Journal of Functional Analysis 287 (2024) 110617

Contents lists available at ScienceDirect

Journal of Functional Analysis

journal homepage: www.elsevier.com/locate/jfa

Regular Article

Well-posedness of the stationary and slowly traveling wave problems for the free boundary incompressible Navier-Stokes equations

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A R T I C L E I N F O

Article history:
Received 28 November 2023
Accepted 30 July 2024
Available online 13 August 2024
Communicated by Michele Coti Zelati

MSC:
primary 35Q30, 35R35, 35S05
secondary 35C07, 76D33, 76D03

Keywords:
Free boundary incompressible Navier-Stokes
Stationary waves
Traveling waves

A B S T R A C T

We establish that solitary stationary waves in three dimensional viscous incompressible fluids are a general phenomenon and that every such solution is a vanishing wave-speed limit along a one parameter family of traveling waves. The setting of our result is a horizontally-infinite fluid of finite depth with a flat, rigid bottom and a free boundary top. A constant gravitational field acts normal to bottom, and the free boundary experiences surface tension. In addition to these gravity-capillary effects, we allow for applied stress tensors to act on the free surface region and applied forces to act in the bulk. These are posited to be in either stationary or traveling form.

In the absence of any applied stress or force, the system reverts to a quiescent equilibrium; in contrast, when such sources of stress or force are present, stationary or traveling waves are generated. We develop a small data well-posedness theory for this problem by proving that there exists a neighborhood of the origin in stress, force, and wave speed data-space in which we obtain the existence and uniqueness of stationary and traveling wave solutions that depend continuously on the stress-force data, wave speed, and other physical parameters.

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[1] N. Stevenson was supported by a National Science Foundation Graduate Research Fellowship.
[2] I. Tice was supported by a National Science Foundation Grant (DMS #2204912).

https://doi.org/10.1016/j.jfa.2024.110617

To the best of our knowledge, this is the first proof of well-posedness of the solitary stationary wave problem and the first continuous embedding of the stationary wave problem into the traveling wave problem. Our techniques are based on vector-valued harmonic analysis, a novel method of indirect symbol calculus, and the implicit function theorem.

1. Introduction

1.1. The free boundary Navier-Stokes system

Our goal in this paper is to study stationary and slowly traveling solutions to the free boundary incompressible Navier-Stokes equations in three dimensions. These equations govern the dynamics of a finite-depth layer of viscous, incompressible fluid lying between a fixed, rigid, flat bottom and an unknown (free) top that evolves with the fluid. In order to properly phrase the equations, we first establish some notation for describing the unknown fluid domain.

The fluids we study will always be assumed to occupy three-dimensional sets of the form

$$\Omega[\eta] = \{(x, y) \in \mathbb{R}^2 \times \mathbb{R} \ : \ 0 < y < b + \eta(x)\}, \tag{1.1}$$

where $b \in \mathbb{R}^+$ is a fixed parameter giving the equilibrium depth of the fluid, and $\eta : \mathbb{R}^2 \to (-b, \infty)$ is the unknown free surface function. We will always have that η is continuous so that the fluid domain $\Omega[\eta]$ is open and connected. The upper free boundary and the fixed lower boundary will be denoted by

$$\Sigma[\eta] = \{(x, y) \in \mathbb{R}^2 \times \mathbb{R} \ : \ y = b + \eta(x)\} \text{ and } \Sigma_0 = \mathbb{R}^2 \times \{0\}. \tag{1.2}$$

Throughout the paper we will also denote the equilibrium sets with the short-hand

$$\Omega = \Omega[0] = \mathbb{R}^2 \times (0, b) \text{ and } \Sigma = \Sigma[0] = \mathbb{R}^2 \times \{b\}. \tag{1.3}$$

The motion of the fluid domain is encoded through the use of a time-dependent free surface function $\zeta(t, \cdot) : \mathbb{R}^2 \to \mathbb{R}$ satisfying $\zeta(t, \cdot) + b > 0$, which then generates the moving fluid domain $\Omega[\zeta(t, \cdot)] \subset \mathbb{R}^3$ and the free upper boundary $\Sigma[\zeta(t, \cdot)]$ as above. The fluid is described by its velocity vector field $w(t, \cdot) : \Omega[\zeta(t, \cdot)] \to \mathbb{R}^n$ and its scalar pressure $r(t, \cdot) : \Omega[\zeta(t, \cdot)] \to \mathbb{R}$. The viscous stress tensor within the fluid is the symmetric tensor

$$S_\mu(r, w) = rI_{3\times 3} - \mu\mathbb{D}w, \tag{1.4}$$

where $\mathbb{D}w = \nabla w + \nabla w^{\mathrm{t}}$ is the symmetrized gradient, and $\mu \in \mathbb{R}^+$ is the fluid viscosity.

In this paper we will assume that there are two sources of bulk force that act on the fluid through vector fields defined on $\Omega[\zeta(t,\cdot)]$ for all t, as well as three sources of stress that act on the fluid through vector fields defined on $\Sigma[\zeta(t,\cdot)]$. The first bulk force is a uniform gravitational field $-\rho\mathfrak{g}e_3$, where $\rho \in \mathbb{R}^+$ is the constant fluid density, $\mathfrak{g} \in \mathbb{R}^+$ is the gravitational constant, and $e_3 = (0,0,1) \in \mathbb{R}^3$. The second is a spatially-varying generic bulk force $\mathcal{F}(t,\cdot) : \Omega[\zeta(t,\cdot)] \to \mathbb{R}^3$. The first of the stresses is due to a constant external pressure $P_{\mathrm{ext}} \in \mathbb{R}$, which then acts via the vector field $P_{\mathrm{ext}}\nu_{\zeta(t,\cdot)}$, where $\nu_{\zeta(t,\cdot)}$ is the unit normal to the free surface at time t. The second is generated by a generic spatially-varying stress tensor $\mathcal{T}(t,\cdot) : \Sigma[\zeta(t,\cdot)] \to \mathbb{R}^{3\times 3}$, which then defines the stress vector $\mathcal{T}(t,\cdot)\,\nu_{\zeta(t,\cdot)}$. We note that in continuum mechanics it is usually the case that $\mathcal{T}(t,\cdot)$ is symmetric, but this condition plays no role in our analysis, so we have allowed for the most general case. The third, and final, source of stress is due to surface tension and is given by the vector field $\kappa\mathscr{H}(\zeta(t,\cdot))\nu_{\zeta(t,\cdot)}$, where $\kappa \in \mathbb{R}^+$ is the coefficient of surface tension and the mean curvature operator is

$$\mathscr{H}(\zeta) = \nabla_{\|} \cdot ((1 + \left|\nabla_{\|}\zeta\right|^2)^{-1/2}\nabla_{\|}\zeta). \tag{1.5}$$

Here we have written $\nabla_{\|} = (\partial_1, \partial_2)$ to refer to the 'tangential gradient.'

The free boundary incompressible Navier-Stokes equations then dictate how ζ, w, and r evolve in time as the result of applied stresses and forces:

$$\begin{cases} \rho(\partial_t w + w\cdot\nabla w) + \nabla\cdot S_\mu(r,w) = -\rho\mathfrak{g}e_3 + \mathcal{F} & \text{in } \Omega[\zeta(t,\cdot)], \\ \nabla\cdot w = 0 & \text{in } \Omega[\zeta(t,\cdot)] \\ -S_\mu(r,w)\nu_\zeta + (P_{\mathrm{ext}} - \kappa\mathscr{H}(\zeta))\nu_\zeta = \mathcal{T}\nu_\zeta & \text{on } \Sigma[\zeta(t,\cdot)], \\ \partial_t\zeta + w\cdot(\nabla_{\|}\zeta, -1) = 0 & \text{on } \Sigma[\zeta(t,\cdot)], \\ w = 0 & \text{on } \Sigma_0. \end{cases} \tag{1.6}$$

The first equation in (1.6) is the momentum equation, and it requires a Newtonian balance of forces in the fluid bulk. Next is the incompressibility constraint, which asserts conservation of mass. After this is the dynamic boundary condition, which enforces a balance of stresses acting on the free surface. The penultimate equation is the kinematic boundary condition, which determines how the free surface evolves according to the fluid velocity. The final equation in (1.6) is simply the no-slip boundary condition for the velocity on the rigid bottom. For the sake of simplicity, we will henceforth assume that $\rho = 1$. This is no loss of generality, as we will continue to track the generic constants $(\mathfrak{g},\mu,\kappa) \in (\mathbb{R}^+)^3$ as well as generic sources of force and stress.

1.2. Equilibria, stationary and traveling reformulations, and the role of stresses and forces

The free boundary incompressible Navier-Stokes equations admit a flat, stationary solution in the absence of external stress or forces, i.e. $\mathcal{F} = 0$ and $\mathcal{T} = 0$; namely, fluid domain $\Omega = \Omega[0]$ and

$$(r_{\text{eq}}, w_{\text{eq}}, \zeta_{\text{eq}}) = (P_{\text{ext}} + \mathfrak{g}(b - \text{id}_{\mathbb{R}^3} \cdot e_3), 0, 0). \tag{1.7}$$

In this paper, we work perturbatively around this equilibrium to study stationary and slowly traveling solutions. To describe these, we let $\gamma \in \mathbb{R}$ denote a fixed speed and make the ansatz that $\mathcal{F}$ and $\mathcal{T}$ are time-independent in the frame moving at velocity γe_1. In turn, we assume that $\zeta(t, \cdot) = \eta(\cdot - \gamma t e_1)$, $w(t, \cdot) = v(\cdot - \gamma t e_1)$, $r(t, \cdot) = P_{\text{ext}} + \mathfrak{g}(b - \text{id}_{\mathbb{R}^3} \cdot e_3) + q(\cdot - \gamma t e_1) + \mathfrak{g}\eta(\cdot - \gamma t e_1)$, for new unknowns $\eta : \mathbb{R}^2 \to (-b, \infty)$, $v : \Omega[\eta] \to \mathbb{R}^3$, and $q : \Omega[\eta] \to \mathbb{R}$. Rewriting the system (1.6) under this ansatz yields:

$$\begin{cases} (v - \gamma e_1) \cdot \nabla v + \nabla \cdot S_\mu(q + \mathfrak{g}\eta, v) = \mathcal{F} & \text{in } \Omega[\eta], \\ \nabla \cdot v = 0 & \text{in } \Omega[\eta], \\ -S_\mu(q, v)\mathcal{N}_\eta - \kappa \mathscr{H}(\eta)\mathcal{N}_\eta = \mathcal{T}\mathcal{N}_\eta & \text{on } \Sigma[\eta], \\ \gamma \partial_1 \eta + v \cdot \mathcal{N}_\eta = 0 & \text{on } \Sigma[\eta], \\ v = 0 & \text{on } \Sigma_0, \end{cases} \tag{1.8}$$

where $\mathcal{N}_\eta = (-\nabla_\| \eta, 1)$. We emphasize three key features of this reformulation. First, the external pressure, P_{ext}, only appears in the hydrostatic background that has been subtracted off and will play no further role in the analysis of (1.8). Second, the constant gravitational force field has been shifted into the term $\mathfrak{g}\eta$ in the first equation. Third, while the set $\Omega[\eta]$ is determined by η, only derivatives of η appear in the equations themselves.

The problem (1.8) lies at the confluence of two distinct lines of inquiry in the mathematical fluid mechanics literature. The first line of inquiry treats the dynamic problem (1.6) as an initial value problem. In this context, the stationary problem ($\gamma = 0$ in (1.8)) arises naturally as a special type of global-in-time solution with stationary sources of force and stress. One then expects solutions to the stationary problem to play an essential role in the study of long-time asymptotics or attractors for the dynamic problem (see, for instance, Robinson [78]). The second line of inquiry, which dates back essentially to the beginning of mathematical fluid mechanics, concerns the search for traveling wave solutions moving with speed $\gamma \neq 0$. In this context, a huge literature exists for the corresponding inviscid problem, but progress on the viscous problem was initiated much more recently in the work of Leoni and Tice [58], and further developed by Stevenson and Tice [93,92], Koganemaru and Tice [55], and Nguyen and Tice [63]. The analysis in [55,58,63,93,92] crucially relies on the condition $\gamma \neq 0$ to provide an estimate for the free surface function in a scale of anisotropic Sobolev spaces. When $\gamma = 0$, this estimate degenerates, and [58] fails to construct solutions within their functional framework. Thus, a natural question is whether there exists an alternate functional framework in which solutions can be constructed for all γ in a neighborhood of 0.

Our main goal in the paper can now be roughly summarized as follows. For every $(\mathfrak{g}, \mu, \kappa) \in (\mathbb{R}^+)^3$ and γ in an open set containing 0 we wish to identify an open set of force and stress data that give rise to locally unique nontrivial solutions. Moreover, we aim to prove well-posedness in the sense of continuity of the solution triple with respect to the force-stress data as well as the various physical parameters and wave speed.

The stated goal suggests that the force and stress should play an essential role in the construction of solutions. This is indeed the case, as we now aim to justify. An elementary formal calculation yields the following balance between dissipation and power for solutions to (1.8):

$$\int_{\Omega[\eta]} \frac{\mu}{2} |\mathbb{D}v|^2 = \int_{\Omega[\eta]} \mathcal{F} \cdot v + \int_{\Sigma[\eta]} \mathcal{T} \nu_\eta \cdot v. \tag{1.9}$$

The physical interpretation of this identity is that if a stationary or traveling wave solution exists, then the power supplied by the forces and stress (the right side of (1.9)) must be in exact balance with the energy dissipation rate due to viscosity (the left side of (1.9)). Identity (1.9) tells us even more if we take $\mathcal{F} = 0$ and $\mathcal{T} = 0$, in which case the L^2-norm of $\mathbb{D}v$ vanishes in $\Omega[\eta]$. By a version of the Korn inequality, this implies that $v = 0$, and in turn, this implies that $q = 0$ and η is constant. Thus, we only expect to be able to generate non-trivial stationary or traveling wave solutions (in a Sobolev-type framework in which (1.9) is valid) via the application of nontrivial $\mathcal{F}$ or $\mathcal{T}$.

1.3. Previous work

We now turn our attention to a brief survey of the mathematical literature associated to (1.6) and (1.8). This is vast, so we will restrict our focus to those results most closely related to ours.

For a thorough review of the fully dynamic problem (1.6) in various geometries we refer to the surveys of Zadrzyńska [99] and Shibata and Shimizu [80]. Beale [16] established local well-posedness with surface tension neglected. With surface tension accounted for, Beale [17] established the existence of global solutions and derived their decay properties with Nishida [18]. Solutions with surface tension were also constructed in other settings by Allain [12], Tani [95], Bae [14], and Shibata and Shimizu [81]. Solutions without surface tension were also constructed in various settings by Abels [6], Guo and Tice [44,45], and Wu [98]. Related analysis of linearized and resolvent problems can be found in the work of Abe and Shibata [1,2], Abels [4,5,7], Abels and Wiegner [8], and Abe and Yamazaki [3].

The inviscid analog of the traveling wave problem, (1.8) with $\gamma \neq 0$, which is also known as the traveling water wave problem, has been the subject of intense work for more than a century. The survey articles of Toland [97], Groves [43], Strauss [94], and Haziot, Hur, Strauss, Toland, Wahlén, Walsh, and Wheeler [48] contain a thorough review. Focusing entirely on the case of solitary (i.e. non-periodic) waves, in analogy with what is studied in this paper, the only positive existence results on the stationary ($\gamma = 0$) inviscid problem we are aware of are the recent constructions by Ehrnström, Walsh, and Zeng [35] and by Matthies, Sewell, and Wheeler [60].

In contrast, progress on the viscous traveling wave problem has only recently commenced. Leoni and Tice [58] developed a well-posedness theory for small forcing and stress data, provided $\gamma \neq 0$. This was generalized to multi-layer, inclined, and periodic configurations by Stevenson and Tice [93] and Koganemaru and Tice [55]. The corresponding well-posedness theory for the compressible analog of (1.8) was developed by Stevenson and Tice [92]. Traveling waves for the Muskat problem were constructed with similar techniques by Nguyen and Tice [63]. There are also experimental studies of viscous traveling waves; for details, we refer to the work of Akylas, Cho, Diorio, and Duncan [29,34], Masnadi and Duncan [59], and Park and Cho [65,66].

We now turn our attention to the viscous, stationary ($\gamma = 0$) literature. To the best of our knowledge, the precise configuration we study in (1.8) - including bulk force, a surface stress, and a non-compact free surface - has not yet appeared in the literature for either the three-dimensional or two-dimensional problem. However, numerous models of similar physical scenarios have been considered.

The non-compactness of the free boundary presents a fundamental difficulty in studying (1.8), as it creates a low-mode degeneracy that simply is not present in, say, the spatially periodic variant or related problems with compact free boundaries. As such, we only briefly review the stationary literature for compact free boundaries. Benjamin [22] studied periodic disturbances to steady flow along an inclined plane in two dimensions. Periodic solutions in two dimensions were also studied by Puhnačev [76]. Solonnikov [85,86], Jean [54], and Ja Jin [53] studied various compact free surface problems with sources and sinks or inflow and outflow conditions in a bounded container with an applied force. In three dimensions, Bemelmans [19–21] studied various stationary droplet problems, considering both the cases with and without surface tension. Abergel [9] gave a geometric approach for studying various configurations in both two and three dimensions, which was expanded on in Abergel and Rouy [10]. We refer also to Solonnikov and Denisova [89] for more references regarding the bounded free surface stationary literature.

Next, we discuss the literature involving unbounded domains and non-compact free surfaces. Much of the attention of the existing work is devoted to steady flows driven by gravity down inclined planes with possibly non-uniform structure. In two dimensions, this configuration was considered by Socolescu [82], Nazarov and Pileckas [62], Pelickas and Socolowsky [67,68], Socolowsky [84], and Pileckas and Solonnikov [69]. In three dimensions Gellrich [37] studied stationary flows with large viscosity and small localized bulk force.

The remaining non-compact literature in two dimensions is primarily devoted to more complicated geometries. For instance, Pileckas [70–73] considered boundary inflows, moving lower boundaries, gliding plates, and flow down a plane making a corner, while Socolowsky [83] described fluid flowing out of a pipe, driven by gravity. In three dimensions the situation is similar: Pileckas [74] studied liquid coming out of a narrow channel onto an incline plane, and Solonnikov [87,88] studied the flow generated by the slow rotation of an immersed rod and flow out of a circular tube.

To conclude, we again emphasize that, to the best of our knowledge, there are no results in the literature that study either: (1) the well-posedness in all parameter regimes of the three dimensional, non-compact stationary wave problem with applied bulk force and surface stress, i.e. system (1.8) with $\gamma = 0$; or (2) the continuous connection between the recent developments in the viscous free boundary traveling wave literature and the stationary wave problem. We address both of these in this paper.

1.4. Flattened reformulation

It will be convenient to reformulate the system (1.8) in the stationary domain $\Omega = \mathbb{R}^2 \times (0, b)$. To this end, we construct a flattening map (also called a Hanzawa transformation in the free boundary literature) from η by way of $\mathfrak{F}_\eta : \Omega \to \Omega[\eta]$ defined via

$$\mathfrak{F}_\eta(x, y) = (x, y + \mathcal{E}\eta(x, y)), \tag{1.10}$$

where $\mathcal{E}$ is the extension operator considered in Proposition 5.17. Note that in Proposition 7.1 we show that the above flattening map is well-defined and enjoys a collection of useful properties on the class of free surface functions considered in this paper.

Given η in an appropriate function space (which will be specified later), and hence $\mathfrak{F}_\eta$, we define two related quantities: the Jacobian $J_\eta : \Omega \to \mathbb{R}^+$ and (when J_η is nowhere vanishing) the geometry matrix $\mathcal{A}_\eta : \Omega \to \mathbb{R}^{3\times3}$, defined respectively via

$$J_\eta = \det(\nabla \mathfrak{F}_\eta) = 1 + \partial_3 \mathcal{E}\eta = \partial_3(\mathfrak{F}_\eta \cdot e_3) \text{ and } \mathcal{A}_\eta = (\nabla \mathfrak{F}_\eta)^{-\mathrm{t}}. \tag{1.11}$$

Provided that $J_\eta > 0$ and $J_\eta, 1/J_\eta \in L^\infty(\Omega)$, we then have that $\mathfrak{F}_\eta(\Omega) = \Omega[\eta]$ and $\mathfrak{F}_\eta$ is a homeomorphism from $\overline{\Omega}$ to $\overline{\Omega[\eta]}$ such that its restriction to Ω defines a smooth diffeomorphism to $\Omega[\eta]$, $\mathfrak{F}_\eta(\Sigma) = \Sigma[\eta]$, and $\mathfrak{F}_\eta$ is the identity on Σ_0. It will also be useful to introduce the map

$$M_\eta = J_\eta \mathcal{A}_\eta^{\mathrm{t}} = \begin{pmatrix} (1 + \partial_3 \mathcal{E}\eta) I_{2\times2} & 0_{2\times1} \\ -\mathcal{E}(\nabla_\| \eta) & 1 \end{pmatrix} : \Omega \to \mathbb{R}^{3\times3} \tag{1.12}$$

when reformulating (1.8).

We then introduce the new unknowns $p = q \circ \mathfrak{F}_\eta : \Omega \to \mathbb{R}$ and $u = M_\eta(v \circ \mathfrak{F}_\eta) : \Omega \to \mathbb{R}^3$. The problem (1.8) then transforms to the following system:

$$\begin{cases} M_\eta^{-\mathrm{t}}((u - \gamma M_\eta e_1) \cdot \nabla(M_\eta^{-1} u)) + \nabla(p + \mathfrak{g}\eta) & \\ -\mu M_\eta^{-\mathrm{t}}(\nabla \cdot ((\mathbb{D}_{\mathcal{A}_\eta}(M_\eta^{-1} u)) M_\eta^{\mathrm{t}})) = J_\eta M_\eta^{-\mathrm{t}} \mathcal{F} \circ \mathfrak{F}_\eta & \text{in } \Omega, \\ \nabla \cdot u = 0 & \text{in } \Omega, \\ -(pI - \mu \mathbb{D}_{\mathcal{A}_\eta}(M_\eta^{-1} u)) M_\eta^{\mathrm{t}} e_3 - \kappa \mathscr{H}(\eta) M_\eta^{\mathrm{t}} e_3 = \mathcal{T} \circ \mathfrak{F}_\eta M_\eta^{\mathrm{t}} e_3 & \text{on } \Sigma, \\ u \cdot e_3 + \gamma \partial_1 \eta = 0 & \text{on } \Sigma, \\ u = 0 & \text{on } \Sigma_0. \end{cases} \tag{1.13}$$

Here we have used the notation $\mathbb{D}_{\mathcal{M}} w = \nabla w \mathcal{M}^{\mathrm{t}} + \mathcal{M} \nabla w^{\mathrm{t}}$, for $\mathcal{M} = \mathcal{A}_\eta$ and $w = M_\eta^{-1} u$.

1.5. Statement of main result and discussion

In order to state and discuss our principal results, we must first introduce the function spaces we will employ in our analysis. We will do so rapidly here, emphasizing that these spaces are more thoroughly developed in Sections 1.6 and 5.

Given an open set $\Gamma \subseteq \mathbb{R}^d$, $k \in \mathbb{N}$, and a normed vector space V, we will write $C^k(\Gamma; V)$ for the k-times continuously differentiable maps from Γ to V. The notation $C_0^k(\Gamma; V)$ denotes the (possible) subspace of functions $f \in C^k(\Gamma; V)$ for which $\sup_{|\alpha| \leqslant k} |\partial^\alpha f(x_n)| \to 0$ along any sequence $\{x_n\}_{n \in \mathbb{N}} \subset \Gamma$ for which $|x_n| \to \infty$ as $n \to \infty$.

Fix $1 < r < 2$. For I denoting either the set $(0, b)$ or else $\mathbb{R}$ and $U = \mathbb{R}^2 \times I$, we define $L_{r,2}(U) = L^r(\mathbb{R}^2; L^2(I))$ to be the mixed-type Lebesgue space. For $s \in \mathbb{N}$ we define the mixed-type Sobolev spaces $H^s_{r,2}(U)$ modeled on $L_{r,2}(U)$ in the natural way (see Definition 5.1). For $t \in [0, \infty)$, we let $H^{t,r}(\mathbb{R}^2) = \{f \in L^r(\mathbb{R}^2) : \langle D \rangle^t f \in L^r(\mathbb{R}^2)\}$ denote the standard Bessel potential Sobolev space (see, e.g., Section 1.3.1 in Grafakos [42] or Section 6.2 in Berg and Löfström [23]) and let $\widetilde{H}^{1+t,r}(\mathbb{R}^2)$ denote the space of $L^{2r/(2-r)}(\mathbb{R}^2)$ functions whose distributional derivatives belong to $H^{t,r}(\mathbb{R}^2)$ (see Definition 5.14).

For $s \in \mathbb{N}$, $1 < r < 2$ we set $\mathbf{X}_{s,r} = H^{1+s}_{r,2}(\Omega) \times H^{2+s}_{r,2}(\Omega; \mathbb{R}^3) \times \widetilde{H}^{5/2+s,r}(\Sigma)$ and $\mathbf{W}_{s,r} = H^{1+s}_{r,2}(\mathbb{R}^3; \mathbb{R}^{3\times 3}) \times H^s_{r,2}(\mathbb{R}^3; \mathbb{R}^3)$. With the notation established, we come to our main theorem, which packages together several results in a fairly concise form but may be briefly summarized as follows: solitary stationary solutions to the free boundary incompressible Navier-Stokes equations are generic and depend continuously on the physical parameters and the data; moreover, every such solution lies along a one parameter family of slowly traveling waves. After the theorem statement we will further unpack and discuss its various statements.

Theorem 1 *(Proved in Section 7: see Theorem 7.13, Proposition 7.1, and Corollary 7.14). Let $1 < r < 2$ and $\mathbb{N} \ni s > 3/r + 1$. Then there exist open sets $W_s \subset \mathbb{R} \times (\mathbb{R}^+)^3 \times \mathbf{W}_{s,r}$ and $\{V_s(\mathfrak{g}, \mu, \kappa)\}_{(\mathfrak{g},\mu,\kappa) \in (\mathbb{R}^+)^3} \subset \mathbf{X}_{s,r}$, satisfying*

$$\{0\} \times (\mathbb{R}^+)^3 \times \{0\} \subset W_s \text{ and } 0 \in \bigcap_{(\mathfrak{g},\mu,\kappa)\in(\mathbb{R}^+)^3} V_s(\mathfrak{g},\mu,\kappa), \tag{1.14}$$

and a continuous map

$$W_s \ni (\gamma, \mathfrak{g}, \mu, \kappa, \mathcal{T}, \mathcal{F}) \mapsto (p, u, \eta) \in V_s(\mathfrak{g}, \mu, \kappa) \subset \mathbf{X}_{s,r} \tag{1.15}$$

such that the following hold.

(1) ***Classical regularity and flattening map diffeomorphism:*** *Let* $k = s - 2 - \lfloor 3/r \rfloor \in \mathbb{N}$*. Then* $\mathbf{X}_{s,r} \hookrightarrow C_0^{2+k}(\Omega) \times C_0^{3+k}(\Omega; \mathbb{R}^3) \times C_0^{4+k}(\Sigma)$*. Moreover, for every* $(p, u, \eta) \in \bigcup_{(\mathfrak{g},\mu,\kappa)\in(\mathbb{R}^+)^3} V_s(\mathfrak{g}, \mu, \kappa)$*, the associated flattening map* $\mathfrak{F}_\eta$ *defined in (1.10) is a smooth diffeomorphism from* Ω *to* $\Omega[\eta]$ *that extends to a* C^{4+k} *diffeomorphism from* $\overline{\Omega}$ *to* $\overline{\Omega[\eta]}$*.*

(2) ***Solution operator:*** *The map (1.15) is a solution operator to the flattened system (1.13) in the sense that for each* $(\gamma, \mathfrak{g}, \mu, \kappa, \mathcal{T}, \mathcal{F}) \in W_s$ *the corresponding pressure, velocity, and free surface* $(p, u, \eta) \in V_s(\mathfrak{g}, \mu, \kappa)$ *is the unique triple in* $V_s(\mathfrak{g}, \mu, \kappa)$ *classically solving (1.13) with stress-force data* $(\mathcal{T}, \mathcal{F})$*, wave speed* γ*, and physical parameters* $(\mathfrak{g}, \mu, \kappa)$*. Moreover, the free surface* η *obeys an extra 'degenerating anisotropic estimate' in the sense that the composition map*

$$W_s \ni (\gamma, \mathfrak{g}, \mu, \kappa, \mathcal{T}, \mathcal{F}) \mapsto (p, u, \eta) \mapsto \gamma \mathcal{R}_1 \eta \in L^r(\Sigma) \tag{1.16}$$

is well-defined and continuous. Here $\mathcal{R}_1$ *refers to the Riesz transform in the* e_1*-direction.*

(3) ***Eulerian transfer:*** *Each solution to the flattened system (1.13) produced by (1.15) gives rise to a classical solution to the stationary-traveling Eulerian formulation of the problem given by system (1.8) by undoing the change of unknowns that led from (1.8) to (1.13).*

We now pause to unpack the content of this theorem with a few comments. The theorem guarantees that for every choice of positive physical parameters $\mathfrak{g}$, μ, and κ there exists a non-empty open neighborhood of the origin in wave-speed, stress, and force data $(\gamma, \mathcal{T}, \mathcal{F})$-space for which we can uniquely solve (1.13), and the solution depends continuous on the data and wave speed, as well as the physical parameters. One should think of this as being analogous to a small-data global existence theory for the corresponding dynamic problem, which is all one should expect due to potential singularity formation in the boundary geometry [28,31].

It is worth highlighting both the superfluous and concrete boundaries of our main theorem. We choose to work in three spatial dimensions for the following two reasons. First, our methods here simply do not work for the two-dimensional variant of (1.13). This is due to the fact that in two spatial dimensions the interface is one dimensional, and hence the container space for the free surface function is degenerate for all choices for tangential integrability parameters $1 < r$. The choice $r = 1$ would be an adequate replacement, but the harmonic analysis methods employed here are unavailable in that setting. The second reason we chose to study the three dimensional problem is physical relevance. The entirety of the theorem can be generalized to handle dimension four and higher, but in this case it would actually be possible to construct solutions in a simpler functional framework utilizing only L^2-based spaces. Another hard boundary in our theorem is seen in the signs of the physical parameters. We crucially use the strict positivity of the coefficient of gravity $\mathfrak{g}$, the viscosity μ, and the surface tension coefficient κ and are simply unable to relax any of these parameters to zero. On the other hand, we believe that the lower regularity threshold of $\mathbb{N} \ni s > 3/r + 1$ in Theorem 1 is a soft boundary. We chose this numerology in an effort to minimize the complexity of the nonlinear analysis, but it could potentially be improved upon with sufficient additional work.

The final remark is on the qualitative nature of the waves produced by Theorem 1. Thanks to the embedding guaranteed by the first item of this theorem, we see that the free surface perturbation η decays to zero at infinity. This means that the waves we construct are solitary waves, to borrow a phrase from the traveling wave literature. Due to the level of generality of our main result, there is not much more we can say about the qualitative nature of our solutions; however, our well-posedness result opens to the door to more detailed qualitative studies given a fixed wave speed and applied stress and force data.

We now state a couple consequences of the main theorem that formalize the above discussion. The first clarifies what we know for a fixed choice of physical parameters $(\mathfrak{g}, \mu, \kappa) \in (\mathbb{R}^+)^3$.

Corollary 2 *(Proved in the third item of Corollary 7.14). Let r and s be as in Theorem 1. Then for each $(\mathfrak{g}, \mu, \kappa) \in (\mathbb{R}^+)^3$ there exists an open set $(0,0,0) \in W_s(\mathfrak{g}, \mu, \kappa) \subset \mathbb{R} \times \mathbf{W}_{s,r}$ with the property that for every triple of wave-speed, stress, and force data $(\gamma, \mathcal{T}, \mathcal{F}) \in W_s(\mathfrak{g}, \mu, \kappa)$ there exists a unique $(p, u, \eta) \in V_s(\mathfrak{g}, \mu, \kappa)$ such that system (1.13) is satisfied classically.*

The second corollary elucidates how we formulate well-posedness of the stationary wave problem.

Corollary 3 *(Proved in the fourth item of Corollary 7.14). Let r and s as in Theorem 1. Then there exists an open set*

$$(\mathbb{R}^+)^3 \times \{0\} \subset Z_s \subset (\mathbb{R}^+)^3 \times \mathbf{W}_{s,r} \tag{1.17}$$

and a continuous mapping

$$Z_s \ni (\mathfrak{g}, \mu, \kappa, \mathcal{T}, \mathcal{F}) \mapsto (p, u, \eta) \in V_s(\mathfrak{g}, \mu, \kappa) \subset \mathbf{X}_{s,r} \tag{1.18}$$

with the property that for each $(\mathfrak{g}, \mu, \kappa, \mathcal{T}, \mathcal{F}) \in Z_s$ *there exists a unique* $(p, u, \eta) \in V_s(\mathfrak{g}, \mu, \kappa)$ *such that the stationary free boundary incompressible Navier-Stokes equations, system* (1.13) *with* $\gamma = 0$*, is satisfied classically.*

We now aim to summarize the principal difficulties in proving Theorem 1 and our strategies for overcoming them. This discussion also serves as an outline of the paper.

As is the case for the traveling wave problems studied in [55,58,63,93,92], the stationary boundary value problem (1.13) lies in an unbounded domain of infinite measure and possesses a non-compact free boundary. The equations are quasilinear and do not enjoy a variational formulation. Consequently, compactness, Fredholm, and variational techniques are unavailable. This suggests that the production of solutions ought to proceed via a perturbative argument, such as the implicit function theorem, which has proved successful in the aforementioned work on traveling waves. As such, we begin our discussion by stating the linearization of (1.13) at zero-wave speed around the equilibrium solution:

$$\begin{cases} \nabla(p + \mathfrak{g}\eta) - \mu\Delta u = f & \text{in } \Omega, \\ \nabla \cdot u = 0 & \text{in } \Omega, \\ -(pI - \mu\mathbb{D}u)e_3 - \kappa\Delta_{\|}\eta e_3 = k & \text{on } \Sigma, \\ u \cdot e_3 = 0 & \text{on } \Sigma, \\ u = 0 & \text{on } \Sigma_0. \end{cases} \tag{1.19}$$

The most natural linear theory for system (1.19) lies within L^2-based Sobolev spaces. In Section 2 we prove that for every choice of $s \in \mathbb{N}$, $f \in H^s(\Omega;\mathbb{R}^3)$, and $k \in H^{1/2+s}(\Sigma;\mathbb{R}^3)$, there exists a solution $p \in H^{1+s}(\Omega)$, $u \in H^{2+s}(\Omega;\mathbb{R}^3)$, and $\eta \in \widetilde{H}^{5/2+s}(\Sigma)$ (meaning $\nabla\eta \in H^{3/2+s}(\Sigma;\mathbb{R}^2)$) that is unique up to modifications of η by constants. Moreover, we have an estimate of the solution (p, u, η) in terms of the data (f, k).

While this basic L^2-based linear theory is encouraging, it is ill-suited for the actual task at hand. The problem is two-fold. First, there is no canonical choice of η, as it is only determined up to a constant, and this is highly problematic in using η to generate the set $\Omega[\eta]$ in which the nonlinear problem (1.8) is posed. Second, and more severe, is that the inclusion $\nabla\eta \in H^{3/2+s}(\Sigma;\mathbb{R}^2)$ can never provide an estimate of $\eta \in L^\infty(\Sigma)$ for any choice of $s \in \mathbb{N}$. This is due to the nature of the critical Sobolev embedding in two dimensions, since $\nabla\eta \in L^2(\Sigma;\mathbb{R}^2)$ only guarantees that $\eta \in \mathrm{BMO}(\Sigma)$. The potential unboundedness of η is an even more severe obstruction in building $\Omega[\eta]$. It is worth noting that for the traveling problem with $\gamma \neq 0$, the papers [55,58,63,93,92] exploit an essential auxiliary estimate of $\gamma\mathcal{R}_1\eta \in L^2(\Sigma)$, where $\mathcal{R}_1$ is the Riesz transform in the e_1 direction, in order to guarantee η belongs to a special anisotropic Sobolev space that embeds into $C^0_0(\Sigma;\mathbb{R})$; this then overcomes the criticality problem and allows for the construction of solutions in the anisotropic space. We see from (1.16) that we obtain an analogous estimate here when $\gamma \neq 0$, but when $\gamma = 0$ there is simply no auxiliary estimate available, and so the anisotropic space is of no use.

Our path forward begins with the following observation. If we could develop a theory for (1.19) that ensured the inclusion $\nabla\eta \in W^{\ell-1,r}(\Sigma;\mathbb{R}^2)$ for some $1 \leqslant r < 2$ and $\mathbb{N} \ni \ell > 2/r$, then the subcritical Sobolev embedding would provide a canonical way to modify η by a constant to guarantee the inclusion $\eta \in L^{2r/(2-r)}(\Sigma)$. Due to the embedding $(L^{2r/(2-r)} \cap \dot{W}^{\ell,r})(\Sigma) \hookrightarrow C^0_0(\Sigma)$, free surface functions in this space are not only admissible for the nonlinear problem, but also enjoy a wealth of nonlinear properties that permit the transition from (1.19) to the full nonlinear problem (1.8).

We thus arrive at the principal task of developing a linear well-posedness theory for (1.19) that yields L^r-estimates (for $r < 2$) on the gradient of the free surface and its derivatives. One possible strategy for this would be to pose the problem in purely L^r-based Sobolev spaces. There are existing techniques in the literature (for instance, [1–5,7,8]) that provide an L^r-based well-posedness theory for the Stokes problems with various boundary conditions. Modifying these to incorporate a coupling to a free surface function η appears to present a number of serious challenges. Rather than start with this L^r-Stokes theory and attempt to build in a coupling to the free surface function, we have instead identified an alternate approach that is more deeply connected to the symmetries of the equilibrium domain and the natural L^2-energy structure of the problem. This technique allows for the simultaneous construction of the solution triple (p, u, η), has clear connections to the relatively simple L^2-existence theory, and has the potential for generalization to other problems with similar symmetries.

Our approach aims only to develop the L^r-theory in the horizontal variables, while maintaining an L^2-theory in the vertical variable. More concretely, we utilize mixed-type Sobolev spaces modeled on the mixed-type Lebesgue spaces $L_{r,2}(\Omega) = L^r(\mathbb{R}^2; L^2(0,b))$ for the bulk unknowns p and u and the bulk data, and we use Bessel potential Sobolev spaces $H^{s,r}(\mathbb{R}^2)$ for the (gradient of the) boundary unknown η and the boundary data.

