Duality for Nonlinear Filtering I: Observability

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***Abstract*— This paper is concerned with the development and use of duality theory for a hidden Markov model (HMM) with white noise observations. The main contribution of this work is to introduce a backward stochastic differential equation (BSDE) as a dual control system. A key outcome is that stochastic observability (resp. detectability) of the HMM is expressed in dual terms: as controllability (resp. stabilizability) of the dual control system. All aspects of controllability, namely, definition of controllable space and controllability gramian, along with their properties and explicit formulae, are discussed. The proposed duality is shown to be an exact extension of the classical duality in linear systems theory. One can then relate and compare the linear and the nonlinear systems. A side-by-side summary of this relationship is given in a tabular form (Table II).**

***Index Terms*— Stochastic systems; Observability; Nonlinear filtering.**

I. Introduction

There is a fundamental dual relationship between estimation and control. The dual relationship is expressed in two inter-related manners:

- Duality between observability and controllability.
- Duality between optimal filtering and optimal control.

The second bullet means expressing one type of problem as another type of problem. In this two-part paper, the main interest is to convert a filtering problem into a control problem.

Duality is coeval with the origin of modern systems and control theory: In [1], Kalman writes "*The analogies [..] between controllability and observability can be expressed cogently by [..] the Principle of Duality*". The two manners of dual relationships are explicitly noted in [1, (62) and (72)]. The original papers [2], [3] describing the Kalman filter contain an extensive mention of duality—between the optimal filter and a certain linear quadratic optimal control problem. Notably, duality explains why the Riccati equation is the fundamental equation for *both* optimal filtering and optimal control.

Sixty years have elapsed since Kalman's original work. One would imagine that duality for the nonlinear stochastic systems (hidden Markov models) is well understood by now. It is a foundational question at the heart of modern systems and control theory, and its modern avatars such as reinforcement learning. However, this is not the case. In his 2008 paper [4], Todorov writes:"*Kalman's duality has been known for half a century and has attracted a lot of attention. If a straightforward generalization to non-LQG settings was possible it would have been discovered long ago. Indeed we will now show that Kalman's duality, although mathematically sound, is an artifact of the LQG setting.*"

Is this to suggest that there is no previous work to extend duality to nonlinear systems? Au contraire! For deterministic models, almost every definition of nonlinear observability, and there have been several notable ones throughout the decades, appeals to duality. Likewise, Mortensen and related minimum energy algorithms, originally invented in 1960s, are standard approaches to construct an estimator. For stochastic systems as well, there have been seminal contributions, notably the work of Mitter and co-authors [5], [6]. Having said that, several reasons are noted in [4] on why the duality described in these prior works are *not* generalizations of the original Kalman-Bucy duality. A comprehensive account of the differences is contained in [7, Ch. 3] and discussed throughout several remarks in this two-part paper.

A. Summary of original contributions

We consider the stochastic filtering problem for a hidden Markov model (HMM) with white noise observations (the mathematical model is introduced in Sec. II). For this filtering problem, we make two types of original contributions:

- Dual controllability characterization of stochastic observability. It is the subject of this present paper (part I).
- Dual (minimum variance) optimal control formulation of the stochastic filtering problem. It is the subject of a companion paper (part II) also submitted to this journal [8].

The focus of this paper is on the dual controllability characterization of stochastic observability of an HMM with white noise observations. The dual control system is introduced in Sec. III. Once the dual control system has been introduced, the ensuing considerations are entirely parallel to linear systems theory: The solution operator of the dual control system is used to define a linear operator whose range space is the controllable subspace. The system is said to be controllable if the range space is dense in a suitable function space. The controllability (resp. stabilizability) of the dual system is shown to be equivalent to stochastic observability (resp. detectability) of the HMM. Several properties of the controllable subspace are noted along with its explicit characterization in the finite state-space case. A formula for the controllability gramian is also described. An upshot of our work is that we can establish parallels between linear and nonlinear models (Table II).

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Part II builds on the results of the part I. In particular, duality is used to transform the minimum variance objective of the nonlinear filtering problem into a stochastic optimal control problem. For the latter problem, the dual control system, introduced in this paper, is shown to arise as the constraint.

B. Relationship to literature

In basic linear systems theory, the following systems are said to be dual to each other:

$$\begin{aligned} &\text{(state-output)} \quad && \dot{x}_t = A^{\mathrm{T}} x_t,\ x_0 = \xi && (1)\\ & && z_t = H^{\mathrm{T}} x_t, \qquad 0 \le t \le T && \\ &\text{(state-input)} \quad && -\dot{y}_t = A y_t + H u_t,\ y_T = 0,\ \ 0 \le t \le T && (2) \end{aligned}$$

The states, x_t and y_t for the two systems, are vector-valued, both of dimension d (the standard dot product in $\mathbb{R}^d$ is denoted by $\langle \cdot, \cdot \rangle_{\mathbb{R}^d}$). The input $u := \{u_t \in \mathbb{R}^m : 0 \le t \le T\}$ and the output $z := \{z_t \in \mathbb{R}^m : 0 \le t \le T\}$ are elements of the function space $L^2([0,T];\mathbb{R}^m) =: \mathcal{U}$ equipped with the inner-product $\langle u, v \rangle_{\mathcal{U}} := \int_0^T u_t^{\mathrm{T}} v_t \,\mathrm{d}t$. For the state-input system (2), the solution map $u \mapsto y_0$ is used to define a linear operator $\mathcal{L} : \mathcal{U} \to \mathbb{R}^d$ as follows:

$$\mathcal{L}u := y_0 = \int_0^T e^{At} H u_t \,\mathrm{d}t$$

Its adjoint is given by

$$(\mathcal{L}^\dagger \xi)(t) = H^{\mathrm{T}} e^{A^{\mathrm{T}} t} \xi, \quad 0 \le t \le T$$

and represents the solution map from the initial condition $\xi \mapsto z$ for the state-output system (1). Mathematically, the dual relationship is expressed as

$$\langle \xi, \mathcal{L}u \rangle_{\mathbb{R}^d} = \langle \mathcal{L}^\dagger \xi, u \rangle_{\mathcal{U}}, \quad \forall\, \xi \in \mathbb{R}^d,\ u \in \mathcal{U}$$

The relationship yields the following important identity [9, Theorem 6.6.1] (also referred to as the closed range theorem)

$$\mathsf{R}(\mathcal{L})^\perp = \mathsf{N}(\mathcal{L}^\dagger)$$

This identity has several important consequences, e.g., controllability (resp. stabilizability) property of the state-input system (2) is equivalent to the observability (resp. detectability) property of the state-output system (1).

