A NEW GRAM DETERMINANT FROM THE MÖBIUS BAND

DIONNE IBARRA AND GABRIEL MONTOYA-VEGA

ABSTRACT. Gram determinants earned traction among knot theorists after E. Witten's presumption about the existence of a 3-manifold invariant connected to the Jones polynomial. Triggered by the creation of such an invariant by N. Reshetikhin and V. Turaev, several mathematicians have explored this line of research ever since. Gram determinants came into play by W. B. Raymond Lickorish's skein theoretic approach to the invariant. The construction of different bilinear forms is possible through changes in the ambient surface of the Kauffman bracket skein module. Hence, different types of Gram determinants have arisen in knot theory throughout the years; some of these determinants are discussed here. In this article, we introduce a new version of such a determinant from the Möbius band and prove some important results about its structure. In particular, we explore its connection to the annulus case and factors of its closed formula.

CONTENTS

ACKNOWLEDGMENTS

1. INTRODUCTION

Gram determinants are named after the Danish mathematician Jørgen Pedersen Gram and they appear in several areas of mathematics including Riemannian geometry, the finite element method, and machine learning. In knot theory, Gram determinants became of interest following Edward Witten's contemplation of a 3-manifold invariant connected to the Jones polynomial [Wit]. In 1991, a construction of such an invariant was presented by Nicolai Reshetikhin and Vladimir Turaev [RT]. Shortly afterwards, W. B. Raymond Lickorish announced a simpler approach to the construction of this invariant; this is considered to be the first modern work on Gram determinants in relation to the mathematical theory of knots [Lic1]. The Gram determinant constructed by Lickorish is known as the Gram determinant of type A and has been extensively studied; see for instance [KS, DiF, Cai, BIMP1]. It is important to remark that in knot theory several matrices arise with a connection to Gram determinants. For instance, the Alexander matrix

of a link L, introduced in 1919 by Alexander to his mentor Veblen, that is used to construct the Alexander polynomial $\Delta_L(t)$, for which $t = -1$ yields the determinant of the link. The Goeritz matrix is also based on the link diagram and yields the determinant of the link; see Chapters 2 and 4 of [PBIMW] for an extensive discussion on the matter and the relation of these two matrices.

By changing the ambient surface of the Kauffman bracket skein module, different bilinear forms can be constructed. Rodica Simion investigated bilinear forms of type B while working on chromatic joins [Sim, Sch]. Józef H. Przytycki and Qi Chen carried on her work in [CP1, CP2]. Paul Martin and Hubert Saleur were the first to consider Gram determinants of type B on their work which, among others, enjoys applications to statistical mechanics [MS1, MS2]. In 2008, besides exploring the type B determinant, Przytycki created the notion of Gram determinant of type Mb which results from defining a bilinear form on the Möbius band [Prz2]. Although a conjecture was presented by Qi Chen during the same year, this line of research was only rigorously pursued about ten years later; see for example [BIMP1, BIMP2, PBIMW] where formal proofs are presented.

The article is organized as follows. In Section 2 we present the definition of the relative Kauffman bracket skein module. Moreover, the Gram determinants of type B and type Mb are defined. We introduce the Gram determinant of type $(Mb)_1$, which arises as a modification of the type Mb, in Section 3. There we present the lollipop method, an innovative tool that helps to prove some important results about the structure of this new Gram determinant. Lastly, an adaptation of Chen's conjecture to this determinant is proposed in Section 4.

2. Gram determinants in knot theory

In order to study Gram determinants in knot theory, we need basis elements of a free module over a commutative ring with unity to be described for a manifold with framed points on its boundary. That is, we need the notion of a relative skein module. Arguably, the most extensively investigated skein module is the Kauffman bracket skein module. In particular, its structure has shown connections between the module and the geometry and topology of the 3-manifold [Prz1]. In this paper, the relative Kauffman bracket skein module plays an important role and is given in Definition 2.1.

Definition 2.1. *Let M be an oriented 3-manifold and $\{x_i\}_1^{2n}$ be the set of $2n$ framed points on ∂M. Let $I = [-1, 1]$, and let $\mathcal{L}^{fr}(2n)$ be the set of all relative framed links (which consists of all framed links in M and all framed arcs, $I \times I$, where $I \times \partial I$ is connected to framed points on the boundary of M) up to ambient isotopy while keeping the boundary fixed in such a way that $L \cap \partial M = \{x_i\}_1^{2n}$. Let R be a commutative ring with unity, $A \in R$ be invertible, and let $S_{2,\infty}^{sub}(2n)$ be the submodule of $R\mathcal{L}^{fr}(2n)$ that is generated by the Kauffman bracket skein relations:*

(i) $L_+ - AL_0 - A^{-1}L_\infty$, and
(ii) $L \sqcup \bigcirc + (A^2 + A^{-2})L$,

where $\bigcirc$ denotes the framed unknot and the skein triple (L_+, L_0, L_∞) denotes the three framed links in M that are identical except in a small 3-ball in M where the difference is shown in Figure 2.1. Then, the **relative Kauffman bracket skein module** *(RKBSM) of M is the quotient:*

$$\mathcal{S}_{2,\infty}(M, \{x_i\}_1^{2n}; R, A) = R\mathcal{L}^{fr}(2n)/S_{2,\infty}^{sub}(2n).$$

Theorem 2.2. *[Prz1] Let F be a surface with $\partial F \neq \varnothing$. If F is orientable let $M = F \times I$, otherwise let $M = F \hat{\times} I$. Let all the points in $\{x_i\}_1^{2n}$ be marked framed points that lie on $\partial F \times \{0\}$. Then $\mathcal{S}_{2,\infty}(M, \{x_i\}_1^{2n}; R, A)$ is a free R-module whose basis is composed of relative links in F without trivial components. When $n = 0$, the empty link is also a generator.*

The following corollary to Theorem 2.2 uses the language of relative skein modules that are used in the definition of Gram determinants in knot theory.
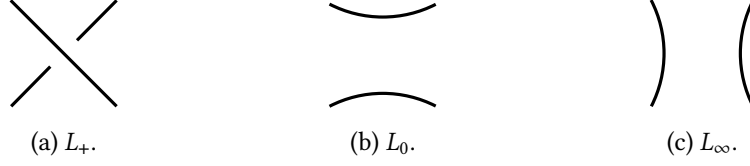
(a) L_+. (b) L_0. (c) L_∞.

FIGURE 2.1. The skein triple.

Corollary 2.3. [Prz1]

(a) $\mathcal{S}_{2,\infty}(D^2 \times I, n) \coloneqq \mathcal{S}_{2,\infty}(D^2 \times I, \{x_i\}_1^{2n}; R, A)$ *is a free R-module with $C_n = \frac{1}{n+1}\binom{2n}{n}$ basis elements.*

(b) $\mathcal{S}_{2,\infty}(Ann \times I, n) \coloneqq \mathcal{S}_{2,\infty}(Ann \times I, \{x_i\}_1^{2n}; R, A)$, *where $\{x_i\}_1^{2n}$ are located in the outer boundary component of the annulus, is a free $R[z]$-module with $D_n = \binom{2n}{n}$ basis elements, where z denotes the homotopically nontrivial curve in the annulus and $d = -A^2 - A^{-2}$ denotes the homotopically trivial curve in the annulus. The basis, denoted by $B_{n,0}$, is the set of all crossingless connections in the annulus with no trivial components or boundary parallel curves.*

(c) $\mathcal{S}_{2,\infty}(Ann_{n,1} \times I) \coloneqq \mathcal{S}_{2,\infty}(Ann \times I, \{y_i^{out}\}_1^{2n} \cup \{y_1^{inn}, y_2^{inn}\}; R, A)$, *where $\{y_i^{out}\}_1^{2n}$ are located in the outer boundary component of the annulus and $\{y_1^{inn}, y_2^{inn}\}$ are located in the inner boundary component of the annulus, is a free R-module. The standard basis is infinite. It contains elements of the form az^i for $i \geq 0$, where z denotes the boundary parallel curve of the annulus and $a \in \mathcal{A}$. The set $\mathcal{A}$ is a finite collection of crossingless connections with no trivial components or boundary parallel curves where outer boundary component of the annulus and y_1^{inn} and y_2^{inn} are connected to each other by a relative link. There are $2\binom{2n}{n}$ such elements in the set. Let X be a finite collection of crossingless connections between $2n-2$ points on the outer boundary such that there exists a path between the two points on the inner boundary and the remaining two points on the outer boundary (i.e. the crossingless connections do not isolate the remaining two points on the outer boundary from a path to the inner boundary points). The rest of the standard basis is from an infinite family of crossingless connections obtained from X, where the remaining outer boundary points are connected to the inner boundary points after wrapping around the inner boundary by πk for $k \in \mathbb{Z}$; an illustration of such connections can be found in Figure 3.1 and Example 2.4.*

(d) $\mathcal{S}_{2,\infty}(Mb \,\hat{\times}\, I, 2n) \coloneqq \mathcal{S}_{2,\infty}(Mb \,\hat{\times}\, I, \{x_i\}_1^{2n}; R, A)$ *is a free R-module. The standard basis contains an infinite number of elements of the form bz^i, bxz^i for $i \geq 0$, where x denotes the simple closed curve that intersects the crosscap once, z denotes the boundary parallel curve of the Möbius band, and b is an element in the set of crossingless connections in the Möbius band with no trivial components or boundary parallel curves for which the arcs do not intersect the crosscap. The rest of the elements in the standard basis are from a finite number of crossingless connections consisting of a collection of $n - k$ arcs for $0 \leq k < n$ that non-trivially intersect the crosscap. Among the finite collection there are $\binom{2n}{k}$ crossingless connections that intersect the crosscap $n - k$ times.*

