

Growing black-hole hair in nonminimally coupled biscalar gravity

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Black holes offer a unique laboratory for fundamental physics and are crucial for understanding theories beyond Einstein's theory of general relativity. In this paper, we focus on 4D effective field theories and string-theory inspired models that include scalar fields. We focus on one such model, axi-dilaton gravity, a quadratic gravity theory with two kinetically coupled scalar fields, an axion and a dilaton. To study the evolution and structure of these fields around black holes, we introduce CANUDA-AxiDil, the first open-source, parametrized numerical relativity code for quadratic and biscalar gravity. Using this code, we perform single black hole simulations to show the dynamical formation of axion and dilaton hairs and quantify the effect of higher-order terms in coupling and spin. Through these simulations, we measure the impact of black hole spin and curvature coupling strength on the profiles of the axion and dilaton and show that including kinetic coupling between the fields increases the observed deviations from general relativity. Furthermore, we simulate the axion and dilaton fields around a binary black hole coalescence demonstrating the growth of axion hair during the inspiral and the production of radiative modes for both fields.

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I. INTRODUCTION

The measurement of gravitational-wave (GW) signals over the last decade has not only confirmed the existence of black-hole binary (BBH) coalescences [1–6], but has also spurred a new interest in using black holes (BHs) as laboratories to test for new signatures of fundamental physics [7–10]. Current, ground-based GW detectors are only the beginning; the potential to search for signatures of new phenomena using BHs becomes increasingly accessible as new experiments come online [11,12].

In addition, collaboration across the gravity and high energy physics communities continues to push our understanding of fundamental physics to new frontiers. One area of interest shared by these communities is using compact objects to probe for corrections to general relativity (GR) by considering additional degrees of freedom coupled to curvature. Recent results by the LIGO-Virgo-KAGRA (LVK) [13–15] and NANOGrav Collaborations [16] and upcoming GW observatories such as the recently approved LISA mission [17] aim to constrain deviations from GR [7–10,18–20]. The prospect of detecting high curvature deviations increases as

we detect smaller BHs; experiments such as LIGO [21] and Cosmic Explorer [22] would be able to detect subsolar mass BHs and probe extreme regions of curvature.

While many beyond GR theories have been proposed, we focus on biscalar-tensor models that are contained in the four-dimensional (4D) string-inspired effective field theory (EFT) of gravity [23,24]. The latter can be derived from the heterotic string effective action [25]. Here, we focus on axi-dilaton gravity that consists of a metric and two scalar fields that are kinetically coupled to each other and nonminimally coupled to spacetime curvature. We adopt a phenomenological approach to these models (despite their high-energy origin), and thus, we do not assume the coupling length to be the Planck scale.

The 4D EFT theory considered in this paper contains dynamical Chern-Simons (dCS) gravity [26,27] and scalar Gauss-Bonnet (sGB) gravity [28] as special cases. They are popular theories in the context of higher curvature corrections to gravity. In both dCS and sGB gravity, an additional scalar degree of freedom couples to a quadratic curvature invariant to form hair [29–40]. In dCS gravity, a pseudoscalar field (or axion) couples to the Pontryagin density that sources axion hair. In sGB gravity, a scalar field known as the dilaton couples to the Gauss-Bonnet invariant, which sources dilaton hair.

Significant work has gone into studying beyond GR effects in either dCS or sGB gravity where a single field is

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coupled to curvature. It may even seem natural to study them separately as the axion is parity odd while the dilaton is parity even, thus probing different phenomena of coupling to curvature. In principle, however, both extensions of GR appear as leading-order corrections in the 4D EFT of gravity [23,24] and therefore should be considered together. Before considering these models together in axi-dilaton gravity, let us first review previous results for BHs in dCS and sGB gravities separately.

The effect of coupling to higher curvature on GWs produced in dCS gravity has been studied with post-Newtonian methods [41–47], numerical simulations using an order-by-order expansion scheme [48–50], and BH perturbation theory [51–58] to probe the inspiral, merger, and ringdown phases of a BBH coalescence, respectively. Furthermore, BBH merger simulations with similar parameters as those from LVK data have been performed to constrain dCS gravity [59].

Single BH simulations have been performed to investigate hairy, rotating BH solutions in dCS gravity [29,60–64]. Previous studies have also considered a massive axion potential around single BHs to explore the quasibound states formed by an ultralight axion field coupled to the dCS curvature invariant [53,65–67].

Similarly, the effect on GW propagation in sGB gravity has also been considered. Post-Newtonian methods have been used to study the inspiral phase of a BBH coalescence [68–73]. Numerical relativity simulations in the order-by-order expansion have been performed to estimate the effect during merger [48,74–77]. Furthermore, recent results of a well-posed formulation of sGB gravity [78,79] have been implemented to perform BBH simulations and compute the GW signal in the full theory [80–83]. The ringdown phase of the BBH coalescence has been studied using BH perturbation theory [84–87].

In the decoupling approximation, single BH simulations have been performed to demonstrate the formation of dilaton hair in nonrotating and rotating BH backgrounds [33,38,74,88,89]. Beyond this approximation, single BH simulations have been performed to include backreaction from dilaton hair growth [90–93]. Other work has determined conditions on predictivity, and loss thereof, during gravitational collapse in sGB gravity [94].

Recent works that include both the axion and dilaton, but not the coupling between them, have explored the fields in the context of BH hair formation [95] and testing the accuracy of slowly-rotating solutions via astrophysical observations [96]. Other works have combined both fields to include a kinetic coupling between them to consider GW propagation in the presence of both parity-violating and parity-conserving corrections [97] and a wider set of cosmological applications [98–101]. Here, we go beyond these studies by including the kinetic coupling between the two fields and by modeling the formation of coupled axion

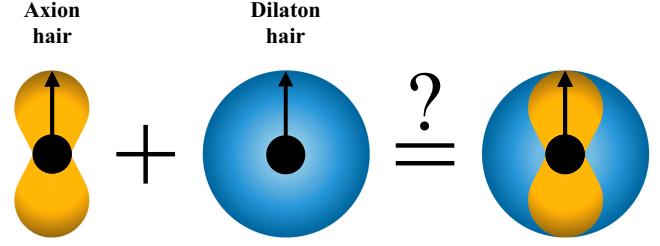


FIG. 1. Sketch of coupled axion (yellow dipole) and dilaton (blue monopole) hairs around a spinning black hole.

and dilaton hair in BH spacetimes that enhances the deviations from GR.

In this paper, we perform numerical relativity simulations to study axi-dilaton gravity, which combines dCS and sGB models through a kinetic coupling between the fields. We simulate both single and binary BH backgrounds and investigate the evolution of the axion and dilaton hairs. The sketch in Fig. 1 illustrates the main idea: On the left-hand side of the pseudoequation, we show the axion and dilaton hairs after they evolve to a stationary profile around a single, spinning BH where the axis of rotation is indicated by the arrow. On the right-hand side of the pseudoequation, we illustrate how the axion and dilaton hairs might evolve together when they are kinetically coupled. We aim to solve this pseudoequation by the end of this paper.

To explore the evolution of the right-hand side of the pseudoequation presented in Fig. 1, we have developed the CANUDA-AxiDil code [102] as a new module of the open-source CANUDA infrastructure, a numerical relativity code for fundamental physics [103]. We implement the evolution equations of the axion and dilaton fields in axi-dilaton gravity in the decoupling approximation; i.e., the fields do not backreact onto the metric. The CANUDA-AxiDil code provides the first, open-source parametrized numerical relativity code for theories of quadratic gravity as the code has been developed to be flexible enough to consider a large class of quadratic gravity theories, including dCS and sGB gravity.

In this paper, we present simulations of the axion and dilaton fields evolving in the background of single and binary BHs. Our simulations show the dynamical formation of the axion and dilaton hairs in both backgrounds. We perform a series of single BH simulations to address the following questions: (i) How does the BH spin impact the evolution of the axion and dilaton hairs? (ii) How does the coupling strength to the curvature invariants affect the axion and dilaton hairs? (iii) What is the effect of the kinetic coupling between the axion and dilaton fields on their respective hairs?

The BBH simulation provides a proof-of-principle computation of the axion and dilaton fields' evolution in axi-dilaton gravity with a BBH background. Through this simulation, we investigate the formation of axion and

dilaton hairs around BBHs and compute the radiative multipoles produced by the inspiraling scalar charges.

The paper is organized as follows: In Sec. II, we define the action and field equations of the 4D EFT. In Sec. III, we describe the (bi)scalar-tensor theories and quadratic gravity theories that are captured by our generalized model. In Sec. IV, we present the time evolution formulation in axi-dilaton gravity. In Sec. V, we introduce our numerical relativity framework CANUDA-AxiDil and summarize the setup of the simulations presented in this paper. In Sec. VI, we present results of single BH simulations, and we show the proof-of-principle BBH simulation in Sec. VII. Finally we conclude in Sec. VIII.

Animations of the simulations presented in this paper are available on the CANUDA YouTube channel [104]. The CANUDA-AxiDil code is publicly available at [102]. A sample simulation is available on the EINSTEIN TOOLKIT website at [105].

In this paper, we adopt the mostly plus signature convention $(-, +, +, +)$ and use geometric units $G = c = 1$.

II. QUADRATIC GRAVITY IN A NUTSHELL

A. Action and field equations

We consider a generalization of biscalar-tensor theories of gravity [23,24] in which a pseudoscalar, Θ , and a scalar, Φ , are nonminimally coupled to gravity. The action is

$$S = \kappa \int d^4x \sqrt{-g} \left({}^{(4)}R + \frac{\alpha_{\text{CS}}}{4} f_\Theta \mathcal{R}_{\text{CS}} + \frac{\alpha_{\text{GB}}}{4} f_\Phi \mathcal{R}_{\text{GB}} - g_\Phi^2 \left(\frac{1}{2} (\nabla\Theta)^2 + V_\Theta \right) - \frac{1}{2} (\nabla\Phi)^2 - V_\Phi \right), \quad (1)$$

where ${}^{(4)}R$ is the four-dimensional Ricci scalar constructed from the metric $g_{\mu\nu}$, $\kappa = (16\pi G)^{-1}$, $f_\Phi \equiv f_\Phi(\Phi)$, $g_\Phi \equiv g_\Phi(\Phi)$, and $f_\Theta \equiv f_\Theta(\Theta)$ are functions of the fields, $V_\Phi \equiv V_\Phi(\Phi)$ and $V_\Theta \equiv V_\Theta(\Theta)$ are the fields' potentials, and α_{GB} and α_{CS} are dimensionful coupling constants with units of length squared, $[\alpha_{\text{GB}}] = [\alpha_{\text{CS}}] = [L]^2$, such that the (pseudo)scalar fields are dimensionless, $[\Phi] = [L]^0$ and $[\Theta] = [L]^0$.

These coupling constants are labeled with the ‘‘Chern-Simons’’ (CS) and ‘‘Gauss-Bonnet’’ (GB) acronyms, respectively, since the individual terms that they multiply correspond to dCS or sGB gravity; see Sec. III for details. The pseudoscalar (or axion) field, Θ , couples to the Pontryagin topological invariant

$$\mathcal{R}_{\text{CS}} = \tilde{R}^{abcd} R_{abcd} = -\frac{1}{2} \epsilon^{cd}{}_{ef} R^{abef} R_{abcd}, \quad (2)$$

via the function f_Θ , where $\tilde{R}^{abcd} := 1/2 \epsilon^{cd}{}_{ef} R^{abef}$ is the dual of the Riemann tensor. The scalar (or dilaton) field, Φ , couples to the Gauss-Bonnet invariant

$$\begin{aligned} \mathcal{R}_{\text{GB}} &= -\tilde{\tilde{R}}^{abcd} R_{abcd} \\ &= R^2 - 4R_{ab}R^{ab} + R_{abcd}R^{abcd}, \end{aligned} \quad (3)$$

where $\tilde{\tilde{R}}^{abcd} = 1/4 \epsilon^{ab}{}_{hg} \epsilon^{cd}{}_{ef} R^{efhg}$ is the double dual of the Riemann tensor. The form of the coupling is determined by the function f_Φ .

The pseudoscalar and scalar fields are coupled to each other via the function $g_\Phi = g_\Phi(\Phi)$. For example, choosing $g_\Phi = \text{constant}$ decouples the fields, while $g_\Phi = 1/f_\Phi$ selects axi-dilaton gravity. Depending on the choice of the (dimensionless) coupling functions, f_Θ and f_Φ , and on the self-interaction potentials, $V_\Theta = V_\Theta(\Theta)$ and $V_\Phi = V_\Phi(\Phi)$, different models of gravity can be selected. In the following derivation, we keep the coupling functions generic and only specify them for the numerical simulations. For brevity, we introduce the notation

$$\cdot = d/d\Theta, \quad \text{and} \quad ' = d/d\Phi, \quad (4)$$

to indicate differentiation with respect to the pseudoscalar Θ and scalar field Φ .

Varying the action, Eq. (1), with respect to the pseudoscalar, scalar, and metric gives the field equations,

$$\square\Theta - \dot{V}_\Theta + \frac{\alpha_{\text{CS}}}{4} \frac{\dot{f}_\Theta}{g_\Phi^2} \mathcal{R}_{\text{CS}} + 2 \frac{g'_\Phi}{g_\Phi} \nabla_\mu \Phi \nabla^\mu \Theta = 0, \quad (5a)$$

$$\square\Phi - V'_\Phi + \frac{\alpha_{\text{GB}}}{4} f'_\Phi \mathcal{R}_{\text{GB}} - g'_\Phi g_\Phi [(\nabla\Theta)^2 + 2V_\Theta] = 0, \quad (5b)$$

$$R_{ab} - \frac{1}{2} g_{ab} R - \frac{1}{2} T_{ab}^{\text{eff}} = 0, \quad (5c)$$

where we define the effective energy-momentum tensor

$$\begin{aligned} T_{ab}^{\text{eff}} &= -2(\alpha_{\text{CS}} \mathcal{C}_{ab}^{\text{CS}} + \alpha_{\text{GB}} \mathcal{C}_{ab}^{\text{GB}}) \\ &\quad + \nabla_a \Phi \nabla_b \Phi - \frac{1}{2} g_{ab} ((\nabla\Phi)^2 + 2V_\Phi) \\ &\quad + g_\Phi^2 \left(\nabla_a \Theta \nabla_b \Theta - \frac{1}{2} g_{ab} ((\nabla\Theta)^2 + 2V_\Theta) \right). \end{aligned} \quad (6)$$

Note that Eq. (5a) has been normalized by dividing by g_Φ , which is strictly allowed only if $g_\Phi \neq 0$. The $\mathcal{C}$ tensors that contain the coupling between the (pseudo)scalar fields and the curvature tensors, are¹

$$\mathcal{C}_{ab}^{\text{CS}} := \mathcal{E}_c \epsilon^{cd}{}_{e(a} \nabla^e R_{b)d} + \mathcal{F}^{cd} (*R_{c(ab)d}), \quad (7a)$$

$$\begin{aligned} \mathcal{C}_{ab}^{\text{GB}} &:= R_{acbd} \mathcal{G}^{cd} + 2R_{c(a} \mathcal{G}^c{}_{b)} - g_{ab} R_{cd} \mathcal{G}^{cd} \\ &\quad - G_{ab} \text{tr}(\mathcal{G}) - \frac{1}{2} R \mathcal{G}_{ab}, \end{aligned} \quad (7b)$$

¹Note that the definition of $\mathcal{C}_{ab}^{\text{GB}}$ differs from the one given in Ref. [74] by a factor 1/8. Here, we absorb the factor in $\mathcal{C}_{ab}^{\text{GB}}$.

where $\text{tr}(\mathcal{G}) = g^{ab}\mathcal{G}_{ab}$, and the auxiliary tensors are

$$\mathcal{E}_a := \dot{f}_\Theta \nabla_a \Theta, \quad (8a)$$

$$\mathcal{F}_{ab} := \dot{f}_\Theta \nabla_a \nabla_b \Theta + \ddot{f}_\Theta \nabla_a \Theta \nabla_b \Theta, \quad (8b)$$

$$\mathcal{G}_{ab} := f'_\Phi \nabla_a \nabla_b \Phi + f''_\Phi \nabla_a \Phi \nabla_b \Phi. \quad (8c)$$

In the next section, we perform the $3+1$ decomposition of the field equations, where we express the curvature invariants in terms of the gravito-electric and gravito-magnetic decomposition of the Weyl tensor

$$W_{abcd} := R_{abcd} - (g_{a[c}R_{d]b} - g_{b[c}R_{d]a}) + \frac{1}{3}g_{a[c}g_{d]b}R, \quad (9)$$

and its dual, ${}^*W^{abcd} := 1/2\epsilon^{cd}{}_{ef}W^{abef}$. Therefore, we here rewrite the topological invariants as

$$\mathcal{R}_{\text{CS}} = {}^*W_{abcd}W^{abcd}, \quad (10a)$$

$$\mathcal{R}_{\text{GB}} = W_{abcd}W^{abcd} - R_{ab}R^{ab} + \frac{2}{3}R^2. \quad (10b)$$

Note that Eqs. (10) can be proven to be always true regardless of the background [106]. Applying the Bianchi identities to Eq. (9), one can derive

$$\nabla_a W^a{}_{bcd} = \nabla_{[c}R_{d]b} + \frac{1}{6}g_{b[c}\nabla_{d]}R, \quad (11)$$

to recast Eq. (7) in terms of the Weyl tensor and its dual,

$$\mathcal{C}_{ab}^{\text{CS}} \equiv 2\mathcal{E}^c \nabla^d {}^*W_{d(ab)c} + \mathcal{F}^{cd} {}^*W_{d(ab)c}, \quad (12a)$$

$$\begin{aligned} \mathcal{C}_{ab}^{\text{GB}} &\equiv W_{acbd}\mathcal{G}^{cd} - \frac{1}{2}g_{ab}R_{cd}\mathcal{G}^{cd} + R_{c(a}\mathcal{G}^c{}_{b)} \\ &\quad - \frac{1}{2}\text{tr}(\mathcal{G})R_{ab} - \frac{1}{3}R(\mathcal{G}_{ab} - g_{ab}\text{tr}(\mathcal{G})). \end{aligned} \quad (12b)$$

One can check that $\mathcal{C}^{\text{CS}} = g^{ab}\mathcal{C}_{ab}^{\text{CS}} = 0$.

B. Decoupling limit

The biscalar-tensor theories of gravity we focus on likely lack a well-posed formulation of the initial value problem when considered as a “complete” theory. This is mainly due to the axion sector of the model, which has been conjectured to be ill-posed outside the EFT treatment [107]. Thus, we work in the decoupling approximation of the theory in which we neglect the backreaction of the axion and dilaton fields onto the spacetime metric. Then, the background spacetime is determined by Einstein’s equations in vacuum and can be a stationary or a time-dependent GR solution. This approximation enables us to investigate the (pseudo)scalar fields’ dynamics in a range of interesting

scenarios: the formation of (pseudo)scalar BH hairs and their scalar radiation generated in BBHs or the onset of BH scalarization and dynamical (de)scalarization in BBHs. In the decoupling approximation, the field equations (5) reduce to

$$\bar{\square}\Theta - \dot{V}_\Theta + \frac{\alpha_{\text{CS}}\dot{f}_\Theta}{4g_\Phi^2}\bar{\mathcal{R}}_{\text{CS}} + 2\frac{g'_\Phi}{g_\Phi}\bar{\nabla}_\mu\Phi\bar{\nabla}^\mu\Theta = 0, \quad (13a)$$

$$\bar{\square}\Phi - V'_\Phi + \frac{\alpha_{\text{GB}}}{4}f'_\Phi\bar{\mathcal{R}}_{\text{GB}} - g'_\Phi g_\Phi[(\bar{\nabla}\Theta)^2 + 2V_\Theta] = 0, \quad (13b)$$

$$\bar{R}_{ab} - \frac{1}{2}\bar{g}_{ab}\bar{R} = 0, \quad (13c)$$

where the bar denotes operators and tensors constructed from the GR background metric, $\bar{g}_{\mu\nu}$, and its derivatives. In the decoupling limit, in a vacuum GR background for which $\bar{R}_{ab} = 0$, one can express the Pontryagin density and Gauss–Bonnet invariant, Eqs. (10), as

$$\bar{\mathcal{R}}_{\text{CS}} = {}^*\bar{W}_{abcd}\bar{W}^{abcd}, \quad \bar{\mathcal{R}}_{\text{GB}} = \bar{W}_{abcd}\bar{W}^{abcd}. \quad (14)$$

For the remainder of the paper, we drop the bar to avoid cumbersome notation. The curvature tensors, derivatives, and metric refer to the background, i.e., vacuum GR quantities.