This classical duality between controllability and observability is useful for *both* analysis and the design of estimation algorithms. For example, most proofs of stability of the Kalman filter (see e.g., [10, Ch. 9]) rely, in direct or indirect manner, on duality theory. Specifically (i) Because of duality, asymptotic stability of the Kalman filter is related to asymptotic stability of the (dual) optimal control system; (ii) necessary and sufficient conditions for the latter are stabilizability for the control model, and (because of duality) detectability for the filtering model; and (iii) analysis of the optimal control problem (convergence of the value function to its stationary limit) yields useful conclusions on stability of the filter (convergence of the error covariance).

This has naturally spurred a large body of work related to:

- Defining observability as a dual property.
- Using the definition to investigate asymptotic stability of optimal and sub-optimal estimators.

The second bullet has by far been the most important reason for defining and studying observability and related concepts. In many studies, the definition of observability is often a sufficient condition that guarantees the stability of the estimator under study, e.g., [11, Defn. 2], [12, Defn. 4.1], [13, Assumption 2], [14, identifiability conditions A-1 and A-2].

We next provide a brief survey of observability and its use for investigating estimator (resp. filter) stability, in the study of deterministic (resp. stochastic) nonlinear models. These have to be separated because the work on these two types of models has little overlap. For deterministic models, an estimator is defined as a dynamical system whose dimension is the same as the dimension of the state and which operates on inputs and outputs. An important class, which also incorporates certain optimality properties, is the minimum energy estimator (MEE) [12, Ch. 4]. (Relationship to MEE is discussed at length in part II [8, Tab. I].) The two bullets above serve as guiding principles around which the discussion is organized.

Deterministic models. In the classical paper [15], Hermann and Krener write "*duality between "controllability" and "observability" [...] is, mathematically, just the duality between vector fields and differential forms*". Similarly, the output-to-state stability (OSS) definitions in [16] are motivated by Wang and Sontag as follows: "*Given the central role often played in control theory by the duality between input/state and state/output behavior, one may reasonably ask what concept obtains if outputs are used instead of inputs in the [input-to-state stability (ISS)] definition*". The OSS definition is important for the following two reasons: (i) For the linear state-output system (1), OSS is equivalent to detectability [17, Excercise 7.3.12]; and (ii) OSS admits a dissipative characterization in terms of an OSS-storage function [16, Thm. 3]. Such characterizations are useful in the study of stability and robustness (several variations of observability definition and their relationship are discussed in [18], [19]). Combining ISS and OSS yields the notion of IOSS which is shown to be equivalent to estimating the norm of the hidden state [18, Sec. 8.5]. Related to detectability, an important notion is the incremental IOSS (i-IOSS) which is standard for asymptotic stability analysis of MEE [12, Thm. 4.10]. Dissipative characterizations of incremental notions are also important, e.g., an i-IOSS Lyapunov function is given in [20].

Stochastic models (HMMs). As in the study of deterministic models, a major impetus to define observability/detectability comes from the question of nonlinear filter stability (now in the sense of asymptotic forgetting of the initial prior). Formally, there are two main cases:

- The case where the Markov process forgets the prior and therefore the filter "inherits" the same property;
- The case where the observation provides sufficient information about the hidden state, allowing the filter to correct its erroneous initialization.

These two cases are referred to as the ergodic and non-ergodic signal cases, respectively. While the two cases are intuitively reasonable, they spurred much work during 1990-2010 with a complete resolution appearing only at the end of this time-period. For the ergodic case, sufficient conditions for filter stability that rely only on the signal model appear in [21, Thm. 5], [22, Assumption 4.3.24], [14, Thm. 4.3], [23, Cor. 2.3.2], [14, Thm. 4.2]. For the non-ergodic signal case, sufficient conditions relying also on the observation model are given in [21, Thm. 7], [14, A-1 and A-2], [24, Rem. 23.1]. All of these conditions are for the HMM with white noise observations, a model which occupies a central place in the nonlinear filtering theory. For a more general class of HMMs, the fundamental definition for stochastic observability and detectability is due to van Handel [25], [26] (see Sec. II-D). There are two notable features: (i) the definition made rigorous the intuition described in the two cases [27, Sec. II-B and Sec. V]; and (ii) the definition led to meaningful conditions that were shown to be necessary and sufficient for filter stability [27, Thm. III.3 and Thm. V.2]. The stochastic observability definition is entirely probabilistic and its information-theoretic extension was given in [28], [29]. For linear stochastic systems, information-theoretic metrics such as relative entropy had earlier been used to define observability [30] (see also [31] where extensions for uncertain linear systems is described). There are also a number of works where observability is defined as a finite memory property (often referred to as uniform observability or reconstructability) whereby only the most recent window of observations is necessary for estimation and/or control [13, Assumption 2], [32, Defn. 2.4] [33, Assumption A2]. Similar considerations also inform [34], [35] where N step observability is defined for an HMM. The definition has many attractive features, e.g., it is relatively easier to verify (as compared to stochastic observability) and is useful for filter and control design. In [35, Fig. 1], the definition is also related to several criteria for filter stability.

In contrast to the deterministic models, duality is conspicuous by its absence both to define stochastic observability and to use it for filter stability. Dissipative characterizations, that are so familiar for deterministic models, are also missing.