Example 2.4. *An illustrative sample of the standard basis of $\mathcal{S}_{2,\infty}(Ann_{2,1} \times I)$, as described in Corollary 2.3(c), is given below.*

(1) *Crossingless connections with no trivial components or boundary parallel curves where the outer boundary component of the annulus and y_1^{inn} and y_2^{inn} are connected to each other by a relative link.*

We denote this set by A;

$$A = \left\{ \vphantom{\Bigg|} \right.$$
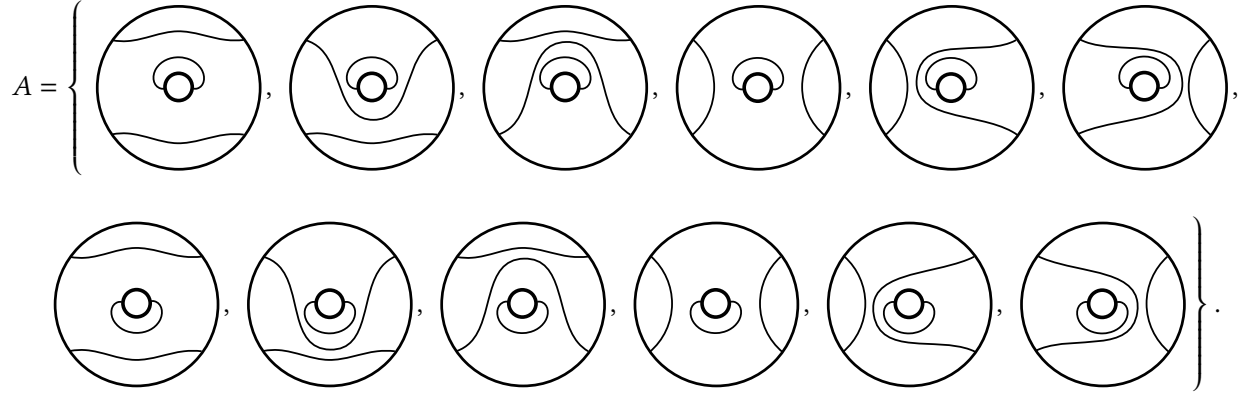
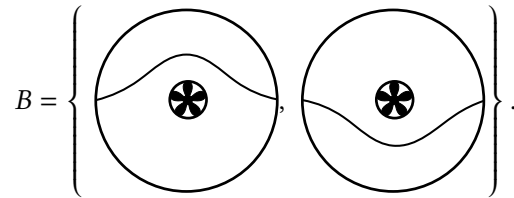*(2) Crossingless connections of the form az^i, for $i \geq 1$, where $a \in A$;*
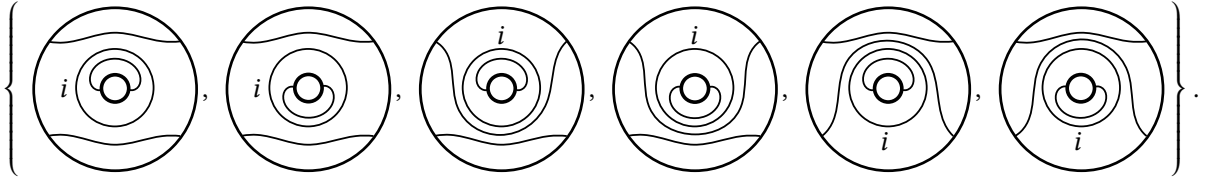
(3) Crossingless connections between two points on the outer boundary that do not isolate the remaining two points on the outer boundary from a path to the inner boundary points where the remaining outer boundary points are connected to the inner boundary points after wrapping around the inner boundary by πk for $k \in \mathbb{Z}$.
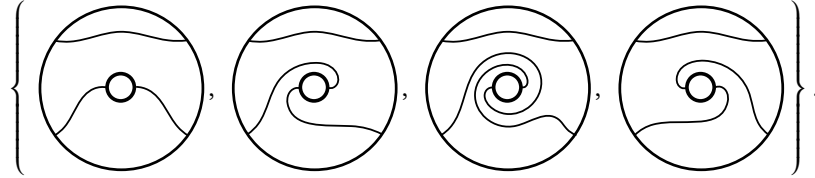

Example 2.5. *Consider the crosscap model of the Möbius band where ✱ denotes a crosscap. By Corollary 2.3(d) the standard basis of $\mathcal{S}_{2,\infty}(Mb \hat{\times} I, 2)$ consists of the following elements:*
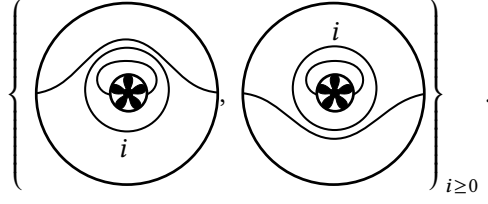
(1) Crossingless connections in the Möbius band with no trivial components or boundary parallel curves for which the arcs do not intersect the crosscap. We will denote the set by B;
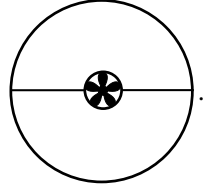
$$B = \left\{ \vphantom{\Bigg|} \right.$$

(2) Crossingless connections of the form bz^i, for $i \geq 1$, where $b \in B$;

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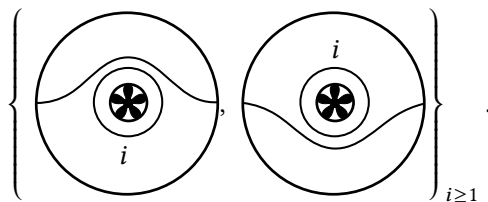
(3) *Crossingless connections of the form* bxz^i *for* $i \geq 0$, *where* $b \in B$;

(4) *Crossingless connections consisting of a collection of* $1 - k$ *arcs for* $0 \leq k < 1$ *that non-trivially intersects the crosscap;*

2.1. **The Gram determinant of type** B. The origins of the Gram determinant of type B can be found in [MS1]. The creation of this type of Gram determinants comes from the blob algebra (see [MS2]) that is associated to the transfer matrix formulation of statistical mechanics on arbitrary lattices. The knot theoretic interest originated from the work of R. Simion [Sim] on chromatic joins when Q. Chen and J. H. Przytycki in [CP1] showed a connection to the Gram matrix of the Temperley-Lieb algebra and the matrix of chromatic joins. Furthermore, Q. Chen and J. H. Przytycki in [CP2, Che] used skein modules, the Jones-Wenzl idempotents, and Chebyshev polynomials to prove a closed formula for type B. The definition we give will use the language of skein modules.

Definition 2.6. *Let* $B_{n,0} = \{b_1, b_2, \ldots, b_{\binom{2n}{n}}\}$ *be the set of all diagrams of crossingless connections between* $2n$ *marked points on the outer boundary of* $Ann \times \{0\}$ *in* $Ann \times I$. *Define the bilinear form* $\langle \ , \ \rangle_B$ *for type* B *as follows:*

$$\langle \ , \ \rangle_B : \mathcal{S}_{2,\infty}(Ann \times I, \{x_i\}_1^{2n}; R, A) \times \mathcal{S}_{2,\infty}(Ann \times I, \{x_i\}_1^{2n}; R, A) \longrightarrow R[d, z].$$

Given $b_i, b_j \in B_{n,0}$, *we glue* b_i *with the inversion of* b_j *along the marked outer boundary, respecting the labels of the marked points. The result is an element in* $Ann \times I$ *containing only disjoint simple closed curves which are either homotopically non-trivial (denoted by* z*), or null homotopic (denoted by* d*). Then,* $\langle b_i, b_j \rangle_B := z^k d^m$, *where* k *and* m *denote the number of these curves, respectively.*

The Gram matrix of type B *is defined as* $G_n^B = (\langle b_i, b_j \rangle_B)_{1 \leq i,j \leq \binom{2n}{n}}$, *and its determinant* D_n^B *is called the Gram determinant of type* B.

Example 2.7. *An example of the bilinear form on two elements in* $B_{5,0}$ *is given below.*
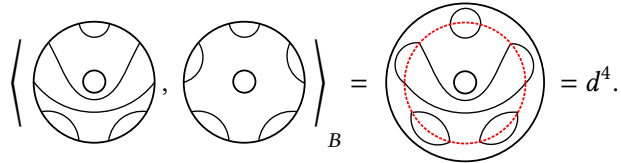
Theorem 2.8. [CP2][MS1] *Let* $R = \mathbb{Z}[A^{\pm 1}]$. *Then*

$$D_n^B = \prod_{i=1}^{n} (T_i(d)^2 - z^2)^{\binom{2n}{n-i}},$$

where $T_i(d)$ *is the Chebyshev polynomial of the first kind recursively defined by* $T_{n+1}(d) = dT_n(d) - T_{n-1}(d)$, *with initial conditions* $T_0(d) = 2$ *and* $T_1(d) = d$, *where* $d = -A^2 - A^{-2}$.

