation 3.46 that $\tau_1$ contains exactly two carrying facets and that we may assume $v_0 = \langle \bar{v}_1 + \bar{v}_2 + \epsilon_3 \bar{e}_i \rangle$, i.e. $\epsilon_2 = +1$. It follows that $r(v_0) = \langle \overline{r(v_1)} + \overline{r(v_2)} - \bar{w} + \epsilon_3 \bar{e}_i \rangle$.

*Case (a)*: Assume that $z = \langle \bar{v}_1 + \bar{v}_2 \rangle$. Observation 3.46 implies that the two carrying facets of $\tau_1$ are the 3-additive facet $\gamma$ and the unique internally 2-additive facet $\beta = \{v_1, v_2, z\}$. Since $\beta$ is carrying, we have that $\overline{r(z)} = \overline{r(v_1)} + \overline{r(v_2)} - \bar{w}$ (compare with Definition 3.16). The two facets $\gamma = \{v_0, v_1, v_2\}$ and $\beta = \{v_1, v_2, z\}$ of $\tau_1$ have been subdivided in $\widehat{\mathsf{Link}}_{\mathrm{TA}_n^m}(w)$. Applying Corollary 3.41 once, it suffices to show that $r$ extends over the subdivision $\operatorname{sd}(\tau_1)$ of $\tau_1$ that extends $\operatorname{sd}(\partial \tau_1)$ as described in Definition 3.47, for the case $r(t(\gamma)) = \langle\overline{r(v_1)} + \overline{r(v_2)} - \bar{w}\rangle$ and $r(t(\beta)) = \langle\overline{r(v_1)} - \bar{w}\rangle$.[9] The following shows that the image of every simplex in $\mathrm{sd}(\tau_1)$ (compare with Definition 3.47) is a simplex of $\widehat{\mathsf{Link}}^{<}_{\mathrm{BAA}_n^m}(w)$,

$$\{r(t(\gamma)), r(t(\beta))\} * \begin{cases} \{r(v_1), r(v_2)\} \text{ is double-triple,} \\ \{r(v_0) = \langle\overline{r(t(\gamma))} + \epsilon_3\bar{e}_i\rangle, r(v_1)\} \text{ is double-double,} \\ \{r(v_0) = \langle\overline{r(t(\gamma))} + \epsilon_3\bar{e}_i\rangle, r(v_2)\} \text{ is double-triple,} \end{cases}$$

and,

$$\{r(t(\beta)), r(z) = \langle\overline{r(v_1)} + \overline{r(v_2)} - \bar{w}\rangle\} * \begin{cases} \{r(v_0), r(v_1)\} \text{ is double-double,} \\ \{r(v_0), r(v_2)\} \text{ is double-triple.} \end{cases}$$

It follows that the map extends the subdivision $\mathrm{sd}(\tau_1)$. By Lemma 3.21, it therefore follows that $r$ extends over any simplex in $\mathrm{sd}(\sigma) = \mathrm{sd}(\tau_1) * \tau_2$.

*Case (b)*: Assume that $z = \langle\bar{v}_k + \epsilon_3\bar{e}_i\rangle$ for $k \in \{1,2\} = \{k, k'\}$. Observation 3.46 implies that the two carrying facets of $\tau_1$ are the 3-additive facet $\gamma$ and the unique internally 2-additive facet $\beta = \{v_0, v_{k'}, z\}$. The two facets $\gamma = \{v_0, v_1, v_2\}$ and $\beta = \{v_0, v_{k'}, z\}$ of $\tau_1$ have been subdivided in $\widehat{\mathsf{Link}}_{\mathrm{TA}_n^m}(w)$. Applying Corollary 3.41 once, it suffices to show that $r$ extends over the subdivision $\mathrm{sd}(\tau_1)$ of $\tau_1$ that extends $\mathrm{sd}(\partial\tau_1)$ as described in Definition 3.47, for the case $r(t(\gamma)) = \langle\overline{r(v_1)} + \overline{r(v_2)} - \bar{w}\rangle$ and $r(t(\beta)) = \langle\overline{r(v_{k'})} - \bar{w}\rangle$.[10] The following shows that the image of every simplex in $\mathrm{sd}(\tau_1)$ is a simplex of $\widehat{\mathsf{Link}}^{<}_{\mathrm{BAA}_n^m}(w)$,

$$\{r(t(\gamma)), r(t(\beta))\} * \begin{cases} \{r(v_1), r(v_2)\} \text{ is double-triple,} \\ \{r(v_0) = \langle\overline{r(t(\gamma))} + \epsilon_3\bar{e}_i\rangle, r(v_{k'})\} \text{ is double-double,} \\ \{r(v_0) = \langle\overline{r(t(\gamma))} + \epsilon_3\bar{e}_i\rangle, r(v_k)\} \text{ is double-triple,} \end{cases}$$

and,

$$\{r(t(\beta)), r(z) = \langle\overline{r(v_k)} + \epsilon_3\bar{e}_i\rangle\} * \begin{cases} \{r(v_k), r(v_{k'})\} \text{ is double-double,} \\ \{r(v_0), r(v_k)\} \text{ is double-triple.} \end{cases}$$

It follows that the map extends the subdivision $\mathrm{sd}(\tau_1)$. By Lemma 3.21, it therefore follows that $r$ extends over any simplex in $\mathrm{sd}(\sigma) = \mathrm{sd}(\tau_1) * \tau_2$. □

In the final step, we prove that the retraction extends over minimal double-triple simplices of dimension 4, i.e. these are internally double-triple simplices. The next observation gives a large class of examples of such double-triple simplices with have the property that every facet is carrying.

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[9] The other choice for $r(t(\beta))$ is $\langle\overline{r(v_2)} - \bar{w}\rangle$.

[10] The other choice for $r(t(\beta))$ is $\langle\overline{r(v_k)} + \epsilon_3\bar{e}_i - \bar{w}\rangle$.

**Observation 3.49.** Let $\tau_1 = \gamma * z$ be a double-triple simplex of dimension 4. Then any face of $\tau_1$ can be a carrying simplex. This is for example the case for

$$\{v_0, v_1, v_2, \langle \bar{v}_0 + \bar{v}_1\rangle, \langle \bar{v}_0 + \bar{v}_1 + \bar{v}_2\rangle\} \text{ with } b_0 + b_1 + b_2 \in [2R, 3R).$$

Up to this point, the construction of the retraction involved explicit subdivisions. For this last case, the complexity is great enough that we will resort to computer calculations. In particular, we will use computers to check high connectivity of the following simplicial complexes, which will aid in our construction of the retraction.

**Definition 3.50.** Let $n \geq 4$ and $\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{w} \in \mathbb{Z}^{m+n}$ be a partial basis such that $\{v_1, v_2, v_3, w\}$ is a simplex of $B_n^m$. Assume that the last coordinate $R$ of $\vec{w}$ is positive and that the last coordinates of $\vec{v}_1, \vec{v}_2, \vec{v}_3$ have absolute value smaller than $R$, that is, $v_i \in \widehat{\mathsf{Link}}^{<}_{\mathrm{BAA}_n^m}(w)$. Let $Q_n^m(\vec{v}_1, \vec{v}_2, \vec{v}_3; \vec{w})$ be the full subcomplex of $\widehat{\mathsf{Link}}^{<}_{\mathrm{BAA}_n^m}(w)$ on the set of lines spanned by vectors of the form

i) $\vec{v}_1 + a_1\vec{w}$,
ii) $\vec{v}_2 + a_2\vec{w}$,
iii) $\vec{v}_3 + a_3\vec{w}$,
iv) $\vec{v}_1 + \vec{v}_2 + a_{12}\vec{w}$,
v) $\vec{v}_1 + \vec{v}_2 + \vec{v}_3 + a_{123}\vec{w}$, or
vi) $\vec{v}_1 + \vec{v}_3 + a_{13}\vec{w}$

for $a_i \in \mathbb{Z}$.

**Theorem 3.51.** *The complexes $Q_n^m(\vec{v}_1, \vec{v}_2, \vec{v}_3; \vec{w})$ are 3-connected for all $m, n \in \mathbb{N}_0$ satisfying $m \geq 0$ and $n \geq 4$.*

This theorem will be shown in Section 4 with the help of computer calculations. We will assume it for now to deal with the last case for defining the retraction:

**Lemma 3.52.** *The map $r$ in Proposition 3.31 extends over all double-triple simplices $\sigma = \tau_1 * \tau_2$ in $\widehat{\mathsf{Link}}_{\mathrm{BAA}_n^m}(w)$ where $\tau_1$ is a minimal double-triple simplex of dimension 4 and $\tau_2$ is a standard simplex.*

More precisely, in the proof of Lemma 3.52 we check that the map

$$r\colon \operatorname{sd}(\widehat{\mathsf{Link}}_{\mathrm{TA}_n^m}(w)) \to \widehat{\mathsf{Link}}^{<}_{\mathrm{BAA}_n^m}(w)$$

in Proposition 3.31 extends over the simplex $\sigma = \tau_1 * \tau_2$ if the simplex is not carrying, and over a subdivision $\operatorname{sd}(\sigma) = \operatorname{sd}(\tau_1) * \tau_2$ of $\sigma$ if the simplex is carrying. On the subdivision $\operatorname{sd}(\tau_1)$ of $\tau_1$ the extension takes values in one of the complexes $Q_n^m(\vec{a}, \vec{b}, \vec{c}; \vec{w})$ introduced in Definition 3.50. The carrying case occurs if and only if $\tau_1 = \{v_0, v_1, v_2, \langle \bar{v}_0 + \bar{v}_1 \pm \bar{v}_2\rangle, \langle \bar{v}_0 + \bar{v}_1\rangle\}$ and $b_0 + b_1 \notin [0, R)$ or $b_0 + b_1 \pm b_2 \notin [0, R)$.

**Proof.** Let $\tau_1$ be a minimal internal double-triple simplex. Let $\operatorname{sd}(\partial\tau_1)$ be the subdivision of $\partial\tau_1$ in $\widehat{\mathsf{Link}}_{\mathrm{TA}_n^m}(w)$. By Proposition 3.31 we obtain a map

$$r\colon \operatorname{sd}(\partial\tau_1) \to \widehat{\mathsf{Link}}^{<}_{\mathrm{BAA}^m_n}(w).$$

Let $\gamma \subset \tau_1$ be the unique 3-additive facet. As in Lemma 3.34 et seq., we fix three lines $\{v_0, v_1, v_2\} \subseteq \gamma$ such that

$$\gamma = \{v_0, v_1, v_2, v_3 = \langle \bar{v}_0 + \bar{v}_1 + (\epsilon_2\bar{v}_2)\rangle\}$$

for the choice of a sign $\epsilon_2 \in \{-1, 1\}$ and where the absolute value of the last coordinate of $\bar{v}_3$ is maximal in $\gamma$. Then, $\tau_1 = \gamma * v_4$ with

$$v_4 = \begin{cases} \langle \bar{v}_0 + \bar{v}_1\rangle, \\ \langle \bar{v}_1 + (\epsilon_2\bar{v}_2)\rangle, \\ \langle \bar{v}_0 + (\epsilon_2\bar{v}_2)\rangle. \end{cases}$$

Firstly, assume that $\tau_1$ is not carrying. Then, $\overline{r(v_3)} = \overline{r(v_0)} + \overline{r(v_1)} + \epsilon_2\overline{r(v_2)}$ and

$$\pm\overline{r(v_4)} = \begin{cases} \overline{r(v_0)} + \overline{r(v_1)}, \\ \overline{r(v_0)} + \epsilon_2\overline{r(v_2)}, \\ \overline{r(v_1)} + \epsilon_2\overline{r(v_2)}. \end{cases}$$

It follows that $r(\tau_1)$ is a double-triple simplex. By Lemma 3.21, it follows that $r(\sigma) = r(\tau_1) * r(\tau_2)$ is a double-triple simplex as well.

Secondly, assume that $\tau_1$ is carrying. Consider the complexes $Q_{01} = Q^m_n(\overline{r(v_0)}, \overline{r(v_1)}, \epsilon_2 \cdot \overline{r(v_2)}; \bar{w})$ and $Q_{10} = Q^m_n(\overline{r(v_1)}, \overline{r(v_0)}, \epsilon_2 \cdot \overline{r(v_2)}; \bar{w})$ introduced in Definition 3.50. We claim that $r(\operatorname{sd}(\partial\tau_1))$ is contained in $Q = Q_{01}$ or $Q = Q_{10}$. To see this, it suffices to check that the image of every vertex in $\operatorname{sd}(\partial\tau_1)$ is contained in this complex. We will explain how to choose between $Q_{01}$ and $Q_{10}$ in the first step of the proof of this claim.

*Step (a)*: Every vertex in $r(\partial\tau_1)$ is contained in $Q$. Indeed, observe that

$$\pm\overline{r(v_4)} = \begin{cases} \overline{r(v_0)} + \overline{r(v_1)} + a_{01}\bar{w}, \\ \overline{r(v_1)} + (\epsilon_2\overline{r(v_2)}) + a_{12}\bar{w}, \\ \overline{r(v_0)} + (\epsilon_2\overline{r(v_2)}) + a_{02}\bar{w}, \end{cases}$$

for $a_{01}, a_{12}, a_{02} \in \{-1, 0, 1\}$. Hence, $r(v_4) \in Q_{10}$ in the first two cases and $r(v_4) \in Q_{01}$ in the third case. Fix this choice of $Q$. Observe furthermore that $r(v_0), r(v_1), r(v_2) \in Q$ and that $r(v_3) \in Q$, since $\overline{r(v_3)} = \overline{r(v_0)} + \overline{r(v_1)} + \epsilon_2\overline{r(v_2)} + a_{012} \cdot \bar{w}$ for $a_{012} \in \{-2, -1, 0, 1\}$ by Lemma 3.34 et seq.

*Step (b)*: Assume that the unique 3-additive facet $\gamma$ of $\tau_1$ is carrying and hence subdivided in $\operatorname{sd}(\partial\tau_1)$ using the new vertex $t(\gamma)$. Lemma 3.34 et seq. shows that $r(t(\gamma)) = r(\langle \bar{v}_0 + \bar{v}_1\rangle)$ is equal to $\langle \overline{r(v_0)} + \overline{r(v_1)} - \bar{w}\rangle$ or $\langle \overline{r(v_0)} + \overline{r(v_1)}\rangle$. In either case, $r(t(\gamma))$ is contained in $Q = Q_{01}$ and $Q = Q_{10}$.

*Step (c)*: Assume that one of the 2-additive facets $\alpha$ of $\tau_1$ is carrying and hence subdivided in $\mathrm{sd}(\partial\tau_1)$. Let $\eta \subset \alpha$ denote the minimal 2-additive simplex that has been barycentrically subdivided using the new vertex $t(\eta)$. Then,

$$r(t(\eta)) = \langle \overline{r(v_l)} - \bar{w} \rangle \text{ for some vertex } v_l \in \eta \subset \tau_1.$$

By Step $(a)$ it holds that $r(v_l) \in Q$ for any vertex $v_l \in \tau_1$ and since the last coordinate of $\overline{r(v_l)} - \bar{w}$ is contained in $(-R, 0)$, it therefore holds that

$$r(t(\eta)) = \langle \overline{r(v_l)} - \bar{w} \rangle = \langle -\overline{r(v_l)} + \bar{w} \rangle \in Q.$$

This completes the proof of the claim that $r(\mathrm{sd}(\partial\tau_1))$ is contained in $Q = Q_{01}$ or $Q = Q_{10}$.

It follows that

$$r\colon \mathrm{sd}(\partial\tau_1) \to Q \hookrightarrow \widehat{\mathsf{Link}}^{<}_{\mathrm{BAA}^m_n}(w).$$

Recall that $\mathrm{sd}(\partial\tau_1)$ is a simplicial 3-sphere. By Theorem 3.51 the complex $Q$ is 3-connected. It follows, that there exists a simplicial pair $(\mathrm{sd}(\tau_1), \mathrm{sd}(\partial\tau_1)) \cong (D^4, S^3)$ and a simplicial extension $r_{\tau_1}\colon \mathrm{sd}(\tau_1) \to Q$ of $r\colon \mathrm{sd}(\partial\tau_1) \to Q$. Subdivide every simplex $\sigma = \tau_1 * \tau_2$ by using the coarsest simplicial structure $\mathrm{sd}(\sigma)$ on $\sigma$ that is compatible with the simplicial structure specified by $\mathrm{sd}(\tau_1)$ on $\tau_1$. I.e. this is defined by replacing the internal double-triple simplex $\sigma = \tau_1 * \tau_2$ by the collection of simplices

$$\{\nu * \tau_2 \mid \nu \text{ a simplex of } \mathrm{sd}(\tau_1)\}.$$

An application of Lemma 3.21 for $\sigma = \tau_1 * \tau_2$ implies that we can extend the map

$$r_{\tau_1}\colon \mathrm{sd}(\tau_1) \to \widehat{\mathsf{Link}}_{\mathrm{BAA}^m_n}(w)$$

to a map

$$r_{\tau_1} * r\colon \mathrm{sd}(\sigma) = \mathrm{sd}(\tau_1) * \tau_2 \to \widehat{\mathsf{Link}}_{\mathrm{BAA}^m_n}(w).$$

This completes the proof. □

Lemma 3.43, Lemma 3.45, Lemma 3.48 and Lemma 3.52 imply Proposition 3.39 and Lemma 3.37, so this concludes our discussion of double-triple simplices. Furthermore, the proof of Proposition 3.39 completes the construction of the retraction map and establishes the main result of this section, Theorem 3.1.

## 4. High connectivity of the complexes $Q^m_n(\vec{v}_1, \vec{v}_2, \vec{v}_3; \vec{w})$

The aim of this section is to prove Theorem 3.51, which states that the complexes $Q^m_n(\vec{v}_1, \vec{v}_2, \vec{v}_3; \vec{w})$ introduced in Definition 3.50 are 3-connected for all $m, n \in \mathbb{N}_0$. This

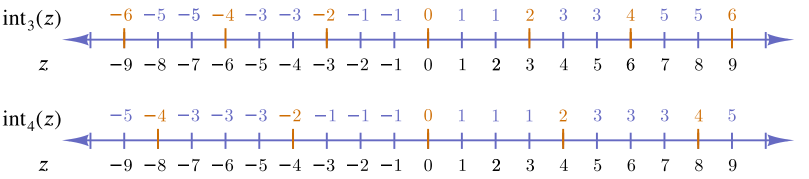


**Fig. 9.** Some values of $\operatorname{int}_3(z)$ and $\operatorname{int}_4(z)$.

was used to define the retraction on double-triple simplices. Throughout this section, we assume that $n \geq 4$.

To prove this theorem, we will first observe that all of these complexes are finite and then show that there is a finite list that contains all of their isomorphism types. Afterwards, we use a computer to verify that the reduced homology of this finite list of finite simplicial complexes vanishes in homological degrees $i \leq 3$ and that each complex is simply connected. The result then follows from Hurewicz's theorem.

### *4.1. Listing the isomorphism types*

We start by introducing notation that will be useful for studying the isomorphism types of $Q_n^m(\vec{v}_1, \vec{v}_2, \vec{v}_3; \vec{w})$. It slightly differs from similar notation used in previous sections, but allows for an easy formalisation on a computer.

**Definition 4.1.** Let $R \in \mathbb{Z}_{\geq 1}$ and $z \in \mathbb{Z}$. We define the *R-interval* of $z$ as

$$\operatorname{int}_R(z) := \begin{cases} 2k, & \text{if } z = kR \text{ for some } k \in \mathbb{Z}; \\ 2k+1, & \text{if } kR < z < (k+1)R \text{ for some } k \geq 0; \\ -(2k+1), & \text{if } -(k+1)R < z < -kR \text{ for some } k \geq 0. \end{cases}$$

If $\vec{v} \in \mathbb{Z}^{m+n}$ is a vector with last coordinate equal to $z \in \mathbb{Z}$, we write $\operatorname{int}_R(\vec{v}) := \operatorname{int}_R(z)$.

In other words, for $k \in \mathbb{Z}$, the $R$-interval of $z$ is $2k+1$ if $z$ lies in the open interval between $kR$ and $(k+1)R$ and it is equal to $2k$ if $z$ is equal to $kR$.

**Example 4.2.** The 3-intervals and 4-intervals of some integers are labelled in Fig. 9.

The next lemma contains three elementary observations about the $R$-interval function (with $R$ fixed) that say that it is close to being linear: It commutes with scalar multiplication by $-1$, it is always close to being additive and it is actually additive if one of the inputs is a multiple of $R$. See Lemma 3.12, which states similar results in a slightly different language.