This paper is drawn from the PhD thesis [7] of the first author. A prior conference version of this paper appeared in [36]. While the focus of conference paper was on the finite state-space case, the present journal version includes the results for the general case. The following additions are noted: A novel extension to stabilizability of the dual control system is described (Sec. III-E) and shown to be the dual to detectability of the HMM (Cor. 2). Remarks 7, 8, and 9 are included to clarify the choice of function spaces. The dual control system described in our work is compared and contrasted to the backward Zakai equation which is how duality is understood in the theory of nonlinear filtering (Rem. 5). Finally, the results are related to both stochastic observability (Thm. 1) and to the linear Gaussian problem (Sec. III-F).

C. Paper outline

The outline of the remainder of this paper is as follows: The problem formulation and background appears in Sec. II. The dual control system is introduced in Sec. III together with the definition of the controllability and related concepts. The explicit formulae for the finite state space case appear in Sec. IV. The paper closes with some conclusions and directions for future research in Sec. V.

Remark 1: An important objective of this paper is to make comparisons with linear systems theory. This objective informs the organization of this paper. In particular, after introducing the dual control system in Sec. III-C, the following sub-sections describe the controllable subspace, controllability gramian (in Sec. III-D), and stabilizability (in Sec. III-E). The following Sec. IV, concerning the finite state-space model, is organized similarly with sub-sections on dual control system (Sec. IV-B), controllable subspace (Sec. IV-C), controllability gramian (Sec. IV-D), and stabilizability (Sec. IV-E). Such a choice of organizing the material is strongly influenced by how these concepts are taught in an introductory course on linear systems theory. The close parallels between the linear and the nonlinear cases are illustrated in Tables I and II, and the linear Gaussian special case is discussed in Sec. III-F. Because of the broad appeal of observability and related concepts to *both* the deterministic and stochastic control communities, it is hoped that the organization is useful to make the paper broadly accessible. For this reason also, the paper begins with a self-contained background on the nonlinear filtering model in Sec. II. For some of the more technical aspects, additional details can be found the references noted.

II. Background and Problem Formulation

A. Notation

For a locally compact Polish space S, the following notation is adopted:

- $\mathcal{B}(S)$ is the Borel σ-algebra on S.
- $\mathcal{M}(S)$ is the space of regular, bounded and finitely additive signed measures (rba measures) on $\mathcal{B}(S)$. The natural norm is the total variation norm denoted $\|\cdot\|_{\text{TV}}$.
- $\mathcal{P}(S)$ is the subset of $\mathcal{M}(S)$ comprising of probability measures.
- $C_b(S)$ is the space of continuous and bounded real-valued functions on S. The natural norm is the sup-norm $\|\cdot\|_\infty$.
- For measure space $(S;\mathcal{B}(S);\lambda)$, $L^2(\lambda) = L^2(S;\mathcal{B}(S);\lambda)$ is the Hilbert space of real-valued functions equipped with the inner product $\langle f,g\rangle_{L^2(\lambda)} = \int_S f(x)g(x)\,\mathrm{d}\lambda(x)$.

For functions $f : S \to \mathbb{R}$ and $g : S \to \mathbb{R}$, the notation fg is used to denote element-wise product of f and g, namely, $(fg)(x) := f(x)g(x)$ for $x \in S$. In particular, $f^2 = ff$. The constant function is denoted by 1 ($1(x) = 1$ for all $x \in S$). For $\mu \in \mathcal{M}(S)$ and $f \in C_b(S)$, $\mu(f) := \int_S f(x)\,\mathrm{d}\mu(x)$ and for $\mu,\nu \in \mathcal{M}(S)$ such that μ is absolutely continuous with respect to ν (denoted $\mu \ll \nu$), the Radon-Nikodym (RN) derivative is denoted by $\frac{\mathrm{d}\mu}{\mathrm{d}\nu}$. For a subset $B \subset C_b(S)$, the annihilator of B, denoted by $B^\perp$, is defined by $B^\perp := \{\mu \in \mathcal{M}(S) : \mu(f) = 0\ \forall f \in B\}$. For a sub σ-algebra $\mathcal{G} \subset \mathcal{B}(S)$, the restriction of the measure μ to $\mathcal{G}$ is denoted by $\mu|_\mathcal{G}$. It is obtained from the defining relation $\mu|_\mathcal{G}(B) = \mu(B)$ for $B \in \mathcal{G}$.

B. Hidden Markov Model

We consider continuous-time stochastic processes on a finite time-horizon $[0,T]$ with a fixed $T < \infty$. Fix the probability space $(\Omega,\mathcal{F}_T,\mathsf{P})$ along with the filtration $\{\mathcal{F}_t : 0 \le t \le T\}$

with respect to which all the stochastic processes are adapted. Of special interest is the pair (X, Z) defined as follows:

- The *state process* $X = \{X_t : \Omega \to \mathbb{S} : 0 \le t \le T\}$ is a Feller-Markov process on the state-space $\mathbb{S}$. Its initial measure (prior) is denoted by $\mu \in \mathcal{P}(\mathbb{S})$ and $X_0 \sim \mu$. The infinitesimal generator is denoted by $\mathcal{A}$.
- The *observation process* $Z = \{Z_t : 0 \le t \le T\}$ satisfies the stochastic differential equation (SDE):

$$Z_t = \int_0^t h(X_s)\,\mathrm{d}s + W_t, \quad 0 \le t \le T$$

  where $h : \mathbb{S} \to \mathbb{R}^m$ is the observation function and $W = \{W_t : 0 \le t \le T\}$ is an m-dimensional Brownian motion (B.M.). We write W is P-B.M. It is assumed that W is independent of X.

The above is referred to as the *white noise observation model* of nonlinear filtering. In the remainder of this paper, the model is referred to as the HMM $(\mathcal{A}, h)$. In the case where $\mathbb{S}$ is not finite, additional assumptions are typically necessary to ensure that the model is well-posed. In applications, the most important examples are as follows:

- $\mathbb{S}$ is finite with cardinality $|\mathbb{S}| = d$.
- $\mathbb{S} = \mathbb{R}^d$ and X is an Itô diffusion.

A historically noteworthy example of an Itô diffusion is the linear Gaussian model.