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Q. Chen and J. H. Przytycki's proof of a closed formula for type B involves the creation of a linear map on the basis $B_{n,0}$ that uses the lollipop method to decorate the inner boundary component with the Jones-Wenzl idempotent, then proving that the image of the linear map under the basis is a subspace of dimension $\binom{2n}{n} - \binom{2n}{n-k}$.

2.2. **The Gram determinant of type Mb.** Józef H. Przytycki constructed the notion of the Gram determinant of type Mb in 2008. This originates from the study of crossingless connections on a Möbius band. Here the bilinear form is defined through the identification of two Möbius bands along their boundaries. This determinant is given in Definition 2.9.

Definition 2.9. *Let $Mb_n = \{m_1, \ldots, m_{\sum_{k=0}^n \binom{2n}{k}}\}$ be the set of all diagrams of crossingless connections between $2n$ marked points on the boundary of the Möbius band $Mb \hat{\times} \{0\}$ in $Mb \hat{\times} I$. Define a bilinear form $\langle\,,\,\rangle_{Mb}$ on the elements of Mb_n as follows:*

$$\langle\,,\,\rangle_{Mb} : \mathcal{S}_{2,\infty}(Mb \hat{\times} I, \{x_i\}_1^{2n}) \times \mathcal{S}_{2,\infty}(Mb \hat{\times} I, \{x_i\}_1^{2n}) \longrightarrow \mathbb{Z}[d, w, x, y, z].$$

Given $m_i, m_j \in Mb_n$, identify the boundary component of m_i with that of the inversion of m_j, respecting the labels of the marked points. The result is an element in $Kb \hat{\times} I$ containing only disjoint simple closed curves. The five homotopically distinct simple closed curves in the Klein bottle, including the homotopically trivial curve, are denoted by x, y, z, w, d as illustrated in Figure 2.2. Then, $\langle m_i, m_j\rangle_{Mb} := d^m x^n y^k z^l w^h$ where m, n, k, l and h denote the number of these curves, respectively.

The Gram matrix of type Mb is defined as $G_n^{Mb} = (\langle m_i, m_j\rangle_{Mb})_{1 \le i,j \le \sum_{k=0}^n \binom{2n}{k}}$ and its determinant D_n^{Mb} is called the Gram determinant of type Mb.
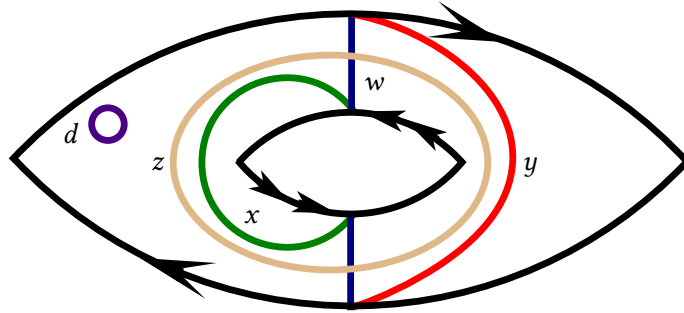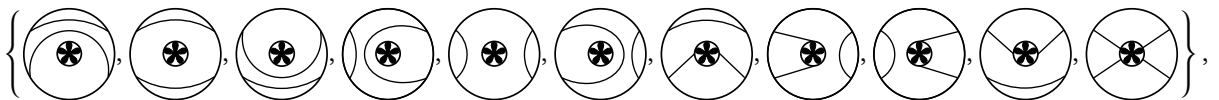

FIGURE 2.2. Klein bottle and its five homotopically distinct simple closed curves.

The following two examples illustrate this definition. In particular, Example 2.10 shows the basis Mb_2, the Gram matrix, and calculates the Gram determinant G_2^{Mb}.

Example 2.10. *For $n = 2$, the set Mb_n is given by*

and the Gram matrix is given in Table 1.

$\langle\,,\,\rangle$											
	d^2	dz	z^2	z	d	z	dy	y	yz	y	z
	dz	d^2	dz	d	z	d	dy	y	dy	y	d
	z^2	dz	d^2	z	d	z	yz	y	dy	y	z
	z	d	z	d^2	dz	z^2	y	yz	y	dy	z
	d	z	d	dz	d^2	dz	y	dy	y	dy	d
	z	d	z	z^2	dz	d^2	y	dy	y	yz	z
	dx	dx	xz	x	x	x	dw	w	xy	w	x
	x	x	x	xz	dx	dx	w	dw	w	xy	x
	xz	dx	dx	x	x	x	xy	w	dw	w	x
	x	x	x	dx	dx	xz	w	xy	w	dw	x
	z	d	z	z	d	z	y	y	y	y	w^2

TABLE 1. The Gram matrix G_2^{Mb}.

The determinant of the Gram matrix is:

$$
\begin{aligned}
D_2^{Mb} &= (d-z)^4[(d+z)w-2xy]^4(d^2(d^2-4))(d^2-2+z) \\
&\quad [(d^2-2-z)(w^2-2)-2(2-z)] \\
&= (T_1(d)-z)^4[(T_1(d)+z)T_1(w)-2xy]^4(T_4(d)-2)(T_2(d)+z) \\
&\quad [(T_2(d)-z)T_2(w)-2(2-z)].
\end{aligned}
$$

Example 2.11. *An example of the bilinear form on two elements in Mb_4 is given below, where* ✹ *denotes a crosscap usually denoted by* .

Q. Chen conjectured the following result for the Gram determinant of type *Mb*. Some work supporting this conjecture can be found in [BIMP1, BIMP2, PBIMW].

Conjecture 2.12 (Chen)**.**

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Let $R = \mathbb{Z}[A^{\pm 1}, w, x, y, z]$. Then the Gram determinant of type Mb for $n \geq 1$, denoted by D_n^{Mb}, is:

$$
\begin{aligned}
D_n^{Mb}(d, w, x, y, z) \ = \ & \prod_{k=1}^{n} (T_k(d) + (-1)^k z)^{\binom{2n}{n-k}} \\
& \prod_{\substack{k=1 \\ k \ odd}}^{n} \left((T_k(d) - (-1)^k z) T_k(w) - 2xy \right)^{\binom{2n}{n-k}} \\
& \prod_{\substack{k=1 \\ k \ even}}^{n} \left((T_k(d) - (-1)^k z) T_k(w) - 2(2-z) \right)^{\binom{2n}{n-k}} \\
& \prod_{i=1}^{n} D_{n,i},
\end{aligned}
$$

where $D_{n,i} = \prod_{k=1+i}^{n} (T_{2k}(d) - 2)^{\binom{2n}{n-k}}$, and i represents the number of curves passing through the crosscap.

Proposition 2.13. [BIMP2] D_n^{Mb} is divisible by $(d - z)^{\binom{2n}{n-1}}$.

Proposition 2.14. [BIMP1] D_n^{Mb} is divisible by $(w(d + z) - 2xy)^{\binom{2n}{n-1}}$.

Proposition 2.15. [PBIMW] D_n^{Mb} is divisible by $((d^2 - 2 - z)(w^2 - 2) - 2(2 - z))^{\binom{2n}{n-2}}$.

3. The Gram determinant of type $(Mb)_1$

The basis of the relative Kauffman bracket skein module of the twisted I-bundle of the Möbius band, as described in Corollary 2.3, is infinite. If we restrict to only basis elements with no z and x curves, then we obtain a finite sub-collection of this basis. The Gram matrix of type Mb was created by this finite sub-collection. However, this construction carries a few disadvantages. First, the number of elements increases exponentially as n increases. In particular, computing the determinant for $n \geq 5$ has not been achieved due to the size of the matrices. Second, the lollipop technique of decorating the crosscap with the Jones-Wenzl idempotent is no longer as straightforward as it was in the case of type B. In fact, attempts to using this technique have not yet been rigorously successful. In this section we propose a new Gram determinant from a sub-collection of Mb_n, give direct connections to a new type B, and explain how the lollipop method can be applied to this case and where it falls short for the case of type Mb.

Definition 3.1. Let $(Mb_n)_1 = Mb_{n,0} \cup Mb_{n,1}$ where $Mb_{n,0} = \{m_1, \ldots, m_{\binom{2n}{n}}\}$ is the set of all diagrams of crossingless connections between $2n$ marked points on the boundary of $Mb \ \hat{\times} \ \{0\}$ whose arcs do not intersect the crosscap and $Mb_{n,1} = \{m_1, \ldots, m_{\binom{2n}{n-1}}\}$ is the set of all diagrams of crossingless connections between $2n$ marked points on the boundary of $Mb \ \hat{\times} \ \{0\}$ with exactly one curve intersecting the crosscap. Define a bilinear form $\langle \, , \, \rangle_{Mb}$ on the elements of $(Mb_n)_1$ by using the same bilinear form as type Mb, as follows:

$$
\langle \, , \, \rangle_{Mb} : \mathcal{S}_{2,\infty}(Mb \ \hat{\times} \ I, \{x_i\}_1^{2n}) \times \mathcal{S}_{2,\infty}(Mb \ \hat{\times} \ I, \{x_i\}_1^{2n}) \longrightarrow \mathbb{Z}[d, w, x, y, z].
$$

Given $m_i, m_j \in (Mb_n)_1$, identify the boundary component of m_i with that of the inversion of m_j, respecting the labels of the marked points. The result is an element in $Kb \ \hat{\times} \ I$ containing only disjoint simple closed curves. Then $\langle m_i, m_j \rangle_{Mb} := d^m x^n y^k z^l w^h$ where m, n, k, l and h denote the number of these curves, respectively.