**Lemma 4.3.** *Let $R \in \mathbb{Z}_{\geq 1}$ and $z, z_1, z_2 \in \mathbb{Z}$. Then*

*i)* $\operatorname{int}_R(-z) = -\operatorname{int}_R(z)$,
*ii)* $\operatorname{int}_R(z_1+z_2) \in \{\operatorname{int}_R(z_1) + \operatorname{int}_R(z_2) - 1, \operatorname{int}_R(z_1) + \operatorname{int}_R(z_2), \operatorname{int}_R(z_1) + \operatorname{int}_R(z_2)+1\}$, *and*
*iii)* $\operatorname{int}_R(z_1 + z_2) = \operatorname{int}_R(z_1) + \operatorname{int}_R(z_2)$ *if at least one of* $\operatorname{int}_R(z_1)$, $\operatorname{int}_R(z_2)$ *is even.*

**Proof.** All three claims follow immediately from the definitions. □

The reason that we use these $R$-intervals is that they allow us to give a formal description of the vertex set of $Q_n^m(\vec{v}_1, \vec{v}_2, \vec{v}_3; \vec{w})$. Using Lemma 4.3, it is easy to deduce the following properties:

**Lemma 4.4.** *Let* $\vec{v} \in \mathbb{Z}^{m+n}$*, let* $\vec{w} \in \mathbb{Z}^{m+n}$ *be a vector with last coordinate equal to* $R \geq 1$ *and* $a \in \mathbb{Z}$*.*

*i) The last coordinate of* $\vec{v}$ *has absolute value strictly smaller than* $R$ *if and only if* $\operatorname{int}_R(\vec{v}) \in \{-1, 0, 1\}$*.*
*ii) If* $\operatorname{int}_R(\vec{v}) = 2k$ *is even, then the last coordinate of* $\vec{v} + a\vec{w}$ *has absolute value strictly smaller than* $R$ *if and only if* $a = -k$*.*
*iii) If* $\operatorname{int}_R(\vec{v}) = 2k + 1$ *is odd, then the last coordinate of* $\vec{v} + a\vec{w}$ *has absolute value strictly smaller than* $R$ *if and only if* $a \in \{-(k + 1), -k\}$*.*

A consequence of Lemma 4.4 is that all the $Q_n^m(\vec{v}_1, \vec{v}_2, \vec{v}_3; \vec{w})$ are finite simplicial complexes on at most 12 vertices. Our next aim is to create an explicit (finite) list of simplicial complexes such that $Q_n^m(\vec{v}_1, \vec{v}_2, \vec{v}_3; \vec{w})$ is isomorphic to one of these for every list of elements $(\vec{v}_1, \vec{v}_2, \vec{v}_3; \vec{w})$. This list will consist of complexes of the following form.

**Definition 4.5.** Let $\vec{e}_1, \vec{e}_2, \vec{e}_3, \vec{e}_4$ denote the standard basis of $\mathbb{Z}^4$. We write $Q(r_1, r_2, r_3, r_{12}, r_{123}, r_{23})$ for the full subcomplex of $\mathsf{Link}_{\mathrm{BAA}_4}(e_4)$ on all lines spanned by vertices of the form

i) $\vec{e}_1 + a_1\vec{e}_4$,
ii) $\vec{e}_2 + a_2\vec{e}_4$,
iii) $\vec{e}_3 + a_3\vec{e}_4$,
iv) $\vec{e}_1 + \vec{e}_2 + a_{12}\vec{e}_4$,
v) $\vec{e}_1 + \vec{e}_2 + \vec{e}_3 + a_{123}\vec{e}_4$, or
vi) $\vec{e}_1 + \vec{e}_3 + a_{13}\vec{e}_4$,

where $a_i = -k$ if $r_i = 2k$ is even and $a_i \in \{-(k + 1), -k\}$ if $r_i = 2k + 1$ is odd.

The following key proposition tells us that the isomorphism type of any complex $Q_n^m(\vec{v}_1, \vec{v}_2, \vec{v}_3; \vec{w})$ is determined by six integers, namely the $R$-intervals of $\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_1 + \vec{v}_2, \vec{v}_1 + \vec{v}_2 + \vec{v}_3$ and $\vec{v}_1 + \vec{v}_3$.

**Proposition 4.6.** *Let* $\vec{v}_1, \vec{v}_2, \vec{v}_3$ *and* $\vec{w}$ *be as in Definition 3.50 and let*

$$(r_1, r_2, r_3, r_{12}, r_{123}, r_{13}) := (\operatorname{int}_R(\vec{v}_1), \operatorname{int}_R(\vec{v}_2), \operatorname{int}_R(\vec{v}_3), \operatorname{int}_R(\vec{v}_1 + \vec{v}_2), \operatorname{int}_R(\vec{v}_1 + \vec{v}_2 + \vec{v}_3),$$

$\mathrm{int}_R(\vec{v}_1 + \vec{v}_3))$.

*Then there is an isomorphism*

$$\phi\colon Q_n^m(\vec{v}_1, \vec{v}_2, \vec{v}_3; \vec{w}) \to Q(r_1, r_2, r_3, r_{12}, r_{123}, r_{13}).$$

*In particular, the isomorphism type of* $Q_n^m(\vec{v}_1, \vec{v}_2, \vec{v}_3; \vec{w})$ *only depends on the six-tuple of integers* $(r_1, r_2, r_3, r_{12}, r_{123}, r_{13})$.

**Proof.** To shorten notation in this proof, we set $\vec{v}_{12} \coloneqq \vec{v}_1 + \vec{v}_2$, $\vec{v}_{123} \coloneqq \vec{v}_1 + \vec{v}_2 + \vec{v}_3$, $\vec{v}_{13} \coloneqq \vec{v}_1 + \vec{v}_3$ and $\vec{e}_{12} \coloneqq \vec{e}_1 + \vec{e}_2$, $\vec{e}_{123} \coloneqq \vec{e}_1 + \vec{e}_2 + \vec{e}_3$, $\vec{e}_{13} \coloneqq \vec{e}_1 + \vec{e}_3$. With this, we have $\mathrm{int}_R(\vec{v}_i) = r_i$ and by Lemma 4.4, the span of $\vec{v}_i + a\vec{w}$ is a vertex in $Q_n^m(\vec{v}_1, \vec{v}_2, \vec{v}_3; \vec{w})$ if and only if the span of $\vec{e}_i + a\vec{e}_4$ is a vertex in $Q(r_1, r_2, r_3, r_{12}, r_{123}, r_{13})$. This gives rise to an obvious bijection $\phi$ between the vertex sets of the two complexes.

We want to show that $\phi$ induces a simplicial isomorphism. Let $l_0, \dots, l_k \in Q_n^m(\vec{v}_1, \vec{v}_2, \vec{v}_3; \vec{w})$. We need to show that $l_0, \dots, l_k$ form a simplex if and only if their images $\phi(l_0), \dots, \phi(l_k)$ do. Spelling out the definitions, one sees that $\{l_0, \dots, l_k\}$ is a simplex in $Q_n^m(\vec{v}_1, \vec{v}_2, \vec{v}_3; \vec{w})$ if and only if $\{l_0, \dots, l_k, w, e_1, \dots, e_m\}$ is a simplex in $\mathrm{BAA}_{m+n}$ and none of the $\vec{l}_i$ is in the span $\langle \vec{e}_1, \dots, \vec{e}_m, \vec{w}\rangle$ (with a slight abuse of notation, we use the same symbols $\vec{e}_i$ to denote the standard basis of $\mathbb{Z}^{m+n}$ and $\mathbb{Z}^4$). A set of vectors gives rise to a simplex in $\mathrm{BAA}_i$ if up to two of them are certain linear combinations of the others and the others from a partial basis; the form of the linear combinations depends on the type of simplex, see the definitions in Section 2. Assume that there is a linear dependency between $\left\{\vec{l}_0, \dots, \vec{l}_k, \vec{w}, \vec{e}_1, \dots, \vec{e}_m\right\}$, i.e. there are $c_i, d_j$ such that

$$\underbrace{\sum_{i=0}^{k} c_i \vec{l}_i + c_{k+1}\vec{w}}_{\in \langle \vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{w}\rangle} + \underbrace{\sum_{j=1}^{m} d_j \vec{e}_j}_{\in \langle \vec{e}_1, \dots, \vec{e}_m\rangle} = 0. \tag{2}$$

By assumption, $\{v_1, v_2, v_3, w\}$ is a simplex in $B_n^m$, which means that $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{w}, \vec{e}_1, \dots, \vec{e}_m\}$ is a partial basis. Hence, Equation (2) implies $\sum_{i=0}^{k} c_i \vec{l}_i + c_{k+1}\vec{w} = 0$. It follows that $\{l_0, \dots, l_k, w, e_1, \dots, e_m\}$ is a simplex in $\mathrm{BAA}_{m+n}$ if and only if $\{l_0, \dots, l_k, w\}$ is a simplex of the same type in the full subcomplex of $\mathrm{BAA}_{m+n}$ on all lines that are contained in $\langle \vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{w}\rangle \cong \mathbb{Z}^4$. The latter is clearly equivalent to saying that $\{\phi(l_0), \dots, \phi(l_k), \phi(w) = e_4\}$ is a simplex in $\mathrm{BAA}_4$, i.e. that $\{\phi(l_0), \dots, \phi(l_k)\}$ is a simplex in $Q(r_1, r_2, r_3, r_{12}, r_{123}, r_{13})$. $\square$

By the above proposition, we can produce a list with all isomorphism types of the complexes $Q_n^m(\vec{v}_1, \vec{v}_2, \vec{v}_3; \vec{w})$ by listing the possible combinations of $R$-intervals of $\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_1 + \vec{v}_2, \vec{v}_1 + \vec{v}_2 + \vec{v}_3$ and $\vec{v}_1 + \vec{v}_3$. Before we do this in Corollary 4.8, we record in the following lemma isomorphisms between these complexes. These are easy to show

**Table 1**
A list of 48 tuples $(r_1, r_2, r_3, r_{12}, r_{123}, r_{13})$ containing (at least) one representative for each isomorphism type $Q(r_1, r_2, r_3, r_{12}, r_{123}, r_{13})$.

| $r_1$ | $r_2$ | $r_3$ | $r_{12}$ | $r_{123}$ | $r_{13}$ |
|---|---|---|---|---|---|
| -1 | -1 | -1 | -3 | -5 | -3 |
| -1 | -1 | -1 | -3 | -4 | -3 |
| -1 | -1 | -1 | -3 | -3 | -3 |
| -1 | -1 | -1 | -3 | -3 | -2 |
| -1 | -1 | -1 | -3 | -3 | -1 |
| -1 | -1 | -1 | -2 | -3 | -2 |
| -1 | -1 | -1 | -2 | -3 | -1 |
| -1 | -1 | -1 | -1 | -3 | -1 |
| -1 | -1 | -1 | -1 | -2 | -1 |
| -1 | -1 | -1 | -1 | -1 | -1 |
| -1 | -1 | 0 | -3 | -3 | -1 |
| -1 | -1 | 0 | -2 | -2 | -1 |
| -1 | -1 | 0 | -1 | -1 | -1 |
| -1 | -1 | 1 | -3 | -3 | -1 |
| -1 | -1 | 1 | -3 | -2 | -1 |
| -1 | -1 | 1 | -3 | -1 | -1 |
| -1 | -1 | 1 | -3 | -1 | 0 |
| -1 | -1 | 1 | -3 | -1 | 1 |
| -1 | -1 | 1 | -2 | -1 | -1 |
| -1 | -1 | 1 | -2 | -1 | 0 |
| -1 | -1 | 1 | -2 | -1 | 1 |
| -1 | -1 | 1 | -1 | -1 | -1 |
| -1 | -1 | 1 | -1 | -1 | 0 |
| -1 | -1 | 1 | -1 | -1 | 1 |
| -1 | -1 | 1 | -1 | 0 | 1 |
| -1 | -1 | 1 | -1 | 1 | 1 |
| -1 | 0 | 0 | -1 | -1 | -1 |
| -1 | 0 | 1 | -1 | -1 | -1 |
| -1 | 0 | 1 | -1 | 0 | 0 |
| -1 | 0 | 1 | -1 | 1 | 1 |
| -1 | 1 | 1 | -1 | -1 | -1 |
| -1 | 1 | 1 | -1 | 0 | -1 |
| -1 | 1 | 1 | -1 | 1 | -1 |
| -1 | 1 | 1 | 0 | 1 | -1 |
| -1 | 1 | 1 | 0 | 1 | 0 |
| -1 | 1 | 1 | 0 | 1 | 1 |
| -1 | 1 | 1 | 1 | 1 | -1 |
| -1 | 1 | 1 | 1 | 1 | 1 |
| -1 | 1 | 1 | 1 | 2 | 1 |
| -1 | 1 | 1 | 1 | 3 | 1 |
| 0 | -1 | -1 | -1 | -3 | -1 |
| 0 | -1 | -1 | -1 | -2 | -1 |
| 0 | -1 | -1 | -1 | -1 | -1 |
| 0 | -1 | 0 | -1 | -1 | 0 |
| 0 | -1 | 1 | -1 | -1 | 1 |
| 0 | -1 | 1 | -1 | 0 | 1 |
| 0 | -1 | 1 | -1 | 1 | 1 |
| 0 | 0 | 0 | 0 | 0 | 0 |

and allow us to reduce the size of the list of isomorphism types, which is helpful for the computer calculations we want to perform.

**Lemma 4.7.** *Let $r_1, r_2, r_3, r_{12}, r_{123}, r_{13} \in \mathbb{Z}$. We have the following identities:*

*i)* $Q(r_1, r_2, r_3, r_{12}, r_{123}, r_{13}) \cong Q(r_1, r_3, r_2, r_{13}, r_{123}, r_{12})$;
*ii)* $Q(r_1, r_2, r_3, r_{12}, r_{123}, r_{13}) \cong Q(-r_1, -r_2, -r_3, -r_{12}, -r_{123}, -r_{13})$;

We would like to remark that these are not the only isomorphisms that exist between complexes $Q(r_1, r_2, r_3, r_{12}, r_{123}, r_{13})$ and $Q(r_1', r_2', r_3', r_{12}', r_{123}', r_{13}')$. However, they are sufficient to reduce the list of isomorphism types to a size that is small enough to allow computer calculations.

**Corollary 4.8.** *Let $\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{w}$ as in Definition 3.50. Then*

$$Q(\mathrm{int}_R(\vec{v}_1), \mathrm{int}_R(\vec{v}_2), \mathrm{int}_R(\vec{v}_3), \mathrm{int}_R(\vec{v}_1 + \vec{v}_2), \mathrm{int}_R(\vec{v}_1 + \vec{v}_2 + \vec{v}_3), \mathrm{int}_R(\vec{v}_1 + \vec{v}_3))$$

*agrees with $Q(r_1, r_2, r_3, r_{12}, r_{123}, r_{13})$ for one of the tuples $(r_1, r_2, r_3, r_{12}, r_{123}, r_{13})$ listed in Table 1.*

**Proof.** Let $\vec{v}_{12} := \vec{v}_1 + \vec{v}_2$, $\vec{v}_{123} := \vec{v}_1 + \vec{v}_2 + \vec{v}_3$, $\vec{v}_{13} := \vec{v}_1 + \vec{v}_3$ and let $z_i$ denote the last coordinate of $v_i$. As the last coordinates of $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are smaller than $R$, we have

$$\mathrm{int}_R(z_i) \in \{-1, 0, 1\} \text{ for } 1 \le i \le 3.$$

Furthermore, by Lemma 4.3, we know that $\mathrm{int}_R(z_{12}) = \mathrm{int}_R(z_1 + z_2)$ is either equal to the sum $\mathrm{int}_R(z_1) + \mathrm{int}_R(z_2)$ or differs from it by at most 1, depending on the parity of $\mathrm{int}_R(z_1)$ and $\mathrm{int}_R(z_2)$. Similarly, $\mathrm{int}_R(z_{123}) = \mathrm{int}_R(z_1 + z_2 + z_3)$ can differ from the sum $\mathrm{int}_R(z_{12}) + \mathrm{int}_R(z_3)$ by at most one. Also, $z_{13} = z_1 + z_3$ can be written both as $(z_1) + (z_3)$ and as $(z_1 + z_2 + z_3) - (z_2)$. Hence, its $R$-interval differs both from $\mathrm{int}_R(z_1) + \mathrm{int}_R(z_3)$ and from $\mathrm{int}_R(z_{123}) - \mathrm{int}_R(z_2)$ by at most one.

These rules allow one to generate a list with all possible tuples that can occur as

$$(\mathrm{int}_R(z_1), \mathrm{int}_R(z_2), \mathrm{int}_R(z_3), \mathrm{int}_R(z_{12}), \mathrm{int}_R(z_{123}), \mathrm{int}_R(z_{13})).$$

The list can be further shortened by using the identities of Lemma 4.7. We did this using computer calculations (available under https://github.com/benjaminbrueck/codim2_cohomology_SLnZ/blob/main/Connectivity%20Q%20complexes.ipynb) and the result was Table 1. □

### 4.2. Computer implementation of the complexes

To show that all the complexes obtained in Corollary 4.8 are indeed 3-connected, we use computer calculations. These are made available under the following link https://github.com/benjaminbrueck/codim2_cohomology_SLnZ.

The core of the calculations is a function, written in python, that takes as an input a set of vectors in $\mathbb{Z}^n$ and returns the subcomplex of $\mathrm{BAA}_n$ that is spanned by these vectors. This simplicial complex is implemented using the `Simplex Tree` module of GUDHI [15]. The GUDHI library was developed for topological data analysis. It allows to conveniently work with filtered simplicial complex and we used the filtration functionality to keep track of the type of the simplices (standard, 2-additive, 3-additive, double-triple or double-double). However, one cannot compute homology with integral coefficients in GUDHI. For performing these homology computations, we use the `SimplicialComplex` class of SageMath [23].

#### 4.2.1. Simplices by facet type

One fact that we used for building subcomplexes of $\mathrm{BAA}_n$ on a computer is that for many simplices, it is sufficient to know what types of simplices their facets form. This is used in the computations to check whether a set of vertices forms a simplex.

**Definition 4.9.** Let $S = \{v_0, \ldots, v_d\}$ be a set of vertices of $\mathrm{BAA}_n$ such that every $d$-element subset of $S$ forms a simplex in BAA. Then the *facet type* of $S$ is the multiset of simplex types that arise among these $d$-element subsets.

With slight abuse of notation, call a subset of size $k$ a facet of $S = \{v_0, \ldots, v_k\}$ (even if it does not necessarily form a simplex in $\mathrm{BAA}_n^m$).

**Example 4.10.** Let $e_1, e_2, e_3, e_4$ be the standard basis of $\mathbb{Z}^4$ and let $S = \{e_1, e_2, e_3, \langle \vec{e}_1 + \vec{e}_2 \rangle, \langle \vec{e}_1 + \vec{e}_3 \rangle, e_4\}$. Then the facet type of $S$ is

$$\{\text{3-additive}, \text{2-additive}, \text{2-additive}, \text{2-additive}, \text{2-additive}, \text{double-triple}\}.$$

**Observation 4.11.** If $\tau$ is one of the types of simplices defined in Section 2 and $d \in \mathbb{Z}$, then every set that forms a $d$-dimensional simplex of type $\tau$ has the same facet type. These types are as follows: Let $S$ be a set of vertices of BAA.

i) If $S$ forms a standard simplex then its facet type is

$$\{\text{standard}, \dots, \text{standard}\}.$$

ii) If $S$ forms a 2-additive simplex, then its facet type is

$$\{\text{standard}, \text{standard}, \text{standard}, \text{2-additive}, \dots, \text{2-additive}\}.$$

iii) If $S$ forms a 3-additive simplex, then its facet type is

$$\{\text{standard}, \text{standard}, \text{standard}, \text{standard}, \text{3-additive}, \dots, \text{3-additive}\}.$$

iv) If $S$ forms a double-triple simplex, then its facet type is

$$\{\text{2-additive}, \text{2-additive}, \text{3-additive}, \text{2-additive}, \text{2-additive}, \text{double-triple}, \dots, \text{double-triple}\}.$$

v) If $S$ forms a double-double simplex, then its facet type is

$$\{\text{2-additive}, \text{2-additive}, \text{2-additive}, \text{2-additive}, \text{2-additive}, \text{2-additive}, \text{double-double}, \dots, \text{double-double}\}.$$

We will see that in most cases, the converse of this is true as well, i.e. if we have a set of vertices whose facet type agrees with one of the types of the list above, then it already forms a simplex of the corresponding type.

**Definition 4.12.** Let $\tau$ be one of the types of simplices defined in Section 2 and let $d \in \mathbb{Z}$. We say that $\tau$ *is determined by its facet type in dimension* $d$ if the following is true: Given a set of vertices $S = \{v_0, \dots, v_d\}$ of $\mathrm{BAA}_n$ such that every $d$-element subset of $S$ forms a simplex in $\mathrm{BAA}_n$. Then $S$ forms a simplex of type $\tau$ if and only if it has the same facet type as a $d$-dimensional simplex of type $\tau$.

It is not hard to check that only 2- and 3-additive simplices are determined by their facet type if they are not minimal and that double-triple and double-double simplices are even determined by their facet types in all possible dimensions. We use these properties for the computer implementation of the complexes. We record them in the following lemma, but omit the (elementary) proofs.

**Lemma 4.13.** *The following hold:*

*i) An $m$-additive simplex is determined by its facet type in all dimensions $d > m$.*
*ii) A double-triple simplex is determined by its facet type in all dimensions $d \geq 4$.*
*iii) A double-double simplex is determined by its facet type in all dimensions $d \geq 5$.*

*4.3. Results of the homology calculations and simple connectivity*

In addition to computing the homology of the complexes $Q_n^m(\vec{v}_1, \vec{v}_2, \vec{v}_3; \vec{w})$, we need to show that they are simply connected. We will do this by showing that they are very "dense" and using the following criterion:

**Lemma 4.14.** *Let $Q$ be a simplicial complex with $k$ vertices. Assume that every pair of vertices forms an edge in $Q$ and that there are only $m$ triples of vertices that do not form a two-simplex. If $m < k - 2$, then $Q$ is simply connected.*

**Proof.** Let $x$ be a vertex of $Q$ and let $\pi_1(Q, x)$ denote the fundamental group of $Q$ with base point $x$. The inclusion of the 1-skeleton $Q^{(1)} \hookrightarrow Q$ induces a surjection $f \colon \pi_1(Q^{(1)}, x) \twoheadrightarrow \pi_1(Q, x)$. Hence, it is sufficient to show that $f$ has trivial image.