C. Nonlinear filtering background

The canonical filtration $\mathcal{F}_t = \sigma\big(\{(X_s, W_s) : 0 \le s \le t\}\big)$. The filtration generated by the observation is denoted by $\mathcal{Z} := \{\mathcal{Z}_t : 0 \le t \le T\}$ where $\mathcal{Z}_t = \sigma\big(\{Z_s : 0 \le s \le t\}\big)$. The *nonlinear (or stochastic) filtering problem* is to compute the conditional expectation for a given test function $f \in C_b(\mathbb{S})$:

$$\pi_t(f) := \mathsf{E}\big(f(X_t) \mid \mathcal{Z}_t\big), \quad 0 \le t \le T$$

The measure-valued process $\pi = \{\pi_t : 0 \le t \le T\}$ is referred to as the nonlinear filter.

A standard approach is based upon the Girsanov change of measure. Suppose the model satisfies the Novikov's condition: $\mathsf{E}\left(\exp\big(\frac{1}{2}\int_0^T |h(X_t)|^2\,\mathrm{d}t\big)\right) < \infty$. Define a new measure $\tilde{\mathsf{P}}$ on $(\Omega, \mathcal{F}_T)$ as follows:

$$\frac{\mathrm{d}\tilde{\mathsf{P}}}{\mathrm{d}\mathsf{P}} = \exp\Big(-\int_0^T h^{\mathrm{T}}(X_t)\,\mathrm{d}W_t - \tfrac{1}{2}\int_0^T |h(X_t)|^2\,\mathrm{d}t\Big) =: D_T^{-1}$$

Then it can be shown that the probability law for X is unchanged but Z is a $\tilde{\mathsf{P}}$-B.M. that is independent of X [23, Lem. 1.1.5]. The expectation with respect to $\tilde{\mathsf{P}}$ is denoted by $\tilde{\mathsf{E}}(\cdot)$. The *un-normalized filter* is a measure-valued process $\sigma = \{\sigma_t : 0 \le t \le T\}$ defined by

$$\sigma_t(f) := \tilde{\mathsf{E}}\big(D_t f(X_t)|\mathcal{Z}_t\big), \quad f \in C_b(\mathbb{S})$$

Because Z is a $\tilde{\mathsf{P}}$-B.M., the equation is unnormalized filter is easily obtained and is in fact the celebrated Zakai equation of nonlinear filtering [10, Thm. 5.5]:

$$\sigma_t(f) = \mu(f) + \int_0^t \sigma_s(hf)^{\mathrm{T}}\,\mathrm{d}Z_t + \int_0^t \sigma_s(\mathcal{A}f)\,\mathrm{d}s, \quad 0 \le t \le T \tag{3}$$

Upon normalization, the nonlinear filter

$$\pi_t(f) = \frac{\sigma_t(f)}{\sigma_t(1)}, \quad 0 \le t \le T \tag{4}$$

This ratio is referred to as the Kallianpur-Striebel formula [10, Thm. 5.3]. Using (3), the equation for the nonlinear filter is readily obtained by a simple application of the Itô formula to the ratio [10, Thm. 5.7].

D. Stochastic observability

In problems concerned with observability of the model $(\mathcal{A}, h)$ or filter stability, the symbol μ is used to denote the true but possibly unknown prior and the symbol ν is used to denote the prior that is used to compute the filter. If μ is exactly known then $\mu = \nu$. In all other cases, it is assumed that $\mu \ll \nu$.

To stress the dependence on the initial measure μ, we use the superscript notation P^μ to denote the probability measure P when $X_0 \sim \mu$. The expectation operator is denoted by $\mathsf{E}^\mu(\cdot)$ and the nonlinear filter $\pi_t^\mu(f) = \mathsf{E}^\mu\big(f(X_t)|\mathcal{Z}_t\big)$. On the common measurable space $(\Omega, \mathcal{F}_T)$, P^ν is used to denote another probability measure such that the transition law of (X, Z) are identical but $X_0 \sim \nu$. (For an explicit construction of P^μ and P^ν, see [37, Sec. 2.2].) The associated expectation operator is denoted by $\mathsf{E}^\nu(\cdot)$ and $\pi_t^\nu(f) = \mathsf{E}^\nu\big(f(X_t)|\mathcal{Z}_t\big)$. The respective un-normalized filters are denoted by σ_t^μ and σ_t^ν. These are solution of the Zakai equation (3) with initialization $\sigma_0^\mu = \mu$ and $\sigma_0^\nu = \nu$, respectively.

The following definition of stochastic observability is introduced in [25]. Although it is identical for a general class of HMMs, we state it for the model $(\mathcal{A}, h)$:

Definition 1 (Defn. 2 in [25]): The model $(\mathcal{A}, h)$ is *observable* if

$$\mathsf{P}^\mu|_{\mathcal{Z}_T} = \mathsf{P}^\nu|_{\mathcal{Z}_T} \implies \mu = \nu, \quad \forall\, \mu, \nu \in \mathcal{P}(\mathbb{S})$$

Remark 2: The definition is contrasted with the definition of observability for deterministic nonlinear models [17, Defn. 6.1.4]. For deterministic models, observability is defined as a property of the map from initial condition to output trajectory. In contrast, stochastic observability is a property of the map from the initial prior to the probability law of the output process.

Consider an equivalence relation on $\mathcal{P}(\mathbb{S})$ as follows:

$$\mu \simeq \nu \quad \text{if} \quad \mathsf{P}^\mu|_{\mathcal{Z}_T} = \mathsf{P}^\nu|_{\mathcal{Z}_T}$$

The following definition naturally arises from this notation:

Definition 2 (Defn. 3 in [25]): The space of *observable functions* $\mathcal{O} = \{f \in C_b(\mathbb{S}) : \mu(f) = \nu(f)\ \forall\, \mu \simeq \nu\}$. The space of *unobservable measures* $\mathcal{N} = \{c(\mu - \nu) \in \mathcal{M}(\mathbb{S}) : c \in \mathbb{R},\ \mu, \nu \in \mathcal{P}(\mathbb{S}) \text{ s.t. } \mu \simeq \nu\}$.