At first glance it might seem that the mixed nature of these spaces will make them cumbersome to work with, but in fact they are a natural and streamlined choice of a functional framework to satisfy our stated goals, as we now aim to justify. First, we observe that the domain Ω is invariant under translations in the two horizontal variables and that the solution operator to system (1.19), denoted

$$T : L^2(\Omega;\mathbb{R}^3) \times H^{1/2}(\Sigma;\mathbb{R}^3) \to H^1(\Omega) \times H^2(\Omega;\mathbb{R}^3) \times (\widetilde{H}^{5/2}(\Sigma)/\mathbb{R}), \tag{1.20}$$

with $(p, u, \eta) = T(f, k)$ solving the PDE, commutes with all horizontal translations. By making the identification $\Sigma \simeq \mathbb{R}^2$ and employing the factorization (see Lemma 5.2)

$$H^s(\Omega;\mathbb{R}^\ell) = H^s(\mathbb{R}^2; L^2((0,b);\mathbb{R}^\ell)) \cap L^2(\mathbb{R}^2; H^s((0,b);\mathbb{R}^\ell)) \text{ for } s,\ell \in \mathbb{N}, \tag{1.21}$$

we see that T is a translation-commuting linear operator acting between certain infinite-dimensional Hilbert-valued Sobolev spaces. Building on some well-established tools in harmonic analysis (see Section 3.1), we deduce from this that T is diagonalized by the Fourier transform in the two horizontal variables. More precisely, this grants us the existence of an operator-valued symbol

$$m : \mathbb{R}^2 \to \mathcal{L}(L^2((0,b);\mathbb{C}^3) \times \mathbb{C}^3; H^1((0,b);\mathbb{C}) \times H^2((0,b);\mathbb{C}^3) \times \mathbb{C}) \tag{1.22}$$

such that $T = m(D)$ and the operator norm of T is equivalent to a certain weighted L^∞-type norm on the symbol m. Harmonic analysis provides numerous frameworks for extending Fourier multiplication operators, such as $m(D)$, from L^2-based spaces to L^r-based spaces for $1 < r < \infty$. A celebrated tool in this area is the Mikhlin-Hörmander multiplier theorem; briefly, this result says that if the derivatives of the symbol obey certain estimates, then the corresponding multiplication operator can be uniquely extended from L^2-based spaces to L^r-based spaces for every $1 < r < \infty$. In our context, with the symbol m and the map T, there is an appropriate vector-valued version of this result to which we appeal and subsequently generalize (see Theorems 3.8 and 3.14). Taking for granted, for the moment, that m satisfies the necessary symbol estimates, we then learn that

$$\begin{aligned} T : L^r(\mathbb{R}^2; L^2((0,b);\mathbb{R}^3)) \\ \times H^{1/2,r}(\Sigma;\mathbb{R}^3) \to (H^{1,r}(\mathbb{R}^2; L^2((0,b);\mathbb{R})) \cap L^r(\mathbb{R}^2; H^1((0,b);\mathbb{R}))) \\ \times (H^{2,r}(\mathbb{R}^2; L^2((0,b);\mathbb{R}^3)) \cap L^r(\mathbb{R}^2; H^2((0,b);\mathbb{R}^3))) \times (\widetilde{H}^{5/2,r}(\Sigma;\mathbb{R})/\mathbb{R}), \end{aligned} \tag{1.23}$$

is a bounded linear extension of (1.20) for any $1 < r < \infty$. The mixed-type Sobolev spaces now simply show up by undoing the factorization (1.21), i.e.

$$H^{s,r}(\mathbb{R}^2; L^2((0,b);\mathbb{R}^\ell)) \cap L^r(\mathbb{R}^2; H^s((0,b);\mathbb{R}^\ell)) = H^s_{r,2}(\Omega;\mathbb{R}^\ell) \text{ for } s, \ell \in \mathbb{N},\ r \in (1,\infty), \tag{1.24}$$

which means that (1.23) rewrites as

$$T : H^0_{r,2}(\Omega;\mathbb{R}^3) \times H^{1/2,r}(\Sigma;\mathbb{R}^3) \to H^1_{r,2}(\Omega) \times H^2_{r,2}(\Omega;\mathbb{R}^3) \times (\widetilde{H}^{5/2,r}(\Sigma)/\mathbb{R}). \tag{1.25}$$

In a similar manner, the mixed-spaces admit a simple Hilbert-valued Littlewood-Paley theory that allows for a rapid development of their properties.

Further evidence of the utility of the mixed-type Sobolev spaces and the tangential-L^r framework is seen in the fact that it allows us to verify that the vector-valued symbol m satisfies the necessary Mikhlin-Hörmander hypotheses in a surprisingly effective and efficient manner. Our proof requires no more than the vector-valued harmonic analysis toolbox of Section 3, paired with the identification of a certain recursive structure present in the L^2 theory for (1.19). In fact, this technique does not rely on any explicit formula for m, nor any specific fluid-dynamical structure of the equations themselves, and so we expect it can serve as a general method for other problems posed in domains with a partial translation symmetry. The main idea of our technique is that derivatives of the symbol are obtained from its difference quotients, which can be computed explicitly in terms of compositions of the solution operator T with modulation operators (see Proposition 4.5). The solution operator T interacts with modulation in a very simple manner due to the product rule, and this allows us to deduce differentiability properties of the symbol m by recursively employing T itself and the correspondence between its operator norm and L^∞-type norms of m. This not only yields the estimates needed to invoke Mikhlin-Hörmander, but also yields analyticity of m away from the origin (see Theorem 4.13).

Section 5 records a number of properties, linear and nonlinear, about the mixed-type Sobolev spaces and the subcritical gradient spaces. We combine these with the L^2-linear theory and the above vector-valued harmonic analysis ideas in Section 6, which culminates in the linear well-posedness result of Theorem 6.6. In Section 7 we then formulate the nonlinear system (1.13) as a nonlinear mapping between appropriate mixed-type spaces and then produce solutions via the implicit function theorem. The proofs of Theorem 1 and Corollaries 2 and 3 are recorded in Section 7.2.

The above discussion has focused entirely on the stationary ($\gamma = 0$) problem, so we conclude with a couple comments about the slowly traveling problem ($\gamma \asymp 0$). Previous work on the traveling problem [55,58,63,93,92] considered linearized operators with general $\gamma \in \mathbb{R} \setminus \{0\}$, but here we only study the case $\gamma = 0$. This explains how our result only ends up handing slowly traveling waves: the solutions with $\gamma \neq 0$ are obtained perturbatively from the $\gamma = 0$ analysis. Based on the L^2 theory, one would expect the free surface function to belong to the obvious L^r-analog of the anisotropic L^2-based Sobolev spaces mentioned above, and this is indeed the case. To handle the mismatch between these anisotropic spaces with $\gamma \neq 0$ and the isotropic space with $\gamma = 0$, we employ a special γ-dependent Fourier multiplier (see Definition 7.6) that reparameterizes the anisotropic function spaces in terms of the stationary isotropic function space. Inverting this operator (see Proposition 7.7) then shows the anisotropic inclusion that is recorded in (1.16).

We emphasize that our work establishes continuity of the solution map into the fixed isotropic space used for the stationary problem, even though the free surface function belongs to a strict subspace (determined by the anisotropic estimate (1.16)) when $\gamma \neq 0$. The limit $\gamma \to 0$ can then be understood as a singular limit, in the sense that this extra anisotropic estimate degenerates when $\gamma = 0$, resulting in a change in the topology of the container space. Another impact of this singular limit is that the anisotropic parameterization operators we use are at best continuous with respect to γ and not differentiable at $\gamma = 0$.

1.6. Notation

The set $\{0, 1, 2, \ldots\}$ is denoted by $\mathbb{N}$; $\mathbb{N}^+ = \mathbb{N} \setminus \{0\}$. The positive real numbers are $\mathbb{R}^+ = (0, \infty)$. $\mathbb{F}$ denotes either $\mathbb{R}$ or $\mathbb{C}$. The notation $\alpha \lesssim \beta$ means that there exists $C \in \mathbb{R}^+$, depending only on the parameters that are clear from context, for which $\alpha \leqslant C\beta$. To highlight the dependence of C on one or more particular parameters $a, \dots, b$, we will occasionally write $\alpha \lesssim_{a,\dots,b} \beta$. We also express that two quantities α, β are equivalent, written $\alpha \asymp \beta$ if both $\alpha \lesssim \beta$ and $\beta \lesssim \alpha$. We shall also employ the bracket notation

$$\langle x \rangle = \sqrt{1 + |x_1|^2 + \cdots + |x_\ell|^2} \text{ for } x \in \mathbb{C}^\ell. \tag{1.26}$$

If $\{X_i\}_{i=1}^\ell$ are normed spaces and X is their product, endowed with any choice of product norm $\|\cdot\|_X$, then we shall write

$$\|x_1, \dots, x_\ell\|_X = \|(x_1, \dots, x_\ell)\|_X \text{ for } (x_1, \dots, x_\ell) \in X. \tag{1.27}$$

We identify the dual of a complex Banach space X, denoted X^*, as the set of antilinear and continuous functionals, so that the dual pairing is sesquilinear (i.e. linear in the right argument) and, in the case that X is Hilbert, the Riesz map is linear.

If $\mathcal{H}$ is a separable Hilbert space, we will denote the Fourier and inverse Fourier transforms (normalized to be unitary on L^2) in the space of $\mathcal{H}$-valued tempered distributions over $\mathbb{R}^d$, $\mathscr{S}^*(\mathbb{R}^d;\mathcal{H})$, via $\mathscr{F}$ and $\mathscr{F}^{-1}$, respectively. For functions defined in the equilibrium domain Ω, we view them as vector-valued tempered distributions on $\mathbb{R}^2$ in the natural way; for example, $L^2(\Omega;\mathbb{C}) = L^2(\mathbb{R}^2; L^2((0,b);\mathbb{C})) \hookrightarrow \mathscr{S}^*(\mathbb{R}^2; L^2((0,b);\mathbb{C}))$. It is in these sense that we are to interpret the Fourier transform acting on functions defined on Ω. We frequently make the natural identification $\Sigma \simeq \mathbb{R}^2$ when performing Fourier analysis for functions defined on Σ.

We write $\nabla = (\partial_1, \dots, \partial_d)$ to denote the gradient on $\mathbb{R}^d$ for $d \in \mathbb{N}^+$. We refer to dimensions 2 and 3 simultaneously, in which case the $\mathbb{R}^2$-gradient is denoted by $\nabla_\| = (\partial_1, \partial_2)$, while the $\mathbb{R}^3$ gradient obeys the aforementioned notation. In $\mathbb{R}^2$ we denote the rotated-gradient operator as $\nabla_\|^\perp = (-\partial_2, \partial_1)$. We also let $D = \nabla/2\pi\mathrm{i}$ or $D = \nabla_\|/2\pi\mathrm{i}$, depending on context. The divergence and tangential divergence operators are written $\nabla \cdot f = \sum_{j=1}^3 \partial_j (f \cdot e_j)$ and $(\nabla_\|, 0) \cdot f = \sum_{j=1}^2 \partial_j (f \cdot e_j)$, for appropriate $\mathbb{R}^3$-valued functions f.

If $\mathcal{H}$ and $\mathcal{K}$ are Hilbert spaces and $m : \mathbb{R}^d \to \mathcal{L}(\mathcal{H};\mathcal{K})$ is a sufficiently nice symbol, we will write $m(D)$ for the linear operator, acting on certain subspaces of tempered distributions, defined via $\mathscr{F}^{-1}[m\mathscr{F}[\cdot]]$. In other words, $m(D)$ is the Fourier multiplication operator corresponding to the symbol m. If $\zeta \in \mathbb{R}^d$, we also let $m(D+\zeta) = (m(\cdot+\zeta))(D)$. The vector of Riesz transforms is $\mathcal{R} = (\mathcal{R}_1, \dots, \mathcal{R}_d)$, where $\mathcal{R}_i = |D|^{-1}\partial_i/2\pi$, $i \in \{1, \dots, d\}$.

We now turn our attention to the notation for various types of standard Sobolev spaces employed in this paper. First, we address certain negative homogeneous type Spaces. For $1 < p < 2$ we define the space

$$\dot{H}^{-1,p}(\mathbb{R}^2;\mathbb{F}) = \{f \in H^{-1,p}(\mathbb{R}^2;\mathbb{F}) \ : \ |D|^{-1} f \in L^p(\mathbb{R}^2;\mathbb{F})\}, \tag{1.28}$$

which is equipped with the norm $[f]_{\dot{H}^{-1,p}} = \||D|^{-1} f\|_{L^p}$. We also define

$$\dot{H}^{-1}(\mathbb{R}^2;\mathbb{F}) = \{f \in H^{-1}(\mathbb{R}^2;\mathbb{F}) \ : \ |\cdot|^{-1}\mathscr{F}[f] \in L^2(\mathbb{R}^2;\mathbb{C})\} \tag{1.29}$$

and endow it the norm $[f]_{\dot{H}^{-1}} = \||\cdot|^{-1}\mathscr{F}[f]\|_{L^2}$. We will sometimes write $\dot{H}^{-1,2} = \dot{H}^{-1}$. We then denote

$$\hat{H}^s(\Omega;\mathbb{F}) = \left\{g \in H^s(\Omega;\mathbb{F}) \ : \ \int_0^b g(\cdot,y)\,\mathrm{d}y \in \dot{H}^{-1}(\mathbb{R}^2;\mathbb{F})\right\} \tag{1.30}$$

which has the norm $\|g\|_{\hat{H}^s} = \left(\|g\|_{H^s}^2 + \left[\int_0^b g(\cdot,y)\,\mathrm{d}y\right]_{\dot{H}^{-1}}^2\right)^{1/2}$.

For $\mathbb{R} \ni s \geqslant 1$ we also define the gradient spaces:

$$\widetilde{H}^{s,p}(\mathbb{R}^2;\mathbb{F}) = \begin{cases} \{f \in L^1_{\mathrm{loc}}(\mathbb{R}^2;\mathbb{F}) \ : \ \nabla f \in H^{s-1,p}(\mathbb{R}^2;\mathbb{F}^2)\} & \text{if } p \geqslant 2, \\ \{f \in L^{2p/(2-p)}(\mathbb{R}^2;\mathbb{F}) \ : \ \nabla f \in H^{s-1,p}(\mathbb{R}^2;\mathbb{F}^2)\} & \text{if } 1 < p < 2. \end{cases} \tag{1.31}$$

The norm (seminorm if $p \geqslant 2$) is given by $\|f\|_{\widetilde{H}^{s,p}} = \|\nabla f\|_{H^{s-1,p}}$. When $p = 2$, we shall again write $\widetilde{H}^s(\mathbb{R}^2;\mathbb{F})$ in place of $\widetilde{H}^{s,2}(\mathbb{R}^2;\mathbb{F})$. The spaces $\widetilde{H}^{s,p}(\mathbb{R}^2;\mathbb{F})$ are complete for $p < 2$ (see Section 5.3 for this and other properties), while the quotient $\widetilde{H}^{s,p}(\mathbb{R}^2;\mathbb{F})/\mathbb{F}$ is complete for $p \geqslant 2$.

Now we consider the classical Bessel-potential Sobolev spaces. Given $\mathcal{H}$ a separable Hilbert space and $\mathbb{R} \ni s \geqslant 0$ we write

$$H^{s,p}(\mathbb{R}^2;\mathcal{H}) = \{f \in L^p(\mathbb{R}^2;\mathcal{H}) \ : \ \langle D\rangle^s f \in L^p(\mathbb{R}^2;\mathcal{H})\} \tag{1.32}$$

and equip this space with the standard norm $\|f\|_{H^{s,p}\mathcal{H}} = \|\langle D\rangle^s f\|_{L^p\mathcal{H}}$. When $p = 2$, we simply write $H^s(\mathbb{R}^2;\mathcal{H})$ in place of $H^{s,2}(\mathbb{R}^2;\mathcal{H})$. Since we are considering the Hilbert-valued case, the theory follows from straightforward adaptations of the scalar theory; for more information on the general Banach-valued cases, we refer the reader to Amann [13] or Section 5.6 in Hytönen, Neerven, Veraar, and Weis [51].

The trace operators on to the hypersurfaces Σ and Σ_0, acting on functions defined on Ω, are denoted by Tr_Σ and Tr_{Σ_0}, respectively. We will utilize the following closed subspace of $H^1(\Omega;\mathbb{F}^3)$:

$${}_0H^1(\Omega;\mathbb{F}^3) = \{u \in H^1(\Omega;\mathbb{F}^3) \ : \ \mathrm{Tr}_{\Sigma_0} u = 0\}. \tag{1.33}$$

For functions like $\eta : \Sigma \to \mathbb{F}$ we can view them as defined on Ω in the natural way, e.g. $\eta(x,y) = \eta(x)$ for $(x,y) \in \Omega$. In particular, the expression of $\nabla\eta$ in the bulk equations of say (1.13) refers to the $\mathbb{R}^3$-vector $(\partial_1\eta, \partial_2\eta, 0)$.

2. Basic linear theory

In this section we are concerned with the well-posedness of the following linear system of equations in the framework of L^2-based Sobolev spaces:

$$\begin{cases} \nabla(p + \mathfrak{g}\eta) - \mu\nabla \cdot \mathbb{D}u = f & \text{in } \Omega, \\ \nabla \cdot u = g & \text{in } \Omega, \\ -(pI - \mu\mathbb{D}u)e_3 - \kappa\Delta_{\|}\eta e_3 = k & \text{on } \Sigma, \\ u \cdot e_3 = h & \text{on } \Sigma, \\ u = 0 & \text{on } \Sigma_0. \end{cases} \tag{2.1}$$

Here the (complex) data are $f : \Omega \to \mathbb{C}^3$, $k : \Sigma \to \mathbb{C}^3$, $h : \Sigma \to \mathbb{C}$, and $g : \Omega \to \mathbb{C}$ while the (complex) unknowns are $u : \Omega \to \mathbb{C}^3$, $p : \Omega \to \mathbb{C}$, and $\eta : \Sigma \to \mathbb{C}$. One of the minor technical issues with this system of equations is that the lowest order term appearing for η is its gradient thus there is a kernel for the differential operator consisting of $p = 0$, $u = 0$, and $\eta = \text{constant} \in \mathbb{C}$. We get around this issue in the following two ways. First, in Sections 2.1 and 2.2, we work in a seminormed space functional framework, rather than a normed one. More precisely, we utilize the $\widetilde{H}^s(\mathbb{R}^2)$ spaces as in (1.31) as the containers for the linearized free surface variable.

As it turns out, working in seminormed spaces is not ideally suited for the next stage of our linear analysis, Section 4, in which we perform operator-valued symbol calculus on a solution operator to the linear problem. Thus, our second way of dealing with the kernel of (2.1) is that in the latter half of Section 2.2, we use an equivalent reformulation of (2.1) for data and solutions both belonging to normed spaces. The reformulation is given by:

$$\begin{cases} \mathfrak{g}(\chi, 0) + \nabla p - \mu\nabla \cdot \mathbb{D}u = f & \text{in } \Omega, \\ \nabla \cdot u = g & \text{in } \Omega, \\ -(pI - \mu\mathbb{D}u)e_3 - \kappa\nabla_{\|} \cdot \chi e_3 = k & \text{on } \Sigma, \\ \nabla_{\|}^{\perp} \cdot \chi = \omega & \text{on } \Sigma, \\ u \cdot e_3 = h & \text{on } \Sigma, \\ u = 0 & \text{on } \Sigma_0. \end{cases} \tag{2.2}$$

Here the data f, k, h are the same as before, and $\omega : \Sigma \to \mathbb{C}$ is a new datum. The solution is (p, u, χ), with p and u as before and $\chi : \Sigma \to \mathbb{C}^2$.

2.1. Weak solutions

The strategy for the theory of weak solutions is to prove a priori estimates and then handle existence via a sequence of approximate problems. The initial bounds allow us to deduce that this approximating sequence is Cauchy and has a limit that solves the equations.

The following definition sets the notation for the weak solution theory.

Definition 2.1 *(Weak formulation operators). We define the linear map*

$$\mathscr{I} : L^2(\Omega;\mathbb{C}) \times {}_0H^1(\Omega;\mathbb{C}^3) \times \widetilde{H}^{3/2}(\Sigma;\mathbb{C}) \to ({}_0H^1(\Omega;\mathbb{C}^3))^*, \tag{2.3}$$

through the action

$$\langle \mathscr{I}(p,u,\eta), v\rangle_{({}_0H^1)^*,{}_0H^1} = \int_\Omega \frac{\mu}{2}\mathbb{D}u : \overline{\mathbb{D}v} - p\overline{\nabla\cdot v} + \mathfrak{g}\nabla\eta\cdot\overline{v} - \kappa\langle\Delta_\|\eta, \mathrm{Tr}_\Sigma v\cdot e_3\rangle_{H^{-1/2},H^{1/2}}. \tag{2.4}$$

Recall that our notational convention is that the $(\cdot)^*$ of a Banach space is its anti-dual and the bracket pairing is antilinear in the right argument.

We also define the following closed subspaces of $H^1(\Omega;\mathbb{C}^3)$:

$$_\natural H^1(\Omega;\mathbb{C}^3) = \{u \in {}_0H^1(\Omega;\mathbb{C}^3) \ : \ \nabla\cdot u = 0,\ \mathrm{Tr}_\Sigma u\cdot e_3 = 0\}, \tag{2.5}$$

and for $\varepsilon \in (0,1)$

$$_\natural H^1_\varepsilon(\Omega;\mathbb{C}^3) = \{u \in {}_\natural H^1(\Omega;\mathbb{C}^3) \ : \ \mathrm{supp}\,\mathscr{F}[u] \subseteq \mathbb{R}^2\setminus B(0,\varepsilon)\}. \tag{2.6}$$

Note that in the above we are interpreting $\mathrm{supp}\mathscr{F}[u] \subseteq \mathbb{R}^2$ *as the support of the vector-valued tempered distribution $\mathscr{F}[u] \in \mathscr{S}^*(\mathbb{R}^2; L^2((0,b);\mathbb{C}^3))$.*

We quote the following construction of a solution operator to the divergence equation with Dirichlet boundary conditions. Recall that a linear map T on a vector space of functions defined on Ω is said to be translation commuting, or tangential translation commuting, if $(TX)(\cdot+h) = T(X(\cdot+h))$ for all functions X and all $h \in \mathbb{R}^3$ with $h\cdot e_3 = 0$.

Lemma 2.2 *(Solution operators to divergence equations). The following hold.*

(1) There exists a bounded, linear, and translation commuting map $\mathcal{B}$ such that for $\ell \in \mathbb{N}$ we have

$$\mathcal{B} : \hat{H}^\ell(\Omega;\mathbb{C}) \to H^1_0(\Omega;\mathbb{C}^3) \cap H^{1+\ell}(\Omega;\mathbb{C}^3) \tag{2.7}$$

and for all $f \in \hat{H}^0(\Omega;\mathbb{C})$ we have

$$\nabla\cdot\mathcal{B}f = f \quad \textit{and} \quad \mathrm{Tr}_{\partial\Omega}\mathcal{B}f = 0. \tag{2.8}$$

(2) There exists a bounded, linear, and translation commuting map $\overline{\mathcal{B}}$ such that for $\ell \in \mathbb{N}$ we have

$$\overline{\mathcal{B}} : H^\ell(\Omega;\mathbb{C}) \to {}_0H^1(\Omega;\mathbb{C}^3) \cap H^{1+\ell}(\Omega;\mathbb{C}^3) \tag{2.9}$$

and for all $f \in H^0(\Omega;\mathbb{C})$ we have $\nabla\cdot\overline{\mathcal{B}}f = f$.

(3) There exists a bounded, linear, and translation commuting map $\mathcal{B}_0$ such that for $\ell \in \mathbb{N}$ we have

$$\mathcal{B}_0 : H^{1/2+\ell}(\Sigma; \mathbb{C}) \cap \dot{H}^{-1}(\Sigma; \mathbb{C}) \to {}_0H^1(\Omega; \mathbb{C}^3) \tag{2.10}$$

and for all $\varphi \in H^{1/2}(\Sigma; \mathbb{C}) \cap \dot{H}^{-1}(\Sigma; \mathbb{C})$ we have

$$\nabla \cdot \mathcal{B}_0\varphi = 0 \quad \text{and} \quad \mathrm{Tr}_\Sigma \mathcal{B}_0\varphi = \varphi e_3. \tag{2.11}$$

Proof. The $\mathbb{R}$-valued variants of these operators are constructed in Proposition C.2 and Corollaries C.3 and C.4 in Stevenson and Tice [92]. Inspection of the proof shows that the solution operators are indeed translation commuting. The $\mathbb{C}$-valued assertions above follow from separate considerations of real and imaginary parts. □

We now prove a priori estimates for system (2.1) in the reduced case that $g = 0$ and $h = 0$.

Proposition 2.3 *(A priori estimates for weak solutions). Suppose that*

$$(p, u, \eta) \in L^2(\Omega; \mathbb{C}) \times {}_\natural H^1(\Omega; \mathbb{C}^3) \times \widetilde{H}^{3/2}(\Sigma; \mathbb{C}) \text{ and } F \in ({}_0H^1(\Omega; \mathbb{C}^3))^* \tag{2.12}$$

satisfy the equation

$$\mathscr{I}(p, u, \eta) = F, \tag{2.13}$$

or in other words, we have a weak solution to (2.1). Then we have the a priori estimate

$$\|p, u, \eta\|_{L^2 \times H^1 \times \widetilde{H}^{3/2}} \lesssim \|F\|_{({}_0H^1)^*}, \tag{2.14}$$

with an implicit constant depending on $\mathfrak{g}$, κ, and μ.

Proof. Fix $\lambda \in (0, 1)$, and let $\eta_\lambda = \mathscr{F}^{-1}[\mathbb{1}_{\mathbb{R}^2 \setminus B(0,\lambda)} \mathscr{F}[\eta]] \in H^{3/2}(\Sigma; \mathbb{C})$. Then (p, u, η_λ) solves the equation

$$\mathscr{I}(p, u, \eta_\lambda) = F - \mathscr{I}(0, 0, \eta - \eta_\lambda). \tag{2.15}$$

Testing this with u and integrating by parts, we acquire the identity

$$\langle F - \mathscr{I}(0, 0, \eta - \eta_\lambda), u \rangle_{({}_0H^1)^*, {}_0H^1} = \int_\Omega \frac{\mu}{2} |\mathbb{D}u|^2 - p\overline{\nabla \cdot u} + \mathfrak{g}\nabla \eta_\lambda \cdot \overline{u} = \int_\Omega \frac{\mu}{2} |\mathbb{D}u|^2. \tag{2.16}$$

Thus, by applying Korn's inequality (see, for instance, Proposition A.3 in Stevenson and Tice [92]) and sending $\lambda \to 0$, we obtain the estimate

$$\|u\|_{H^1} \lesssim \|F\|_{({}_0H^1)^*}. \tag{2.17}$$

We now derive an estimate on η. With η_λ as before, we define $v_\lambda \in {}_0H^1(\Omega;\mathbb{C}^3)$ via $v_\lambda = -\mathcal{B}_0(\langle\nabla_\|\rangle^{-1}\Delta_\|\eta_\lambda)$, with $\mathcal{B}_0$ from Lemma 2.2. The lemma provides the bound $\|v_\lambda\|_{H^1} \lesssim \|\eta\|_{\widetilde{H}^{3/2}}$. With the understanding that duality pairings are antilinear in the right argument, we then test (2.15) with v_λ and integrate by parts to learn that

$$\begin{aligned}
&\langle F - \mathscr{I}(0,0,\eta-\eta_\lambda), v_\lambda\rangle_{({}_0H^1)^*,{}_0H^1} \\
&\quad = \int_\Omega \frac{\mu}{2}\mathbb{D}u : \overline{\mathbb{D}v_\lambda} + \mathfrak{g}\nabla\eta_\lambda \cdot \overline{v_\lambda} + \kappa\langle\Delta_\|\eta_\lambda, \langle\nabla_\|\rangle^{-1}\Delta_\|\eta_\lambda\rangle_{H^{-1/2},H^{1/2}} \\
&\quad = \int_\Omega \frac{\mu}{2}\mathbb{D}u : \overline{\mathbb{D}v_\lambda} + \langle(\mathfrak{g}-\kappa\Delta_\|)\eta_\lambda, -\langle\nabla_\|\rangle^{-1}\Delta_\|\eta_\lambda\rangle_{H^{-1/2},H^{1/2}},
\end{aligned} \tag{2.18}$$

from which we deduce the estimate

$$\|\eta_\lambda\|^2_{\widetilde{H}^{3/2}} \lesssim \|\eta\|_{\widetilde{H}^{3/2}}(\|u\|_{H^1} + \|F - \mathscr{I}(0,0,\eta-\eta_\lambda)\|)_{({}_0H^1)^*}. \tag{2.19}$$

By sending $\lambda \to 0$ and combining with the already established estimate on u, we then derive the bound $\|\eta\|_{\widetilde{H}^{3/2}} \lesssim \|F\|_{({}_0H^1)^*}$.

Finally, we derive an estimate on p. For this, we test (2.13) with $\overline{\mathcal{B}}p \in {}_0H^1(\Omega;\mathbb{C}^3)$, where $\overline{\mathcal{B}}$ is again from Lemma 2.2, to see that

$$\langle F, v\rangle_{({}_0H^1)^*,{}_0H^1} = \int_\Omega \frac{\mu}{2}\mathbb{D}u : \overline{\mathbb{D}v} + \mathfrak{g}\nabla\eta\cdot\overline{v} - |p|^2, \tag{2.20}$$

which then implies the estimate

$$\|p\|_{L^2} \lesssim \|u,\eta,F\|_{H^1\times\widetilde{H}^{3/2}\times({}_0H^1)^*} \lesssim \|F\|_{({}_0H^1)^*}. \tag{2.21}$$

Synthesizing the above estimates then completes the proof. □

Our next result examines the existence of weak solutions.

Proposition 2.4 *(Existence and uniqueness of weak solutions). For any $F \in ({}_0H^1(\Omega;\mathbb{C}^3))^*$ there exists a $(p,u,\eta) \in L^2(\Omega;\mathbb{C}) \times {}_\natural H^1(\Omega;\mathbb{C}^3) \times \widetilde{H}^{3/2}(\Sigma;\mathbb{C})$ satisfying (2.13). The triple (p,u,η) is unique modulo changes of η by constant functions.*

Proof. Uniqueness, modulo constants in the linearized free surface variable, is a consequence of estimate (2.14) from Proposition 2.3. To prove existence let $\varepsilon \in (0,1)$ and consider the sesquilinear form $B : {}_\natural H^1_\varepsilon(\Omega;\mathbb{C}^3) \times {}_\natural H^1_\varepsilon(\Omega;\mathbb{C}^3) \to \mathbb{C}$ given by

$$B(u,v) = \int_\Omega \frac{\mu}{2}\mathbb{D}u : \overline{\mathbb{D}v}. \tag{2.22}$$

B is bounded and also coercive thanks to the Korn inequality (see, e.g., Proposition A.3 in [92]). Thus, the Lax-Milgram lemma shows that for every $F \in ({}_0H^1(\Omega;\mathbb{C}^3))^*$ (which defines an element of $({}_\natural H^1_\varepsilon(\Omega;\mathbb{C}^3))^*$ via restriction), there exists a unique $u_\varepsilon \in {}_\natural H^1_\varepsilon(\Omega;\mathbb{C}^3)$ such that $B(u_\varepsilon, v) = \langle F, v\rangle$ for all $v \in {}_\natural H^1_\varepsilon(\Omega;\mathbb{C}^3)$.

Next, we introduce the Hilbert spaces

$$\begin{aligned} L^2_\varepsilon(\Omega;\mathbb{C}) &= \{g \in L^2(\Omega;\mathbb{C}) \ : \ \operatorname{supp}\mathscr{F}[g] \subseteq \mathbb{R}^2 \setminus B(0,\varepsilon)\}, \\ {}_0H^1_\varepsilon(\Omega;\mathbb{C}^3) &= \{v \in H^1_0(\Omega;\mathbb{C}^3) \ : \ \operatorname{supp}\mathscr{F}[v] \subseteq \mathbb{R}^2 \setminus B(0,\varepsilon)\}, \\ H^s_\varepsilon(\Sigma;\mathbb{C}) &= \{h \in H^s(\Sigma;\mathbb{C}) \ : \ \operatorname{supp}\mathscr{F}[h] \subseteq \mathbb{R}^2 \setminus B(0,\varepsilon)\} \text{ for } s \in \mathbb{R}, \end{aligned} \tag{2.23}$$

which are all endowed with the inner-product from their defining container spaces. Clearly, we have the embeddings $L^2_\varepsilon(\Omega;\mathbb{C}) \hookrightarrow \hat{H}^0(\Omega;\mathbb{C})$, and $H^s_\varepsilon(\Sigma;\mathbb{C}) \hookrightarrow \dot{H}^{-1}(\Sigma;\mathbb{C})$ for any $s \geqslant -1$. Hence, Lemma 2.2 allows us to consider the bounded antilinear functional

$$L^2_\varepsilon(\Omega;\mathbb{C})\times H^{1/2}_\varepsilon(\Sigma;\mathbb{C}) \ni (g,h) \mapsto G_\varepsilon(g,h) = B(u_\varepsilon, \mathcal{B}g+\mathcal{B}_0h) - \langle F, \mathcal{B}g+\mathcal{B}_0h\rangle \in \mathbb{C} \tag{2.24}$$

and apply the Riesz-representation theorem to acquire $(q_\varepsilon, \zeta_\varepsilon) \in L^2_\varepsilon(\Omega;\mathbb{C}) \times H^{1/2}_\varepsilon(\Sigma;\mathbb{C})$ such that

$$G_\varepsilon(g,h) = (q_\varepsilon, g)_{L^2} + (\langle D\rangle^{1/2}\zeta_\varepsilon, \langle D\rangle^{1/2}h)_{L^2(\Sigma)} \text{ for all } (g,h) \in L^2_\varepsilon(\Omega;\mathbb{C}) \times H^{1/2}_\varepsilon(\Sigma;\mathbb{C}). \tag{2.25}$$

For any $v \in {}_0H^1_\varepsilon(\Omega;\mathbb{C}^3)$ we can use Lemma 2.2 to decompose $v = \mathcal{P}v + \mathcal{Q}v$ via

$$\mathcal{P}v = v - \mathcal{B}(\nabla\cdot v) - \mathcal{B}_0(\mathrm{Tr}_\Sigma v \cdot e_3) \text{ and } \mathcal{Q}v = \mathcal{B}(\nabla \cdot v) + \mathcal{B}_0(\mathrm{Tr}_\Sigma v\cdot e_3), \tag{2.26}$$

for bounded, linear, and translation commuting maps

$$\mathcal{P} : {}_0H^1_\varepsilon(\Omega;\mathbb{C}^3) \to {}_\natural H^1_\varepsilon(\Omega;\mathbb{C}^3) \text{ and } \mathcal{Q} : {}_0H^1_\varepsilon(\Omega;\mathbb{C}^3) \to {}_0H^1_\varepsilon(\Omega;\mathbb{C}^3). \tag{2.27}$$

Now, by the construction of u_ε we know that for any $v \in {}_0H^1_\varepsilon(\Omega;\mathbb{C}^3)$ we have the identity $B(u_\varepsilon, \mathcal{P}v) - \langle F, \mathcal{P}v\rangle = 0$, and hence, by the definition of G_ε and identity (2.25), we have

$$\begin{aligned} B(u_\varepsilon, v) - \langle F, v\rangle = B(u_\varepsilon, \mathcal{Q}v) - \langle F, \mathcal{Q}v\rangle = G_\varepsilon(\nabla\cdot v, \mathrm{Tr}_\Sigma v \cdot e_3) \\ = \int_\Omega q_\varepsilon \overline{\nabla\cdot v} + (\langle D\rangle^{1/2}\zeta_\varepsilon, \langle D\rangle^{1/2}\mathrm{Tr}_\Sigma v\cdot e_3)_{L^2(\Sigma)}. \end{aligned} \tag{2.28}$$

We then set

$$\eta_\varepsilon = \langle D\rangle(-\mathfrak{g} + \kappa\Delta_{\|})^{-1}\zeta_\varepsilon \in H^{3/2}(\Sigma;\mathbb{C}), \tag{2.29}$$

$p_\varepsilon = q_\varepsilon - \mathfrak{g}\eta_\varepsilon \in L^2_\varepsilon(\Omega;\mathbb{C})$, and $F_\varepsilon = \mathbb{1}_{\mathbb{R}^2\setminus B(0,\varepsilon)}(D)F \in ({}_0H^1(\Omega;\mathbb{C}^3))^*$ to learn from this identity and Definition 2.1 that

$$\mathscr{I}(p_\varepsilon, u_\varepsilon, \eta_\varepsilon) = F_\varepsilon. \tag{2.30}$$

Pick a sequence $\{\varepsilon_n\}_{n\in\mathbb{N}} \subset (0,1)$ such that $\varepsilon_n \to 0$ as $n \to \infty$. We claim that $\{(p_{\varepsilon_n}, u_{\varepsilon_n}, \eta_{\varepsilon_n})\}_{n\in\mathbb{N}} \subseteq L^2(\Omega;\mathbb{C}) \times {}_\natural H^1(\Omega;\mathbb{C}^3) \times \widetilde{H}^{3/2}(\Sigma;\mathbb{C})$ is Cauchy. To this end, we first note that (2.30) shows that

$$\mathscr{I}(p_{\varepsilon_n} - p_{\varepsilon_m}, u_{\varepsilon_n} - u_{\varepsilon_m}, \eta_{\varepsilon_n} - \eta_{\varepsilon_m}) = (\mathbb{1}_{\mathbb{R}^2\setminus B(0,\varepsilon_n)} - \mathbb{1}_{\mathbb{R}^2\setminus B(0,\varepsilon_m)})(D)F. \tag{2.31}$$

Then the a priori estimates for weak solutions in Proposition 2.3 grant the estimate

$$\begin{aligned}\|p_{\varepsilon_1} - p_{\varepsilon_0}, u_{\varepsilon_1} - u_{\varepsilon_0}, \eta_{\varepsilon_1} - \eta_{\varepsilon_0}\|_{L^2\times H^1\times \widetilde{H}^{3/2}} &\lesssim \|(\mathbb{1}_{\mathbb{R}^2\setminus B(0,\varepsilon_n)} - \mathbb{1}_{\mathbb{R}^2\setminus B(0,\varepsilon_m)})(D)F\|_{({}_0H^1)^*}\\ &\lesssim \|(\mathbb{1}_{\mathbb{R}^2\setminus B(0,\varepsilon_n)} - \mathbb{1}_{\mathbb{R}^2\setminus B(0,\varepsilon_m)})(D)(-\Delta)^{-1}F\|_{{}_0H^1},\end{aligned} \tag{2.32}$$

where $(-\Delta)^{-1}$ is the (translation commuting) inverse to the Σ_0-Dirichlet Σ-Neumann Laplacian in Ω. The claim is then proved by noting that

$$\limsup_{n,m\to\infty} \|(\mathbb{1}_{\mathbb{R}^2\setminus B(0,\varepsilon_n)} - \mathbb{1}_{\mathbb{R}^2\setminus B(0,\varepsilon_m)})(D)(-\Delta)^{-1}F\|_{{}_0H^1} = 0, \tag{2.33}$$

which follows from Lemma 5.2, Plancherel's theorem, and the monotone convergence theorem. The claim is proved.