The 1-skeleton $Q^{(1)}$ is the full graph with vertex set $Q^{(0)}$. This implies that $\pi_1(Q^{(1)}, x)$ is a free group with generating set given by $\{\Delta(x, u, v) | u, v \in Q^{(0)} \setminus \{x\}\}$, where $\Delta(x, u, v)$ is the triangle consisting of the three (oriented) edges from $x$ to $u$, $u$ to $v$ and $v$ to $x$. We need to show that $\Delta(x, u, v)$ is trivial in $\pi_1(Q, x)$. This is definitely true if $\{x, u, v\}$ forms a 2-simplex in $Q$, so we can assume that this is not the case. It then suffices to show that there is a vertex $w$ such that $\{x, u, w\}$, $\{x, v, w\}$ and $\{u, v, w\}$ are all 2-simplices in $Q$; such a $w$ would form a cone point for the triangle $\Delta(x, u, v)$, showing that it is trivial in $\pi_1(Q, x)$.

Now by assumption, there are at most $m - 1$ triples other than $\{x, u, v\}$ that do not form a 2-simplex. Hence, if there are more than $(m - 1 + 3) = m + 2$ vertices, there is at least one $w$ with the desired properties. □

The preceding Lemma 4.14 also follows from [2, Lemma 2.1].

**Lemma 4.15.** *For all tuples $(r_1, r_2, r_3, r_{12}, r_{123}, r_{13})$ in Table 1, the complex $Q(r_1, r_2, r_3, r_{12}, r_{123}, r_{13})$ is simply connected.*

**Proof.** This follows from Lemma 4.14 using the computer calculations in the jupyter notebook https://github.com/benjaminbrueck/codim2_cohomology_SLnZ/blob/main/Connectivity%20Q%20complexes.ipynb. □

We are now ready to show that every complex $Q_n^m(\vec{v}_1, \vec{v}_2, \vec{v}_3; \vec{w})$ is 3-connected.

**Proof of Theorem 3.51.** By Proposition 4.6 and Corollary 4.8, it suffices to find for every tuple $(r_1, r_2, r_3, r_{12}, r_{123}, r_{13})$ in Table 1 vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{w} \in \mathbb{Z}^4$ with last entries $z_1, z_2, z_3, R$ such that $(r_1, r_2, r_3, r_{12}, r_{123}, r_{13})$ is given by

$$(\mathrm{int}_R(z_1), \mathrm{int}_R(z_2), \mathrm{int}_R(z_3), \mathrm{int}_R(z_1 + z_2), \mathrm{int}_R(z_1 + z_2 + z_3), \mathrm{int}_R(z_1 + z_3))$$

and to show that $Q_4^0(\vec{v}_1, \vec{v}_2, \vec{v}_3; \vec{w})$ is 3-connected.

As this complex is always simply connected (Lemma 4.15), Hurewicz's theorem implies that it is sufficient to show that its integral homology vanishes in degrees 2 and 3. This reduces the proof to computing the homology of a finite list of finite simplicial complexes. We performed these calculations with a computer, the results can be found in the following notebook https://github.com/benjaminbrueck/codim2_cohomology_SLnZ/blob/main/Connectivity%20Q%20complexes.ipynb. □

### 4.4. *Resource consumption, runtime and verifiability of the computer calculations*

All of the used algorithms are exact and guaranteed to terminate. The entire computations take less than one minute on a mid-class laptop and have negligible memory consumption.

There are four steps in this section where we use computer calculations. Firstly, to find the list of isomorphism types of the complexes $Q(r_1, r_2, r_3, r_{12}, r_{123}, r_{13})$ given in Corollary 4.8. Finding this list, i.e. creating Table 1, amounts in a simple application of the relations given in Lemma 4.3 and Lemma 4.7. This is done by a sequence of case distinctions. While this is a tedious task and we believe that the computer is less likely to make mistakes, this can also be verified by hand. Secondly, to find a representative for each such isomorphism type, i.e. to find for each tuple of integers $(r_1, r_2, r_3, r_{12}, r_{123}, r_{13})$ in Table 1 a basis $\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{w}$ of $\mathbb{Z}^4$ such that

$$\begin{aligned}(\mathrm{int}_R(\vec{v}_1), \mathrm{int}_R(\vec{v}_2), \mathrm{int}_R(\vec{v}_3), \mathrm{int}_R(\vec{v}_1 + \vec{v}_2), \mathrm{int}_R(\vec{v}_1 + \vec{v}_2 + \vec{v}_3), \mathrm{int}_R(\vec{v}_1 + \vec{v}_3))\\ = (r_1, r_2, r_3, r_{12}, r_{123}, r_{13}).\end{aligned}$$

For this, the computer needs to calculate $R$-intervals and to check whether a set of vectors forms a basis of $\mathbb{Z}^4$. It is easy to verify by hand (also just in examples) that the vectors given by the computer actually form a basis and do have the correct $R$-intervals. Thirdly, to calculate the set of simplices for each of the 48 complexes $Q_4^0(\vec{v}_1, \vec{v}_2, \vec{v}_3; \vec{w})$. This is done by iterating through increasingly big subsets of the vertex set and for each such subset checking whether it forms a simplex in $\mathrm{BAA}_4^0$. Using Lemma 4.13, it is sufficient to do these checks for standard, 2-additive and 3-additive simplices. This amounts in verifying whether a set of lines in $\mathbb{Z}^4$ forms a partial basis or satisfies a certain linear relation. The code for this, together with comments giving further explanations, is contained in the files `complex_constructor.py` and `simplex_constructor.py` in the repository https://github.com/benjaminbrueck/codim2_cohomology_SLnZ. Lastly, the computer calculates the homology of the 48 given complexes $Q_4^0(\vec{v}_1, \vec{v}_2, \vec{v}_3; \vec{w})$ and counts their simplices in order to show that they are simply connected by Lemma 4.14. These homology calculations are performed with established software (the `SimplicialComplex` class of SageMath [23]) and can also be verified with different existing or self-written code. The efficiency of the used software here is not very important as the complexes are all comparably small (they each have between 62 and 1097 simplices).

## 5. Towards the connectivity of $\mathrm{BAA}_n^m$

In this section, we prepare for proving that the complexes $\mathrm{BAA}_n^m$ are Cohen–Macaulay (Theorem 2.11). We study links and certain subcomplexes of the links. We prove the case $n = 1$, which will be our induction base case, and we show some auxiliary results that will be used in the induction step. Throughout this section, we assume that $n \geq 1$ and $n + m \geq 3$.

### 5.1. *Description of* $\mathsf{Link}$, $\widehat{\mathsf{Link}}$ *and the Cohen–Macaulay property*

In this subsection, we show that the complexes $\mathrm{BAA}_n^m$ are Cohen–Macaulay, provided that they are highly-connected.

**Definition 5.1.** Let $\sigma$ be 3-additive simplex of $\mathrm{BAA}_n^m$. We can write $\sigma = \{v_0, v_1, \ldots, v_k\}$, where $\{\vec{v}_1, \ldots, \vec{v}_k, \vec{e}_1, \ldots, \vec{e}_m\}$ is a partial basis and $\vec{v}_0 = \vec{w}_1 + \vec{w}_2 + \vec{w}_3$ for certain $w_1, w_2, w_3 \in \{v_1, \ldots, v_k, e_1, \ldots, e_m\}$. Let $J(\sigma)$ be the set of vertices of $\mathrm{BAA}_n^m$ that are lines spanned by a vector of the form $\{\vec{w}_1 + \vec{w}_2, \vec{w}_1 + \vec{w}_3, \vec{w}_2 + \vec{w}_3\}$.

Note that $J(\sigma)$ might contain less than three elements (e.g. if $\vec{v}_0 = \vec{v}_1 + \vec{e}_1 + \vec{e}_2$, because $\{\langle \vec{e}_1 + \vec{e}_2 \rangle\} \notin \mathrm{BAA}_m^n$), but it is always nonempty. Going through the definitions of different simplex types, one obtains the following:

**Lemma 5.2.** *Let $\sigma$ be a simplex of* $\mathrm{BAA}_m^n$.

*i) If $\sigma$ is a standard simplex of dimension $k$, there is an isomorphism* $\widehat{\mathsf{Link}}_{\mathrm{BAA}_n^m}(\sigma) \cong \mathrm{BAA}_{n-k-1}^{m+k+1}$.

*ii) If $\sigma$ is a 2-additive simplex, we can write $\sigma = \{v_0, v_1, \ldots, v_k\}$ with $\{\vec{v}_1, \ldots, \vec{v}_k\}$ a partial basis. We then have* $\widehat{\mathsf{Link}}_{\mathrm{BAA}_n^m}(\sigma) = \widehat{\mathsf{Link}}_{\mathrm{BA}_n^m}(\{v_1, \ldots, v_k\})$.

*iii) If $\sigma$ is a 3-additive simplex, we can write $\sigma = \{v_0, v_1, \ldots, v_k\}$ with $\{\vec{v}_1, \ldots, \vec{v}_k\}$ a partial basis. We then have*

$$\widehat{\mathsf{Link}}_{\mathrm{BAA}_n^m}(\sigma) = \mathsf{Link}_{\mathrm{B}_n^m}(\{v_1, \ldots, v_k\}) \text{ and}$$
$$\mathsf{Link}_{\mathrm{BAA}_n^m}(\sigma) = \mathsf{Link}_{B_n^m}(\{v_1, \ldots, v_k\}) * J(\sigma),$$

*where $J(\sigma)$ is seen as a 0-dimensional complex.*

*iv) If $\sigma$ is a double-triple or double-double simplex, we can write $\sigma = \{v_0, v_1, \dots, v_k\}$ with $\{\vec{v}_2, \dots, \vec{v}_k\}$ a partial basis. We then have*

$$\mathsf{Link}_{\mathrm{BAA}_n^m}(\sigma) = \widehat{\mathsf{Link}}_{\mathrm{BAA}_n^m}(\sigma) = \mathsf{Link}_{\mathrm{B}_n^m}(\{v_2, \dots, v_k\}).$$

The description of the links in the following lemma is easy to see and will both be used in Section 6 and in the proof of Proposition 5.5 below.

**Lemma 5.3.** *Let $\sigma$ be a simplex of $\mathrm{BAA}_n^m$ and $\tau \in \mathsf{Link}_{\mathrm{BAA}_n^m}(\sigma)$ such that no vertex of $\tau$ is in $\widehat{\mathsf{Link}}_{\mathrm{BAA}_n^m}(\sigma)$. Then $\mathsf{Link}_{\mathsf{Link}_{\mathrm{BAA}_n^m}(\sigma)}(\tau) \cap \widehat{\mathsf{Link}}_{\mathrm{BAA}_n^m}(\sigma) = \widehat{\mathsf{Link}}_{\mathrm{BAA}_n^m}(\sigma \cup \tau)$.*

Once we show connectivity of $\mathrm{BAA}_n^m$, Proposition 5.5 below implies that the complex is Cohen–Macaulay. To prove this proposition, we will use the following lemma.

**Lemma 5.4** *(Galatius–Randal-Williams [10, Proposition 2.5]). Let $X$ be a simplicial complex, and $Y \subseteq X$ be a full subcomplex. Let $N$ be an integer with the property that for each $p$-simplex $\tau$ in $X$ having no vertex in $Y$, the complex $Y \cap \mathrm{Lk}_X(\tau)$ is $(N-p-1)$-connected. Then the inclusion $|Y| \to |X|$ is $N$-connected.*

**Proposition 5.5.** *If $\mathrm{BAA}_n^m$ is $n$-connected for all $n \geq 1$ and $m+n \geq 3$, then for every $k$-simplex $\sigma$, the link $\mathsf{Link}_{\mathrm{BAA}_n^m}(\sigma)$ is $(n-k-1)$-connected.*

**Proof. Case 1: $\sigma$ is a 3-additive, double-double or double-triple simplex.** By the work of Church–Putman, there is an isomorphism $\mathsf{Link}_{B_n^m}(\{v_0, \dots, v_\ell\}) \cong \mathrm{B}_{n-\ell-1}^{m+\ell+1}$ [6, Lemma 4.3] and this complex is $(n-\ell-3)$-connected for all $n, m \geq 0$ and $0 \leq \ell \leq n-1$ [6, Theorem 4.2]. Combining this with Lemma 5.2, we obtain the claim if $\sigma$ is a 3-additive, double-double or double-triple simplex.

**Case 2: $\sigma$ is a 2-additive simplex.** Next assume that $\sigma$ is 2-additive. If $k = n$, then we must check that $\mathsf{Link}_{\mathrm{BAA}_n^m}(\sigma)$ is nonempty. First suppose $\sigma$ has the form $\{\langle \vec{v}_1 + \vec{v}_2 \rangle, v_1, \dots, v_n\}$, where $\{\vec{v}_1, \dots, \vec{v}_n, \vec{e}_1, \dots \vec{e}_m\}$ is a basis for $\mathbb{Z}^{m+n}$. Since $m+n \geq 3$, either $m \geq 1$ or $m = 0$ and $n \geq 3$. In these two cases, either $\langle \vec{v}_1 + \vec{e}_1 \rangle$ or $\langle \vec{v}_1 + \vec{v}_3 \rangle$, respectively, is an element of $\mathsf{Link}_{\mathrm{BAA}_n^m}(\sigma)$. Alternatively suppose $\sigma$ has the form $\{\langle \vec{v}_1 + \vec{e}_1 \rangle, v_1, \dots, v_n\}$, where $\{\vec{v}_1, \dots, \vec{v}_n, \vec{e}_1, \dots \vec{e}_m\}$ is a basis for $\mathbb{Z}^{m+n}$. Now $m+n \geq 3$ implies either $m \geq 2$ or $n \geq 2$. But then at least one of $\langle \vec{v}_1 + \vec{e}_2 \rangle$ or $\langle \vec{v}_1 + \vec{v}_2 \rangle$ must be a vertex in $\mathsf{Link}_{\mathrm{BAA}_n^m}(\sigma)$.

Now suppose that $k \neq n$. By Lemma 5.2, we can write $\sigma = \{v_0, v_1, \dots, v_k\}$ with $\{\vec{v}_1, \dots, \vec{v}_k\}$ a partial basis and

$$\widehat{\mathsf{Link}}_{\mathrm{BAA}_n^m}(\sigma) = \widehat{\mathsf{Link}}_{\mathrm{BA}_n^m}(\{v_1, \dots, v_k\}).$$

By [6, Lemma 4.12(b)], $\widehat{\mathsf{Link}}_{\mathrm{BA}_n^m}(\{v_1, \dots, v_k\})$ is isomorphic to $\mathrm{BA}_{n-k}^{m+k}$ and this complex is $(n-k-1)$-connected by [6, Theorem C']. We want to extend this to

$\mathsf{Link}_{\mathrm{BAA}_n^m}(\sigma) \supseteq \widehat{\mathsf{Link}}_{\mathrm{BAA}_n^m}(\sigma)$. We will apply Lemma 5.4 with $N = n-k-1$. We need to show that for every $\tau \in \mathsf{Link}_{\mathrm{BAA}_n^m}(\sigma)$ that has no vertex in $\widehat{\mathsf{Link}}_{\mathrm{BAA}_n^m}(\sigma)$, the intersection

$$\mathsf{Link}_{\mathsf{Link}_{\mathrm{BAA}_n^m}(\sigma)}(\tau) \cap \widehat{\mathsf{Link}}_{\mathrm{BAA}_n^m}(\sigma)$$

is $(n-k-\dim(\tau)-2))$-connected. By Lemma 5.3, this complex is equal to $\widehat{\mathsf{Link}}_{\mathrm{BAA}_n^m}(\sigma \cup \tau)$. Every vertex $v$ of $\tau$ satisfies $\vec{v} \in \langle \vec{v}_0, \dots, \vec{v}_k, \vec{e}_1, \dots, \vec{e}_m \rangle$, so in particular, it is contained in the additive core of $\sigma \cup \tau$. Hence, $\sigma \cup \tau$ is a double-double or double-triple simplex of dimension $k+\dim(\tau)+1$.[11] As observed in Lemma 5.2, $\widehat{\mathsf{Link}}_{\mathrm{BAA}_n^m}(\sigma \cup \tau) = \mathsf{Link}_{\mathrm{BAA}_n^m}(\sigma \cup \tau)$ for such simplices. In Case 1 we showed that this link is $(n-k-\dim(\tau)-2)$-connected as claimed.

**Case 3: $\sigma$ is a standard simplex.** If $\sigma$ is a standard simplex, we apply the same argument in two steps. By Lemma 5.2, $\widehat{\mathsf{Link}}_{\mathrm{BAA}_n^m}(\sigma)$ is isomorphic to $\mathrm{BAA}_{n-k-1}^{m+k+1}$ and this is $(n-k-1)$-connected by our assumption. Furthermore, every vertex in $\mathsf{Link}_{\mathrm{BAA}_n^m}(\sigma) \setminus \widehat{\mathsf{Link}}_{\mathrm{BAA}_n^m}(\sigma)$ is either of the form $\langle \vec{w}_1 + \vec{w}_2 \rangle$ or $\langle \vec{w}_1 + \vec{w}_2 + \vec{w}_3 \rangle$ for some $w_1, w_2, w_3 \in \{v_0, \dots, v_k, e_1, \dots, e_m\}$. We will apply Lemma 5.4 twice to the chain subcomplexes

$$\widehat{\mathsf{Link}}_{\mathrm{BAA}_n^m}(\sigma) \subseteq Z \subseteq \mathsf{Link}_{\mathrm{BAA}_n^m}(\sigma),$$

where $Z$ is spanned by $\widehat{\mathsf{Link}}_{\mathrm{BAA}_n^m}(\sigma)$ and all vertices of the form $\langle \vec{w}_1 + \vec{w}_2 \rangle$ as above. We first consider the inclusion $\widehat{\mathsf{Link}}_{\mathrm{BAA}_n^m}(\sigma) \hookrightarrow Z$ and consider Lemma 5.4 with $N = n-k-1$. Let $\tau$ be a simplex of $Z$ that has no vertex in $\widehat{\mathsf{Link}}_{\mathrm{BAA}_n^m}(\sigma)$. Then using Lemma 5.3,

$$\mathsf{Link}_Z(\tau) \cap \widehat{\mathsf{Link}}_{\mathrm{BAA}_n^m}(\sigma) = \widehat{\mathsf{Link}}_{\mathrm{BAA}_n^m}(\sigma \cup \tau).$$

Depending on the form of $\tau$, the simplex $\sigma \cup \tau$ is 2-additive, a double-triple simplex or a double-double simplex. For each possibility, we have already seen in Case 2 that $\widehat{\mathsf{Link}}_{\mathrm{BAA}_n^m}(\sigma \cup \tau)$ is $(n-k-\dim(\tau)-2)$-connected. It follows that $Z$ is $(n-k-1)$-connected.

Now we apply Lemma 5.4 to the inclusion $Z \hookrightarrow \mathsf{Link}_{\mathrm{BAA}_n^m}(\sigma)$ and again let $N = n-k-1$. Let $\tau$ be a simplex of $\mathsf{Link}_{\mathrm{BAA}_n^m}(\sigma)$ that has no vertex in $Z$. Then $\tau$ has the form $\{\langle w_1 + w_2 + w_3 \rangle\}$ and $\sigma \cup \tau$ is 3-additive. It follows from Lemma 5.3 that

$$\mathsf{Link}_{\mathsf{Link}_{\mathrm{BAA}_n^m}(\sigma)}(\tau) \cap Z = \mathsf{Link}_{\mathrm{BAA}_n^m}(\sigma \cup \tau).$$

We already demonstrated in Case 1 that this 3-additive simplex's link is $(n-k-\dim(\tau)-2)$-connected. This implies that $\mathsf{Link}_{\mathrm{BAA}_n^m}(\sigma)$ is $(n-k-1)$-connected, and concludes the final case in the proof. □

[11] In fact, it follows that $\dim(\tau) = 0$ here.

**Remark 5.6.** The preceding proof shows that for a standard, 2-additive, double-triple and double-double simplex $\sigma$, not only $\mathsf{Link}_{\mathrm{BAA}_n^m}(\sigma)$, but also $\widehat{\mathsf{Link}}_{\mathrm{BAA}_n^m}(\sigma)$ is $(n-\dim(\sigma)-1)$-connected. The latter is however not the case for 3-additive simplices.

### 5.2. Description of $\mathsf{Link}^<$, $\widehat{\mathsf{Link}}^<$

We next describe the structure of certain links in $\mathrm{BAA}_n^m$. As before, we omit the proofs of a few statements that simply follow by spelling out the definitions.

**Lemma 5.7.** *Let $v \in \mathrm{BAA}_n^m$ be a vertex with nonzero last coordinate. Then*

*i)* $\mathsf{Link}^<_{\mathrm{BAA}_n^m}(v) = \widehat{\mathsf{Link}}^<_{\mathrm{BAA}_n^m}(v)$, *and*
*ii)* $\mathsf{Link}^<_{\mathrm{BAA}_n^m}(\{v, \langle \vec{v} \pm \vec{e}_i \rangle\}) = \widehat{\mathsf{Link}}^<_{\mathrm{BAA}_n^m}(\{v, \langle \vec{v} \pm \vec{e}_i \rangle\})$ *for all* $1 \le i \le m$.