To further simplify the discussion, we refer to both the pseudoscalar, Θ , and scalar, Φ , as “scalars.”

III. BISCALAR-TENSOR MODELS OF GRAVITY

The field equations in the decoupling approximation, Eqs. (13), represent a generalized scalar-tensor model of gravity in which (up to two) scalars are minimally or nonminimally coupled to gravity. The generalized equations reduce to well-known models of modified gravity through simple parameter choices. In the following, we briefly describe the different models, determine their field equations, and identify the appropriate choice of parameters and coupling functions for the model selection; see also Table II in Appendix A on how to select them in the CANUDA–AxiDil code.

A. Single minimally coupled scalar field

The simplest case included in Eqs. (13) is that of a single scalar field Φ , with a potential $V_\Phi(\Phi)$, minimally coupled to gravity. This corresponds to setting

$$\Theta = 0, \quad f_\Phi = 0, \quad f_\Theta = V_\Theta = 0. \quad (15)$$

Then, the action in Eq. (1) reduces to

$$S = \kappa \int d^4x \sqrt{-g} \left({}^{(4)}R - \frac{1}{2}(\nabla\Phi)^2 - V_\Phi \right), \quad (16)$$

and the scalar's equation of motion reduces to the Klein-Gordon equation,

$$\square\Phi - V'_\Phi = 0. \quad (17)$$

B. Dynamical Chern-Simons gravity

Staying with single scalar field models, we now consider the special case of dCS gravity. Here, the axion, Θ , is nonminimally coupled to gravity via the Pontryagin density $\mathcal{R}_{\text{CS}}$, while the dilaton field vanishes. This corresponds to setting

$$\Phi = 0, \quad f_\Phi = V_\Phi = 0, \quad g_\Phi = 1. \quad (18)$$

With this choice, the action in Eq. (1) reduces to

$$S = \kappa \int d^4x \sqrt{-g} \left({}^{(4)}R - \frac{1}{2}(\nabla\Theta)^2 - V_\Theta + \frac{\alpha_{\text{CS}}}{4} f_\Theta \mathcal{R}_{\text{CS}} \right). \quad (19)$$

The axion's field equation becomes

$$\square\Theta - \dot{V}_\Theta + \frac{\alpha_{\text{CS}}}{4} \dot{f}_\Theta \mathcal{R}_{\text{CS}} = 0. \quad (20)$$

The choices of coupling function f_Θ available in the numerical implementation are

$$\text{Shift-symmetric: } f_\Theta = \Theta, \quad (21a)$$

$$\text{Cubic: } f_\Theta = \Theta^3. \quad (21b)$$

The first choice yields a shift-symmetric coupling function and is the standard for dCS gravity [27]. In principle, one could also consider higher-order polynomials as in the second choice, Eq. (21b), that might yield new phenomena like spontaneous scalarization of BHs [64].

Typically, the potential V_Θ is set to zero. However, symmetry breaking in the axion section may introduce a small mass term [65–67]. The latter choice is also available in our numerical implementation.

C. Scalar-Gauss-Bonnet gravity

Our model of quadratic gravity, furthermore, includes sGB gravity as a special case. Here, a single scalar field is nonminimally coupled to gravity via the Gauss-Bonnet invariant, $\mathcal{R}_{\text{GB}}$. This corresponds to the choice

$$\Theta = 0, \quad f_\Theta = V_\Theta = 0. \quad (22)$$

Then, the action in Eq. (1) reduces to

$$S = \kappa \int d^4x \sqrt{-g} \left({}^{(4)}R - \frac{1}{2}(\nabla\Phi)^2 - V_\Phi + \frac{\alpha_{\text{GB}}}{4} f_\Phi \mathcal{R}_{\text{GB}} \right). \quad (23)$$

The resulting scalar field equation of motion is

$$\square\Phi - V'_\Phi + \frac{\alpha_{\text{GB}}}{4} f'_\Phi \mathcal{R}_{\text{GB}} = 0. \quad (24)$$

This includes the original Einstein-dilaton-Gauss-Bonnet (EdGB) gravity model [28,108–111], the widely used shift-symmetric sGB model [112,113], as well as models with quadratic coupling functions that allow for scalarized BHs [34,35]. The respective choices are

$$\text{Dilaton: } f_\Phi = e^{\lambda_{\text{GB}}\Phi}, \quad (25a)$$

$$\text{Shift-symmetric: } f_\Phi = \Phi, \quad (25b)$$

$$\text{Quadratic: } f_\Phi = \Phi^2, \quad (25c)$$

where λ_{GB} is a dimensionless constant. Note that the sign conventions in Refs. [28,114] correspond to $\lambda_{\text{GB}} = +1$ in our model. Typically, the potential is set to $V_\Phi = 0$, although including a mass term may yield new interesting phenomena [115].

D. Minimally coupled biscalar field model

Next, let us consider a model in which both the axion Θ and dilaton Φ fields are present and coupled to each other, but they are minimally coupled to gravity. The latter is achieved by setting

$$f_\Theta = 0, \quad f_\Phi = 0. \quad (26)$$

Then, the action becomes

$$S = \kappa \int d^4x \sqrt{-g} \left({}^{(4)}R - \frac{1}{2}(\nabla\Phi)^2 - V_\Phi - g_\Phi^2 \left[\frac{1}{2}(\nabla\Theta)^2 + V_\Theta \right] \right), \quad (27)$$

where the function g_Φ determines the coupling between the two fields. The resulting field equations are

$$\square\Theta - \dot{V}_\Theta + 2 \frac{g'_\Phi}{g_\Phi} \nabla_\mu \Phi \nabla^\mu \Theta = 0, \quad (28a)$$

$$\square\Phi - V'_\Phi - g'_\Phi g_\Phi [(\nabla\Theta)^2 + 2V_\Theta] = 0. \quad (28b)$$

The action (27) is a special case of the more general multiscalar-tensor theories [116,117]. In these models, multiple scalars are coupled via a target-space metric, which reduces to $\text{diag}(1, g_\Phi^2)$ in our case {e.g., compare with Eq. (2.1) in [117]}. Furthermore, the biscalar field action can describe dark energy models investigated in cosmological contexts [98–101].

E. Axi-dilaton gravity

The generalized quadratic gravity action, Eq. (1), also includes axi-dilaton gravity [23,24] and corresponds to

$$\begin{aligned} f_\Phi &= e^{\lambda_{\text{GB}}\Phi}, & g_\Phi &= e^{-\lambda_{\text{AD}}\Phi}, & f_\Theta &= \Theta, \\ V_\Phi &= 0, & V_\Theta &= 0, \end{aligned} \quad (29)$$

where λ_{GB} and λ_{AD} are dimensionless constants. We typically set $\lambda_{\text{GB}} = \lambda_{\text{AD}} = \lambda = +1$ in our simulations [111,118]. In axi-dilaton gravity, the coupling constants are related as $\alpha_{\text{CS}} = \alpha_{\text{GB}} = \alpha_s$.

Inserting the model parameters into Eq. (1), and keeping the parameter λ , the axi-dilaton action becomes

$$\begin{aligned} S = \kappa \int d^4x \sqrt{-g} & \left({}^{(4)}R - \frac{1}{2}(\nabla\Phi)^2 - \frac{1}{2}e^{-2\lambda\Phi}(\nabla\Theta)^2 \right. \\ & \left. + \frac{\alpha_s}{4}(e^{\lambda\Phi}\mathcal{R}_{\text{GB}} + \Theta\mathcal{R}_{\text{CS}}) \right). \end{aligned} \quad (30)$$

Note that α_s is a dimensionful coupling constant with units of length squared, $[\alpha_s] = [L]^2$. For the numerical implementation, it is useful to introduce dimensionless parameter, $\hat{\alpha}_s = (\alpha_s/M^2)$. Then, the axi-dilaton field equations evaluated on a GR background are given by

$$\square\Theta + \frac{\hat{\alpha}_s M^2}{4} e^{2\lambda\Phi} \mathcal{R}_{\text{CS}} - 2\lambda \nabla_\mu \Phi \nabla^\mu \Theta = 0, \quad (31a)$$

$$\square\Phi + \frac{\hat{\alpha}_s M^2 \lambda}{4} e^{\lambda\Phi} \mathcal{R}_{\text{GB}} + \lambda e^{-2\lambda\Phi} (\nabla\Theta)^2 = 0. \quad (31b)$$

The present paper focuses on the evolution of single and binary BHs in axi-dilaton gravity; i.e., we adopt Eqs. (31) for the remainder of this paper unless stated otherwise.

IV. TIME EVOLUTION FORMULATION

In this section, we present the time evolution formulation for our numerical simulations. We describe the background spacetime in Sec. IV A, provide the scalar field initial data in Sec. IV B, and present the 3 + 1 decomposition of the field equations in Sec. IV C.

A. Background spacetime

To conduct the numerical simulations of the scalar fields, we rewrite Eqs. (13) as a time evolution problem. We obtain the time evolution formulation by foliating the four-dimensional spacetime $(\mathcal{M}, g_{ab})$ into a set of three-dimensional, spatial hypersurfaces (Σ_t, γ_{ij}) . Each hypersurface Σ_t is labeled by the time parameter t , and its

²We note that the model in Eq. (29) reduces to the one derived in [24] when $2\alpha_{\text{GB}} = -2\alpha_{\text{CS}} = \alpha'$ and $\lambda = -1$ as the authors use the signature $(+, -, -, -)$.

geometry is determined by the induced (or spatial) metric $\gamma_{ab} = g_{ab} + n_a n_b$. Here, n^a is the timelike unit vector normal to the hypersurface, $\gamma^a_b n^b = 0$, with normalization $n^a n_a = -1$. Furthermore, the spatial metric defines a projection operator

$$\gamma^a_b = \delta^a_b + n^a n_b. \quad (32)$$

The line element takes the form

$$\begin{aligned} ds^2 &= g_{ab} dx^a dx^b \\ &= -(\alpha^2 - \beta^k \beta_k) dt^2 + 2\gamma_{ij} \beta^i dt dx^j + \gamma_{ij} dx^i dx^j, \end{aligned} \quad (33)$$

where α and β^i are, respectively, the lapse function and shift vector. We denote the covariant derivative and Ricci tensor associated with the 3-metric γ_{ij} as D_i and R_{ij} . The extrinsic curvature K_{ab} describes how a hypersurface is embedded in the spacetime manifold and is defined as

$$K_{ab} = -\gamma^c_a \gamma^d_b \nabla_c n_d = -\frac{1}{2} \mathcal{L}_n \gamma_{ab}, \quad (34)$$

where $\mathcal{L}_n$ is the Lie derivative along n^a .

We simulate the formation and time evolution of axion and dilaton fields in the background of (1) a single rotating BH and (2) a dynamical spacetime consisting of a BBH coalescence. Please refer to Sec. II B of Ref. [67] and Sec. III B of Ref. [74], respectively, for details.

B. Scalar field initial data

In this paper, we focus on two types of initial data for the axion and dilaton fields. They are either initialized as zero to follow the formation of axion and dilaton hairs, or they are initialized by the small-spin, small-coupling approximation [24] to determine their new end states for large BH spins and large coupling strengths. We also demonstrate the robustness of the end state and its independence from the initial data.

The CANUDA-AxiDil code provides additional choices, such as Gaussian profiles or quasibound states, and we present a complete list in Appendix A 2. Furthermore, the initial data profiles can be mixed and matched for the axion and dilaton fields; i.e., the fields do not have to have the same type of initial data.

Type I: Zero fields The first choice of initial data initializes both the axion and dilaton to be zero,

$$\Theta|_{t=0} = 0, \quad K_\Theta|_{t=0} = 0, \quad (35a)$$

$$\Phi|_{t=0} = 0, \quad K_\Phi|_{t=0} = 0. \quad (35b)$$

Type II: Axion and dilaton hairs in the small-spin, small-coupling approximation The second choice of initial data for the axion and dilaton implements the analytical solutions around single BHs in axi-dilaton gravity in the

small-spin, small-coupling approximation [24]. The fields' conjugated momenta vanish initially, as follows from their definition (41) and the symmetries of the background spacetime.

Then, the initial axion profile in spherical-polar coordinates $\{r, \theta, \phi\}$ is given by

$$\Theta|_{t=0} \simeq 2\hat{\alpha}_s \chi \cos \theta [\vartheta_1 - \chi^2(\vartheta_2 + \vartheta_3 \cos^2 \theta)], \quad (36a)$$

$$K_\Theta|_{t=0} = 0, \quad (36b)$$

where we neglected terms of order $\mathcal{O}(\hat{\alpha}_s \chi^5)$ and $\mathcal{O}(\hat{\alpha}_s^2 \chi)$ and introduced

$$\vartheta_1 = \left(\frac{5M^2}{16r_{\text{BL}}^2} + \frac{5M^3}{8r_{\text{BL}}^3} + \frac{9M^4}{8r_{\text{BL}}^4} \right), \quad (37a)$$

$$\vartheta_2 = \left(\frac{M^2}{32r_{\text{BL}}^2} + \frac{M^3}{16r_{\text{BL}}^3} + \frac{3M^4}{40r_{\text{BL}}^4} + \frac{M^5}{20r_{\text{BL}}^5} \right), \quad (37b)$$

$$\vartheta_3 = \left(\frac{3M^4}{8r_{\text{BL}}^4} + \frac{3M^5}{2r_{\text{BL}}^5} + \frac{25M^6}{6r_{\text{BL}}^6} \right). \quad (37c)$$

Here, r_{BL} denotes the Boyer-Lindquist radial coordinate. It is related to the isotropic radial coordinate,

$$r_{\text{BL}} = r \left(1 + \frac{r_{\text{BL},+}}{4r} \right)^2. \quad (38)$$

Here, $r_{\text{BL},+}/M = 1 + \sqrt{1 - \chi^2}$ denotes the outer horizon in Boyer-Lindquist coordinates, and it corresponds to $r_+ = r_{\text{BL},+}/4$ in isotropic radial coordinates. The inner horizon, $r_{\text{BL},-}/M = 1 - \sqrt{1 - \chi^2}$, is not covered by the isotropic coordinate chart.

The dilaton's initial profile is given by

$$\Phi|_{t=0} \simeq 2\lambda \hat{\alpha}_s [\varphi_0 - \chi^2(\varphi_1 + \varphi_2 \cos^2 \theta)] \quad (39a)$$

$$- \chi^4(\varphi_3 - \varphi_4 \cos^2 \theta - \varphi_5 \cos^4 \theta)], \quad (39b)$$

$$K_\Phi|_{t=0} = 0, \quad (39c)$$

where we neglected terms of order $\mathcal{O}(\hat{\alpha}_s \chi^6)$ and $\mathcal{O}(\hat{\alpha}_s^2)$ and introduced

$$\varphi_0 = \left(\frac{M}{4r_{\text{BL}}} + \frac{M^2}{4r_{\text{BL}}^2} + \frac{M^3}{3r_{\text{BL}}^3} \right), \quad (40a)$$

$$\varphi_1 = \left(\frac{M}{16r_{\text{BL}}} + \frac{M^2}{16r_{\text{BL}}^2} + \frac{M^3}{20r_{\text{BL}}^3} + \frac{M^4}{40r_{\text{BL}}^4} \right), \quad (40b)$$

$$\varphi_2 = \left(\frac{7M^3}{20r_{\text{BL}}^3} + \frac{21M^4}{20r_{\text{BL}}^4} + \frac{12M^5}{5r_{\text{BL}}^5} \right), \quad (40c)$$

$$\varphi_3 = \left(\frac{M}{32r_{\text{BL}}} + \frac{M^2}{32r_{\text{BL}}^2} + \frac{3M^3}{112r_{\text{BL}}^3} + \frac{M^4}{56r_{\text{BL}}^4} + \frac{M^5}{140r_{\text{BL}}^5} \right), \quad (40d)$$

$$\varphi_4 = \left(\frac{M^3}{56r_{\text{BL}}^3} + \frac{3M^4}{56r_{\text{BL}}^4} + \frac{3M^5}{35r_{\text{BL}}^5} + \frac{M^6}{14r_{\text{BL}}^6} \right), \quad (40e)$$

$$\varphi_5 = \left(\frac{11M^5}{28r_{\text{BL}}^5} + \frac{55M^6}{28r_{\text{BL}}^6} + \frac{45M^7}{7r_{\text{BL}}^7} \right). \quad (40f)$$

As a reminder, we introduce the dimensionless constant λ in Eq. (30) and typically set $\lambda = +1$. In passing, we also note that Eqs. (36a) and (39a) can be generalized by introducing dimensionless coupling parameters $\hat{\alpha}_{\text{CS}} = \alpha_{\text{CS}}/M^2$ and $\hat{\alpha}_{\text{GB}} = \alpha_{\text{GB}}/M^2$ such that $\hat{\alpha}_{\text{CS}} \neq \hat{\alpha}_{\text{GB}}$; please refer to Appendix C for more details.

C. Axi-dilaton scalar evolution equations

In this project, we adopt the axi-dilaton model of gravity presented in Sec. III E. We rewrite the scalar field equations, Eqs. (5a) and (5b), in terms of the 3 + 1 variables introduced in the previous subsection and the conjugated momenta of the scalar fields

$$K_\Theta := -\mathcal{L}_n \Theta, \quad K_\Phi := -\mathcal{L}_n \Phi. \quad (41)$$

This procedure yields evolution equations for the scalars and their conjugated momenta,

$$d_t \Theta = -\alpha K_\Theta, \quad (42a)$$

$$d_t \Phi = -\alpha K_\Phi, \quad (42b)$$

$$\begin{aligned} d_t K_\Theta &= -\alpha D^i D_i \Theta - D^i \alpha D_i \Theta \\ &\quad + \alpha \left(K K_\Theta - \frac{\hat{\alpha}_s M^2 \lambda}{4} e^{2\lambda \Phi} \mathcal{R}_{\text{CS}} \right) \\ &\quad - 2\alpha \lambda \left(K_\Theta K_\Phi - D_i \Theta D^i \Phi \right), \end{aligned} \quad (42c)$$

$$\begin{aligned} d_t K_\Phi &= -\alpha D^i D_i \Phi - D^i \alpha D_i \Phi \\ &\quad + \alpha \left(K K_\Phi - \frac{\hat{\alpha}_s M^2 \lambda}{4} e^{\lambda \Phi} \mathcal{R}_{\text{GB}} \right) \\ &\quad + \alpha \lambda e^{-2\lambda \Phi} \left(K_\Theta^2 - D_i \Theta D^i \Theta \right), \end{aligned} \quad (42d)$$

where $d_t = (\partial_t - \mathcal{L}_\beta)$, and $\mathcal{L}_\beta$ is the Lie derivative along the shift vector. For the evolution equations with general coupling functions f_Θ, f_Φ, g_Φ , see Appendix C.

The topological invariants of the background spacetime, $\mathcal{R}_{\text{GB}}$ and $\mathcal{R}_{\text{CS}}$, can be written as

$$\mathcal{R}_{\text{CS}} = -16E^{ij}B_{ij}, \quad (43a)$$

$$\mathcal{R}_{\text{GB}} = 8(E^{ij}E_{ij} - B^{ij}B_{ij}), \quad (43b)$$

where we have introduced the gravito-electric, $E_{ij} = \gamma^a{}_i \gamma^b{}_j n^c n^d W_{acbd}$, and the gravito-magnetic, $B_{ij} = \gamma^a{}_i \gamma^b{}_j n^c n^d {}^*W_{acbd}$, parts of the Weyl tensor.

In the decoupling approximation, and focusing on a vacuum GR background, the gravito-electro and gravito-magnetic fields, E_{ij} and B_{ij} , become

$$E_{ij} = R_{ij}^{\text{tf}} + \frac{1}{3}A_{ij}K - A_i{}^k A_{jk} + \frac{1}{3}\gamma_{ij}A_{kl}A^{kl}, \quad (44a)$$

$$B_{ij} = -\epsilon_{(i}{}^{kl}D_{l}A_{j)k}. \quad (44b)$$

The components of the effective energy-momentum tensor associated with biscalar-tensor models of gravity are summarized in Appendix C.

V. NUMERICAL RELATIVITY FRAMEWORK

In this section, we provide a detailed description of our open-source numerical relativity framework CANUDA–AxiDil and describe the single and binary BH simulations that we perform.

A. Code description

The core infrastructure for our numerical experiments is provided by the EINSTEIN TOOLKIT [119–121], an open-source numerical relativity code for computational astrophysics. The EINSTEIN TOOLKIT is based on the Cactus computational toolkit [122,123] and the Carpet driver [124,125] to provide box-in-box adaptive mesh refinement and MPI parallelization. We pair the EINSTEIN TOOLKIT with CANUDA [74,103,126,127], our open-source code for fundamental physics.