The Gram matrix of type $(Mb)_1$ is defined as $G_n^{(Mb)_1} = (\langle m_i, m_j \rangle_{Mb})_{1 \leq i, j \leq \binom{2n}{n-1} + \binom{2n}{n}}$ and its determinant $D_n^{(Mb)_1}$ is called the Gram determinant of type $(Mb)_1$.

In Example 3.2 we show the smallest Gram matrix of type $(Mb)_1$ that differs from type Mb (compare Example 2.10).

Example 3.2. *For n = 2, the set $(Mb_n)_1$ is given by*
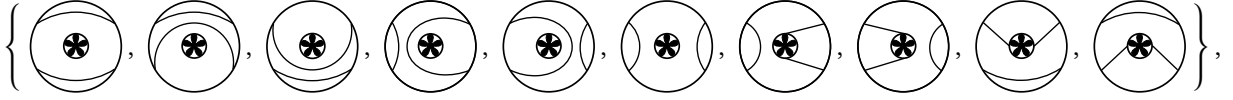
and the Gram matrix is shown in Table 2.

$\langle\ ,\ \rangle$										
	d^2	dz	z^2	z	d	z	dy	y	yz	y
	dz	d^2	dz	d	z	d	dy	y	dy	y
	z^2	dz	d^2	z	d	z	yz	y	dy	y
	z	d	z	d^2	dz	z^2	y	yz	y	dy
	d	z	d	dz	d^2	dz	y	dy	y	dy
	z	d	z	z^2	dz	d^2	y	dy	y	yz
	dx	dx	xz	x	x	x	dw	w	xy	w
	x	x	x	xz	dx	dx	w	dw	w	xy
	xz	dx	dx	x	x	x	xy	w	dw	w
	x	x	x	dx	dx	xz	w	xy	w	dw

TABLE 2. The Gram matrix $G_2^{(Mb)_1}$.

$$
\begin{aligned}
D_2^{(Mb)_1} &= (-2+d)d^2(2+d)(d-z)^4(-2+d^2-z)(-2+d^2+z)(-dw+2xy-wz)^4 \\
&= (d-z)^4((d^2-2)+z)((d+z)w-2xy)^4((d^2-2)-z)(d^2(d^2-4)+2-2) \\
&= (T_1(d)-z)^4(T_2(d)^2-z^2)((d+z)w-2xy)^4(T_4(d)-2).
\end{aligned}
$$

Observe that $D_2^{(Mb)_1}$ does not divide D_2^{Mb}. However, the following proposition is a direct result from a proof given in [BIMP1] for type Mb.

Proposition 3.3. [BIMP1] $D_n^{(Mb)_1}$ *is divisible by* $(w(d+z)-2xy)^{\binom{2n}{n-1}}$.

Consider an annulus with $2n$ points on the outer boundary and 2 points on the inner boundary. Fix a crossingless connection between $2n-2$ points on the outer boundary that do not isolate the remaining two points on the outer boundary from a path to the inner boundary points. Then there are an infinite number of ways to connect the remaining outer boundary points to the inner boundary points without introducing a crossing. For example, if we start by connecting two arcs, each on the outer boundary, then we may wrap the arcs around the annular boundary by πk where $k \in \mathbb{Z}$ before connecting to the inner boundary points. An illustration is given in Figure 3.1.

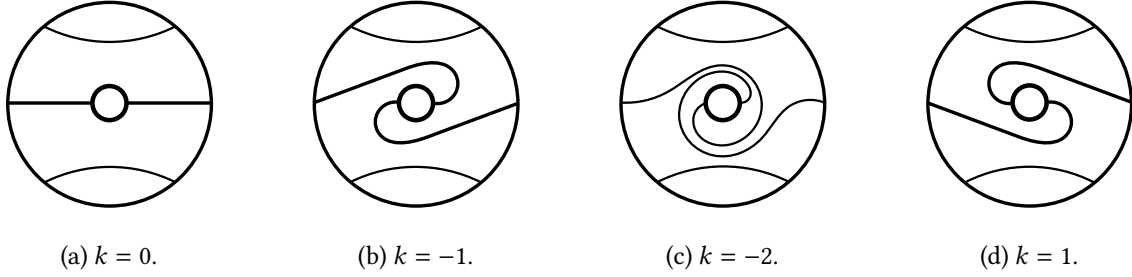(a) $k = 0$. (b) $k = -1$. (c) $k = -2$. (d) $k = 1$.

FIGURE 3.1. Examples of distinct crossingless connections from an infinite family where two arcs attached to the inner and outer boundary wrap around the annular boundary $k/2$ times for $k \in \mathbb{Z}$.

Fix a line segment connected between the 1st and $2n^{th}$ marked point of the outer boundary and between the two inner boundary points. Call this segment the **lollipop**. As before, consider a fixed crossingless connection between $2n-2$ points on the outer boundary that do not isolate the remaining four points. If the two arcs bounding the remaining four marked points are not allowed to intersect the lollipop, then there is only one way, up to isotopy, to connect the arcs from the outer boundary points to the inner boundary points, without introducing crossings.
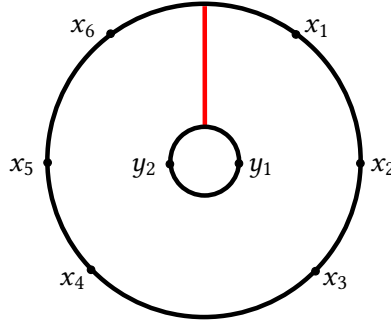

FIGURE 3.2. An illustration of a lollipop in an annulus with 6 marked points on the outer boundary and two marked points on the inner boundary.

The following lemma is a generalization of the children playing a game proof given in [Prz1] and explained in [PBIMW]. Also, one can find a detailed proof of the lemma in [Iba].

Lemma 3.4. [PBIMW, Prz1] *Consider an annulus with $2n$ marked points in the outer boundary, 2 marked points in the inner boundary, and suppose it contains a lollipop S. Let $B_{n,1} = \{b'_1, \cdots, b'_N\}$ be the set of crossingless connections, up to isotopy, with the following properties:*

(1) *There are $n - 1$ arcs connected to the marked points in the outer boundary.*
(2) *These $n-1$ arcs do not isolate the remaining two marked points in the outer boundary from the marked points in the inner boundary.*
(3) *The two remaining arcs are disjoint from the lollipop and*
(4) *each are connected to the inner and outer boundary components of the annulus, respectively.*

Then $N = \binom{2n}{n-1}$ and there is a one-to-one correspondence between $B_{n,1}$ and $(Mb_n)_1$.

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Example 3.5. *The set $B_{2,1}$ consists of 4 elements as illustrated below:*

$$B_{2,1} = \left\{ \quad , \quad , \quad , \quad \right\}.$$
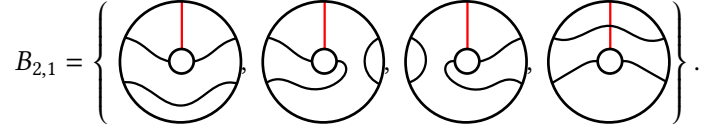
Definition 3.6. *Consider $\mathcal{S}_{2,\infty}(Ann \times I, \{x_i\}_1^{2n}) \oplus \mathcal{S}_{2,\infty}(Ann, \{y_i^{out}\}_1^{2n} \cup \{y_1^{inn}, y_2^{inn}\})$ and its submodule, $\mathcal{R}((B_n)_1) = \mathcal{R}(B_{n,0}) \oplus \mathcal{R}(B_{n,1})$[1], where $\mathcal{R}(B_{n,0})$ and $\mathcal{R}(B_{n,1})$ are submodules of $\mathcal{S}_{2,\infty}(Ann \times I, \{x_i\}_1^{2n})$ and $\mathcal{S}_{2,\infty}(Ann \times I, \{y_i^{out}\}_1^{2n} \cup \{y_1^{inn}, y_2^{inn}\})$, generated by $B_{n,0}$ and $B_{n,1}$, respectively. In particular, $(B_n)_1 = B_{n,0} \sqcup B_{n,1}$. Furthermore, let $\mathcal{R}((Mb_n)_1)$ be a submodule of $\mathcal{S}_{2,\infty}(Mb \,\hat{\times}\, I, \{x_i\}_1^{2n})$ generated by the elements of $(Mb_n)_1$. Define a linear map $\varphi : \mathcal{R}((Mb_n)_1) \to \mathcal{R}((B_n)_1)$ on the basis as follows:*

Recall $(Mb_n)_1 = Mb_{n,0} \cup Mb_{n,1}$,

(1) *If $m \in Mb_{n,0}$, then there exists a unique element in $B_{n,0}$, say $b_{n,0}$, obtained from m by replacing the crosscap with a table. In this case $\varphi(m) = b_{n,0} \in B_{n,0}$.*

(2) *If $m \in Mb_{n,1}$, then there exists a unique element in $B_{n,1}$, say $b_{n,1}$, obtained from matching the arcs whose boundary is disjoint from the inner boundary. In this case $\varphi(m) = b_{n,1} \in B_{n,1}$.*

The next lemma gives a direct connection between type $(Mb)_1$ and a new Gram determinant of type B constructed by the set $(B_n)_1$ and using the same bilinear form as type B. Type $(Mb)_1$ is a special case of it when the distinction of the two curves attached to the inner boundary (or outer boundary) of the annulus is ignored as shown in Figure 3.3.