Later on, we will need to know that $\mathsf{Link}^<_{\mathrm{BAA}_n^m}(\{v, \langle \vec{v} \pm \vec{e}_i \rangle\}) = \widehat{\mathsf{Link}}^<_{\mathrm{BAA}_n^m}(\{v, \langle \vec{v} \pm \vec{e}_i \rangle\})$ is highly-connected. To prepare for this, we compare this complex to $\widehat{\mathsf{Link}}^<_{\mathrm{BA}_n^m}(v)$:

**Lemma 5.8.** *Let $v \in \mathrm{BAA}_n^m$ be a vertex with nonzero last coordinate and $1 \le i \le m$. Let $\sigma$ be a set of vectors in $\mathbb{Z}^{m+n}$. Then the following hold.*

*i) The simplex $\sigma$ is a standard simplex of $\widehat{\mathsf{Link}}^<_{\mathrm{BA}_n^m}(v)$ if and only if it is a 2-additive simplex of $\widehat{\mathsf{Link}}^<_{\mathrm{BAA}_n^m}(\{v, \langle \vec{v} \pm \vec{e}_i \rangle\})$.*
*ii) The simplex $\sigma$ is a 2-additive simplex of $\widehat{\mathsf{Link}}^<_{\mathrm{BA}_n^m}(v)$ such that the additive core of $\{e_1, \dots, e_m\} \cup \{v\} \cup \sigma$ does not contain $v$ or $e_i$ if and only if $\sigma$ is a double-double simplex of $\widehat{\mathsf{Link}}^<_{\mathrm{BAA}_n^m}(\{v, \langle \vec{v} \pm \vec{e}_i \rangle\})$.*
*iii) If $\sigma$ is a 2-additive simplex of $\widehat{\mathsf{Link}}^<_{\mathrm{BA}_n^m}(v)$ such that the additive core of $\{e_1, \dots, e_m\} \cup \{v\} \cup \sigma$ contains $v$ or $e_i$, then $\sigma$ is a double-triple simplex of $\widehat{\mathsf{Link}}^<_{\mathrm{BAA}_n^m}(\{v, \langle \vec{v} \pm \vec{e}_i \rangle\})$.*
*iv) If $\sigma$ is a double-triple simplex of $\widehat{\mathsf{Link}}^<_{\mathrm{BAA}_n^m}(\{v, \langle \vec{v} \pm \vec{e}_i \rangle\})$, then it is a simplex of $\widehat{\mathsf{Link}}^<_{\mathrm{BA}_n^m}(v)$ except if it is of the form $\sigma = \{w, \langle \vec{v} \pm \vec{e}_i + \vec{w} \rangle, v_2, \dots, v_k\}$.*
*v) No simplex of $\widehat{\mathsf{Link}}^<_{\mathrm{BAA}_n^m}(\{v, \langle \vec{v} \pm \vec{e}_i \rangle\})$ is 3-additive.*

*In particular, $\widehat{\mathsf{Link}}^<_{\mathrm{BA}_n^m}(v)$ is a subcomplex of $\widehat{\mathsf{Link}}^<_{\mathrm{BAA}_n^m}(\{v, \langle \vec{v} \pm \vec{e}_i \rangle\}) \subseteq \mathsf{Link}_{\mathrm{BAA}_n^m}(\{v, \langle \vec{v} \pm \vec{e}_i \rangle\})$ and every simplex of $\widehat{\mathsf{Link}}^<_{\mathrm{BAA}_n^m}(\{v, \langle \vec{v} \pm \vec{e}_i \rangle\})$ that is not contained in $\widehat{\mathsf{Link}}^<_{\mathrm{BA}_n^m}(v)$ is of type double-triple.*

**Proof.** Throughout this proof, we will use the observation that for $1 \le j, k \le m$, the lines $\langle \vec{v} \pm \vec{e}_j \rangle$, $\langle \vec{v} \pm \vec{e}_i \pm \vec{e}_j \rangle$, $\langle \vec{e}_j \pm \vec{e}_k \rangle$ or $\langle \vec{e}_i \pm \vec{e}_j \pm \vec{e}_k \rangle$ are **not** vertices of $\widehat{\mathsf{Link}}^<_{\mathrm{BA}_n^m}(v)$ or

$\widehat{\mathsf{Link}}^{<}_{\mathrm{BAA}^m_n}(v)$. This follows because their last coordinate is equal to that of $v$ or they lie in $\langle \vec{e}_1, \dots, \vec{e}_m \rangle$.

Part i) is immediate. Part ii) follows because, if the additive core of $\{e_1, \dots, e_m\} \cup \{v\} \cup \sigma$ does not contain $v$ or $e_i$, then $\{e_1, \dots, e_m\} \cup \{v, \langle \vec{v} \pm \vec{e}_i \rangle\ \} \cup \sigma$ contains two disjoint 2-additive faces.

If $\sigma$ is a 2-additive simplex of $\widehat{\mathsf{Link}}^{<}_{\mathrm{BA}^m_n}(v)$ such that the additive core of $\{e_1, \dots, e_m\} \cup \{v\} \cup \sigma$ contains $v$ or $e_i$, then this additive core must be of the form $\{v, w, \langle \vec{v} + \vec{w} \rangle\}$ or $\{e_i, w, \langle \vec{e}_i + \vec{w} \rangle\}$ for some $w \in \sigma$. This implies Part iii).

For Part iv) note that if $\sigma$ is a double-triple simplex of $\widehat{\mathsf{Link}}^{<}_{\mathrm{BAA}^m_n}(\{v, \langle \vec{v} \pm \vec{e}_i \rangle\})$, then the additive core of $\{e_1, \dots, e_m\} \cup \{v, \langle \vec{v} \pm \vec{e}_i \rangle\ \} \cup \sigma$ is of the form

$$\{e_i, v, \langle \vec{v} \pm \vec{e}_i \rangle\} \cup \tau,$$

where $\tau = \{w, \langle \vec{v} + \vec{w} \rangle\}$, $\{w, \langle \pm\vec{e}_i + \vec{w} \rangle\}$ or $\{w, \langle \vec{v} \pm \vec{e}_i + \vec{w} \rangle\}$ for some $w \in \sigma$. In the first two cases, $\sigma = \tau \cup \{v_2, \dots, v_k\}$ is a simplex of $\widehat{\mathsf{Link}}^{<}_{\mathrm{BA}^m_n}(v)$. If however $\{w, \langle \vec{v} \pm \vec{e}_i + \vec{w} \rangle\}$, then $\{e_1, \dots, e_m\} \cup \{v\} \cup \sigma$ contains the 3-additive simplex $\{e_i, v, w, \langle \vec{v} \pm \vec{e}_i + \vec{w} \rangle\}$, so $\sigma$ is not a simplex in $\widehat{\mathsf{Link}}^{<}_{\mathrm{BA}^m_n}(v)$.

Finally, Part v) follows because for any simplex $\sigma$ in $\widehat{\mathsf{Link}}^{<}_{\mathrm{BAA}^m_n}(\{v, \langle \vec{v} \pm \vec{e}_i \rangle\})$, the simplex $\{e_1, \dots, e_m\} \cup \{v, \langle \vec{v} \pm \vec{e}_i \rangle\ \} \cup \sigma$ contains the 2-additive face $\{e_i, v, \langle \vec{v} \pm \vec{e}_i \rangle\}$. As every face of a 3-additive simplex is either standard or 3-additive (see Observation 4.11), this implies that $\sigma$ cannot be 3-additive. □

The following will be used to describe the link of 3-additive simplices during the proof that $\mathrm{BAA}^m_n$ is spherical.

**Lemma 5.9.** *Let $\sigma$ be a 3-additive simplex in $\mathrm{BAA}^m_n$ and $R > 0$ the highest absolute value of the last coordinates of all of its vertices. As in Definition 5.1, write $\sigma = \{v_0, v_1, \dots, v_k\}$, where $\vec{v}_0 = \vec{w}_1 + \vec{w}_2 + \vec{w}_3$, let $J(\sigma)$ be as in Definition 5.1 and let $J^< \subseteq J(\sigma)$ be the subset of all vertices with last coordinate* smaller *in absolute value than $R$.*

*Then $J^<$ is empty if and only if the last coordinate of $v_0$ is $\pm R$ and there are $1 \le l \le k$, $1 \le i \ne j \le m$ such that $\vec{v}_0 = \vec{v}_l \pm \vec{e}_i \pm \vec{e}_j$.*

### *5.3. Induction beginning*

The following is an adaptation of [6, Proof of Theorem C', Base Case].

**Lemma 5.10.** *Let $m \ge 2$. The complex $\mathrm{BAA}^m_1$ is 1-connected.*

**Proof.** We show this by successively describing the structures of $\mathrm{B}^m_1, \mathrm{BA}^m_1$ and $\mathrm{BAA}^m_1$. All of these complexes have the same vertex set. Every vertex is a line $v$ that is spanned by a vector $\vec{v}$ of the form $(a_1, \dots, a_m, 1)$, which we will write as $\vec{v} = (\vec{a}, 1)$ for $\vec{a} \in \mathbb{Z}^m$.

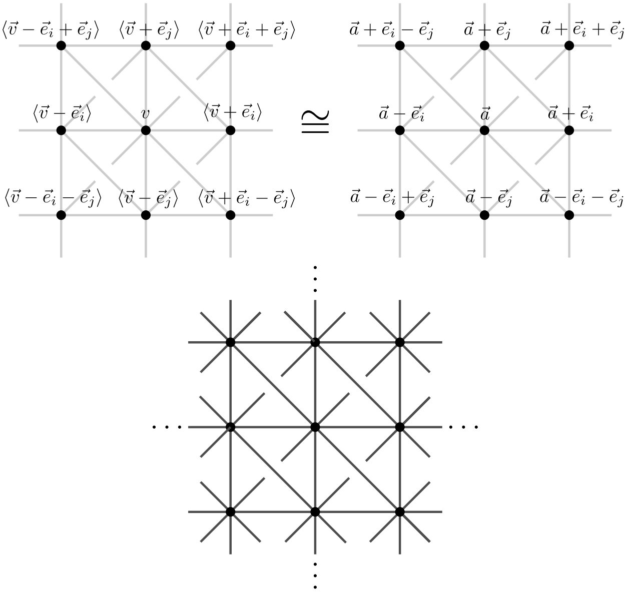


**Fig. 10.** We identify the 1-skeleton of $\mathrm{BAA}_1^m$ (left) with a graph obtained by gluing edges onto the Cayley graph of $\mathbb{Z}^m$ (right). Here $v$ is spanned by $\vec{v}=(\vec{a},1)$.

This gives an identification of the vertex set with $\mathbb{Z}^m$. The complex $\mathrm{B}_1^m$ has dimension zero, so it has no simplices other than these vertices.

The complex $\mathrm{BA}_1^m$ has dimension one. In [6, Proof of Theorem C', Base Case], Church–Putman show that it is isomorphic to the Cayley graph of $\mathbb{Z}^m$ with respect to the generating set given by $\vec{e}_1,\dots,\vec{e}_m$: Every edge in $\mathrm{BA}_1^m$ can be written in the form $\sigma=\{v,\langle\vec{v}+\vec{e}_i\rangle\}$ for some $v\in B_1^m$ and $1\le i\le m$; such an edge comes from the 2-additive simplex $\{v,\langle\vec{v}+\vec{e}_i\rangle\}\cup\{e_1,\dots,e_m\}$ in $\mathrm{BA}_{1+m}$. For $\vec{v}=(\vec{a},1)$, this edge gets identified with the edge $\{\vec{a},\vec{a}+\vec{e}_i\}$ of the Cayley graph. (We slightly abuse notation here by writing $\vec{e}_i$ both for elements in $\mathbb{Z}^{1+m}$ and in $\mathbb{Z}^m$.)

The complex $\mathrm{BAA}_1^m$ has dimension two. It is obtained from $\mathrm{BA}_1^m$ by attaching simplices $\sigma$ such that $\sigma\cup\{e_1,\dots,e_m\}$ is either 3-additive or of type double-triple in $\mathrm{BAA}_{1+m}$. (No double-double simplices can occur in this low dimensional case.) Concretely, the double-triple simplices in $\mathrm{BAA}_1^m$ are all of the form

$$\sigma=\{v,\langle\vec{v}\pm\vec{e}_i\rangle,\langle\vec{v}\pm\vec{e}_i\pm\vec{e}_j\rangle\}\,,\quad \sigma=\{v,\langle\vec{v}\pm\vec{e}_j\rangle,\langle\vec{v}\pm\vec{e}_i\pm\vec{e}_j\rangle\}\,,$$
$$\sigma=\{v,\langle\vec{v}\pm\vec{e}_i\rangle,\langle\vec{v}\pm\vec{e}_j\rangle\}\,,\ \text{or}\ \sigma=\{\langle\vec{v}\pm\vec{e}_i\rangle,\langle\vec{v}\pm\vec{e}_j\rangle,\langle\vec{v}\pm\vec{e}_i\pm\vec{e}_j\rangle\}\,,$$

for some $v\in\mathrm{B}_1^m$ and $1\le i<j\le m$. The 3-additive simplices arise as faces of these. Fig. 10 shows the 1-skeleton of $\mathrm{BAA}_1^m$ and its relationship to the Cayley graph of $\mathbb{Z}^m$ with respect to the standard generators. In fact, the 1-skeleton of $\mathrm{BAA}_1^m$ is isomorphic to the Cayley graph of $\mathbb{Z}^m$ with respect to generators of the form $\vec{e}_i$ and $\vec{e}_i\pm\vec{e}_j$.

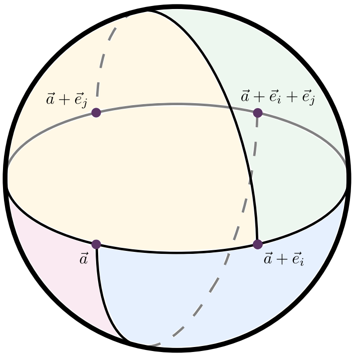


**Fig. 11.** The subcomplex spanned by $\{\vec{a}, \vec{a}+\vec{e}_i, \vec{a}+\vec{e}_j, \vec{a}+\vec{e}_i+\vec{e}_j\}$.

It follows that $\mathrm{BAA}_1^m$ is isomorphic to a complex that is obtained as follows. Start with the Cayley graph of $\mathbb{Z}^m$ with respect to the generating set $\vec{e}_1, \ldots, \vec{e}_m$. Every minimal cycle in this graph has length four and vertices $\vec{a}$, $\vec{a}+\vec{e}_i$, $\vec{a}+\vec{e}_j$, $\vec{a}+\vec{e}_i+\vec{e}_j$ for some $\vec{a} \in \mathbb{Z}^m$ and $1 \leq i < j \leq m$.

Now attach to each such cycle two quadrilaterals along their boundaries. Both quadrilaterals are composed of two triangles, the first one of

$$\{\vec{a}, \vec{a}+\vec{e}_i, \vec{a}+\vec{e}_i+\vec{e}_j\} \text{ and } \{\vec{a}, \vec{a}+\vec{e}_j, \vec{a}+\vec{e}_i+\vec{e}_j\},$$

the second one of

$$\{\vec{a}, \vec{a}+\vec{e}_i, \vec{a}+\vec{e}_j\} \text{ and } \{\vec{a}+\vec{e}_i, \vec{a}+\vec{e}_j, \vec{a}+\vec{e}_i+\vec{e}_j\}.$$

See Fig. 11. The fundamental group of this Cayley graph (with base point the identity) is generated by loops of the form $p_{\vec{a}} \cdot l_{\vec{a},i,j} \cdot p_{\vec{a}}^{-1}$, where $p_{\vec{a}}$ is a path from the identity to $\vec{a}$, $p_{\vec{a}}$ is its inverse and $l_{\vec{a},i,j}$ is the 4-edges loop around the square $\{\vec{a}, \vec{a}+\vec{e}_i, \vec{a}+\vec{e}_j, \vec{a}+\vec{e}_i+\vec{e}_j\}$. Our complex is constructed by gluing a 2-disk (in fact, two 2-disks) to each such square, the resulting complex is 1-connected. We conclude $\mathrm{BAA}_1^m$ is 1-connected as claimed. □

## 6. Proof of Theorem 2.11

In this section, we will finish the proof of Theorem 2.11, which states that $\mathrm{BAA}_n^m$ is Cohen–Macaulay of dimension $(n+1)$ whenever $n \geq 1$ and $m+n \geq 3$. By Proposition 5.5, to prove this it suffices to show $\mathrm{BAA}_n^m$ is $n$-connected whenever $n \geq 1$ and $m+n \geq 3$. Our proof roughly follows the strategy of Church–Putman [6, Proof of Theorem C', Steps 1-4]. The analogue of [6, Proof of Theorem C', Step 2] does not work in our context but fortunately it is not essential here or in [6, Proof of Theorem C']. Step 1 in our proof is roughly speaking a combination of Step 1 and Step 3 of the proof of Church–Putman while our Step 2 corresponds to their Step 4.

Let $n \geq 1$ and $m + n \geq 3$. By Lemma 5.10, $\mathrm{BAA}_1^m$ is 1-connected for all $m \geq 2$. We use this as a base case for an induction on $n$. Now assume that $n \geq 2$ and that by induction, $\mathrm{BAA}_{n-1}^{m+1}$ is $(n-1)$-connected. For $d \leq n$, let $f\colon S^d \to \mathrm{BAA}_n^m$ be a map that is simplicial with respect to some simplicial structure on $S^d$. Here and from now on, we will assume that all simplicial structures on manifolds (possibly with boundary) are chosen to be combinatorial. This ensures that links of simplices are homeomorphic to spheres of the appropriate dimension. Let $R$ be the maximum of the absolute value of the last coordinate of $f(x)$ over all vertices $x \in S^d$. If $R = 0$, then $f$ can be extended to a disk via coning its image with the vertex $e_{m+n}$. Thus, we are done if we can show that we can homotope $f$ to lower $R$. A visual outline of the proof is shown in Fig. 12.

This homotopy is done in two steps: In Step 1, we isolate vertices in $S^d$ that get mapped to vertices with last entry $\pm R$, i.e. we homotope $f$ such that if $x, y$ form an edge in $S^d$ and $f(x)$, $f(y)$ have last entry $\pm R$, then $f(x) = f(y)$. In Step 2, we then successively replace all of these "bad" vertices by vertices whose last coordinate has absolute value less than $R$. Only this second step uses our inductive hypothesis. In order to perform these two steps, we will perform a sequence of homotopies that step-by-step replace $f$ by "better" maps. Before we start with these, we make some definitions that help us to keep track of the progress we make and describe a Procedure 1 that we will repeatedly use during Step 1.

**Definition 6.1.** A simplex $\sigma$ of $S^d$ is called *edgy* if $f(\sigma) = \{v_0, v_1\}$ is an edge with the last coordinates of $v_0$ and $v_1$ equal to $\pm R$.

If $f\colon S^d \to \mathrm{BAA}_n^m$ has no edgy simplices, then the bad vertices are isolated in the above sense. So removing all edgy simplices is the aim of Step 1.

Our method for removing edgy simplices only works if we can control the stars of such simplices. For this, we need to make sure that there are no simplices of the following type:

**Definition 6.2.** A simplex $\sigma$ of $S^d$ is called $(a, b, c)$*-over-augmented*, $a, b, c \in \mathbb{N}_0$, if

- $f(\sigma)$ is a 3-additive, double-triple, or double-double simplex,
- every vertex of $f(\sigma)$ either has last coordinate $\pm R$ or is contained in the additive core,
- $\sigma$ contains exactly $a \geq 1$ vertices $x$ such that $f(x)$ has last coordinate $\pm R$,
- $\sigma$ contains exactly $b \geq 0$ vertices $x$ such that $f(x)$ is contained in the additive core of a 3-additive face of $f(\sigma)$,
- $\dim(\sigma) = c$, and
- if $f(\sigma)$ is 3-additive, then for all $v_0 \in f(\sigma)$ with last coordinate $\pm R$, there does **not** exist $v_1 \in f(\sigma)$ and $1 \leq i \neq j \leq m$ such that $\vec{v}_0 = \vec{v}_1 \pm \vec{e}_i \pm \vec{e}_j$.

▨ **Implicit homotopy:** Accomplished by verifying Claim 7.4, Claim 7.5, Claim 7.6 and using a highly-connected subcomplex

△ **Explicit homotopy:** Accomplished through subdividing 'bad' simplices by introducing a new vertex at the barycentre

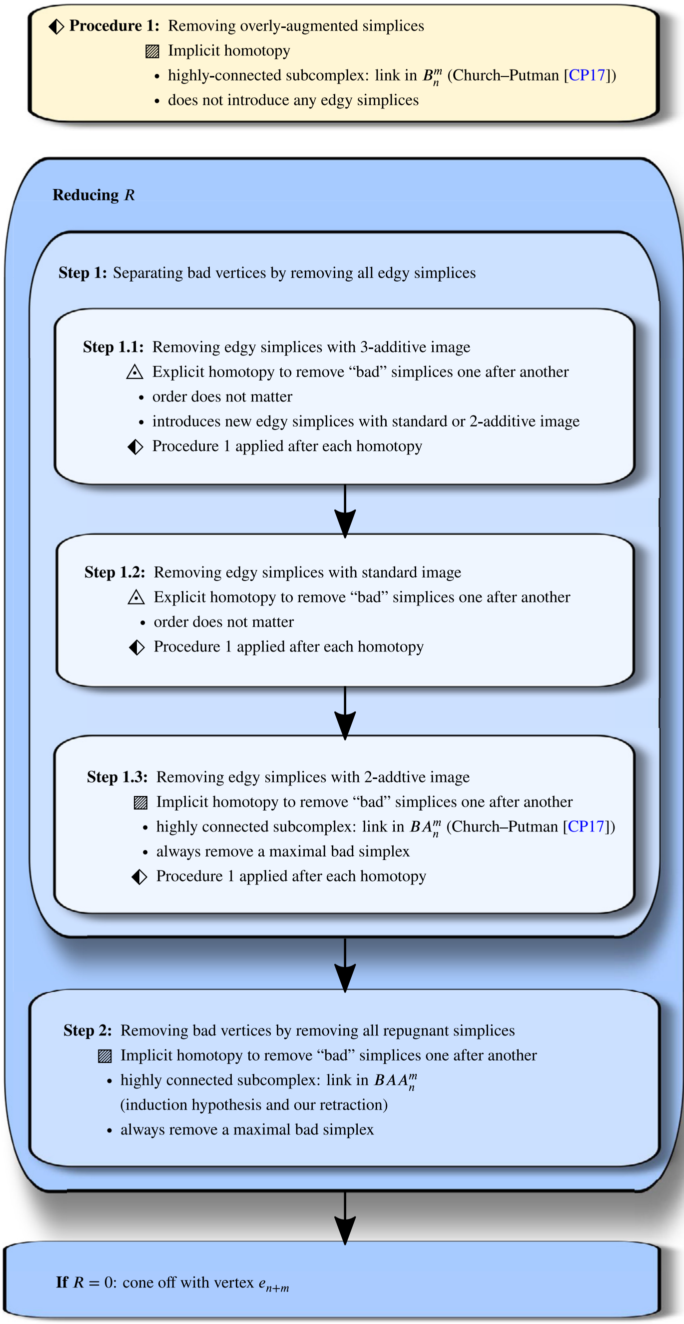


**Fig. 12.** A schematic of the proof of Theorem 2.11.

We call a simplex *overly augmented* if it is $(a, b, c)$-over-augmented for some $a \geq 1$ and $b, c \geq 0$. Suppose $\sigma$ is an $(a, b, c)$-over-augmented simplex and $\tau$ is a $(a', b', c')$-over-augmented simplex. We call $\tau$ *better* than $\sigma$ if $(a, b, c) < (a', b', c')$ lexicographically.