With the claim in hand, we send $n \to \infty$ to obtain (p, u, η) belonging to the same space as the sequence. Testing against $v \in {}_0H^1(\Omega;\mathbb{C}^3)$ in identity (2.30), and sending $n \to \infty$, we then conclude that the limit (p, u, η) satisfies (2.13). □

2.2. Strong solutions

The purpose of this subsection is to obtain estimates in strong norms of solutions to the equations (2.1) and (2.2). We begin with the former.

Theorem 2.5 *(Analysis of strong solutions, I). Let $s \in \mathbb{N}$. For every*

$$(g, f, k, h) \in H^{1+s}(\Omega;\mathbb{C}) \times H^s(\Omega;\mathbb{C}^3) \times H^{1/2+s}(\Sigma;\mathbb{C}^3) \times H^{3/2+s}(\Sigma;\mathbb{C}) \tag{2.34}$$

satisfying

$$h - \int_0^b g(\cdot, y)\,\mathrm{d}y \in \dot{H}^{-1}(\Sigma;\mathbb{C}) \tag{2.35}$$

there exists a unique (again, with the understanding that η is only unique modulo constant functions)

$$(p, u, \eta) \in H^{1+s}(\Omega; \mathbb{C}) \times H^{2+s}(\Omega; \mathbb{C}^3) \times \widetilde{H}^{5/2+s}(\Sigma; \mathbb{C}) \tag{2.36}$$

such that system (2.1) *is solved with data* (2.34) *and solution* (2.36)*; moreover, we have the estimate*

$$\|p, u, \eta\|_{H^{1+s} \times H^{2+s} \times \widetilde{H}^{5/2+s}} \lesssim \|g, f, k, h\|_{H^{1+s} \times H^s \times H^{1/2+s} \times H^{3/2+s}} + \left[h - \int_0^b g(\cdot, y) \, \mathrm{d}y\right]_{\dot{H}^{-1}}. \tag{2.37}$$

Proof. We begin by introducing a useful linear operator. Let

$$\widetilde{\mathcal{B}} : \{(g, h) \in H^{1+s}(\Omega; \mathbb{C}) \times H^{3/2+s}(\Omega; \mathbb{C}) \ : \ (2.35) \text{ is satisfied}\} \to {}_0H^{2+s}(\Omega; \mathbb{C}^3) \tag{2.38}$$

be defined via

$$\widetilde{\mathcal{B}}(g, h) = \overline{\mathcal{B}}g + \mathcal{B}_0(h - \mathrm{Tr}_\Sigma \overline{\mathcal{B}}g \cdot e_3), \tag{2.39}$$

where $\overline{\mathcal{B}}$ and $\mathcal{B}_0$ are from Lemma 2.2; $\widetilde{\mathcal{B}}$ is well-defined thanks to the lemma and the fact that

$$h - \mathrm{Tr}_\Sigma \overline{\mathcal{B}}g \cdot e_3 = \Big(h - \int_0^b g(\cdot, y) \, \mathrm{d}y\Big) + \Big(\int_0^b g(\cdot, y) \, \mathrm{d}y - \mathrm{Tr}_\Sigma \overline{\mathcal{B}}g \cdot e_3\Big) \in \dot{H}^{-1}(\Sigma; \mathbb{C}). \tag{2.40}$$

Given a data tuple (2.34), we set $\widetilde{f} = f + \mu \nabla \cdot \mathbb{D}\widetilde{\mathcal{B}}(g, h) \in H^s(\Omega; \mathbb{C}^3)$ and $\widetilde{k} = k - \mu \mathrm{Tr}_\Sigma \mathbb{D}\widetilde{\mathcal{B}}(g, h) e_3 \in H^{1/2+s}(\Sigma; \mathbb{C}^3)$. Thanks to the mapping properties of $\widetilde{\mathcal{B}}$, the reduced data satisfy the estimate

$$\|\widetilde{f}, \widetilde{k}\|_{H^s \times H^{1/2+s}} \lesssim \|g, f, k, h\|_{H^{1+s} \times H^s \times H^{1/2+s} \times H^{3/2+s}} + \left[h - \int_0^b g(\cdot, y) \, \mathrm{d}y\right]_{\dot{H}^{-1}}. \tag{2.41}$$

We then consider the reduced problem of finding

$$(p, w, \eta) \in H^{1+s}(\Omega; \mathbb{C}) \times ({}_\natural H^1(\Omega; \mathbb{C}^3) \cap H^{2+s}(\Omega; \mathbb{C}^3)) \times \widetilde{H}^{5/2+s}(\Sigma; \mathbb{C}) \tag{2.42}$$

solving

$$\begin{cases} \nabla(p + \mathfrak{g}\eta) - \mu\nabla \cdot \mathbb{D}w = \widetilde{f} & \text{in } \Omega, \\ \nabla \cdot w = 0 & \text{in } \Omega, \\ -(pI - \mu\mathbb{D}w)e_3 - \kappa\Delta_{\|}\eta e_3 = \widetilde{k} & \text{on } \Sigma, \\ w \cdot e_3 = 0 & \text{on } \Sigma, \\ w = 0 & \text{on } \Sigma_0, \end{cases} \tag{2.43}$$

where the reduced data $(\widetilde{f}, \widetilde{k})$ are determined as above by a data tuple (2.34). We claim that the data (g, f, k, h) uniquely determine solutions to the reduced problem, provided they exist. Indeed, if (p, w, η) solve the reduced problem with data $(g, f, k, h) = 0$, then $(\widetilde{f}, \widetilde{k}) = 0$ and (p, w, η) satisfy $\mathscr{I}(p, w, \eta) = 0$; then the a priori estimates of Proposition 2.3 imply that $(p, w, \eta) = 0$. This proves the claim.

The connection between the reduced problem and the original is as follows. Given (p, u, η) as in (2.36) solving (2.1), then upon setting $w = u - \widetilde{\mathcal{B}}(g, h) \in {}_{\natural}H^1(\Omega; \mathbb{C}^3) \cap H^{2+s}(\Omega; \mathbb{C}^3)$ we arrive at a solution (p, w, η) to the reduced system. Conversely, if (p, w, η) as in (2.42) solve the reduced problem, then we obtain a solution to the original problem by setting $u = w + \widetilde{\mathcal{B}}(g, h)$; moreover,

$$\|u\|_{H^{2+s}} \lesssim \|w\|_{H^{2+s}} + \|g, f, k, h\|_{H^{1+s} \times H^s \times H^{1/2+s} \times H^{3/2+s}} + \left[h - \int_0^b g(\cdot, y)\, \mathrm{d}y\right]_{\dot{H}^{-1}}. \tag{2.44}$$

We thus reduce to solving the reduced problem and deriving the high regularity bounds

$$\|p, w, \eta\|_{H^{1+s} \times H^{2+s} \times \widetilde{H}^{5/2+s}} \lesssim \|\widetilde{f}, \widetilde{k}\|_{H^s \times H^{1/2+s}}. \tag{2.45}$$

Now, with $(\widetilde{f}, \widetilde{k})$ in hand, we define $F \in ({}_0H^1(\Omega; \mathbb{C}^3))^*$ via

$$\langle F, v \rangle = \int_\Omega \widetilde{f} \cdot \overline{v} + \int_\Sigma \widetilde{k} \cdot \overline{v} \in \mathbb{C}, \tag{2.46}$$

and use Proposition 2.4 to obtain a weak solution (p, w, η) to the reduced system. To complete the proof, it is thus sufficient to prove that for every $s \in \mathbb{N}$ and every $(\widetilde{f}, \widetilde{k}) \in H^s(\Omega; \mathbb{C}^3) \times H^{1/2+s}(\Sigma; \mathbb{C})$ the associated unique weak solution $(p, w, \eta) \in L^2(\Omega; \mathbb{C}) \times {}_{\natural}H^1(\Omega; \mathbb{C}^3) \times \widetilde{H}^{3/2}(\Sigma; \mathbb{C})$ to (2.43) satisfies the higher regularity bounds (2.45).

We proceed via induction. The case $s = 0$ is handled first. We let $\lambda \in (0, 1)$ and apply $|D|\mathbb{1}_{A_\lambda}(D)$ to weak solution identity (here $D = \nabla_{\|}/2\pi\mathrm{i}$ and $A_\lambda = B(0, \lambda^{-1}) \setminus \overline{B(0, \lambda)}$) and obtain that

$$\mathscr{I}(|D|\mathbb{1}_{A_\lambda}(D)p, |D|\mathbb{1}_{A_\lambda}(D)u, |D|\mathbb{1}_{A_\lambda}(D)\eta) = |D|\mathbb{1}_{A_\lambda}(D)F, \tag{2.47}$$

where F is as in (2.46). Thus we may invoke the a priori estimates of Proposition 2.3 to bound

$$\||D|p, |D|w, |D|\eta\|_{L^2\times H^1\times \widetilde{H}^{3/2}} \lesssim \limsup_{\lambda\to 0}\||D|\mathbb{1}_{A_\lambda}(D)F\|_{({}_0H^1)^*} \lesssim \|\widetilde{f},\widetilde{k}\|_{L^2\times H^{1/2}}, \quad (2.48)$$

which is the desired tangential regularity. To establish normal regularity, we note that

$$\partial_3 p = \mu\Delta_{\|} w\cdot e_3 - \mu(\nabla_{\|},0)\cdot\partial_3 w + \widetilde{f}\cdot e_3 \quad (2.49)$$

and

$$\mu\partial_3^2 w = -\mu\Delta_{\|} w + \nabla(p+\mathfrak{g}\eta) - \widetilde{f}. \quad (2.50)$$

Identity (2.49) (paired with (2.48)) establishes that $\partial_3 p \in L^2(\Omega;\mathbb{C})$. Then we use identity (2.50) to establish that $\partial_3^2 w \in L^2(\Omega;\mathbb{C}^3)$ as well. This completes the proof of the base case.

Now suppose that $s\in\mathbb{N}$ and assume the induction hypothesis at s. Further suppose

$$(\widetilde{f},\widetilde{k}) \in H^{1+s}(\Omega;\mathbb{R}^3)\times H^{3/2+s}(\Sigma;\mathbb{R}^3). \quad (2.51)$$

By the induction hypothesis and a tangential regularity argument similar to the one used in the base case, we obtain the estimate

$$\|p,w,\eta\|_{H^{1+s}\times H^{2+s}\times\widetilde{H}^{5/2+s}} + \sum_{j=1}^{2}\|\partial_j p,\partial_j w,\partial_j\eta\|_{H^{1+s}\times H^{2+s}\times\widetilde{H}^{5/2+s}} \lesssim \|\widetilde{f},\widetilde{k}\|_{H^{1+s}\times H^{3/2+s}}. \quad (2.52)$$

To complete the proof, we once more employ identities (2.49) and (2.50) to estimate $\partial_3 p$ and $\partial_3^2 w$ as before, which then proves the induction hypothesis at level $s+1$. □

Our final result of this subsection reformulates the previous result in an equivalent way that avoids the use of seminormed spaces. This will be the main take away of our linear analysis for utilization in the next section.

Theorem 2.6 *(Analysis of strong solutions, II). Let $s\in\mathbb{N}$. For every*

$$(g,f,k,h,\omega)\in H^{1+s}(\Omega;\mathbb{C})\times H^s(\Omega;\mathbb{C}^3)\times H^{1/2+s}(\Sigma;\mathbb{C}^3)\times H^{3/2+s}(\Sigma;\mathbb{C})\times H^{1/2+s}(\Sigma;\mathbb{C}) \quad (2.53)$$

satisfying

$$h - \int_0^b g(\cdot,y)\,\mathrm{d}y \in \dot{H}^{-1}(\Sigma;\mathbb{C}) \text{ and } \omega\in\dot{H}^{-1}(\Sigma;\mathbb{C}) \quad (2.54)$$

there exists a unique

$$(p,u,\chi)\in H^{1+s}(\Omega;\mathbb{C})\times H^{2+s}(\Omega;\mathbb{C}^3)\times H^{3/2+s}(\Sigma;\mathbb{C}^2) \quad (2.55)$$

such that the equations (2.2) *are satisfied. Moreover, we have the estimate*

$$\|p,u,\chi\|_{H^{1+s}\times H^{2+s}\times H^{3/2+s}} \lesssim \|g,f,k,h,\omega\|_{H^{1+s}\times H^{s}\times H^{1/2+s}\times H^{3/2+s}\times H^{1/2+s}} + \left[h - \int_0^b g(\cdot,y)\,\mathrm{d}y, \omega\right]_{\dot{H}^{-1}\times\dot{H}^{-1}} \tag{2.56}$$

Proof. For any $\chi \in H^{3/2+s}(\Sigma;\mathbb{C}^2)$ we have that

$$\chi = \Delta_{\|}^{-1}\nabla_{\|}\nabla_{\|}\cdot\chi + \Delta_{\|}^{-1}\nabla_{\|}^{\perp}\nabla_{\|}^{\perp}\cdot\chi. \tag{2.57}$$

Hence, given a solution to (2.2), we can set $\eta = \Delta_{\|}^{-1}\nabla_{\|}\cdot\chi \in \widetilde{H}^{5/2+s}(\Sigma;\mathbb{C})$ and observe that

$$\begin{cases} \nabla(p+\mathfrak{g}\eta) - \mu\nabla\cdot\mathbb{D}u = f - \mathfrak{g}(\Delta_{\|}^{-1}\nabla_{\|}^{\perp}\omega, 0) & \text{in } \Omega, \\ \nabla\cdot u = g & \text{in } \Omega, \\ -(pI - \mu\mathbb{D}u)e_3 - \kappa\Delta_{\|}\eta e_3 = k & \text{on } \Sigma, \\ u\cdot e_3 = h & \text{on } \Sigma, \\ u = 0 & \text{on } \Sigma_0. \end{cases} \tag{2.58}$$

On the other hand, given a data tuple (g,f,k,h,ω), we may use Theorem 2.5 to obtain the existence of a solution triples (p,u,η) to (2.58), which is unique modulo constants in the free surface variable. The solution also obeys the estimate

$$\|p,u,\eta\|_{H^{1+s}\times H^{2+s}\times \widetilde{H}^{5/2+s}} \lesssim \|g,f,k,h,\omega\|_{H^{1+s}\times H^{s}\times H^{1/2+s}\times H^{3/2+s}\times H^{1/2+s}} + \left[h - \int_0^b g(\cdot,y)\,\mathrm{d}y, \omega\right]_{\dot{H}^{-1}\times\dot{H}^{-1}}. \tag{2.59}$$

We then obtain the unique solution to (2.2) upon setting $\chi = \nabla_{\|}\eta + \Delta_{\|}^{-1}\nabla_{\|}^{\perp}\omega$. The bound (2.56) follows easily from (2.59). □

3. Vector-valued harmonic analysis

This section is a necessary step back from the main PDE line of the story into abstract, vector-valued harmonic analysis. Our goal moving forward is to take the solution operator to the reformulated linear system (2.2) granted by Theorem 2.6 and prove that we can extend it from its domain of L^2-based Sobolev spaces to some kind of L^r-based Sobolev spaces for an integrability parameter $1 < r < 2$. The reason for doing so is potentially opaque at this point, but it is exactly this change in integrability parameter to below the threshold 2 that makes it possible to come back to system (2.1) and pose it in normed spaces, rather than seminormed ones in such a way that the linkage with the nonlinear theory of Section 7 becomes possible.

Implementing the above program requires both old and new ideas in vector-valued harmonic analysis. It is thus the goal of this section of the document to record variants of classical results in harmonic analysis adapted to the vector-valued setting relevant for this paper and to showcase our new tools in the subject, which are Theorem 3.5, Corollary 3.6, and Theorem 3.14. We make an effort to include as many abbreviated proofs and external references as possible, striving for a concise treatment.

3.1. Translation commuting linear maps

This section is devoted to the diagonalization, via the Fourier transform, of vector-valued translation commuting linear maps on L^2-based Sobolev spaces. In the finite-dimensional vector-valued case, we have the following formulation.

Theorem 3.1 *(Translation commuting linear maps, finite dimensional case). Let V_0, V_1 be finite dimensional complex Hilbert spaces. The following are equivalent for a bounded linear map $T : L^2(\mathbb{R}^d; V_0) \to L^2(\mathbb{R}^d; V_1)$.*

(1) T commutes with translations in the sense that $(Tf)(\cdot + h) = T(f(\cdot + h))$ for all $f \in L^2(\mathbb{R}^d; V_0)$ and all $h \in \mathbb{R}^d$.
(2) There exists $m \in L^\infty(\mathbb{R}^d; \mathcal{L}(V_0, V_1))$ such that $T = m(D)$ in the sense that $\mathscr{F}^{-1} T \mathscr{F} = m$, where the right hand side is simply a multiplication operator.

In either case, we have that the operator norm of T coincides with the essential supremum of m, i.e.

$$\|T\|_{\mathcal{L}(L^2 V_0; L^2 V_1)} = \|m\|_{L^\infty \mathcal{L}(V_0; V_1)} \tag{3.1}$$

Proof. The proof with $V_0 = V_1 = \mathbb{C}$ is standard; see, for instance, Theorem 2.5.10 in Grafakos [41]. The general finite dimensional case follows easily from this using orthonormal bases. □

We require an infinite dimensional generalization of Theorem 3.1. To formulate this we first need an appropriate notion of measurable maps taking values in a space of bounded linear operators. This can be found in Hille and Phillips [50] (Definition 3.5.4, the subsequent remark applied for σ-finite measure spaces, and Definition 3.5.5), and we record it now in the second item in the definition below.

Definition 3.2 *(Some notions of measurability). Let X be a complete and σ-finite measure space.*

(1) ***Bochner measurability:*** *Let Y be a Banach space. We say that a function $g : X \to Y$ is Bochner measurable if it is the almost everywhere limit of a sequence of finitely-valued measurable (simple) functions.*
(2) ***Operator-valued strong measurability:*** *Let V_0, V_1 be Banach spaces over $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$. We say that a function $f : X \to \mathcal{L}(V_0; V_1)$ is strongly measurable if for all $v \in V_0$ we have that the map $fv : X \to V_1$ is Bochner measurable.*

Next we record some essential properties of this notion of measurability.

Theorem 3.3 *(Properties of operator-valued strongly measurable functions). Let V_0 and V_1 be separable Hilbert spaces over $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$ and X be a complete and σ-finite measure space. Let $f : X \to \mathcal{L}(V_0; V_1)$. Then the following hold.*

(1) *f is strongly measurable in the sense of the second item of Definition 3.2 if and only if for all $v_0 \in V_0$ and $v_1 \in V_1$ we have that $\langle fv_0, v_1\rangle : X \to \mathbb{F}$ is measurable.*
(2) *If f is strongly measurable and $g : X \to V_0$ is Bochner measurable in the sense of the first item of Definition 3.2, then $fg : X \to V_1$ is also Bochner measurable.*
(3) *If f is strongly measurable and $g : X \to \mathcal{L}(V_0; V_1)$ is such that $f = g$ almost everywhere, then g is strongly measurable.*

Proof. The first item follows from Theorem 3.5.5 in [50]. To prove the second item, we note that Theorem 3.5.4 in [50] shows that if $\psi : X \to V_0$ is simple, then $f\psi : X \to V_1$ is Bochner measurable. Thus, fg is the almost everywhere limit of a sequence of Bochner measurable functions and is then measurable, again by Theorem 3.5.4 in [50]. The third item follows by noting that if $h \in V_0$, then $f(x)h = g(x)h$ for almost every $x \in X$, and hence $g(\cdot)h$ is Bochner measurable. □

The notion of operator-valued strong measurability leads us to the following space of essentially bounded functions (see Blasco and van Neerven [24] for the generalization to $p < \infty$ and applications in the study of multiplication operators between vector-valued Lebesgue spaces).

Definition 3.4 *(Space of essentially bounded and strongly measurable operator-valued functions). Let V_0 and V_1 be separable Hilbert spaces over $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$ and X be a complete and σ-finite a measure space. We define*

$$L^\infty_*(X; \mathcal{L}(V_0; V_1)) = \{[f] \mid f : X \to \mathcal{L}(V_0; V_1) \text{ is strongly measurable and essentially bounded}\} \tag{3.2}$$

where $[f]$ denotes the usual equivalence class formed via almost everywhere equality. Note, though, that as per usual we will dispense with the equivalence class notation in what follows. For $f \in L^\infty_*(X; \mathcal{L}(V_0; V_1))$ we write

$$\|f\|_{L^\infty_*\mathcal{L}(V_0;V_1)} = \operatorname*{ess\,sup}_{x\in X} \|f(x)\|_{\mathcal{L}(V_0;V_1)} = \inf\{C \in \mathbb{R}^+ \,:\, \|f\|_{\mathcal{L}(V_0;V_1)} \leqslant C \text{ a.e.}\}. \quad (3.3)$$

We emphasize that although the notion of almost everywhere is used in the definition of the essential supremum, it does not actually require the map f to be measurable; consequently, the essential supremum is well-defined even on the space of strongly measurable $\mathcal{L}(V_0;V_1)$-valued maps.

We can now state and prove our first infinite dimensional generalization of Theorem 3.1. To the best of our knowledge, the following result does not appear in the literature.

Theorem 3.5 *(Translation commuting linear maps, infinite dimensional case I). Suppose that V_0 and V_1 are separable infinite dimensional Hilbert spaces over $\mathbb{C}$ and that*

$$T : L^2(\mathbb{R}^d;V_0) \to L^2(\mathbb{R}^d;V_1) \quad (3.4)$$

is a bounded linear map. Then the following are equivalent.

(1) T commutes with translations in the sense that $(Tf)(\cdot + h) = T(f(\cdot + h))$ for all $h \in \mathbb{R}^d$ and $f \in L^2(\mathbb{R}^d;V_0)$.
(2) There exists $m \in L^\infty_*(\mathbb{R}^d;\mathcal{L}(V_0;V_1))$ such that

$$Tf = \mathscr{F}^{-1}[m\mathscr{F}[f]] \text{ for every } f \in L^2(\mathbb{R}^d;V_0). \quad (3.5)$$

In either case, we have the equality

$$\|m\|_{L^\infty_*\mathcal{L}(V_0,V_1)} = \|T\|_{\mathcal{L}(L^2V_0;L^2V_1)}. \quad (3.6)$$

Proof. If T is given by (3.5), then it is a trivial matter to verify that it commutes with translations, so we only need to prove the converse and (3.6). We begin with the proof of the latter, assuming that $T = m(D)$ for $m \in L^\infty_*(\mathbb{R}^d;\mathcal{L}(V_0;V_1))$.

Let $f \in L^2(\mathbb{R}^d;V_0)$. Since for any null set $E \subset \mathbb{R}^d$ and $\xi \in \mathbb{R}^d \setminus E$ we have $\|m(\xi)\mathscr{F}[f](\xi)\|_{V_1} \leqslant \sup_{\mathbb{R}^d\setminus E}\|m\|_{\mathcal{L}(V_0;V_1)}\|\mathscr{F}[f](\xi)\|_{V_0}$, we are free to integrate the square of this, take the infimum over such E, and apply Plancherel's theorem to deduce that $\|Tf\|^2_{L^2V_1} = \|mf\|^2_{L^2V_1} \leqslant \|m\|^2_{L^\infty_*}\|f\|^2_{L^2V_0}$. Thus, $\|T\| \leqslant \|m\|$.

For the opposite inequality, we let $\varphi_\lambda = \mathbb{1}_{B(0,\lambda)}/\sqrt{|B(0,\lambda)|} \in L^2(\mathbb{R}^d;\mathbb{R})$. Then for any $x \in V_0$, $\xi_0 \in \mathbb{R}^d$, and $\lambda > 0$ we have that

$$\begin{aligned}\frac{1}{|B(0,\lambda)|}\int_{B(\xi_0,\lambda)} \|m(\xi)x\|^2_{V_1}\,\mathrm{d}\xi &= \int_{\mathbb{R}^d} \|m(\xi)x\|^2_{V_1}|\varphi_\lambda(\xi-\xi_0)|^2\,\mathrm{d}\xi \\ &= \|T\mathscr{F}^{-1}(x\varphi_\lambda(\cdot-\xi_0))\|^2_{L^2V_1} \leqslant \|T\|^2\|x\|^2_{V_0}.\end{aligned} \quad (3.7)$$

Since $m \in L^\infty_*(\mathbb{R}^d; \mathcal{L}(V_0; V_1))$, the map $\xi \mapsto \|m(\xi)x\|_{V_1}^2$ is locally integrable; thus, by Lebesgue's differentiation theorem, for each $x \in V_0$ we obtain a full measure set $E_x \subset \mathbb{R}^d$ such that $\xi_0 \in E_x$ implies that $\|m(\xi_0)x\|_{V_1} \leqslant \|T\|\|x\|_{V_0}$. Since V_0 is separable, we can let $\{x_n\}_{n\in\mathbb{N}}$ be a dense subset of V_0 and set $E = \bigcap_{n\in\mathbb{N}} E_{x_n} \subseteq \mathbb{R}^d$. It follows that E has full measure and $\xi_0 \in E$ implies that $\|m(\xi_0)\|_{\mathcal{L}(V_0,V_1)} \leqslant \|T\|$. Thus, $\|m\| \leqslant \|T\|$, and the proof of (3.6) is complete.

We now turn to the construction of the multiplier m from the map T, assuming it commutes with translations. Since V_0 and V_1 are separable Hilbert spaces, we can find sequences $\{\Pi^i_N\}_{N=0}^\infty$ of orthogonal projection operators on V_i such that for $i \in \{0, 1\}$ we have that $\dim(\Pi^i_N V_i) = N$ and $\Pi^i_N V_i \subset \Pi^i_{N+1} V_i$ for all $N \in \mathbb{N}$, and $\lim_{N\to\infty} \Pi^i_N x =$ for all $x \in V_i$.

For $N \in \mathbb{N}$, define the maps $S_N : L^2(\mathbb{R}^d; \Pi^0_N V_0) \to L^2(\mathbb{R}^d; \Pi^1_N V_1)$ via $S_N f = \Pi^1_N T f$. It is a simple matter to check that each S_N commutes with translations. Since $\Pi^0_N V_0$ and $\Pi^1_N V_1$ are finite dimensional, Theorem 3.1 then provides $\mu_N \in L^\infty(\mathbb{R}^d; \mathcal{L}(\Pi^0_N V_0; \Pi^1_N V_1))$ such that $S_N = \mu_N(D)$ and $\|S_N\|_{\mathcal{L}} = \|\mu_N\|_{L^\infty}$. We then define the symbols $\{m_N\}_{N\in\mathbb{N}} \subset L^\infty(\mathbb{R}^d; \mathcal{L}(V_0, V_1))$ via $m_N = \mu_N \Pi^0_N$, which means that

$$m_N(D) = S_N \Pi^0_N = \Pi^1_N T \Pi^0_N, \quad \|m_N\|_{L^\infty \mathcal{L}(V_0,V_1)} \leqslant \|T\|, \text{ and } \Pi^1_N m_{N+1} \Pi^0_N = m_N. \tag{3.8}$$

Note that we are free to modify each symbol in the sequence on a set of measure zero and obtain the pointwise inequality $\|m_N(\xi)\|_{\mathcal{L}(V_0,V_1)} \leqslant \|T\|$ for all $\xi \in \mathbb{R}^d$.

Given $\xi_0 \in \mathbb{R}^d$, $x \in V_0$, $y \in V_1$, and $N, M \in \mathbb{N}$, we have that

$$\langle y, (m_{N+M}(\xi_0) - m_N(\xi_0))x\rangle = \langle y, m_{N+M}(\xi_0)(1 - \Pi^0_N)x\rangle + \langle y, (1 - \Pi^1_N) m_{N+M}(\xi_0)\Pi^0_N x\rangle, \tag{3.9}$$

and hence

$$\begin{aligned} \limsup_{N,M\to\infty} |\langle y, (m_{N+M}(\xi_0) - m_N(\xi_0))x\rangle| \leqslant \lim_{N\to\infty} &\|T\|(\|y\|_{V_1}\|(1 - \Pi^0_N)x\|_{V_0} \\ &+ \|(1 - \Pi_N)y\|_{V_1}\|x\|_{V_0}) = 0. \end{aligned} \tag{3.10}$$

Thus, $\{\langle y, m_N(\xi_0)x\rangle\}_{N\in\mathbb{N}} \subset \mathcal{L}(V_0, V_1)$ is Cauchy, and hence convergent. Using this and the established bounds on m_N together with Theorem VI.1 of Reed and Simon [77], we acquire $m(\xi_0) \in \mathcal{L}(V_0, V_1)$ such that $\|m(\xi_0)\| \leqslant \|T\|$ and $\langle y, m_N(\xi_0)x\rangle \to \langle y, m(\xi_0)x\rangle$ as $N \to \infty$ for all $x \in V_0$ and $y \in V_1$. It then follows from Theorem 3.3 that $\xi_0 \mapsto m(\xi_0)$ is strongly measurable, since $\langle y, mx\rangle$ is the pointwise limit of measurable functions for every $y \in V_1$ and $x \in V_0$. Synthesizing this information, we find that $m \in L^\infty_*(\mathbb{R}^d; \mathcal{L}(V_0; V_1))$.

To complete the proof, it only remains to check that $m(D) = T$. For this, we use Parseval's theorem for fixed $f \in L^2(\mathbb{R}^d; V_0)$ and $g \in L^2(\mathbb{R}^d; V_1)$ to write

$$\langle g, Tf\rangle = \int_\Omega \langle \mathscr{F}[g](\xi), m_N(\xi)\mathscr{F}[f](\xi)\rangle \, \mathrm{d}\xi + \langle (1-\Pi^1_N)g, T\Pi^0_N f\rangle + \langle g, T(1-\Pi^0_N)f\rangle. \tag{3.11}$$

We then apply the dominated convergence theorem while sending $N \to \infty$ to deduce that $\langle g, Tf\rangle = \langle \mathscr{F}[g], m\mathscr{F}[f]\rangle$. Hence, $m(D) = T$. $\square$

The following generalization will also be of use to us. In fact, this corollary is the main workhorse of Section 4 in that it is the dictionary that allows us to transfer operator bounds derived from solving PDEs to bounds on the derivatives of the symbol of a special vector-valued Fourier multiplication operator.

Corollary 3.6 *(Translation commuting linear maps, infinite dimensional case II). Suppose that W, V_0, V_1 are separable Hilbert spaces over $\mathbb{C}$ with $V_1 \hookrightarrow V_0$, and suppose that for some $s_0, s_1, s \in \mathbb{R}$ we have a translation commuting and continuous linear map*

$$T : H^{s_0}(\mathbb{R}^d; V_0) \cap H^{s_1}(\mathbb{R}^d; V_1) \to H^s(\mathbb{R}^d; W). \tag{3.12}$$

Then, there exists a unique (up to modification on sets of measure zero) locally essentially bounded and strongly measurable (in the sense of the second item of Definition 3.2) function $m : \mathbb{R}^d \to \mathcal{L}(V_1; W)$ such that $T = m(D)$; moreover, m obeys the estimate

$$\langle\xi\rangle^s \|m(\xi)x\|_W \lesssim \|T\|(\langle\xi\rangle^{s_0}\|x\|_{V_0} + \langle\xi\rangle^{s_1}\|x\|_{V_1}), \quad \forall\, x \in V_1 \tag{3.13}$$

for almost every $\xi \in \mathbb{R}^d$. The above implicit constant only depends on d, s, s_0, and s_1.

Proof. For $\ell \in \mathbb{N}$ we let

$$A_\ell = \begin{cases} B(0,1) & \text{if } \ell = 0, \\ B(0,2^\ell) \setminus B(0,2^{\ell-1}) & \text{if } \ell \geqslant 1, \end{cases} \tag{3.14}$$

$T_\ell = T\mathbb{1}_{A_\ell}(D)$, and

$$\|y\|_{W^{(\ell)}} = \langle 2^\ell\rangle^s \|y\|_W, \quad \|x\|_{V_1^{(\ell)}} = \sqrt{\langle 2^\ell\rangle^{2s_0}\|x\|_{V_0}^2 + \langle 2^\ell\rangle^{2s_1}\|x\|_{V_1}^2} \tag{3.15}$$

for $y \in W$ and $x \in V_1$. We consider W equipped with the norm $\|\cdot\|_{W^{(\ell)}}$, denoted $W^{(\ell)}$, and V_1 equipped with the norm $\|\cdot\|_{V_1^{(\ell)}}$, similarly denoted $V_1^{(\ell)}$, and apply Theorem (3.5) to T_ℓ, viewed as a map $T_\ell : L^2(\mathbb{R}^d; W_1^{(\ell)}) \to L^2(\mathbb{R}^d; W^{(\ell)})$. The hypothesis (3.12) and the usual Fourier characterization of H^r Sobolev norms then provide the estimate

$$\|T_\ell\|_{\mathcal{L}(L^2 V_1^{(\ell)}; L^2 W^{(\ell)})} \lesssim \|T\|_{\mathcal{L}(H^{s_0}V_0 \cap H^{s_1}V_1; H^s W)}, \tag{3.16}$$

where the implicit constant depends on d, s, s_0, and s_1 but not on ℓ. Then the associated multiplier $m_\ell \in L^\infty_*(\mathbb{R}^d; \mathcal{L}(V_1^{(\ell)}, W^{(\ell)}))$ granted from Theorem 3.5 obeys the bounds

$$\langle 2^\ell\rangle^s \|m_\ell(\xi)x\|_W \lesssim \|T\|(\langle 2^\ell\rangle^{s_0}\|x\|_{H^{s_0}} + \langle 2^\ell\rangle^{s_1}\|x\|_{V_1}) \tag{3.17}$$

for every $x \in V_1$ and almost every $\xi \in A_\ell$. Note that m_ℓ can be modified on a set of measure zero and made to have support contained in A_ℓ. To conclude, we take $m = \sum_{\ell=0}^{\infty} m_\ell$. It is then straightforward to check that $m(D) = T$ and, by using (3.17), that estimate (3.13) holds. □

3.2. Classical results in vector-valued Harmonic analysis

We now turn our attention to a collection of classical results in vector-valued harmonic analysis. We showcase these here for two reasons. First, in the subsequent subsection we will develop some variants and generalizations that will play a crucial role in our study of the linear PDEs (2.1) and (2.2). Second, we will need them in our development of some Sobolev-type function spaces in Section 5.1.

We begin by recording a pair of well-known multiplier theorems. The first up is the scalar-valued Marcinkiewicz theorem, for which a proof can be found in Corollary 6.25 of Grafakos [41].

Theorem 3.7 *(Marcinkiewicz). Let $m : \mathbb{R}^d \to \mathbb{C}$ be a bounded function that is d-times continuously differentiable away from the coordinate axes in $\mathbb{R}^d$. Assume that there exists a constant $A \geqslant 0$ such that for all $k \in \{1, \ldots, d\}$, each choice of distinct $j_1, \ldots, j_k \in \{1, \ldots, d\}$, and every $\xi \in \mathbb{R}^d$ such that $\xi_r \neq 0$ for $r \notin \{j_1, \ldots, j_k\}$ we have that*

$$|(\partial_{j_1} \cdots \partial_{j_k} m)(\xi)| \leqslant A|\xi_{j_1}|^{-1} \cdots |\xi_{j_k}|^{-1}. \tag{3.18}$$

Then the map $m(D) = L^2(\mathbb{R}^d; \mathbb{C}) \to L^2(\mathbb{R}^d; \mathbb{C})$ uniquely extends to a bounded linear map $m(D) : L^p(\mathbb{R}^d; \mathbb{C}) \to L^p(\mathbb{R}^d; \mathbb{C})$ for every $1 < p < \infty$, and

$$\|m(D)\|_{\mathcal{L}(L^p)} \leqslant C_{p,d}(A + \|m\|_{L^\infty}) \tag{3.19}$$

for a constant $C_{p,d} > 0$ depending only on d and p. If, in addition, we have that $m(-\xi) = \overline{m(\xi)}$ for a.e. $\xi \in \mathbb{R}^d$, then m is reality preserving in the sense that $m(D) : L^p(\mathbb{R}^d; \mathbb{R}) \to L^p(\mathbb{R}^d; \mathbb{R})$.

We next record a vector-valued version of the celebrated Mikhlin-Hörmander multiplier theorem, originally due to Schwartz [79]. For a proof of the following formulation we refer to Proposition 6.16 in Bergh and Löfström [23].

Theorem 3.8 *(Mikhlin-Hörmander). Let V_0 and V_1 be two separable complex Hilbert spaces and let $\mathbb{N} \ni \ell > d/2$. Suppose that $m \in C^\ell(\mathbb{R}^d \setminus \{0\}; \mathcal{L}(V_0, V_1))$ satisfies*

$$\max_{|\alpha| \leqslant \ell} \sup_{\xi \neq 0} |\xi|^{|\alpha|} \|\partial^\alpha m(\xi)\|_{\mathcal{L}(H_0, H_1)} \leqslant C_\ell \tag{3.20}$$

for a constant $C_\ell \in \mathbb{R}^+$. Then the map $m(D) : L^2(\mathbb{R}^d; V_0) \to L^2(\mathbb{R}^d; V_1)$ uniquely extends to a bounded linear map $m(D) : L^p(\mathbb{R}^d; V_0) \to L^p(\mathbb{R}^d; V_1)$, and

$$\|m(D)\|_{\mathcal{L}(L^p V_0, L^p V_1)} \lesssim C_\ell, \tag{3.21}$$

where the implicit constant depends only on d and p.

We now record an important maximal inequality due to Fefferman and Stein [36] (see also Theorem 1 in Chapter 2 of Stein [91]). In what follows $\mathcal{M}$ denotes the usual Hardy-Littlewood maximal function.