Remark 3: An HMM is observable if and only if $\mathcal{N} = \{0\}$, and because $\mathcal{O}^\perp = \mathcal{N}$, an HMM is observable if and only if $\mathcal{O}$ is dense in $C_b(\mathbb{S})$ [25, p. 42]. Because $1 \in \mathcal{O}$, the space of observable functions $\mathcal{O}$ is always non-trivial. This also means $\mathcal{N} \subseteq \mathcal{M}_0(\mathbb{S}) := \{\mu \in \mathcal{M}(\mathbb{S}) : \mu(1) = 0\}$.

III. Dual Control System

A. Function spaces

It is noted that Z is a $\tilde{\mathsf{P}}$-B.M.. For a $\mathcal{Z}_T$-measurable random vector, the following definition of Hilbert space is standard: $L^2_{\mathcal{Z}_T}(\Omega;\mathbb{R}^m) := L^2(\Omega;\mathcal{Z}_T;\,\mathrm{d}\tilde{\mathsf{P}})$ [38, Ch. 5.1.1]. For a $\mathcal{Z}$-adapted vector-valued stochastic process, the Hilbert space is $L^2_{\mathcal{Z}}\big([0,T];\mathbb{R}^m\big) := L^2\big(\Omega\times[0,T];\mathcal{Z}\otimes\mathcal{B}([0,T]);\,\mathrm{d}\tilde{\mathsf{P}}\,\mathrm{d}t\big)$ where $\mathcal{B}([0,T])$ is the Borel sigma-algebra on $[0,T]$, $\mathcal{Z}\otimes\mathcal{B}([0,T])$ is the product sigma-algebra and $\mathrm{d}\tilde{\mathsf{P}}\,\mathrm{d}t$ denotes the product measure on it. The inner product for these spaces are

$$\langle F,G\rangle_{L^2_{\mathcal{Z}_T}} = \tilde{\mathsf{E}}\big(F^\intercal G\big), \quad \langle U,V\rangle_{L^2_{\mathcal{Z}}} = \tilde{\mathsf{E}}\Big(\int_0^T U_t^\intercal V_t\,\mathrm{d}t\Big)$$

These Hilbert spaces suffice if the state-space $\mathbb{S}$ is finite. In general settings, let $\mathcal{Y}$ denote a suitable Banach space of real-valued functions on $\mathbb{S}$, equipped with the norm $\|\cdot\|_{\mathcal{Y}}$. Then

- For a random function, the Banach space $L^2_{\mathcal{Z}_T}(\Omega;\mathcal{Y}) := \big\{F:\Omega\to\mathcal{Y} : F \text{ is } \mathcal{Z}_T\text{-measurable, } \tilde{\mathsf{E}}\big(\|F\|^2_{\mathcal{Y}}\big)<\infty\big\}$.
- For a function-valued stochastic process, the Banach space is $L^2_{\mathcal{Z}}([0,T];\mathcal{Y}) := \Big\{Y : \Omega\times[0,T]\to\mathcal{Y} : Y \text{ is } \mathcal{Z}\text{-adapted, } \tilde{\mathsf{E}}\Big(\int_0^T \|Y_t\|^2_{\mathcal{Y}}\,\mathrm{d}t\Big)<\infty\Big\}$.

In this paper, examples of $\mathcal{Y}$ are: (i) $C_b(\mathbb{S})$ equipped with sup norm denoted by $\|\cdot\|_\infty$, and (ii) $L^2(\lambda)$ where λ is a positive reference measure on $\mathbb{S}$.

B. Stochastic observability and Zakai equation

For a white noise observation model, a quantitative analysis of stochastic observability is possible based on the following formula for relative entropy [37, Thm. 3.1]:

$$\mathsf{D}\big(\mathsf{P}^\mu|_{\mathcal{Z}_T}\mid \mathsf{P}^\nu|_{\mathcal{Z}_T}\big) = \tfrac{1}{2}\mathsf{E}^\mu\Big(\int_0^T |\pi_t^\mu(h)-\pi_t^\nu(h)|^2\,\mathrm{d}t\Big) \tag{5}$$

where $\mathsf{D}(\cdot\mid\cdot)$ is the Kullback-Leibler (KL) divergence. The formula is used to obtain the following result:

Theorem 1: T.F.A.E.:

1) The model $(\mathcal{A},h)$ is observable.
2) For $\mu,\nu\in\mathcal{P}(\mathbb{S})$,
$$\pi_t^\mu(h)=\pi_t^\nu(h),\quad t\text{-a.e., } \mathsf{P}^\mu|_{\mathcal{Z}_T}\text{-a.s.}\quad\Longrightarrow\quad \mu=\nu$$
3) For $\mu,\nu\in\mathcal{P}(\mathbb{S})$,
$$\sigma_t^\mu(h)=\sigma_t^\nu(h),\quad t\text{-a.e., } \mathsf{P}^\mu|_{\mathcal{Z}_T}\text{-a.s.}\quad\Longrightarrow\quad \mu=\nu$$

Proof: See Appendix A. ■

The value of Thm. 1 is that the un-normalized filter is the solution to the Zakai equation which is linear. A linear operator $\mathcal{L}^\dagger:\mathcal{M}(\mathbb{S})\to L^2_{\mathcal{Z}}\big([0,T];\mathbb{R}^m\big)\times\mathbb{R}$ is defined as follows:

$$\mathcal{L}^\dagger\mu = \big(\{\sigma_t^\mu(h):0\le t\le T\},\mu(1)\big) \tag{6}$$

The notation is suggestive: In this paper, we will define a linear operator $\mathcal{L}$ such that the operator defined by (6) is its adjoint.

Corollary 1: The model $(\mathcal{A},h)$ is observable if and only if $\mathsf{N}(\mathcal{L}^\dagger)=\{0\}$.