Lemma 3.7. *The map φ is a bijection between the bases $(Mb_n)_1$ and $(B_n)_1$. Furthermore, the Gram determinant is preserved up to an appropriate labelling of the elements.*

Proof. By construction, φ is a bijection between $(Mb_n)_1$ and $(B_n)_1$. To show that φ preserves the Gram determinant it suffices to prove that the bilinear form is preserved. Since $\langle \varphi(m_i), \varphi(m_j) \rangle_B$ belongs to

$$\mathcal{S}_{2,\infty}(Ann \times I) \quad \oplus \quad \mathcal{S}_{2,\infty}(Ann \times I, \{y_1^{inn}, y_2^{inn}\})$$
$$\oplus \quad \mathcal{S}_{2,\infty}(Ann \times I, \{y_1^{out}, y_2^{out}\}) \oplus \mathcal{S}_{2,\infty}(Ann \times I, \{y_1^{inn}, y_2^{inn}\} \cup \{y_1^{out}, y_2^{out}\}),$$

then we only need to choose the following labelling illustrated in Figure 3.3. That is, for example, if one arc is attached to the inner and outer boundary then there exists a corresponding arc that also intersects the inner and outer boundary; we label the element with these pair of arcs by w. This corresponds to the element in $\mathcal{S}_{2,\infty}(Kb \,\hat{\times}\, I)$ that intersects the two crosscaps once, namely the w curve. Furthermore, if an arc is attached to the outer boundary then we label it y; this corresponds to the y curve in the Klein bottle that intersects the outer crosscap. If it is attached to only the inner boundary we label it x. In particular, the element with two arcs, one only attached to the inner boundary and one only attached to the outer boundary, is labeled xy.
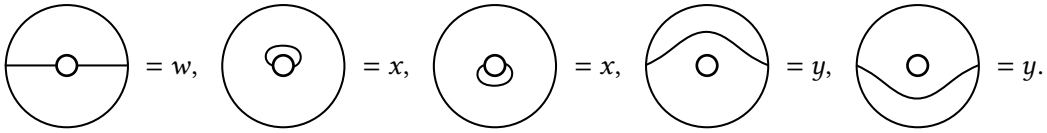

$= w, \qquad = x, \qquad = x, \qquad = y, \qquad = y.$

FIGURE 3.3. A labeling of the curves attached to the boundary of the annulus.

$\square$

Remark 3.8. *Even though the bijection φ can be extended to Mb_n it will no longer preserve the bilinear form. Indeed this can be seen in Example 2.11; when two arcs intersect the crosscap you might obtain a d or z for the Klein bottle while for the annulus case it would just be x^2.*

---

[1]Note that $\mathcal{R}((B_n)_1)$ is not a skein module.

3.1. **White markers.** We introduce elements whose closure remains unlinked from all simple closed curves or arcs in $Ann \times I$. These elements will be used to adapt Chen and Przytycki's proof in [CP2] to prove that a factor of the Gram determinant of type B divides the Gram determinant of type $(Mb)_1$.

Definition 3.9. *Let $I = [-1, 1]$, $M = Ann \times I$ with $2n$ framed marked points attached to the outer boundary of $Ann \times \{0\}$, and let M be decorated with a lollipop $S = S \times I$. Consider $\mathcal{L} \sqcup \{w_j\}_{j=1}^k \subset M$; a relative framed link $\mathcal{L}$ and a pair of k labelled white marked framed points $\{w_j\}_{j=1}^k$ disjoint from $\mathcal{L} \cup S$ and distinct from the $2n$ marked points, called **white markers**.*

White markers are either attached and fixed to the boundary of $(Ann - S) \times \{0\}$ of $M - S$, or attached to a second white marker in the interior of $M - S$ as illustrated in Figure 3.4.

(a) White Markers attached to the boundary.   (b) White Markers attached to each other.

FIGURE 3.4. Illustrations of white markers.

The labelled white markers are identified in M with framed arcs according to their labelling if, up to ambient isotopy, there exists a collection of k pairwise disjoint curves $\{\gamma_j\}_{j=1}^k$ such that for each j, γ_j is connected to the core of the framed points $pt_{1,j}$ and $pt_{2,j}$ of the pair of white markers $w_j = \{pt_{1,j}, pt_{2,j}\}$ under the following condition:

There exists a neighborhood of each arc γ_j, $U(\gamma_j) \subset Ann \times \{0\}$, such that $(\cup_{i=1}^k U(\gamma_j) \times I) \cap (\mathcal{L} \cup S) = \varnothing$.

If the condition is not satisfied then the white markers remain as white marked framed points with attaching information intact but no arcs attached. Furthermore, the sign of each labelling assigned to a white marker will change under an inversion operation.

Example 3.10.

(a) *Let $I = [-1, 1]$, $M = Ann \times I$ with $2n$ framed marked points attached to the outer boundary of $Ann \times \{0\}$, and let M be decorated with a lollipop $S = S \times I$. Suppose that $\mathcal{L}$ consists of only relative framed links with no crossings and suppose $\mathcal{L} \cup w_1 \subset M$ where the pair of white markers $w_1 = pt_{1,1} \cup pt_{2,1}$ are attached to each other in the interior of M. Then there exists a disk $D \subset Ann \times \{0\}$ such that $w_1 \subset D$ and $(D \times I) \cap (\mathcal{L} \cup S) = \varnothing$. In this case the pair of labelled white markers are identified by γ_1 in D to produce a simple closed curve. An illustration is given in Figure 3.5a.*

(b) *Now suppose $\mathcal{L} \cup \{w_1, w_2\} \subset M$ where each white marker from w_1 is attached to a white marker from w_2 in the interior of M. Then there exists a disk $D \subset Ann \times \{0\}$ such that $\{w_1 \cup w_2\} \subset D$ and $(D \times I) \cap (\mathcal{L} \cup S) = \varnothing$. In this case the pair of white markers are identified by arcs in D. An illustration is given in Figure 3.5b.*

So far this notion results in simple closed curves and the closure always exists. However, if we consider white markers attached to the boundary of M, as shown in Figure 3.6, we find that there exist relative framed links that obstruct the closure of the white markers. Furthermore, we find obstructions from the position of the white markers regardless of the relative link.
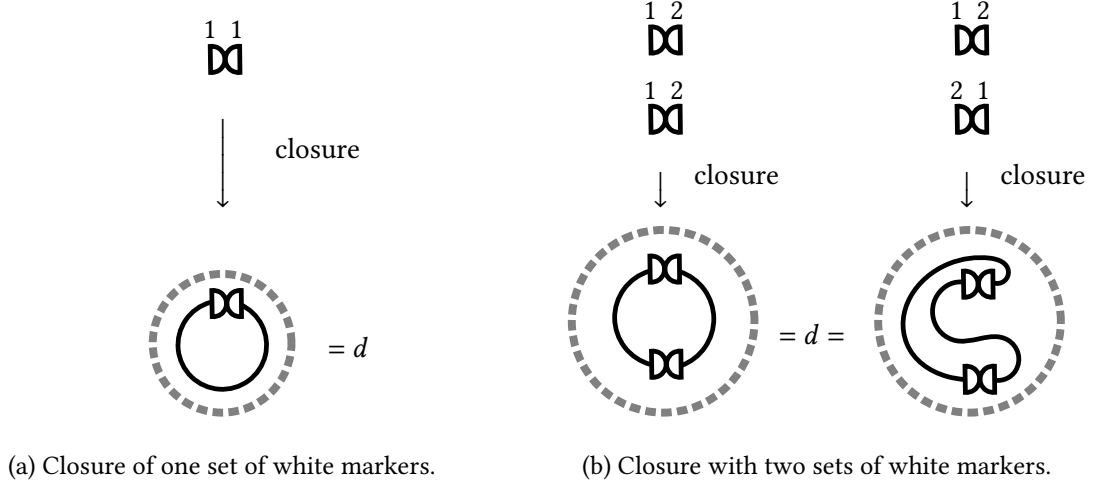
(a) Closure of one set of white markers.

(b) Closure with two sets of white markers.

FIGURE 3.5. A local illustration of the closure of arcs in $D \times \{0\} \subset Ann \times [-1, 1]$ obtained by identifying arcs attached to white markers with the same label.
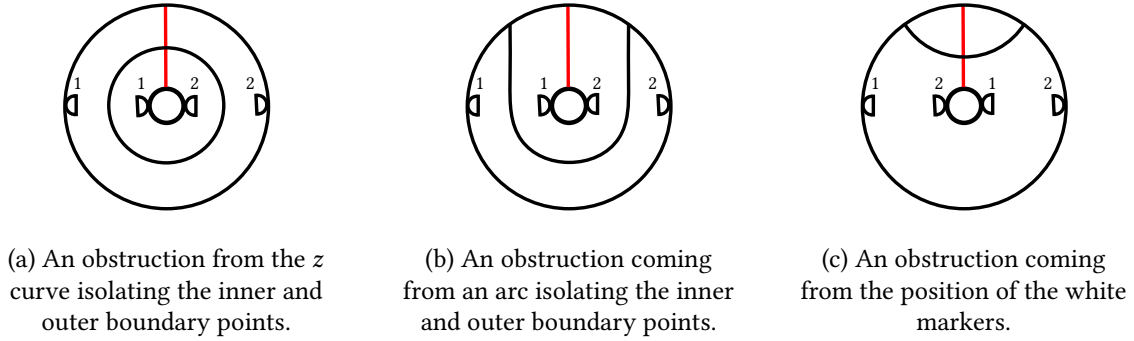

(a) An obstruction from the z curve isolating the inner and outer boundary points.