As a new extension to CANUDA’s capabilities, we present the CANUDA–AxiDil [102] module here. It provides a framework for parametrized numerical relativity that incorporates a large class of quadratic gravity models and of biscalar fields that are minimally or nonminimally coupled to gravity. CANUDA–AxiDil implements the field equations (13a) and (13b) and enables the selection of the different models presented in Sec. II via simple parameter choices; see details in Table II of Appendix A. The evolution equations are implemented using up to sixth-order finite-differences for the spatial derivatives. Our standard choice for the time integration is a fourth-order Runge-Kutta scheme. Up to seventh order Kreiss-Oliger dissipation is employed to reduce the numerical noise produced at the refinement boundaries.

The CANUDA–AxiDil module consists of three thorns. The AxiDil-Base thorn sets up the necessary grid functions. The AxiDil-Init thorn provides a variety of initial axion and dilaton profiles. They represent zero fields or approximate analytical solutions as introduced in Sec. IV B. In Appendix A 2, we list additional initial data choices that are available in the code but have not been used in this paper. The time evolution of the scalar field sector is conducted by the AxiDil-Evol thorn, which implements the equations (13a) and (13b) in the Baumgarte-Shapiro-Shibata-Nakamura (BSSN) formulation given in Appendix C. This thorn provides all models described in Sec. III. The parameter settings to select a specific model are summarized in Appendix A 1.

For simulations of single BHs, the background metric is fixed and consists of a stationary Kerr BH in quasi-isotropic coordinates. The KerrQuasIsotropic thorn from the CANUDA library provides the appropriate metric and curvature initial data. For fixed backgrounds, we introduce a parameter called `set_lapse_to_zero` in the AxiDil-Init thorn that enables setting the lapse to zero inside the horizon, $\alpha(r \leq r_+) = 0$. We typically set this parameter to “yes” to minimize numerical growth arising from nonphysical gauge effects.

For the BBH simulations, the BH initial data is produced by TwoPunctures, a spectral solver for puncture data [128]. In this case, the metric is dynamical and evolved according to Einstein’s equations in vacuum. Specifically, we evolve Einstein’s equations in the BSSN formulation as implemented in CANUDA’s spacetime evolution solver LeanBSSNMOL [103], adapted from the Lean code [129]. For both the scalars and the dynamical metric, we apply radiative boundary conditions using the NewRad thorn [130].

To compute the (l, m) multipoles of Θ and Φ , we use the Multipole thorn. The grid functions representing the axion or dilaton field are interpolated onto spheres of fixed extraction radii r_{ex} and projected onto $s = 0$ spherical harmonics $Y_{lm}(\theta, \phi)$,

$$\Theta_{lm}(t, r_{\text{ex}}) = \int d\Omega \Theta(t, r_{\text{ex}}, \theta, \phi) Y_{lm}^*(\theta, \phi), \quad (45)$$

and likewise for the field Φ . We visualize our numerical data with KUBIT [131], a set of tools for postprocessing data generated with the EINSTEIN TOOLKIT.

B. Simulation setup

In Table I, we list all simulations performed in the background of a single BH with dimensionless spin parameter χ . We give the axion and dilaton initial data (see Sec. IV B), the model of gravity considered (see Sec. III), and the dimensionless coupling constant $\hat{\alpha}_s$.

For single BH simulations with a coupling strength $\hat{\alpha}_s < 0.1$, we employ a fully Cartesian box-in-box mesh refinement with the BH placed at the origin. In this case, the

TABLE I. Simulations of axion and dilaton fields about a single BH with dimensionless spin χ . We initialize the fields either as zero (Type I in Sec. IV B) or as the approximate analytical solution (Type II in Sec. IV B). The first series of runs evolves axi-dilaton gravity; i.e., the axion and dilaton are kinetically coupled (Model E in Sec. III) with coupling strength $\hat{\alpha}_s$. The second series simulates the axion and dilaton decoupled from each other according to, respectively, dCS gravity with a linear coupling function and sGB gravity with a dilaton coupling function (Models B and C in Sec. III).

Run	Initial data	Model	$\hat{\alpha}_s$	χ
IDI_AD_chi01	Type I	E	0.1	0.1
IDI_AD_chi03	Type I	E	0.1	0.3
IDI_AD_chi05	Type I	E	0.1	0.5
IDI_AD_chi07	Type I	E	0.1	0.7
IDI_AD_chi09	Type I	E	0.1	0.9
IDII_AD_chi01	Type II	E	{0.001, 0.01, 0.1}	0.1
IDII_AD_chi03	Type II	E	{0.001, 0.01, 0.1}	0.3
IDII_AD_chi05	Type II	E	{0.001, 0.01, 0.1}	0.5
IDII_AD_chi07	Type II	E	{0.001, 0.01, 0.1}	0.7
IDII_AD_chi09	Type II	E	{0.001, 0.01, 0.1}	0.9
IDII_dec_chi01	Type II	B&C	0.1	0.1
IDII_dec_chi03	Type II	B&C	0.1	0.3
IDII_dec_chi05	Type II	B&C	0.1	0.5
IDII_dec_chi07	Type II	B&C	0.1	0.7
IDII_dec_chi09	Type II	B&C	0.1	0.9

outermost level has a radius of $128 M$ and a grid spacing of $1 M$. We have a total of seven refinement levels with each inner refinement level halving the grid spacing. The finest grid has a radius of $0.75 M$ and a grid spacing of $1/64 M$.

The remaining single BH simulations with a larger coupling strength of $\hat{\alpha}_s = 0.1$ are more demanding, so we use a box-in-box mesh refinement combined with a spherical multipatch grid provided by the `Llama thorn` [132,133]. We set up a total of eight Cartesian refinement levels with the coarsest level having an outer radius of $24 M$ and a grid spacing of $1 M$. The innermost (finest) refinement level has a radius of $0.75 M$ and a grid spacing of $1/128 M$. The inner boundary of the spherical grid is located at $24 M$, and the outer boundary is at $200 M$ with an angular resolution of 64 points.

For the BBH simulation, we set our initial punctures to be separated by $6 M$ and use `TwoPunctures` [128] to solve for the initial data. The momenta are selected to produce a quasicircular orbit. The initial BHs are nonspinning and have equal masses, $m_{1,2} = 0.5$, where $m_{1,2}$ refer to the individual ADM masses of the punctures as defined in [128]. The grid is fully Cartesian with a total radius of $256 M$ and a coarsest grid spacing of $\sim 1 M$. We use a total of eight refinement levels with the finest refinement level having a radius of $0.75 M$ and a grid spacing of $\sim 1/128 M$.

C. Code verification

To verify our code, we benchmark simulations in dCS gravity with the `CANUDA-dCS` module [67] and benchmark simulations in EdGB gravity with the `CANUDA-EdGB` module [74,127]. For the same physical parameters and simulation setup, we find that our simulations in dCS gravity agree within 0.04% with the `CANUDA-dCS` module, and our simulations in EdGB gravity agree within 0.3% with the `CANUDA-EdGB` module. Note that we adopt the default parameter setting `set_lapse_to_zero = no` to match the `CANUDA-EdGB` module.

To confirm the validity of the simulations and to estimate their numerical error, we perform convergence tests for both the single and binary BHs. For the single BH results, we show convergence for the run `IDII_AD_chi09` with coupling strength $\hat{\alpha}_s = 0.1$ and present our results in Appendix B 3. We find third order convergence, and discretization errors at late times ($t > 150 M$) of $\Delta\Theta_{10}/\Theta_{10,h} < 0.1\%$ and $\Delta\Phi_{00}/\Phi_{00,h} < 0.02\%$ for the leading-order contributions to the axion and dilaton fields, respectively. For the BBH results, our convergence tests are shown in Appendix B 4. We find approximately third order convergence for the BBH simulation as well.

VI. FORMATION OF HAIRY BLACK HOLES

In this section, we present our numerical results of the axion and dilaton fields evolving in the background of a single BH in axi-dilaton gravity. We work in the decoupling approximation and capture the dynamical growth of the axion and dilaton hairs as proposed in sketch 1. The simulations presented here assist us to address the following questions: (i) What is the effect of BH spin on the axion and dilaton hairs in axi-dilaton gravity? (ii) How does the strength of the couplings to curvature in axi-dilaton gravity affect the axion and dilaton hairs? (iii) How does the kinetic coupling between the two fields affect their respective evolution as compared to their evolution in isolation? Sections VI A–VI C address each of these questions, respectively.

Many of the results presented in the following sections focus on the evolution of the Θ_{10} and Φ_{00} multipoles, defined in Eq. (45), because they are the leading-order contribution to the axion and dilaton hairs. This can be seen by rewriting the perturbative solutions of Eqs. (36) and (39) in terms of their leading harmonic content,

$$\Theta \simeq \sqrt{\frac{16\pi}{3}} [\Theta_{10}(r)Y_{10} + \Theta_{30}(r)Y_{30} + \dots], \quad (46a)$$

$$\Phi \simeq \sqrt{16\pi\lambda} [\Phi_{00}(r)Y_{00} + \Phi_{20}(r)Y_{20} + \dots], \quad (46b)$$

where $Y_{lm} \equiv Y_{lm}(\theta, \phi)$ are spherical harmonics. The fields' leading-order multipoles are

$$\Theta_{10} = \hat{\alpha}_s \chi \left[\vartheta_1 - \chi^2 \left(\vartheta_2 + \frac{3}{5} \vartheta_3 \right) \right] + \mathcal{O}(\hat{\alpha}_s \chi^4), \quad (47a)$$

$$\Theta_{30} = -\frac{2}{5} \sqrt{\frac{3}{7}} \hat{\alpha}_s \chi^3 \vartheta_3 + \mathcal{O}(\hat{\alpha}_s \chi^4), \quad (47b)$$

$$\Phi_{00} = \hat{\alpha}_s \left[\varphi_0 - \chi^2 \left(\varphi_1 + \frac{1}{3} \varphi_2 \right) \right] + \mathcal{O}(\hat{\alpha}_s \chi^4), \quad (47c)$$

$$\Phi_{20} = -\frac{2}{3\sqrt{5}} \hat{\alpha}_s \chi^2 \varphi_2 + \mathcal{O}(\hat{\alpha}_s \chi^4), \quad (47d)$$

where $\vartheta_i(r)$, $\varphi_i(r)$ are given by Eqs. (37) and (40).

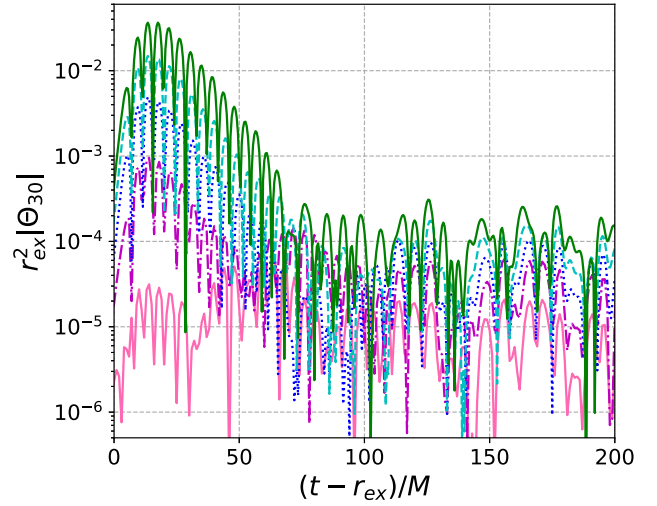
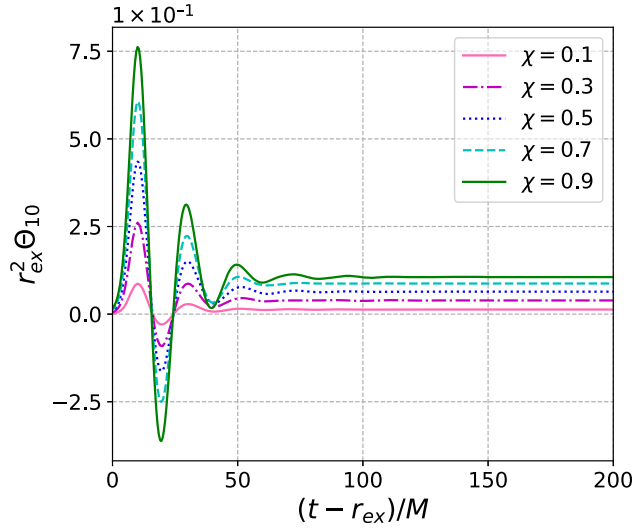


FIG. 2. Evolution of the axion's $l = 1, m = 0$ (left panel) and $l = 3, m = 0$ (right panel) multipoles extracted at $r_{\text{ex}} = 20M$. The axion is initialized as zero, coupled to gravity with coupling parameter $\hat{\alpha}_s = 0.1$ and evolves in the background of a rotating BH with dimensionless spin $\chi = 0.1$ (solid pink), $\chi = 0.3$ (dash-dotted magenta), $\chi = 0.5$ (dotted blue), $\chi = 0.7$ (dashed cyan), and $\chi = 0.9$ (solid green).

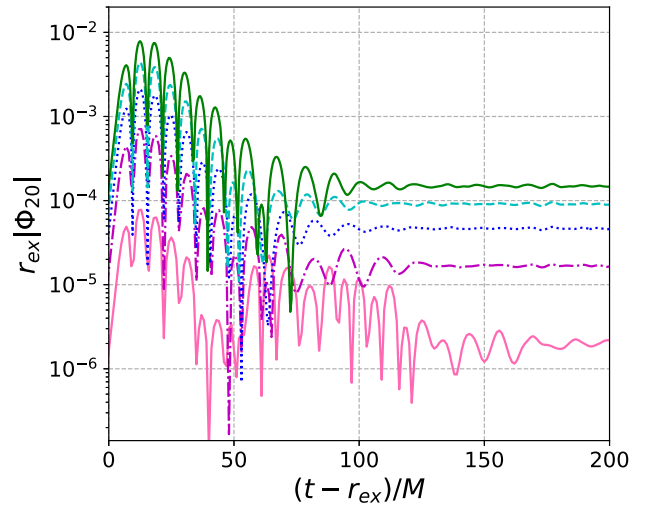
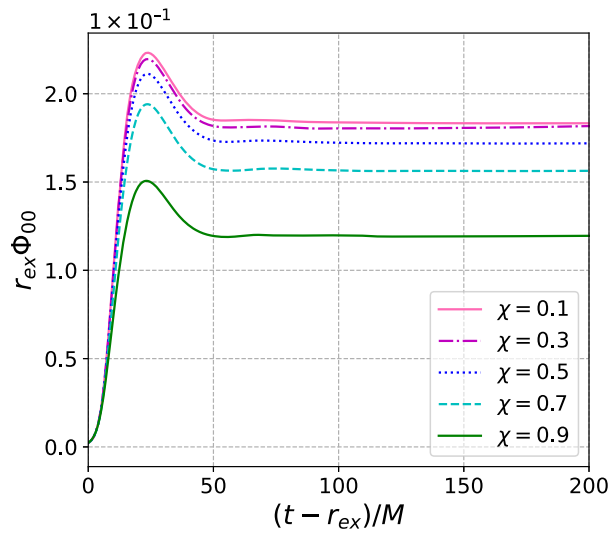


FIG. 3. Same as Fig. 2 but for the dilaton's $l = m = 0$ (left panel) and $l = 2, m = 0$ (right panel) multipoles.

A. Effect of black-hole spin on final hairs

In this section, we investigate how BH spin affects the growth, evolution, and end state of the axion and dilaton hairs. We concentrate on simulations with initially vanishing axion and dilaton fields. We show additional robustness checks using the approximate analytical initial data at the end of the section. We present the fields' time evolution in Figs. 2 and 3 and two-dimensional snapshots of their late-time profile in Figs. 4 and 5.

In Figs. 2 and 3, we show the evolution of the leading-order multipoles of the axion and dilaton hairs, Θ_{10} and Φ_{00} , in the left panels and the next-to-leading-order modes, Θ_{30} and Φ_{20} , in the right panels. We also evolved

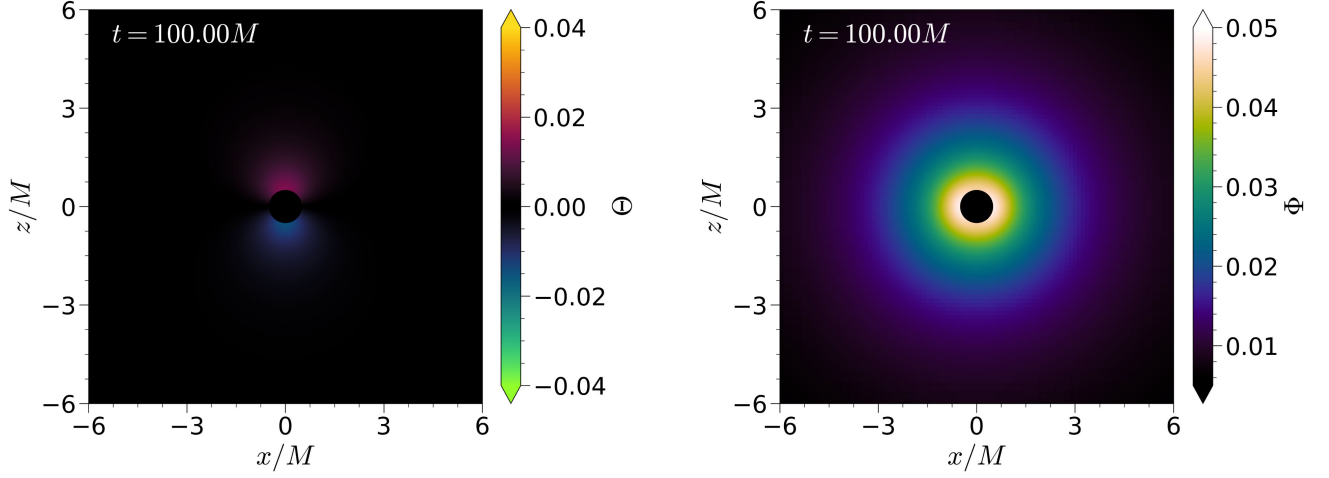


FIG. 4. Snapshot of axion (left) and dilaton (right) fields with coupling strength $\hat{\alpha}_s = 0.1$ evolving around a single BH with dimensionless spin $\chi = 0.3$ in the xz plane, i.e., the slice perpendicular to the equatorial plane, at $t = 100M$. The black circle indicates the BH's apparent horizon with a radius $r_h \sim 0.49M$ (in isotropic coordinates).

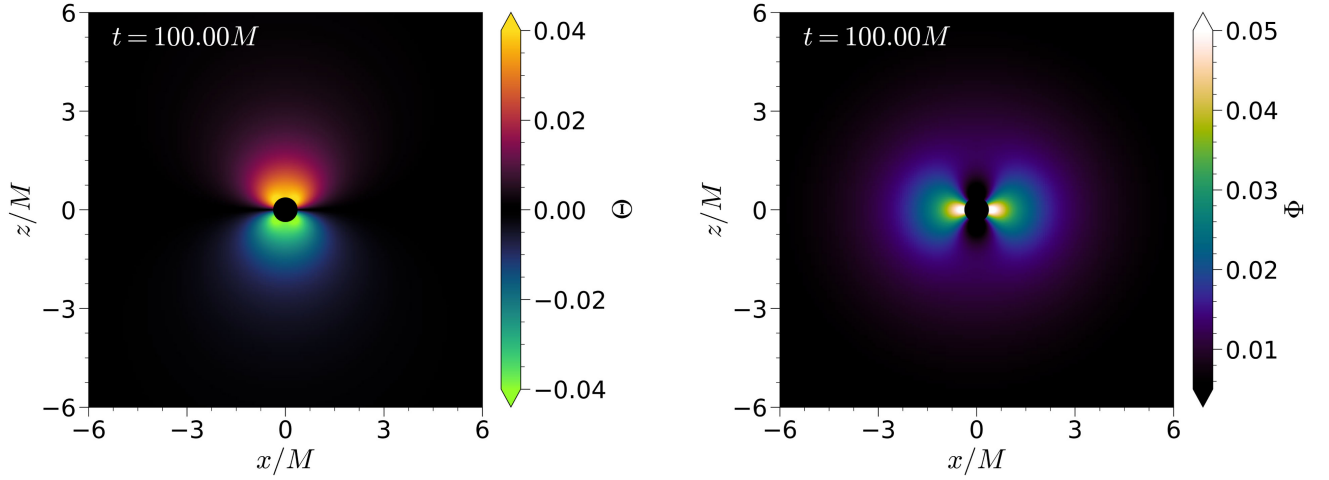


FIG. 5. Same as Fig. 4 but for a BH of dimensionless spin $\chi = 0.9$ and BH radius of $r_h \sim 0.36M$.

multipoles up to $l = 4$, but they are subdominant by several orders of magnitude, so we omit presenting them here. We fix the coupling strength $\hat{\alpha}_s = 0.1$ and show results for a range of BH spin parameters $\chi \in \{0.1, 0.3, 0.5, 0.7, 0.9\}$. The multipoles are extracted at $r_{\text{ex}} = 20M$, and we shift time by r_{ex} so the origin coincides with the start of the growth of the hairs.