Theorem 3.9 *(Fefferman-Stein maximal inequality). Suppose that* $\{f_\ell\}_{\ell\in\mathbb{Z}} \subseteq L^1_{\mathrm{loc}}(\mathbb{R}^d)$. *Then for* $1 < p < \infty$ *we have the inequality*

$$\Big\|\Big(\sum_{\ell\in\mathbb{Z}} |\mathcal{M}(f_\ell)|^2\Big)^{1/2}\Big\|_{L^p} \lesssim \Big\|\Big(\sum_{\ell\in\mathbb{Z}} |f_\ell|^2\Big)^{1/2}\Big\|_{L^p}. \tag{3.22}$$

Next, we record some vector-valued adaptations of well-known estimates from Littlewood-Paley theory. The first is based on Theorem 6.1.2 and Proposition 6.1.4 of Grafakos [41].

Theorem 3.10 *(Annular Littlewood-Paley, I). Let V be a separable complex Hilbert space, and let* $s \in [0,\infty)$, $1 < p < \infty$, *and* $m \in \mathbb{N}$. *Suppose that* $\{\varphi_j\}_{j\in\mathbb{Z}} \in C^\infty_c(\mathbb{R}^d)$ *are such that for all* $j \in \mathbb{Z}$ *we have that* $\mathscr{F}[\varphi_j]$ *is supported in the annulus* $B(0, 2^{m+j})\setminus\overline{B(0, 2^{-m+j})}$ *and there exists constants* $\{C_\alpha\}_{\alpha\in\mathbb{N}^d}$ *such that*

$$|\partial^\alpha \varphi_j(\xi)| \leqslant C_\alpha 2^{-j|\alpha|} \text{ for all } \xi \in \mathbb{R}^d \text{ and } \alpha \in \mathbb{N}^d. \tag{3.23}$$

Then for every $f \in H^{s,p}(\mathbb{R}^d; E)$ *we have the inequality*

$$\Big\|\Big(\sum_{j\in\mathbb{Z}} \langle 2^j\rangle^{2s} \|\varphi_j(D) f\|_V^2\Big)^{1/2}\Big\|_{L^p} \lesssim \|f\|_{H^{s,p}V}, \tag{3.24}$$

with implicit constants depending only on φ, s, p, d, m, *and finitely many of the* $\{C_\alpha\}_\alpha$.

Proof. It suffices to prove that for $f \in L^p(\mathbb{R}^d; V)$ we have the inequality

$$\Big\|\Big(\sum_{j\in\mathbb{Z}} \langle 2^j\rangle^{2s} \|\varphi_j(D)\langle D\rangle^{-s} f\|_V^2\Big)^{1/2}\Big\|_{L^p} \lesssim \|f\|_{L^p V}. \tag{3.25}$$

But this is a direct application of the Mikhlin-Hörmander multiplier theorem, Theorem 3.8, with the smooth multiplier $m : \mathbb{R}^d \to \mathcal{L}(V; \ell^2(\mathbb{Z}; V))$ given by

$$m(\xi)x = \{\langle 2^j\rangle^s \varphi_j(\xi)x/\langle\xi\rangle^s\}_{j\in\mathbb{Z}} \text{ for } \xi \in \mathbb{R}^d \text{ and } x \in V. \quad \square \tag{3.26}$$

Our next two results are loosely based on Lemmas 2.1.F and 2.1.G from Taylor [96].

Theorem 3.11 *(Annular Littlewood-Paley, II). Let V be a separable complex Hilbert space, and fix $s \in [0,\infty)$, $1 < p < \infty$, and $m \in \mathbb{N}^+$. For $j \in \mathbb{Z}$ let $A_j = B(0,2^{j+m}) \setminus \overline{B(0,2^{j-m})}$. Suppose that $\{f_j\}_{j\in\mathbb{Z}} \subset L^p(\mathbb{R}^d;V)$ satisfy $\operatorname{supp}\mathscr{F}[f_j] \subset A_j$ for every $j \in \mathbb{Z}$. Then*

$$\sup_{\substack{F\subset\mathbb{Z}\\ F \text{ finite}}} \Big\| \sum_{j\in F} f_j \Big\|_{H^{s,p}V} \lesssim \Big\| \Big(\sum_{j\in\mathbb{Z}} \langle 2^j\rangle^{2s} \|f_j\|_V^2 \Big)^{1/2} \Big\|_{L^p}, \tag{3.27}$$

with the implicit constant depending only on d, s, p, and m. Moreover, if the right hand side is finite, then the series $\sum_{j\in\mathbb{Z}} f_j$ converges unconditionally in $H^{s,p}(\mathbb{R}^d;V)$.

Proof. Let $1 < q < \infty$ satisfy $1/p + 1/q = 1$, and let $\varphi \in C_c^\infty(A_{-1} \cup A_0 \cup A_1)$ be such that $\varphi = 1$ on $\overline{A_0}$. Given $g \in L^q(\mathbb{R}^d;V)$ and a finite set $F \subset \mathbb{Z}$, we compute

$$\int_{\mathbb{R}^d} \Big\langle \langle D\rangle^s \sum_{j\in F} f_j, g \Big\rangle = \sum_{j\in F} \int_{\mathbb{R}^d} \big\langle \langle 2^j\rangle^s f_j, \widetilde{\varphi}_j(D) g \big\rangle \tag{3.28}$$

where $\widetilde{\varphi}_j(\xi) = \langle\xi\rangle^s \langle 2^j\rangle^{-s} \varphi(\xi/2^j)$. Hence, we can apply Cauchy-Schwartz, Hölder, Theorem 3.10, and duality (See, e.g., Theorem 5 of Section 4 in Chapter 12 of Dinculeanu [33]) to acquire the bound

$$\Big\| \sum_{j\in F} f_j \Big\|_{H^{s,p}V} = \sup_{\|g\|_{L^qV}\leqslant 1} \Big| \int_{\mathbb{R}^d} \Big\langle \langle D\rangle^s \sum_{j\in F} f_j, g \Big\rangle \Big| \lesssim \Big\| \Big(\sum_{j\in\mathbb{Z}} \langle 2^j\rangle^{2s} \|f_j\|_V^2 \Big)^{1/2} \Big\|_{L^p}. \tag{3.29}$$

Estimate (3.27) follows. A similar strategy shows that if the right hand side of (3.27) is finite, and $\{F_n\}_{n\in\mathbb{N}}$ is any increasing sequence of finite subsets of $\mathbb{Z}$ such that $\mathbb{Z} = \bigcup_{n\in\mathbb{N}} F_n$, then $\{\sum_{j\in F_n} f_j\}_{n\in\mathbb{N}}$ is Cauchy (and hence convergent) in $H^{s,p}(\mathbb{R}^d;V)$, and the limit is independent of the choice of the sequence. This implies the unconditional convergence of the series. We omit further details for the sake of brevity. □

Finally, we record a non-annular Littlewood-Paley estimate. Note that in this result it is important that we are considering spaces of positive regularity.

Theorem 3.12 *(Ball Littlewood-Paley). Let V be a separable complex Hilbert space, and fix $s \in \mathbb{R}^+$, $1 < p < \infty$, and $m \in \mathbb{N}^+$. For $j \in \mathbb{N}$ set $B_j = B(0,2^{j+m})$. Suppose that $\{f_j\}_{j\in\mathbb{N}} \subset L^p(\mathbb{R}^d;V)$ satisfy $\operatorname{supp}\mathscr{F}[f_j] \subset B_j$ for every $j \in \mathbb{N}$. Then*

$$\sup_{\substack{F\subset\mathbb{N}\\ F \text{ finite}}} \Big\| \sum_{j\in F} f_j \Big\|_{H^{s,p}V} \lesssim \Big\| \Big(\sum_{j\in\mathbb{N}} 4^{js} \|f_j\|_V^2 \Big)^{1/2} \Big\|_{L^p}, \tag{3.30}$$

with the implicit constant depending only on d, s, p, and m. Moreover, if the right hand side is finite, then the series $\sum_{j\in\mathbb{N}} f_j$ converges unconditionally in $H^{s,p}(\mathbb{R}^d;V)$.

Proof. Let $\{\varphi_k\}_{k=0}^{\infty}$ be an inhomogeneous Littlewood-Paley partition of unity with $\sum_{k=0}^{\infty} \varphi_k = 1$, $\varphi_k = \varphi_1(\cdot/2^k)$ for $k \geqslant 1$, $\operatorname{supp} \varphi_0 \subseteq B(0,2)$, $\operatorname{supp} \varphi_k \subseteq B(0,2^{k+2}) \setminus \overline{B(0,2^{k-2})}$. We also write $\widetilde{\varphi}_k = 2^{-ks}\langle\cdot\rangle^s \varphi_k$.

Let $1 < q < \infty$ satisfy $1/p + 1/q = 1$, and let $g \in L^q(\mathbb{R}^d; V)$. Then for any finite $F \subset \mathbb{N}$ we have that

$$\begin{aligned} \int_{\mathbb{R}^d} \left\langle \langle D\rangle^s \sum_{j\in F} f_j, g \right\rangle &= \sum_{j\in F} \sum_{k=0}^{j+3+m} \int_{\mathbb{R}^d} \langle \langle D\rangle^s f_j, \varphi_k(D) g\rangle \\ &= \sum_{j\in F} \sum_{k=0}^{j+3+m} \int_{\mathbb{R}^d} \langle 2^{js} f_j, 2^{(k-j)s} \widetilde{\varphi}_k(D) g\rangle. \end{aligned} \tag{3.31}$$

Hence, we may use Cauchy-Schwarz, Young's convolution inequality, and Hölder's inequality to bound

$$\begin{aligned} \left| \int_{\mathbb{R}^d} \left\langle \langle D\rangle^s \sum_{j\in F} f_j, g \right\rangle \right| &\leqslant \int_{\mathbb{R}^d} \Big(\sum_{j\in F} 4^{js} \|f_j\|_V^2 \Big)^{1/2} \Big(\sum_{j\in F} \Big(\sum_{k=0}^{j+3+m} 2^{s(k-j)} \|\widetilde{\varphi}_k(D) g\|_V \Big)^2 \Big)^{1/2} \\ &\lesssim \Big\| \Big(\sum_{j=0}^{\infty} 4^{js} \|f_j\|_V^2 \Big)^{1/2} \Big\|_{L^p} \Big\| \Big(\sum_{k=0}^{\infty} \|\widetilde{\varphi}_k(D) g\|_V^2 \Big)^{1/2} \Big\|_{L^q}. \end{aligned} \tag{3.32}$$

Theorem 3.10 provides the bound

$$\Big\| \Big(\sum_{k=0}^{\infty} \|\widetilde{\varphi}_k(D) g\|_V^2 \Big)^{1/2} \Big\|_{L^q} \lesssim \|g\|_{L^q V}. \tag{3.33}$$

Taking the supremum over g such that $\|g\|_{L^q V} \leqslant 1$ in (3.32) then gives the desired estimate. The unconditional convergence of the series then follows from a variant of this bound as in the proof of Theorem 3.11. □

3.3. On a novel variant of the Mikhlin-Hörmander multiplier theorem

We now return to the topic of multipliers, with the aim of deriving a generalized version of Mikhlin-Hörmander (the classical vector-valued version is Theorem 3.8). The task of generalizing Mikhlin-Hörmander is, of course, not new, and there are many works in the existing literature that do so. We pause here briefly to review these. In unpublished work, Pisier showed that if a vector-valued Mikhlin-Hörmander theorem holds for an $\mathcal{L}(E)$-valued symbol, then E is isomorphic to a Hilbert space. Note that Lancien, Lancien, and Le Merdy [56] provide a proof of Pisier's unpublished result in Remark 6.4, as a consequence of another result. The takeaway is that there is an obstruction to versions of Mikhlin-Hörmander for maps valued in $\mathcal{L}(B_0; B_1)$ if the B_i spaces are only Banach.

The work of Bourgain [25], Burkholder [27], McConnell [61], Zimmermann [100] found a workaround for scalar-valued symbol multipliers if the Banach space is an unconditional martingale difference (UMD) space and the Mikhlin-Hörmander hypotheses are strengthened a bit. However, it is known that UMD spaces are reflexive. The scalar UMD extension was later strengthened to operator-valued symbols by Amann [13], Hieber [49], Haller, Heck, and Noll [47], and Giardi and Weis [39]. We also refer to Chapter 4 of Prüss and Simonett [75] and the survey of Giardi and Weis [40] for more information.

The Banach obstruction can also be overcome by changing from L^p spaces to others. Amann [13] developed a version of Mikhlin-Hörmander in the context of vector-valued Besov spaces, which was subsequently extended by Giardi and Weis [38]. The setting of Triebel-Lizorkin spaces was considered by Bu and Kim [26].

For our purposes in this paper, we can restrict to multipliers that take values in $\mathcal{L}(V;W)$, where V and W are separable Hilbert spaces. However, we need a version of Mikhlin-Hörmander that allows us to replace a single L^p space with the more general setting of $H^{s_0,p}(\mathbb{R}^d;V_0)\cap H^{s_1,p}(\mathbb{R}^d;V_1)$, the intersection of different Bessel potential spaces with values in different Hilbert spaces such that $V_1\hookrightarrow V_0$. We will prove this new variant by combining three main ingredients. The first is the annular Littlewood-Paley results of Theorems 3.10 and 3.11, while the second is the Fefferman-Stein maximal inequality of Theorem 3.9. The final ingredient we need, which we record in the next result, gives pointwise bounds via maximal functions. It is a generalization of Lemma 2.2 in Bahouri, Chemin, and Danchin [15].

Theorem 3.13 *(Pointwise bounds for spectrally localized Fourier multipliers). Suppose V, V_0, and V_1 are separable complex Hilbert spaces such that $V_1\hookrightarrow V_0$. Let $s,s_0,s_1\in\mathbb{R}$ and $\mu,\ell\in\mathbb{Z}$ with $\mu\leqslant 0$ and $d<\ell$. Suppose that $m\in C^{\ell}(\mathbb{R}^d\setminus\{0\};\mathcal{L}(V_1,V))$ satisfies*

$$\langle\xi\rangle^s\max_{|\alpha|\leqslant\ell}|\xi|^{|\alpha|}\|\partial^\alpha m(\xi)x\|_V\leqslant C_\ell|\xi|^\mu(\langle\xi\rangle^{s_0}\|x\|_{V_0}+\langle\xi\rangle^{s_1}\|x\|_{V_1}) \tag{3.34}$$

for all $\xi\in\mathbb{R}^d\setminus\{0\}$ and $x\in V_1$. Let $\varphi,\widetilde{\varphi}\in C_c^\infty(B(0,16)\setminus\overline{B(0,1/16)})$ be such that $\widetilde{\varphi}=1$ on the support of φ. Then for $f\in\mathscr{S}(\mathbb{R}^d;V_1)$, $\lambda\in\mathbb{R}^+$, and $z\in\mathbb{R}^d$ we have the pointwise estimate

$$\begin{aligned}\langle\lambda\rangle^s\|m(D)\varphi(D/\lambda)f(z)\|_V\lesssim C_\ell|\lambda|^\mu\,[&\langle\lambda\rangle^{s_0}\mathcal{M}(\|\widetilde{\varphi}(D/\lambda)f\|_{V_0})(z)\\&+\langle\lambda\rangle^{s_1}\mathcal{M}(\|\widetilde{\varphi}(D/\lambda)f\|_{V_1})(z)]\,.\end{aligned} \tag{3.35}$$

Proof. We let $K_\lambda:\mathbb{R}^d\to\mathcal{L}(V_1;V)$ be defined via

$$K_\lambda(z)=\int_{\mathbb{R}^d}e^{2\pi\mathrm{i}z\cdot\xi}\varphi(\xi)m(\lambda\xi)\,\mathrm{d}\xi, \tag{3.36}$$

and note that

$$m(D)\varphi(D/\lambda)f = \lambda^d K_\lambda(\lambda(\cdot)) * (\widetilde{\varphi}(D/\lambda)f) \tag{3.37}$$

for all $f \in \mathscr{S}(\mathbb{R}^d; V_1)$.

Fix $x \in V_1$. Hypothesis (3.34) implies the rescaled form

$$\|\partial^\beta(m(\lambda\cdot))(\xi)x\|_V \leqslant C_\ell|\lambda|^\mu\langle\lambda\xi\rangle^{-s}|\xi|^{\mu-|\beta|}(\langle\lambda\xi\rangle^{s_0}\|x\|_{V_0} + \langle\lambda\xi\rangle^{s_1}\|x\|_{V_1}) \tag{3.38}$$

for every $|\beta| \leqslant \ell$. This, integration by parts, and an application of the Leibniz rule then provide the following kernel bound for $|\alpha| \leqslant \ell$:

$$\begin{aligned}\|(2\pi \mathrm{i} z)^\alpha K_\lambda(z)x\|_V &= \int_{\mathbb{R}^d} \|\partial^\alpha(\varphi m(\lambda(\cdot)))(\xi)x\|_V^2 \, \mathrm{d}\xi \\ &\lesssim C_\ell\langle\lambda\rangle^{-s}|\lambda|^\mu(\langle\lambda\rangle^{s_0}\|x\|_{V_0} + \langle\lambda\rangle^{s_1}\|x\|_{V_1}).\end{aligned} \tag{3.39}$$

Summing over $|\alpha| \leqslant \ell$, we deduce from this that

$$\sup_{z\in\mathbb{R}^d}\langle z\rangle^\ell\|K_\lambda(z)x\|_V \lesssim C_\ell\langle\lambda\rangle^{-s}|\lambda|^\mu(\langle\lambda\rangle^{s_0}\|x\|_{V_0} + \langle\lambda\rangle^{s_1}\|x\|_{V_1}). \tag{3.40}$$

Now we return to formula (3.37) to obtain the claimed pointwise bounds. Let $f \in \mathscr{S}(\mathbb{R}^d; V_1)$, and set $g = \widetilde{\varphi}(D/\lambda)f$. Write

$$A_j(y,\lambda) = \begin{cases} B(y, 1/\lambda) & \text{if } j = 0, \\ B(y, 2^j/\lambda) \setminus B(y, 2^{j-1}/\lambda) & \text{if } j \geqslant 1. \end{cases} \tag{3.41}$$

Then the above estimates allow us to bound

$$\begin{aligned}\|(\lambda^d K_\lambda(\lambda\cdot) * g)(y)\|_V &\leqslant \sum_{j=0}^\infty \int_{A_j(y,\lambda)} \lambda^d\|K_\lambda(\lambda(y-z))g(z)\|_V \, \mathrm{d}z \\ &\lesssim C_\ell \sum_{j=0}^\infty \int_{A_j(y,\lambda)} \frac{\lambda^d}{\langle\lambda(y-z)\rangle^\ell}\langle\lambda\rangle^{-s}|\lambda|^\mu(\langle\lambda\rangle^{s_0}\|g(z)\|_{V_0} + \langle\lambda\rangle^{s_1}\|g(z)\|_{V_1}) \, \mathrm{d}z \\ &\lesssim C_\ell\langle\lambda\rangle^{-s}|\lambda|^\mu\sum_{j=0}^\infty\langle 2^j\rangle^{d-\ell}\Big(\frac{1}{|B(y,2^j/\lambda)|}\int_{B(y,2^j/\lambda)}(\langle\lambda\rangle^{s_0}\|g(z)\|_{V_0} + \langle\lambda\rangle^{s_1}\|g(z)\|_{V_1}) \, \mathrm{d}z\Big) \\ &\lesssim C_\ell\langle\lambda\rangle^{-s}|\lambda|^\mu\left[\langle\lambda\rangle^{s_0}\mathcal{M}(\|g\|_{V_0})(y) + \langle\lambda\rangle^{s_1}\mathcal{M}(\|g\|_{V_1})(y)\right].\end{aligned} \tag{3.42}$$

This yields the desired bound upon substituting in $g = \widetilde{\varphi}(D/\lambda)f$. □

We now have all the tools needed to prove our generalization of the Mikhlin-Hörmander multiplier theorem.

Theorem 3.14 *(Mikhlin-Hörmander, novel form). Suppose V, V_0, and V_1 are separable complex Hilbert spaces such that $V_1 \hookrightarrow V_0$. Let $s, s_0, s_1 \in \mathbb{R}$ and $\mu, \ell \in \mathbb{Z}$ with $\mu \leqslant 0$ and $d < \ell$. Suppose that $m \in C^\ell(\mathbb{R}^d \setminus \{0\}; \mathcal{L}(V_1; V))$ satisfies*

$$\langle \xi \rangle^s \max_{|\beta| \leqslant \ell} |\xi|^{|\beta|} \|\partial^\beta m(\xi) x\|_V \leqslant C_\ell |\xi|^\mu (\langle \xi \rangle^{s_0} \|x\|_{V_0} + \langle \xi \rangle^{s_1} \|x\|_{V_1}) \tag{3.43}$$

for all $\xi \in \mathbb{R}^d \setminus \{0\}$ and $x \in V_1$. Let $\alpha \in \mathbb{N}^d$ satisfy $|\alpha| = -\mu$. Then the bounded linear map

$$m(D) D^\alpha : H^{s_0}(\mathbb{R}^d; V_0) \cap H^{s_1}(\mathbb{R}^d; V_1) \to H^s(\mathbb{R}^d; V), \tag{3.44}$$

which is well-defined and bounded in light of the $\beta = 0$ estimate from (3.43), uniquely extends to a bounded linear map

$$m(D) D^\alpha : H^{s_0,p}(\mathbb{R}^d; V_0) \cap H^{s_1,p}(\mathbb{R}^d; V_1) \to H^{s,p}(\mathbb{R}^d; V) \tag{3.45}$$

for every $1 < p < \infty$.

Proof. Let $\varphi, \widetilde{\varphi}, \widetilde{\widetilde{\varphi}} \in C_c^\infty(\mathbb{R}^d)$ be such that $\operatorname{supp} \varphi \subset B(0,8) \setminus \overline{B(0,1/8)}$, $\operatorname{supp} \widetilde{\varphi}, \operatorname{supp} \widetilde{\widetilde{\varphi}} \subset B(0,16) \setminus \overline{B(0,1/16)}$, $\sum_{j \in \mathbb{Z}} \varphi(\cdot/2^j) = \mathbb{1}_{\mathbb{R}^d \setminus \{0\}}$, $\widetilde{\varphi} = 1$ on $\operatorname{supp} \varphi$, and $\widetilde{\widetilde{\varphi}} = 1$ on $\operatorname{supp} \widetilde{\varphi}$.

Suppose that $f \in \mathscr{S}(\mathbb{R}^d; V_1)$. We first use the Littlewood-Paley estimate of Theorem 3.11 to bound

$$\|m(D) D^\alpha f\|_{H^{s,p} V} \lesssim \left\| \Big(\sum_{j \in \mathbb{Z}} \langle 2^j \rangle^{2s} \|m(D) D^\alpha \varphi(D/2^j) f\|_V^2 \Big)^{1/2} \right\|_{L^p}. \tag{3.46}$$

Next, for each $j \in \mathbb{Z}$ we apply Theorem 3.13 with $\lambda = 2^j$ to see that

$$\begin{aligned} \langle 2^j \rangle^s \|m(D) D^\alpha \varphi(D/2^j) f\|_V \lesssim C_\ell 2^{-j|\alpha|} (&\langle 2^j \rangle^{s_0} \mathcal{M}(\|D^\alpha \widetilde{\varphi}(D/2^j) f\|_{V_0}) \\ &+ \langle 2^j \rangle^{s_1} \mathcal{M}(\|D^\alpha \widetilde{\varphi}(D/2^j) f\|_{V_1})). \end{aligned} \tag{3.47}$$

For $i \in \{0,1\}$, we may apply Theorem 3.13 again (using the trivial multiplier $m = 1$ and parameters $s = s_0 = s_1 = 0$ with V_i used for all three Hilbert spaces), to acquire the bound

$$\|D^\alpha \widetilde{\varphi}(D/2^j) f\|_{V_i} \lesssim 2^{j|\alpha|} \mathcal{M}(\|\widetilde{\widetilde{\varphi}}(D/2^j) f\|_{V_i}). \tag{3.48}$$

We then combine the estimates (3.47) and (3.48):

$$\begin{aligned} \frac{1}{C_\ell} \langle 2^j \rangle^s \|m(D) D^\alpha \varphi(D/2^j) f\|_V \lesssim &\langle 2^j \rangle^{s_0} \mathcal{M}\mathcal{M}(\|\widetilde{\widetilde{\varphi}}(D/2^j) f\|_{V_0}) \\ &+ \langle 2^j \rangle^{s_1} \mathcal{M}\mathcal{M}(\|\widetilde{\widetilde{\varphi}}(D/2^j) f\|_{V_1}). \end{aligned} \tag{3.49}$$

From this, estimate (3.46), two applications of the Fefferman-Stein maximal inequality of Theorem 3.9, and an application of the Littlewood-Paley estimate of Theorem 3.10 we deduce that

$$\frac{1}{C_\ell}\|m(D)D^\alpha f\|_{H^{s,p}V} \lesssim \Big\|\Big(\sum_{j\in\mathbb{Z}}\langle 2^j\rangle^{2s_0}[\mathcal{M}\mathcal{M}(\|\widetilde{\widetilde{\varphi}}(D/2^j)f\|_{V_0})]^2\Big)^{1/2}\Big\|_{L^p}$$
$$+\Big\|\Big(\sum_{j\in\mathbb{Z}}\langle 2^j\rangle^{2s_1}[\mathcal{M}\mathcal{M}(\|\widetilde{\widetilde{\varphi}}(D/2^j)f\|_{V_1})]^2\Big)^{1/2}\Big\|_{L^p}$$
$$\lesssim \Big\|\Big(\sum_{j\in\mathbb{Z}}\langle 2^j\rangle^{2s_0}\|\widetilde{\widetilde{\varphi}}(D/2^j)f\|_{V_0}^2\Big)^{1/2}\Big\|_{L^p}+\Big\|\Big(\sum_{j\in\mathbb{Z}}\langle 2^j\rangle^{2s_1}\|\widetilde{\widetilde{\varphi}}(D/2^j)f\|_{V_1}^2\Big)^{1/2}\Big\|_{L^p}$$
$$\lesssim \|f\|_{H^{s_0,p}V_0}+\|f\|_{H^{s_1,p}V_1}. \quad (3.50)$$

This estimate then allows us to extend $m(D)D^\alpha$ as stated. □

4. Vector-valued symbol calculus for the solution map

The goal of this section is to prove that the solution operator associated to the linear system (2.2) is given by an operator-valued Fourier multiplier. Once this is established, we then show that the symbol of the operator is a smooth function of frequency away from zero and satisfies bounds of the type appearing in the hypotheses of the Mikhlin-Hörmander multiplier theorem (see Theorems 3.8 and 3.14).

4.1. Preliminaries

The following definition enumerates the translation commuting maps we are interested in extending to an L^p theory. That the following is well-defined is a consequence of Theorem 2.6.

Definition 4.1 *(Spaces and translation commuting maps). We set the following notation.*

(1) For $s \in \mathbb{N}$ we define the spaces

$$\mathbb{X}_s = H^{1+s}(\Omega;\mathbb{C}) \times H^{2+s}(\Omega;\mathbb{C}^3) \times H^{3/2+s}(\Sigma;\mathbb{C}^2), \quad (4.1)$$

$$\mathbb{Y}_s = H^{s}(\Omega;\mathbb{C}^3) \times H^{1/2+s}(\Sigma;\mathbb{C}^3) \times H^{5/2+s}(\Sigma;\mathbb{C}^2), \quad (4.2)$$

$$\widetilde{\mathbb{Y}}_s = \Big\{(g,f,k,h,\omega) \text{ as in (2.53)} \;:\; \omega,\; h - \int_0^b g(\cdot,y)\,\mathrm{d}y \in \dot{H}^{-1}(\Sigma;\mathbb{C})\Big\}. \quad (4.3)$$

(2) We define the bounded linear map $\Phi : \widetilde{\mathbb{Y}}_s \to \mathbb{X}_s$ via $\Phi(g,f,k,h,\omega) = (p,u,\chi)$, where the latter tuple is the unique solution to (2.2) with data (g,f,k,h,ω) provided by

Theorem 2.6. We also define the bounded linear map $\Psi : \mathbb{Y}_s \to \mathbb{X}_s$ via $\Psi(f,k,H) = \Phi(0,f,k,\nabla_\| \cdot H, 0)$.

It is clear from the existence and uniqueness result of Theorem 2.6 and the fact that the equations (2.2) have constant coefficients that the operators Φ and Ψ of the previous definition commute with translations of the tangential variables. Thus, we expect that they are given by Fourier multipliers. The following definition gives the class of the vector-valued Fourier multipliers with which we are concerned. As we will show later, Ψ has a multiplier belonging to this admissible class.

Definition 4.2 *(Admissible class of vector-valued Fourier multipliers). We make the following definitions for $s \in \mathbb{N}$.*

(1) We say that a symbol

$$m : \mathbb{R}^2 \to \mathcal{L}(H^s((0,b);\mathbb{C}^3) \times \mathbb{C}^3 \times \mathbb{C}^2; H^{1+s}((0,b);\mathbb{C}) \times H^{2+s}((0,b);\mathbb{C}^3) \times \mathbb{C}^2), \quad (4.4)$$

belongs to the admissible class $\mathfrak{A}(s)$ if it is strongly measurable in the sense of the second item of Definition 3.2, and upon writing m in 'matrix form'

$$m = \begin{pmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{pmatrix}, \quad (4.5)$$

we have that there exists a constant $C < \infty$ and a full measure set $E \subseteq \mathbb{R}^2$ such that for $\xi \in E$:

(a) $m_{11}(\xi) \in \mathcal{L}(H^s((0,b);\mathbb{C}^3); H^{1+s}((0,b);\mathbb{C}))$ obeys the bound

$$\|m_{11}(\xi)\phi\|_{H^{1+s}} + \langle\xi\rangle^{1+s}\|m_{11}(\xi)\phi\|_{L^2} \leqslant C(\|\phi\|_{H^s} + \langle\xi\rangle^s\|\phi\|_{L^2}) \quad (4.6)$$

for $\phi \in H^s((0,b);\mathbb{C}^3)$;

(b) $m_{12}(\xi) \in \mathcal{L}(\mathbb{C}^3; H^{1+s}((0,b);\mathbb{C}))$ obeys the bound

$$\|m_{12}(\xi)\|_{H^{1+s}} + \langle\xi\rangle^{1+s}\|m_{12}(\xi)\|_{L^2} \leqslant C\langle\xi\rangle^{1/2+s}; \quad (4.7)$$

(c) $m_{13} \in \mathcal{L}(\mathbb{C}^2; H^{1+s}((0,b);\mathbb{C}))$ obeys the bound

$$\|m_{13}(\xi)\|_{H^{1+s}} + \langle\xi\rangle^{1+s}\|m_{13}(\xi)\|_{L^2} \leqslant C\langle\xi\rangle^{5/2+s}; \quad (4.8)$$

(d) $m_{21}(\xi) \in \mathcal{L}(H^s((0,b);\mathbb{C}^3), H^{2+s}((0,b);\mathbb{C}^3))$ obeys the bound

$$\|m_{21}(\xi)\phi\|_{H^{2+s}} + \langle\xi\rangle^{2+s}\|m_{21}(\xi)\phi\|_{L^2} \leqslant C(\|\phi\|_{H^s} + \langle\xi\rangle^s\|\phi\|_{L^2}) \quad (4.9)$$

for $\phi \in H^s((0,b);\mathbb{C}^3)$;

(e) $m_{22}(\xi) \in \mathcal{L}(\mathbb{C}^3; H^{2+s}((0,b);\mathbb{C}^3))$ *obeys the bound*

$$\|m_{22}(\xi)\|_{H^{2+s}} + \langle\xi\rangle^{2+s}\|m_{22}(\xi)\|_{L^2} \leqslant C\langle\xi\rangle^{1/2+s}; \tag{4.10}$$

(f) $m_{23}(\xi) \in \mathcal{L}(\mathbb{C}^2; H^{2+s}((0,b);\mathbb{C}^3))$ *obeys the bound*

$$\|m_{23}(\xi)\|_{H^{2+s}} + \langle\xi\rangle^{2+s}\|m_{23}(\xi)\|_{L^2} \leqslant C\langle\xi\rangle^{5/2+s}; \tag{4.11}$$

(g) $m_{31}(\xi) \in \mathcal{L}(H^s((0,b);\mathbb{C}^3);\mathbb{C}^2)$ *obeys the bound*

$$\langle\xi\rangle^{3/2+s}|m_{31}(\xi)\phi| \leqslant C(\|\phi\|_{H^s} + \langle\xi\rangle^s\|\phi\|_{L^2}) \tag{4.12}$$

for $\phi \in H^s((0,b);\mathbb{C}^3)$*;*

(h) $m_{32}(\xi) \in \mathcal{L}(\mathbb{C}^3;\mathbb{C}^2)$ *obeys the bound*

$$\langle\xi\rangle^{3/2+s}|m_{32}(\xi)| \leqslant C\langle\xi\rangle^{1/2+s}; \tag{4.13}$$

(i) $m_{33}(\xi) \in \mathcal{L}(\mathbb{C}^2;\mathbb{C}^2)$ *obeys the bound*

$$\langle\xi\rangle^{3/2+s}|m_{33}(\xi)| \leqslant C\langle\xi\rangle^{5/2+s}. \tag{4.14}$$

(2) If $m \in \mathfrak{A}(s)$*, then we write* $[\![m]\!]_s \in [0,\infty)$ *to be the infimum over the constants* C *for which the bounds (4.6), (4.7), (4.8), (4.9), (4.10), (4.11), (4.12), (4.13), and (4.14) hold over a full measure set of frequencies. This makes* $\mathfrak{A}(s)$ *into a Banach space.*

Before continuing, we remark that the $[\![\cdot]\!]_s$-norm on $\mathfrak{A}(s)$ locally controls essential supremum norm.

Lemma 4.3 *(Local essentially uniform control). Fix* $s \in \mathbb{N}$*. For any* $R \in \mathbb{R}^+$ *there exists a constant* $C_R \in \mathbb{R}^+$*, depending only on* s *and* R*, such that for all* $m \in \mathfrak{A}(s)$ *we have the estimate*

$$\|\mathbb{1}_{B(0,R)}m\|_{L^\infty_*\mathcal{L}} \leqslant C_R[\![m]\!]_s, \tag{4.15}$$

where here $\mathcal{L}$ *refers to the space of linear operators on the right hand side of (4.4) and* $L^\infty_*\mathcal{L}$ *is the norm from Definition 3.4. Moreover, the map* $R \mapsto C_R$ *is bounded on bounded sets.*

Proof. An inspection of the $[\![\cdot]\!]_s$ norm in Definition 4.2 shows that we can replace the ξ-dependent bounds on the right hand sides of (4.6)–(4.14) by their value at R to derive (4.15). $\square$

The utility of Definition 4.2 is seen in the next result.

Proposition 4.4 *(Symbols and translation commuting maps). The following are equivalent for $s \in \mathbb{N}$ and a bounded linear map $T : \mathbb{Y}_s \to \mathbb{X}_s$:*

(1) T commutes with the collection of tangential translation operators in the sense that for all $Y \in \mathbb{Y}_s$ and all $h \in \mathbb{R}^3$ satisfying $h \cdot e_3 = 0$, it holds that $(TY)(\cdot + h) = T(Y(\cdot + h))$.
(2) There exists $m \in \mathfrak{A}(s)$ such that $T = m(D)$.

In either case, m is unique up to modification on a set of measure zero, and we have that

$$[\![m]\!]_s \asymp \|T\|_{\mathcal{L}(\mathbb{Y}_s;\mathbb{X}_s)} \tag{4.16}$$

with implicit constants depending only on s.

Proof. Thanks to the norm equivalence of Lemma 5.2 with $p = 2$, each of the nine components of the 'matrix' of T satisfies the hypotheses of Corollary 3.6. Thus, by enumerating the estimates on each component, we find that we are granted a multiplier $m \in \mathfrak{A}(s)$ such that $T = m(D)$. Estimate (3.13) implies that $[\![m]\!]_s \lesssim \|T\|_{\mathcal{L}(\mathbb{Y}_s;\mathbb{X}_s)}$. The opposite inequality and the fact that the second item implies the first are immediate from Plancherel's theorem. □

Our next result relates translations of the symbol to conjugation of the operator by complex exponentials. This is the key that allows us to recast questions of smoothness for multipliers in the language of PDE on the spatial side.

Proposition 4.5 *(Symbol translation). Let $s \in \mathbb{N}$, and suppose that $T : \mathbb{Y}_s \to \mathbb{X}_s$ is a tangentially translation commuting bounded linear map with associated symbol $m \in \mathfrak{A}(s)$. Then for all $Y = (f, k, H) \in \mathbb{Y}_s$ and all $\zeta \in \mathbb{R}^2$ we have the identity*

$$m(D + \zeta)Y = \mathbf{e}_{-\zeta}(T(\mathbf{e}_\zeta Y)), \tag{4.17}$$

as an equality in the space $\mathbb{X}_s$, where $\mathbf{e}_\zeta(\xi) = e^{2\pi i \xi \cdot \zeta}$ for $\xi \in \mathbb{R}^2$, and $m(D + \zeta) = (m(\cdot + \zeta))(D)$.

Proof. With Y and ζ as in the statement, we have that the second item of Proposition 4.4 applies and we have $T(\mathbf{e}_\zeta Y) = m(D)(\mathbf{e}_\zeta Y)$. On the other hand, it is clear by properties of the Fourier transform

$$\mathscr{F}[\mathbf{e}_\zeta m(D + \zeta)Y] = \mathscr{F}[m(D + \zeta)Y](\cdot - \zeta) = m\mathscr{F}[Y](\cdot - \zeta) = m\mathscr{F}[\mathbf{e}_\zeta Y]. \tag{4.18}$$

By taking inverse Fourier transforms, we reveal identity (4.17). □

4.2. Derivative estimates for the symbol

The main symbols of concern in this subsection are given by the following definition. The notation here is from Definitions 4.1 and 4.2.