Note that the last condition of the definition coincides with the one given in the last bullet point of Lemma 5.9. It excludes the case of edgy simplices $\sigma$ whose image $f(\sigma) = \{\langle \vec{v}_1 \pm \vec{e}_i \pm \vec{e}_j \rangle, v_1\}$ is 3-additive. These are considered in detail later on (Step 1.2). For later reference, we record the following observation. It describes the stars of edgy simplices in the case where $f$ has no overly augmented simplices.

**Observation 6.3.** If $f$ has no overly augmented simplices, then the following is true: Let $\sigma$ be a simplex of $S^d$ such that $f(\sigma)$ contains two vertices with last coordinate $\pm R$. Then $f(\sigma)$ is neither a double-triple nor a double-double simplex. If it is 3-additive, it can be written in the form $f(\sigma) = \{\langle \vec{v}_1 \pm \vec{e}_i \pm \vec{e}_j \rangle, v_1, v_2, \ldots, v_k\}$, where the last coordinate of $v_1$ is $\pm R$; in particular, $\sigma$ then contains an edgy simplex with 3-additive image $\{\langle \vec{v}_1 \pm \vec{e}_i \pm \vec{e}_j \rangle, v_1\}$.

**Procedure 1: Removing overly augmented simplices.** We will now describe a procedure that allows us to remove overly augmented simplices from $f$. Let $\sigma$ be an $(a, b, c)$-over-augmented simplex with $(a, b, c)$ as large as possible lexicographically. Our goal is to homotope $f$ to have one less $(a, b, c)$-over-augmented simplex while only adding better simplices and no new edgy simplices. In order to do so, we will modify $f|_{\mathsf{Star}_{S^d}(\sigma)}$ such that image of the result lies in $f(\partial\sigma) * K(\sigma)$, where $K(\sigma)$ is a certain subcomplex of $\mathrm{BAA}_n^m$ whose vertices have more desirable properties than those of $f(\sigma)$. The same type of argument will be used several times in this article (Step 1.3, Step 2, Proposition 9.3). We spell it out in detail here and will use this as a blueprint for later occurrences. This is a standard procedure that has been used by many authors to prove various simplicial complexes are highly-connected. This proof strategy is often called a "bad simplex" argument.

We start by defining $K(\sigma)$. If $f(\sigma)$ is a double-triple or double-double simplex, we can write $f(\sigma) = \{v_0, v_1, \ldots, v_k\}$, where $\{\vec{v}_2, \ldots, \vec{v}_k\}$ is a partial basis. We define

$$K(\sigma) := \mathsf{Link}^<_{B_n^m}(\{v_2, \ldots, v_k\}) \qquad [f(\sigma) \text{ double-triple or double-double}].$$

If $f(\sigma)$ is a 3-additive simplex, we can write $f(\sigma) = \{v_0, v_1, \ldots, v_k\}$ as in Definition 5.1, i.e. such that $\{\vec{v}_1, \ldots, \vec{v}_k, \vec{e}_1, \ldots, \vec{e}_m\}$ is a partial basis and $\vec{v}_0 = \vec{w}_1 + \vec{w}_2 + \vec{w}_3$ for $w_1, w_2, w_3 \in \{v_1, \ldots, v_k, e_1, \ldots, e_m\}$. Let $J(\sigma)$ be the set of vertices of $\mathrm{BAA}_n^m$ that are lines spanned by a vector of the form $\{\vec{w}_1 + \vec{w}_2, \vec{w}_1 + \vec{w}_3, \vec{w}_2 + \vec{w}_3\}$, as in Definition 5.1 and $J^< \subseteq J(\sigma)$ the subset of all vertices with last coordinate smaller in absolute value than $R$. By the last assumption in the definition of overly augmented simplices and Lemma 5.9, the set $J^<$ is nonempty. We view $J^<$ as a 0-dimensional simplicial complex and define

$$K(\sigma) := \mathsf{Link}^{<}_{\mathrm{B}^m_n}(\{v_1, \dots, v_k\}) * J^{<} \qquad [f(\sigma) \text{ 3-additive}].$$

**Claim 6.4.** $K(\sigma)$ is a subcomplex of $\mathsf{Link}_{\mathrm{BAA}^m_n}(f(\sigma))$ and $f\left(\mathsf{Link}_{S^d}(\sigma)\right) \subseteq K(\sigma)$.

As $f$ is simplicial, we have $f(\mathsf{Link}_{S^d}(\sigma)) \subseteq \mathsf{Star}_{\mathrm{BAA}^m_n}(f(\sigma))$. Since $\sigma$ is maximally over-augmented, every $x \in \mathsf{Link}_{S^d}(\sigma)$ gets mapped to a vertex $f(x) \in \mathsf{Link}_{\mathrm{BAA}^m_n}(f(\sigma))$ with last coordinate smaller in absolute value than $R$. Hence, we actually have $f(\mathsf{Link}_{S^d}(\sigma)) \subseteq \mathsf{Link}^{<}_{\mathrm{BAA}^m_n}(f(\sigma))$ and it suffices to show that $K(\sigma) = \mathsf{Link}^{<}_{\mathrm{BAA}^m_n}(f(\sigma))$. This follows immediately from Lemma 5.2.

**Claim 6.5.** $K(\sigma)$ is $(\dim \mathsf{Link}_{S^d}(\sigma))$-connected.

By the work of Church–Putman, $\mathsf{Link}^{<}_{B^m_n}(\{v_0, \dots, v_\ell\})$ is $(n - \ell - 3)$-connected [6, Theorem 4.2 and Lemma 4.5; see the first paragraph on p. 1016]. This implies that $K(\sigma)$ is $(n - k - 1)$-connected in all cases under consideration; note when $\sigma$ is 3-additive, we know $J^{<} \neq \emptyset$ by Lemma 5.9. The claim follows because $\dim \mathsf{Link}_{S^d}(\sigma) = d - \dim(\sigma) - 1 \leq n - k - 1$.

These two claims allow us to modify $f$ up to homotopy on $\mathsf{Star}(\sigma)$: By Claim 6.4, $f$ restricts to a map

$$\mathsf{Link}_{S^d}(\sigma) \to K(\sigma)$$

whose domain $\mathsf{Link}_{S^d}(\sigma)$ is isomorphic to a triangulated sphere. By Claim 6.5, this map can be extended to a map

$$g \colon \mathsf{Cone}(\mathsf{Link}_{S^d}(\sigma)) \to K(\sigma)$$

that is simplicial with respect to some simplicial structure on $\mathsf{Cone}(\mathsf{Link}_{S^d}(\sigma))$. Again by Claim 6.4, $K(\sigma)$ is a subcomplex of $\mathsf{Link}_{\mathrm{BAA}^m_n}(f(\sigma))$. This implies that $g$ extends to

$$f|_\sigma * g \colon \sigma * \mathsf{Cone}(\mathsf{Link}_{S^d}(\sigma)) \to f(\sigma) * K(\sigma) \subset \mathrm{BAA}^m_n .$$

Topologically, $\sigma * \mathsf{Cone}(\mathsf{Link}_{S^d}(\sigma))$ is a ball whose boundary can be decomposed as

$$\partial(\sigma * \mathsf{Cone}(\mathsf{Link}_{S^d}(\sigma))) = (\partial\sigma * \mathsf{Cone}(\mathsf{Link}_{S^d}(\sigma))) \cup \mathsf{Star}_{S^d}(\sigma).$$

Note that $f|_{\mathsf{Link}_{S^d}(\sigma)} = g|_{\mathsf{Link}_{S^d}(\sigma)}$. It follows that the restriction of $f$ to $\mathsf{Star}_{S^d}(\sigma)$ is homotopic to a simplicial map $h \colon \partial\sigma * \mathsf{Cone}(\mathsf{Link}_{S^d}(\sigma)) \to f(\sigma) * K(\sigma)$ that agrees with $f$ on $\partial\sigma * \mathsf{Link}_{S^d}(\sigma)$.

**Claim 6.6.** The map $h \colon \partial\sigma * \mathsf{Cone}(\mathsf{Link}_{S^d}(\sigma)) \to f(\sigma) * K(\sigma)$ has only simplices that are better than $\sigma$. Furthermore, every edgy simplex of $h$ is contained in $\partial\sigma$.

Every simplex in $\partial\sigma * \mathsf{Cone}(\mathsf{Link}_{S^d}(\sigma))$ is of the form $\sigma' = \tilde{\sigma} \cup \tau$, where $\emptyset \subseteq \tilde{\sigma} \subset \sigma$ is a proper face of $\sigma$. Such a simplex gets mapped to $h(\sigma') = f(\tilde{\sigma}) \cup g(\tau)$, where $g(\tau) \subseteq K(\sigma)$. Observe that every vertex of $K(\sigma)$ has last entry of absolute value smaller than $R$. This implies that every edgy simplex of $h$ must be contained in $\partial\sigma$. Now let $\sigma' = \tilde{\sigma} \cup \tau$ be a simplex in the domain of $h$ that is $(a', b', c')$-over-augmented. We need to show that $(a', b', c') < (a, b, c)$ lexicographically. That $a' \leq a$ follows immediately because every vertex of $K(\sigma)$ has last entry of absolute value smaller than $R$.

Assume that $f(\sigma)$ is a double-triple or double-double simplex. In this case, the definition of $K(\sigma)$ implies that no vertex of $g(\tau)$ can be contained in the additive core of $h(\sigma')$. This and the assumption that $\sigma'$ is overly augmented imply firstly that $b' \leq b$ and secondly that $\tau$ is the empty simplex, i.e. $\sigma' = \tilde{\sigma} \subset \sigma$. But then, as $c' = \dim(\sigma') < \dim(\sigma) = c$, we have $(a', b', c') < (a, b, c)$.

Next assume that $f(\sigma)$ is 3-additive. Here, we defined $K(\sigma) = \mathsf{Link}^{<}_{\mathrm{B}_n^m}(\{v_1, \ldots, v_k\}) * J^{<}$. The vertices of $g(\tau)$ that are contained in $\mathsf{Link}^{<}_{\mathrm{B}_n^m}(\{v_1, \ldots, v_k\})$ can neither be in the additive core of $h(\sigma')$ nor do they have last coordinate of absolute value $\pm R$. Hence, as $\sigma' = \tilde{\sigma} \cup \tau$ is overly augmented, we have $g(\tau) \cap \mathsf{Link}^{<}_{\mathrm{B}_n^m}(\{v_1, \ldots, v_k\}) = \emptyset$. In other words, either $\tau$ is the empty simplex and $\sigma' = \tilde{\sigma} \subset \sigma$ or $h(\sigma') = f(\tilde{\sigma}) \cup \{j\}$ for some $j \in J^{<}$ and $\tilde{\sigma} \subset \sigma$. In the first case, we have $(a', b', c') < (a, b, c)$ for the same reasons as in the situation of double-triple or double-double simplices. For the second case, note that although $j$ might be contained in the additive core of $h(\sigma')$, it cannot be contained in the additive core of a 3-additive face: We know that $f(\sigma) \cup \{j\}$ is a double-triple simplex containing $h(\sigma')$ and that $f(\sigma)$ is a 3-additive face of it. But a double-triple simplex has exactly one 3-additive face (see Observation 4.11). Hence, $b' \leq b$. As $\sigma$ is overly augmented, every vertex of it is either mapped to a vertex with last coordinate $\pm R$ or to the additive core of the 3-additive simplex $f(\sigma)$. This implies that every vertex contributes either to $a$ or $b$ (or to both). On the other hand, the vertices of $\tau$ are mapped to $j$, which neither has rank $R$ nor is it contained in the additive core of a 3-additive face of $h(\sigma')$. It follows that these vertices neither contribute to $a'$ nor to $b'$. Consequently, we have $a' + b' < a + b$, which implies $(a', b', c') < (a, b, c)$.

We can now replace $f$ by the homotopic map $f' \colon S^d \to \mathrm{BAA}_n^m$ that is obtained by replacing $\mathsf{Star}(\sigma)$ with $\partial\sigma * \mathsf{Cone}(\mathsf{Link}_{S^d}(\sigma))$ and setting $f'$ to be equal to $h$ on this subset of $S^d$. By Claim 6.6, the map $f'$ has one less $(a, b, c)$-over-augmented simplex than $f$, no worse simplices and no additional edgy simplices. Iterating this shows that we may replace $f$ by a map that has no overly augmented simplices and no other edgy simplices than those of $f$.

We now proceed with the process of reducing $R$, the maximum of the absolute values of the last coordinate of vectors in the image of $f$.

**Step 1: Separating bad vertices.** In this first step, we will remove all edgy simplices. If $\sigma$ is edgy, then its image $f(\sigma) = \{v_0, v_1\}$ is either a standard simplex (if $\{\vec{v}_0, \vec{v}_1, \vec{e}_1, \ldots, \vec{e}_m\}$ is a partial basis) or a 2-additive simplex (if $\vec{v}_0 = \vec{v}_1 \pm \vec{e}_i$) or a 3-additive simplex (if

$\vec{v}_0 = \vec{v}_1 \pm \vec{e}_i \pm \vec{e}_j$). We will now successively remove edgy simplices by first removing those with 3-additive image, then those with standard image and finally those with 2-additive image. While doing so, we will repeatedly apply Procedure 1.

*Step 1.1 Removing edgy simplices with* 3*-additive image.* Let $\sigma$ be an edgy simplex such that $f(\sigma) = \{v_0, v_1\}$ is 3-additive. We can find representatives $\vec{v}_0$ and $\vec{v}_1$ such that their last coordinates are equal to $R$ and $\vec{v}_0 = \vec{v}_1 \pm \vec{e}_i \pm \vec{e}_j$ for some $1 \leq i \neq j \leq m$. Define $v = \langle \vec{v}_1 \pm \vec{e}_i \rangle$, where the sign $\pm$ of $e_i$ agrees with its sign in the sum $\vec{v}_0 = \vec{v}_1 \pm \vec{e}_i \pm \vec{e}_j$. Our aim is to use $v$ to replace $f$ by a map $f'$ that avoids the simplex $\sigma$ and has no further edgy simplices with 3-additive image than those of $f$.

Consider a simplex $\tau \supsetneq \sigma$ of $S^d$ that contains $\sigma$. Then its image $f(\tau) = \{v_0, v_1, \dots, v_k\}$ contains $v_0$ and $v_1$, which have last coordinates $\pm R$. As $f$ has no overly augmented simplices, this implies that $f(\tau)$ cannot be a double-triple or double-double simplex (see Observation 6.3). On the other hand, $f(\tau)$ contains the 3-additive edge $f(\sigma)$, so it must be 3-additive itself, with additive core $\{v_0, v_1, e_i, e_j\}$ (see Observation 4.11). Hence, $f(\tau) \cup \{v\} = \{\langle \vec{v}_1 \pm \vec{e}_i \rangle, \langle \vec{v}_1 \pm \vec{e}_i \pm \vec{e}_j \rangle, v_1, \dots, v_k\}$ is a double-triple simplex in $\mathrm{BAA}_n^m$ with additive core $\{v, v_0, v_1, e_i, e_j\}$. This implies that $f$ maps $\mathsf{Star}_{S^d}(\sigma)$ to $\mathsf{Star}_{\mathrm{BAA}_n^m}(\{v_0, v_1, v\})$.

Let $(S^d)'$ be the coarsest subdivision of $S^d$ that subdivides $\sigma$ by adding a new vertex $t$ at its barycentre. Let $f' \colon (S^d)' \to \mathrm{BAA}_n^m$ be the map that agrees with $f$ on vertices of $S^d$ and sends $t$ to $v$. The previous paragraph proves that $f'$ is simplicial, and $f$ and $f'$ are homotopic. The structure of $f'$ can be described as follows: To obtain $f'$ from $f$, subdivide every simplex $\tau \supseteq \sigma$ that contains $\sigma$ into $(\dim(\sigma)+1)$-many simplices of the same dimension as $\tau$. Each such new simplex is obtained by replacing one vertex of $\sigma \subseteq \tau$ with the newly added $t$. This vertex gets mapped to $f'(t) = v$ and $f'$ agrees with $f$ on the remaining vertices of $\tau'$. Every simplex of $S^d$ that does not contain $\sigma$ is also a simplex in $(S^d)'$ and the maps $f$ and $f'$ agree on these simplices.

Clearly, $f'$ does not contain the edgy simplex $\sigma$ anymore. We claim that furthermore, no new edgy simplices with 3-additive image were created when passing from $f$ to $f'$. To see this, assume that $\sigma'$ is an edgy simplex of $f'$ that is not an edgy simplex of $f$. Then $\sigma'$ must contain the newly added vertex $t$ and hence is a face of some $\tau'$ that was obtained by subdividing a simplex $\tau \supseteq \sigma$. This implies that the image $f(\sigma')$ must be of one of the forms

$$\{v, v_0\} = \{\langle \vec{v}_1 \pm \vec{e}_i \rangle, \langle \vec{v}_1 \pm \vec{e}_i \pm \vec{e}_j \rangle\}, \{v, v_1\} = \{\langle \vec{v}_1 \pm \vec{e}_i \rangle, v_1\}, \text{ or}$$
$$\{v, v_l\} \text{ for some } v_l \text{ such that } \{\vec{v}, \vec{v}_l, \vec{e}_1, \dots, \vec{e}_m\} \text{ is a partial basis.}$$

But $f(\sigma')$ is not 3-additive in any of these cases.[12]

The subdivision mentioned above might have introduced new overly augmented simplices. Before we can remove another edgy simplex, we need to get rid of these simplices.

[12] The subdivision created new edgy simplices with 2-additive image though, e.g. of the form $\{\langle \vec{v}_1 \pm \vec{e}_i \rangle, v_1\}$. These will be removed in the next Step 1.2.

To do so, we apply Procedure 1 again. This removes all overly augmented simplices without introducing new edgy simplices. Afterwards, we can remove another edgy simplex whose image is 3-additive. Iterating this procedure leads to a map in which the image of every edgy simplex is either standard or 2-additive.

*Step 1.2: Removing edgy simplices with standard image.* After the previous step, we can assume that $f$ has no edgy simplices with 3-additive image and (after possibly applying Procedure 1 again) also has no overly augmented simplices. In this step, we will also remove all edgy simplices with standard image.

Let $\sigma$ be an edgy simplex such that $f(\sigma) = \{v_0, v_1\}$ is standard. We will use a procedure that is very similar to the one described in Step 1.1 in order to replace $f$ by a map $f'$ that avoids $\sigma$. Choose representatives $\vec{v}_0$ and $\vec{v}_1$ such that their last coordinates are equal to $R$ and define $\vec{v} := \vec{v}_0 - \vec{v}_1$. Clearly, $v$ is a vertex in $\mathrm{BAA}_n^m$ and has last coordinate equal to 0.

Let $\tau \supseteq \sigma$ be a simplex of $S^d$ that contains $\sigma$. Then its image $f(\tau) = \{v_0, v_1, \dots, v_k\}$ contains $v_0$ and $v_1$, which have last coordinates $\pm R$. As $f$ has no overly augmented simplices and no edgy simplices with 3-additive image, this implies that $f(\tau)$ cannot be a 3-additive, double-triple or double-double simplex (see Observation 6.3). Hence, it must be either standard or 2-additive. In either case, $f(\tau) \cup \{v\} = \{\langle \vec{v}_0 - \vec{v}_1 \rangle, v_0, v_1, \dots, v_k\}$ is a simplex in $\mathrm{BAA}_n^m$. Here, we use the observation that $v = \langle \vec{v}_0 - \vec{v}_1 \rangle$ might be contained in $f(\tau)$, but $\langle \vec{v}_0 + \vec{v}_1 \rangle$ cannot: the last coordinate of $\vec{v}_0 + \vec{v}_1$ is $2R$, which would contradict the definition of $R$. This implies that $f$ maps $\mathsf{Star}_{S^d}(\sigma)$ to $\mathsf{Star}_{\mathrm{BAA}_n^m}(\{v_0, v_1, v\})$.

Let $(S^d)'$ be the coarsest subdivision of $S^d$ that subdivides $\sigma$ by adding a new vertex $t$ at its barycentre. Let $f' \colon (S^d)' \to \mathrm{BAA}_n^m$ be the map that agrees with $f$ on vertices of $S^d$ and sends $t$ to $v$. By the observations of the previous paragraph, this map is simplicial and $f$ and $f'$ are homotopic. Just as in Step 1.1, every edgy simplex of $f'$ is either also an edgy simplex of $f$ or it contains the vertex $t$. However, the latter is impossible here as $t$ gets mapped to the vertex $v$. This has last coordinate 0, whereas every vertex in the image of an edgy simplex must have last coordinate $\pm R$.