(b) An obstruction coming from an arc isolating the inner and outer boundary points.

(c) An obstruction coming from the position of the white markers.

FIGURE 3.6. Examples of obstructions to the closure of white markers.

Definition 3.11. *Let $I = [-1, 1]$, $k \leq n$, $M = Ann \times I$ with $2n$ framed marked points attached to the outer boundary of $Ann \times \{0\}$, and let M be decorated with a lollipop $S = S \times I$. Let $\mathcal{L}^{fr}(2n, k)$ be the set of all k white markers fixed to the boundary of $Ann \times \{0\}$ along with all relative framed links $\mathcal{L}$ disjoint from the white markers, $\mathcal{L} \sqcup \{w_j\}_{j=1}^{k}$, up to ambient isotopy, while keeping all marked points on the boundary fixed. Furthermore, we restrict the k white markers to finitely many possible placements in the inner boundary and also restrict to allowing at most one white marker between framed points and also between the 1st and $2n^{th}$ framed points and the lollipop on the outer boundary. Let R be a commutative ring with unity, $A \in R$ be invertible, and let $S_{2,\infty}^{sub}(2n, k)$ be the submodule of $R\mathcal{L}^{fr}(2n, k)$ that is generated by the Kauffman bracket skein relations. The **relative Kauffman bracket skein module of M with white markers** is the quotient*

$$\mathcal{S}_{2,\infty}(Ann \times I, \{x_i\}_1^{2n} \cup \{w_j\}_{j=1}^{k}; R, A) = R\mathcal{L}^{fr}(2n, k)/S_{2,\infty}^{sub}(2n, k).$$

For simplicity we will denote this skein module by $\mathcal{S}(Ann_k^n)$.

Example 3.12. *Consider the following relative link with two white markers in $\mathcal{L}^{fr}(4, 2)$;*

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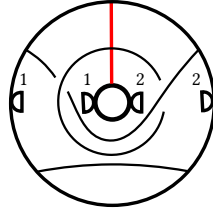
This link as an element in $\mathcal{S}_{2,\infty}(Ann \times I, \{x_i\}_1^2 \cup \{w_j\}_{j=1}^2; R, A)$ is equal to

(1)
$$\text{} = (1 + A^{-4}) \text{} + A^2 \text{} .$$

Corollary 3.13. *The relative Kauffman bracket skein module of $Ann \times I$ with white markers, $\mathcal{S}(Ann_k^n)$, is a free R-module. The basis contains an infinite number of elements described as follows.*

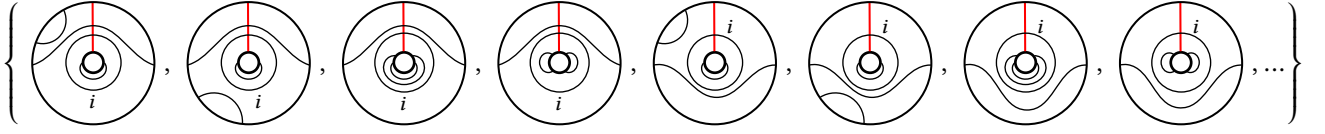(1) *Crossingless connections between $2n + 2k$ framed points in the outer boundary of $Ann \times I$ and i number of boundary parallel curves where $i \geq 0$. In particular, the arcs connected to the $2k$ framed points do not intersect the lollipop.*

(2) *Crossingless connections between $2n + 2k$ framed points in $\partial(Ann \times \{0\})$ and i number of boundary parallel curves where $i \geq 0$, $2l$ framed points are in the inner boundary for $l \leq k$, and the arcs attached to the $2l$ framed points are not connected to the outer boundary.*

(3) *Crossingless connections between $2n + 2k$ framed points in $\partial(Ann \times \{0\})$ where l framed points are in the inner boundary for $l < 2k$ and at least one arc that is connected to one of the l framed points is also connected to one of the $2k - l$ framed points lying in the outer boundary.*

(4) *Crossingless connections between $2n$ framed points in the outer boundary of $Ann \times \{0\}$, i boundary parallel curves where $i > 0$, and k white markers $\{w_j\}_{j=1}^k$ in $\partial(Ann \times \{0\})$, where at least one white marker, say $pt_{1,l}$ of $w_l = \{pt_{1,l}, pt_{2,l}\}$, lies in the inner boundary of $Ann \times \{0\}$ and $pt_{2,l}$ lies in the outer boundary.*

(5) *Crossingless connections between $2n$ framed points on the outer boundary $\partial(Ann \times \{0\})$ and k white markers $\{w_j\}_1^k$ in $\partial(Ann \times \{0\})$, where for at least one white marker a path connecting it is obstructed by an arc connected to one pair of framed points or by the lollipop; see Figure 3.6. For each element, say b, in this set of crossingless connections with white markers we also have bz^i where $i > 0$.*

Example 3.14. *Examples of the basis elements of $\mathcal{S}(Ann_2^1)$ as described in Corollary 3.13 are illustrated below.*
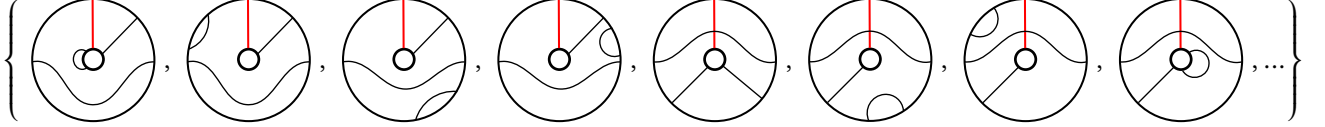
(1) *Crossingless connections between 6 framed points in the outer boundary of $Ann \times I$ and i number of boundary parallel curves where $i \geq 0$.*
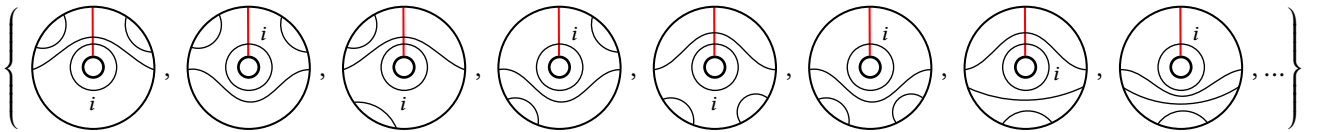
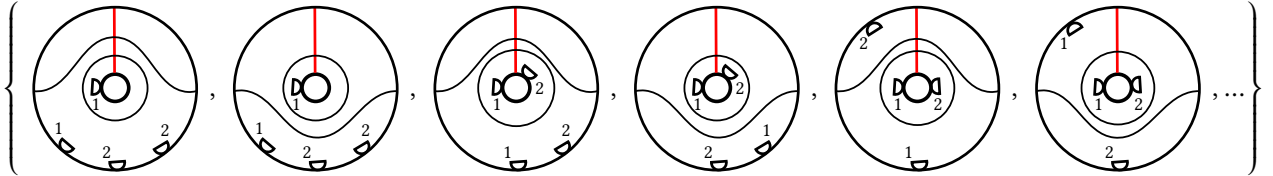(2) *Crossingless connections between 6 framed points in $\partial(Ann \times \{0\})$ and i number of boundary parallel curves where $i \geq 0$, $2l$ framed points are in the inner boundary for $l \leq 2$, and the arcs attached to the $2l$ framed points are not connected to the outer boundary.*

(3) *Crossingless connections between* 6 *framed points in* $\partial(Ann \times \{0\})$ *where* l *framed points are in the inner boundary for* $l < 4$ *and at least one arc that is connected to one of the* l *framed points is also connected to one of the* $4 - l$ *framed points lying in the outer boundary.*

(4) *Crossingless connections between* 2 *framed points in the outer boundary of* $Ann \times \{0\}$, i *boundary parallel curves where* $i > 0$, *and* 2 *white markers* $\{w_j\}_{j=1}^2$ *in* $\partial(Ann \times \{0\})$, *where at least one white marker, say* $pt_{1,l}$ *of* $w_l = \{pt_{1,l}, pt_{2,l}\}$, *lies in the inner boundary of* $Ann \times \{0\}$ *and* $pt_{2,l}$ *lies in the outer boundary.*

(5) *Crossingless connections between* 2 *framed points on the outer boundary* $\partial(Ann \times \{0\})$ *and* 2 *white markers* $\{w_j\}_1^k$ *in* $\partial(Ann \times \{0\})$, *where for at least one white marker a path connecting it is obstructed by an arc connected to one pair of framed points or by the lollipop. For each element, say* b, *in this set of crossingless connections with white markers we also have* bz^i *where* $i > 0$. *Notice that some elements of the form* bz^i, *for* $i > 0$, *are also described in (4).*
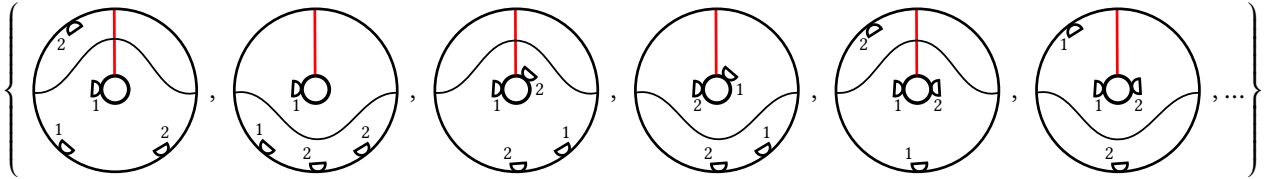
3.2. **Lollipop method.** In this section we use the lollipop method discussed in [CP2]. In the same spirit as Chen and Przytycki, in this section we define several maps. In particular, we define the maps ψ_k and β_k that involve the use of a lollipop, and the bilinear map H on the Hopf link with zero framing. These maps are modified by using white markers so that the inner boundary is left intact and the x, y, and w curves remain unchanged under the maps and the bilinear form. Furthermore, ξ_k is a modification of ϕ_k in [CP2] where we decorate the z curve in the Klein bottle instead of the annulus.