Definition 4.6 *(Main symbol). Let $s \in \mathbb{N}$. We denote by $\mathbf{m} \in \mathfrak{A}(s)$ the symbol associated with the translation commuting bounded linear map $\Psi : \mathbb{Y}_s \to \mathbb{X}_s$ whose existence is guaranteed by Proposition 4.4.*

Our aim now is to study the differentiability of $\mathbf{m}$ in $\mathbb{R}^2 \backslash \{0\}$. In doing so, it will be very convenient to introduce some minor abuse of notation in order to make various expressions easier to read. This abuse, the use of which we confine to this subsection, is to view any vector $\theta \in \mathbb{C}^2$ as being contained in $\mathbb{C}^3$ via $(\theta, 0) \in \mathbb{C}^3$. For example, this shorthand means that for $v \in \mathbb{C}^3$ and $\theta \in \mathbb{C}^2$,

$$\theta \cdot v = (\theta, 0) \cdot v = \sum_{j=1}^{2} \theta_j v_j, \tag{4.19}$$

and so on. This abuse will only be used in equations that are naturally understood to be posed in $\mathbb{C}^3$.

Our next result explores the manifestations of symbol translation on the PDE side.

Proposition 4.7 *(Spatial realization of symbol translation). Suppose that $s \in \mathbb{N}$, $(f, k, H) \in \mathbb{Y}_s$, and $\zeta \in \mathbb{R}^2$. Then $\mathbf{m}(D + \zeta)(f, k, H) = (p_\zeta, u_\zeta, \chi_\zeta) \in \mathbb{X}_s$ obeys the following equations*

$$\begin{cases} \mathfrak{g}\chi_\zeta + \nabla p_\zeta + 2\pi\mathrm{i}\zeta p_\zeta - \mu\nabla \cdot \mathbb{D}u_\zeta - \mu 2\pi\mathrm{i}\nabla(\zeta \cdot u_\zeta) & \\ -\mu 4\pi\mathrm{i}\zeta \cdot \nabla u_\zeta + \mu 4\pi^2 |\zeta|^2 u_\zeta = f & \text{in } \Omega, \\ \nabla \cdot u_\zeta + 2\pi\mathrm{i}\zeta \cdot u_\zeta = 0 & \text{in } \Omega, \\ -(p_\zeta I - \mu\mathbb{D}u_\zeta)e_3 + \mu 2\pi\mathrm{i}\zeta(u_\zeta \cdot e_3) - \kappa\nabla_\| \cdot \chi_\zeta e_3 - \kappa 2\pi\mathrm{i}\zeta \cdot \chi_\zeta e_3 = k & \text{on } \Sigma, \\ \nabla_\|^\perp \cdot \chi_\zeta + 2\pi\mathrm{i}\zeta^\perp \cdot \chi_\zeta = 0 & \text{on } \Sigma, \\ u_\zeta \cdot e_3 = \nabla_\| \cdot H + 2\pi\mathrm{i}\zeta \cdot H & \text{on } \Sigma, \\ u_\zeta = 0 & \text{on } \Sigma_0. \end{cases} \tag{4.20}$$

Proof. We invoke Proposition 4.5 and Definition 4.6 to see that $(p_\zeta, u_\zeta, \chi_\zeta) = \mathbf{e}_{-\zeta}\Psi(\mathbf{e}_\zeta f, \mathbf{e}_\zeta k, \mathbf{e}_\zeta H)$, and hence $(\mathbf{e}_\zeta p_\zeta, \mathbf{e}_\zeta u_\zeta, \mathbf{e}_\zeta \chi_\zeta)$ is a solution to the equations (2.2) with data $(0, \mathbf{e}_\zeta f, \mathbf{e}_\zeta k, \nabla_\| \cdot (\mathbf{e}_\zeta H), 0)$. We can then derive equations for $(p_\zeta, u_\zeta, \chi_\zeta)$ by expanding with the Leibniz rule and then multiplying by $\mathbf{e}_\zeta$. This results in the system (4.20). □

The following lemma gives a simple estimate for the solution to (4.20) and, more importantly, a relationship between the translated symbol and the operators Φ and Ψ from Definition 4.1.

Lemma 4.8 *(Estimate and identity for the translated symbol PDE). The following hold for $s \in \mathbb{N}$, $(f,k,H) \in \mathbb{Y}_s$, $\zeta \in \mathbb{R}^2$, and $(p_\zeta, u_\zeta, \chi_\zeta) = \mathbf{m}(D+\zeta)(f,k,H)$.*

(1) We have the estimate

$$\|p_\zeta, u_\zeta, \chi_\zeta\|_{\mathbb{X}_s} \leqslant C_\zeta \|\Psi\|_{\mathcal{L}} \|f,k,H\|_{\mathbb{Y}_s}, \tag{4.21}$$

where the constant $\mathbb{R}^2 \ni \zeta \mapsto C_\zeta \in \mathbb{R}^+$ is bounded on bounded sets.

(2) If $0 \notin \operatorname{supp} \mathscr{F}(f,k,H)$, then

$$\begin{aligned}(p_\zeta, u_\zeta, \chi_\zeta) &= \Psi(f,k,H) \\ &\quad + \Phi \begin{pmatrix} -2\pi\mathrm{i}\zeta \cdot u_\zeta \\ -2\pi\mathrm{i}\zeta p_\zeta + \mu 2\pi\mathrm{i}\nabla(\zeta \cdot u_\zeta) + \mu 4\pi\mathrm{i}\zeta \cdot \nabla u_\zeta - \mu 4\pi^2|\zeta|^2 u_\zeta \\ \kappa 2\pi\mathrm{i}\zeta \cdot \chi_\zeta e_3 - \mu 2\pi\mathrm{i}\zeta(u_\zeta \cdot e_3) \\ 2\pi\mathrm{i}\zeta \cdot H \\ -2\pi\mathrm{i}\zeta^\perp \cdot \chi_\zeta \end{pmatrix}.\end{aligned} \tag{4.22}$$

Proof. For the first item, we note that $(\mathbf{e}_\zeta p_\zeta, \mathbf{e}_\zeta u_\zeta, \mathbf{e}_\zeta \chi_\zeta) = \Psi(\mathbf{e}_\zeta f, \mathbf{e}_\zeta k, \mathbf{e}_\zeta H)$ and hence

$$\|p_\zeta, u_\zeta, \chi_\zeta\|_{\mathbb{X}_s} \lesssim_\zeta \|\mathbf{e}_\zeta p_\zeta, \mathbf{e}_\zeta u_\zeta, \mathbf{e}_\zeta \chi_\zeta\|_{\mathbb{X}_s} \leqslant \|\Psi\|_{\mathcal{L}} \|\mathbf{e}_\zeta f, \mathbf{e}_\zeta k, \mathbf{e}_\zeta H\|_{\mathbb{Y}_s} \lesssim_\zeta \|\Psi\|_{\mathcal{L}} \|f,k,H\|_{\mathbb{Y}_s}. \tag{4.23}$$

Next, we use the system (4.20) from Proposition 4.7 along with Definition 4.1 to derive the equation (4.22). Note that the hypotheses $0 \notin \operatorname{supp} \mathscr{F}(f,k,H)$ ensure that the argument of Φ in (4.22) belongs to its domain $\widetilde{\mathbb{Y}}_s$. $\square$

Motivated by formula (4.22), we now construct the translation commuting linear maps whose symbols will turn out to be the derivatives of $\mathbf{m}$. In what follows $\mathcal{S}_\ell$ denotes the symmetric group on the set $\{1,\dots,\ell\}$.

Definition 4.9 *(Iterative derivative construction). Let $(f,k,H) \in \mathbb{Y}_s$ satisfy $0 \notin \operatorname{supp} \mathscr{F}(f,k,H)$ and set $(p,u,\chi) = \Psi(f,k,H) \in \mathbb{X}_s$. For $j \in \mathbb{N}^+$ we define the j-multilinear and symmetric maps*

$$(\mathbb{R}^2)^j \ni (\zeta_1,\dots,\zeta_j) \mapsto (p^{(j)}, u^{(j)}, \chi^{(j)})[\zeta_1,\dots,\zeta_j] \in \mathbb{X}_s \tag{4.24}$$

via the following inductive procedure. If $j=1$, we set

$$(p^{(1)}, u^{(1)}, \chi^{(1)})[\zeta] = \Phi \begin{pmatrix} -2\pi\mathrm{i}\zeta \cdot u \\ -2\pi\mathrm{i}\zeta p + \mu 2\pi\mathrm{i}\nabla(\zeta \cdot u) + \mu 4\pi\mathrm{i}\zeta \cdot \nabla u \\ \kappa 2\pi\mathrm{i}\zeta \cdot \chi e_3 - \mu 2\pi\mathrm{i}\zeta(u \cdot e_3) \\ 2\pi\mathrm{i}\zeta \cdot H \\ -2\pi\mathrm{i}\zeta^\perp \cdot \chi \end{pmatrix}. \tag{4.25}$$

If $j = 2$, we set

$$(p^{(2)}, u^{(2)}, \chi^{(2)})[\zeta_1, \zeta_2] = \sum_{\sigma \in \mathcal{S}_2} \Phi$$
$$\times \begin{pmatrix} -2\pi\mathrm{i}\zeta_{\sigma 1} \cdot u^{(1)}[\zeta_{\sigma 2}] \\ -2\pi\mathrm{i}\zeta_{\sigma 1} p^{(1)}[\zeta_{\sigma 2}] + \mu 2\pi\mathrm{i}\nabla(\zeta_{\sigma 1} \cdot u^{(1)}[\zeta_{\sigma 2}]) + \mu 4\pi\mathrm{i}\zeta_{\sigma 1} \cdot \nabla u^{(1)}[\zeta_{\sigma 2}] - \mu 4\pi^2(\zeta_{\sigma 1} \cdot \zeta_{\sigma 2}) u \\ \kappa 2\pi\mathrm{i}\zeta_{\sigma 1} \cdot \chi^{(1)}[\zeta_{\sigma 2}] e_3 - \mu 2\pi\mathrm{i}\zeta_{\sigma 1}(u^{(1)}[\zeta_{\sigma 2}] \cdot e_3) \\ 0 \\ -2\pi\mathrm{i}\zeta_{\sigma 1}^\perp \cdot \chi^{(1)}[\zeta_{\sigma 2}] \end{pmatrix} \tag{4.26}$$

and if $j \geqslant 3$, we take

$$(p^{(j)}, u^{(j)}, \chi^{(j)})[\zeta_1, \ldots, \zeta_j] = \frac{1}{(j-1)!} \sum_{\sigma \in \mathcal{S}_j} \Phi$$
$$\times \begin{pmatrix} -2\pi\mathrm{i}\zeta_{\sigma 1} \cdot u^{(j-1)} \\ -2\pi\mathrm{i}\zeta_{\sigma 1} p^{(j-1)} + \mu 2\pi\mathrm{i}\nabla(\zeta_{\sigma 1} \cdot u^{(j-1)}) + \mu 4\pi\mathrm{i}\zeta_{\sigma 1} \cdot \nabla u^{(j-1)} \\ \kappa 2\pi\mathrm{i}\zeta_{\sigma 1} \cdot \chi^{(j-1)} e_3 - \mu 2\pi\mathrm{i}\zeta_{\sigma 1}(u^{(j-1)} \cdot e_3) \\ 0 \\ -2\pi\mathrm{i}\zeta_{\sigma 1}^\perp \cdot \chi^{(j-1)} \end{pmatrix} [\zeta_{\sigma 2}, \ldots, \zeta_{\sigma j}]$$
$$+ \frac{1}{(j-2)!} \sum_{\sigma \in \mathcal{S}_j} \Phi \begin{pmatrix} 0 \\ -\mu 4\pi^2(\zeta_{\sigma 1} \cdot \zeta_{\sigma 2}) u^{(j-2)} \\ 0 \\ 0 \\ 0 \end{pmatrix} [\zeta_{\sigma 3}, \ldots, \zeta_{\sigma j}]. \tag{4.27}$$

Our next result studies the previous construction more carefully.

Proposition 4.10 *(Properties of the derivative construction). The following hold.*

(1) For every $j \in \mathbb{N}^+$ the map

$$\{(f, k, H) \in \mathbb{Y}_s \;:\; 0 \notin \operatorname{supp} \mathscr{F}(f, k, H)\} \ni (f, k, H) \mapsto |D|^j (p^{(j)}, u^{(j)}, \chi^{(j)}) \in \mathcal{L}^j(\mathbb{R}^2; \mathbb{X}_s) \tag{4.28}$$

is continuous and tangentially translation commuting, and hence extends uniquely to a bounded linear map defined on all of $\mathbb{Y}_s$.

(2) *For every $j \in \mathbb{N}^+$ there exists a unique multilinear mapping into the space of symbols, $(\zeta_1, \ldots, \zeta_j) \mapsto \mathbf{m}^{(j)}[\zeta_1, \ldots, \zeta_j]$, such that for all $(f, k, H) \in \mathbb{Y}_s$ satisfying $0 \notin \operatorname{supp} \mathscr{F}(f, k, H)$ we have the identity*

$$\mathbf{m}^{(j)}[\zeta_1, \ldots, \zeta_j](D)(f, k) = (p^{(j)}, u^{(j)}, \chi^{(j)})[\zeta_1, \ldots, \zeta_j]. \tag{4.29}$$

(3) *For all $j \in \mathbb{N}$ and all $(\zeta_1, \ldots, \zeta_j) \in (\mathbb{R}^2)^j$ we have that the map $\xi \mapsto |\xi|^j \mathbf{m}^{(j)}[\zeta_1, \ldots, \zeta_j](\xi)$ belongs to the space $\mathfrak{A}(s)$ from Definition 4.2; moreover, we have the estimate*

$$[\![|\cdot|^j \mathbf{m}^{(j)}[\zeta_1, \ldots, \zeta_j]]\!]_s \lesssim j! \cdot C^j \prod_{i=1}^{j} |\zeta_i|, \tag{4.30}$$

with an implicit constant independent of j.

Proof. We begin with the first item in the case $j = 1$. The key point is that the operator norm of $\Phi : \widetilde{\mathbb{Y}}_s \to \mathbb{X}_s$ depends on a few quantities belonging to $\dot{H}^{-1}(\Sigma; \mathbb{C})$. The appearance of the operator $|D|$ is thus crucial, as it permits the bounds

$$\Big[2\pi \mathrm{i} \zeta \cdot |D| H + 2\pi \mathrm{i} \int_0^b \zeta \cdot (|D| u)(\cdot, y) \, \mathrm{d}y \Big]_{\dot{H}^{-1}} + [2\pi \mathrm{i} \zeta^\perp \cdot |D| \chi]_{\dot{H}^{-1}} \lesssim |\zeta| \| \chi, u, H \|_{L^2 \times L^2 \times L^2}. \tag{4.31}$$

With this observation in hand, we appeal directly to Definition 4.9 and the mapping properties of Φ established in Theorem 2.6 and Definition 4.1, to verify that

$$\| |D| (p^{(1)}, u^{(1)}, \chi^{(1)})[\zeta] \|_{\mathbb{X}_s} \lesssim (\| p, u, \chi \|_{\mathbb{X}_s} + \| f, k, H \|_{\mathbb{Y}_s}) |\zeta| \lesssim \| f, k, H \|_{\mathbb{Y}_s} |\zeta|. \tag{4.32}$$

In a similar manner to the above, we may derive the estimate

$$\begin{aligned} \sup_{|\zeta_1|, |\zeta_2| \leqslant 1} \| |D|^2 (p^{(2)}, u^{(2)}, \chi^{(2)})[\zeta_1, \zeta_2] \|_{\mathbb{X}_s} &\lesssim \sup_{|\zeta| \leqslant 1} \| |D| (p^{(1)}, u^{(1)}, \chi^{(1)})[\zeta] \|_{\mathbb{X}_s} + \| p, u, \chi \|_{\mathbb{X}_s} \\ &\lesssim \| f, k \|_{\mathbb{Y}_s} \end{aligned} \tag{4.33}$$

and, for $j \geqslant 3$,

$$\begin{aligned} &\sup_{|\zeta_1|, \ldots, |\zeta_j| \leqslant 1} \| |D|^j (p^{(j)}, u^{(j)}, \chi^{(j)})[\zeta_1, \ldots, \zeta_j] \|_{\mathbb{X}_s} \lesssim \\ &\quad j \sum_{\sigma=1}^{2} ((j-2)\sigma + 3 - j) \sup_{|\zeta_1|, \ldots, |\zeta_{j-\sigma}| \leqslant 1} \| |D|^{j-\sigma} (p^{(j-\sigma)}, u^{(j-\sigma)}, \chi^{(j-\sigma)})[\zeta_1, \ldots, \zeta_{j-\sigma}] \|_{\mathbb{X}_s}. \end{aligned} \tag{4.34}$$

Upon iterating these estimates, we deduce the boundedness assertion of the first item.

As a consequence of the first item and Proposition 4.4, we find that for each $j \in \mathbb{N}^+$ and $(\zeta_1, \dots, \zeta_j) \in (\mathbb{R}^2)^j$, the map $(f, k, H) \mapsto |D|^j(p^{(j)}, u^{(j)}, \chi^{(j)})[\zeta_1, \dots, \zeta_j]$ is given by a symbol $\widetilde{\mathbf{m}}^{(j)}[\zeta_1, \dots, \zeta_j] \in \mathfrak{A}(s)$. By uniqueness, we must have that $\widetilde{\mathbf{m}}^{(j)}$ is a j-multilinear and symmetric function of the $\zeta_1, \dots, \zeta_j$. We then set $\mathbf{m}^{(j)} = |\cdot|^{-j}\widetilde{\mathbf{m}}^{(j)}$, which implies that (4.29) holds, and hence the second item is satisfied.

The third item follows since $|\cdot|^j\mathbf{m}^{(j)} = \widetilde{\mathbf{m}}^{(j)}$, with the symbol on the right a j-multilinear function of the $\zeta_1, \dots, \zeta_j$ into $\mathfrak{A}(s)$. □

We pause to note that estimate (4.30) would tell us that $\mathbf{m}$ obeys estimates of Mikhlin-Hörmander type if we knew that $\mathbf{m}$ were smooth away from the origin with the j^{th}-derivative equal to $\mathbf{m}^{(j)}$. It is thus our goal now to prove that these are, in fact the derivatives of the symbol $\mathbf{m}$. We first require a definition and a technical lemma about remainders. Note that the following is well-defined thanks to Proposition 4.10.

Definition 4.11 *(Remainders). Given $(f, k, H) \in \mathbb{Y}_s$ satisfying $0 \notin \operatorname{supp} \mathscr{F}(f, k, H)$ and $\zeta \in \mathbb{R}^2$, we define the following elements of $\mathbb{X}_s$:*

$$\mathcal{R}_0(f, k, H)[\zeta] = (\mathbf{m}(D + \zeta) - \mathbf{m}(D))(f, k), \tag{4.35}$$

and for $j \in \mathbb{N}^+$

$$\mathcal{R}_j(f, k, H)[\zeta] = \mathcal{R}_0(f, k, H)[\zeta] - \sum_{i=1}^{j} \frac{1}{i!}\mathbf{m}^{(i)}[\zeta^{\otimes i}](D)(f, k, H), \tag{4.36}$$

where in the above we write $\zeta^{\otimes i} \in (\mathbb{R}^2)^i$ to refer to the i-tuple of vectors with each entry equal to ζ.

Now we derive estimates for these remainder terms.

Lemma 4.12 *(Remainder estimates). There exists a constant $C \geqslant 0$ such that for all $(f, k, H) \in \mathbb{Y}_s$ satisfying $0 \notin \operatorname{supp} \mathscr{F}(f, k, H)$ and $\zeta \in \mathbb{R}^2$ satisfying $|\zeta| < \min\{\operatorname{dist}(0, \operatorname{supp} \mathscr{F}(f, k, H)), 1\}$, we have the following estimates for $j \in \mathbb{N}$:*

$$\| |D|^{j+1}\mathcal{R}_j(f, k, H)[\zeta] \|_{\mathbb{X}_s} \leqslant C^{j+1}|\zeta|^{j+1}\|f, k, H\|_{\mathbb{Y}_s}. \tag{4.37}$$

Proof. Throughout the proof, we will employ the following convenient notation: we set $(p_\zeta, u_\zeta, \chi_\zeta) = \mathbf{m}(D + \zeta)(f, k, H)$, and for $j \in \mathbb{N}$ we set

$$\mathcal{R}_j(f, k, H)[\zeta] = (\mathcal{R}_j p[\zeta], \mathcal{R}_j u[\zeta], \mathcal{R}_j \chi[\zeta]). \tag{4.38}$$

We first establish (4.37) when $j = 0$. From identity (4.22), we find that

$$\mathcal{R}_0(f,k,H)[\zeta] = \Phi \begin{pmatrix} -2\pi\mathrm{i}\zeta \cdot u_\zeta \\ -2\pi\mathrm{i}\zeta p_\zeta + \mu 2\pi\mathrm{i}\nabla(\zeta \cdot u_\zeta) + \mu 4\pi\mathrm{i}\zeta \cdot \nabla u_\zeta - \mu 4\pi^2|\zeta|^2 u_\zeta \\ \kappa 2\pi\mathrm{i}\zeta \cdot \chi_\zeta e_3 - \mu 2\pi\mathrm{i}\zeta(u_\zeta \cdot e_3) \\ 2\pi\mathrm{i}\zeta \cdot H \\ -2\pi\mathrm{i}\zeta^\perp \cdot \chi_\zeta \end{pmatrix}. \tag{4.39}$$

We then apply $|D|$ to the above and take the norm in $\mathbb{X}_s$. Thanks to the continuity properties of $\Phi : \widetilde{\mathbb{Y}}_s \to \mathbb{X}_s$ (see the proof of Proposition 4.10) and Lemma 4.8, we acquire the bound

$$\||D|\mathcal{R}_0(f,k,H)[\zeta]\|_{\mathbb{X}_s} \lesssim |\zeta|(\|p_\zeta, u_\zeta, \eta_\zeta\|_{\mathbb{X}_s} + \|f,k,H\|_{\mathbb{Y}_s}) \lesssim |\zeta|\|f,k,H\|_{\mathbb{Y}_s}, \tag{4.40}$$

which completes the proof in the $j = 0$ case.

Next up is $j = 1$. We subtract $\mathbf{m}^{(1)}[\zeta](D)(f,k,H)$ from both sides of (4.39) and recall (4.25). This yields the equation

$$\mathcal{R}_1(f,k,H)[\zeta] = \Phi \begin{pmatrix} -2\pi\mathrm{i}\zeta \cdot \mathcal{R}_0 u[\zeta] \\ -2\pi\mathrm{i}\zeta\mathcal{R}_0 p[\zeta] + \mu 2\pi\mathrm{i}\nabla(\zeta \cdot \mathcal{R}_0 u[\zeta]) + \mu 4\pi\mathrm{i}\zeta \cdot \nabla\mathcal{R}_0 u[\zeta] - \mu 4\pi^2|\zeta|^2 u_\zeta \\ \kappa 2\pi\mathrm{i}\zeta \cdot \mathcal{R}_0\chi[\zeta]e_3 - \mu 2\pi\mathrm{i}\zeta(\mathcal{R}_0 u[\zeta] \cdot e_3) \\ 0 \\ -2\pi\mathrm{i}\zeta^\perp \cdot \mathcal{R}_0\chi[\zeta] \end{pmatrix}. \tag{4.41}$$

We now apply $|D|^2$ to (4.41) and then take the norm in $\mathbb{X}_s$. The mapping properties of Φ, together with Lemma 4.8 and estimate (4.40), then show that

$$\||D|^2\mathcal{R}_1(f,k,H)[\zeta]\|_{\mathbb{X}_s} \lesssim |\zeta|\||D|\mathcal{R}_0(f,k,H)[\zeta]\|_{\mathbb{X}_s} + |\zeta|^2\||D|^2 u_\zeta\|_{H^s} \lesssim |\zeta|^2\|f,k\|_{\mathbb{Y}_s}. \tag{4.42}$$

This completes the proof in the case $j = 1$.

Now we claim the following identity holds for all $j \geqslant 2$:

$$\mathcal{R}_j(f,k,H)[\zeta] = \Phi \begin{pmatrix} -2\pi\mathrm{i}\zeta \cdot \mathcal{R}_{j-1} u[\zeta] \\ -2\pi\mathrm{i}\zeta\mathcal{R}_{j-1} p[\zeta] + \mu 2\pi\mathrm{i}\nabla(\zeta \cdot \mathcal{R}_{j-1} u[\zeta]) + \mu 4\pi\mathrm{i}\zeta \cdot \nabla\mathcal{R}_{j-1} u[\zeta] - \mu 4\pi^2|\zeta|^2 \mathcal{R}_{j-2} u[\zeta] \\ \kappa 2\pi\mathrm{i}\zeta \cdot \mathcal{R}_{j-1}\chi[\zeta]e_3 - \mu 2\pi\mathrm{i}\zeta(\mathcal{R}_{j-1} u[\zeta] \cdot e_3) \\ 0 \\ -2\pi\mathrm{i}\zeta^\perp \cdot \mathcal{R}_{j-1}\chi[\zeta] \end{pmatrix}. \tag{4.43}$$

We prove this via induction. For the base case $j = 2$, we look to Definition 4.9 to see that

$$\frac{1}{2}\mathbf{m}^{(2)}[\zeta^{\otimes 2}](D)(f,k,H) = \begin{pmatrix} -2\pi\mathrm{i}\zeta \cdot u^{(1)}[\zeta] \\ -2\pi\mathrm{i}\zeta p^{(1)}[\zeta] + \mu 2\pi\mathrm{i}\nabla(\zeta \cdot u^{(1)}[\zeta]) + \mu 4\pi\mathrm{i}\zeta \cdot \nabla u^{(1)}[\zeta] - \mu 4\pi^2|\zeta|^2 u \\ \kappa 2\pi\mathrm{i}\zeta \cdot \chi^{(1)}[\zeta]e_3 - \mu 2\pi\mathrm{i}\zeta(u^{(1)}[\zeta]\cdot e_3) \\ 0 \\ -2\pi\mathrm{i}\zeta^{\perp} \cdot \chi^{(1)}[\zeta] \end{pmatrix}. \tag{4.44}$$

We subtract the above from (4.41) to see that (4.43) is true for $j = 2$. Proceeding inductively, suppose that this identity holds for some $\mathbb{N} \ni j \geqslant 2$. We again look to Definition 4.9 to acquire the identity

$$\begin{aligned}\frac{1}{(j+1)!}\mathbf{m}^{(j+1)}[\zeta^{\otimes(j+1)}](D)(f,k,H) &= \frac{1}{j!}\Phi\begin{pmatrix} -2\pi\mathrm{i}\zeta \cdot u^{(j)}[\zeta^{\otimes j}] \\ -2\pi\mathrm{i}\zeta p^{(j)}[\zeta^{\otimes j}] + \mu 2\pi\mathrm{i}\nabla(\zeta\cdot u^{(j)}[\zeta^{\otimes(j)}]) + \mu 4\pi\mathrm{i}\zeta\cdot\nabla u^{(j)}[\zeta^{\otimes(j)}] \\ \kappa 2\pi\mathrm{i}\zeta\cdot\chi^{(j)}[\zeta^{\otimes(j)}]e_3 - \mu 2\pi\mathrm{i}\zeta(u^{(j)}[\zeta^{\otimes j}]\cdot e_3) \\ 0 \\ -2\pi\mathrm{i}\zeta^{\perp}\cdot\chi^{(j)}[\zeta^{\otimes j}]e_3 \end{pmatrix} \\ &\quad + \frac{1}{(j-1)!}\Phi\begin{pmatrix} 0 \\ -\mu 4\pi^2|\zeta|^2 u^{(j-1)}[\zeta^{\otimes(j-1)}] \\ 0 \\ 0 \\ 0 \end{pmatrix}.\end{aligned} \tag{4.45}$$

We then simply subtract (4.45) from the induction hypothesis (4.43) to prove the stated identity in the $j+1$ case. Thus (4.43) holds for all $\mathbb{N} \ni j \geqslant 2$, and the claim is proved.

With the claim in hand, we apply $|D|^{j+1}$ in the case $\mathbb{N} \ni j \geqslant 2$ to identity (4.43), take the norm in $\mathbb{X}_s$, and utilize the mapping properties of Φ to deduce the inductive estimate

$$\||D|^{j+1}\mathcal{R}_j(f,k,H)[\zeta]\|_{\mathbb{X}_s} \lesssim \sum_{\sigma=1}^{2}|\zeta|^{\sigma}\||D|^{j-\sigma+1}\mathcal{R}_{j-\sigma}(f,k,H)[\zeta]\|_{\mathbb{X}_s}. \tag{4.46}$$

Iteratively applying (4.46) and employing (4.42) and (4.40), we then readily conclude that (4.37) holds. □

At last, we prove analyticity of $\mathbf{m}$ away from the origin.

Theorem 4.13 *(Analyticity of the symbol). The symbol $\mathbf{m} \in \mathfrak{A}(s)$ from Definition 4.6 has a representative that is analytic as a mapping*

$$\mathbf{m} : \mathbb{R}^2 \setminus \{0\} \to \mathcal{L}(H^s((0,b);\mathbb{C}^3) \times \mathbb{C}^3 \times \mathbb{C}^2; H^{1+s}((0,b);\mathbb{C}) \times H^{2+s}((0,b);\mathbb{C}^3) \times \mathbb{C}^2). \tag{4.47}$$

Moreover, the derivatives of the above symbol obey the following Mikhlin-Hörmander type bounds: for every $\alpha \in \mathbb{N}^d$ there exists $C_\alpha < \infty$ such that

$$[\![|\cdot|^{|\alpha|} \partial^\alpha \mathbf{m}]\!]_s \leqslant C_\alpha. \tag{4.48}$$

Proof. It suffices to prove that $\mathbf{m}$ is analytic as claimed and that $\nabla^j \mathbf{m} = \mathbf{m}^{(j)}$ for $j \in \mathbb{N}^+$. Indeed, once this is established, the bound (4.48) is a consequence of the third item of Proposition 4.10 and Definition 4.6. We divide the proof into several steps.

Step 1 - Setup and basic estimates:

For the sake of brevity, define the Hilbert spaces $V_0 = H^s((0,b);\mathbb{C}^3) \times \mathbb{C}^3 \times \mathbb{C}^2$ and $V_1 = H^{1+s}((0,b);\mathbb{C}) \times H^{2+s}((0,b);\mathbb{C}^3) \times \mathbb{C}^2$. We will write $\mathcal{L} = \mathcal{L}(V_0;V_1)$ throughout the proof.

Fix $\varepsilon \in (0,1/4)$ and define the annuli $A_2 = B(0,2\varepsilon^{-1}) \setminus \overline{B(0,\varepsilon/2)}$, $A_1 = B(0,\varepsilon^{-1}) \setminus \overline{B(0,\varepsilon)} \subset A_2$, and $A_0 = B(0,\varepsilon^{-1}/2) \setminus \overline{B(0,2\varepsilon)} \subset A_1$. Thanks to Definition 4.11, for $j \in \mathbb{N}$ and $|\zeta| < \varepsilon/2$ the linear map

$$\mathbb{Y}_s \ni (f,k,H) \overset{T_j^\varepsilon[\zeta]}{\mapsto} \mathcal{R}_j(f,k,H)[\zeta] \mathbb{1}_{A_2}(D)(f,k) \in \mathbb{X}_s \tag{4.49}$$

is bounded, translation commuting, and satisfies $T_j^\varepsilon[\zeta] = \mathbf{m}_j^\varepsilon[\zeta](D)$ for the symbol

$$\mathbf{m}_j^\varepsilon[\zeta](\xi) = \Big(\mathbf{m}(\xi+\zeta) - \sum_{i=0}^{j} \frac{1}{i!} \mathbf{m}^{(i)}[\zeta^{\otimes i}](\xi) \Big) \mathbb{1}_{A_2}(\xi), \tag{4.50}$$

with the understanding that $\frac{1}{0!}\mathbf{m}^{(0)}[\zeta^{\otimes 0}](\xi) = \mathbf{m}(\xi)$. Lemma 4.12 provides the estimate

$$\| |D|^{j+1} T_j^\varepsilon[\zeta] \|_{\mathcal{L}(\mathbb{Y}_s;\mathbb{X}_s)} \leqslant C^{j+1} |\zeta|^{j+1}, \tag{4.51}$$

and so Proposition 4.4 then yields multiplier bound

$$[\![|\cdot|^{j+1} \mathbf{m}_j^\varepsilon[\zeta]]\!]_s \leqslant C^{j+1} |\zeta|^{j+1} \text{ for all } |\zeta| < \varepsilon/2. \tag{4.52}$$

Now, since the multiplier $\mathbf{m}_j^\varepsilon$ is supported in the set A_2, we are free to apply Lemma 4.3 and find that there exists a constant c_ε, depending only ε and s, such that

$$\operatorname*{ess\,sup}_{\xi \in A_2} |\xi|^{j+1} \Big\| \mathbf{m}(\xi+\zeta) - \sum_{i=0}^{j} \frac{1}{i!} \mathbf{m}^{(i)}[\zeta^{\otimes i}](\xi) \Big\|_{\mathcal{L}} \leqslant c_\varepsilon [\![|\cdot|^{j+1} \mathbf{m}_j^\varepsilon[\zeta]]\!]_s \leqslant c_\varepsilon C^{j+1} |\zeta|^{j+1}, \tag{4.53}$$

where we recall that $\mathcal{L} = \mathcal{L}(V_0;V_1)$, and we deduce from this that if $|\zeta| < \varepsilon/2$, then

$$\operatorname*{ess\,sup}_{\xi \in A_2} \Big\| \mathbf{m}(\xi+\zeta) - \sum_{i=0}^{j} \frac{1}{i!} \mathbf{m}^{(i)}[\zeta^{\otimes i}](\xi) \Big\|_{\mathcal{L}} \leqslant c_\varepsilon \Big(\frac{2C|\zeta|}{\varepsilon} \Big)^{j+1}. \tag{4.54}$$

A similar application of Definition 4.6 and Lemma 4.3 shows that

$$\operatorname*{ess\,sup}_{\xi\in A_2} \|\mathbf{m}(\xi)\|_{\mathcal{L}} \leqslant c_\varepsilon. \tag{4.55}$$

Step 2 - Lipschitz continuity in A_1:

We now aim to show that $\mathbf{m} : A_1 \to \mathcal{L}$ has a Lipschitz continuous representative. To this end, let $\varphi \in C_c^\infty(\mathbb{R}^n)$ be such that $\varphi \geqslant 0$, $\int \varphi = 1$, and $\operatorname{supp}(\varphi) \subseteq B(0,1)$. For $0 < \delta < \varepsilon/4$ and $\xi \in A_1$ we define $\mathbf{n}_\varepsilon(\xi) : V_0 \to V_1$ via

$$\mathbf{n}_\delta(\xi)v = \int_{B(0,\delta)} \frac{1}{\delta^2}\varphi(\omega/\delta)\mathbf{m}(\xi-\omega)v \,\mathrm{d}\omega, \tag{4.56}$$

which is well-defined since $B(0,\delta) \ni \omega \mapsto \mathbf{m}(\xi-\omega)v \in V_2$ is measurable and essentially bounded for each $v \in V_0$ and $\xi \in A_1$. From this it's easy to see that $\mathbf{n}_\delta(\xi) \in \mathcal{L}$ for each $\xi \in A_1$ and that the induced map $\mathbf{n}_\delta : A_1 \to \mathcal{L}$ is continuous. In fact, the mollified sequence is uniformly Lipschitz in δ. Indeed, from (4.54) with $j = 0$ we know that there exists a null set $N \subset A_2$ such that if $\xi \in A_1$ and $\omega \in B(0,\delta)$ are such that $\xi - \omega \in A_2 \backslash N$, then

$$\sup_{\|v\|_{V_0}\leqslant 1} \|\mathbf{m}(\xi-\omega+\eta)v - \mathbf{m}(\xi-\omega)v\|_{V_1} \leqslant \sup_{\theta\in A_2\backslash N} \|\mathbf{m}(\theta+\eta)-\mathbf{m}(\theta)\|_{\mathcal{L}} \leqslant \frac{2c_\varepsilon C}{\varepsilon}|\eta|. \tag{4.57}$$

Consequently, for any given $\xi \in A_1$ and $|\eta| < \varepsilon/2$ we have that

$$\sup_{\|v\|_{V_0}\leqslant 1} \|\mathbf{m}(\xi-\omega+\eta)v - \mathbf{m}(\xi-\omega)v\|_{V_1} \leqslant \frac{2c_\varepsilon C}{\varepsilon}|\eta| \text{ for a.e. } \omega \in B(0,\delta), \tag{4.58}$$

and hence

$$\|\mathbf{n}_\delta(\xi+\eta) - \mathbf{n}_\delta(\xi)\|_{\mathcal{L}} \leqslant \frac{2c_\varepsilon C}{\varepsilon}|\eta| \int_{\mathbb{R}^2} \frac{1}{\delta^2}\varphi(\omega/\delta)\,\mathrm{d}\omega = \frac{2c_\varepsilon C}{\varepsilon}|\eta|. \tag{4.59}$$

Since $\xi \in A_1$ was arbitrary, we deduce that

$$\sup_{\xi\in A_1} \|\mathbf{n}_\delta(\xi+\eta) - \mathbf{n}_\delta(\xi)\|_{\mathcal{L}} \leqslant \frac{2c_\varepsilon C}{\varepsilon}|\eta| \tag{4.60}$$

for all $|\eta| < \varepsilon/2$. By similar considerations, we may estimate

$$\|\mathbf{n}_\delta - \mathbf{m}\|_{L^\infty_*(A_1;\mathcal{L})} \leqslant \frac{2c_\varepsilon C}{\varepsilon} \int_{B(0,\delta)} \frac{1}{\delta^2}\varphi(\omega/\delta)|\omega|\,\mathrm{d}\omega \leqslant \frac{2c_\varepsilon C}{\varepsilon}\delta. \tag{4.61}$$

Sending $\delta \to 0$ and appealing to (4.60) and (4.55), we find that the restriction of $\mathbf{m}$ to A_1 is almost everywhere equal to a Lipschitz continuous function from A_1 to $\mathcal{L}$. From now on, we shall use the continuous representative of $\mathbf{m}$ in A_1.

Step 3 - Local convergence of the power series in A_2:

We now claim that there exists a constant $C_1 \geqslant 2$ and a set $\mathfrak{E} \subseteq A_2$ with $|A_2 \setminus \mathfrak{E}| = 0$ such that for any $\xi \in \mathfrak{E}$ the power series

$$\mathbb{R}^2 \supset B(\xi, \varepsilon/C_1) \ni \zeta \mapsto \sum_{j=0}^{\infty} \mathbf{m}^{(j)}[\zeta^{\otimes j}](\xi) \in \mathcal{L} \tag{4.62}$$

converges uniformly absolutely and thus defines an analytic $\mathcal{L}$-valued function in $B(\xi, \varepsilon/N)$.