Remark 4: $\mathsf{N}(\mathcal{L}^\dagger)$ is identical to the space of unobservable measure (see Defn. 2). Suppose $\mathsf{N}(\mathcal{L}^\dagger)$ is not trivial and let $\tilde{\mu}\in\mathsf{N}(\mathcal{L}^\dagger)$ be a non-zero element. For $\mu\in\mathcal{P}(\mathbb{S})$, and then choose $\epsilon\neq 0$ such that $\nu=\mu+\epsilon\tilde{\mu}\in\mathcal{P}(\mathbb{S})$. Then owing to the linearity of (3), $\sigma_t^\mu(h)=\sigma_t^\nu(h)$ for $0\le t\le T$. From Thm. 1 then $\mathsf{P}^\mu|_{\mathcal{Z}_T}=\mathsf{P}^\nu|_{\mathcal{Z}_T}$.

Remark 5 (Prior work on adjoint of the Zakai equation): Because the Zakai equation is linear, its adjoint has previously been considered in literature. There are two types of equivalent constructions:

- The most direct route relies on the pathwise or the robust representation of the nonlinear filter. In this approach, by using a log transformation, the stochastic partial differential equation (PDE) is transformed into a linear deterministic PDE with random coefficients [39, Ch. VI.11]. Because the transformed PDE is deterministic, the formula for its adjoint is obtained by standard means [40, Eq. 4.17-4.18].
- The second type of adjoint is the *backward Zakai equation*
$$\begin{aligned} -\mathrm{d}\eta_t(x) &= (\mathcal{A}\eta_t)(x)\,\mathrm{d}t + (h\eta_t)(x)\overleftarrow{\mathrm{d}Z_t} \\ \eta_T(x) &= f(x),\quad x\in\mathbb{S}\end{aligned} \tag{7}$$
where $\overleftarrow{\mathrm{d}Z_t}$ denotes the backward Itô integral: its construction is based on choosing the right-endpoints in the partial sum approximation of the stochastic integral [41, Rem. 3.3]. The backward and forward Zakai equation are said to be dual because of the following formula [42, Thm. 4.7.5]:
$$\sigma_T(f)=\mu(\eta_0)$$
The formula is used to prove the uniqueness of the solution to the (forward) Zakai equation [10, Sec. 6.5].

The two constructions are equivalent because, using the log transformation, the backward Zakai equation is transformed to the pathwise adjoint [43, Sec. 2.3]. In nonlinear filtering, the forward and backward Zakai equation are both classical [44]. The two equations together yield the solution of the smoothing problem [41, Thm. 3.8].

Despite the well known duality between forward and backward Zakai equations, it is distinct from the controllability–observability duality (in linear systems theory) because of the following aspects:

- The dual equation (7) does not have a control input term.
- The stochastic process $\eta=\{\eta_t:0\le t\le T\}$ is not adapted to any forward-in-time filtration. In particular, η_0 is a $\mathcal{Z}_T$-measurable random variable.

The dual control system described in the following section is original and distinct from these prior adjoints.

C. Dual control system

The goal is to define a linear operator $\mathcal{L}$ whose adjoint $\mathcal{L}^\dagger$ is given by (6). Because $\mathcal{L}^\dagger:\mathcal{M}(\mathbb{S})\to L^2_{\mathcal{Z}}\big([0,T];\mathbb{R}^m\big)\times\mathbb{R}$, the domain of $\mathcal{L}$ is $L^2_{\mathcal{Z}}\big([0,T];\mathbb{R}^m\big)\times\mathbb{R}$. We set $\mathcal{U} := L^2_{\mathcal{Z}}\big([0,T];\mathbb{R}^m\big)$ and refer to it as the space of *admissible* *control inputs*. Next, because of duality pairing between $C_b(\mathbb{S})$ and $\mathcal{M}(\mathbb{S})$, the co-domain of $\mathcal{L}$ is $C_b(\mathbb{S})$. We set $\mathcal{Y} = C_b(\mathbb{S})$.

The main result (Thm. 2 below) is to show that the operator $\mathcal{L}:\mathcal{U}\times\mathbb{R}\to\mathcal{Y}$ is defined by the solution operator of the linear backward stochastic differential equation (BSDE):

$$-\,\mathrm{d}Y_t(x) = \big((\mathcal{A}Y_t)(x) + h^{\intercal}(x)(U_t + V_t(x))\big)\,\mathrm{d}t - V_t^{\intercal}(x)\,\mathrm{d}Z_t \tag{8a}$$

$$Y_T(x) = c, \quad \forall\, x \in \mathbb{S} \tag{8b}$$

where the control input $U := \{U_t : 0 \le t \le T\} \in \mathcal{U}$ and $c \in \mathbb{R}$ is a deterministic constant. The solution of the BSDE $(Y,V) := \{(Y_t,V_t) : 0 \le t \le T\} \in L^2_{\mathcal{Z}}\big([0,T];\mathcal{Y}\times\mathcal{Y}^m\big)$ is (forward) adapted to the filtration $\mathcal{Z}$. Existence, uniqueness, and regularity theory for linear BSDEs is standard and throughout the paper, we assume that the solution of BSDE (Y,V) is uniquely determined in $L^2_{\mathcal{Z}}\big([0,T];\mathcal{Y}\times\mathcal{Y}^m\big)$ for each given $Y_T \in L^2_{\mathcal{Z}_T}(\Omega;\mathcal{Y})$ and $U \in L^2_{\mathcal{Z}}\big([0,T];\mathbb{R}^m\big)$. The well-posedness results for finite state-space can be found in [45, Ch. 7] and for the Euclidean state space in [46] (see also Rem. 8 below).

The linear operator $\mathcal{L}:\mathcal{U}\times\mathbb{R}\to\mathcal{Y}$ is defined as follows:

$$\mathcal{L}(U,c) = Y_0 \tag{9}$$

where $Y_0 \in \mathcal{Y}$ is the solution at time 0 to the BSDE (8). Note that Y_0 is a deterministic function.

Controllability is now defined in the same way as linear systems theory. Note however that the target set (at time T) is the space of constant functions (see also Rem. 7).