Definition 3.15. *Define a linear map* $\psi_k : \mathcal{R}((B_n)_1) \to \mathcal{R}((B_{n+k})_1)$ *on the basis as follows. For* $b \in (B_n)_1$:

   (1) *Decorate* b *with a lollipop* S *and let* $\mathbf{L}$ *denote the arcs whose minimum intersection points with the lollipop is equal to one.*
   (2) *Include* k *marked points, denoted by* ℓ_k, *between the* $2n^{th}$ *marked point and the lollipop.*
   (3) *Include* k *marked points, denoted by* r_k, *between the lollipop and the* 1st *marked point.*
   (4) *Add parallel curves denoted by* $U_{b,k}$ *that connects* ℓ_k *to* r_k *in such a way that each curve crosses over* $\mathbf{L}$ *on the left of the lollipop, intersects the lollipop once, and crosses under* $\mathbf{L}$ *on the right of the lollipop.*

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An illustration is shown in Figure 3.7.

(a) ψ_k under an element in $B_{5,1}$.
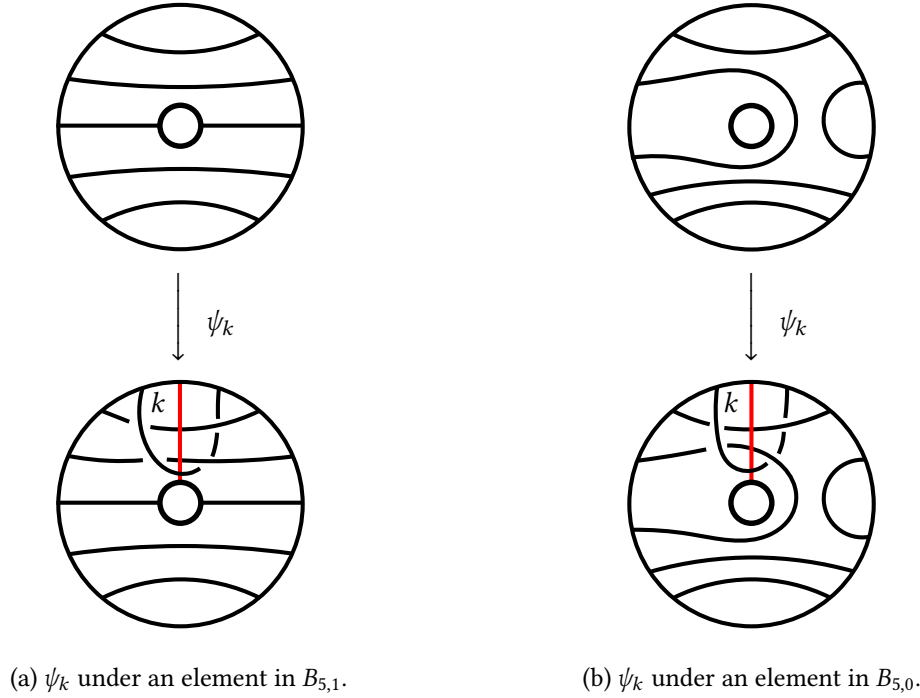
(b) ψ_k under an element in $B_{5,0}$.

FIGURE 3.7. An illustration of the map ψ_k under two basis elements of $\mathcal{R}((B_5)_1)$.

Definition 3.16. *Define a linear map* $\beta_k : \mathcal{R}((B_{n+k})_1) \rightarrow \mathcal{R}(B_{n+k,0}) \oplus \mathcal{S}(Ann_2^{n+k})$ *on the basis as follows. For* $b \in (B_{n+k})_1$:

    (1) If $b \in B_{n+k,0}$, *then*

        (a) decorate b *with a lollipop* S *and let* **L** *denote the arcs whose minimum intersection points with the lollipop is equal to one. Denote by* ℓ_k *the first* k *marked points from the left of the lollipop and* r_k *by the first* k *points from the right of the lollipop.*

        (b) Push **L** *to the other side of the inner boundary of* $Ann \times \{0\}$.

        (c) Insert a copy of the kth *Jones-Wenzl idempotent,* f_k, *close to* ℓ_k, *into the arcs connected to* ℓ_k, *and another copy close to* r_n *into the arcs connected to* r_k.

    (2) If $b \in B_{n+k,1}$, *then*

        (a) Decorate the arc attached to y_1 *with a white marker labelled* 1.

        (b) Decorate the arc attached to y_2 *with a white marker labelled* 2.

        (c) Push **L** $\cup U_{b,k}$ *to the other side of the inner boundary of* $Ann \times \{0\}$.

        (d) Insert two copies of the kth *Jones-Wenzl idempotent,* f_k, *into* $U_{b,k}$, *one close to* ℓ_n *and another close to* r_n.
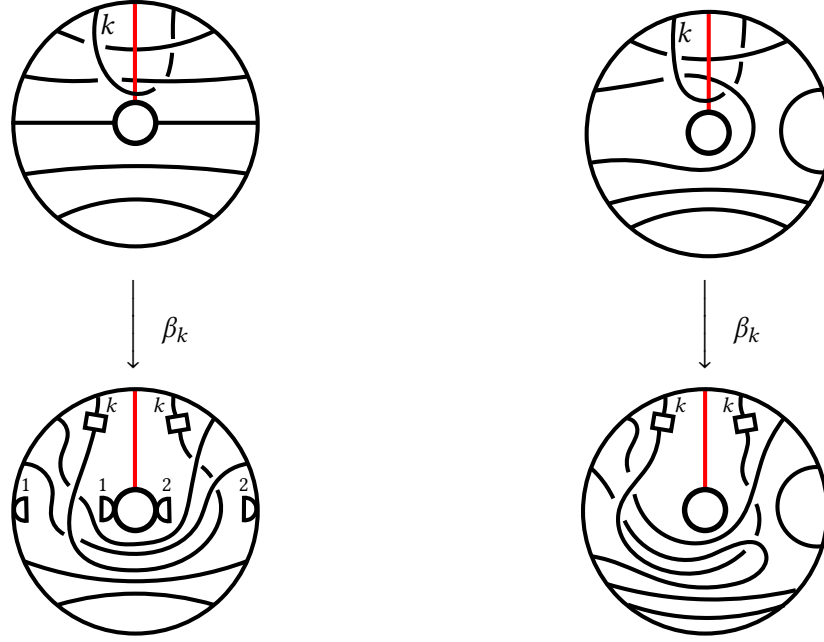
An illustration is shown in Figure 3.8.

The next lemma is a direct result of the previous definition.

Lemma 3.17. *Let* $b_i, b_j \in (B_n)_1$. *If* $\langle b_i, b_j \rangle_B = x^p y^m w^h d^n z^r$ *for* $p, m, h, n, r \geq 0$, *then*

$$\langle \beta_0 \circ \psi_0(b_i), \beta_0 \circ \psi_0(b_j) \rangle_B = x^p y^m w^h d^{n+r}.$$

That is, all z curves become homotopically trivial.

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(a) β_k under an element in $B_{5+k,1}$.

(b) β_k under an element in $B_{5+k,0}$.

FIGURE 3.8. An illustration of the map β_k under two elements of $(B_{5+k})_1$.

Definition 3.18. *Let H be a bilinear map associated to the Hopf link with zero framing and ordered components $\{\boldsymbol{K}_1, \boldsymbol{K}_2\}$,*

$$H : \mathcal{S}_{2,\infty}(\boldsymbol{K}_1 \times I; \mathbb{Z}[A^{\pm 1}]) \times \mathcal{S}_{2,\infty}(\boldsymbol{K}_2 \times I; \mathbb{Z}[A^{\pm 1}]) \to \mathcal{S}_{2,\infty}(S^3; \mathbb{Z}[A^{\pm 1}]),$$

defined on the basis as follows:

For each $1 \le i \le 2$, if the ith component is equal to z^h for $h > 0$ then replace K_i with h parallel copies, and if the ith component is equal to 1 then remove K_i from the diagram. Evaluate the resulting link in $\mathcal{S}_{2,\infty}(S^3; \mathbb{Z}[A^{\pm 1}])$.

We define a linear map $\xi_k : \mathcal{S}_{2,\infty}(Kb \mathbin{\hat{\times}} I) \to \mathbb{Z}[A^{\pm 1}, x, y, w]$ such that ξ_k is the identity on x, y, w, and d, $d = -A^2 - A^{-2}$, and

$$\xi_k(z) := H(z, S_n(z)),$$

where $S_n(z)$ is the nth Chebyshev polynomial of the second kind in the variable z. In particular, $S_n(z) = tr_{Ann}(f_n)$ (See [Lic2, PBIMW] for more details about the connections to the Jones-Wenzl idempotents and Chebyshev polynomials).