Let us first focus on the growth of the Θ_{10} multipole in the left panel of Fig. 2 where Θ_{10} has been rescaled by $r_{\text{ex}}^2 = (20M)^2$ to show the leading-order contribution to the axion hair independent of the extraction radius. As the spin increases, the axion hair initially fluctuates more than in simulations with smaller BH spins because the initial data is further from the end state. Additionally, as the BH spin increases, the final axion hair increases, indicating a proportional relationship between the axion hair and the BH spin that is consistent with the approximate analytical solution given by Eq. (47a).

Next we consider the dilaton field and focus on the evolution of the Φ_{00} multipole in the left panel of Fig. 3. We rescale Φ_{00} by $r_{\text{ex}} = 20M$ to present the leading-order contribution to the dilaton hair independent of the extraction radius. After the initial adjustment, the dilaton hair settles inversely proportional to the BH spin; as the BH spin increases, the dilaton hair decreases. This numerical result is consistent with the analytical approximation for the dilaton hair given by Eq. (47c) as spin appears at second order with a minus sign, indicating a decrease in dilaton hair with increasing spin.

Combining the discussion of Figs. 2 and 3, we find that the axion and dilaton fields—for sufficiently small BH spins—dynamically reach configurations that are consistent with the perturbative solutions. For larger BH spins, the fields dynamically adjust to a new end state that captures next-to-leading-order effects in both the spin and coupling strength.

Up to this point, we have focused on the evolution of the dominant Θ_{10} and Φ_{00} multipoles. Let us now consider the evolution of the next-to-leading-order modes, Θ_{30} and Φ_{20} , in the right panels of Figs. 2 and 3.

In the right panel of Fig. 2, we see that, rather than briefly adjusting before settling close to the final value, the Θ_{30} mode rings down before reaching its final state. After the initial decay, we find that the magnitude of Θ_{30} increases with BH spin and roughly settles to $r_{\text{ex}}^2 \Theta_{30} \lesssim 10^{-4}$ for all BH spins. Thus, the Θ_{30} multipole is subdominant to the Θ_{10} multipole, and their ratio is consistent with Eq. (47) as $\Theta_{30}/\Theta_{10} \propto \chi^2 M^2 / r_{\text{BL}}^2 \lesssim 10^{-3}$.

Observing the evolution of Φ_{20} in the right panel of Fig. 3, we find that the magnitude of Φ_{20} increases with increasing BH spin. This behavior differs from the leading-order contribution to the dilaton field as $|\Phi_{00}|$ decreases with increasing spin as seen in the left panel of Fig. 3. This behavior is consistent with the small-spin, small-coupling approximation in Eqs. (47c) and (47d), respectively, but we here also capture the evolution for high BH spins. Since $|\Phi_{00}|$ decreases and $|\Phi_{20}|$ increases as BH spin increases, the higher mode $|\Phi_{20}|$ contributes more to the final dilaton hair of a highly rotating BH than that of a slowly rotating BH. Once the modes have settled, we compare the $|\Phi_{00}|$ mode with the $|\Phi_{20}|$ mode at $t = 150M$ and find $\Phi_{00}/\Phi_{20} = 1.08 \times 10^5$ for BH spin $\chi = 0.1$ and $\Phi_{00}/\Phi_{20} = 7.91 \times 10^2$ for BH spin $\chi = 0.9$. This not only visibly changes the dilaton's profile as we will see in the two-dimensional snapshots below, but it may also have important implications for the dilaton radiation sourced by spinning BBHs. In particular, the Φ_{00} and Φ_{20} multipoles source the dipolar and $l = 3$ radiative modes of the dilaton. Our results suggest a strong dependence of their excitation on the BHs' spins.

In the introduction, Sec. I, we wondered what profiles the axion and dilaton fields will evolve to; see the sketch in Fig. 1. We present two-dimensional snapshots from the numerical simulations in Figs. 4 and 5 to illustrate the fields' final states. The snapshots depict the axion (left panels) and dilaton (right panels) in the xz plane, i.e., along the BH's axis of rotation, evolved around a BH with spin $\chi = \{0.3, 0.9\}$ in Figs. 4 and 5, respectively.

For visualization purposes, we add a black circle indicating the BH, whose radius is given by the mean BH apparent horizon $r_h = \{0.49M, 0.36M\}$ (corresponding to $\chi = \{0.3, 0.9\}$). Comparing the left panels of Figs. 4 and 5, we see that the magnitude of the axion field, given by the green-orange colorbar, increases with increasing BH spin. The dipole shape along the axis of rotation remains, indicating the dominance of the $l = 1, m = 0$ mode in the axion's final profile in the xz plane.

We now compare the dilaton profiles in the xz plane in the right panels of Figs. 4 and 5. As the spin increases, the magnitude of the dilaton field, given by the black-white colorbar, decreases, consistent with the results presented in

Fig. 3. The field's shape also changes with increasing BH spin due to the relative magnitude of the subdominant Φ_{20} increasing with BH spin; see also our discussion of Fig. 3. For smaller BH spins such as $\chi = 0.3$, the dilaton field appears primarily as a monopole, i.e., dominated by the $l = m = 0$ mode. For higher BH spins, this monopole appears deformed as the $l = 2, m = 0$ mode becomes more excited and leaves an imprint in the dilaton field along the axis of rotation; for reference, the $l = 2, m = 0$ mode appears as a dipole along the axis of rotation with a donut in the equatorial plane.

To test the robustness of our results, we initialize a set of simulations with the approximate analytical solution given by initial data Type II in Sec. IV B and compare the results to those with zero initial data (Type I). We present the results in Figs. 8(c) and 9(c) for a range of BH spins $\chi \in \{0.1, 0.3, 0.5, 0.7, 0.9\}$. Note, that Figs. 8 and 9 also show results for smaller coupling constants that are discussed in the next section; for the comparison done here, we focus on the coupling strength of $\hat{\alpha}_s = 0.1$.

Compared to the simulations where the fields are initialized as zero (Type I), those with the approximate, analytic initial data (Type II) exhibit less adjustment before settling to their final state. Despite the differences in their early-time behavior, the end state for both types of initial data fall within 0.6% of each other, comparable to our numerical error. This indicates that our results are consistent and that the final axion and dilaton hairs are independent of the choice of initial data.

In summary, the results presented in this section demonstrate that BHs in axi-dilaton gravity do indeed acquire two scalar charges in the form of an axion and a dilaton hair. We show that the two hairs form dynamically even when they are absent in the initial data. Investigating the hairs' dependence on the BH spin, we find that both the leading-order and next-to-leading-order contributions to the axion hair, Θ_{10} and Θ_{30} , increase with increasing BH spin. In the case of the dilaton field, we find that the dilaton charge, $|\Phi_{00}|$, decreases as BH spin increases, while the next-to-leading-order contribution, $|\Phi_{20}|$, increases with the BH spin. This inverse relationship between Φ_{00} and Φ_{20} becomes most apparent for higher BH spins, as shown by the deformed "monopole" profile in the right panel of Fig. 5.

B. Effect of coupling strength on final hairs

In this section, we consider the effect of the dimensionless coupling strength, $\hat{\alpha}_s$, on the evolution of the axion and dilaton fields. We perform three sets of simulations in the range $\hat{\alpha}_s = \{0.001, 0.01, 0.1\}$ and vary the BH's spin parameter $\chi \in \{0.1, 0.3, 0.5, 0.7, 0.9\}$ for each value of the coupling strength. We initialize the axion and dilaton fields with the analytic result in the small-spin, small-coupling approximation, corresponding to Type II initial data in Sec. IV B. The analytic expressions given in

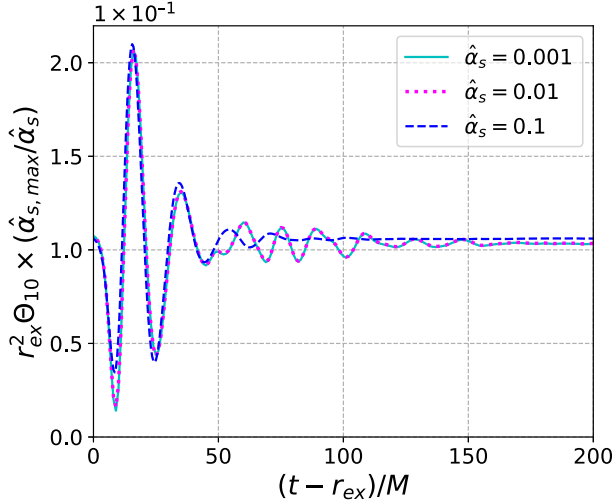


FIG. 6. Evolution of the Θ_{10} multipole in the background of a BH with spin $\chi = 0.9$. Θ_{10} is rescaled by $r_{\text{ex}}^2 = (20M)^2$ as well as the ratio between the maximum coupling strength $\hat{\alpha}_s = 0.1$ and the coupling strength used in the simulation, $\hat{\alpha}_s = 0.001$ (solid cyan), $\hat{\alpha}_s = 0.01$ (dotted magenta), and $\hat{\alpha}_s = 0.1$ (dashed blue).

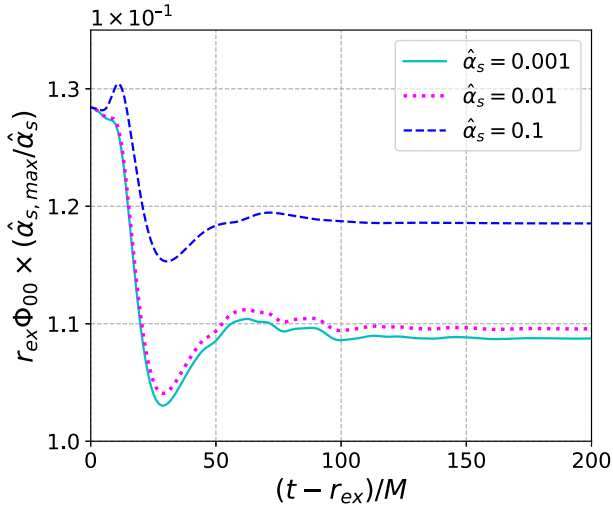


FIG. 7. Same as Fig. 6 but for Φ_{00} rescaled by $r_{\text{ex}} = 20M$.

Eqs. (36) and (39) scale linearly with the coupling strength. In the remainder of this section, we assess the significance of higher-order terms in coupling.

In Figs. 6 and 7, we illustrate the dependence of the axion and dilaton hairs on the coupling strength, focusing on simulations with a BH spin of $\chi = 0.9$. Then in Figs. 8 and 9, we present the hairs' dependence on both the coupling strength and the BH spin.

Figure 6 shows the evolution of Θ_{10} , rescaled by $\hat{\alpha}_{s,\text{max}}/\hat{\alpha}_s$ with $\hat{\alpha}_{s,\text{max}} = 0.1$, for all three coupling strengths. For the simulations with small couplings, $\hat{\alpha}_s \in \{0.001, 0.01\}$, we see that the final axion hairs overlap nearly perfectly. This is consistent with the linear scaling of the small-coupling approximation. However, the simulation

with the largest coupling strength, $\hat{\alpha}_s = 0.1$, differs by approximately 2%. While this difference is small, it is larger than the expected numerical error (see left panel of Fig. 18) and, thus, captures next-to-leading-order effects in the coupling strength, $\mathcal{O}(\hat{\alpha}_s^2)$. A more pronounced deviation from linear scaling with the coupling strength is seen in the dilaton shown in Fig. 7. Here, we present Φ_{00} , rescaled by $\hat{\alpha}_{s,\text{max}}/\hat{\alpha}_s$. Comparing the simulations with small coupling strengths $\hat{\alpha}_s = 0.001$ and $\hat{\alpha}_s = 0.01$, we find that the final dilaton hairs differ by less than a percent. Then, comparing the simulations with coupling strengths $\hat{\alpha}_s = 0.01$ and $\hat{\alpha}_s = 0.1$, we find that Φ_{00} differs by approximately 8%. This difference indicates the significance of the next-to-leading-order effects in the coupling. Our results also indicate that effects of the order $\mathcal{O}(\hat{\alpha}_s^2)$ are more significant for the dilaton than for the axion.

We now turn to Figs. 8 and 9, where we show the dominant axion and dilaton multipoles simulated with $\hat{\alpha}_s \in \{0.001, 0.01, 0.1\}$ and varying the BH spin. We see that both the axion and dilaton hairs increase with the coupling strength as is expected from Eqs. (36) and (39), which are linear in the coupling. Moreover, in Figs. 8 and 9, we can measure the true value of the scalar hair and quantify the effect of higher-order terms by comparing the end state to the approximate analytic solution.

To isolate the effect of high coupling, we do this comparison for simulations with small BH spin $\chi = 0.1$. In the case of the axion, we find the deviation between the final hair and the approximate analytic solution to be $\sim 0.2\%$, 0.02% , and 2.4% for coupling strengths $\hat{\alpha}_s = 0.001$, 0.01 , and 0.1 , with the approximation overestimating the scalar charge for $\hat{\alpha}_s = 0.001$ and underestimating the scalar charge for $\hat{\alpha}_s = \{0.01, 0.1\}$. Similarly for the dilaton, we find the deviation to be $\sim 0.2\%$, 0.5% , and 3.3% for coupling strengths $\hat{\alpha}_s = 0.001$, 0.01 , and 0.1 , with the approximation underestimating the true value for all cases. The differences at small coupling are comparable to numerical error, as expected, while the differences at high coupling capture higher-order effects.

Since the approximate analytic solution we use for our initial data employs both the small-spin and small-coupling approximation, we also study the interplay between the coupling and the spin. As shown in Figs. 8 and 9, in the highest spin case ($\chi = 0.9$), we note that for both the axion and dilaton, the final hairs differ from the initial approximations more for small-coupling simulations ($\hat{\alpha}_s \leq 0.01$) than for the high-coupling simulation ($\hat{\alpha}_s = 0.1$), counter to intuition. Given that higher coupling pushes the final hair value up and higher spin values appear to push the final hair down relative to the analytic approximation, we can understand this trend to be the effects of higher-order spin and higher-order coupling corrections “canceling” each other out. This effect is even more apparent for the evolution of the dilaton hair in the $\chi = 0.7$ case, as seen in Fig. 9. It can be seen that the approximate initial data is overestimating the

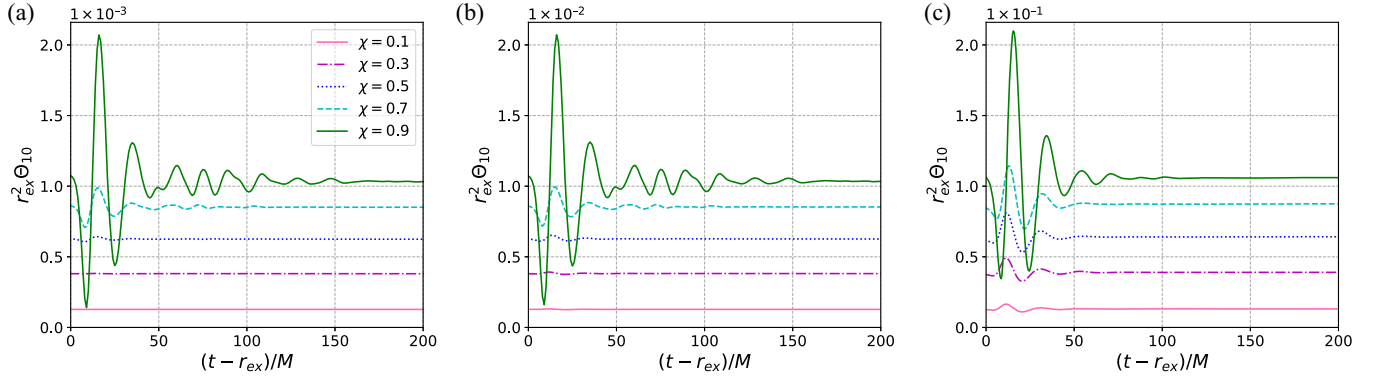


FIG. 8. Evolution of the $l = 1, m = 0$ multipole of the axion field Θ extracted at $r_{\text{ex}} = 20M$ with coupling strengths (a) $\hat{\alpha}_s = 0.001$, (b) $\hat{\alpha}_s = 0.01$, (c) $\hat{\alpha}_s = 0.1$ in the background of a rotating BH with spin parameter $\chi = 0.1$ (solid pink), $\chi = 0.3$ (dash-dotted magenta), $\chi = 0.5$ (dotted blue), $\chi = 0.7$ (dashed cyan), $\chi = 0.9$ (solid green). The axion is initialized by the small-coupling, small-spin solution, i.e., Type II initial data in Sec. IV B.

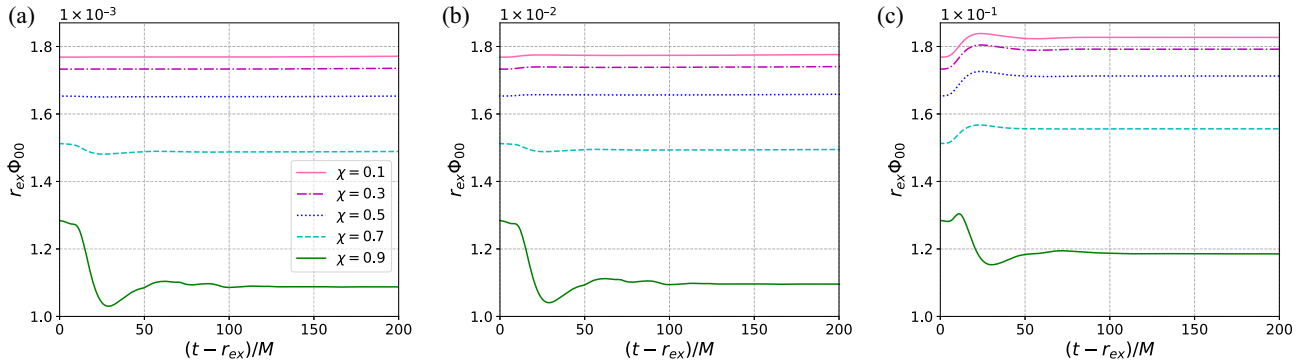


FIG. 9. Same as Fig. 8 but for the evolution of the $l = m = 0$ multipole of dilaton field Φ . The dilaton field is initialized by Type II initial data in Sec. IV B. Same as Fig. 8 but for the evolution of the $l = m = 0$ multipole of dilaton field Φ for coupling strengths (a) $\hat{\alpha}_s = 0.001$, (b) $\hat{\alpha}_s = 0.01$, (c) $\hat{\alpha}_s = 0.1$. The dilaton field is initialized by Type II initial data in Sec. IV B.

dilaton hair for the first two panels and is underestimating it for the rightmost panel. This shows the higher-order spin terms dominating in the first two panels and the higher-order coupling terms dominating in the rightmost panel.

Combining our results for Θ_{10} and Φ_{00} , we find that the axion and dilaton hairs form dynamically for different coupling strengths. We also show that the linear scaling in coupling strength fails for large coupling, indicating the importance of higher-order coupling terms. Finally, we note the interplay of higher-order spin and coupling terms and anticipate that they enter with opposite signs.

C. Effect of kinetic coupling on final hairs

To gain further insight into the effect of the kinetic coupling between the axion and dilaton fields on the final hairs, we compare simulations in axi-dilaton gravity to simulations in which the fields are not coupled to each other. The latter correspond to the axion determined by dCS gravity and the dilaton determined by EdGB gravity. The results are shown in Figs. 10 and 11. All simulations presented in this section were performed using

the CANUDA–AxiDil thorns with suitable parameter choices; see Appendix A 1. For cases in which the axion and dilaton evolve separately according to dCS and EdGB gravity, we simulate them in the same run and realize the decoupling through the appropriate parameter choice.