Definition 3: For the BSDE (8), the *controllable subspace* $\mathcal{C}_T := \mathsf{R}(\mathcal{L})$. Explicitly,

$$\mathcal{C}_T = \big\{y_0 \in \mathcal{Y} : \exists\, c \in \mathbb{R} \text{ and } U \in \mathcal{U} \text{ s.t. } Y_0 = y_0 \text{ and } Y_T = c1\big\} \tag{10}$$

The BSDE (8) is said to be *controllable* if $\mathcal{C}_T$ is dense in $\mathcal{Y}$.

Remark 6: If the control input U is deterministic then $V = 0$. If U is stochastic then V must be non-zero to obtain an adapted Y. It is possible to relate V to Y in terms of a certain Malliavin derivative (see [47, Sec. 5] for the finite state-space case and [46, Sec. 7] for the Euclidean case). However, such formulae are not particularly intuitive or computationally tractable. Additional insight may well be possible in the optimal control setting of the problem (which is the subject of the part II of this paper). However, as of yet, both the meaning of the process V and its relationship to the process Y remains an important topic of future study.

The duality between observability of the model $(\mathcal{A},h)$ and the controllability of the BSDE (8) is described in the following theorem:

Theorem 2: $\mathcal{L}^\dagger$ is the adjoint operator of $\mathcal{L}$. Consequently, the HMM $(\mathcal{A},h)$ is observable if and only if the BSDE (8) is controllable.

Proof: See Appendix B. ■

We refer to the BSDE (8) as the *dual control system* for the HMM $(\mathcal{A},h)$. We make some remarks on function and measure spaces.

Remark 7: In the definition of $\mathcal{L}$, the domain space is $L^2_{\mathcal{Z}}\big([0,T];\mathbb{R}^m\big)\times\mathbb{R}$. For the purposes of this study, it also suffices to consider a restriction of $\mathcal{L}^\dagger$ on the subspace $\mathcal{M}_0(\mathbb{S})$ (for example, the null-space of $\mathcal{L}^\dagger$ is a subspace of $\mathcal{M}_0(\mathbb{S})$). An advantage of considering such a restriction is that the co-domain space for $\mathcal{L}^\dagger$, and therefore the domain of $\mathcal{L}$, now is $L^2_{\mathcal{Z}}\big([0,T];\mathbb{R}^m\big)$. The dual space of $\mathcal{M}_0(\mathbb{S})$ is the quotient space $C_b(\mathbb{S})/\{c1 : c \in \mathbb{R}\}$ and therefore $\mathcal{L} : L^2_{\mathcal{Z}}\big([0,T];\mathbb{R}^m\big) \to C_b(\mathbb{S})/\{c1 : c \in \mathbb{R}\}$. Although such a change will make duality between controllability and observability somewhat terser, we prefer to keep the measure space as $\mathcal{M}(\mathbb{S})$ and the function space as $C_b(\mathbb{S})$. This has an advantage of not having to deal with the solutions of the BSDE on the quotient space.

Remark 8: The choice of function space $\mathcal{Y} = C_b(\mathbb{S})$ is guided by duality pairing between $C_b(\mathbb{S})$ and measure space $\mathcal{M}(\mathbb{S})$ [48, Thm. IV.6.2]. An important reason is to relate with [25] who defines observable functions as a subspace of $C_b(\mathbb{S})$. However, this may place restriction on the model (e.g., $\mathbb{S}$ is finite or compact) for the linear operator $\mathcal{L}:\mathcal{U}\times\mathbb{R}\to\mathcal{Y}$ to be bounded. Alternatively, one may consider linear operator on a suitable Hilbert space. An important case is when a $\mathbb{S}$ admits a positive reference measure λ. In this case, provided these are well-defined, one may consider

$$\mathcal{L} \;:\; \mathcal{U}\times\mathbb{R}\to L^2(\lambda), \qquad \mathcal{L}^\dagger \;:\; L^2(\lambda)\to\mathcal{U}\times\mathbb{R}$$

where the domain for the adjoint $\mathcal{L}^\dagger$ is the space of absolutely continuous measures $\nu \in \mathcal{M}(\mathbb{S})$ whose density $\dfrac{\mathrm{d}\nu}{\mathrm{d}\lambda} \in L^2(\lambda)$. Examples are (i) finite state-space where λ is the counting measure; and (ii) the Euclidean state-space where λ is the Lebesgue measure. In fact, the well-posedness results for the BSDE in Euclidean settings are for $\mathcal{Y} = L^2(\lambda)$ [46]. For linear Gaussian problems, one may take λ to be the Gaussian prior.

Remark 9: If $\mathbb{S}$ is finite, $C_b(\mathbb{S})$ and $\mathcal{M}(\mathbb{S})$ are isomorphic to $\mathbb{R}^d$ and the controllability definition reduces to $\mathsf{R}(\mathcal{L}) = \mathcal{Y}$. If $\mathbb{S} = \mathbb{R}^d$, $\mathcal{Y}$ is an infinite-dimensional space. In the study of infinite-dimensional linear control systems, it is known that the range space of an integral solution operator $\mathcal{L}:\mathcal{U}\times\mathbb{R}\to\mathcal{Y}$ can never be the entire space $\mathcal{Y}$ [49, Thm. 4.1.5]. Therefore, the best one can hope for is $\overline{\mathsf{R}(\mathcal{L})} = \mathcal{Y}$ which is how controllability is defined (see also [25, Rem. 5]).

In the remainder of this paper, we let $\mathcal{Y}$ to be a suitable function space and $\mathcal{Y}^\dagger$ is the dual space. A reader may replace $\mathcal{Y} = C_b(\mathbb{S})$ and $\mathcal{Y}^\dagger = \mathcal{M}(\mathbb{S})$ and such a choice is always possible if $\mathbb{S}$ is finite or compact.

D. Controllable subspace and controllability gramian

The following proposition describes an important property of the controllable subspace which is useful for computations.