Informally, $\xi_k(z)$ is defined by decorating the first component of the Hopf link with one parallel curve and decorating the second component with the kth Jones-Wenzl idempotent f_k.

Recall from W. B. R. Lickorish in [Lic2] that

(2)
$$\underset{m}{} \ \underset{k}{} = (-A^{2(k+1)} - A^{-2(k+1)})^m \Delta_k = ((-1)^k T_{k+1}(d))^m \Delta_k.$$

Therefore,

$$\xi_k(z^m) = H(z^m, f_k) = \underset{m}{\bigcirc\!\!\!\bigcirc}\!\!\square_k = (-A^{2(k+1)} - A^{-2(k+1)})^m \Delta_k.$$

Example 3.19. *Let $d = -A^2 - A^{-2}$. Then*

$$\xi_k\left(\vphantom{\Bigg|} \right) = dH(z^3, f_k) = d(-A^{2(k+1)} - A^{-2(k+1)})^3 \Delta_k = d((-1)^k T_{k+1}(d))^3 \Delta_k,$$

$$\xi_k\left(\vphantom{\Bigg|} \right) = \xi_k(yz^2) = yH(z^2, f_k) = y((-1)^k T_{k+1}(d))^2 \Delta_k,$$

and

$$\xi_k\left(\vphantom{\Bigg|} \right) = \xi_k(dxyz^3) = dxy((-1)^k T_{k+1}(d))^3 \Delta_k.$$

Definition 3.20. *Define the matrix $F_{n,k}$ using ξ_k as follows. For $m_i, m_j \in (Mb_n)_1$,*

$$F_{n,k} = (\xi_{k-1}(\langle m_i, m_j \rangle_{Mb}))_{1 \le i,j \le \binom{2n}{n-1} + \binom{2n}{n}}.$$

Remark 3.21. *By the definition, if $d = -A^2 - A^{-2}$, then*

$$G_n^{(Mb)_1}(d, (-1)^{k-1} T_k(d), x, y, w) = \frac{1}{\Delta_{k-1}} F_{n,k}.$$

Furthermore, if we define $F'_{n,k}$ on Mb_n in a similar way, $F'_{n,k} = (\xi_{k-1}(\langle m'_i, m'_j \rangle_{Mb}))_{1 \le i,j \le |Mb_n|}$, then

$$G_n^{Mb}(d, (-1)^{k-1} T_k(d), x, y, w) = \frac{1}{\Delta_{k-1}} F'_{n,k}.$$

The following lemma is a direct result of the construction of the maps β_k, ψ_k, and φ. Notice that φ is a bijection between the sets $(Mb_n)_1$ and $(B_n)_1$ which preserves the bilinear form, ψ_k introduces k trivial link components linked to the z curves under the bilinear form of type B, and $\beta_k \circ \psi_k$, under the bilinear form of type B, inserts the kth Jones-Wenzl idempotent into the k trivial link components. Additionally, $\beta_k \circ \psi_k$ pushes all boundary parallel curves through the inner boundary of the annulus while preserving the number of x, y, and w curves.

Lemma 3.22. *Let $\eta_k = \beta_k \circ \psi_k \circ \varphi$. For $m_i, m_j \in (Mb_n)_1$,*

$$\xi_k(\langle m_i, m_j \rangle_{Mb}) = \langle \eta_k(m_i), \eta_k(m_j) \rangle_B.$$

The next theorem is our main result regarding the structure of the closed formula for the Gram determinant of type $(Mb)_1$.

Theorem 3.23.

$$\prod_{k=1}^{n} (T_k(d) + (-1)^k z)^{\binom{2n}{n-k}} \text{ divides } D_n^{(Mb)_1}(d, z, x, y, w).$$

Proof. We modify and build upon Chen and Przytycki's proof in [CP2] by showing that for $k \geq 1$, the nullity of $G_n^{(Mb)_1}(d, (-1)^{k-1}T_k(d), x, y, w)$ for $d = -A^2 - A^{-2}$ is at least $\binom{2n}{n-k}$.

Recall that for $F_{n,k} = (\xi_{k-1}(\langle m_i, m_j \rangle_{Mb}))_{1 \leq i,j \leq \binom{2n}{n-1} + \binom{2n}{n}}$ we have

$$G_n^{(Mb)_1}(d, (-1)^{k-1}T_k(d), x, y, w) = \frac{1}{\Delta_{k-1}}F_{n,k}.$$

Furthermore, by Lemma 3.22, we have $\xi_{k-1}(\langle m_i, m_j \rangle_{Mb}) = \langle \eta_{k-1}(m_i), \eta_{k-1}(m_j) \rangle_B$ where $\eta_k = \beta_k \circ \psi_k \circ \varphi$. Since φ is a bijection between $(Mb_n)_1$ and $(B_n)_1$, then it suffices to show that $\beta_{k-1} \circ \psi_{k-1}((B_n)_1)$ is contained in a subspace of dimension $\binom{2n}{n-1} + \binom{2n}{n} - \binom{2n}{n-k}$ in $\mathcal{S}_{2,\infty}(Ann_4^{n+k-1}) \oplus \mathcal{S}_{2,\infty}(D^2 \times I, n+k-1)$. As in [CP2], it suffices to show that

$$\dim(\mathrm{Im}(\beta_{k-1})) \leq \binom{2n}{n-1} + \binom{2n}{n} - \binom{2n}{n-k}.$$

Recall that $\mathcal{R}((B_n)_1) = \mathcal{R}(B_{n,0}) \oplus \mathcal{R}(B_{n,1})$ where $\mathcal{R}(B_{n,0})$ and $\mathcal{R}(B_{n,1})$ are free R-modules generated by $B_{n,0}$ and $B_{n,1}$, respectively. Since $B_{n,0} \cap B_{n,1} = \varnothing$, then

$$\beta_{k-1} \circ \psi_{k-1}(B_{n,0} \oplus B_{n,1}) = \beta_{k-1} \circ \psi_{k-1}(B_{n,0} \oplus \{0\}) \oplus \beta_{k-1} \circ \psi_{k-1}(\{0\} \oplus B_{n,1}).$$

Furthermore, $|B_{n,1}| = \binom{2n}{n-1}$. Therefore,

$$\dim(\beta_{k-1} \circ \psi_{k-1}((B_n)_1)) \leq \dim(\beta_{k-1} \circ \psi_{k-1}(B_{n,0} \oplus \{0\})) + \binom{2n}{n-1}.$$

The set $\beta_{k-1} \circ \psi_{k-1}(B_{n,0} \oplus \{0\})$ can be viewed as a set of crossingless connections between $2(n+k-1)$ points in the disc by cutting along the lollipop. Therefore, it suffices to prove that $\beta_{k-1} \circ \psi_{k-1}(B_{n,0} \oplus \{0\})$ is contained in a subspace of dimension $\binom{2n}{n} - \binom{2n}{n-k}$ in $\mathcal{S}_{2,\infty}(D \times I, n+k-1)$ which was already proven in [CP2]. $\square$

4. Future directions

We present a conjectured formula for type $(Mb)_1$. This conjecture has been verified for $n \leq 3$.

Conjecture 4.1.
Let $R = \mathbb{Z}[A^{\pm 1}, w, x, y, z]$. Then, the Gram determinant of type $(Mb)_1$ for $n \geq 1$, is:

$$D_n^{(Mb)_1} = [(d-z)((d+z)w - 2xy)]^{\binom{2n}{n-1}} \prod_{k=2}^{n}(T_k(d)^2 - z^2)^{\binom{2n}{n-k}} \prod_{k=2}^{n}(T_{2k}(d) - 2)^{\binom{2n}{n-k}},$$

where $T_k(d)$ is the kth Chebyshev polynomial of the first kind and $d = -A^2 - A^{-2}$.

The lollipop method was verbally proposed by J. H. Przytycki, and first explicitly stated in writing in [PBIMW], to be a suitable method to apply to type Mb for a proof of the same factor given in our main theorem. In fact, the concluding arguments of the proof of Theorem 3.23 suggest that the new techniques given in this paper along with the lollipop method can be modified and applied to type Mb; albeit in a not so straightforward way.

Lemma 3.7 gives insight into direct connections between type $(Mb)_1$ and a Gram determinant from a matrix created by using the set $(B_n)_1$ and the bilinear form of type B. We will call this determinant type $(B)_1$. As shown in the proof of the lemma, type $(Mb)_1$ is a special case of type $(B)_1$ where no distinction is made between arcs whose boundary components are both connected to the inner boundary (or outer boundary). Another future direction is to focus on the Gram determinant of type $(B)_1$ and obtain a closed formula.

Furthermore, potential connections to statistical mechanics are suggested, by investigating the blob algebra discussed in [MS2], when the y and x variables are made equal as to obtain a determinant from a symmetric matrix.

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School of Mathematics, Monash University, VIC 3800, Australia.
Email address: dionne.ibarra@monash.edu

Department of Mathematics, The Graduate Center CUNY, NY, USA, and
Department of Mathematics, University of Puerto Rico-Río Piedras, San Juan, PR
Email address: gabrielmontoyavega@gmail.com