To prove the claim, we first appeal to the bounds (4.30) from Proposition 4.10 and Lemma 4.3 to find a constant $R > 0$ such that for every $j \in \mathbb{N}$ we have the bound

$$\frac{1}{j!} \operatorname*{ess\,sup}_{\xi \in A_2} |\xi|^j \|\mathbf{m}^{(j)}[\zeta^{\otimes j}](\xi)\|_{\mathcal{L}} \leqslant c_\varepsilon R^j |\zeta|^j \text{ for every } |\zeta| < \frac{\varepsilon}{2}. \tag{4.63}$$

This estimate can be improved by virtue of multilinearity. To see how, set $C_1 = 2(1+2R)$ and let $\{\zeta_n\}_{n\in\mathbb{N}} \subset B(0, \varepsilon/C_1)$ be dense. Then by (4.63), the fact that a countable union of null sets is again null, the pointwise continuity of the multilinear maps involved, and the fact that $\operatorname{dist}(0, A_2) = \varepsilon/2$, we have that

$$\begin{aligned} \operatorname*{ess\,sup}_{\xi \in A_2} \sup_{|\zeta| < \varepsilon/C_1} \|\mathbf{m}^{(j)}[\zeta^{\otimes j}](\xi)\|_{\mathcal{L}} &\leqslant \operatorname*{ess\,sup}_{\xi \in A_2} \sup_{n \in \mathbb{N}} \frac{2^j |\xi|^j}{\varepsilon^j} \|\mathbf{m}^{(j)}[\zeta_n^{\otimes j}](\xi)\|_{\mathcal{L}} \\ &\leqslant j! \cdot c_\varepsilon \left(\frac{2R}{C_1}\right)^j \leqslant j! \cdot c_\varepsilon 2^{-j}. \end{aligned} \tag{4.64}$$

Summing over $j \in \mathbb{N}$, we may thus bound

$$\sum_{j=0}^{\infty} \frac{1}{j!} \operatorname*{ess\,sup}_{\xi \in A_2} \sup_{|\zeta| < \varepsilon/C_1} \|\mathbf{m}^{(j)}[\zeta^{\otimes j}](\xi)\|_{\mathcal{L}} \leqslant 2c_\varepsilon, \tag{4.65}$$

and the claim now follows directly from this.

Step 4 - Analyticity in A_0:

Finally, we aim to prove that the multiplier $\mathbf{m}$ in (4.47) is analytic in the annulus A_0. The strategy is to combine estimates (4.65) and (4.54). Let $C_2 = \max\{C_1, 2(1+2C)\}$, where C_1 is from Step 3 and $C > 0$ is the constant in (4.54). By letting $\{\zeta_n\}_{n\in\mathbb{N}} \subset B(0, \varepsilon/C_2)$ be dense, we may use (4.54) to produce a set $\mathfrak{F} \subseteq A_2$ with the property that $|A_2 \setminus \mathfrak{F}| = 0$ and if $\xi \in \mathfrak{F}$ and $j \in \mathbb{N}$ then

$$\sup_{n \in \mathbb{N}} \left\| \mathbf{m}(\xi + \zeta_n) - \sum_{i=0}^{j} \mathbf{m}^{(i)}[\zeta_n^{\otimes i}](\xi) \right\|_{\mathcal{L}} \leqslant c_\varepsilon 2^{-(j+1)}. \tag{4.66}$$

In particular, if $\xi \in A_0 \cap \mathfrak{E}$, then the continuity assertion of Step 2 allows us to bound

$$\sup_{|\zeta|<\varepsilon/C_2} \left\| \mathbf{m}(\xi+\zeta) - \sum_{i=0}^{j} \mathbf{m}^{(i)}[\zeta^{\otimes i}](\xi) \right\|_{\mathcal{L}} \leqslant c_\varepsilon 2^{-(j+1)}. \tag{4.67}$$

Let $\mathfrak{F}$ be the set from Step 3 and note that $A_0 \cap \mathfrak{E} \cap \mathfrak{F}$ is a set of full measure in A_0. We can then send $j \to \infty$ in (4.67) to see that

$$\mathbf{m}(\xi+\zeta) = \sum_{i=0}^{\infty} \mathbf{m}^{(i)}[\zeta^{\otimes i}](\xi) \text{ for } \xi \in A_0 \cap \mathfrak{E} \cap \mathfrak{F} \text{ and } \zeta \in B(0, \varepsilon/C_2). \tag{4.68}$$

In light of the continuity of $\mathbf{m}$ in A_1, we learn from this that the power series produced in Step 3 agree on the intersection of their balls of convergence. Consequently, we may produce a single $\mathcal{L}$-valued analytic function on A_0 that is equal to $\mathbf{m}$ a.e., and since $\varepsilon > 0$ was arbitrary we conclude that $\mathbf{m}$ has an analytic representative in $\mathbb{R}^2 \backslash \{0\}$. Finally, we learn from (4.68) that $\nabla^j \mathbf{m} = \mathbf{m}^{(j)}$ almost everywhere. $\square$

5. On some Sobolev-type spaces

In Section 4 we constructed an operator-valued symbol $\mathbf{m}$ such that the corresponding Fourier multiplication operator $\mathbf{m}(D)$ is a particular solution map for the PDE (2.2). We know from Theorem 4.13 that $\mathbf{m}$ obeys certain inequalities of Mikhlin-Hörmander type, and so we expect to be able to employ Theorem 3.14 to obtain the boundedness of $\mathbf{m}(D)$ on certain vector-valued Sobolev spaces. The first goal of this section is to define and study these mixed-type spaces for use in this manner. This is done in Section 5.1. In Section 5.2 we record a number of nonlinear tools that we will use in working with the mixed-type spaces.

The second purpose of this section, which is the content of Section 5.3, is to study what we call subcritical gradient spaces. These are spaces of functions whose distributional derivatives belong to $H^{s-1,p}(\mathbb{R}^d)$ for $1 < p < d$ and $s \in \mathbb{N}^+$, and they arise naturally in our analysis of the free surface function in (1.13). Their properties will play a crucial role in our subsequent PDE analysis.

Throughout this section, we have phrased the results in a more general manner than what is precisely needed in Sections 6 and 7, as we believe that the analysis here may be of independent interest.

5.1. Mixed-type Sobolev spaces

Throughout this subsection we consider a generic open interval $I \subseteq \mathbb{R}$ and define the set $U = \mathbb{R}^d \times I$ for $d \in \mathbb{N}^+$.

Definition 5.1 *(Mixed-type Sobolev spaces). Let $1 < p < \infty$ and V be a finite dimensional normed space over $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$. Let $U = \mathbb{R}^d \times I$ for $d \in \mathbb{N}^+$.*

(1) We define the mixed type Lebesgue space

$$L_{p,2}(U;V)=L^p(\mathbb{R}^d;L^2(I;V)),\quad \|f\|_{L_{p,2}}=\Big(\int_{\mathbb{R}^d}\Big(\int_I\|f(x,y)\|_V^2\,\mathrm{d}y\Big)^{p/2}\,\mathrm{d}x\Big)^{1/p},\tag{5.1}$$

which is a Banach space when endowed with the obvious norm. Moreover, the Fubini-Tonelli theorem shows that $L_{p,2}(U;V)\hookrightarrow L^{\min\{2,p\}}_{\mathrm{loc}}(U;V)$*. We model Sobolev spaces on these mixed Lebesgue-spaces in the natural way.*

(2) For $s\in\mathbb{N}$ *we define*

$$H^s_{p,2}(U;V)=\{f\in L_{p,2}(U;V)\ :\ \partial^\alpha f\in L_{p,2}(U;V)\ \textit{for all}\ \alpha\in\mathbb{N}^3\ \textit{with}\ |\alpha|\leqslant s\},\tag{5.2}$$

and endow this space with the norm

$$\|f\|_{H^s_{p,2}}=\Big(\sum_{|\alpha|\leqslant s}\|\partial^\alpha f\|^p_{L_{p,2}}\Big)^{1/p}.\tag{5.3}$$

Minor variants of the usual Sobolev-theoretic arguments apply to show that these spaces are Banach and that the restrictions to U *of elements of* $C^\infty_c(\mathbb{R}^{d+1};V)$ *form a dense subspace. When we write* $H^s_{p,2}(U)$ *the understanding is that* $V=\mathbb{R}$*.*

Our first lemma about these spaces provides a useful equivalent norm obtained via factorization.

Lemma 5.2 *(Equivalent norm on the mixed type Sobolev spaces). Let* $s\in\mathbb{N}$, $1<p<\infty$, *and* V *be a finite dimensional normed space over* $\mathbb{F}\in\{\mathbb{R},\mathbb{C}\}$*. Then we have that*

$$H^s_{p,2}(U;V)=L^p(\mathbb{R}^d;H^s(I;V))\cap H^{s,p}(\mathbb{R}^d;L^2(I;V))\tag{5.4}$$

with norm equivalence

$$\|f\|_{H^s_{p,2}}\asymp\|f\|_{L^pH^s}+\|f\|_{H^{s,p}L^2}\,.\tag{5.5}$$

Proof. Since we can always complexify a real normed space V, it suffices to prove the result when $\mathbb{F}=\mathbb{C}$. Assume this. The continuous embedding of the space of the left of (5.4) into the right is obvious; the reverse inclusion requires work. Suppose that f belongs to the space on the right and $k\in\{1,\dots,s-1\}$. Let $\varphi\in C^\infty_c(\mathbb{R}^d)$ be a generator for a homogeneous Littlewood-Paley partition of unity. We then use Gagliardo-Nirenberg interpolation on the space $H^k(I;V)\hookrightarrow H^s(I;V)$, together with Young's inequality for products, namely $a^{1-k/s}b^{k/s}\lesssim a+b$, the triangle inequality, and Theorem 3.10 to estimate

$$\begin{aligned}
&\Big\|\Big(\sum_{j\in\mathbb{Z}}\langle 2^j\rangle^{2(s-k))}\|\varphi(D/2^j)f\|_{H^k}^2\Big)^{1/2}\Big\|_{L^p} \\
&\qquad\lesssim \Big\|\Big(\sum_{j\in\mathbb{Z}}\langle 2^j\rangle^{2(s-k)}\|\varphi(D/2^j)f\|_{L^2}^{2(1-k/s)}\|\varphi(D/2^j)f\|_{H^s}^{2s/k}\Big)^{1/2}\Big\|_{L^p} \\
&\qquad\lesssim \Big\|\Big(\sum_{j\in\mathbb{Z}}\langle 2^j\rangle^{2s}\|\varphi(D/2^j)f\|_{L^2}^2\Big)^{1/2}\Big\|_{L^p} + \Big\|\Big(\sum_{j\in\mathbb{Z}}\|\varphi(D/2^j)f\|_{H^s}^2\Big)^{1/2}\Big\|_{L^p} \\
&\qquad\lesssim \|f\|_{H^{s,p}L^2} + \|f\|_{L^pH^s}. \qquad (5.6)
\end{aligned}$$

In turn, we may use Theorem 3.11 to conclude that $f = \sum_{j\in\mathbb{Z}}\varphi(D/2^j)f$ and

$$\|f\|_{H^{s-k,p}H^k} \lesssim \Big\|\Big(\sum_{j\in\mathbb{Z}}\langle 2^j\rangle^{2(s-k))}\|\varphi(D/2^j)f\|_{H^k}^2\Big)^{1/2}\Big\|_{L^p} \lesssim \|f\|_{H^{s,p}L^2} + \|f\|_{L^pH^s}. \qquad (5.7)$$

Summing over $k \in \{1, \dots, s-1\}$ then completes the proof of the reverse embedding in (5.4). □

Next we turn our attention to extension operators.

Proposition 5.3 *(Extensions on mixed-type Sobolev spaces). Let $s \in \mathbb{N}$, $1 < p < \infty$, and V be a finite dimensional normed space over $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$. There exists a bounded linear extension operator*

$$\mathfrak{E}_U : H^s_{p,2}(U; V) \to H^s_{p,2}(\mathbb{R}^{d+1}; V) \qquad (5.8)$$

such that $\mathfrak{R}_U\mathfrak{E}_U = \mathrm{id}$ on $H^s_{p,2}(U)$, where $\mathfrak{R}_U$ denotes the restriction operator.

Proof. Since V is assumed to be finite dimensional over $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$, we may use a basis to reduce to the case $V = \mathbb{F}$. The notion of a Stein extension operator is given in Section 3.1 in Chapter VI of Stein [90]. For the domain $U = \mathbb{R}^d \times I$, one can select a Stein-extension operator that is tangentially translation commuting. In fact, we only need the Stein-extension operator $I \to \mathbb{R}$ and view it as acting on functions on $\mathbb{R}^d \times I$ via carrying along the first d-variables as parameters. It is then immediate that

$$\mathfrak{E}_U : L^p(\mathbb{R}^d; H^s(I; \mathbb{F})) \to L^p(\mathbb{R}^d; H^s(\mathbb{R}; \mathbb{F})) \qquad (5.9)$$

is a bounded linear map. From tangential translation invariance, we get

$$\begin{aligned}
\|\mathfrak{E}_U f\|_{H^{s,p}L^2} = \|\mathfrak{E}_U\langle D\rangle^s f\|_{L^pL^2} &\lesssim \|\|\langle D\rangle^s f\|_{L^2(I)}\|_{L^p(\mathbb{R}^d)} \\
&= \|f\|_{H^{s,p}L^2} \text{ for } f \in H^{s,p}(\mathbb{R}^d; L^2(I; \mathbb{F})),
\end{aligned} \qquad (5.10)$$

and hence $\mathfrak{E}_U : H^{s,p}(\mathbb{R}^d; L^2(I; \mathbb{F})) \to H^{s,p}(\mathbb{R}^d; L^2(\mathbb{R}; \mathbb{F}))$ is bounded. Thus, (5.8) follows from Lemma 5.2. □

Next, we discuss traces. Note that in this result the regularity loss caused by taking a trace is $1/2$ rather than $1/p$. This is due to the L^2-based regularity spaces used in the 'normal' direction.

Proposition 5.4 *(Traces of mixed-type Sobolev spaces). Let $s \in \mathbb{N}^+$ and $1 < p < \infty$, and V be a finite dimensional normed space over $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$. For $b \in \mathbb{R}^+$ define $\Omega = \mathbb{R}^d \times (0, b)$ and $\Sigma = \mathbb{R}^d \times \{b\}$. Then there exists a bounded and linear trace map*

$$\mathrm{Tr}_\Sigma : H^s_{p,2}(\Omega; V) \to H^{s-1/2,p}(\Sigma; V). \tag{5.11}$$

Proof. Using a basis of V, we reduce to proving the result with $V = \mathbb{F}$. The key observation is the following interpolation inequality for functions $\phi \in H^s((0, b))$:

$$|\phi(b)| \lesssim \|\phi\|_{L^2}^{1-1/2s} \|\phi\|_{H^s}^{1/2s}, \tag{5.12}$$

where the implicit constant depends on b and s. A proof may be found, for instance, in Lemmas 4.9 and 4.10 of Constantin and Foias [30].

Let $\varphi \in C_c^\infty(\mathbb{R}^d)$ generate a homogeneous Littlewood-Paley partition of unity. Then, given $f \in H^s_{p,2}(\Omega; \mathbb{F})$ we use the above interpolation inequality with Young's inequality, namely $a^{1-1/2s} b^{1/2s} \lesssim a + b$, and Theorem 3.10 to bound

$$\begin{aligned}
&\Big\| \Big(\sum_{j\in\mathbb{Z}} \langle 2^j \rangle^{2s-1} |\varphi(D/2^j) \mathrm{Tr}_\Sigma f|^2 \Big)^{1/2} \Big\|_{L^p} \\
&\qquad \lesssim \Big\| \Big(\sum_{j\in\mathbb{Z}} \langle 2^j \rangle^{2s-1} \|\varphi(D/2^j) f\|_{L^2}^{2-1/s} \|\varphi(D/2^j) f\|_{H^s}^{1/s} \Big)^{1/2} \Big\|_{L^p} \\
&\qquad \lesssim \Big\| \Big(\sum_{j\in\mathbb{Z}} \langle 2^j \rangle^{2s} \|\varphi(D/2^j) f\|_{L^2}^2 \Big)^{1/2} \Big\|_{L^p} + \Big\| \Big(\sum_{j\in\mathbb{Z}} \|\varphi(D/2^j) f\|_{H^s}^2 \Big)^{1/2} \Big\|_{L^p} \\
&\qquad \lesssim \|f\|_{L^p H^s} + \|f\|_{H^{s,p} L^2}.
\end{aligned} \tag{5.13}$$

This, Lemma 5.2, and Theorem 3.11 then provide the estimate

$$\|\mathrm{Tr}_\Sigma f\|_{H^{s-1/2,p}} \lesssim \|f\|_{L^p H^s} + \|f\|_{H^{s,p} L^2} \lesssim \|f\|_{H^s_{p,2}}, \tag{5.14}$$

so the trace operator is bounded as stated. □

Now we discuss lifting maps that complement the trace map.

Proposition 5.5 *(Lifting in mixed-type Sobolev spaces). Let $s \in \mathbb{N}^+$, and $1 < p < \infty$, and V be a finite dimensional normed space over $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$. For $b \in \mathbb{R}^+$ let $\Omega = \mathbb{R}^d \times (0, b)$ and $\Sigma = \mathbb{R}^d \times \{b\}$. There exists a bounded linear extension map $\mathrm{L}_\Omega : H^{s-1/2,p}(\Sigma; V) \to H^s_{2,p}(\Omega; V)$ such that $\mathrm{Tr}_\Sigma \mathrm{L}_\Omega = \mathrm{id}_{H^{s-1/2,p}(\Sigma)}$.*

Proof. We will prove the result when $V = \mathbb{C}$; the general case can be deduced from this. Given $f \in \mathscr{S}(\Sigma;\mathbb{C})$, we define $\mathrm{L}_\Omega f : \Omega \to \mathbb{C}$ via

$$\mathscr{F}[\mathrm{L}_\Omega f](\xi, y) = \exp(\langle\xi\rangle(y-b))\mathscr{F}[f](\xi) \text{ for } \xi \in \mathbb{R}^d \text{ and } y \in (0,b). \tag{5.15}$$

Now define $m \in C^\infty(\Omega;\mathbb{C})$ via $m(\xi, y) = \exp(\langle\xi\rangle(y-b))$. Thanks to the Leibniz rule and Faà di Bruno's formula, we have that

$$|\partial_y^j D_\xi^k m(\xi,y)| = |D_\xi^k(\langle\xi\rangle^j \exp(\langle\xi\rangle(y-b)))| \lesssim \sum_{i=0}^{k} \langle\xi\rangle^{j-k+i} |D_\xi^i(\exp(\langle\xi\rangle(y-b)))|$$
$$\lesssim \sum_{i=0}^{k} \langle\xi\rangle^{j-k+i} \sum_{\ell=1}^{i} \langle\xi\rangle^{-i+\ell} |y-b|^\ell \exp(\langle\xi\rangle(y-b)) \lesssim \langle\xi\rangle^{j-k} \max_{1\leqslant\ell\leqslant k} |y-b|^\ell \langle\xi\rangle^\ell \exp(\langle\xi\rangle(y-b)). \tag{5.16}$$

We readily deduce from this that for any $\alpha \in \mathbb{N}^d$ there exists a constant $C_\alpha > 0$ such that

$$\sup_{\xi\in\mathbb{R}^d} \langle\xi\rangle^{|\alpha|+1/2} \|\partial_\xi^\alpha m(\xi,\cdot)\|_{L^2} + \sup_{\xi\in\mathbb{R}^d} \langle\xi\rangle^{|\alpha|+1/2-s} \|\partial_\xi^\alpha m(\xi,\cdot)\|_{H^s} \leqslant C_\alpha. \tag{5.17}$$

Let $\ell \in \{0,s\}$. Employing the canonical isometric identification $H^\ell((0,b);\mathbb{C}) = \mathcal{L}(\mathbb{C}; H^\ell((0,b);\mathbb{C}))$, we define the vector-valued Fourier multiplier with symbol $\mu \in C^\infty(\mathbb{R}^d; \mathcal{L}(\mathbb{C}; H^\ell((0,b);\mathbb{C})))$ given by $\mu(\xi) = m(\xi,\cdot)$. In light of (5.17), we can then invoke Theorem 3.14 twice to deduce that

$$\mu(D) : H^{s-1/2,p}(\Sigma;\mathbb{C}) \to H^{s,p}(\mathbb{R}^d; L^2((0,b);\mathbb{C})) \cap L^p(\mathbb{R}^d; H^s((0,b);\mathbb{C})) = H^s_{2,p}(\Omega;\mathbb{C}) \tag{5.18}$$

is a bounded linear map (in the last equality we have employed Lemma 5.2). To conclude, we simply note that for $f \in \mathscr{S}(\Sigma;\mathbb{C})$ we have that $\mu(D)f(x) = \mathrm{L}_\Omega f(x,\cdot)$ for all $x \in \mathbb{R}^d$. □

Finally, we record a mixed-type Sobolev spaces variant of Proposition C.1 in Stevenson and Tice [92].

Proposition 5.6 *(Divergence compatibility in mixed-type Sobolev spaces). Let $1 < p < \infty$, $b \in \mathbb{R}^+$, and $\Omega = \mathbb{R}^d \times (0,b)$. Then there exists a constant C such that for all $u \in H^1_{p,2}(\Omega;\mathbb{F}^{d+1})$ we have the estimate*

$$\Big[\int_0^b (\nabla\cdot u)(\cdot,y)\,\mathrm{d}y - \mathrm{Tr}_\Sigma u\cdot e_{d+1} + \mathrm{Tr}_{\Sigma_0} u\cdot e_{d+1}\Big]_{\dot{H}^{-1,p}} \leqslant C\|u\|_{L_{p,2}}. \tag{5.19}$$

Proof. By density, it suffices to consider the case that $0 \notin \mathrm{supp}\mathscr{F}[u]$. Thanks to the fundamental theorem of calculus, we have

$$\int_0^b (\nabla \cdot u)(\cdot, y)\, \mathrm{d}y - \mathrm{Tr}_\Sigma u \cdot e_{d+1} + \mathrm{Tr}_{\Sigma_0} u \cdot e_{d+1} = (\partial_1, \dots, \partial_d, 0) \cdot \int_0^b u(\cdot, y)\, \mathrm{d}y \tag{5.20}$$

Applying $|D|^{-1}$ and using the definition of $\dot{H}^{-1,p}$ from (1.28) and (1.29) along with the boundedness of Riesz transforms yields

$$\Big[\int_0^b (\nabla \cdot u)(\cdot, y)\, \mathrm{d}y - \mathrm{Tr}_\Sigma u \cdot e_{d+1} + \mathrm{Tr}_{\Sigma_0} u \cdot e_{d+1}\Big]_{\dot{H}^{-1,p}} \lesssim \Big\| \int_0^b u(\cdot, y)\, \mathrm{d}y \Big\|_{L^p}. \tag{5.21}$$

We conclude after noting that the embedding $L^2((0,b)) \hookrightarrow L^1((0,b))$ allows us to bound the right hand side of (5.21) by $\|u\|_{L_{p,2}}$. □

5.2. Some nonlinear analysis in mixed-type Sobolev spaces

The goal of this subsection is to record a series of useful results related to the nonlinear use of mixed-type spaces. As in Section 5.1, we will let $I \subseteq \mathbb{R}$ be an open interval and set $U = \mathbb{R}^d \times I$.

Our first result gives a product estimates for the mixed-type Sobolev spaces.

Lemma 5.7 *(Product estimates in mixed type Sobolev spaces). Suppose $s \in \mathbb{N}^+$, $1 < p < \infty$, and V is a finite dimensional normed space over $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$. Then we have the estimate*

$$\|fg\|_{H^s_{p,2}} \lesssim \|f\|_{L^\infty \cap L_{p,2}} \|g\|_{H^s_{p,2}} + \|f\|_{H^s_{p,2}} \|g\|_{L^\infty \cap L_{p,2}} \tag{5.22}$$

for all $f \in H^s_{p,2}(U; \mathbb{F}) \cap L^\infty(U; \mathbb{F})$ and $g \in H^s_{p,2}(U; V) \cap L^\infty(U; V)$.

Proof. It suffices to prove the result for $\mathbb{F} = \mathbb{C}$, so we will assume this in the proof. To check the product belongs to the correct space, we will use the norm from Lemma 5.2. We first recall the well-known (see e.g. Theorem D.6 in [92]) high-low product estimate

$$\|FG\|_{H^s(I;V)} \lesssim \|F\|_{L^\infty(I;\mathbb{F})} \|G\|_{H^s(I;V)} + \|F\|_{H^s(I;\mathbb{F})} \|G\|_{L^\infty(I;V)} \tag{5.23}$$

for all $F \in H^s(I; \mathbb{F}) \cap L^\infty(I; \mathbb{F})$ and $G \in H^s(I; V) \cap L^\infty(I; V)$. Applying this almost everywhere in $\mathbb{R}^d$ and integrating, we then derive the bound

$$\|fg\|_{L^p H^s} \lesssim \|f\|_{L^\infty} \|g\|_{L^p H^s} + \|f\|_{L^p H^s} \|g\|_{L^\infty}. \tag{5.24}$$

The bounds for tangential derivatives are more involved. To handle them we let $\varphi \in C_c^\infty(\mathbb{R}^d)$ be generator for a homogeneous Littlewood-Paley partition of unity, we set $\Phi = \sum_{j\leqslant 0} \varphi(\cdot/2^j)$, and we introduce the (tangential, homogeneous) paraproduct decomposition

$$\begin{aligned} fg = \sum_{j\in\mathbb{Z}} \varphi(D/2^j)f\ \Phi(D/2^{j-3})g \\ + \sum_{j\in\mathbb{Z}}\sum_{k=j-2}^{j+2} \varphi(D/2^j)f\ \varphi(D/2^k)g + \sum_{k\in\mathbb{Z}} \Phi(D/2^{k-3})f\ \varphi(D/2^k)g \\ = \pi_{\mathrm{hl}}(f,g) + \pi_{\mathrm{hh}}(f,g) + \pi_{\mathrm{lh}}(f,g). \end{aligned} \tag{5.25}$$

For the π_{hl} term, we use the annular Littlewood-Paley estimates from Theorems 3.10 and 3.11 together with the bound

$$\sup_{j\in\mathbb{N}} \|\Phi(D/2^j)g\|_{L^\infty(\Omega)} \lesssim \|g\|_{L^\infty}, \tag{5.26}$$

which follows from Young's convolution inequality, in order to estimate

$$\|\pi_{\mathrm{hl}}(f,g)\|_{H^{s,p}L^2} \lesssim \left\| \Big(\sum_{j\in\mathbb{Z}} \langle 2^j\rangle^{2s} \|\varphi(D/2^j)f\ \Phi(D/2^{j-3})g\|_{L^2}^2 \Big)^{1/2} \right\|_{L^p} \lesssim \|f\|_{H^{s,p}L^2}\|g\|_{L^\infty}. \tag{5.27}$$

By an entirely symmetric argument, we have that $\|\pi_{\mathrm{lh}}(f,g)\|_{H^{s,p}L^2} \lesssim \|f\|_{L^\infty}\|g\|_{H^{s,p}L^2}$. For the remaining high-high paraproduct, we split again:

$$\pi_{\mathrm{hh}}(f,g) = \Big(\sum_{j\in\mathbb{N}}\sum_{k=j-2}^{j+2} + \sum_{j<0}\sum_{k=j-2}^{j+2} \Big) \varphi(D/2^j)f\varphi(D/2^k)g = \pi_{\mathrm{hhh}}(f,g) + \pi_{\mathrm{hhl}}(f,g). \tag{5.28}$$

We now use the ball Littlewood-Paley estimate of Theorem 3.12 to handle π_{hhh}:

$$\begin{aligned} \|\pi_{\mathrm{hhh}}(f,g)\|_{H^{s,p}L^2} &\lesssim \sum_{|c|\leqslant 2} \left\| \Big(\sum_{j=0}^{\infty} 4^{sj} \|\varphi(D/2^j)f\ \varphi(D/2^{j+c})g\|_{L^2}^2 \Big)^{1/2} \right\|_{L^p} \\ &\lesssim \|f\|_{H^{s,p}L^2}\|g\|_{L^\infty}, \end{aligned} \tag{5.29}$$

where in the final inequality we have pulled g out in $L^\infty(U;V)$ and estimated f in $H^{s,p}(\mathbb{R}^d; L^2(I;\mathbb{F}))$ as above. To handle π_{hhl} we first note that it is band limited; indeed, it is supported in $B(0,16)$ by construction. Hence the L^pL^2 norm controls $H^{s,p}L^2$:

$$\|\pi_{\mathrm{hhl}}(f,g)\|_{H^{s,p}L^2} \lesssim \|\pi_{\mathrm{hhl}}(f,g)\|_{L^pL^2} \leqslant \sum_{|c|\leqslant 2} \left\| \sum_{j<0} \varphi(D/2^j)f\varphi(D/2^{j+c})g \right\|_{L^pL^2}. \tag{5.30}$$

For the final term above, we then use Cauchy-Schwartz, Hölder, and Theorem 3.10:

$$
\begin{aligned}
&\Big\| \sum_{j<0} \varphi(D/2^j) f \varphi(D/2^{j+c}) g \Big\|_{L^p L^2} \\
&\qquad\lesssim \Big\| \Big(\sum_{j<0} \|\varphi(D/2^j) f\|_{H^1}^2 \Big)^{1/2} \Big(\sum_{j<0} \|\varphi(D/2^{j+c}) g\|_{L^2}^2 \Big)^{1/2} \Big\|_{L^p} \\
&\lesssim \Big\| \Big(\sum_{j<0} \|\varphi(D/2^j)\Phi(D/2^3) f\|_{H^1}^2 \Big)^{1/2} \Big\|_{L^{2p}} \Big\| \Big(\sum_{j<0} \|\varphi(D/2^j)\Phi(D/2^{3+c}) g\|_{L^2}^2 \Big)^{1/2} \Big\|_{L^{2p}} \\
&\qquad\qquad \lesssim \|\Phi(D/2^3) f\|_{L^{2p} H^1} \|\Phi(D/2^{3+c}) g\|_{L^{2p} L^2}. \qquad (5.31)
\end{aligned}
$$

Now we invoke Young's inequality and the fact that $\Phi(D/2^3)f$ and $\Phi(D/2^{3+c})g$ are given via (tangential) convolution with a Schwartz function to bound

$$
\|\Phi(D/2^3) f\|_{L^{2p} H^1} \lesssim \|f\|_{L^p H^1} \text{ and } \|\Phi(D/2^{3+c}) g\|_{L^{2p} L^2} \lesssim \|g\|_{L^p L^2}. \qquad (5.32)
$$

Upon synthesizing these estimates, we arrive at

$$
\|\pi_{\mathrm{hhl}}(f,g)\|_{H^{s,p} L^2} \lesssim \|f\|_{L^p H^1} \|g\|_{L^p L^2}. \qquad (5.33)
$$

Since $\mathbb{N} \ni s \geqslant 1$, we see that this completes the proof. □

As a consequence of the previous result and a supercritical embedding, we find sufficient conditions under which the mixed-type Sobolev spaces are an algebra.

Proposition 5.8 *(Supercritical Sobolev embeddings in mixed-type Sobolev spaces). The following hold.*

(1) $H^s_{p,2}(U;\mathbb{F}) \hookrightarrow C^k_0(U;\mathbb{F})$ *for* $s > k + (d+1)/\min\{2,p\}$, $k \in \mathbb{N}$.
(2) For $s > (d+1)/\min\{2,p\}$, *the mixed type Sobolev* $H^s_{p,2}(U;\mathbb{F})$ *space is an algebra.*

Proof. For the first item we note that the restriction $H^s_{p,2}(U;\mathbb{F}) \to W^{s,\min\{2,p\}}(\mathbb{R}^d \times J;\mathbb{F})$ is continuous, for every $J \subset I$ of finite length and hence the claim follows from standard Sobolev embeddings. The second item follows from the first and Lemma 5.7. □

We now rapidly record three useful consequences of the previous results.

Proposition 5.9 *(Products, I). Suppose that* $1 < p < \infty$ *and* $\mathbb{N} \ni s > (d+1)/\min\{2,p\}$. *Then the pointwise product map*

$$
H^s_{p,2}(U) \times H^s_{p,2}(U) \times W^{s,\infty}(U) \ni (f,g,h) \mapsto f \cdot (g+h) \in H^s_{p,2}(U) \qquad (5.34)
$$

is well-defined and smooth.

Proof. That $H^s_{p,2}(U) \times W^{s,\infty}(U) \ni (f,h) \mapsto fh \in H^s_{p,2}(U)$ is well-defined and smooth is clear from bilinearity and the Leibniz rule. On the other hand, that these properties as also true for the assignment $H^s_{p,2}(U) \times H^s_{p,2}(U) \ni (f,g) \mapsto fg \in H^s_{p,2}(U)$ is a consequence of Lemma 5.7 and Proposition 5.8. □

Proposition 5.10 *(Products, II). Suppose that $1 < p < \infty$ and $\mathbb{N} \ni s > (d+1)/\min\{2,p\}$. Then the Banach sum space*

$$(H^s_{p,2} + W^{s,\infty})(U) = \{f \in L^1_{\mathrm{loc}}(U) \;:\; f = f_0 + f_1,\; f_0 \in H^s_{p,2}(U),\; f_1 \in W^{s,\infty}(U)\}, \tag{5.35}$$

equipped with the norm

$$\|f\|_{H^s_{2,p}+W^{s,\infty}} = \inf\{\|f_0\|_{H^s_{2,p}} + \|f_1\|_{W^{s,\infty}} \;:\; f = f_0 + f_1\}, \tag{5.36}$$

is a Banach algebra under pointwise multiplication.

Proof. That this space is Banach is straightforward to see, so we only prove that it is an algebra. Let $f, g \in (H^s_{p,2} + W^{s,\infty})(U)$ and decompose $f = f_0 + f_1$, $g = g_0 + g_1$ with $f_0, g_0 \in H^s_{p,2}(U)$ and $f_1, g_1 \in W^{s,\infty}(U)$. Then we use that $W^{s,\infty}(U)$ is an algebra along with Proposition 5.9 to estimate

$$\begin{aligned} \|fg\|_{H^s_{p,2}+W^{s,\infty}} &\leqslant \|f_0g_0 + f_1g_0 + f_0g_1\|_{H^s_{p,2}} + \|f_1g_1\|_{W^{s,\infty}} \\ &\lesssim (\|f_0\|_{H^s_{p,2}} + \|f_1\|_{W^{s,\infty}})(\|g_0\|_{H^s_{p,2}} + \|g_1\|_{W^{s,\infty}}). \end{aligned} \tag{5.37}$$

The result follows by taking the infimum over all decompositions of f and g. □

Remark 5.11 *(Products, III). For $1 < p < \infty$ and $\mathbb{N} \ni s > (d+1)/\min\{2,p\}$, we have that the pointwise product map*

$$H^s_{p,2}(U) \times (H^s_{p,2} + W^{s,\infty})(U) \ni (f,g) \mapsto fg \in H^s_{p,2}(U) \tag{5.38}$$

is smooth. This follows directly from Proposition 5.9 and definitions (5.35) and (5.36).

Our next result is meant to handle the reciprocal Jacobian of the flattening map.

Corollary 5.12 *(Smoothness of pointwise inversion). For $1 < p < \infty$ and $\mathbb{N} \ni s > (d+1)/\min\{2,p\}$, we have that there exists a constant $\rho \in \mathbb{R}^+$, depending on U, s, d, and p, such that the map*

$$(H^s_{p,2} + W^{s,\infty})(U) \supset B(0,\rho) \ni f \mapsto (1+f)^{-1} \in (H^s_{p,2} + W^{s,\infty})(U) \tag{5.39}$$

is well-defined and smooth.

Proof. Proposition 5.10 established that $(H^s_{p,2} + W^{s,\infty})(U)$ is a Banach algebra. Consequently, we can use the usual theory of power series on Banach algebras to pick ρ sufficiently small such that the power series

$$B(0,\rho) \ni f \mapsto \sum_{j=0}^{\infty} (-1)^j f^j \tag{5.40}$$

is uniformly absolutely convergent and defines an analytic map. It is then elementary to verify that this power series is pointwise equal to $f \mapsto (1+f)^{-1}$. □

Now we study the composition appearing in the interaction between the flattening map and the data. The following is a modification of the main argument presented in Inci, Kappeler, and Topalov [52].

Proposition 5.13 *(On composition). Assume that $2 \leqslant d \in \mathbb{N}$, and let $1 < p < \infty$ and $\mathbb{N} \ni s > 1 + d/\min\{2,p\}$. There exists a constant $\lambda \in \mathbb{R}^+$, depending only on p, d, and s such that the following hold.*

(1) If $f \in B_{(H^s_{p,2}+W^{s,\infty})(\mathbb{R}^d)}(0,\lambda)$, then the map $\mathrm{id}_{\mathbb{R}^d} + fe_d$ is a C^m diffeomorphism of $\mathbb{R}^d$ onto itself for every $\mathbb{N} \ni m < s - d/\min\{2,p\}$.
(2) If $k \in \mathbb{N}$, then the map Λ defined by

$$H^{s+k}_{p,2}(\mathbb{R}^d) \times B_{(H^s_{p,2}+W^{s,\infty})(\mathbb{R}^d)}(0,\lambda) \ni (F,f) \mapsto \Lambda(F,f) = F(\mathrm{id}_{\mathbb{R}^d} + fe_d) \in H^s_{p,2}(\mathbb{R}^d) \tag{5.41}$$

is well-defined and C^k.

Proof. Thanks to Corollary 5.12, there exists a $\lambda \in \mathbb{R}^+$ for which the map

$$B_{(H^s_{p,2}+W^{s,\infty})(\mathbb{R}^d)}(0,\lambda) \ni f \mapsto K_f = (1+\partial_d f)^{-1} \in (H^{s-1}_{p,2} + W^{s-1})(\mathbb{R}^d) \tag{5.42}$$

is analytic, and $K_f > 0$ in $\mathbb{R}^d$ for each f in the domain of the map. In turn, from the above and the fact that $s - 1 > d/\min\{2,p\}$, we find that the map $\mathfrak{F}_f : \mathbb{R}^d \to \mathbb{R}^d$ given by $\mathfrak{F}_f = \mathrm{id}_{\mathbb{R}^d} + fe_d$ is a C^1 diffeomorphism; indeed the condition that $1 + \partial_d f$ is bounded and bounded away from zero guarantees that for every $x \in \mathbb{R}^d$ the map $\mathbb{R} \ni y \overset{\psi_{f,x}}{\mapsto} y + f(x,y) \in \mathbb{R}$ is a diffeomorphism $\mathbb{R} \to \mathbb{R}$. Thus $\mathfrak{F}_f^{-1}(x,y) = (x, \psi^{-1}_{f,x}(y))$, for $(x,y) \in \mathbb{R}^d \times \mathbb{R}$.