In the top panels of Figs. 10 and 11, we show the evolution of $r_{\text{ex}}^2 \Theta_{10}$ and $r_{\text{ex}} \Phi_{00}$ in axi-dilaton gravity (opaque lines), together with their evolution in dCS and EdGB gravity (faded lines), respectively. We set the coupling strength to $\hat{\alpha}_s = 0.1$ and consider a range of BH spins $\chi \in \{0.1, 0.3, 0.5, 0.7, 0.9\}$. We initialize the axion and dilaton as the approximate analytical solution, i.e., Type II initial data in Sec. IV B. At this order in $\hat{\alpha}_s$, the initial data given by Eq. (36) is the same for the axion in dCS and in axi-dilaton gravity, and Eq. (39) is the same for the dilaton in EdGB and in axi-dilaton gravity.

For both the axion and the dilaton, we observe that the magnitude of the final hair in axi-dilaton gravity is larger than that in dCS and EdGB gravity individually. Since the interaction between the axion and dilaton fields is the only alteration between the two sets of simulations, this difference in magnitude highlights the effect of the kinetic

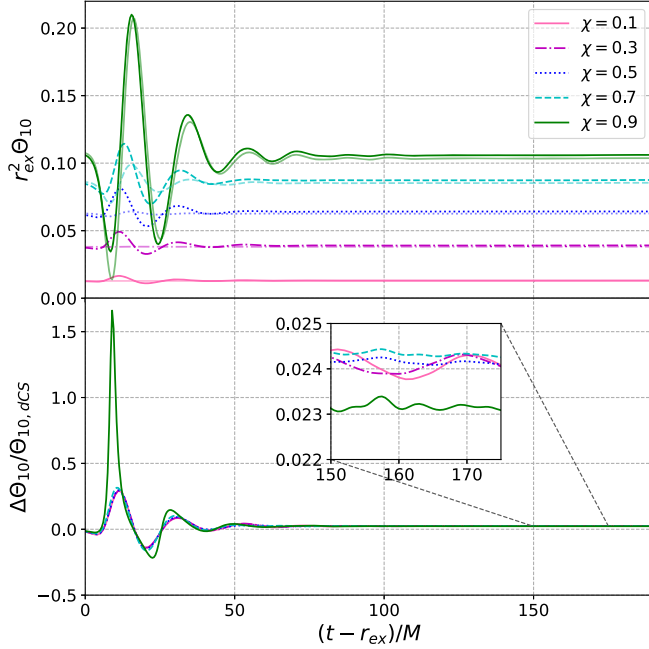


FIG. 10. Top panel: Evolution of Θ_{10} rescaled by $r_{\text{ex}}^2 = (20M)^2$ in axi-dilaton gravity (coupled to the dilaton; Model E in Sec. III, opaque lines) and in dCS gravity (no dilaton; Model B in Sec. III, faded lines) for coupling strength $\hat{\alpha}_s = 0.1$ and a range of BH spins. Bottom panel: Relative difference between Θ_{10} evolved in axi-dilaton gravity and in dCS gravity.

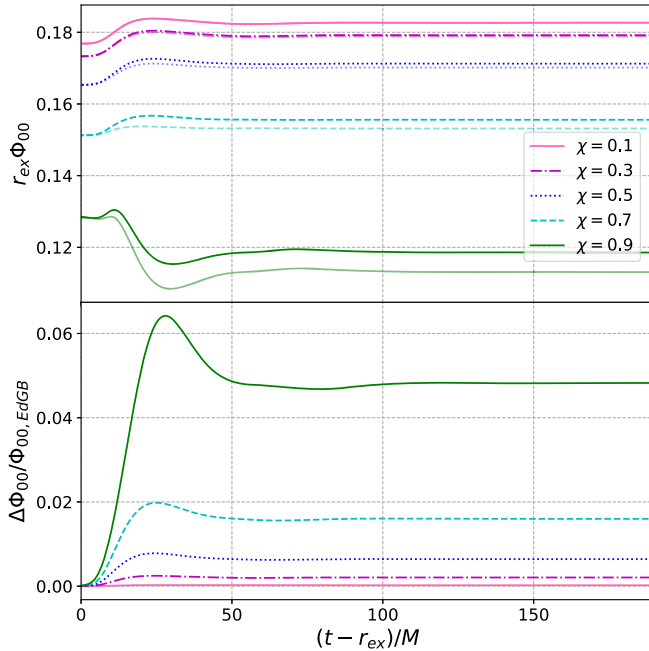


FIG. 11. Top panel: Evolution of Φ_{00} rescaled by $r_{\text{ex}} = 20M$ in axi-dilaton gravity (coupled to the axion; Model E in Sec. III, opaque lines) and in EdGB gravity [no axion; Model C in Sec. III with Eq. (25a), faded lines] for coupling strength $\hat{\alpha}_s = 0.1$ and a range of BH spins. Bottom panel: Relative difference between Φ_{00} evolved in axi-dilaton gravity and in EdGB gravity.

coupling between the fields. In particular, this makes it clear that the kinetic coupling between the axion and dilaton fields comprises some of the higher-order coupling effects observed in Sec. VIB.

To quantify the effect of the kinetic coupling between the fields on the axion hair, we present the relative difference between the axion in axi-dilaton and dCS gravity, $(\Theta_{10,\text{AxiDil}} - \Theta_{10,\text{dCS}})/\Theta_{10,\text{dCS}}$, in the bottom panel of Fig. 10. After the initial perturbation decays, we see that the relative difference, and thus, the impact of the coupling between the fields, is the same for all BH spins.

Turning to the effect of the fields' kinetic coupling on the dilaton hair, we show the relative difference between the dilaton in axi-dilaton and EdGB gravity, $(\Phi_{00,\text{AxiDil}} - \Phi_{00,\text{EdGB}})/\Phi_{00,\text{EdGB}}$ in the bottom panel of Fig. 11. Here, we see a clear dependence on the BH spin.

We quantify the relationship between this relative difference and the BH spin in Fig. 12. As mentioned above, $\Delta\Theta_{10}/\Theta_{10,\text{dCS}}$ does not depend on BH spin, so we fit it as a constant and find $\Delta\Theta_{10}/\Theta_{10,\text{dCS}} = 0.024$. In the case of the dilaton, we fit $\Delta\Phi_{00}/\Phi_{00,\text{EdGB}}$ with

$$\Delta\Phi_{00}/\Phi_{00,\text{EdGB}} = a(\chi + b)^2 + c(\chi + b)^4 + d,$$

and find $a = 0.014$, $b = -0.139$, $c = 0.115$, and $d = 0.001$. While not presented here, we note that for small BH spins $\chi = \{0.1, 0.3, 0.5\}$, $\Delta\Phi_{00}/\Phi_{00,\text{EdGB}}$ fits extremely well with only a quadratic function (i.e., $c = 0$), while the χ^4 term is necessary to accurately fit high spins.

The spin dependence of the effect of the kinetic coupling is understood by combining the fields' evolution equations

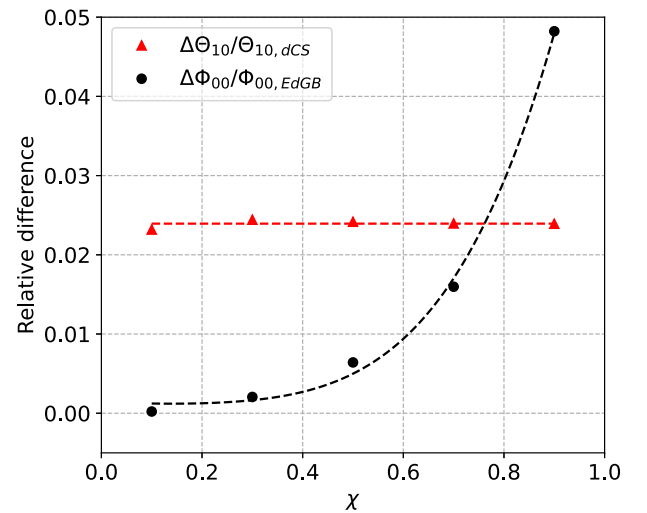


FIG. 12. Relative difference at late times ($t = 150 M$) between Θ_{10} evolved in axi-dilaton and dCS gravity (red triangles) and Φ_{00} evolved in axi-dilaton and EdGB gravity (black circles) as a function of BH spin χ with coupling strength $\hat{\alpha}_s = 0.1$. We give lines of fit for the axion field (dashed red) and dilaton field (dashed black).

with their approximate analytical solutions. In Eqs. (31), the axion and dilaton are determined in part by the kinetic coupling terms $\square\Theta \propto \nabla_i\Theta\nabla^i\Phi$ and $\square\Phi \propto \nabla_i\Theta\nabla^i\Theta$. Inserting the leading-order contributions to the approximate analytical solutions $\Theta \propto \hat{\alpha}_s\chi$ [see Eq. (36)] and $\Phi \propto \hat{\alpha}_s$ [see Eq. (39)], into these pseudoequations, we recognize that for the axion, the spin cancels on both sides, while the dilaton picks up a factor of χ^2 from the axion source on the right-hand-side.

In summary, we find that the effect of the kinetic coupling between the axion and dilaton fields is comparable for all BH spins for the final axion hair and increases (at least) quadratically with BH spin for the dilaton hair. For both the axion and dilaton hairs, the effect of the kinetic coupling leads to larger end states in comparison to the hairs evolved only in dCS or EdGB gravity. We note that since the corrections are $\lesssim 5\%$, the results remain within the regime of validity of the EFT. Extending to BBH mergers, we expect the effect of the kinetic coupling to give stronger deviations from GR BHs than in dCS and EdGB gravity.

VII. AXION AND DILATON EVOLUTION AROUND BINARY BLACK HOLES

In this section, we present first results of a BBH merger in axi-dilaton gravity, in the decoupling approximation. We perform a short, proof-of-principle simulation to (i) study the formation of the axion and dilaton hairs around the binary and (ii) compute the axion's and dilaton's radiative multipoles produced by the inspiraling hairs that represent orbiting scalar charges.

The background spacetime, described by GR, consists of an equal-mass, nonspinning BBH system with a total BH mass of $M = m_1 + m_2 = 1$, where m_1 and m_2 are the ADM masses of the individual punctures, as defined in [128]. The BHs have an initial coordinate separation of $d = 6M$. They inspiral for about two orbits before merging at $t \sim 108M$ (determined by the formation of the common apparent horizon). The newly formed BH rings down to a rotating remnant BH with dimensionless spin parameter $\chi = 0.688$.

We illustrate the evolution of the BHs in the top panels of Fig. 13. In Fig. 13(a), we show the evolution of the BHs' coordinate separation to guide our understanding of the axion's and dilaton's behavior throughout the coalescence. The dashed vertical lines indicate times during the late inspiral, the formation of the common apparent horizon that signifies the merger, and after the ringdown. They correspond to the times of the two-dimensional snapshots of the axion and dilaton presented later in Fig. 14. In the top panel of Fig. 13(b), we present the dominant gravitational waveform, i.e., the quadrupole of the Newman-Penrose scalar $\Psi_{4,22}$. After a short burst of unphysical "junk" radiation, the waveform exhibits the characteristic

morphology of the late inspiral, merger peak, and exponentially decaying ringdown.

We now turn to the evolution of the axion and dilaton whose dynamics are driven by the binary. They are initialized as zero according to initial data Type I in Sec. IV B. We set the dimensionless coupling strength $\hat{\alpha}_s = 0.1$. For BBH simulations, we define the dimensionless coupling parameter $\hat{\alpha}_s = \alpha_s/(m_1 + m_2)^2$.

We illustrate the evolution of the axion's and dilaton's multipoles in Fig. 13. In the left panels, we show the dominant multipoles making up the hair of each field, and in the right panels, we show their radiative modes. We extract our data at $r_{\text{ex}} = 80M$ with the origin placed at the center of mass of the BBH, thus capturing the behavior of the fields around both BHs.

In the bottom panel of Fig. 13(a), we show the evolution of the dominant mode of the dilaton hair, $r_{\text{ex}}\Phi_{00}$. The dilaton hair initially grows and settles to a nearly constant value as the BHs coalesce. After the BHs merge, the dilaton charge decreases and settles to a value lower than that during the inspiral. This is predominantly due to the remnant BH's larger mass, i.e., lower curvature, as well as due to the remnant's higher spin; see Eq. (47c).

In the middle panel of Fig. 13(a), we present the evolution of the leading-order multipole of the axion hair, $r_{\text{ex}}^2\Theta_{10}$. At the beginning of the simulation, the axion hair oscillates and settles close to zero. This behavior is expected during the inspiral of *nonspinning* BHs as the axion hair scales with the BH spin; see Eq. (36). We observe, however, that the axion starts to grow in the late inspiral, but before the final, rotating BH is born. The origin of this growth around the nonspinning BHs may be twofold: First, the orbital angular momentum in the late inspiral may generate a nonzero Pontryagin density that sources the axion; see Eq. (31a). This is consistent with results of the axion's dynamics in dCS gravity [48], i.e., in the absence of the dilaton. Second, the kinetic coupling to the dilaton may enhance the growth of the axion hair [see the last term in Eq. (31a)]. The axion hair peaks as the BHs merge. As the newly formed BH settles to a Kerr BH with a spin of $\chi \sim 0.688$, the axion rings down and settles to its final hair profile. A detailed analysis of the pre-merger axion growth is forthcoming.

Since each BH carries a dilaton and, in the late inspiral, axion charge, their inspiral generates axion and dilaton radiation. In Fig. 13(b) we present the dominant radiative modes, Θ_{32} and Φ_{22} , respectively. In the first $t \sim 50M$, both fields show an initial oscillation as they adjust from the zero initial data to the binary configuration. The axion's radiation, Θ_{32} , shown in the middle panel of Fig. 13(b), remains close to zero during the inspiral which is consistent with the zero axion hair. Only when the latter starts growing in the late inspiral does it generate axion radiation that is increasing in amplitude and frequency. It peaks during the BHs' merger and then decays exponentially. The dilaton's

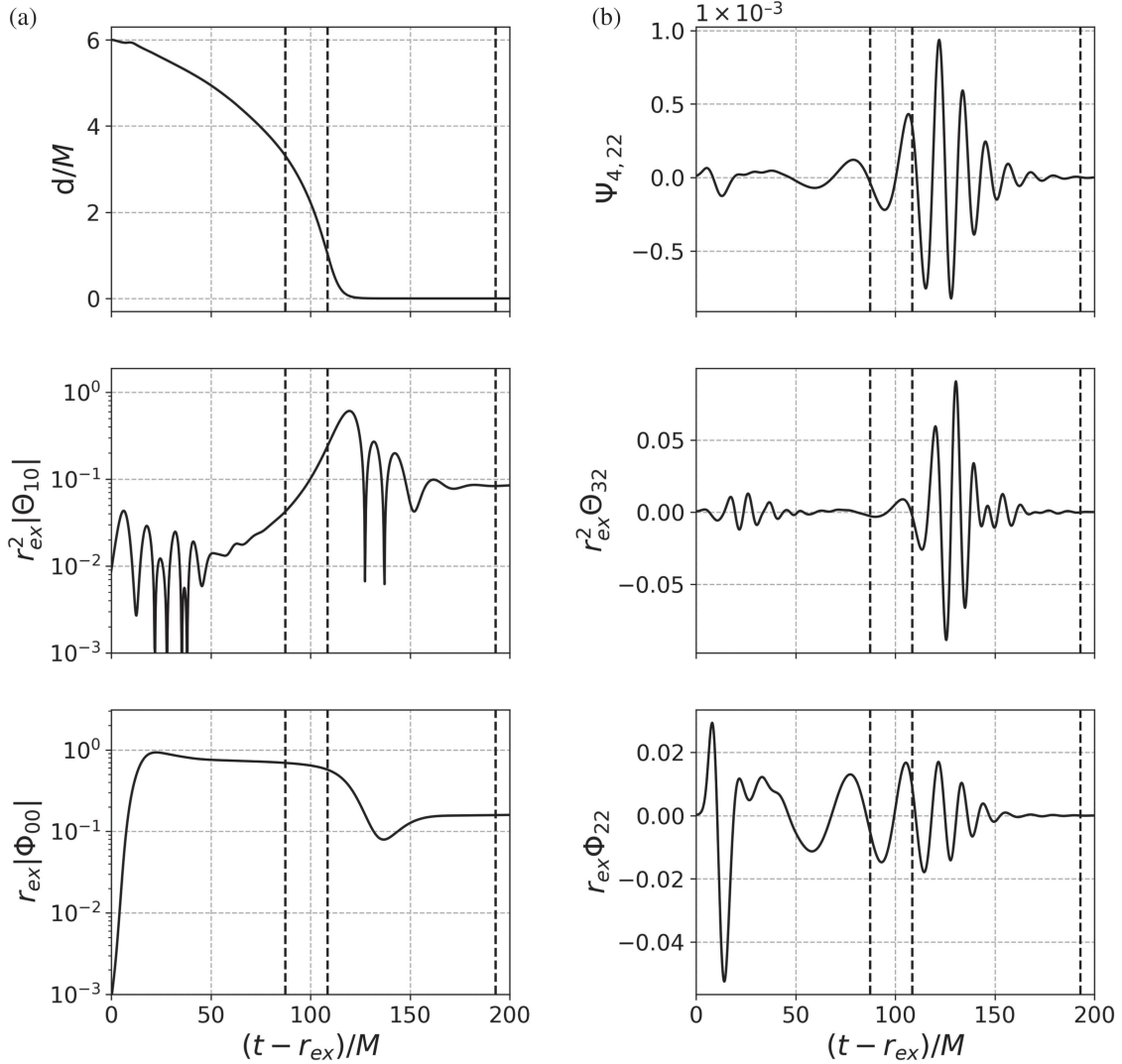


FIG. 13. Evolution of axion and dilaton fields in the background of an equal mass, nonspinning BBH merger. The shown multipoles are extracted at a radius of $r_{\text{ex}} = 80 M$. The coupling strength is $\hat{\alpha}_s = 0.1$, and the fields are set to zero at the beginning of the simulation. The remnant BH has a spin of $\chi = 0.688$. The dashed, vertical lines denote each of the timestamps shown in Fig. 14, the middle of which is the first appearance of the common horizon at $t = 108 M$. (a) Top panel: evolution of the BBH's coordinate separation. Middle panel: evolution of dominant axion multipole, Θ_{10} . Bottom panel: evolution of dominant dilaton multipole, Φ_{00} . (b) Top panel: dominant $l = 2, m = 2$ mode of the gravitational waveform, $\Psi_{4,22}$ of the BBH background. Middle panel: dominant radiative multipole, Θ_{32} , of the axion. Bottom panel: dominant radiative multipole, Φ_{22} , of the dilaton.

radiation, Φ_{22} , shown in the bottom panel of Fig. 13(b), oscillates with increasing amplitude and frequency until merger. After the BHs' merger, the dilaton radiation decays to zero.

In Fig. 14, we show two-dimensional snapshots of the axion and dilaton fields at $t = 87.36 M$ (during the inspiral after junk has settled), $t = 108.48 M$ (just after the common apparent horizon is found), and $t = 192.96 M$ (after the final BH has settled). We see that the strength of the axion field is significantly higher after the merger, consistent with the nonzero spin of the remnant BH, which leads to growth of axion hair. The dilaton field changes in both magnitude and shape between the inspiral and the

postmerger. Due to the larger mass of the remnant BH relative to that of the initial, individual BHs, the dilaton's magnitude decreases after the merger. This is compounded by the nonzero spin of the remnant, which also drives down the overall amplitude of the dilaton field. Since the dilaton's next-to-leading-order multipole increases with spin, its relative amplitude as compared to the dominant mode increases, and we observe a deviation from the monopole structure of the dilaton field around the BH. This behavior is consistent with the results shown in Fig. 5 for the highly spinning single BH.