It follows that $f'$ has one less edgy simplex than $f$ (namely $\sigma$, which got subdivided) and that every edgy simplex of $f'$ also forms an edgy simplex of $f$. In particular, as $f$ does not have any edgy simplex with 3-additive image, neither does $f'$. It might be that $f'$ has overly augmented simplices.[13] However, we can use Procedure 1 again to remove those without introducing new edgy simplices. Afterwards, we can remove another edgy simplex with standard image. After finitely many iterations, we obtain a map that has only edgy simplices with 2-additive image.

*Step 1.3: Removing edgy simplices with 2-additive image.* We can now assume that $f$ has no edgy simplices whose image is standard or 3-additive. After performing Procedure

[13] If $\sigma$ is contained in $\tau$ and $f(\tau)$ is 2-additive, it might be that the image of $f'$ contains a 3-additive simplex with a vertex that has last coordinate $\pm R$. For example if $f(\tau) = \{v_0, v_1, \langle \vec{v}_1 + \vec{v}_2 \rangle, v_2\}$, then there is a simplex $\tau'$ with $f'(\tau') = \{v_0, \langle \vec{v}_0 - \vec{v}_1 \rangle, \langle \vec{v}_1 + \vec{v}_2 \rangle, v_2\}$.

1, we can also assume that is has no overly augmented simplices. What remains to be done for completing Step 1 is to remove edgy simplices with 2-additive image. Let $\sigma$ be a maximal such simplex, i.e. $f(\sigma) = \{v_0, v_1\}$ is a 2-additive simplex, the last coordinates of $v_0$ and $v_1$ are equal to $\pm R$ and if $\tau \supset \sigma$, then $f(\tau) \neq f(\sigma)$. As $f(\sigma) = \{v_0, v_1\}$ is 2-additive, we have $\vec{v}_0 = \vec{v}_1 \pm \vec{e}_i$ for some $1 \leq i \leq m$. Here, we cannot proceed as in the case of standard simplices (Step 1.2), because if $\vec{v}_0, \vec{v}_1$ have last coordinate $R$, then $\vec{v}_0 - \vec{v}_1 = \pm\vec{e}_i$ is a not vertex in $\mathrm{BAA}_n^m$. What we will do instead is to apply an argument similar to the one of Procedure 1: We will define a complex $K(\sigma)$ and homotope $f$ such that it maps $\mathsf{Star}_{S^d}(\sigma)$ to $f(\partial\sigma) * K(\sigma)$.

Define $K(\sigma) := \widehat{\mathsf{Link}}^{<}_{\mathrm{BA}_n^m}(v_0)$. In order to perform an argument similar to Procedure 1, we need to verify the analogues of Claim 6.4, Claim 6.5 and Claim 6.6.

That $K(\sigma)$ is a subcomplex of $\mathsf{Link}_{\mathrm{BAA}_n^m}(f(\sigma))$ is a part of Lemma 5.8. Furthermore, $f$ maps $\mathsf{Link}_{S^d}(\sigma)$ to $K(\sigma)$: As $f$ is simplicial, we have $f(\mathsf{Link}_{S^d}(\sigma)) \subseteq \mathsf{Star}_{\mathrm{BAA}_n^m}(f(\sigma))$ and because we assumed $\sigma$ to be maximal with respect to inclusion, $f(\mathsf{Link}_{S^d}(\sigma)) \subseteq \mathsf{Link}_{\mathrm{BAA}_n^m}(f(\sigma))$. Next, we show that the image of every vertex of $\mathsf{Link}_{S^d}(\sigma)$ has last coordinate of absolute value less than $R$. Assume for contradiction that there is a vertex $x \in \mathsf{Link}_{S^d}(\sigma)$ such that the last coordinate of $f(x)$ is $\pm R$. As we assumed $\sigma$ to be maximal, the image of the simplex $\sigma \cup \{x\}$ has three vertices. Its image $f(\sigma \cup \{x\})$ contains vertices with last coordinate $\pm R$ and has the 2-additive simplex $f(\sigma)$ as a (proper) face. As $f$ has no overly augmented simplices, this implies that $f(\sigma \cup \{x\})$ is 2-additive as well (see Observation 6.3 and Observation 4.11). But then it has a face that is a standard edge. As all three vertices of $f(\sigma\cup\{x\})$ have last coordinate $\pm R$, this shows that $f$ needs to have an edgy simplex whose image is standard. This is a contradiction to our assumption. Hence, we have $f(\mathsf{Link}_{S^d}(\sigma)) \subseteq \mathsf{Link}^{<}_{\mathrm{BAA}_n^m}(f(\sigma))$.

By Lemma 5.7, we know that

$$\mathsf{Link}^{<}_{\mathrm{BAA}_n^m}(f(\sigma)) = \widehat{\mathsf{Link}}^{<}_{\mathrm{BAA}_n^m}(f(\sigma))$$

and by Lemma 5.8, every simplex of $\widehat{\mathsf{Link}}^{<}_{\mathrm{BAA}_n^m}(f(\sigma))$ is either contained in $\widehat{\mathsf{Link}}^{<}_{\mathrm{BA}_n^m}(v_0) = K(\sigma)$ or is of type double-triple. However, as $v_0$ has last coordinate $\pm R$ (as does $v_1$) and there are no overly augmented simplices, there are no double-triple simplices in $f(\mathsf{Link}_{S^d}(\sigma))$ (see Observation 6.3). This finishes the proof of our claim that $f(\mathsf{Link}_{S^d}(\sigma)) \subseteq K(\sigma)$.

The analogue of Claim 6.5 is to show that $K(\sigma)$ is $(\dim \mathsf{Link}_{S^d}(\sigma))$-connected. Here, we can use again a result of Church–Putman. By [6, Section 4.5, third paragraph after Step 4 on p. 1029], $K(\sigma) = \widehat{\mathsf{Link}}^{<}_{\mathrm{BA}_n^m}(v_0)$ is $(n-2)$-connected. The claim follows because $\dim \mathsf{Link}_{S^d}(\sigma) = d - \dim(\sigma) - 1 \leq n - 1 - 1$.

As in Procedure 1, it follows that the restriction of $f$ to $\mathsf{Star}_{S^d}(\sigma)$ is homotopic to a simplicial map

$$h\colon \partial\sigma * \mathsf{Cone}(\mathsf{Link}_{S^d}(\sigma)) \to f(\sigma) * K(\sigma)$$

that agrees with $f$ on $\partial\sigma * \mathsf{Link}_{S^d}(\sigma)$ and has the property that $h(\mathsf{Cone}(\mathsf{Link}_{S^d}(\sigma))) \subseteq K(\sigma)$. We next verify the analogue of Claim 6.6, namely that every edgy simplex of $h$ is contained in $\partial\sigma$. This is immediate here because every vertex of $K(\sigma)$ has last coordinate of absolute value smaller than $R$. Hence, a simplex can only be edgy if $h$ maps it to $f(\sigma)$. This is only the case for simplices in $\partial\sigma$.

We can now replace $\mathsf{Star}(\sigma)$ with $\partial\sigma * \mathsf{Cone}(\mathsf{Link}_{S^d}(\sigma))$ and replace $f$ by a homotopic map $f'$ that agrees with $f$ outside $\mathsf{Star}(\sigma)$ and is equal to $h$ on $\partial\sigma * \mathsf{Cone}(\mathsf{Link}_{S^d}(\sigma))$. As every edgy simplex of $h$ is contained in $\partial\sigma$, every edgy simplex of $f'$ is also an edgy simplex of $f$. Hence, no new edgy simplices are created when passing from $f$ to $f'$. In particular, $f'$ still has only edgy simplices whose image is 2-additive. However, $f'$ has one less of these simplices than $f$ (namely $\sigma$).

After applying Procedure 1 again to remove overly augmented simplices, we can go on and remove another edgy simplex of the resulting map. Iterating this leads to a map that has no edgy simplices (with 2-additive, standard, or 3-additive image).

**Step 2: Removing bad vertices.** We can now assume that $f$ has no edgy simplices. Call a simplex $\sigma$ of $S^d$ *bad* if $f(\sigma) = \{v\}$ with the last coordinates of $v$ equal to $\pm R$. Recall that our aim is to the replace $f$ by a map whose image has only vertices with last entries of absolute value less than $R$. Hence, we are done if we can remove all bad simplices. Let $\sigma$ be a bad simplex that is maximal with respect to inclusion among all bad simplices. We define $K(\sigma) := \mathsf{Link}^{<}_{\mathrm{BAA}_n^m}(v)$ and proceed as in Procedure 1 above, verifying in the following three paragraphs the analogues of Claim 6.4, Claim 6.5 and Claim 6.6.

First note that $f$ maps $\mathsf{Link}_{S^d}(\sigma)$ to $K(\sigma)$: As $f$ is simplicial and $\sigma$ is maximal among bad simplices, we have $f(\mathsf{Link}_{S^d}(\sigma)) \subseteq \mathsf{Link}_{\mathrm{BAA}_n^m}(f(\sigma))$. Assume that there was $x \in \mathsf{Link}_{S^d}(\sigma)$ that gets mapped to a line with last entry $\pm R$. Then, as there are no edgy simplices, we have $f(x) = v$ and $\sigma \cup \{x\}$ gets mapped to $\{v\}$. This contradicts $\sigma$ being maximal. Consequently, we have $f(\mathsf{Link}_{S^d}(\sigma)) \subseteq \mathsf{Link}^{<}_{\mathrm{BAA}_n^m}(f(\sigma)) = K(\sigma)$.

Next, we want to verify that $K(\sigma)$ is $(\dim \mathsf{Link}_{S^d}(\sigma))$-connected. For this, we finally use the inductive hypothesis and the retraction defined in Section 3: First note that by the first item of Lemma 5.7, $K(\sigma)$ actually coincides with $\widehat{\mathsf{Link}}^{<}_{\mathrm{BAA}_n^m}(v)$. Hence, it suffices to show that this complex is $(\dim \mathsf{Link}_{S^d}(\sigma)) = (d - \dim(\sigma) - 1)$-connected. As noted in Lemma 5.2, there is an isomorphism

$$\widehat{\mathsf{Link}}_{\mathrm{BAA}_n^m}(v) \cong \mathrm{BAA}_{n-1}^{m+1},$$

so by induction, $\widehat{\mathsf{Link}}_{\mathrm{BAA}_n^m}(v)$ is $(n-1)$-connected. By Theorem 3.1, $\widehat{\mathsf{Link}}^{<}_{\mathrm{BAA}_n^m}(v)$ is as highly-connected as $\widehat{\mathsf{Link}}_{\mathrm{BAA}_n^m}(v)$ and hence is also $(n-1)$-connected. The claimed connectivity of $K(\sigma)$ now follows because $(n-1) \geq (d - \dim(\sigma) - 1)$.

As in Procedure 1 and Step 1.3, it follows that the restriction of $f$ to $\mathsf{Star}_{S^d}(\sigma)$ is homotopic to a simplicial map $h\colon \partial\sigma * \mathsf{Cone}(\mathsf{Link}_{S^d}(\sigma)) \to f(\sigma) * K(\sigma)$ that agrees with $f$ on $\partial\sigma * \mathsf{Link}_{S^d}(\sigma)$ and such that $h(\mathsf{Cone}(\mathsf{Link}_{S^d}(\sigma))) \subseteq K(\sigma)$. For the analogue of Claim 6.6, observe that every bad simplex of $h$ is contained in $\partial\sigma$ and that $h$ does not have any edgy simplices: This follows similarly to Step 1.3 because every vertex of $K(\sigma)$ has last coordinate of absolute value smaller than $R$.

We now replace $\mathsf{Star}(\sigma)$ with $\partial\sigma * \mathsf{Cone}(\mathsf{Link}_{S^d}(\sigma))$ and $f$ by the map $f'$ that agrees with $f$ outside $\mathsf{Star}(\sigma)$ and is equal to $h$ on $\partial\sigma * \mathsf{Cone}(\mathsf{Link}_{S^d}(\sigma))$. This removes the bad simplex $\sigma$ without introducing any new bad or edgy simplices. Iterating this, we obtain a map that has no bad simplices and hence maps every vertex of $S^d$ to a line with last entry of absolute value less than $R$. $\square$

## 7. Maps of posets

In this section, we recall Quillen's map of posets spectral sequence [20] and some of its corollaries. In this and the following sections, we use posets as they are closely related to simplicial complexes. In fact, to each poset $\mathbb{A}$, we associate a simplicial complex of chains in $\mathbb{A}$, i.e. its vertices are the elements of $\mathbb{A}$ and a set $\{a_0, \dots, a_p\}$ forms a $p$-simplex if it is a chain $a_0 < \cdots < a_p$ in $\mathbb{A}$. Vice versa, given a simplicial complex $X$, we denote by $\mathbb{P}(X)$ the poset of simplices of $X$. The associated simplicial complex to $\mathbb{P}(X)$ is the barycentric subdivision of $X$.

We begin by fixing some terminology concerning posets.

**Definition 7.1.** Let $\mathbb{A}$ be a poset and $a \in \mathbb{A}$. Define

$$\mathrm{ht}(a) := \min(\{k \mid \exists\, a_1 < a_2 < \cdots < a_k < a\}).$$

We call $\mathrm{ht}(a)$ the *height* of $a$.

**Definition 7.2.** Let $\mathbb{A}$ be a poset and let $a \in \mathbb{A}$. Let $\mathbb{A}_{>a}$ be the subposet of $\mathbb{A}$ of elements $x$ with $x > a$.

**Definition 7.3.** Let $\phi\colon \mathbb{A} \to \mathbb{B}$ be a map of posets and $b \in \mathbb{B}$. Let $\phi^{\leq b}$ be the subposet of $\mathbb{A}$ of elements $a$ with $\phi(a) \leq b$.

When we speak about the homology of a poset $\mathbb{A}$, we mean the homology of the geometric realisation of its associated simplicial complex, which we will just refer to as the geometric realisation of $\mathbb{A}$. Similarly, when we say that a poset is $d$-connected, $d$-dimensional, etc., we mean its geometric realisation has this property.

We now define a more general notion of homology of posets.

**Definition 7.4.** Let $\mathsf{Ab}$ denote the category of abelian groups. Let $\mathbb{A}$ be a poset (viewed as a category with objects the elements of $\mathbb{A}$ and exactly one morphism $a_1 \to a_2$ if $a_1 \leq a_2$ and none otherwise) and let $T\colon \mathbb{A} \to \mathsf{Ab}$ be a functor. For $p \geq 0$, let

$$C_p(\mathbb{A}; T) := \bigoplus_{a_0 < \cdots < a_p} T(a_0).$$

Define maps

$$d_i : C_p(\mathbb{A}; T) \to C_{p-1}(\mathbb{A}; T), \qquad (0 \le i \le p)$$

as follows. For $i > 0$ let $d_i$ be given by the identity map $T(a_0) \to T(a_0)$ from the summand indexed by $a_0 < \cdots < a_p$ to the summand indexed by $a_0 < \cdots < a_{i-1} < a_{i+1} < \cdots < a_p$. Let $d_0 : C_p(\mathbb{A}; T) \to C_{p-1}(\mathbb{A}; T)$ be given by $T(a_0 \to a_1): T(a_0) \to T(a_1)$ from the summand indexed by $a_0 < \cdots < a_p$ to the summand indexed by $a_1 < \cdots < a_p$. Let

$$d = \sum_i (-1)^i d_i : C_p(\mathbb{A}; T) \to C_{p-1}(\mathbb{A}; T).$$

Since $d \circ d = 0$, these groups and maps form a chain complex which we denote by $C_*(\mathbb{A}; T)$. Let

$$H_i(\mathbb{A}; T) = H_i(C_*(\mathbb{A}; T)).$$

One of the most basic examples of a functor is the constant functor $\mathbb{Z}$ which sends every object to $\mathbb{Z}$ and every morphism to the identity map. Note that $H_i(\mathbb{A}; \mathbb{Z})$ is isomorphic to the homology of the geometric realisation of $\mathbb{A}$. Another class of functors that we will consider is the following.

**Definition 7.5.** Let $\phi: \mathbb{A} \to \mathbb{B}$ be a map of posets. Let $H_i(\phi): \mathbb{B} \to \mathsf{Ab}$ be the functor sending $b \in \mathbb{B}$ to $H_i(\phi^{\le b})$ and $b_1 \le b_2$ to $H_i(\phi^{\le b_1}) \to H_i(\phi^{\le b_2})$ induced by the inclusion $\phi^{\le b_1} \subset \phi^{\le b_2}$.

The following spectral sequence is due to Quillen [20].

**Theorem 7.6** *(Quillen). Let $\phi: \mathbb{A} \to \mathbb{B}$ be a map of posets. There is a homologically graded spectral sequence:*

$$E^2_{pq} = H_p(\mathbb{B}; H_q(\phi)) \implies H_{p+q}(\mathbb{A}).$$

See Charney [5, Lemma 1.3] or [17, Lemma 3.2] for a proof of the following.

**Lemma 7.7.** *Let $\mathbb{A}$ be a poset, let $T: \mathbb{A} \to \mathsf{Ab}$ be a functor, and $m \in \mathbb{N}$. Suppose $T(a) \cong 0$ if $\mathrm{ht}(a) \neq m$. Then there is a natural isomorphism:*

$$H_i(\mathbb{A}; T) \cong \bigoplus_{\mathrm{ht}(a)=m} \widetilde{H}_{i-1}(\mathbb{A}_{>a}; T(a)).$$

Here $\widetilde{H}_{i-1}(\mathbb{A}_{>a}; T(a))$ means the reduced homology of the geometric realisation with (untwisted) coefficients $T(a)$. This lemma gives the following corollary (see e.g. [18, Lemma 3.7]).

**Proposition 7.8.** *Let $\phi\colon \mathbb{A} \to \mathbb{B}$ be a map of posets and let $E^r_{p,q}$ denote the map of posets spectral sequence. Assume for some fixed $d, e, r \geq 0$, the following holds for all $V \in \mathbb{B}$:*

- $\widetilde{H}_i(\phi^{\leq V}) \cong 0$ *for all* $i \notin [\mathrm{ht}(V) + d - r, \mathrm{ht}(V) + d]$.
- $\widetilde{H}_i(\mathbb{B}_{>V}) \cong 0$ *for all* $i \neq e - \mathrm{ht}(V) - 1$.

*Then for all $a \geq 0$ and $b \geq 1$ satisfying $a + b \notin [d + e - r, d + e]$, we have that $E^2_{a,b} \cong 0$.*

A poset is called Cohen–Macaulay of dimension $d$ if its associated simplicial complex is Cohen–Macaulay. A map $f\colon A \to B$ is called $k$-acyclic if it induces an isomorphism on $H_i$ for $i < k$ and a surjection for $k = i$.

**Proposition 7.9** *([6, Proposition 2.3]). Fix $m \geq 0$ and let $\phi\colon \mathbb{A} \to \mathbb{B}$ be a map of posets. Assume that $\mathbb{B}$ is Cohen–Macaulay of dimension $d$ and that for all $b \in \mathbb{B}$ and $q \neq \mathrm{ht}(b) + m$, we have $\widetilde{H}_q(\phi^{\leq b}) = 0$. Then $\phi$ is $(d + m)$-acyclic.*

## 8. Proof of Theorem A and Theorem B

The goal of this section is to prove Theorem A, which describes the relations among the relations in Steinberg modules and use this to prove Theorem B, which states that the codimension-2 rational homology of $\mathrm{SL}_n(\mathbb{Z})$ vanishes for $n \geq 3$. Throughout this section, we will assume that $n \geq 3$.

For a field $\mathbb{F}$, we write $\mathbb{T}_n(\mathbb{F})$ for the poset of proper nonzero subspaces of $\mathbb{F}^n$. As in the introduction, the geometric realisation of this poset is the Tits building associated to $\mathrm{SL}_n(\mathbb{F})$, denoted by $\mathcal{T}_n(\mathbb{F})$. It is elementary to see that $\mathbb{T}_n(\mathbb{Q})$ is isomorphic to the following poset.

**Definition 8.1.** We write $\mathbb{T}_n(\mathbb{Z})$ (or simply $\mathbb{T}_n$) for the poset of proper nonzero direct summands of $\mathbb{Z}^n$ under inclusion. We write its geometric realisation as $\mathcal{T}_n(\mathbb{Z})$.

We prove Theorem A and Theorem B using $n$-connectivity of $\mathrm{BAA}_n$. The proof here works very similarly to [6, Proof of Theorem A and B]; we largely follow [6, Section 3].

**Definition 8.2.** For $\Lambda = \mathbb{Z}$ or $\Lambda = \mathbb{F}_p$, let $\mathrm{BAA}^{\pm}_n(\Lambda)'$ be the subcomplex of $\mathrm{BAA}^{\pm}_n(\Lambda)$ consisting of all simplices $\{v_0, \ldots, v_k\}$ such that $\langle \vec{v}_0, \ldots, \vec{v}_k \rangle_\Lambda$ is a proper subgroup of $\Lambda^n$. Let $\mathrm{BAA}'_n = \mathrm{BAA}^{\pm}_n(\mathbb{Z})'$.

In other words, the simplices of $\mathrm{BAA}^{\pm}_n(\Lambda)$ that are not contained in $\mathrm{BAA}^{\pm}_n(\Lambda)'$ are precisely

- the standard simplices of dimension $n - 1$,
- the 2-additive and 3-additive simplices of dimension $n$, and
- the double-triple and double-double simplices of dimension $n + 1$.