That the maps $\mathfrak{F}_f$ and $\mathfrak{F}_f^{-1}$ are class C^m for every $\mathbb{N} \ni m < s - d/\min\{2,p\}$ follows from Proposition 5.8 and the inverse function theorem. This completes the proof of the first item.

With $\lambda \in \mathbb{R}^+$ in hand, we now turn to the proof of the second item. First, we prove (5.41) in the case $k = 0$ via an induction argument. For $j \in \{0,1,\ldots,s\}$ let $\mathbb{P}_j$ denote the proposition that

$$\Lambda : H^j_{p,2}(\mathbb{R}^d) \times B_{(H^s_{p,2}+W^{s,\infty})(\mathbb{R}^d)}(0,\lambda) \to H^j_{p,2}(\mathbb{R}^d) \tag{5.43}$$

is well-defined and continuous.

For $\mathbb{P}_0$, we perform a change of variables $z = y + f(x,y)$ on each fiber to estimate

$$\|\Lambda(F,f)\|_{H^0_{p,2}} = \Big(\int\limits_{\mathbb{R}^{d-1}} \Big(\int\limits_{\mathbb{R}} |F(x, y + f(x,y))|^2 \,\mathrm{d}y \Big)^{p/2} \,\mathrm{d}x \Big)^{1/p} \leqslant \|K_f\|^{1/2}_{L^\infty} \|F\|_{H^0_{p,2}}. \tag{5.44}$$

Thus well-definedness is established. For continuity, we estimate

$$\|\Lambda(F,f) - \Lambda(G,g)\|_{H^0_{p,2}} \leqslant \|\Lambda(F-G,f)\|_{H^0_{p,2}} + \|\Lambda(G,f) - \Lambda(G,g)\|_{H^0_{p,2}}. \tag{5.45}$$

The former term is made small when (G,g) are close to (F,f) via estimate (5.44), while the latter term is made small by approximating G via a smooth compactly supported function and using that $g \to f$ in $(H^s_{p,2} + W^{s,\infty})(\mathbb{R}^d)$ implies that $\mathfrak{F}_g \to \mathfrak{F}_f$ uniformly. This gives $\mathbb{P}_0$.

Now suppose that $j \in \{0, \dots, s-1\}$ is such that $\mathbb{P}_\ell$ is true for all $\ell \in \{1, \dots, j\}$. We compute

$$\partial_q(\Lambda(F,f)) = \Lambda(\partial_q F, f) + \Lambda(\partial_d F, f)\partial_q f, \tag{5.46}$$

for $q \in \{1, \dots, d\}$. By combining Remark 5.11 and the induction hypothesis, we have that

$$H^{j+1}_{p,2}(\mathbb{R}^d) \times B_{(H^s_{p,2}+W^{s,\infty})(\mathbb{R}^d)}(0,\lambda) \ni (F,f) \mapsto \partial_q(\Lambda(F,f)) \in H^j_{p,2}(\mathbb{R}^d) \tag{5.47}$$

is a well-defined and continuous map. This paired with $\mathbb{P}_0$ gives $\mathbb{P}_{j+1}$. Thus the induction is complete, and we have proved (5.41) in the case $k = 0$.

We now turn to the proof of (5.41) for the remaining cases of $k \in \mathbb{N}^+$. For this we shall use the converse to Taylor's theorem: see, for instance, Theorem 2.4.15 and Supplement 2.4B in Abraham, Marsden, and Ratiu [11]. For $r \in \{1, \dots, k\}$, we define

$$\begin{aligned} \Lambda^{(r)} : H^{s+k}_{p,2}(\mathbb{R}^d) \times B_{(H^s_{p,2}+W^{s,\infty})(\mathbb{R}^d)}(0,\lambda) \to \mathcal{L}^r_{\mathrm{sym}}(&H^s_{p,2}(\mathbb{R}^d) \\ &\times (H^s_{p,2} + W^{s,\infty})(\mathbb{R}^d); H^s_{p,2}(\mathbb{R}^d)) \end{aligned} \tag{5.48}$$

via the formula

$$\Lambda^{(r)}(F,f)[(F_1,f_1), \dots, (F_r,f_r)] = \Lambda(\partial^r_d F, f) \prod_{\ell=1}^{r} f_\ell + \sum_{m=1}^{r} \Lambda(\partial^{r-1}_d F_m, f) \prod_{\ell \neq m} f_\ell. \tag{5.49}$$

In the above $\mathcal{L}^r_{\mathrm{sym}}$ refers to the space of symmetric r-multilinear maps. By using the already established continuity properties of Λ along with Remark 5.11, we see that $\Lambda^{(r)}$ is continuous for $r \in \{1, \dots, k\}$. By similar considerations, we find that the map

$$\mathcal{R}_k : \mathcal{U} \to \mathcal{L}^k_{\mathrm{sym}}(H^s_{p,2}(\mathbb{R}^d) \times (H^s_{p,2} + W^{s,\infty})(\mathbb{R}^d); H^s_{p,2}(\mathbb{R}^d)) \tag{5.50}$$

defined via

$$\mathcal{R}_k((F,f),(G,g)) = \int_0^1 \frac{(1-t)^{k-1}}{(k-1)!}(\Lambda^{(k)}(F+tG, f+tg) - \Lambda^{(k)}(F,f))\,\mathrm{d}t \tag{5.51}$$

is continuous, where

$$\begin{aligned}\mathcal{U} = \{((F,f),(G,g)) \in (H^{s+k}_{p,2} \times B(0,\lambda))^2 \ : \ \forall\, t \in [0,1]\ (F+tG, f+tg) \\ \in H^{s+k}_{p,2} \times B(0,\lambda)\}.\end{aligned} \tag{5.52}$$

Let $((F,f),(G,g)) \in \mathcal{U}$. Now, according to Taylor's theorem with integral remainder (used pointwise), we have that

$$\Lambda(F, f+g) - \Lambda(F,f) = \sum_{r=1}^k \frac{1}{r!}\Lambda(\partial^r_d F, f) g^r + \int_0^1 \frac{(1-t)^{k-1}}{(k-1)!}(\Lambda(\partial^k_d F, f+tg) - \Lambda(\partial^k_d F, f)) g^k \,\mathrm{d}t. \tag{5.53}$$

We can also apply the same result to express for $h(s) = s\Lambda(G, f+sg)$

$$\begin{aligned}\Lambda(G, f+g) = h(1) - h(0) &= \sum_{r=1}^k \frac{1}{r!}(\partial^r h)(0) + \int_0^1 \frac{(1-t)^{k-1}}{(k-1)!}((\partial^k h)(t) - (\partial^k h)(0))\,\mathrm{d}t \\ &= \sum_{r=1}^k \frac{1}{(r-1)!}\Lambda(\partial^{r-1}_d G, f) g^{r-1} \\ &+ \int_0^1 \frac{(1-t)^{k-1}}{(k-1)!}(t\Lambda(\partial^k_d G, f+tg) g^k + k(\Lambda(\partial^{k-1}_d G, f+tg) - \Lambda(\partial^{k-1}_d G, f)) g^{k-1})\,\mathrm{d}t.\end{aligned} \tag{5.54}$$

Adding (5.53) and (5.54) and using definitions (5.49) and (5.51) yields

$$\Lambda(F+G, f+g) - \Lambda(F,f) = \sum_{r=1}^k \frac{1}{r!}\Lambda^{(r)}(F,f)[(G,g)^{\otimes r}] + \mathcal{R}_k((F,f),(G,g))[(G,g)^{\otimes k}]. \tag{5.55}$$

Therefore, by the converse to Taylor's theorem, we deduce that Λ is C^k on its domain. □

5.3. Subcritical gradient spaces

We now turn our attention to subcritical gradient spaces.

Definition 5.14 *(Subcritical gradient spaces). For $1 < p < d$, $\mathbb{R} \ni s \geqslant 1$, and $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$ we define the space*

$$\widetilde{H}^{s,p}(\mathbb{R}^d; \mathbb{F}) = \{f \in L^{dp/(d-p)}(\mathbb{R}^d; \mathbb{F}) \ : \ \nabla f \in H^{s-1,p}(\mathbb{R}^d; \mathbb{F}^d)\} \tag{5.56}$$

and equip it with a norm $\|f\|_{\widetilde{H}^{s,p}} = \|\nabla f\|_{H^{s-1,p}}$. When $\mathbb{F} = \mathbb{R}$ we will typically abbreviate $\widetilde{H}^{s,p}(\mathbb{R}^d) = \widetilde{H}^{s,p}(\mathbb{R}^d; \mathbb{R})$.

Next, we verify that these spaces are complete.

Proposition 5.15 *(Properties of subcritical gradient spaces). Let $\mathbb{R} \ni s \geqslant 1$, $1 < p < d$, and $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$. Then the space $\widetilde{H}^{s,p}(\mathbb{R}^d; \mathbb{F})$ defined in (5.56) is Banach, and we have the bound*

$$\|f\|_{H^{s-1,dp/(d-p)}} \lesssim \|f\|_{\widetilde{H}^{s,p}} \text{ for all } f \in \widetilde{H}^{s,p}(\mathbb{R}^d; \mathbb{F}). \tag{5.57}$$

Moreover, $\mathscr{S}(\mathbb{R}^d; \mathbb{F}) \subset \widetilde{H}^{s,p}(\mathbb{R}^d; \mathbb{F})$ is dense.

Proof. It suffices to prove the result when $\mathbb{F} = \mathbb{R}$, so we will assume this. We first consider the case $s = 1$ and note that in this case $\widetilde{H}^{1,p}(\mathbb{R}^d) = \dot{W}^{1,p}(\mathbb{R}^d) \cap L^{dp/(d-p)}(\mathbb{R}^d)$ for the homogeneous Sobolev space $\dot{W}^{1,p}(\mathbb{R}^d) = \{f \in L^1_{\text{loc}}(\mathbb{R}^d) \ : \ \nabla f \in L^p(\mathbb{R}^d; \mathbb{R}^d)\}$. Next, we note that a density result of Hajłasz and Kałamajska [46] shows that if $f \in \dot{W}^{1,p}(\mathbb{R}^d)$, then there exists a sequence $\{f_n\}_{n \in \mathbb{N}} \subset C_c^\infty(\mathbb{R}^d)$ such that $\|\nabla f_n - \nabla f\|_{L^p} \to 0$ as $n \to \infty$. Additionally, Theorem 12.9 in Leoni [57], which is crucially based on this density assertion, provides a constant $C_p > 0$ such that for each $f \in \dot{W}^{1,p}(\mathbb{R}^d)$ there exists a unique constant $c_f \in \mathbb{R}$ such that $\|f - c_f\|_{L^{dp/(d-p)}} \leqslant C_p \|\nabla f\|_{L^p}$. Now, if $f \in \widetilde{H}^{1,p}(\mathbb{R}^d)$ then upon applying this bound to f and noting that $f \in L^{dp/(d-p)}(\mathbb{R}^d)$, we deduce that $c_f = 0$. Hence, (5.57) holds when $s = 1$, and with this bound in hand it is a routine matter to verify that $\widetilde{H}^{1,p}(\mathbb{R}^d)$ is complete. The above density assertion shows that $\mathscr{S}(\mathbb{R}^n; \mathbb{R})$ is dense in $\widetilde{H}^{1,p}(\mathbb{R}^d)$. To complete the proof for general $s \in \mathbb{N}^+$ we simply note that the map $\langle D \rangle^{s-1} : \widetilde{H}^{s,p}(\mathbb{R}^d) \to \widetilde{H}^{1,p}(\mathbb{R}^d)$ is an isometric isomorphism that maps $\mathscr{S}(\mathbb{R}^n; \mathbb{R})$ to itself. □

Next we record a frequency splitting result.

Lemma 5.16 *(Frequency splitting in gradient spaces). Let $\mathbb{R} \ni s \geqslant 1$, $1 < p < d$, and $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$. Suppose that $\varphi \in C_c^\infty(\mathbb{R}^d)$ is an even function that satisfies $\varphi = 1$ on $B(0,1)$ and $\operatorname{supp}(\varphi) \subseteq B(0,2)$. Then $(1-\varphi)(D) : \widetilde{H}^{s,p}(\mathbb{R}^d; \mathbb{F}) \to H^{s,p}(\mathbb{R}^d; \mathbb{F})$ is a bounded linear map, and $\varphi(D) : \widetilde{H}^{s,p}(\mathbb{R}^d; \mathbb{F}) \to W^{k,dp/(d-p)}(\mathbb{R}^d; \mathbb{F})$ is a bounded linear map for every $k \in \mathbb{N}$.*

Proof. The proof is straightforward and we only sketch it. The mapping properties for $\varphi(D)$ follow directly from the embedding $\widetilde{H}^{s,p}(\mathbb{R}^d; \mathbb{F}) \hookrightarrow L^{dp/(d-p)}(\mathbb{R}^d; \mathbb{F})$ and that

$\varphi(D)$ is convolution with a band-limited Schwartz function. The mapping properties of $(1-\varphi(D))$ are verified via the Littlewood-Paley characterizations from Theorems 3.10 and (3.11). The evenness of φ guarantees that real-valued maps stay real-valued when either $\varphi(D)$ or $(1-\varphi)(D)$ is applied. $\square$

By using the lifting map of Proposition 5.5, we can build a useful extension operator.

Proposition 5.17 *(Extension operator). Let $s \in \mathbb{N}^+$, $1 < p < d$, and $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$. For $b \in \mathbb{R}^+$ write $\Omega = \mathbb{R}^d \times (0,b)$ and $\Sigma = \mathbb{R}^d \times \{b\}$. There exists a bounded linear operator $\mathcal{E}_0 : \widetilde{H}^{3/2+s,p}(\Sigma;\mathbb{F}) \to H^{2+s}_{p,2}(\Omega;\mathbb{F})$ and a linear operator $\mathcal{E}_1 : \widetilde{H}^{3/2+s,p}(\Sigma;\mathbb{F}) \to W^{k,\infty}(\Omega;\mathbb{F})$ that is bounded for every $k \in \mathbb{N}$ such that the linear extension operator $\mathcal{E} = \mathcal{E}_0 + \mathcal{E}_1$ satisfies*

$$\mathrm{Tr}_{\Sigma_0}\mathcal{E}\eta = 0 \text{ and } \mathrm{Tr}_{\Sigma}\mathcal{E}\eta = \eta \text{ for all } \eta \in \widetilde{H}^{3/2+s,p}(\Sigma;\mathbb{F}). \tag{5.58}$$

Proof. Let $\varphi \in C^\infty_c(\mathbb{R}^d)$ be as in Lemma 5.16. We define $\mathcal{E}_0$ via

$$(\mathcal{E}_0\eta)(x,y) = \phi(b-y)(\mathrm{L}_\Omega(1-\varphi)(D)\eta)(x,y) \text{ for } (x,y) \in \mathbb{R}^d \times (0,b), \tag{5.59}$$

where L_Ω is the lifting operator from Proposition 5.5 and $\phi \in C^\infty_c(\mathbb{R})$ satisfies $\phi(0) = 1$ and $\mathrm{supp}(\phi) \subseteq (-b/2, b/2)$. We also define $\mathcal{E}_1$ via

$$(\mathcal{E}_1\eta)(x,y) = (y/b)(\varphi(D)\eta)(x) \text{ for } (x,y) \in \mathbb{R}^d \times (0,b). \tag{5.60}$$

The claimed mapping properties then follow immediately from Lemma 5.16 and Proposition 5.5. $\square$

6. Linear analysis in mixed-type Sobolev spaces

In this penultimate section, we conclude our linear theory for systems (2.2) and (2.1). In Section 6.1, we synthesize our multiplier analysis of Section 4 with our generalization of the Mikhlin-Hörmander, Theorem 3.14, and produce a linear well-posedness result for (2.2) posed in the mixed-type Sobolev spaces of Section 5.1. In Section 6.2 we then return to the main linear equations, system (2.1), and port over the extended linear theory for (2.2). This lands us a linear well-posedness theory for (2.1) that employs both the mixed-type Sobolev spaces of Section 5.1 and the subcritical gradient spaces of Section 5.3. Armed with this result, we will then be ready to turn to the nonlinear analysis in Section 7.

6.1. Existence and uniqueness

We begin by setting some notation for the spaces in which we wish to extend our existence theory. Note that the mixed-type Sobolev spaces, which were introduced in Section 5.1, are used here.

Definition 6.1 *(Mixed-type function spaces, I). For $s \in \mathbb{N}$ and $1 < r < \infty$, we define the function spaces*

$$\mathbb{X}_{s,r} = H^{1+s}_{r,2}(\Omega;\mathbb{C}) \times H^{2+s}_{r,2}(\Omega;\mathbb{C}^3) \times H^{3/2+s,r}(\Sigma;\mathbb{C}^2), \tag{6.1}$$

$$\mathbb{Y}_{s,r} = H^{s}_{r,2}(\Omega;\mathbb{C}^3) \times H^{1/2+s,r}(\Sigma;\mathbb{C}^3) \times H^{5/2+s,r}(\Sigma;\mathbb{C}^2). \tag{6.2}$$

We are now ready to state and prove our main existence result of this subsection. We remark that $\mathbb{Y}_s \cap \mathbb{Y}_{s,r}$ is dense in $\mathbb{Y}_{s,r}$. It is in this sense we use the word 'extension' in what follows.

Theorem 6.2 *(Extension to mixed type spaces). For $s \in \mathbb{N}$ and $1 < r < \infty$ the linear map $\Psi : \mathbb{Y}_s \to \mathbb{X}_s$ from Definition 4.1 has a unique bounded extension*

$$\Psi : \mathbb{Y}_{s,r} \to \mathbb{X}_{s,r}. \tag{6.3}$$

Moreover, the above extension retains the property that if $(p, u, \chi) = \Psi(f, k, H)$, then system (2.2) is solved by the former tuple with data $(0, f, k, \nabla_{\|} \cdot H, 0)$.

Proof. Ψ is given by the Fourier multiplier $\mathbf{m}$ from Definition 4.6, and $\mathbf{m}$ satisfies Mikhlin-Hörmander bounds on its derivatives thanks to estimate (4.48) from Theorem 4.13 and the definition of $[\![\cdot]\!]_s$ from Definition 4.2. We then apply our Mikhlin-Hörmander variant, Theorem 3.14 (with $\mu = 0$), to each of the component maps $\mathbf{m}_{jk}$ for $j, k \in \{1, 2, 3\}$ to deduce the existence of the stated bounded extension. It is straightforward to check by density that these extensions of Ψ are still solution operators to (2.2). $\square$

Our linear analysis for the system (2.2) is synthesized with the following theorem.

Theorem 6.3 *(Well-posedness of the linearization in mixed-type Sobolev spaces, I). Let $s \in \mathbb{N}$ and $1 < r \leqslant 2$. For every $(f, k, H) \in \mathbb{Y}_{s,r}$, there exists a unique $(p, u, \chi) \in \mathbb{X}_{s,r}$ such that (2.2) is satisfied in the strong sense with data $(0, f, k, \nabla_{\|} \cdot H, 0)$.*

Proof. Existence follows from Theorem 6.2, so it remains to prove uniqueness. So suppose that $(p, u, \chi) \in \mathbb{X}_{s,r}$ solve (2.2) with trivial data. Let $\varphi \in C^\infty_c(\mathbb{R}^2)$ be such that $\varphi = 1$ in $B(0, 1)$. Then for every N we have the inclusion $\varphi(D/N)(p, u, \chi) \in \mathbb{X}_s$ thanks to Young's inequality and the fact that $\varphi(D/N)$ is a tangential convolution with a Schwartz function; moreover, the triple $\varphi(D/N)(p, u, \chi)$ remains a solution to (2.2) with trivial data. We may thus invoke Theorem 2.6 to deduce that $\varphi(D/N)(p, u, \chi) = 0$. As this holds for every $N \in \mathbb{N}$, we necessarily have that $(p, u, \chi) = 0$. Uniqueness is proved. $\square$

6.2. Reformulated well-posedness

We now aim to make the transition from system (2.2) back to the original linearization (2.1). The previous subsection gave us the well-posedness of the former system in the mixed-type Sobolev spaces. The goal of this subsection is to port these result to the latter, and specialize to $\mathbb{R}$-valued functions. Note that the following definition implements the notions of subcritical gradient spaces, which were introduced in Section 5.3.

Definition 6.4 *(Mixed-type function spaces, II). For $s \in \mathbb{N}$ and $1 < r < 2$, we define the Banach spaces*

$$\mathbf{X}_{s,r} = H^{1+s}_{r,2}(\Omega;\mathbb{R}) \times H^{2+s}_{r,2}(\Omega;\mathbb{R}^3) \times \widetilde{H}^{5/2+s,r}(\Sigma;\mathbb{R}), \tag{6.4}$$

$$\mathbf{Y}_{s,r} = H^{s}_{r,2}(\Omega;\mathbb{R}^3) \times H^{1/2+s,r}(\Sigma;\mathbb{R}^3) \times (H^{3/2+s,r} \cap \dot{H}^{-1,r})(\Sigma;\mathbb{R}). \tag{6.5}$$

Note that the space $\dot{H}^{-1,r}(\Sigma;\mathbb{R})$ is defined in (1.28).

Provided that $s \in \mathbb{N}$ is sufficiently large relative to $r \in (1,2)$, the spaces $\mathbf{X}_{s,r}$ enjoy classical regularity.

Proposition 6.5. *Let $1 < r < 2$ and $\mathbb{N} \ni s > 3/r - 1$. Then*

$$\mathbf{X}_{s,r} \hookrightarrow C^k_0(\Omega) \times C^{1+k}_0(\Omega;\mathbb{R}^3) \times C^{2+k}_0(\Sigma) \tag{6.6}$$

for $k = s - \lfloor 3/r \rfloor \in \mathbb{N}$.

Proof. The embedding of the first two factors follows from the first item of Proposition 5.8. For the third factor, the embedding follows from Lemma 5.16, the standard embedding of Bessel-Sobolev spaces, and the observation that $\lfloor 3/r \rfloor \geqslant 1 + \lfloor 2/r - 1/2 \rfloor$ for $1 < r < 2$. □

We can now state our well-posedness result.

Theorem 6.6 *(Well-posedness of the linearization in mixed-type Sobolev spaces, II). For every $s \in \mathbb{N}$, $1 < r < 2$, and $(f,k,h) \in \mathbf{Y}_{s,r}$ there exists a unique $(p,u,\eta) \in \mathbf{X}_{s,r}$ such that system (2.1) is solved with data $(0,f,k,h)$.*

Proof. We begin by proving uniqueness. Suppose that $(p,u,\eta) \in \mathbf{X}_{s,r}$ solve system (2.1) with trivial data. Then we set $\chi = \nabla_{\|}\eta$ and observe that $(p,u,\chi) \in \mathbb{X}_{s,r}$ solves (2.2) with trivial data. Then Theorem 6.3 applies, and we learn that $(p,u,\chi) = 0$ and hence η is constant. However, $\eta \in L^{2r/(2-r)}(\Sigma;\mathbb{R})$, so $\eta = 0$. This completes the proof of uniqueness.

We now prove existence. Suppose that $(f,k,h) \in \mathbf{Y}_{s,r}$. Using Mikhlin-Hörmander Theorem 3.8, we may define $H = \nabla_{\|}\Delta_{\|}^{-1} h \in H^{5/2+s,r}(\Sigma;\mathbb{R}^2)$, which obeys the estimate $\|H\|_{H^{5/2+s,r}} \lesssim \|h\|_{\dot{H}^{-1,r} \cap H^{3/2+s,r}}$ as well as the identity $\nabla_{\|} \cdot H = h$. We may then use Theorem 6.3 to acquire $(p,u,\chi) = \Psi(f,k,H) \in \mathbb{X}_{s,r}$. Next, we define $\widetilde{\eta} = |\nabla_{\|}|\Delta_{\|}^{-1}\nabla_{\|} \cdot \chi$ and note that $\widetilde{\eta} \in H^{3/2+s,r}(\Sigma;\mathbb{C})$ thanks to another application of Mikhlin-Hörmander.

In turn, this allows us to employ the Hardy-Littlewood-Sobolev inequality (see, for instance, Theorem 1 in Section 1.2 in Chapter V of Stein [90]) to set $\eta = |\nabla_{\|}|^{-1}\widetilde{\eta} \in L^{2r/(2-r)}(\Sigma;\mathbb{C})$. Since $\nabla_{\|}^{\perp} \cdot \chi = 0$, we then have that $\nabla_{\|}\eta = \chi \in H^{3/2+s,r}(\Sigma;\mathbb{C}^2)$, and so $\eta \in \widetilde{H}^{5/2+s,r}(\Sigma;\mathbb{C})$.

At this point we have established that (p, u, η) is a solution to (2.1) with the correct regularity and integrability properties, but the construction we have used does not guarantee a priori that the solution is real-valued. To see that this is actually the case, we note that since (f, k, H) have vanishing imaginary part, we can take the imaginary part of the equations and use the fact that its coefficients are all real to deduce that $(\mathrm{Im}p, \mathrm{Im}u, \mathrm{Im}\chi) = \Psi(0, 0, 0)$. Thus, Theorem 6.3's uniqueness assertion shows that $(\mathrm{Im}p, \mathrm{Im}u, \mathrm{Im}\chi) = 0$, and the existence proof is complete. □

7. Nonlinear analysis

In this section we complete the proof of our main result, Theorem 1, by synthesizing our previous analysis and appealing to the implicit function theorem. Section 7.1 sets up the nonlinear framework, and then our main results are proved in Section 7.2.

7.1. Operators and mapping properties

The goal of this subsection is to define a nonlinear operator associated with system (1.13) and study its mapping properties. We begin by studying the flattening map $\eta \mapsto \mathfrak{F}_\eta$, which we recall is defined in (1.10).

Proposition 7.1 *(Properties of the flattening map). For $1 < r < 2$ and $\mathbb{N} \ni 1 + s > 3/r$, there exists $\varrho \in \mathbb{R}^+$ such that the following hold.*

(1) For $\eta \in B(0,\varrho) \subset \widetilde{H}^{3/2+s,r}(\Sigma)$ the flattening map $\mathfrak{F}_\eta = \mathrm{id}_{\mathbb{R}^3} + \mathcal{E}\eta e_3$ is a smooth diffeomorphism from Ω to $\Omega[\eta]$ that extends to a C^n diffeomorphism from $\overline{\Omega}$ to $\overline{\Omega[\eta]}$ for $\mathbb{N} \ni n < s + 2 - 3/r$.

(2) Let V be a finite dimensional real normed space. For $\ell, m \in \mathbb{N}$ with $m \leqslant 2 + s$ the map

$$H^{\ell+m}_{r,2}(\mathbb{R}^3;V) \times B_{\widetilde{H}^{3/2+s,r}(\Sigma)}(0,\varrho) \ni (F,\eta) \mapsto F \circ \mathfrak{F}_\eta \in H^m_{r,2}(\Omega;V) \tag{7.1}$$

is well-defined and C^ℓ.

Proof. Let $\mathfrak{E}_\Omega$ denote the extension operator granted by Proposition 5.3, and consider the extended flattening map $\overline{\mathfrak{F}}_\eta : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $\overline{\mathfrak{F}}_\eta = \mathrm{id}_{\mathbb{R}^3} + \mathfrak{E}_\Omega\mathcal{E}\eta e_3$. By the mapping properties of $\mathfrak{E}_\Omega$ and $\mathcal{E}$ from Propositions 5.3 and 5.17, we find that the map E defined by

$$\widetilde{H}^{3/2+s,r}(\Sigma) \ni \eta \mapsto E\eta = \mathfrak{E}_\Omega \mathcal{E}\eta \in (H^{2+s}_{r,2} + W^{2+s,\infty})(\mathbb{R}^3) \tag{7.2}$$

is bounded and linear. As such, there exists $\varrho \in \mathbb{R}^+$ such that $E(B(0,\varrho)) \subseteq B(0,\lambda)$, where $\lambda \in \mathbb{R}^+$ is the constant from Proposition 5.13. We may then invoke the conclusions of this proposition to deduce the mapping properties asserted in (7.1). This completes the proof of the second item.

To prove the first item we only need to make a trio of observations. First, Proposition 5.13 shows that $\overline{\mathfrak{F}}_\eta$ is a C^n diffeomorphism from $\mathbb{R}^3$ to itself when $\mathbb{N} \ni n < s+2-3/r$. Second, $\mathfrak{F}_\eta$ is the restriction of $\overline{\mathfrak{F}}_\eta$ to $\overline{\Omega}$, and by construction $\mathfrak{F}_\eta$ maps $\overline{\Omega}$ to $\overline{\Omega[\eta]}$. Third, the construction of $\mathcal{E}\eta$ shows that its restriction to Ω itself is smooth. Together, these prove the first item. □

Next, we analyze quantities derived from the flattening map. Recall that J_η, $\mathcal{A}_\eta$, and M_η are defined in (1.11) and (1.12).

Proposition 7.2 *(Properties of the Jacobian and geometry matrices). Let $1 < r < 2$, $\mathbb{N} \ni 1+s > 3/r$, and $\varrho \in \mathbb{R}^+$ (depending on s and r) be as in Proposition 7.1. Then the following mapping properties hold.*

(1) For $\eta \in \widetilde{H}^{3/2+s,r}(\Sigma)$, we have that $J_\eta > 0$ and both of the maps

$$\widetilde{H}^{3/2+s,r}(\Sigma) \supset B(0,\varrho) \ni \eta \mapsto J_\eta, 1/J_\eta \in (H^{1+s}_{r,2} + W^{1+s,\infty})(\Omega) \tag{7.3}$$

are smooth.

(2) The maps

$$\widetilde{H}^{3/2+s,r}(\Sigma) \supset B(0,\varrho) \ni \eta \mapsto \mathcal{A}_\eta, \mathcal{A}_\eta^{-1} \in (H^{1+s}_{r,2} + W^{1+s,\infty})(\Omega;\mathbb{R}^{3\times 3}) \tag{7.4}$$

are smooth.

(3) The maps

$$\widetilde{H}^{3/2+s,r}(\Sigma) \supset B(0,\varrho) \ni \eta \mapsto M_\eta, M_\eta^{-1} \in (H^{1+s}_{r,2} + W^{1+s,\infty})(\Omega;\mathbb{R}^{3\times 3}) \tag{7.5}$$

are smooth.

Proof. The maps $\eta \mapsto J_\eta$ and $\eta \mapsto M_\eta$ are affine, and thus smooth thanks to Proposition 5.17. By invoking Corollary 5.12, we have that $\eta \mapsto 1/J_\eta$ is smooth. This fact combined with Proposition 5.10 implies that $\eta \mapsto \mathcal{A}_\eta = M_\eta^{\mathrm{t}}/J_\eta$ is smooth. For the smoothness of the pointwise inversion in the third item, we appeal to Proposition 5.10 again and the adjugate formula $\eta \mapsto M_\eta^{-1} = \mathrm{adj}(M_\eta)/J_\eta^2$. The remaining assertion, the smoothness of pointwise inversion in the second item is then handled via the formula $\eta \mapsto \mathcal{A}_\eta^{-1} = J_\eta M_\eta^{-\mathrm{t}}$. □

Now, working towards a synthesis, we make the following definitions. First we define some spaces.

Definition 7.3 *(Spaces for the nonlinear analysis). We make the following definitions for $s \in \mathbb{N}$, $1 < r < 2$, and $\rho \in \mathbb{R}^+$:*

(1) ${}_\diamond H^{2+s}_{r,2}(\Omega;\mathbb{R}^3) = \{u \in H^{2+s}_{r,2}(\Omega;\mathbb{R}^3) \;:\; \nabla\cdot u = 0,\ \mathrm{Tr}_{\Sigma_0} u = 0\}$,
(2) ${}_\diamond\mathbf{X}_{s,r} = H^{1+s}_{r,2}(\Omega) \times {}_\diamond H^{2+s}_{r,2}(\Omega;\mathbb{R}^3) \times \widetilde{H}^{5/2+s,r}(\Sigma)$,
(3) $\mathcal{O}_{s,r}(\rho) = \{(p,u,\eta) \in {}_\diamond\mathbf{X}_{s,r} \;:\; \eta \in B(0,\rho)\}$,
(4) $\mathbf{W}_{s,r} = H^{1+s}_{r,2}(\mathbb{R}^3;\mathbb{R}^{3\times 3}) \times H^s_{r,2}(\mathbb{R}^3;\mathbb{R}^3)$.

Next, we define some maps.

Definition 7.4 *(Maps for the nonlinear analysis, I). For $1 < r < 2$, $\mathbb{N} \ni s > 3/r$, and $\varrho \in \mathbb{R}^+$ as in Proposition 7.1 we make the following definitions.*

(1) $\Xi_1 : \mathcal{O}_{s,r}(\varrho) \times \mathbb{R} \times \mathbb{R}^+ \times \mathbb{R}^+ \to H^s_{r,2}(\Omega;\mathbb{R}^3)$ *is defined via*

$$\begin{aligned}\Xi_1(p,u,\eta,\gamma,\mathfrak{g},\mu) = {}& M_\eta^{-\mathrm{t}}((u - \gamma M_\eta e_1)\cdot\nabla(M_\eta^{-1}u)) + \nabla(p + \mathfrak{g}\eta) \\ & - \mu M_\eta^{-\mathrm{t}}(\nabla\cdot((\mathbb{D}_{\mathcal{A}_\eta}(M_\eta^{-1}u))M_\eta^{\mathrm{t}})).\end{aligned} \tag{7.6}$$

(2) $\Xi_2 : \mathcal{O}_{s,r}(\varrho) \times \mathbb{R}^+ \times \mathbb{R}^+ \to H^{1/2+s,r}(\Sigma;\mathbb{R}^3)$ *is defined via*

$$\Xi_2(p,u,\eta,\mu,\kappa) = \mathrm{Tr}_\Sigma[-(pI - \mu\mathbb{D}_{\mathcal{A}_\eta}(M_\eta^{-1}u))M_\eta^{\mathrm{t}}e_3 - \kappa\mathscr{H}(\eta)M_\eta^{\mathrm{t}}e_3]. \tag{7.7}$$

(3) For $m \in \mathbb{N}$ and ϱ as in Proposition 7.1, $\Upsilon_1 : H^{m+s}_{r,2}(\Omega;\mathbb{R}^3) \times B_{\widetilde{H}^{3/2+s,r}}(0,\varrho) \to H^s_{r,2}(\Omega;\mathbb{R}^3)$ is defined via

$$\Upsilon_1(\mathcal{F},\eta) = -J_\eta M_\eta^{-\mathrm{t}}(\mathcal{F}\circ\mathfrak{F}_\eta). \tag{7.8}$$

(4) For $m \in \mathbb{N}$ and ϱ as in Proposition 7.1, $\Upsilon_2 : H^{m+1+s}_{r,2}(\Omega;\mathbb{R}^{3\times 3}) \times B_{\widetilde{H}^{3/2+s,r}}(0,\varrho) \to H^{1/2+s,r}(\Sigma;\mathbb{R}^3)$ is defined via

$$\Upsilon_2(\mathcal{T},\eta) = -\mathrm{Tr}_\Sigma[(\mathcal{T}\circ\mathfrak{F}_\eta)M_\eta^{\mathrm{t}}e_3]. \tag{7.9}$$

Our next two results study the smoothness of the nonlinear differential operators in the momentum equation and dynamic boundary condition in system (1.13).

Proposition 7.5 *(Mapping properties of the nonlinearities). For $1 < r < 2$ and $\mathbb{N} \ni s > 3/r$, the following mapping properties hold.*

(1) Ξ_1 and Ξ_2, as defined in the first and second items of Definition 7.4, are smooth.

(2) Υ_1 and Υ_2, as defined in the third and fourth items of Definition 7.4 are C^m.

Proof. The first item follows from Propositions 7.2 and 5.9 along with Remark 5.11, since all of the nonlinearities in Ξ_1 are various products of the derivatives of the velocity field with the geometry matrices and parameters.

The analysis of Ξ_2 follows similarly, with the exception of the mean curvature term $\mathscr{H}(\eta)$. For this we use that $\mathbb{R}^2 \ni x \mapsto x\langle x\rangle^{-1} \in \mathbb{R}^2$ is everywhere analytic and vanishing at zero and hence the map

$$H^{3/2+s,r}(\Sigma)^2 \ni (\partial_1\eta, \partial_2\eta) \mapsto \langle \nabla_{\|}\eta\rangle^{-1}(\partial_1\eta, \partial_2\eta) \in H^{3/2+s,r}(\Sigma)^2 \tag{7.10}$$

is smooth since $H^{3/2+s,r}(\Sigma)$ is an algebra. In turn, we find that the map $\eta \mapsto \mathscr{H}(\eta)$ is also smooth. This completes the proof of the first item.

The second item follow by similar considerations, supplemented with the second item of Proposition 7.1. □

The remainder of this subsection's nonlinear analysis is meant to deal with the technicalities arising in the slowly traveling limit ($\gamma \to 0$) in system (1.13). As the equations are currently formulated, there is a change of natural function spaces that occurs in this limit, which suggests that the stationary problem is a (low-mode) singular limit of traveling problems. The effect of this is that the formulation (1.13) works fine for $\gamma = 0$ and $\gamma \in \mathbb{R} \setminus \{0\}$ separately, but is ill-suited for capturing the slowly traveling limit $\gamma \to 0$. To overcome this issue, we make a change of unknowns in the free surface.

Definition 7.6 *(Anisotropic parameterization operators). For $\gamma \in \mathbb{R}$ we let P_γ be the Fourier multiplication operator with the following symbol*

$$P_\gamma = \mathbf{p}_\gamma(D), \quad \mathbf{p}_\gamma(\xi) = \frac{4\pi^2|\xi|^2}{4\pi^2|\xi|^2 + 2\pi\mathrm{i}\gamma\xi_1}. \tag{7.11}$$

Note that P_0 is the identity operator.

The relevant properties of the maps P_γ are enumerated in the following result. Recall that the spaces $\dot{H}^{-1,r}(\mathbb{R}^2;\mathbb{R})$ are defined in (1.28).