In summary, using CANUDA-AxiDil, we evolve axion and dilaton fields coupled to each other in a BBH background,

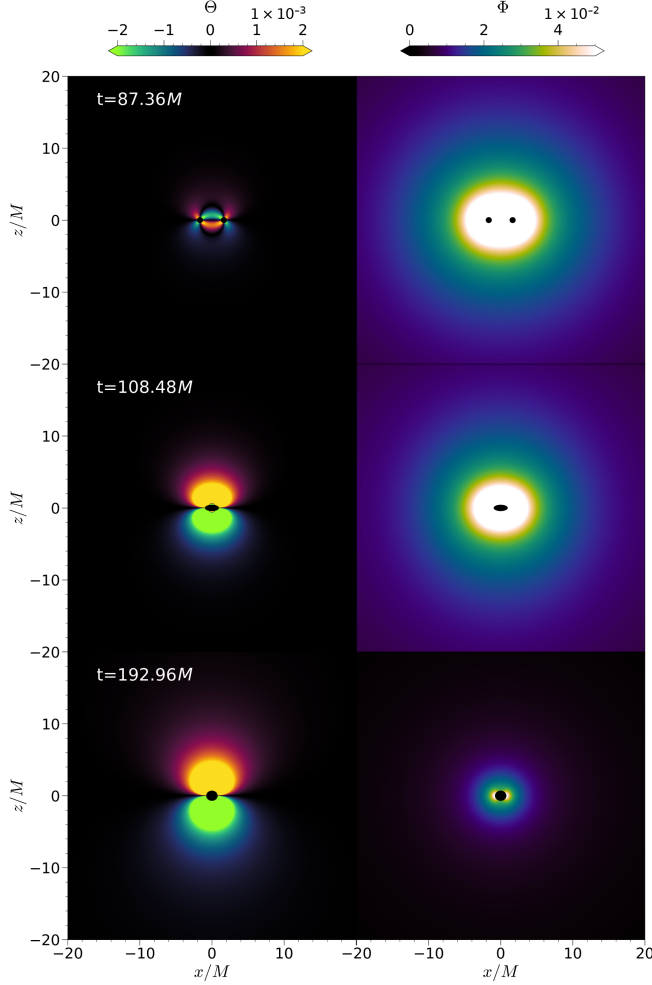


FIG. 14. Axion (left) and dilaton (right) fields in the xz plane (a slice perpendicular to the orbital plane) during the inspiral at $t = 87.36M$ (top), the merger at $t = 108.48M$ (middle), and postmerger at $t = 192.96M$ (bottom) for a BBH system with equal mass, nonspinning initial BHs. The coupling strength is $\hat{\alpha}_s = 0.1$, and the fields are set to zero at the beginning of the simulation.

showing how their profiles change throughout the evolution. We present the growth of axion hair during the inspiral before the formation of a spinning remnant, an effect that has only been captured by numerical relativity simulations and has never before been shown in axi-dilaton gravity. This growth may come from several sources including a possible nonzero Pontryagin density caused by the orbital angular momentum or the coupling with the dilaton field. We also show the presence of radiative modes for both the axion and dilaton fields which carry energy away from the system. Such energy loss leads to changes in the GWs, which may be observable with future GW observatories. As a consequence of the increased axion and dilaton hairs observed for axi-dilaton gravity versus dCS and EdGB gravity, it is likely that this radiation, and thus energy loss, will be larger for axi-dilaton gravity as well. In future work,

we will quantify this energy loss and expand this analysis to include more coverage of the BBH parameter space.

VIII. CONCLUSIONS

In this paper, we probed the dynamical evolution of an axion and a dilaton, kinetically coupled to each other, around single and binary BHs in axi-dilaton gravity. To accomplish this, we created CANUDA-AxiDil, the first open-source, parametrized numerical relativity code to perform simulations in (bi)scalar-tensor theories of gravity. Through this parametrization, the code has the flexibility to study BHs in a variety of theories beyond GR, including axi-dilaton gravity as well as popular models such as dCS and sGB gravity.

Using our new code, we performed a series of simulations to investigate how axion and dilaton fields evolve in axi-dilaton gravity in the background of BHs. Our results present the first simulations of the dynamical formation of axion and dilaton hair. In the case of simulations with high BH spin or high coupling strength, we captured a new end-state solution of the axion and dilaton hairs.

We then addressed three questions: (i) How do the axion and dilaton hairs depend on the BH spin, (ii) How does the strength of the coupling to curvature affect the axion and dilaton hairs, and (iii) How does the kinetic coupling between the axion and dilaton fields impact these hairs?

We showed the dynamical growth of the axion and dilaton hairs from zero initial data for a variety of BH spins. For backgrounds with small BH spin, we confirmed consistency with the approximate analytical solution valid in the small-spin, small-coupling limit. For larger BH spins, we captured a new end-state solution for the axion and dilaton hairs. By varying the spin of the BH, we found that the final axion hair increases with increasing spin. This holds true for both the leading-order and next-to-leading-order multipoles. For the dilaton hair, we found that the magnitude of the dominant mode decreases with increasing spin, while the next-to-leading-order multipole increases with increasing spin. For high spins, this deformation away from the monopole is visible in the dilaton profile on a slice perpendicular to the equatorial plane.

To continue our analysis, we varied the strength of the fields' coupling to their respective curvature invariants. For small coupling strengths, the magnitude of the axion and dilaton hairs increase linearly with the coupling parameter, which is consistent with the approximate analytical solution. In simulations with large coupling strengths, however, the increase in the fields' magnitude exceeds a linear scaling, thus indicating the importance of higher-order effects in the coupling.

We investigated one such higher-coupling effect in detail, namely the kinetic coupling between the axion and dilaton fields. We found that the kinetic coupling yields axion and dilaton end states with a larger magnitude in comparison to simulations in which the fields are not

coupled to each other. To quantify the impact of the kinetic coupling on the final axion and dilaton hairs, we performed simulations where the coupling between the fields was turned on ($g_\Phi = e^{-\Phi}$) and simulations where it was turned off ($g_\Phi = \text{constant}$). For the axion field, switching on the kinetic coupling increased the final axion hair by $\approx 2.4\%$, independent of the spin of the BH. The kinetic coupling between the fields also caused an increase in the magnitude of the dilaton hair. We observed that, in this case, the increase depends on the BH spin and found a quadratic scaling for small to moderate spins, while for high spins, quartic terms also become important. For a BH of spin $\chi = 0.9$, the dilaton hair was $\approx 5\%$ larger in simulations with the kinetic coupling switched on as compared to simulations without it. For both the axion and the dilaton, the difference due to the kinetic coupling between the fields accounts, in part, for the higher-order coupling effects observed earlier.

We also performed a simulation of the axion and dilaton fields evolving around a BBH system in axi-dilaton gravity, which nicely captured several of the trends observed in the single BH simulations. We began with two nonspinning BHs with both the axion and dilaton fields set to zero and observed their evolution.

Although the initial BHs are nonspinning, and therefore their Pontryagin density vanishes initially, we observed a growth of the axion hair in the late inspiral before it settled to a static hair around the spinning remnant BH. The growth of the axion hair before the merger could be indicative of a nonzero Pontryagin density caused by the orbital angular momentum, or it could be the result of coupling with the dilaton field, which is present throughout the inspiral. We note that the premerger growth of the axion has only been captured with numerical relativity simulations, and we plan to investigate this effect further in upcoming work.

The opposite trend could be seen for the dilaton field, which began at a larger value around the nonspinning initial BHs and decreased in magnitude post merger due to the higher mass and, in part, due to spin of the remnant BH. The final dilaton hair profile again displayed a slight deviation away from a monopole profile due to higher multipoles being present in the stationary solution around the spinning remnant BH.

Radiative modes were present for both the axion and dilaton fields, carrying away energy from the binary. Given the effect of the kinetic coupling increasing the axion and dilaton hairs, we anticipate that this radiation will be larger for axi-dilaton gravity than for dCS or EdGB gravity alone. Such energy dissipation would impact the BBH system itself as well as its gravitational radiation, effects that we will study in future works.

This work used the open-source softwares `xTensor` [134,135], the `EINSTEIN TOOLKIT` [119,121], `CANUDA` [103], and `KUIBIT` [131].

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DATA AVAILABILITY

The `CANUDA-AxiDil` code developed to conduct the simulations in this work is open source and available in a git repository [102].

APPENDIX A: CANUDA-AxiDil CODE DESCRIPTION

In this section, we complete the description of the `CANUDA-AxiDil` code [102]. `CANUDA-AxiDil` is a parametrized numerical relativity code for (bi)scalar-tensor theories of gravity in which the scalars can be coupled to each other, and they can be minimally or nonminimally coupled to gravity. This includes scalar fields in GR as well as popular quadratic gravity models such as sGB, dCS and axi-dilaton gravity. In Appendix A 1, we describe the parameter choices in `CANUDA-AxiDil` that need to be set to select the models presented in Sec. III. In Appendix A 2, we summarize the choices of axion and dilaton initial data that are available in the `CANUDA-AxiDil` code.

1. Model selection parameters

Table II lists the parameters, and their appropriate settings, to select the models described in Sec. III. Note that the dimensionless coupling parameter $\hat{\alpha}_s$ introduced in Sec. III E is given by the (dimensionless) parameters a_{GB} and a_{CS} in the code. That is, the user may choose a coupling such that $a_{\text{GB}} \neq a_{\text{CS}}$ (e.g., when sGB and dCS gravity are simulated in the same run). This is not to be confused with the dimensionful coupling constants α_{GB} and α_{CS} used throughout the paper, all of which are related via $a_{\text{GB}} = \hat{\alpha}_{\text{GB}} = \alpha_{\text{GB}}/M^2$ and $a_{\text{CS}} = \hat{\alpha}_{\text{CS}} = \alpha_{\text{CS}}/M^2$, where M is a characteristic mass scale of the considered problem.

TABLE II. Parameter choices in the CANUDA–AxiDil code to select different models of gravity as outlined in Sec. III. `axi_coupling` and `aCS` describe the coupling function f_Θ and coupling strength between the axion and the Pontryagin density. `dil_coupling` and `aGB` describe the coupling function f_Φ and coupling strength between the dilaton and the Gauss-Bonnet invariant. `dil_lambda` is the dimensionless constant λ_{GB} in the coupling function $f_\Phi = e^{\lambda_{\text{GB}}\Phi}$. `AD_coupling` determines the coupling function g_Φ between the fields. `AD_lambda` is the dimensionless constant λ_{AD} in the coupling function $g_\Phi = e^{-\lambda_{\text{AD}}\Phi}$.

Model of gravity	<code>axi_coupling</code>	<code>aCS</code>	<code>dil_coupling</code>	<code>aGB</code>	<code>dil_lambda</code>	<code>AD_coupling</code>	<code>AD_lambda</code>
Minimally coupled scalar	None	0.0	None	0.0	...	None	...
Shift-symmetric dCS	Linear	10^{-3} – 10^{-1}	None	0.0	...	None	...
Cubic dCS	Cubic	10^{-3} – 10^{-1}	None	0.0	...	None	...
EdGB	None	0.0	Exponential	10^{-3} – 10^{-1}	1.0	None	...
Shift-symmetric sGB	None	0.0	Linear	10^{-3} – 10^{-1}	...	None	...
Quadratic sGB	None	0.0	Quadratic	10^{-3} – 10^{-1}	...	None	...
Minimally coupled biscalar	None	0.0	None	0.0	...	Exponential	Any value
Axi-dilaton	Linear	10^{-3} – 10^{-1}	Exponential	10^{-3} – 10^{-1}	1.0	Exponential	1.0

The parameters `axi_coupling` and `dil_coupling` specify the coupling functions, f_Θ and f_Φ , of the axion and dilaton fields. The parameters `dil_lambda` and `AD_lambda` refer to λ_{GB} and λ_{AD} ; see Eqs. (25a) and (29). Finally, the parameter `AD_coupling` refers to the function g_Φ and determines the coupling function connecting the axion and dilaton fields. Setting it to “none” decouples the fields and allows the user to individually evolve sGB and dCS gravity with the same simulation.

2. Scalar field initial data, continued

The CANUDA–AxiDil code offers a suite of initial data choices for the axion and dilaton fields. In Sec. IV B, we presented zero initial data and the approximate analytic solution in axi-dilaton gravity around a single BH. The latter coincides with the small-spin, small-coupling solutions in dCS and sGB gravity. The initial data thorn also provides axion and dilaton profiles around BBHs through a superposition of the single BH solutions.³ Here, we list additional options and their applicability.

Type III: Gaussian We provide the option to initialize the axion and dilaton fields as Gaussian profiles centered around $r = r_0$ as

$$\Theta|_{t=0} = \Theta_0 \exp\left(-\frac{(r-r_0)^2}{\sigma_\Theta^2}\right) \Sigma_\Theta(\theta, \phi), \quad (\text{A1a})$$

$$K_\Theta|_{t=0} = 0, \quad (\text{A1b})$$

³Note that all scalar initial data are currently treated in the decoupling approximation, i.e., do not backreact onto the metric. The construction of consistent metric-(bi)scalar initial data is beyond the scope of this paper. CANUDA provides constraint-satisfying initial data of the Einstein–Klein–Gordon equations in its Scalar module.

and

$$\Phi|_{t=0} = \Phi_0 \exp\left(-\frac{(r-r_0)^2}{\sigma_\Phi^2}\right) \Sigma_\Phi(\theta, \phi), \quad (\text{A2a})$$

$$K_\Phi|_{t=0} = 0, \quad (\text{A2b})$$

where σ_Θ and σ_Φ are the Gaussians’ width, and Θ_0 and Φ_0 are their maximum amplitude. The angular profiles $\Sigma_\Theta(\theta, \phi)$ and $\Sigma_\Phi(\theta, \phi)$ are given by a superposition of spherical harmonics, $Y_{lm}(\theta, \phi)$, that can be chosen by the user. The code also offers a superposition of two Gaussians in Θ and Φ that can be used to set up field profiles around the individual BHs of a binary. This type of initial data is suitable for all models of gravity in the CANUDA–AxiDil code.

Type IV: quasibound state of a massive scalar field in general relativity We also implement the quasibound state of a massive scalar field, with mass parameter μ_S , around a single BH in GR. The implementation is available for both the axion and dilaton. Here, we summarize the construction exemplarily for the axion. The massive quasibound state is a solution of the Klein-Gordon equation

$$(\square - \mu_S^2)\Theta = 0, \quad (\text{A3})$$

with the d’Alembertian determined by the Kerr metric. We implemented the solution found by Dolan [136] that is valid for any mass parameter μ_S and BH spin $0 \leq |\chi| < 1$. The full scalar field takes the form

$$\Theta_{lm} = \exp(-i\omega_\Theta t) \exp(im\phi) \mathcal{S}_{lm}(\theta) \mathcal{R}_{lm}(r_{\text{BL}}), \quad (\text{A4})$$

where ω_Θ is the field's frequency, $\mathcal{S}_{lm}(\theta)$ are spheroidal harmonics, and $\mathcal{R}_{lm}(r_{\text{BL}})$ is the radial profile that we construct numerically.

The corresponding initial data is given by

$$\Theta_{lm}(t=0) = \exp(\text{im}\phi) \mathcal{S}_{lm}(\theta) \mathcal{R}_{lm}(r_{\text{BL}}), \quad (\text{A5a})$$

$$K_{\Theta lm}(t=0) = i\omega_\Theta \Theta_{lm}(t=0). \quad (\text{A5b})$$

We apply the same procedure to Φ .

Type V: Dilaton bound state in quadratic sGB gravity
The CANUDA-AxiDil code facilitates simulations of BHs in different types of sGB gravity that are determined by the coupling function f_Φ ; see Eq. (25).

In particular, shift-symmetric sGB or EdGB yield unique BHs with dilaton hair given by Type II initial data in Sec. IV B. Instead, quadratic sGB, which is determined by the coupling function $f_\Phi \propto \Phi^2$, has a richer phase space of solutions that includes both the Kerr metric and scalarized solutions [34,35]. The latter refers to a bound state of the scalar Φ around a single BH located at $r = R$. The scalar's profile is given by

$$\Phi(t=0) = \frac{MR}{\zeta^2} \left(c_1 + c_2 \frac{MR}{\zeta^2} + c_3 \frac{(MR)^2}{\zeta^4} \right), \quad (\text{A6a})$$

$$K_\Phi(t=0) = 0, \quad (\text{A6b})$$

where $\zeta = M + 2R$, and the fitting parameters are $c_1 = 3.68375$, $c_2 = 4.97242$, and $c_3 = 229.938$ [77]. The CANUDA-AxiDil code also provides initial data around BBHs through a superposition of the scalar bound states.

Recall that the initial data choices for the axion and dilaton fields can be chosen to be different from each other.

Since the initial data here only describes the dilaton field and its momenta, the axion field and its respective momenta can be chosen to be any of the other initial data choices described in this paper.

APPENDIX B: NUMERICAL TESTS

In this section, we present several numerical tests we performed to validate our results. In Appendix B 1, we compare our results with analytic approximations in the small-spin, small-coupling limit. In Appendix B 2, we compare the results produced with our new CANUDA-AxiDil module to those obtained with our CANUDA-dCS [67] and CANUDA-EdGB [74] codes, which have been thoroughly validated. Finally, in Appendices B 3 and B 4, we present convergence tests for single and binary BHs.

1. Comparison with approximate analytic solution

In Fig. 15, we compare our numerical results against the approximate analytical solutions for the axion and dilaton hair [see Eqs. (36) and (39)] and find excellent agreement. Specifically, we show the profiles of the axion, Θ , and dilaton, Φ , evolved in the background of slowly-rotating BH with spin $\chi = 0.1$ along the axis of rotation (z-axis) at $t = 200 M$. We rescale the fields by $\hat{\alpha}_s^{\text{max}}/\hat{\alpha}_s$ with $\hat{\alpha}_s^{\text{max}} = 0.1$ and coupling strengths $\hat{\alpha}_s = \{0.001, 0.01, 0.1\}$. Figure 15 shows excellent agreement between the numerical simulations around slowly rotating BHs and the small-spin analytic solutions.

2. Code comparison

To verify that our newly developed code, CANUDA-AxiDil, obtains physically sound results, we benchmark it against our CANUDA-dCS [67] and CANUDA-EdGB [74]

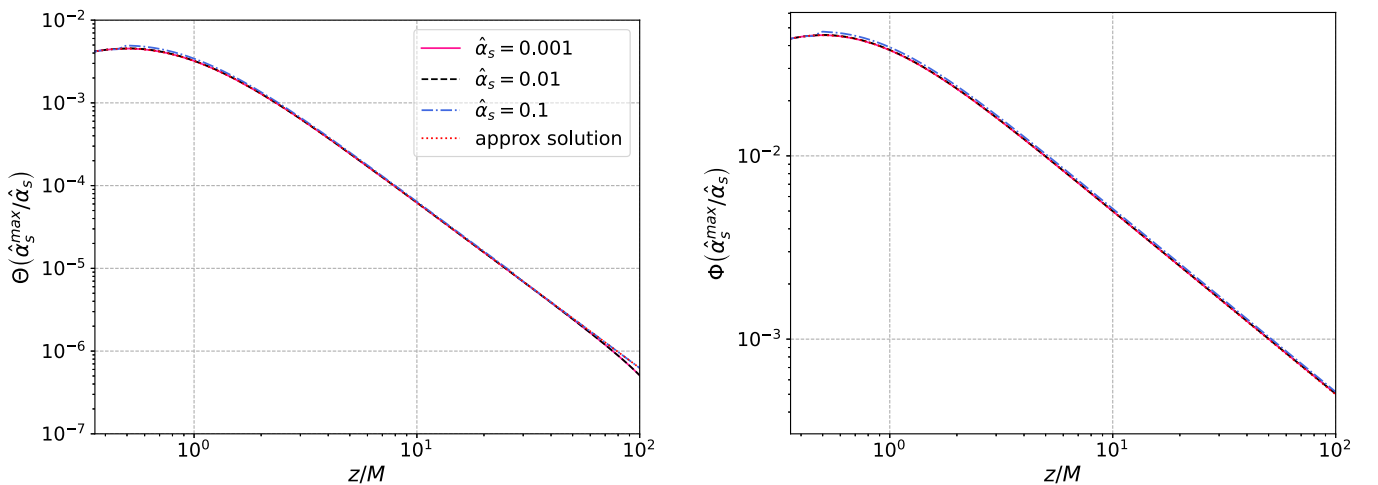


FIG. 15. Profile of the axion (left panel) and dilaton (right panel) fields along the z axis, at $t = 200 M$. The fields are evolved around a BH with spin $\chi = 0.1$. The numerical data is rescaled by $\hat{\alpha}_{s,\text{max}}/\hat{\alpha}_s$, with $\hat{\alpha}_{s,\text{max}} = 0.1$, for $\hat{\alpha}_s = 0.001$ (pink solid), $\hat{\alpha}_s = 0.01$ (black dashed), and $\hat{\alpha}_s = 0.1$ (blue dash dotted). We validate the numerical data against the analytic solution in the small-spin, small-coupling approximation (red dotted line).

TABLE III. Simulations for convergence test around a single BH. We ran the configuration IDII_AD_chi09 in Table I with three different resolutions. We list the resolution dx/M on the outermost refinement level and the resolution $h = dx/2^{n-1}$ (with $n = 8$) on the innermost refinement level.

Run	dx/M	h/M
Low	1.1	8.6×10^{-3}
Med	1.0	7.8×10^{-3}
High	0.9	7.0×10^{-3}

codes. They had been designed to simulate scalar fields in dCS and sGB gravity, respectively, both of which are special cases in our new code.