In particular,

| | |
|---|---|
| $C_k(\mathrm{BAA}_n, \mathrm{BAA}'_n)$ | vanishes for $k \leq (n-2)$, |
| $C_{n-1}(\mathrm{BAA}_n, \mathrm{BAA}'_n)$ | is spanned by standard simplices, |
| $C_n(\mathrm{BAA}_n, \mathrm{BAA}'_n)$ | is spanned by 2-additive and 3-additive simplices, |
| $C_{n+1}(\mathrm{BAA}_n, \mathrm{BAA}'_n)$ | is spanned by double-triple and double-double simplices. |

(Note that in the case $n = 3$, the complex $\mathrm{BAA}_n$ contains no double-double simplices. So in this case, $C_{n+1}(\mathrm{BAA}_n, \mathrm{BAA}'_n)$ is spanned by double-triple simplices.)

The connectivity of $\mathrm{BAA}_n$ gives us isomorphisms between the homology of these relative chains and the homology of $\mathrm{BAA}'_n$:

**Lemma 8.3.** *Let $n \geq 3$. There are isomorphisms*

$$H_{n-2}(\mathrm{BAA}'_n) \cong H_{n-1}(\mathrm{BAA}_n, \mathrm{BAA}'_n) \text{ and } H_{n-1}(\mathrm{BAA}'_n) \cong H_n(\mathrm{BAA}_n, \mathrm{BAA}'_n).$$

**Proof.** This follows immediately from the long exact sequence of the pair $(\mathrm{BAA}_n, \mathrm{BAA}'_n)$ because $\mathrm{BAA}_n$ is $n$-connected (Theorem 2.10). □

We use Proposition 7.9 to get an explicit description of the homology of $\mathrm{BAA}'_n$ in high degrees.

**Lemma 8.4.** *Let $n \geq 3$. There are isomorphisms*

$$H_{n-2}(\mathrm{BAA}'_n) \cong \mathrm{St}_n(\mathbb{Q}) \text{ and } H_{n-1}(\mathrm{BAA}'_n) \cong 0.$$

**Proof.** Let $\mathbb{P}(\mathrm{BAA}'_n)$ denote the poset of simplices of $\mathrm{BAA}'_n$ under inclusion, and consider the map of posets

$$\begin{aligned} \phi\colon \mathbb{P}(\mathrm{BAA}'_n) &\longrightarrow \mathbb{T}_n \\ \{v_0, \ldots, v_k\} &\longmapsto \langle \vec{v}_0, \ldots, \vec{v}_k \rangle. \end{aligned}$$

We want to apply Proposition 7.9 with $m = 2$. As $\mathbb{T}_n$ is Cohen–Macaulay of dimension $(n-2)$, we have to verify that for every proper direct summand $\{0\} \neq V \subset \mathbb{Z}^n$, the fibre $\phi_{\leq V}$ has vanishing reduced homology in all degrees except $(\mathrm{ht}(V) + 2) = (\mathrm{rank}(V) + 1)$. But we have

$$\phi^{\leq V} = \{\sigma \in \mathbb{P}(\mathrm{BAA}'_n) \mid \phi(\sigma) \leq V\} \cong \mathbb{P}(\mathrm{BAA}(V)).$$

The complex $\mathrm{BAA}(V)$ has dimension at most[14] $\mathrm{rank}(V) + 1$ and is $\mathrm{rank}(V)$-connected by Theorem 2.10.

[14] In fact, its dimension is equal to $\mathrm{rank}(V) + 1$ if $\mathrm{rank}(V) \geq 3$, see the comments after Theorem 2.10.

It follows that the map $\phi$ induces isomorphisms

$$H_{n-2}(|\mathbb{P}(\mathrm{BAA}'_n)|) \to H_{n-2}(\mathcal{T}_n) \cong \mathrm{St}_n(\mathbb{Q}) \text{ and } H_{n-1}(|\mathbb{P}(\mathrm{BAA}'_n)|) \to H_{n-1}(\mathcal{T}_n) \cong 0.$$

The claim follows because $|\mathbb{P}(\mathrm{BAA}'_n)|$ is the geometric realisation of the barycentric subdivision of $|\,\mathrm{BAA}'_n\,|$. □

**Proposition 8.5.** *Let $n \geq 3$. The sequence*

$$C_{n+1}(\mathrm{BAA}_n, \mathrm{BAA}'_n) \xrightarrow{\partial_{n+1}} C_n(\mathrm{BAA}_n, \mathrm{BAA}'_n) \xrightarrow{\partial_n} C_{n-1}(\mathrm{BAA}_n, \mathrm{BAA}'_n) \xrightarrow{q}$$
$$\xrightarrow{q} H_{n-1}(\mathrm{BAA}_n, \mathrm{BAA}'_n) \cong \mathrm{St}_n(\mathbb{Q}) \longrightarrow 0$$

*is exact.*

**Proof.** Firstly, the map $q$ is surjective by the definition of homology, so we have exactness at $H_{n-1}(\mathrm{BAA}_n, \mathrm{BAA}'_n)$. Secondly, we noted above that $C_{n-2}(\mathrm{BAA}_n, \mathrm{BAA}'_n)$ is trivial. Hence,

$$H_{n-1}(\mathrm{BAA}_n, \mathrm{BAA}'_n) \cong C_{n-1}(\mathrm{BAA}_n, \mathrm{BAA}'_n)/\operatorname{im}(\partial_n),$$

which shows that the sequence is exact at $C_{n-1}(\mathrm{BAA}_n, \mathrm{BAA}'_n)$. Lastly, exactness at $C_n(\mathrm{BAA}_n, \mathrm{BAA}'_n)$ is equivalent to the vanishing of the homology group $H_n(\mathrm{BAA}_n, \mathrm{BAA}'_n)$. By Lemma 8.3, this group is isomorphic to $H_{n-1}(\mathrm{BAA}'_n)$, which vanishes by Lemma 8.4.

The isomorphism $H_{n-1}(\mathrm{BAA}_n, \mathrm{BAA}'_n) \cong \mathrm{St}_n(\mathbb{Q})$ is also an immediate consequence of Lemma 8.3 and Lemma 8.4. □

This proposition implies Theorem A, our partial resolution of $\mathrm{St}_n(\mathbb{Q})$.

**Proof of Theorem A.** If $n = 2$, the group $\mathcal{M}_2$ is trivial and the result was shown by Church–Putman [6, Theorem B]. If $n \geq 3$, for $i = 0, 1$ or $2$, the groups $\mathcal{M}_i$ in the statement of Theorem A are isomorphic to the relative chain groups $C_{n-1+i}(\mathrm{BAA}_n, \mathrm{BAA}'_n)$. The claim now follows from Proposition 8.5. □

To deduce Theorem B from this, we need the following well-known lemma. The proof is adapted from Church–Putman [6, Lemma 3.2] using Putman–Studenmund [19, Lemma 2.2]. Also see Putman–Studenmund [19, Lemma 2.3].

**Lemma 8.6.** *Let $G$ be a group and let $Y \hookrightarrow X$ be an inclusion of $G$-simplicial complexes. Assume that the setwise stabiliser subgroup of every $k$-simplex of $X$ that is not contained in the image of $Y$ is finite. Let $R$ be a ring such that the orders of all stabiliser groups of such simplices are invertible in $R$. Then $C_k(X, Y)$ is a projective $R[G]$-module.*

**Proof.** Let $\sigma$ be a $k$-simplex of $X$ not contained in $Y$ and pick an orientation on $\sigma$. Let $G_\sigma \subset G$ be the stabiliser of $\sigma$. By abuse of notation, also view $\sigma$ as an element of $C_k(X, Y; R)$. Let $M_\sigma \subset C_k(X, Y; R)$ be the $R[G]$-submodule generated by $\sigma$. Let $R_\sigma$ be the $R[G_\sigma]$-module whose underlying $R$-module is just $R$ but an element of $G$ acts by $\pm 1$ depending on whether it reverses the orientation on $\sigma$ or not. As in Church–Putman [6, Lemma 3.2], we have that

$$M_\sigma \cong \operatorname{Ind}_{G_\sigma}^{G} R_\sigma.$$

Putman–Studenmund [19, Lemma 2.2] states that $R_\sigma$ is a projective $R[G_\sigma]$-module and hence a summand of a free $R[G_\sigma]$-module. Since

$$\operatorname{Ind}_{G_\sigma}^{G} R[G_\sigma] \cong R[G],$$

it follows that $M_\sigma$ is a summand of a free $R[G]$-module and hence $M_\sigma$ is projective. Since $C_k(X, Y; R)$ is a direct sum of modules of the form $M_\sigma$, the module $C_k(X, Y; R)$ is also projective. □

**Lemma 8.7.** *Let $R$ be a ring and let $\Gamma$ be a subgroup of $\mathrm{SL}_n(\mathbb{Z})$. Assume that for any $g \in \Gamma$ of finite order $j < \infty$, the element $j$ is a unit in $R$. Then $C_k(\mathrm{BAA}_n, \mathrm{BAA}'_n; R)$ is projective as an $R[\Gamma]$-module.*

**Proof.** Note that the groups $C_k(\mathrm{BAA}_n, \mathrm{BAA}'_n; R)$ vanish unless $k \in \{n-1, n, n+1\}$, so we shall restrict attention to those cases.

In order to apply Lemma 8.6, we will first show that for every $k \in \{n-1, n, n+1\}$ and for every $k$-simplex $\sigma$ of $\mathrm{BAA}_n$ that is not contained in $\mathrm{BAA}'_n$, the setwise stabiliser $\mathrm{SL}_n(\mathbb{Z})_\sigma$ of $\sigma$ under the action of $\mathrm{SL}_n(\mathbb{Z})$ is finite. Let $\sigma = \{v_0, \dots, v_k\}$ be such a simplex. Then by definition, we have $\langle \vec{v}_0, \dots, \vec{v}_k \rangle = \mathbb{Z}^n$ and we can assume that $\{\vec{v}_0, \dots, \vec{v}_{n-1}\}$ is a basis. An element $\phi \in \mathrm{SL}_n(\mathbb{Z})$ that stabilises $\sigma$ induces a signed permutation of the set $\{\vec{v}_0, \dots, \vec{v}_k\}$. Furthermore, any such $\phi$ is uniquely determined by the images of $\vec{v}_0, \dots, \vec{v}_{n-1}$ because these form a basis of $\mathbb{Z}^n$. It follows that $\mathrm{SL}_n(\mathbb{Z})_\sigma$ is a subgroup of the group of signed permutations of a set with $k+1$ elements. This is the Coxeter group of type $\mathtt{B}_{k+1}$, a finite group of order $2^{k+1} \cdot (k+1)!$.

This implies that the stabiliser $\Gamma_\sigma$ is finite and by assumption, the orders of all its elements are invertible in $R$. It follows from Cauchy's Theorem that the order of $\Gamma_\sigma$ is invertible in $R$ as well, so we can apply Lemma 8.6. □

**Remark 8.8.** In particular, this implies that $C_k(\mathrm{BAA}_n, \mathrm{BAA}'_n; \mathbb{Q})$ is projective as a $\mathbb{Q}[\mathrm{SL}_n(\mathbb{Z})]$-module and $C_k(\mathrm{BAA}_n, \mathrm{BAA}'_n; \mathbb{Z})$ is projective as a $\mathbb{Z}[\Gamma]$-module if $\Gamma$ is torsion-free.

We use the concrete description of $C_{n+1}(\mathrm{BAA}_n, \mathrm{BAA}'_n)$ in terms of double-double and double-triple simplices to show the following.

**Lemma 8.9.** *The group* $C_{n+1}(\mathrm{BAA}_n, \mathrm{BAA}'_n; \mathbb{Q}) \otimes_{\mathrm{SL}_n(\mathbb{Z})} \mathbb{Q}$ *vanishes for all* $n \geq 3$.

**Proof.** The group $C_{n+1}(\mathrm{BAA}_n, \mathrm{BAA}'_n; \mathbb{Q})$ is generated by all oriented $(n+1)$-dimensional double-triple and double-double simplices of $\mathrm{BAA}_n$. (As noted above, the double-double simplices only occur if $n \geq 4$.) Let $\sigma = \{v_0, \ldots, v_{n+1}\}$ be such an $(n+1)$-simplex, where $\vec{v}_2, \ldots, \vec{v}_{n+1}$ is a basis of $\mathbb{Z}^n$. We need to show that for any $q \in \mathbb{Q}$, the element $\sigma \otimes q$ is trivial in $C_{n+1}(\mathrm{BAA}_n, \mathrm{BAA}'_n; \mathbb{Q}) \otimes_{\mathrm{SL}_n(\mathbb{Z})} \mathbb{Q}$. There are two cases to consider.

First suppose that $\sigma$ is a double-double simplex. Then for suitable choices of signs, we have $\vec{v}_0 = \vec{v}_2 + \vec{v}_3$ and $\vec{v}_1 = \vec{v}_4 + \vec{v}_5$. Let $\phi\colon \mathbb{Z}^n \to \mathbb{Z}^n$ be the automorphism defined by

$$\phi(\vec{v}_2) = \vec{v}_4,\ \phi(\vec{v}_4) = \vec{v}_2,\ \phi(\vec{v}_3) = \vec{v}_5,\ \phi(\vec{v}_5) = \vec{v}_3,\ \phi(\vec{v}_i) = \vec{v}_i \text{ for } i > 5.$$

The automorphism $\phi$ is contained in $\mathrm{SL}_n(\mathbb{Z})$ because it acts as an even permutation on the basis $\vec{v}_2, \ldots, \vec{v}_{n+1}$. On the other hand, it acts as an odd permutation on the vertices of $\sigma$, as

$$\phi((v_0, \ldots, v_{n+1})) = (v_1, v_0, v_4, v_5, v_2, v_3, v_6, \ldots, v_{n+1}).$$

Hence, in $C_{n+1}(\mathrm{BAA}_n, \mathrm{BAA}'_n; \mathbb{Q}) \otimes_{\mathrm{SL}_n(\mathbb{Z})} \mathbb{Q}$, we have $\sigma \otimes q = \phi(\sigma) \otimes q = -\sigma \otimes q$ for any $q \in \mathbb{Q}$. This implies that $\sigma \otimes q$ is trivial.

Next suppose that $\sigma$ is a double-triple simplex. In this case, we can choose signs such that $\vec{v}_0 = \vec{v}_2 + \vec{v}_3$ and $\vec{v}_1 = \vec{v}_2 + \vec{v}_4$. We define $\psi\colon \mathbb{Z}^n \to \mathbb{Z}^n$ as the automorphism given by

$$\psi(\vec{v}_2) = -\vec{v}_2,\ \psi(\vec{v}_3) = \vec{v}_2 + \vec{v}_3,\ \psi(\vec{v}_4) = -\vec{v}_4,\ \psi(\vec{v}_i) = \vec{v}_i \text{ for } i > 4.$$

It is easy to see that $\psi$ has determinant 1 and hence is contained in $\mathrm{SL}_n(\mathbb{Z})$. Noting that $\psi(\vec{v}_0) = \psi(\vec{v}_2 + \vec{v}_3) = \vec{v}_3$ and $\psi(\vec{v}_1) = \psi(\vec{v}_2 + \vec{v}_4) = -\vec{v}_1$, one sees that $\psi$ acts as an odd permutation on the vertices of $\sigma$, namely

$$\psi((v_0, \ldots, v_{n+1})) = (v_3, v_1, v_2, v_0, v_4, v_5, \ldots, v_{n+1}).$$

As before, it follows that $\sigma \otimes q$ is trivial. □

That the codimension-2 rational cohomology of $\mathrm{SL}_n(\mathbb{Z})$ vanishes for $n \geq 3$ is an easy consequence of the previous results:

**Proof of Theorem B.** Because of Borel–Serre duality (see Equation (1)), it is sufficient to show that $H_2(\mathrm{SL}_n(\mathbb{Z}); \mathrm{St}_n(\mathbb{Q}) \otimes \mathbb{Q})$ is trivial. Proposition 8.5 and Lemma 8.7 give us a partial projective resolution of $\mathrm{St}_n(\mathbb{Q}) \otimes \mathbb{Q}$ as follows:

$$C_{n+1}(\mathrm{BAA}_n, \mathrm{BAA}'_n; \mathbb{Q}) \longrightarrow C_n(\mathrm{BAA}_n, \mathrm{BAA}'_n; \mathbb{Q}) \longrightarrow C_{n-1}(\mathrm{BAA}_n, \mathrm{BAA}'_n; \mathbb{Q}) \longrightarrow \mathrm{St}_n(\mathbb{Q}) \otimes \mathbb{Q} \longrightarrow 0.$$

As this partial resolution can be extended to a projective resolution, it suffices to show that the second homology of the chain complex

$$\cdots \longrightarrow C_{n+1}(\mathrm{BAA}_n, \mathrm{BAA}'_n; \mathbb{Q}) \otimes_{\mathrm{SL}_n(\mathbb{Z})} \mathbb{Q} \longrightarrow C_n(\mathrm{BAA}_n, \mathrm{BAA}'_n; \mathbb{Q}) \otimes_{\mathrm{SL}_n(\mathbb{Z})} \mathbb{Q} \longrightarrow \longrightarrow C_{n-1}(\mathrm{BAA}_n, \mathrm{BAA}'_n; \mathbb{Q}) \otimes_{\mathrm{SL}_n(\mathbb{Z})} \mathbb{Q} \longrightarrow 0$$

vanishes. This is an immediate consequence of Lemma 8.9. □

**Remark 8.10.** Church–Putman [6, Theorem A] also proved a vanishing result for the codimension-1 cohomology of $\mathrm{SL}_n(\mathbb{Z})$ with coefficients in rational representations of $\mathrm{GL}_n(\mathbb{Q})$. The analogous result is true for the codimension-2 cohomology.

## 9. Proof of Theorem C

We now shift attention to the codimension-1 cohomology congruence subgroups and prove Theorem C.

### *9.1. Relevant simplicial complexes and connectivity results*

We will deduce our results about congruence subgroups by studying connectivity properties of $\mathrm{BAA}^{\pm}_n(\mathbb{F}_p)$. To prove the following result, it is not difficult to adapt the proofs of [18, Lemmas 2.35 and 2.43].

**Proposition 9.1.** *For $p$ an odd prime, $\mathrm{BAA}_n / \Gamma_n(p) \cong \mathrm{BAA}^{\pm}_n(\mathbb{F}_p)$ and $\mathrm{BAA}'_n / \Gamma_n(p) \cong \mathrm{BAA}^{\pm}_n(\mathbb{F}_p)'$.*

An argument identical to [18, Lemma 3.23] gives the following corollary.

**Proposition 9.2.** *Let $p$ be an odd prime. There is a natural isomorphism*

$$H_1(\Gamma_n(p); \mathrm{St}_n(\mathbb{Q})) \cong H_n(\mathrm{BAA}^{\pm}_n(\mathbb{F}_p), \mathrm{BAA}^{\pm}_n(\mathbb{F}_p)').$$

**Proof.** Proposition 8.5 states that

$$C_{n+1}(\mathrm{BAA}_n, \mathrm{BAA}'_n) \xrightarrow{\partial_{n+1}} C_n(\mathrm{BAA}_n, \mathrm{BAA}'_n) \xrightarrow{\partial_n} C_{n-1}(\mathrm{BAA}_n, \mathrm{BAA}'_n) \to \mathrm{St}_n(\mathbb{Q}) \to 0$$

is exact. Note that for $p$ odd $\Gamma_n(p)$ is torsion-free. Thus Lemma 8.7 implies that the groups $C_k(\mathrm{BAA}_n, \mathrm{BAA}'_n)$ are projective $\mathbb{Z}[\Gamma_n(p)]$-modules and hence that this is a partial projective resolution of $\mathbb{Z}[\Gamma_n(p)]$-modules. Therefore, $H_1(\Gamma_n(p); \mathrm{St}_n(\mathbb{Q}))$ is the homology of the sequence

$$C_{n+1}(\mathrm{BAA}_n, \mathrm{BAA}'_n)_{\Gamma_n(p)} \to C_n(\mathrm{BAA}_n, \mathrm{BAA}'_n)_{\Gamma_n(p)} \to C_{n-1}(\mathrm{BAA}_n, \mathrm{BAA}'_n)_{\Gamma_n(p)}.$$

This sequence agrees with

$$C_{n+1}(\mathrm{BAA}_n /\Gamma_n(p), \mathrm{BAA}'_n /\Gamma_n(p)) \to C_n(\mathrm{BAA}_n /\Gamma_n(p), \mathrm{BAA}'_n /\Gamma_n(p)) \to$$
$$\to C_{n-1}(\mathrm{BAA}_n /\Gamma_n(p), \mathrm{BAA}'_n /\Gamma_n(p)).$$

Using Proposition 9.1, this is exactly

$$C_{n+1}(\mathrm{BAA}^{\pm}_n(\mathbb{F}_p)', \mathrm{BAA}^{\pm}_n(\mathbb{F}_p)') \to C_n(\mathrm{BAA}^{\pm}_n(\mathbb{F}_p)', \mathrm{BAA}^{\pm}_n(\mathbb{F}_p)')$$
$$\to C_{n-1}(\mathrm{BAA}^{\pm}_n(\mathbb{F}_p)', \mathrm{BAA}^{\pm}_n(\mathbb{F}_p)').$$

Thus, the homology of this sequence is $H_n(\mathrm{BAA}^{\pm}_n(\mathbb{F}_p), \mathrm{BAA}^{\pm}_n(\mathbb{F}_p)')$. □

**Proposition 9.3.** *For all $p$, the inclusion $\mathrm{BA}^{\pm}_n(\mathbb{F}_p) \to \mathrm{BAA}^{\pm}_n(\mathbb{F}_p)$ induces a surjective map on $\pi_d$, $d \leq n$.*

**Proof.** Fix $d \leq n$ and let $f\colon S^d \to \mathrm{BAA}^{\pm}_n(\mathbb{F}_p)$ be a map that is simplicial with respect to some simplicial structure on $S^d$. It suffices to show that $f$ is homotopic to a map $\tilde{f}\colon S^d \to \mathrm{BAA}^{\pm}_n(\mathbb{F}_p)$ that factors through the inclusion $\mathrm{BA}^{\pm}_n(\mathbb{F}_p) \hookrightarrow \mathrm{BAA}^{\pm}_n(\mathbb{F}_p)$.