Proposition 7.7 *(Properties of the anisotropic parameterization operators). The following hold for $s \in \mathbb{N}^+$ and $1 < r < 2$.*

(1) For each $\gamma \in \mathbb{R}$ we have that $P_\gamma \in \mathcal{L}(\widetilde{H}^{s,r}(\mathbb{R}^2;\mathbb{R}))$. Moreover, the map $\mathbb{R} \ni \gamma \mapsto P_\gamma \in \mathcal{L}(\widetilde{H}^{s,r}(\mathbb{R}^2;\mathbb{R}))$ is bounded, and for any $\eta \in \widetilde{H}^{s,r}(\mathbb{R}^2;\mathbb{R})$ the map $\mathbb{R} \ni \gamma \mapsto P_\gamma\eta \in \widetilde{H}^{s,r}(\mathbb{R}^2;\mathbb{R})$ is continuous.

(2) For each $\gamma \in \mathbb{R}$ we have that $\gamma\partial_1 P_\gamma \in \mathcal{L}(\widetilde{H}^{s,r}(\mathbb{R}^2;\mathbb{R}); \dot{H}^{-1,r}(\mathbb{R}^2;\mathbb{R}))$. Moreover, the map $\mathbb{R} \ni \gamma \mapsto \gamma\partial_1 P_\gamma \in \mathcal{L}(\widetilde{H}^{s,r}(\mathbb{R}^2;\mathbb{R}))$ is bounded, and for any $\eta \in \widetilde{H}^{s,r}(\mathbb{R}^2;\mathbb{R})$ the map $\mathbb{R} \ni \gamma \mapsto \gamma\partial_1 P_\gamma\eta \in \dot{H}^{-1,r}(\mathbb{R}^2;\mathbb{R})$ is continuous.

(3) The mappings

$$\mathbb{R} \times \widetilde{H}^{s,r}(\mathbb{R}^2;\mathbb{R}) \ni (\gamma, \eta) \mapsto \begin{cases} P_\gamma \eta \in \widetilde{H}^{s,r}(\mathbb{R}^2;\mathbb{R}), \\ \gamma \partial_1 P_\gamma \eta \in \dot{H}^{-1,r}(\mathbb{R}^2;\mathbb{R}), \\ \gamma \mathcal{R}_1 P_\gamma \eta \in H^{s,r}(\mathbb{R}^2;\mathbb{R}) \end{cases} \tag{7.12}$$

are continuous, where $\mathcal{R}_1$ refers to the Riesz transform in the e_1 direction.

Proof. The third item follows from the first two, so we turn our attention to proving these.

We claim that the multiplier $\mathbf{p}_\gamma(\xi) = \frac{4\pi^2|\xi|^2}{4\pi^2|\xi|^2 + 2\pi \mathrm{i}\gamma\xi_1}$ is of Marcinkiewicz type (see Theorem 3.7) and the defining inequalities (3.18) are satisfied uniformly over $\gamma \in \mathbb{R}$. To prove the claim, we first note that

$$|\mathbf{p}_\gamma(\xi)| \leqslant 1 \tag{7.13}$$

and that $\mathbf{p}_\gamma$ is smooth away from the coordinate axes. Next, we compute

$$\partial_1 \mathbf{p}_\gamma(\xi) = \frac{2\pi \mathrm{i}\gamma(\xi_1^2 - \xi_2^2)}{(2\pi|\xi|^2 + \mathrm{i}\gamma\xi_1)^2}, \quad \partial_2 \mathbf{p}_\gamma(\xi) = \frac{4\pi \mathrm{i}\gamma\xi_1\xi_2}{(2\pi|\xi|^2 + \mathrm{i}\gamma\xi_1)^2}, \tag{7.14}$$

and

$$\partial_1\partial_2 \mathbf{p}_\gamma(\xi) = -\frac{4\pi \mathrm{i}\gamma\xi_2(6\pi\xi_1^2 - 2\pi\xi_2^2 + \mathrm{i}\gamma\xi_1)}{(2\pi|\xi|^2 + \mathrm{i}\gamma\xi_1)^3}. \tag{7.15}$$

Thus, we have the following estimates from Cauchy's inequality:

$$|\xi_1\partial_1 \mathbf{p}_\gamma(\xi)| \leqslant \frac{|\gamma\xi_1| 2\pi|\xi|^2}{4\pi^2|\xi|^4 + \gamma^2|\xi_1|^2} \leqslant \frac{1}{2}, \quad |\xi_2\partial_2 \mathbf{p}_\gamma(\xi)| \leqslant \frac{4\pi|\gamma\xi_1||\xi|^2}{4\pi^2|\xi|^4 + \gamma^2|\xi_1|^2} \leqslant 1, \tag{7.16}$$

and

$$|\xi_1\xi_2\partial_1\partial_2 \mathbf{p}_\gamma(\xi)| \leqslant \frac{12\pi|\gamma\xi_1||\xi|^2|2\pi|\xi|^2 + \mathrm{i}\gamma\xi_1|}{(4\pi^2|\xi|^4 + \gamma^2|\xi_1|^2)^{3/2}} \leqslant \frac{12\pi|\gamma\xi_1||\xi|^2}{4\pi^2|\xi|^4 + \gamma^2|\xi_1|^2} \leqslant 3. \tag{7.17}$$

This completes the proof of the claim, and so we may invoke Theorem 3.7 (with the observation that $\mathbf{p}_\gamma(-\xi) = \overline{\mathbf{p}_\gamma(\xi)}$ for $\xi \in \mathbb{R}^2 \setminus \{0\}$) to see that

$$\|P_\gamma\|_{\mathcal{L}(L^r(\mathbb{R}^2))} \leqslant C_r(3+1) = 4C_r \tag{7.18}$$

for a constant C_r depending only on r.

Armed with (7.18), we are now ready to prove the first item. If $\eta \in \widetilde{H}^{s,r}(\mathbb{R}^2;\mathbb{R})$, then we have

$$\sup_{\gamma\in\mathbb{R}}\|P_\gamma\eta\|_{\widetilde{H}^{s,r}} = \sup_{\gamma\in\mathbb{R}}\|\mathbf{p}_\gamma(D)\langle D\rangle^{s-1}\nabla_\|\eta\|_{L^r} \lesssim \|\langle D\rangle^{s-1}\nabla_\|\eta\|_{L^r} \lesssim \|\eta\|_{\widetilde{H}^{s,r}}. \tag{7.19}$$

Next, we prove that $\gamma \mapsto P_\gamma$ is continuous for the strong operator topology. Fix $\eta \in \widetilde{H}^{s,r}(\mathbb{R}^2;\mathbb{R}^2)$ and $\varepsilon \in \mathbb{R}^+$. Thanks to density of $H^{s,r}(\mathbb{R}^2;\mathbb{R}) \hookrightarrow \widetilde{H}^{s,r}(\mathbb{R}^2;\mathbb{R})$ (see Proposition 5.15), there exists $\zeta \in H^{s,r}(\mathbb{R}^2;\mathbb{R})$ such that $\|\zeta-\eta\|_{\widetilde{H}^{s,r}} \leqslant \varepsilon$. Thus, we learn from (7.18) that if $\gamma,\gamma_0 \in \mathbb{R}$ then

$$\|(P_\gamma - P_{\gamma_0})\eta\|_{\widetilde{H}^{s,r}} \lesssim \varepsilon + \|(P_\gamma - P_{\gamma_0})\zeta\|_{\widetilde{H}^{s,r}}. \tag{7.20}$$

We compute via the fundamental theorem of calculus that

$$\mathbf{p}_\gamma(\xi) - \mathbf{p}_{\gamma_0}(\xi) = (\gamma-\gamma_0)\frac{\mathrm{i}\xi_1}{2\pi|\xi|^2}\mathbf{q}_{\gamma,\gamma_0}(\xi) \text{ for } \mathbf{q}_{\gamma,\gamma_0}(\xi) = \int_0^1 (\mathbf{p}_{t\gamma+(1-t)\gamma_0}(\xi))^2\,\mathrm{d}t. \tag{7.21}$$

Estimates (7.13), (7.16), and (7.17) and the Leibniz rule show that

$$\sup_{\gamma,\gamma_0}\left(|\mathbf{q}_{\gamma,\gamma_0}(\xi)| + |\xi_1\partial_1\mathbf{q}_{\gamma,\gamma_0}(\xi)| + |\xi_2\partial_2\mathbf{q}_{\gamma,\gamma_0}(\xi)| + |\xi_1\xi_2\partial_1\partial_2\mathbf{q}_{\gamma,\gamma_0}(\xi)|\right) \lesssim 1, \tag{7.22}$$

and we also have that $\mathbf{q}_{\gamma,\gamma_0}(-\xi) = \overline{\mathbf{q}_{\gamma,\gamma_0}(\xi)}$, so we may once more appeal to Theorem 3.7 to learn that

$$\sup_{\gamma,\gamma_0}\|\mathbf{q}_{\gamma,\gamma_0}(D)\|_{\mathcal{L}(L^p(\mathbb{R}^2))} \lesssim 1. \tag{7.23}$$

Writing $\mathcal{R} = (\mathcal{R}_1,\mathcal{R}_2)$ for the vector of Riesz transforms, we then have that

$$\begin{aligned}\|(P_\gamma - P_{\gamma_0})\zeta\|_{\widetilde{H}^{s,r}} = \|\nabla(P_\gamma - P_{\gamma_0})\zeta\|_{H^{s-1,r}} = \frac{|\gamma-\gamma_0|}{2\pi}\|\mathbf{q}_{\gamma,\gamma_0}(D)\mathcal{R}\mathcal{R}_1\langle D\rangle^{s-1}\zeta\|_{L^r} \\ \lesssim |\gamma-\gamma_0|\|\langle D\rangle^{s-1}\zeta\|_{L^r} = |\gamma-\gamma_0|\|\zeta\|_{H^{s-1,r}}. \end{aligned}\tag{7.24}$$

By combining (7.20) and (7.24), we get

$$\limsup_{\gamma_0\to\gamma}\|(P_\gamma - P_{\gamma_0})\eta\|_{\widetilde{H}^{s,r}} \lesssim \varepsilon, \tag{7.25}$$

so the continuity claim follows. This completes the proof of the first item.

The second item is proved by similar considerations thanks to the identity

$$\gamma\mathcal{R}_1 P_\gamma = (1-P_\gamma)2\pi|D| = (P_\gamma - 1)\mathcal{R}\cdot\nabla, \tag{7.26}$$

which shows that for $\eta \in \widetilde{H}^{s,r}(\mathbb{R}^2;\mathbb{R})$

$$\sup_\gamma[\gamma\partial_1 P_\gamma\eta]_{\dot{H}^{-1,r}} = \sup_\gamma\|\gamma\mathcal{R}_1 P_\gamma\eta\|_{L^r} = \sup_\gamma\|(P_\gamma-1)\mathcal{R}\cdot\nabla\eta\|_{L^r} \lesssim \|\eta\|_{\widetilde{H}^{s,r}}, \tag{7.27}$$

where the last inequality follows from the (7.18) and the L^r boundedness of Riesz transforms. For continuity, we again fix $\eta \in \widetilde{H}^{s,r}(\mathbb{R}^2;\mathbb{R})$, $\gamma_0, \gamma \in \mathbb{R}$ and then use (7.26) to deduce that

$$[(\gamma\partial_1 P_\gamma - \gamma_0\partial_1 P_{\gamma_0})\eta]_{\dot{H}^{-1,r}} \lesssim \|(P_{\gamma_0} - P_\gamma)\mathcal{R}\cdot\nabla\eta\|_{L^r} \lesssim \|(P_{\gamma_0} - P_\gamma)\eta\|_{\widetilde{H}^{s,r}}, \tag{7.28}$$

which means that the continuity assertion of the second item follows from that of the first. This completes the proof of the second item. □

The operators of Definition 7.6 permit us to make a change of unknowns in (1.13) to overcome the aforementioned singular limit issue. We consider $\eta = P_\gamma\eta$, and view η as our new unknown. The main theorem of this section's nonlinear analysis, which implements this change of unknowns crucially, is now given as follows.

Theorem 7.8 *(Mapping properties). For $1 < r < 2$, $\mathbb{N} \ni s > 3/r$, and $\varrho \in \mathbb{R}^+$ as in Proposition 7.1, there exists $C \in \mathbb{R}^+$ such that the map $\Xi : \mathcal{O}_{s,r}(\varrho/C) \times \mathbb{R} \times (\mathbb{R}^+)^3 \times \mathbf{W}_{1+s,r} \to \mathbf{Y}_{s,r}$ given by*

$$\begin{aligned}&\Xi(p, u, \eta, \gamma, \mathfrak{g}, \mu, \kappa, \mathcal{T}, \mathcal{F})\\ &= (\Xi_1(p, u, P_\gamma\eta, \gamma, \mathfrak{g}, \mu)+\Upsilon_1(\mathcal{F}, P_\gamma\eta), \Xi_2(p, u, P_\gamma\eta, \mu, \kappa)+\Upsilon_2(\mathcal{T}, P_\gamma\eta), \mathrm{Tr}_\Sigma u\cdot e_3+\gamma\partial_1 P_\gamma\eta)\end{aligned} \tag{7.29}$$

is well-defined and continuous. Moreover, the Fréchet derivative with respect to the first factor, $D_1\Xi : \mathcal{O}_{s,r}(\varrho/C)\times\mathbb{R}\times(\mathbb{R}^+)^3\times\mathbf{W}_{1+s,r} \to \mathcal{L}({}_\diamond\mathbf{X}_{s,r};\mathbf{Y}_{s,r})$, exists and is continuous.

Proof. The uniform boundedness assertions in the first item of Proposition 7.7 guarantee that for some $C \in \mathbb{R}^+$ we have $(p, u, P_\gamma\eta) \in \mathcal{O}_{s,r}(\varrho)$ for all $(p, u, \eta) \in \mathcal{O}_{s,r}(\varrho/C)$. Thus, by composition, linearity of P_γ, and the third item of Proposition 7.7 we may invoke is Proposition 7.5 to reach the desired conclusions for the first and second components of the map Ξ. The third component of Ξ is handled via the linearity of $\gamma\partial_1 P_\gamma$, together with the second and third items of Proposition 7.7 and the divergence compatibility estimate of Proposition 5.6. □

7.2. Well-posedness

We are now ready to prove our main theorem. This subsection is split into four main results and then a list of corollaries, which combine to prove Theorem 1. In the first main result, we invoke the implicit function theorem at a fixed tuple of positive physical parameters and obtain a solution map. In the next, we show that we can glue these together across all parameter values. One slight issue that remains after this is done is that the resulting solution map loses a derivative relative to what one would expect. This fact stems from the numerology of higher order smoothness of composition-type nonlinearities (see e.g. Proposition 5.13). We are thus lead to the third result in this subsection, in which we show a posteriori that the solution map actually obeys the optimal derivative counting. In the fourth, and final, main result of this subsection, we recast the previous results into a more physically relevant formulation by anisotropically parameterizing the free surface variable with the operators in Proposition 7.7.

We shall use the following version of the inverse function theorem, as formulated as in Theorem A in Crandall and Rabinowitz [32] (for a verbose proof see Theorem 2.7.2 in Nirenberg [64], but note that there is slight misstatement of the uniqueness assertion in the first item there that is correct in [32]).

Theorem 7.9 *(Implicit function theorem). Let X, Y, Z be Banach spaces and f a continuous mapping of an open set $U \subset X \times Y \to Z$. Assume that f has a Fréchet derivative with respect to the first factor, $D_1 f : U \to \mathcal{L}(X; Z)$ that is continuous. Suppose that $(x_0, y_0) \in U$ and $f(x_0, y_0) = 0$. If $D_1 f(x_0, y_0)$ is an isomorphism of X onto Z, then there exist balls $B(y_0, r_Y) \subset Y$ and $B(x_0, r_X) \subset X$ such that $B(x_0, r_X) \times B(y_0, r_Y) \subset U$ and a continuous unique function $u : B(y_0, r_Y) \to B(x_0, r_X)$ such that $u(y_0) = x_0$ and $f(u(y), y) = 0$ for all $y \in B(y_0, r_Y)$. Moreover, the implicit function u is continuous.*

We now apply Theorem 7.9 in our first well-posedness result.

Theorem 7.10 *(Well-posedness, I). Let $1 < r < 2$, $\mathbb{N} \ni s > 3/r$, and $\varrho, C \in \mathbb{R}^+$ be as in Theorem 7.8. For each $\mathbf{v} = (\overline{\mathfrak{g}}, \overline{\mu}, \overline{\kappa}) \in (\mathbb{R}^+)^3$ there exists $\rho_{s,\mathbf{v}}, \rho'_{s,\mathbf{v}} \in \mathbb{R}^+$ and a unique mapping*

$$\iota_{\mathbf{v}} : B((0, \mathbf{v}, 0), \rho_{s,\mathbf{v}}) \subset \mathbb{R} \times (\mathbb{R}^+)^3 \times \mathbf{W}_{1+s,r} \times \mathbf{Y}_{s,r} \to B(0, \rho'_{s,\mathbf{v}}) \subset \mathcal{O}_{s,r}(\varrho/C) \tag{7.30}$$

with the property that for all data $(\gamma, \mathfrak{g}, \mu, \kappa, \mathcal{T}, \mathcal{F}, f, k, h) \in B((0, \mathbf{v}, 0), \rho_{s,\mathbf{v}})$ we have that $(p, u, \eta) = \iota_{\mathbf{v}}(\gamma, \mathfrak{g}, \mu, \kappa, \mathcal{T}, \mathcal{F}, f, k, h) \in \mathcal{O}_{s,r}(\varrho/C)$ is a solution to

$$\begin{cases} M^{-t}_{P_\gamma \eta}((u - \gamma M_{P_\gamma \eta} e_1) \cdot \nabla(M^{-1}_{P_\gamma \eta} u)) + \nabla(p + \mathfrak{g} P_\gamma \eta) \\ \quad - \mu M^{-t}_{P_\gamma \eta}(\nabla \cdot ((\mathbb{D}_{\mathcal{A}_{P_\gamma \eta}}(M^{-1}_{P_\gamma \eta} u)) M^{t}_{P_\gamma \eta})) = f + J_{P_\gamma \eta} M^{-t}_{P_\gamma \eta} \mathcal{F} \circ \mathfrak{F}_{P_\gamma \eta} & \text{in } \Omega, \\ \nabla \cdot u = 0 & \text{in } \Omega, \\ -(p - \mu \mathbb{D}_{\mathcal{A}_{P_\gamma \eta}}(M^{-1}_{P_\gamma \eta} u)) M^{t}_{P_\gamma \eta} e_3 - \kappa \mathscr{H}(P_\gamma \eta) M^{t}_{P_\gamma \eta} e_3 = k + \mathcal{T} \circ \mathfrak{F}_{P_\gamma \eta} M^{t}_{P_\gamma \eta} e_3 & \text{on } \Sigma, \\ u \cdot e_3 + \gamma \partial_1 P_\gamma \eta = h & \text{on } \Sigma, \\ u = 0 & \text{on } \Sigma_0. \end{cases} \tag{7.31}$$

Moreover, $\iota_{\mathbf{v}}$ in (7.30) is continuous.

Proof. Consider the map $\overline{\Xi} : \mathcal{O}_{s,r}(\varrho/C) \times \mathbb{R} \times (\mathbb{R}^+)^3 \times \mathbf{W}_{1+s,r} \times \mathbf{Y}_{s,r} \to \mathbf{Y}_{s,r}$ defined via

$$\overline{\Xi}(p, u, \eta, \gamma, \mathfrak{g}, \mu, \kappa, \mathcal{T}, \mathcal{F}, f, k, h) = \Xi(p, u, \eta, \gamma, \mathfrak{g}, \mu, \kappa, \mathcal{T}, \mathcal{F}) - (f, k, h). \tag{7.32}$$

Thanks to Theorem 7.8, this map is well-defined, continuous, and the Fréchet derivative with respect to the first factor, $D_1\overline{\Xi} : \mathcal{O}_{s,r}(\varrho/C) \times \mathbb{R} \times (\mathbb{R}^+)^3 \times \mathbf{W}_{1+s,r} \times \mathbf{Y}_{s,r} \to \mathcal{L}({}_\diamond\mathbf{X}_{s,r}; \mathbf{Y}_{s,r})$, exists and is continuous. Moreover, we have that $\overline{\Xi}(0, \mathbf{v}, 0) = 0$. Theorem 6.6 then shows that $D_1\overline{\Xi}(0, \mathbf{v}, 0)$ is an isomorphism from ${}_\diamond\mathbf{X}_{s,r}$ to $\mathbf{Y}_{s,r}$. We may then invoke the version of the implicit function theorem given in Theorem 7.9 to obtain parameters $\rho_{s,\mathbf{v}}, \rho'_{s,\mathbf{v}} \in \mathbb{R}^+$ along with the map $\iota_{\mathbf{v}}$. □

We can recast the previous theorem into the following more general statement via a gluing argument.

Theorem 7.11 *(Well-posedness, II). Let $1 < r < 2$, $\mathbb{N} \ni s > 3/r$. There exists and open set*

$$\{0\} \times (\mathbb{R}^+)^3 \times \{0\} \times \{0\} \subset \mathscr{U}_s \subset \mathbb{R} \times (\mathbb{R}^+)^3 \times \mathbf{W}_{1+s,r} \times \mathbf{Y}_{s,r} \tag{7.33}$$

and a continuous map

$$\iota : \mathscr{U}_s \to \bigcup_{\mathbf{v}\in(\mathbb{R}^+)^3} B(0, \rho'_{s,\mathbf{v}}) \subset \mathcal{O}_{s,r}(\varrho/C), \tag{7.34}$$

where the radii $\rho'_{s,\mathbf{v}} > 0$ are as in Theorem 7.10, with the property that for all $\mathbf{U} = (\gamma, \mathfrak{g}, \mu, \kappa, \mathcal{T}, \mathcal{F}, f, k, h) \in \mathscr{U}_s$ we have that $\iota(\mathbf{U}) = (p, u, \eta) \in B(0, \rho'_{s,(\mathfrak{g},\mu,\kappa)})$ is the unique solution to (7.31) with data $\mathbf{U}$.

Proof. We set

$$\mathscr{U}_s = \bigcup_{\mathbf{v}\in(\mathbb{R}^+)^3} B((0, \mathbf{v}, 0), \rho_{s,\mathbf{v}}), \tag{7.35}$$

where $\rho_{s,\mathbf{v}}$ are the radii granted by Theorem 7.10. We propose defining the map (7.34) via

$$\iota = \iota_{\mathbf{v}} \quad \text{on the set } B((0, \mathbf{v}, 0), \rho_{s,\mathbf{v}}) \text{ whenever } \mathbf{v} \in (\mathbb{R}^+)^3, \tag{7.36}$$

where the maps $\iota_{\mathbf{v}}$ are the solution operators granted by Theorem 7.10. This is well-defined since if $\mathbf{v}, \mathbf{w} \in (\mathbb{R}^+)^3$ are such that $B((0, \mathbf{v}, 0), \rho_{s,\mathbf{v}}) \cap B((0, \mathbf{w}, 0), \rho_{s,\mathbf{w}}) \neq \varnothing$, then the maps $\iota_{\mathbf{v}}$ and $\iota_{\mathbf{w}}$ agree on this intersection since, according to the aforementioned theorem, they are both the unique solution operators to the PDE (7.31). Once we know that ι is well-defined, continuity follows from the continuity of each $\iota_{\mathbf{v}}$. The remaining uniqueness assertions are just a restatement of those from Theorem 7.10. □

The next well-posedness result gains an extra derivative on the solution to reach the optimal counting.

Theorem 7.12 *(Well-posedness, III). Let $1 < r < 2$, $\mathbb{N} \ni s > 3/r + 1$. There exists an open set*

$$\{0\} \times (\mathbb{R}^+)^3 \times \{0\} \subset \mathscr{W}_s \subset \mathbb{R} \times (\mathbb{R}^+)^3 \times \mathbf{W}_{s,r} \tag{7.37}$$

such that $\mathscr{W}_s \times \{0\} \subset \mathscr{U}_{s-1}$, where $\mathscr{U}_{s-1}$ is as in Theorem 7.11, and a continuous map

$$\widetilde{\iota} : \mathscr{W}_s \to \mathbf{X}_{s,r} \cap \bigcup_{\mathbf{v} \in (\mathbb{R}^+)^3} B_{\mathbf{X}_{s-1,r}}(0, \rho'_{s-1,\mathbf{v}}) \subset \mathcal{O}_{s,r}(\varrho/C) \tag{7.38}$$

with the property that for all $\mathbf{W} = (\gamma, \mathfrak{g}, \mu, \kappa, \mathcal{T}, \mathcal{F}) \in \mathscr{W}_s$ we have that $\widetilde{\iota}(\mathbf{W}) = (p, u, \eta) \in \mathbf{X}_{s,r} \cap B_{\mathbf{X}_{s,r}}(0, \rho'_{s-1,(\mathfrak{g},\mu,\kappa)})$ is the unique solution to (7.31) with data $(\mathbf{W}, 0)$.

Proof. We set $\widetilde{\iota}(\gamma, \mathfrak{g}, \mu, \kappa, \mathcal{T}, \mathcal{F}) = \iota(\gamma, \mathfrak{g}, \mu, \kappa, \mathcal{T}, \mathcal{F}, 0, 0, 0)$, where ι is the solution operator granted by Theorem 7.11. A priori, we only know that

$$\widetilde{\iota} : \bigcup_{\mathbf{v} \in (\mathbb{R}^3)^+} B((0, \mathbf{v}, 0), \rho_{s,\mathbf{v}}) \subset \mathbb{R} \times (\mathbb{R}^+)^3 \times \mathbf{W}_{s,r} \to \bigcup_{\mathbf{v} \in (\mathbb{R}^+)^3} B_{\mathbf{X}_{s-1,r}}(0, \rho'_{s-1,\mathbf{v}}) \tag{7.39}$$

is a continuous mapping.

To complete the proof, we claim that by shrinking the domain a little bit, if necessary, we can gain an additional derivative on the solution. To see this let $0 < \sigma \leqslant 1$ and set

$$\mathscr{W}_s(\sigma) = \bigcup_{\mathbf{v} \in (\mathbb{R}^3)^+} B((0, \mathbf{v}, 0), \sigma \rho_{s,\mathbf{v}}) \subset \mathbb{R} \times (\mathbb{R}^+)^3 \times \mathbf{W}_{s,r}. \tag{7.40}$$

Denote $(p, u, \eta) = \widetilde{\iota}(\gamma, \mathfrak{g}, \mu, \kappa, \mathcal{T}, \mathcal{F})$. By the second item of Proposition 7.5, the continuity of the solution map in (7.39), and the continuity of composition we have that the map

$$\mathscr{W}_s(\sigma) \ni (\gamma, \mathfrak{g}, \mu, \kappa, \mathcal{T}, \mathcal{F}) \mapsto (\Upsilon_1(\mathcal{F}, P_\gamma \eta), \Upsilon_2(\mathcal{T}, P_\gamma \eta), 0) \in \mathbf{Y}_{s,r} \tag{7.41}$$

is continuous, vanishes whenever $\mathcal{T}$ and $\mathcal{F}$ are zero, and is independent of $(\mathfrak{g}, \mu, \kappa)$. Thus, by taking $0 < \sigma_\star \leqslant 1$ sufficiently small, we guarantee that

$$\mathscr{W}_s(\sigma_\star) \ni (\gamma, \mathfrak{g}, \mu, \kappa, \mathcal{T}, \mathcal{F}) \mapsto (\gamma, \mathfrak{g}, \mu, \kappa, 0, 0, \Upsilon_1(\mathcal{F}, P_\gamma \eta), \Upsilon_2(\mathcal{T}, P_\gamma \eta), 0) \in \mathscr{U}_s. \tag{7.42}$$

Since the map in (7.42) actually has its range in the domain of the map ι, which is given in (7.34), we have verified the following key identity:

$$(p, u, \eta) = \iota(\gamma, \mathfrak{g}, \mu, \kappa, 0, 0, \Upsilon_1(\mathcal{F}, P_\gamma \eta), \Upsilon_2(\mathcal{T}, P_\gamma \eta), 0). \tag{7.43}$$

Then the mapping properties of ι reveal that the solution

$$(p, u, \eta) \in B_{\mathbf{X}_{s,r}}(0, \rho'_{s,(\mathfrak{g},\mu,\kappa)}) \cap B_{\mathbf{X}_{s-1,r}}(0, \rho'_{s-1,(\mathfrak{g},\mu,\kappa)}) \tag{7.44}$$

varies continuously with the data $(\gamma, \mathfrak{g}, \mu, \kappa, \mathcal{T}, \mathcal{F}) \in \mathscr{W}_s = \mathscr{W}_s(\sigma_\star)$. This proves the claim. $\square$

This section's list of main theorems is concluded with the following.

Theorem 7.13 *(Well-posedness, IV). Let $1 < r < 2$, $\mathbb{N} \ni s > 3/r + 1$, and $\mathscr{W}_s$ be the open set from Theorem 7.12. There exists a map*

$$\mathscr{W}_s \ni (\gamma, \mathfrak{g}, \mu, \kappa, \mathcal{T}, \mathcal{F}) \mapsto (p, u, \eta) \in \mathcal{O}_{s,r}(\varrho) \tag{7.45}$$

such that the following hold.

(1) The map is a continuous solution operator to the nonlinear system (1.13).
(2) The map is locally unique in the sense that there exist open sets $W_s \subseteq \mathscr{W}_s$ and $\{V_s(\mathbf{v})\}_{\mathbf{v} \in (\mathbb{R}^+)^3} \subseteq \mathcal{O}_{s,r}(\varrho)$, obeying the non-degeneracy conditions of (1.14), for which the following two conditions hold.

(i) The image of W_s under (7.45) is contained within $\bigcup_{\mathbf{v} \in (\mathbb{R}^+)^3} V_s(\mathbf{v})$.
(ii) For each $\mathbf{v} \in (\mathbb{R}^+)^3$ the restriction of (7.45) to the preimage of $V_s(\mathbf{v})$, thought of as a mapping to $V_s(\mathbf{v})$, is the unique function that is a solution operator to (1.13).

(3) We have an extra 'anisotropic' estimate on the free surface in the sense that the composition map

$$\mathscr{W}_s \ni (\gamma, \mathfrak{g}, \mu, \kappa, \mathcal{T}, \mathcal{F}) \mapsto (p, u, \eta) \mapsto \gamma \mathcal{R}_1 \eta \in L^r(\Sigma) \tag{7.46}$$

is well-defined and continuous, where $\mathcal{R}_1$ is the Riesz transform in the e_1 direction.

Proof. We take (7.45) to be the composition $(\gamma, \mathfrak{g}, \mu, \kappa, \mathcal{T}, \mathcal{F}) \overset{\tilde{\iota}}{\mapsto} (p, u, \eta) \mapsto (p, u, \eta)$, with $\eta = P_\gamma \eta$. Thanks to Theorem 7.12 and Proposition 7.7, this is a continuous solution operator for (1.13). The final item of the aforementioned proposition also guarantees that the mapping of (7.46) is well-defined and continuous.

It remains to establish local uniqueness. Suppose that $(p, u, \eta), (p', u', \eta') \in \mathcal{O}_{s,p}(\varrho)$ are such that there exists $(\gamma, \mathfrak{g}, \mu, \kappa, \mathcal{T}, \mathcal{F}) \in \mathscr{W}_s$ and both (p, u, η) and (p', u', η') are solutions to (1.13) with the same data $(\mathcal{T}, \mathcal{F})$, same wave speed γ, and same physical constants $(\mathfrak{g}, \mu, \kappa)$. Integrating the divergence free constraint over $y \in (0, b)$ and appealing to the kinematic boundary condition and the no slip condition yields the identity

$$\gamma \partial_1 \zeta = (\nabla_\parallel, 0) \cdot \int_0^b w(\cdot, y) \, \mathrm{d}y \text{ for } (\zeta, w) \in \{(\eta, u), (\eta', u')\}. \tag{7.47}$$

Therefore, by (7.47) and the identity $P_\gamma^{-1}\nabla = \nabla + \mathcal{R}\gamma\mathcal{R}_1$, where $\mathcal{R} = (\mathcal{R}_1, \mathcal{R}_2)$ is the vector of Riesz transforms, we have the estimate

$$\begin{aligned} \|P_\gamma^{-1}\zeta\|_{\widetilde{H}^{5/2+s,r}} &\lesssim \|\zeta\|_{\widetilde{H}^{5/2+s,r}} + \left\| |D|^{-1}(\nabla_\|, 0) \cdot \int_0^b w(\cdot, y)\,\mathrm{d}y \right\|_{H^{3/2+s,r}} \\ &\lesssim \|w, \zeta\|_{H^{2+s}_{r,2} \times \widetilde{H}^{5/2+s,r}}, \end{aligned} \tag{7.48}$$

where in both of the inequalities we have applied the boundedness of Riesz transforms. If (p, u, η) and (p', u', η') are sufficiently small, we therefore guarantee that $(p, u, P_\gamma^{-1}\eta), (p', u', P_\gamma^{-1}\eta') \in \mathbf{X}_{s,r} \cap B_{\mathbf{X}_{s-1,r}}(0, \rho'_{s-1,(\mathfrak{g},\mu,\kappa)})$, but we can then invoke the uniqueness assertion of Theorem 7.12 to find that $(p, u, P_\gamma^{-1}\eta) = (p', u', P_\gamma^{-1}\eta')$, which implies $(p, u, \eta) = (p', u', \eta')$.

Thus, we take $V_s(\mathfrak{g}, \mu, \kappa)$ to be an open ball about the origin of a positive radius that obeys the above smallness requirements. The set W_s is then defined to be the union of the preimages of these balls under the map (7.45). □

We now enumerate some important consequences.

Corollary 7.14 *(Some further conclusions). For $1 < r < 2$ and $\mathbb{N} \ni s > 3/r + 1$ the following hold.*

(1) ***Classical solutions:*** *Each triple (p, u, η) produced by Theorem 7.13 is a classical solution to system (1.13). More precisely, whenever $(\gamma, \mathfrak{g}, \mu, \kappa, \mathcal{T}, \mathcal{F}) \in \mathscr{W}_s$ we have that the associated solution satisfies*

$$(p, u, \eta) \in C_0^{2+k}(\Omega) \times C_0^{3+k}(\Omega; \mathbb{R}^3) \times C_0^{4+k}(\Sigma), \tag{7.49}$$

for $k = s - 2 - \lfloor 3/r \rfloor \in \mathbb{N}$.

(2) ***Eulerian transfer:*** *Each solution (p, u, η) to system (1.13), produced by Theorem 7.13 gives rise to a corresponding classical solution*

$$(q, v, \eta) \in C_0^{2+k}(\Omega[\eta]) \times C_0^{3+k}(\Omega[\eta]) \times C_0^{4+k}(\Omega[\eta]), \quad k = s - 2 - \lfloor 3/r \rfloor \tag{7.50}$$

to the stationary-traveling Eulerian formulation of the problem given by system (1.8) via unflattening.

(3) ***Fixed physical parameters, variable wave speed:*** *For each $(\mathfrak{g}, \mu, \kappa) \in (\mathbb{R}^+)^3$, there exists an open set $(0, 0, 0) \in W_s(\mathfrak{g}, \mu, \kappa) \subset \mathbb{R} \times \mathbf{W}_{s,r}$ and a unique function*

$$W_s(\mathfrak{g}, \mu, \kappa) \ni (\gamma, \mathcal{T}, \mathcal{F}) \mapsto (p, u, \eta) \in V_s(\mathfrak{g}, \mu, \kappa) \tag{7.51}$$

with the property that for all $(\gamma, \mathcal{T}, \mathcal{F})$ belonging to the domain, the corresponding (p, u, η) solves (1.13) with wave speed γ, physical parameters $(\mathfrak{g}, \mu, \kappa)$, and stress-force data $(\mathcal{T}, \mathcal{F})$. Moreover, the map (7.51) is continuous.

(4) ***Well-posedness of the stationary wave problem:*** *There exists an open set* $Z_s \subset (\mathbb{R}^+)^3 \times \mathbf{W}_{s,r}$ *satisfying* (1.17) *and continuous a map*

$$Z_s \ni (\mathcal{T}, \mathcal{F}, \mathfrak{g}, \mu, \kappa) \mapsto (p, u, \eta) \in \bigcup_{\mathbf{v} \in (\mathbb{R}^+)^3} V_s(\mathbf{v}) \tag{7.52}$$

with the property that for all $(\mathcal{T}, \mathcal{F}, \mathfrak{g}, \mu, \kappa) \in Z_s$, *the corresponding* (p, u, η) *belongs to the set* $V_s(\mathfrak{g}, \mu, \kappa)$ *and is the unique solution to* (1.13) *with* $\gamma = 0$ *in this set with data in* Z_s.

Proof. The first item follows from Proposition 6.5 and the condition $s > 3/r + 1$. We continue by proving the second item. Let $(p, u, \eta) \in \mathcal{O}_{s,r}(\varrho)$ be a solution generated by the data $(\gamma, \mathfrak{g}, \mu, \kappa, \mathcal{T}, \mathcal{F}) \in \mathscr{W}_s$. We set $v : \Omega[\eta] \to \mathbb{R}^3$ and $q : \Omega[\eta] \to \mathbb{R}$ via $v = (M_\eta^{-1} u) \circ \mathfrak{F}_\eta^{-1}$ and $q = p \circ \mathfrak{F}_\eta^{-1}$. Proposition 7.1 verifies that the map $\mathfrak{F}_\eta : \Omega \to \Omega[\eta]$ is a smooth diffeomorphism that is sufficiently regular up to the boundary as to preserve the notion of classical solution. It is then elementary to verify that (q, v, η) classically solve (1.8) with wave speed γ, physical parameters $(\mathfrak{g}, \mu, \kappa) \in (\mathbb{R}^+)^3$, and stress-force data $(\mathcal{T}, \mathcal{F})$

The third and fourth items are just particular 'restrictions' of the map in (7.45) from Theorem 7.13, so long as we define

$$W_s(\mathfrak{g}, \mu, \kappa) = \{(\gamma, \mathcal{T}, \mathcal{F}) \ : \ (\gamma, \mathfrak{g}, \mu, \kappa, \mathcal{T}, \mathcal{F}) \in W_s\} \tag{7.53}$$

and

$$Z_s = \{(\mathfrak{g}, \mu, \kappa, \mathcal{T}, \mathcal{F}) \ : \ (0, \mathfrak{g}, \mu, \kappa, \mathcal{T}, \mathcal{F}) \in W_s\}, \tag{7.54}$$

and define the mappings (7.51) and (7.52) via (7.45) and the 'slice' identifications $W_s(\mathfrak{g}, \mu, \kappa), Z_s \subset W_s$. $\square$

Declaration of competing interest

There does not exist conflict of interest in this document.

Data availability

No data was used for the research described in the article.

Acknowledgments

The authors would like to express their gratitude to the anonymous referees for their helpful suggestions.

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