We simulate the axion and dilaton fields around a single Kerr BH with a moderate spin of $\chi = 0.5$. They are initialized as the approximate analytical solution (see Type II in Sec. IV B), which also represents the small-spin, small-coupling solutions in dCS and sGB gravity.

We run a simulation with CANUDA–AxiDil where the fields are “decoupled”; i.e., the axion and dilaton fields evolve at the same time but without any coupling between them. See Secs. III B and III C. We compare the fields thus evolved with CANUDA–AxiDil to the analogous simulations of the axion with CANUDA–dCS and of the dilaton with CANUDA–EdGB. After the fields have settled to their final hairs, the decoupled CANUDA–AxiDil results are in excellent agreement with those obtained from the CANUDA–dCS code for the axion hair and the CANUDA–EdGB thorn for the dilaton hair.

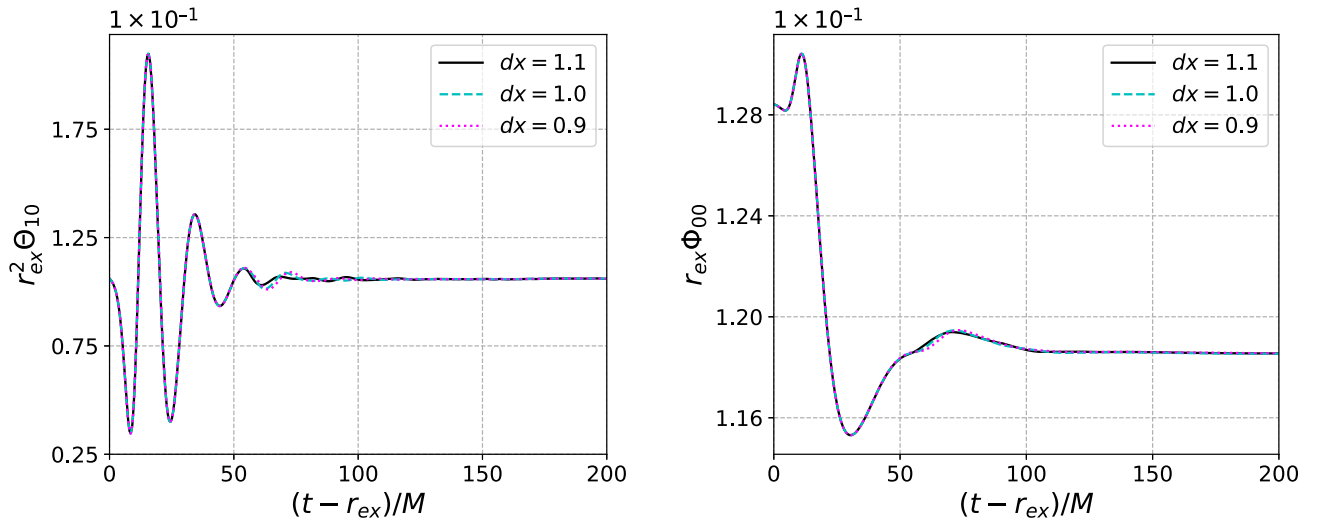


FIG. 16. Evolution of the dominant multipoles of the axion (left panel) and dilaton (right panel) fields for different resolutions.

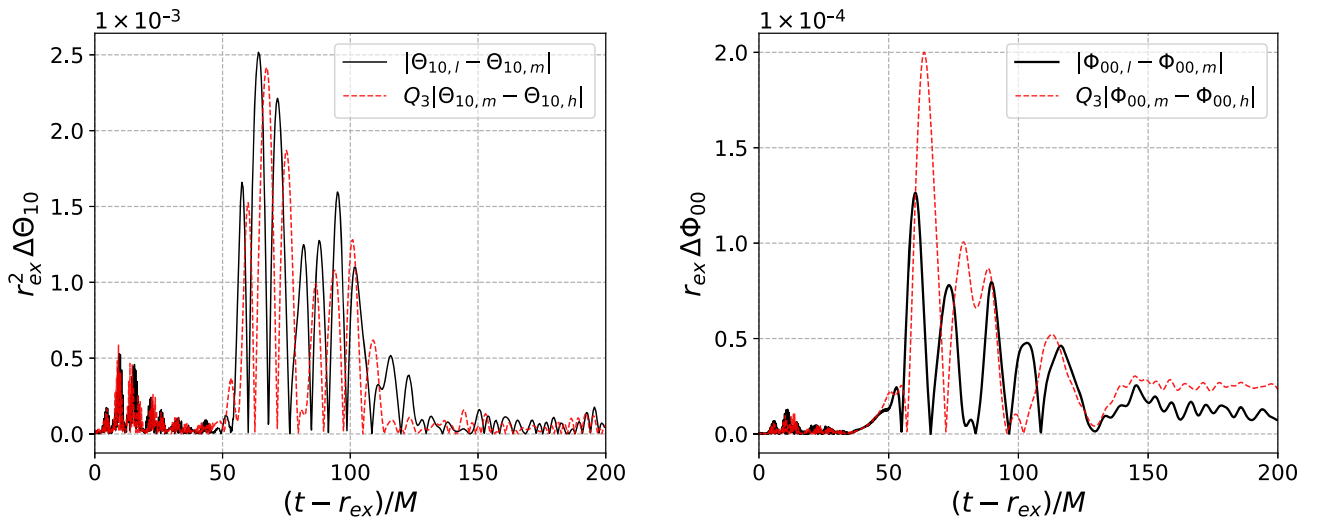


FIG. 17. Convergence tests of the dominant multipoles of the axion (left panel) and dilaton (right panel) fields. We show the residual between the low and medium resolution runs (black solid line), and the residual between the medium and high resolution runs rescaled by Q_3 (red dashed line). The latter indicates third order convergence.

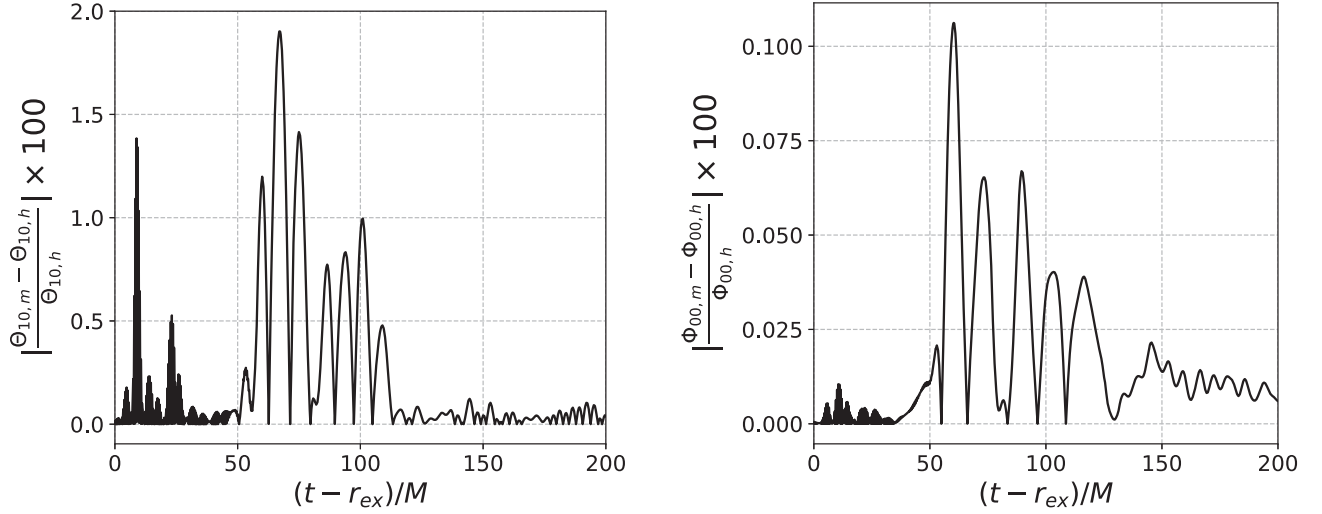


FIG. 18. Percent error in the dominant modes of the axion (left) and dilaton (right) fields computed between the medium and high resolutions.

3. Single black hole convergence tests

In this section, we present convergence tests of the axion and dilaton fields evolved around a single BH. For codes that use finite differencing, the numerical (i.e., discretized) data can be expressed in terms of the true values as $\Theta_i = \Theta + c dx_i^n$, where n is the convergence rate, i identifies a given resolution, dx is the grid spacing, and c is some function of time that does not depend on resolution. Since we use mesh refinement, we use dx to refer to the grid spacing on the coarsest grid. The residuals between different resolutions (low, medium, high) are then related by

$$\Theta_l - \Theta_m = Q_n(\Theta_m - \Theta_h), \quad (\text{B1})$$

where the convergence factor is

$$Q_n := \frac{|dx_l^n - dx_m^n|}{|dx_m^n - dx_h^n|}. \quad (\text{B2})$$

We perform convergence tests with our most demanding parameters, BH spin $\chi = 0.9$ and coupling strength $\hat{\alpha}_s = 0.1$ with the same grid setup as described in Sec. VB, corresponding to run IDII_AD_chi09 in Table I but with the resolutions listed in Table III. Figure 16 shows the dominant multipoles of the axion and dilaton fields for three different resolutions. They are in excellent agreement, demonstrating that the data obtained with different resolutions is consistent. Figure 17 presents the convergence tests for the axion (left panel) and dilaton (right panel) fields. Specifically, it shows the residuals between the low and medium resolutions, and the medium and high resolutions scaled by $Q_3 = 1.22$, indicating third order convergence. Figure 18 shows the percent error between the medium and high resolutions for each of the fields. At late times $t > 150M$, the fields settle to their final profiles, and

we find a numerical error of $\Delta\Theta_{10}/\Theta_{10,h} < 0.1\%$ (left panel) and $\Delta\Phi_{00}/\Phi_{00,h} < 0.02\%$ (right panel).

4. Binary black hole convergence tests

To verify the accuracy of the BBH results, we also perform a convergence study for a BBH simulation with equal-mass, nonrotating BHs with coupling strength $\hat{\alpha}_s = 0.1$. We use three resolutions with grid spacing on the coarsest refinement level of $dx_l = 1.6$, $dx_m = 1.28$, and $dx_h = 1.067$. In Figs. 19–23, we test convergence for the Weyl scalar $\Psi_{4,22}$, the axion field Θ_{10} and Θ_{32} , and the

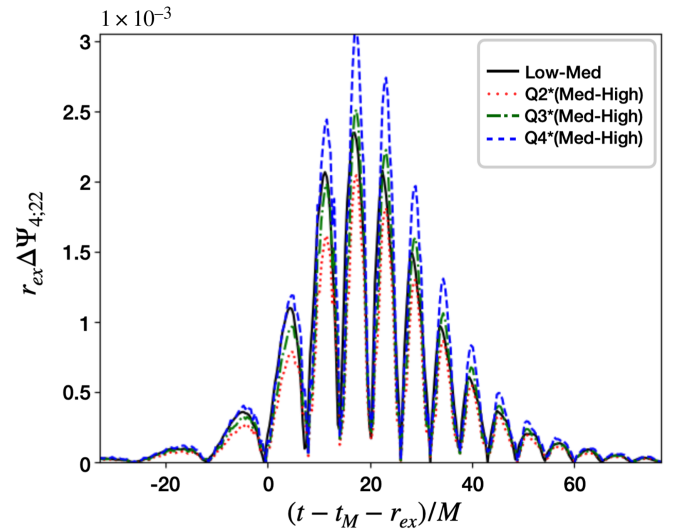
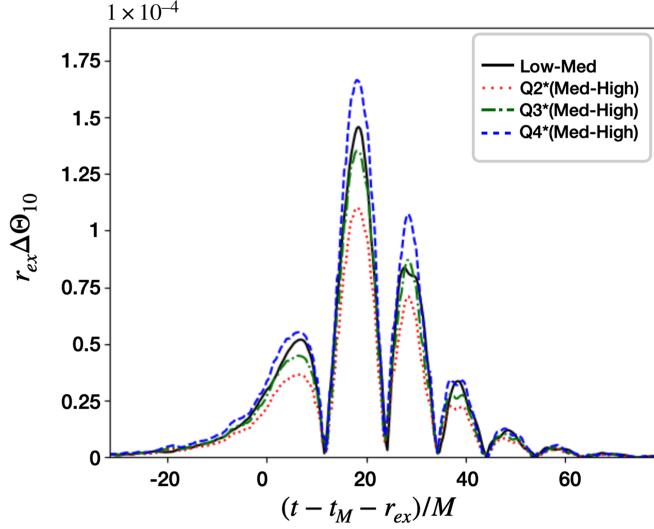
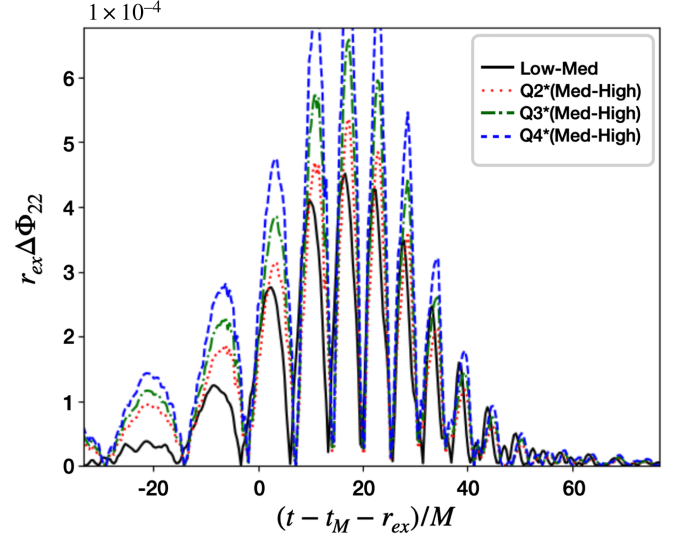
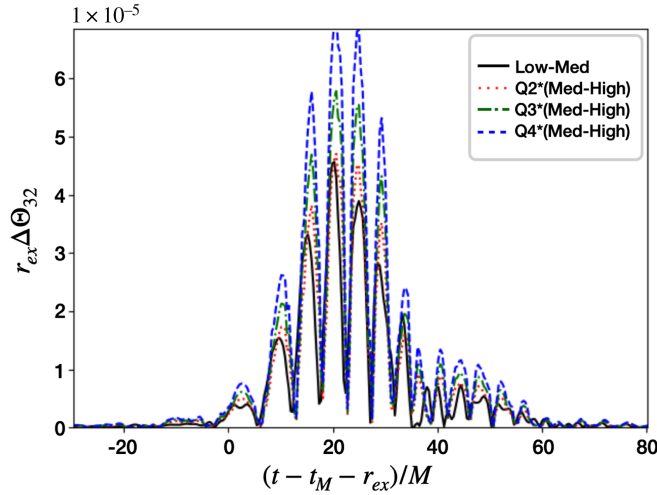
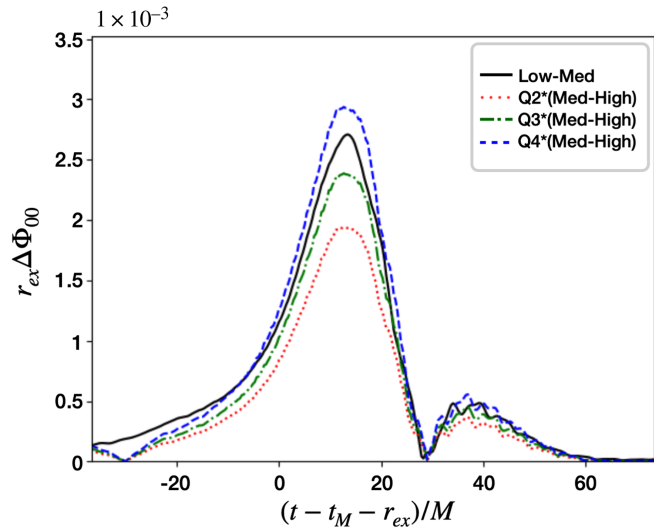


FIG. 19. Convergence plots of $\Psi_{4,22}$. Binary simulation of equal-mass nonrotating BHs with coupling strength $\hat{\alpha}_s = 0.1$. We compare the difference between the low and medium resolution run (black) with the medium and high resolution run. The latter is rescaled by Q_2 (red), Q_3 (green), and Q_4 (blue) indicating second, third, and fourth order convergence, respectively.

FIG. 20. Same as Fig. 19 but for Θ_{10} .FIG. 23. Same as Fig. 19 but for Φ_{22} .FIG. 21. Same as Fig. 19 but for Θ_{32} .FIG. 22. Same as Fig. 19 but for Φ_{00} .

dilaton field Φ_{00} and Φ_{22} . For each quantity, we plot the difference between the low resolution and medium resolution simulations as well as the difference between the medium and high resolution simulations scaled by the factor Q_n . By varying n , we find the convergence factor that best aligns the data. The plots include $Q_2 = 1.84$, $Q_3 = 2.26$, and $Q_4 = 2.78$ indicating second, third, and fourth order convergence, respectively. The convergence order differs slightly for each of the quantities plotted, but typically falls between three and four.

APPENDIX C: 3+1 FORMULATION FOR QUADRATIC GRAVITY

In this section, we present the general evolution equations for the axion and dilaton fields in quadratic or biscalar-tensor gravity. We continue to work in the decoupling approximation, i.e., considering the axion and dilaton fields in the background of a vacuum GR spacetime. To present the most general evolution equations (without the assumption that $\alpha_{\text{GB}} = \alpha_{\text{CS}} = \alpha_s$), we use dimensionless coupling parameters $\hat{\alpha}_{\text{CS}} = \alpha_{\text{CS}}/M^2$ and $\hat{\alpha}_{\text{GB}} = \alpha_{\text{GB}}/M^2$, in analogy to $\hat{\alpha}_s = \alpha_s/M^2$ in the main text. In the CANUDA-AxiDil code, $\hat{\alpha}_{\text{CS}}$ and $\hat{\alpha}_{\text{GB}}$ correspond to the parameters aCS, aGB.

We present vacuum Einstein's equations in their BSSN formulation in Appendix C 1, followed by the time evolution formulation of Eqs. (13a) and (13b) in ADM-York form in Appendix C 2 and their BSSN form in Appendix C 3.

1. Einstein's equations in BSSN form

For completeness, we present the evolution and constraint equations of the GR background spacetime here. We employ the BSSN formulation [137,138] that consists of a clever

constraint addition and conformal rescaling of the evolution variables such that the resulting PDEs are strongly hyperbolic. In CANUDA, we use the W version of the BSSN formulation, with variables

$$W = \gamma^{-\frac{1}{6}}, \quad \tilde{\gamma}_{ij} = W^2 \gamma_{ij}, \quad (\text{C1a})$$

$$K = \gamma^{ij} K_{ij}, \quad \tilde{A}_{ij} = W^2 A_{ij}, \quad (\text{C1b})$$

$$\tilde{\Gamma}^i = \tilde{\gamma}^{kl} \tilde{\Gamma}^i_{kl} = -\partial_j \tilde{\gamma}^{ij}, \quad (\text{C1c})$$

where $\gamma \equiv \det \gamma_{ij}$, $\tilde{\gamma}_{ij}$, and W are the conformal metric and factor, chosen such that $\tilde{\gamma} \equiv \det \tilde{\gamma}_{ij} = 1$, $\tilde{A}_{ij}$ is the conformal tracefree part of the extrinsic curvature, and $\tilde{\Gamma}^i$ is the conformal connection function.

Constraints of the general relativity background: As is common in numerical relativity, we adopt a free evolution scheme. That is, we solve the constraints only on the initial time slice to obtain initial data.