This can be shown similarly to Procedure 1, which was used in Section 6 to show that $\mathrm{BAA}^m_n$ is highly-connected. We will follow this procedure very closely and keep the notation as similar as possible to make it easier to follow. We first define the "bad" simplices that we want to remove: A simplex $\sigma$ of $S^d$ is called $(b, c)$-*over-augmented*, $b, c \in \mathbb{N}_0$, if

- $f(\sigma)$ is a 3-additive, double-triple, or double-double simplex,
- every vertex of $f(\sigma)$ is contained in the additive core,
- $\sigma$ has exactly $b \geq 0$ vertices $x$ such that $f(x)$ is contained in the additive core of a 3-additive face of $f(\sigma)$,
- $\dim(\sigma) = c$.

We call a simplex *overly augmented* if it is $(b, c)$-over-augmented for some $b, c \geq 0$. We say that a $(b, c)$-over-augmented simplex $\sigma$ is *better* than a $(b', c')$-over-augmented simplex $\tau$ if $(b, c) < (b', c')$ lexicographically. If $f$ has no overly augmented simplices, then its image lies in $\mathrm{BA}_n^{\pm}(\mathbb{F}_p)$. To obtain such a map, we will successively replace $f$ with homotopic maps that have less such simplices that are maximally over-augmented.

Let $\sigma$ be a $(b, c)$-over-augmented simplex with $(b, c)$ as large as possible lexicographically. We want to remove $\sigma$ from $f$. To do so, we first define, just as in Procedure 1, a complex $K(\sigma)$ and then verify adapted versions of Claim 6.4, Claim 6.5 and Claim 6.6.

If $f(\sigma)$ is a double-triple or double-double simplex, it can be written as $\{v_0, v_1, \dots v_k\}$, where $\{\vec{v}_2, \dots, \vec{v}_k\}$ is a partial basis. As we assumed that every vertex of $f(\sigma)$ is contained in the additive core, we here have $k = 4$ for a double-triple and $k = 5$ for a double-double simplex. Define $K(\sigma) := \mathsf{Link}_{\mathrm{B}_n^{\pm}(\mathbb{F}_p)}(\{v_2, \dots, v_k\})$. If $f(\sigma)$ is 3-additive, we can write $f(\sigma) = \{\langle \vec{v}_1 + \vec{v}_2 + \vec{v}_3\rangle, v_1, v_2, v_3\}$, where $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is a partial basis. Let $J := \{\langle \vec{v}_1 + \vec{v}_2\rangle, \langle \vec{v}_1 + \vec{v}_3\rangle, \langle \vec{v}_2 + \vec{v}_3\rangle\}$. Note that all elements of $J$ are vertices of $\mathrm{BAA}_n^{\pm}(\mathbb{F}_p)$. We view $J$ as a 0-dimensional simplicial complex and define $K(\sigma) := \mathsf{Link}_{\mathrm{B}_n^{\pm}(\mathbb{F}_p)}(\{v_1, v_2, v_3\}) * J$.

As $f$ is simplicial and $\sigma$ is maximally over-augmented, we have $f(\mathsf{Link}_{S^d}(\sigma)) \subseteq \mathsf{Link}_{\mathrm{BAA}_n^{\pm}(\mathbb{F}_p)}(f(\sigma))$. So to prove the analogue of Claim 6.4, it suffices to see that $K(\sigma) = \mathsf{Link}_{\mathrm{BAA}_n^{\pm}(\mathbb{F}_p)}(f(\sigma))$. This can be checked easily just as in Procedure 1. In Proposition 5.5, we describe the links of simplices in $\mathrm{BAA}_n$ and an analogous statement is true for $\mathrm{BAA}_n^{\pm}(\mathbb{F}_p)$.

To see that $K(\sigma)$ is $\dim \mathsf{Link}_{S^d}(\sigma)$-connected, note that by a result of Miller–Patzt–Putman the complex $\mathsf{Link}_{\mathrm{B}_n^{\pm}(\mathbb{F}_p)}(\{v_0, \dots, v_l\})$ is $(n - l - 3)$-connected [18, Proposition 2.45]. Hence, $K(\sigma)$ is $(n - 5)$-connected if $f(\sigma)$ is a double-triple simplex, $(n - 6)$-connected if $f(\sigma)$ is a double-double simplex and $(n - 5 + 1) = (n - 4)$-connected if $f(\sigma)$ is 3-additive. We have $\dim \mathsf{Link}_{S^d}(\sigma) \leq n - \dim(f(\sigma)) - 1$. The claim follows because $f(\sigma)$ is a double-triple, double-double or 3-additive simplex with all vertices contained in the additive core and hence has dimension 4, 5 or 3, respectively.

Consequently, the restriction $f|_{\mathsf{Star}_{S^d}(\sigma)}$ is homotopic to a simplicial map $h \colon \partial\sigma * \mathsf{Cone}(\mathsf{Link}_{S^d}(\sigma)) \to f(\sigma) * K(\sigma)$ that agrees with $f$ on $\partial\sigma * \mathsf{Link}_{S^d}(\sigma)$ and such that $h(\mathsf{Cone}(\mathsf{Link}_{S^d}(\sigma))) \subseteq K(\sigma)$. We will now verify that $h$ has only simplices that are better than $\sigma$. This is very similar to the proof of Claim 6.6 in Procedure 1, so we will be slightly briefer here:

Every simplex in $\partial\sigma * \mathsf{Cone}(\mathsf{Link}_{S^d}(\sigma))$ is of the form $\tilde{\sigma} \cup \tau$, where $\emptyset \subsetneq \tilde{\sigma} \subset \sigma$ is a proper face of $\sigma$. It gets mapped to $h(\sigma') = f(\tilde{\sigma}) \cup g(\tau)$, where $g(\tau) \subseteq K(\sigma)$. Let $\sigma' = \tilde{\sigma} \cup \tau$ be a simplex in the domain of $h$ that is $(b', c')$-over-augmented. We need to show that $(b', c') < (b, c)$ lexicographically. If $f(\sigma)$ is a double-triple or double-double simplex, no vertex of $g(\tau) \subseteq K(\sigma)$ can be contained in the additive core of $h(\sigma')$. As $\sigma'$ is overly augmented, this implies that $b' \leq b$ and that $\tau$ is the empty simplex. Hence, $c' = \dim(\sigma') < \dim(\sigma) = c$ and we have $(b', c') < (b, c)$. Next assume that $f(\sigma)$ is 3-additive. In this case, $K(\sigma) := \mathsf{Link}_{\mathrm{B}_n^{\pm}(\mathbb{F}_p)}(\{v_1, v_2, v_3\}) * J$. As $\sigma'$ is overly augmented and no vertex of $\mathsf{Link}_{\mathrm{B}_n^{\pm}(\mathbb{F}_p)}(\{v_1, v_2, v_3\})$ can be contained in the additive core of $h(\sigma')$, all vertices of $\tau$ get mapped to $J$. This means that either $\tau$ is the empty simplex and $\sigma' = \tilde{\sigma} \subset \sigma$ or

$h(\sigma') = f(\tilde{\sigma}) \cup \{j\}$ for some $j \in J$ and $\tilde{\sigma} \subset \sigma$. In the first case, we have $(b', c') < (b, c)$ with the same argument as in the situation of double-triple or double-double simplices. In the second case, $f(\sigma) \cup \{j\}$ is a double-triple simplex that contains $h(\sigma')$ and has $f(\sigma)$ as its unique 3-additive face. Hence, $j$ cannot be contained in the additive core of a 3-additive face of $h(\sigma')$. It follows that $b' \leq \dim(\tilde{\sigma}) < \dim(\sigma) = b$. In particular, $(b', c') < (b, c)$.

We now replace $\mathsf{Star}(\sigma)$ with $\partial\sigma * \mathsf{Cone}(\mathsf{Link}_{S^d}(\sigma))$ and $f$ by the map $f'$ that agrees with $f$ outside $\mathsf{Star}(\sigma)$ and is equal to $h$ on $\partial\sigma * \mathsf{Cone}(\mathsf{Link}_{S^d}(\sigma))$. This removes $\sigma$ without introducing any other simplices that are $(b, c)$-over-augmented or worse. Iterating this procedure, we obtain a map that has no overly augmented simplices and hence factors through $\mathrm{BA}_n^{\pm}(\mathbb{F}_p)$. □

**Corollary 9.4.** *For $p = 3$ or $5$, the complex $\mathrm{BAA}_n^{\pm}(\mathbb{F}_p)$ is $(n-1)$-connected.*

**Proof.** By Proposition 9.3, there is a surjection $\pi_d(\mathrm{BA}_n^{\pm}(\mathbb{F}_p)) \twoheadrightarrow \pi_d(\mathrm{BAA}_n^{\pm}(\mathbb{F}_p))$ for $d \leq n$. The claim follows because by [18, Proposition 2.50], the complex $\mathrm{BA}_n^{\pm}(\mathbb{F}_p)$ is $(n-1)$-connected for $p = 3$ or 5. □

**Corollary 9.5.** *For $p = 3$ or $5$, there is a surjection $H_1(\Gamma_n(p); \mathrm{St}_n(\mathbb{Q})) \to H_{n-1}(\mathrm{BAA}_n^{\pm}(\mathbb{F}_p)')$.*

**Proof.** This follows from the long exact sequence of the pair $(\mathrm{BAA}_n^{\pm}(\mathbb{F}_p), \mathrm{BAA}_n^{\pm}(\mathbb{F}_p)')$, Proposition 9.2 and Corollary 9.4. □

**Proposition 9.6.** *For $p$ an odd prime, $H_2(\mathrm{BAA}_2^{\pm}(\mathbb{F}_p)) \cong \mathbb{Z}$.*

**Proof.** Note that the inclusion $\mathrm{BA}_2^{\pm}(\mathbb{F}_p) \to \mathrm{BAA}_2^{\pm}(\mathbb{F}_p)$ is an isomorphism. The claim follows from [18, Lemma 2.44] which identifies $\mathrm{BA}_2^{\pm}(\mathbb{F}_p)$ with the compactified modular curve of level $p$, a compact surface of genus $(p+2)(p-3)(p-5)/24$. □

**Remark 9.7.** Proposition 9.6 shows that $\mathrm{BAA}_n^{\pm}(\mathbb{F}_p)$ is not always $n$-connected. This may come as a surprise. This fact is not just due to our restriction that the determinant of bases be equal to $\pm 1$ as this condition is vacuous for $p = 3$. If the reader is interested in defining a complex $\mathrm{BAA}_n(\mathbb{F})$ to determine the relations among the relations in $\mathrm{St}_n(\mathbb{F})$ for $\mathbb{F}$ an arbitrary field, we expect that extra types of additive simplices will be needed. For example, simplices of the form $\{v_1, v_2, \langle a\vec{v}_1 + b\vec{v}_2\rangle, \langle c\vec{v}_1 + d\vec{v}_2\rangle\}$ with $ad - bc \neq \pm 1$ might be needed in the $n = 2$ case.

We now describe a model for $\mathcal{T}_n(\mathbb{Q})/\Gamma_n(p)$.

**Definition 9.8.** A $\pm$-*orientation* on a rank $k$ submodule $V \subset \mathbb{F}_p^n$ is an equivalence class of generators of $\wedge^k V \cong \mathbb{F}_p$ up to sign.

Let $Gr^n_k(\mathbb{F}_p)^{\pm}$ denote the set of $\pm$-oriented summands of rank $k$ in $\mathbb{F}_p^n$. We let $\mathbb{T}_n^{\pm}(\mathbb{F}_p)$ denote the poset whose elements are all proper nonzero $\pm$-oriented summands of $\mathbb{F}_p^n$ with order induced by proper inclusion. Let $\mathcal{T}_n^{\pm}(\mathbb{F}_p)$ denote its geometric realisation.

Note that the $\pm$-orientations play no role in deciding if summands of different ranks are comparable and differently oriented subspaces of the same rank are never comparable. The following results are due to Miller–Patzt–Putman [18].

**Proposition 9.9** *([18, Proposition 3.16]). For $p$ an odd prime, the natural map $\mathcal{T}_n(\mathbb{Z})/\Gamma_n(p) \to \mathcal{T}_n^{\pm}(\mathbb{F}_p)$ is an isomorphism.*

**Proposition 9.10** *([18, Lemma 3.15]). For all $p$, the complex $\mathcal{T}_n^{\pm}(\mathbb{F}_p)$ is Cohen–Macaulay of dimension $n-2$.*

*9.2. Lower bounds on the codimension-1 cohomology of certain congruence subgroups*

In this subsection, we use the map-of-poset spectral sequence and the fact that $\mathrm{BAA}_2^{\pm}(\mathbb{F}_p)$ is not 2-connected to produce cohomology classes in the codimension-1 cohomology of level 3 and 5 congruence subgroups.

The following is a categorified version of Theorem C.

**Theorem 9.11.** *For $p = 3$ or $5$ and $n \geq 3$, $H^{\binom{n}{2}-1}(\Gamma_n(p))$ surjects onto*

$$\mathbb{Z}[Gr^n_2(\mathbb{F}_p)^{\pm}] \otimes \widetilde{H}_{n-4}(\mathcal{T}_{n-2}^{\pm}(\mathbb{F}_p)).$$

**Proof.** Since $\Gamma_n(p)$ is torsion-free for $p$ an odd prime, Borel–Serre duality holds with integral coefficients. In particular,

$$H^{\binom{n}{2}-1}(\Gamma_n(p)) \cong H_1(\Gamma_n(p); \mathrm{St}_n(\mathbb{Q})).$$

Thus, by Corollary 9.5, it suffices to produce a surjection

$$H_{n-1}(\mathrm{BAA}_n^{\pm}(\mathbb{F}_p)') \to \mathbb{Z}[Gr^n_2(\mathbb{F}_p)^{\pm}] \otimes \widetilde{H}_{n-4}(\mathcal{T}_{n-2}^{\pm}(\mathbb{F}_p)).$$

Let $\phi\colon \mathbb{P}(\mathrm{BAA}_n^{\pm}(\mathbb{F}_p)') \to \mathbb{T}_n^{\pm}(\mathbb{F}_p)$ be the map sending a simplex $\sigma = \{v_0, \ldots, v_k\}$ to $\langle \vec{v}_0, \ldots, \vec{v}_k\rangle_{\mathbb{F}_p}$ with the orientation given by $\vec{v}_{i_0} \wedge \vec{v}_{i_1} \wedge \ldots \wedge \vec{v}_{i_m}$, where $\{\vec{v}_{i_0}, \vec{v}_{i_1}, \ldots, \vec{v}_{i_m}\}$ is any maximal partial basis contained in $\sigma$. Observe that this orientation does not depend on the choice of the maximal partial basis in $\sigma$: For example, if $\sigma = \{\vec{v}_1 + \vec{v}_2, \vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\}$ is 2-additive, then

$$\vec{v}_1 \wedge \vec{v}_2 \wedge \ldots \wedge \vec{v}_k = (\vec{v}_1 + \vec{v}_2) \wedge \vec{v}_2 \wedge \vec{v}_3 \wedge \ldots \wedge \vec{v}_k = (\vec{v}_1 + \vec{v}_2) \wedge \vec{v}_1 \wedge \vec{v}_3 \wedge \ldots \wedge \vec{v}_k.$$

Reordering the vectors of the partial basis introduces a sign but does not change the equivalence class of the orientation.

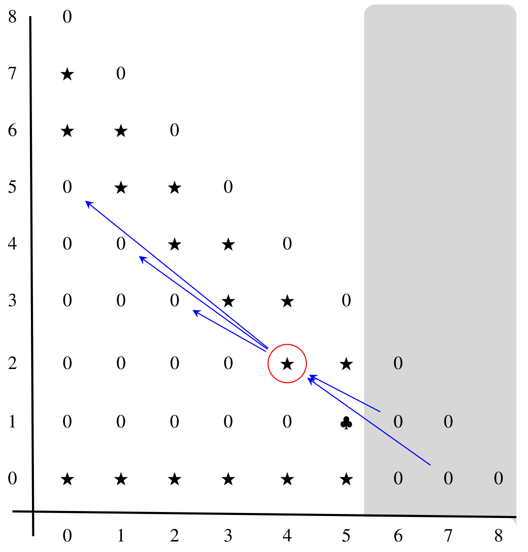

**Fig. 13.** The page $E^2_{a,b}$ when $n = 7$. The domains of all subsequent differentials into $E^2_{n-3,2}$ are 0, as are the codomains of all subsequent differentials out of $E^2_{n-3,2}$. Thus $E^2_{n-3,2} \cong E^\infty_{n-3,2}$.

Let $E^r_{a,b}$ denote the associated map-of-poset spectral sequence associated to $\phi$ described in Theorem 7.6. For $V$ a proper nonzero rank $k$ $\pm$-summand of $\mathbb{F}_p^n$, note that $\mathrm{ht}(V) = k - 1$, $\phi^{\leq V} \cong \mathbb{P}(\mathrm{BAA}_k^\pm(\mathbb{F}_p))$, and $\mathbb{T}_n^\pm(\mathbb{F}_p)_{>V} \cong \mathbb{T}^\pm_{n-k}(\mathbb{F}_p)$. Applying Proposition 7.8 with $e = n-2$, $d = 2$, and $r = 1$, we find that for $b \geq 1$, $E^2_{a,b} \cong 0$ unless $a + b = n - 1$ or $a + b = n$. See Fig. 13.

Since $\mathcal{T}_n^\pm(\mathbb{F}_p^n)$ is $(n-2)$-dimensional, $E^2_{a,b} \cong 0$ for $a > n-2$. This region is shaded grey in Fig. 13. Thus $E^2_{n-3,2} \cong E^\infty_{n-3,2}$ as all higher differentials to or from $E^*_{n-3,2}$ vanish, as in Fig. 13.

Observe that the group $E^\infty_{n-2,1}$ (marked by ♣ in Fig. 13) must vanish, since $H_1(\mathrm{BAA}_1^\pm(\mathbb{F}_p)) = H_1(\mathrm{BA}_1^\pm(\mathbb{F}_p)) \cong 0$. Thus the abutment of the spectral sequence surjects onto $E^\infty_{n-3,2}$. All that remains is to identify $E^2_{n-3,2}$ with $[\mathbb{Z}[Gr_2^n(\mathbb{F}_p)^\pm] \otimes \widetilde{H}_{n-4}(\mathcal{T}^\pm_{n-2}(\mathbb{F}_p))$. We will apply Lemma 7.7. Observe that the functor $V \mapsto H_2(\phi^{\leq V})$ is supported on vector spaces $V$ of dimension 2, equivalently, of height 1 in the poset $\mathbb{T}_n^\pm(\mathbb{F}_p)$. By Lemma 7.7,

$$E^2_{n-3,2} \cong H_{n-3}(\mathbb{T}_n^\pm(\mathbb{F}_p); H_2(\phi)) \cong \bigoplus_{\mathrm{ht}(V)=1} \widetilde{H}_{n-4}(\mathbb{T}^\pm_{n-2}(\mathbb{F}_p); H_2(\mathrm{BAA}_2^\pm(\mathbb{F}_p)).$$

The set of height-1 elements of $\mathbb{T}_n^\pm(\mathbb{F}_p)$ is isomorphic to $Gr_2^n(\mathbb{F}_p)^\pm$, and $H_2(\mathrm{BAA}_2^\pm(\mathbb{F}_p)) \cong \mathbb{Z}$ by Proposition 9.6. Thus $E^2_{n-3,2} \cong \mathbb{Z}[Gr_2^n(\mathbb{F}_p)^\pm] \otimes \widetilde{H}_{n-4}(\mathcal{T}^\pm_{n-2}(\mathbb{F}_p))$ and the result follows. □

We now prove Theorem C which gives a numerical lower bound for $H^{\binom{n}{2}-1}(\Gamma_n(p)); \mathbb{Q})$.

**Proof of Theorem C.** We must show that if $p = 3$ or $5$, then

$$\dim_{\mathbb{Q}} H^{\binom{n}{2}-1}(\Gamma_n(p); \mathbb{Q}) \geq p^{\binom{n-2}{2}} |Gr_2^n(\mathbb{F}_p)| \left(\frac{p-1}{2}\right)^{n-2}.$$

When $n = 2$, both sides of the inequality are equal to 1. Assume $n \geq 3$. By Theorem 9.11,

$$\dim_{\mathbb{Q}} H^{\binom{n}{2}-1}(\Gamma_n(p); \mathbb{Q}) \geq |Gr_2^n(\mathbb{F}_p)^{\pm}| \dim_{\mathbb{Q}} \widetilde{H}_{n-4}(\mathcal{T}_{n-2}^{\pm}(\mathbb{F}_p); \mathbb{Q}).$$

Observe that the order of $Gr_2^n(\mathbb{F}_p)^{\pm}$ is $\frac{p-1}{2}$ times the order of $Gr_2^n(\mathbb{F}_p)$. Furthermore, [18, Page 5] contains a proof that $\widetilde{H}_{n-4}(\mathcal{T}_{n-2}^{\pm}(\mathbb{F}_p))$ is a free abelian group of rank at least $p^{\binom{n-2}{2}} \left(\frac{p-1}{2}\right)^{n-3}$. We deduce Theorem C. □

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