During the evolution, we monitor the constraints to verify that they are satisfied. Thus, they provide an excellent first benchmark test for the simulations. In terms of the BSSN variables, Eqs. (C1), the Hamiltonian and momentum constraints in vacuum are given by

$$\mathcal{H} = R - \tilde{A}^{ij} \tilde{A}_{ij} + \frac{2}{3} K^2 = 0, \quad (\text{C2a})$$

$$\mathcal{M}_i = \tilde{D}^j \tilde{A}_{ij} - \frac{2}{3} \tilde{D}_i K - \frac{3}{W} \tilde{A}^j_i \tilde{D}_j W = 0, \quad (\text{C2b})$$

where $\tilde{D}_i$ and $\tilde{R}_{ij}$ are the covariant derivative and Ricci tensor associated with the conformal metric $\tilde{\gamma}_{ij}$, and we use the physical Ricci scalar as auxiliary variable,

$$R = W^2 \tilde{R} + 4W \tilde{D}^i \tilde{D}_i W - 6 \tilde{D}^i W \tilde{D}_i W. \quad (\text{C3})$$

Evolution equations of the GR background:

After appropriate constraint addition to eliminate the Ricci scalar and divergence of the (tracefree part of the) extrinsic curvature, we obtain

$$d_t W = \frac{1}{3} \alpha W K, \quad (\text{C4a})$$

$$d_t \tilde{\gamma}_{ij} = -2\alpha \tilde{A}_{ij}, \quad (\text{C4b})$$

$$d_t K = -D^i D_i \alpha + \alpha \left(\tilde{A}^{ij} \tilde{A}_{ij} + \frac{1}{3} K^2 \right), \quad (\text{C4c})$$

$$d_t \tilde{A}_{ij} = -W^2 [D_i D_j \alpha]^{\text{tf}} + \alpha (W^2 R^{\text{tf}}_{ij} + K \tilde{A}_{ij} - 2 \tilde{A}_{ik} \tilde{A}^k_j), \quad (\text{C4d})$$

$$d_t \tilde{\Gamma}^i = -2 \tilde{A}^{ik} \partial_k \alpha + \tilde{\gamma}^{kl} \partial_k \partial_l \beta^i + \frac{1}{3} \tilde{\gamma}^{ik} \partial_k \partial_l \beta^l + 2\alpha \left(\tilde{\Gamma}^i_{kl} \tilde{A}^{kl} - \frac{3}{W} \tilde{A}^{ik} \partial_k W - \frac{2}{3} \tilde{D}^i K \right), \quad (\text{C4e})$$

where $X^{\text{tf}}_{ij} = X_{ij} - \frac{1}{3} \gamma_{ij} \gamma^{kl} X_{kl}$ denotes the trace-free part of a tensor taken with respect to the *physical* metric, $d_t = \partial_t - \mathcal{L}_\beta$, and $\mathcal{L}_\beta$ denotes the Lie derivative along the shift vector. We introduced the abbreviations

$$R_{ij} = \tilde{R}_{ij} + \frac{1}{W} (\tilde{D}_i \tilde{D}_j W + \tilde{\gamma}_{ij} \tilde{D}^k \tilde{D}_k W) - \frac{2}{W^2} \tilde{\gamma}_{ij} \tilde{D}^k W \tilde{D}_k W, \quad (\text{C5a})$$

$$D_i D_j \alpha = \tilde{D}_i \tilde{D}_j \alpha + \frac{2}{W} \tilde{D}_{(i} W \tilde{D}_{j)} \alpha - \frac{1}{W} \tilde{\gamma}_{ij} \tilde{D}^k W \tilde{D}_k \alpha, \quad (\text{C5b})$$

and we denote $D^i D_i \alpha = \gamma^{ij} D_i D_j \alpha$. These equations are evolved in conjunction with the moving puncture gauge [139,140].

2. Scalar equations in ADM-York form

We now rewrite the axion's and dilaton's equations of motion, Eqs. (13a) and (13b), as a set of time evolution equations in ADM-York form. We follow the same procedure as discussed in Sec. IV C and find

$$d_t \Theta = -\alpha K_\Theta, \quad (\text{C6a})$$

$$d_t \Phi = -\alpha K_\Phi, \quad (\text{C6b})$$

$$\begin{aligned} d_t K_\Theta &= -\alpha D^i D_i \Theta - D^i \alpha D_i \Theta \\ &\quad + \alpha \left(K K_\Theta + \dot{V}_\Theta - \frac{\hat{\alpha}_{\text{CS}} M^2 \dot{f}_\Theta}{4 g_\Phi^2} \mathcal{R}_{\text{CS}} \right) \\ &\quad + 2\alpha \frac{g'_\Phi}{g_\Phi} \left(K_\Theta K_\Phi - D_i \Theta D^i \Phi \right), \end{aligned} \quad (\text{C6c})$$

$$\begin{aligned} d_t K_\Phi &= -\alpha D^i D_i \Phi - D^i \alpha D_i \Phi \\ &\quad + \alpha \left(K K_\Phi + V'_\Phi - \frac{\hat{\alpha}_{\text{GB}} M^2 f'_\Phi}{4} \mathcal{R}_{\text{GB}} \right) \\ &\quad - \alpha g'_\Phi g_\Phi \left(K_\Theta^2 - 2V_\Theta - D_i \Theta D^i \Theta \right). \end{aligned} \quad (\text{C6d})$$

The topological invariants, $\mathcal{R}_{\text{CS}}$ and $\mathcal{R}_{\text{GB}}$, appearing in the right-hand-side of the above equations, are defined in Eqs. (43) in terms of the gravito-electromagnetic tensors, E_{ij} and B_{ij} . The latter are defined in Eqs. (44).

To estimate the energy-momentum due to the axion and dilaton, it is useful to compute the projections of the effective energy momentum tensor given in Eq. (6). Before deriving those expressions, we rewrite the auxiliary tensors $\mathcal{E}_a$, $\mathcal{F}_{ab}$, and $\mathcal{G}_{ab}$ [see Eqs. (8)] in terms of the induced metric and extrinsic curvature.

The projections of the auxiliary tensor $\mathcal{F}_{ab}$ appear in the dCS modifications and are given by

$$\begin{aligned}\mathcal{F}_{nn} = & \ddot{f}_\Theta K_\Theta^2 + \frac{\hat{\alpha}_{\text{CS}} M^2 \dot{f}_\Theta^2}{4 g_\Phi^2} \mathcal{R}_{\text{CS}} \\ & + \dot{f}_\Theta (D_k D^k \Theta - K_\Theta K - \dot{V}_\Theta) \\ & - 2 \dot{f}_\Theta \frac{g'_\Phi}{g_\Phi} (K_\Theta K_\Phi - D_k \Phi D^k \Theta),\end{aligned}\quad (\text{C7a})$$

$$\mathcal{F}_i = \ddot{f}_\Theta K_\Theta D_i \Theta + \dot{f}_\Theta (D_i K_\Theta - K^j_i D_j \Theta), \quad (\text{C7b})$$

$$\mathcal{F}_{ij} = \ddot{f}_\Theta D_i \Theta D_j \Theta + \dot{f}_\Theta (D_i D_j \Theta - K_\Theta K_{ij}), \quad (\text{C7c})$$

where we used Eq. (C6c) to derive $\mathcal{F}_{nn}$. In a vacuum GR background, $R_{ab} = 0 = \nabla^e R_{ab}$, so terms multiplying auxiliary tensor $\mathcal{E}_a$ vanish and thus do not contribute to the effective energy-momentum tensor. Nonetheless, we state its projections here for completeness:

$$\mathcal{E}_{nn} = \dot{f}_\Theta K_\Theta, \quad \mathcal{E}_i = \dot{f}_\Theta D_i \Theta. \quad (\text{C8})$$

The projections of the auxiliary tensor $\mathcal{G}_{ab}$, appearing in the contributions from the Gauss-Bonnet term, are

$$\begin{aligned}\mathcal{G}_{nn} = & f''_\Phi K_\Phi^2 + \frac{\hat{\alpha}_{\text{GB}} M^2}{4} f'_\Phi{}^2 \mathcal{R}_{\text{GB}} \\ & + f'_\Phi (D_k D^k \Phi - V'_\Phi - K_\Phi K) \\ & + f'_\Phi g'_\Phi g_\Phi (K_\Theta^2 - 2V_\Theta - D_k \Theta D^k \Theta)\end{aligned}\quad (\text{C9a})$$

$$\mathcal{G}_i = f''_\Phi K_\Phi D_i \Phi + f'_\Phi (D_i K_\Phi - K^j_i D_j \Phi), \quad (\text{C9b})$$

$$\mathcal{G}_{ij} = f''_\Phi D_i \Phi D_j \Phi + f'_\Phi (D_i D_j \Phi - K_\Phi K_{ij}), \quad (\text{C9c})$$

where we used the scalar field evolution equation, Eq. (C6d), to derive $\mathcal{G}_{nn}$. In the following, we denote the traces $\mathcal{F} = \gamma^{ij} \mathcal{F}_{ij}$ and $\mathcal{G} = \gamma^{ij} \mathcal{G}_{ij}$.

With all auxiliary tensors in place, we can write down the projections of the effective energy-momentum tensor, Eq. (6). The (effective) energy density $\rho^{\text{eff}} = n^a n^b T_{ab}^{\text{eff}}$, energy-momentum flux $j_i^{\text{eff}} = -\gamma^a_i n^b T_{ab}^{\text{eff}}$, and spatial stress-tensor $S_{ij}^{\text{eff}} = \gamma^a_i \gamma^b_j T_{ab}^{\text{eff}}$ become

$$\rho^{\text{eff}} = 2M^2 (\hat{\alpha}_{\text{CS}} B^{ij} \mathcal{F}_{ij} - \hat{\alpha}_{\text{GB}} E^{ij} \mathcal{G}_{ij}) + \frac{1}{2} (K_\Phi^2 + 2V_\Phi + D_i \Phi D^i \Phi) + \frac{g_\Phi^2}{2} (K_\Theta^2 + 2V_\Theta + D_i \Theta D^i \Theta), \quad (\text{C10a})$$

$$j_i^{\text{eff}} = 2\hat{\alpha}_{\text{CS}} M^2 (B_{ij} \mathcal{F}^j - \epsilon_{ijk} E^{lj} \mathcal{F}_l^k) - 2\hat{\alpha}_{\text{GB}} M^2 (E_{ij} \mathcal{G}^j + \epsilon_{ijk} B^{lj} \mathcal{G}_l^k) + (K_\Phi D_i \Phi + g_\Phi^2 K_\Theta D_i \Theta), \quad (\text{C10b})$$

$$\begin{aligned}S_{ij}^{\text{eff}} = & 2\hat{\alpha}_{\text{CS}} M^2 (B_{ij} (\mathcal{F} + \mathcal{F}_{nn}) + \gamma_{ij} B^{kl} \mathcal{F}_{kl} - 2B^k_{(i} \mathcal{F}_{j)k} + 2\epsilon_{(i|}{}^{kl} \mathcal{F}_k E_{|j)l}) \\ & - 2\hat{\alpha}_{\text{GB}} M^2 (E_{ij} (\mathcal{G} + \mathcal{G}_{nn}) + \gamma_{ij} E^{kl} \mathcal{G}_{kl} - 2E^k_{(i} \mathcal{G}_{j)k} - 2\epsilon_{(i|}{}^{kl} \mathcal{G}_k B_{|j)l}) \\ & + g_\Phi^2 \left(D_i \Theta D_j \Theta + \frac{1}{2} \gamma_{ij} (K_\Theta^2 - 2V_\Theta - D_k \Theta D^k \Theta) \right) + \left(D_i \Phi D_j \Phi + \frac{1}{2} \gamma_{ij} (K_\Phi^2 - 2V_\Phi - D_k \Phi D^k \Phi) \right).\end{aligned}\quad (\text{C10c})$$

3. Scalar equations in BSSN form

We implement the axion and dilaton equations in their BSSN form in CANUDA-AxiDiL,

$$d_t \Theta = -\alpha K_\Theta, \quad (\text{C11a})$$

$$d_t \Phi = -\alpha K_\Phi, \quad (\text{C11b})$$

$$d_t K_\Theta = -W^2 \tilde{D}^k \alpha \tilde{D}_k \Theta + \alpha \left(K K_\Theta - W^2 \tilde{D}^i \tilde{D}_i \Theta + W \tilde{D}^i \Theta \tilde{D}_i W + \dot{V}_\Theta - \frac{\hat{\alpha}_{\text{CS}} M^2 \dot{f}_\Theta}{4 g_\Phi^2} \mathcal{R}_{\text{CS}} \right) + 2\alpha \frac{g'_\Phi}{g_\Phi} (K_\Theta K_\Phi - W^2 \tilde{D}^k \Phi \tilde{D}_k \Theta), \quad (\text{C11c})$$

$$\begin{aligned}d_t K_\Phi = & -W^2 \tilde{D}^k \alpha \tilde{D}_k \Phi + \alpha \left(K K_\Phi - W^2 \tilde{D}^i \tilde{D}_i \Phi + W \tilde{D}^i \Phi \tilde{D}_i W + V'_\Phi - \frac{\hat{\alpha}_{\text{GB}} M^2}{4} f'_\Phi \mathcal{R}_{\text{GB}} \right) \\ & - \alpha g'_\Phi g_\Phi (K_\Theta^2 - 2V_\Theta - W^2 \tilde{D}^k \Theta \tilde{D}_k \Theta).\end{aligned}\quad (\text{C11d})$$

The curvature invariants are expressed as

$$\mathcal{R}_{\text{CS}} = -16\tilde{E}^{ij}\tilde{B}_{ij}, \quad (\text{C12a})$$

$$\mathcal{R}_{\text{GB}} = 8(\tilde{E}^{ij}\tilde{E}_{ij} - \tilde{B}^{ij}\tilde{B}_{ij}), \quad (\text{C12b})$$

where

$$\tilde{E}_{ij} = W^2 E_{ij} = W^2 R_{ij}^{\text{tf}} + \frac{1}{3}(\tilde{\gamma}_{ij}\tilde{A}^{kl}\tilde{A}_{kl} + K\tilde{A}_{ij} - 3\tilde{A}_i^k\tilde{A}_{kj}), \quad (\text{C13a})$$

$$\tilde{B}_{ij} = W^2 B_{ij} = \tilde{\epsilon}_{(i}{}^{kl}\tilde{A}_{|j)k}\tilde{D}_l W + W\tilde{\epsilon}_{(i}{}^{kl}\tilde{D}_k\tilde{A}_{|j)l}. \quad (\text{C13b})$$

The auxiliary tensors become

$$\tilde{\mathcal{F}}_{nn} = \ddot{f}_\Theta K_\Theta^2 + \frac{\hat{\alpha}_{\text{CS}} M^2 \dot{f}_\Theta^2}{4 g_\Phi^2} \mathcal{R}_{\text{CS}} + \dot{f}_\Theta (W^2 \tilde{D}^k \tilde{D}_k \Theta - W \tilde{D}^k W \tilde{D}_k \Theta - K K_\Theta - \dot{V}_\Theta) + 2 \frac{\dot{f}_\Theta g'_\Phi}{g_\Phi} (W^2 \tilde{D}^k \Theta \tilde{D}_k \Phi - K_\Theta K_\Phi), \quad (\text{C14a})$$

$$\tilde{\mathcal{F}}_i = \ddot{f}_\Theta K_\Theta \tilde{D}_i \Theta + \dot{f}_\Theta \left(\tilde{D}_i K_\Theta - \tilde{A}^j{}_i \tilde{D}_j \Theta - \frac{1}{3} K \tilde{D}_i \Theta \right), \quad (\text{C14b})$$

$$\tilde{\mathcal{F}}_{ij} = W^2 \mathcal{F}_{ij} = W^2 \dot{f}_\Theta \tilde{D}_i \tilde{D}_j \Theta + W^2 \ddot{f}_\Theta \tilde{D}_i \Theta \tilde{D}_j \Theta + W \dot{f}_\Theta (2 \tilde{D}_{(i} W \tilde{D}_{j)} \Theta - \tilde{\gamma}_{ij} \tilde{D}^k W \tilde{D}_k \Theta) - \dot{f}_\Theta K_\Theta \left(\tilde{A}_{ij} + \frac{1}{3} \tilde{\gamma}_{ij} K \right), \quad (\text{C14c})$$

and

$$\tilde{\mathcal{G}}_{nn} = f''_\Phi K_\Phi^2 + \frac{\hat{\alpha}_{\text{GB}} M^2}{4} f'_\Phi{}^2 \mathcal{R}_{\text{GB}} + g_\Phi g'_\Phi f'_\Phi (K_\Theta^2 - W^2 \tilde{D}^k \Theta \tilde{D}_k \Theta - 2V_\Theta) + f'_\Phi (W^2 \tilde{D}^k \tilde{D}_k \Phi - W \tilde{D}^k W \tilde{D}_k \Phi - K K_\Phi - V'_\Phi), \quad (\text{C15a})$$

$$\tilde{\mathcal{G}}_i = f''_\Phi K_\Phi \tilde{D}_i \Phi + f'_\Phi \left(\tilde{D}_i K_\Phi - \tilde{A}^j{}_i \tilde{D}_j \Phi - \frac{1}{3} K \tilde{D}_i \Phi \right), \quad (\text{C15b})$$

$$\tilde{\mathcal{G}}_{ij} = W^2 \mathcal{G}_{ij} = W^2 f'_\Phi \tilde{D}_i \tilde{D}_j \Phi + W^2 f''_\Phi \tilde{D}_i \Phi \tilde{D}_j \Phi + W f'_\Phi (2 \tilde{D}_{(i} \Phi \tilde{D}_{j)} W - \tilde{\gamma}_{ij} \tilde{D}^k W \tilde{D}_k \Phi) - f'_\Phi K_\Phi \left(\tilde{A}_{ij} + \frac{1}{3} \tilde{\gamma}_{ij} K \right). \quad (\text{C15c})$$

We denote the traces $\tilde{\mathcal{F}} = \tilde{\gamma}^{kl} \tilde{\mathcal{F}}_{kl}$ and $\tilde{\mathcal{G}} = \tilde{\gamma}^{kl} \tilde{\mathcal{G}}_{kl}$.

Finally, the projections of the effective energy-momentum tensor are

$$\rho^{\text{eff}} = \frac{1}{2} (W^2 \tilde{D}^k \Phi \tilde{D}_k \Phi + K_\Phi^2 + 2V_\Phi) + \frac{g_\Phi^2}{2} (W^2 \tilde{D}^k \Theta \tilde{D}_k \Theta + K_\Theta^2 + 2V_\Theta) + 2M^2 (\hat{\alpha}_{\text{CS}} \tilde{B}^{kl} \tilde{\mathcal{F}}_{kl} - \hat{\alpha}_{\text{GB}} \tilde{E}^{kl} \tilde{\mathcal{G}}_{kl}), \quad (\text{C16a})$$

$$j_i^{\text{eff}} = K_\Phi \tilde{D}_i \Phi + g_\Phi^2 K_\Theta \tilde{D}_i \Theta + 2\hat{\alpha}_{\text{CS}} M^2 \left(\tilde{B}_i{}^j \tilde{\mathcal{F}}_j - \frac{1}{W} \tilde{\epsilon}_i{}^{jk} \tilde{E}'_j \tilde{\mathcal{F}}_{kl} \right) - 2\hat{\alpha}_{\text{GB}} M^2 \left(\tilde{E}_i{}^j \tilde{\mathcal{G}}_j + \frac{1}{W} \tilde{\epsilon}_i{}^{jk} \tilde{B}^l{}_j \tilde{\mathcal{G}}_{kl} \right), \quad (\text{C16b})$$

$$\begin{aligned} S_{ij}^{\text{eff}} &= \tilde{D}_i \Phi \tilde{D}_j \Phi + \frac{\tilde{\gamma}_{ij}}{2W^2} (K_\Phi^2 - 2V_\Phi - W^2 \tilde{D}_k \Phi \tilde{D}^k \Phi) + g_\Phi^2 \left(\tilde{D}_i \Theta \tilde{D}_j \Theta + \frac{\tilde{\gamma}_{ij}}{2W^2} (K_\Theta^2 - 2V_\Theta - W^2 \tilde{D}_k \Theta \tilde{D}^k \Theta) \right) \\ &\quad + 2 \frac{\hat{\alpha}_{\text{CS}} M^2}{W^2} (\tilde{B}_{ij} (\tilde{\mathcal{F}} + \tilde{\mathcal{F}}_{nn}) - 2\tilde{B}^k{}_{(i} \tilde{\mathcal{F}}_{j)k} + \tilde{\gamma}_{ij} \tilde{B}^{kl} \tilde{\mathcal{F}}_{kl} + 2W \tilde{\epsilon}_{(i}{}^{kl} \tilde{E}_{|j)k} \tilde{\mathcal{F}}_l) \\ &\quad - 2 \frac{\hat{\alpha}_{\text{GB}} M^2}{W^2} (\tilde{E}_{ij} (\tilde{\mathcal{G}} + \tilde{\mathcal{G}}_{nn}) - 2\tilde{E}^k{}_{(i} \tilde{\mathcal{G}}_{j)k} + \tilde{\gamma}_{ij} \tilde{E}^{kl} \tilde{\mathcal{G}}_{kl} - 2W \tilde{\epsilon}_{(i}{}^{kl} \tilde{B}_{|j)k} \tilde{\mathcal{G}}_l). \end{aligned} \quad (\text{C16c})$$

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