Proposition 1: The controllable subspace $\mathcal{C}_T$ is the smallest such subspace $\mathcal{C} \subset \mathcal{Y}$ that satisfies the following two properties:

1) The constant function $1 \in \mathcal{C}$; and
2) If $g \in \mathcal{C}$ then $\mathcal{A}g \in \mathcal{C}$ and $gh \in \mathcal{C}$.

Proof: See Appendix C. ■

The *controllability gramian* $\mathsf{W}:\mathcal{Y}^\dagger\to\mathcal{Y}$ is a deterministic linear operator defined as follows:

$$\mathsf{W}:=\mathcal{L}\mathcal{L}^\dagger$$

Explicitly, for a measure $\mu\in\mathcal{Y}^\dagger$, $\mathsf{W}\mu=Y_0$ where Y_0 is obtained for solving the BSDE

$$-\,\mathrm{d}Y_t(x)=\big(\mathcal{A}Y_t(x)+h^{\intercal}(x)(\sigma_t^\mu(h)+V_t(x))\big)\,\mathrm{d}t-V_t^{\intercal}(x)\,\mathrm{d}Z_t$$
$$Y_T(x)=\mu(1),\quad x\in\mathbb{S}$$

The gramian yields an explicit control input as follows:

Proposition 2: Suppose $f\in\mathsf{R}(\mathsf{W})$, i.e., there exists $\mu\in\mathcal{Y}^\dagger$ such that $f=\mathsf{W}\mu$. Then the control

$$U_t=\sigma_t^\mu(h),\quad 0\le t\le T$$

transfers the system (8) from $Y_T=\mu(1)1$ to $Y_0=f$. Suppose $\tilde{U}$ is another control which also transfers $Y_T=c1$ to $Y_0=f$ for some $c\in\mathbb{R}$. Then

$$\tilde{\mathsf{E}}\Big(\int_0^T|\tilde{U}_t|^2\,\mathrm{d}t\Big)+c^2\ge\tilde{\mathsf{E}}\Big(\int_0^T|U_t|^2\,\mathrm{d}t\Big)+\big(\mu(1)\big)^2 \tag{11}$$

Proof: See Appendix D. ■

E. Stabilizability and detectability

Consider the solution $\{\mu_t\in\mathcal{Y}^\dagger:t\ge0\}$ to the forward Kolmogorov equation:

$$\mu_t(f)=\mu_0(f)+\int_0^t\mu_s(\mathcal{A}f)\,\mathrm{d}s,\qquad t\ge0$$

The *stable complement* of $\mathcal{A}$ is defined as follows:

$$S_s:=\big\{\mu_0\in\mathcal{Y}^\dagger:\mu_T(f)\to0\text{ as }T\to\infty,\forall f\in\mathcal{Y}\big\}$$

Observe that a constant function is $\mathcal{A}$-invariant and therefore $\mu_T(1)=\mu_0(1)$. Consequently, $S_s\subset\mathcal{M}_0(\mathbb{S})$. It is natural to define stabilizability and detectability as dual properties as follows:

Definition 4: The BSDE (8) is *stabilizable* if $\mathsf{R}(\mathcal{L})^\perp\subset S_s$.

Definition 5: The HMM $(\mathcal{A},h)$ is *detectable* if $\mathsf{N}(\mathcal{L}^\dagger)\subset S_s$.

Corollary 2: The HMM $(\mathcal{A},h)$ is detectable if and only if the BSDE (8) is stabilizable.

Remark 10: We compare with the detectability definition in [27, Definition V.1]. An HMM is said to be detectable if for any $\mu,\nu\in\mathcal{P}(\mathbb{S})$,

$$\text{Either }\ \mathsf{P}^\mu|_{\mathcal{Z}_T}\ne\mathsf{P}^\nu|_{\mathcal{Z}_T}\quad\text{or}\quad\|\mu_T-\nu_T\|_{\text{TV}}\overset{(T\to\infty)}{\longrightarrow}0$$

By Thm. 1, this statement is identical to the Def. 5.

Remark 11: A Markov process is said to be *ergodic* if there exists an invariant measure $\bar{\mu}$ such that for all $\mu_0\in\mathcal{P}(\mathbb{S})$, $\|\mu_T-\bar{\mu}\|_{\text{TV}}\overset{(T\to\infty)}{\longrightarrow}0$. For an ergodic process, consider $\tilde{\mu}_0\in\mathcal{M}_0(\mathbb{S})$ and pick $\mu_0^{(1)},\mu_0^{(2)}\in\mathcal{P}(\mathbb{S})$ and $c\in\mathbb{R}$ such that $\tilde{\mu}_0=c(\mu_0^{(1)}-\mu_0^{(2)})$. Then

$$\|\tilde{\mu}_T\|_{\text{TV}}\le c\|\mu_T^{(1)}-\bar{\mu}\|_{\text{TV}}+c\|\mu_T^{(2)}-\bar{\mu}\|_{\text{TV}}\overset{(T\to\infty)}{\longrightarrow}0$$

Therefore, if the state process is ergodic then $S_s=\mathcal{M}_0(\mathbb{S})$ and the BSDE is stabilizable (and from duality the HMM is detectable) irrespective of the observation function h.

F. Linear Gaussian model

Consider the linear Gaussian model:

$$\mathrm{d}X_t=A^{\intercal}X_t\,\mathrm{d}t+\sigma\,\mathrm{d}B_t,\quad X_0\sim N(m_0,\Sigma_0) \tag{12a}$$
$$\mathrm{d}Z_t=H^{\intercal}X_t\,\mathrm{d}t+\,\mathrm{d}W_t \tag{12b}$$

where the prior $N(m_0,\Sigma_0)$ is a Gaussian density with mean $m_0\in\mathbb{R}^d$ and variance $\Sigma_0\succeq0$, and the process noise $B=\{B_t:0\le t\le T\}$ is a B.M. It is assumed that X_0,B,W are mutually independent. The model parameters $A\in\mathbb{R}^{d\times d}$, $H\in\mathbb{R}^{d\times m}$, and $\sigma\in\mathbb{R}^{d\times n}$.

We impose the following restrictions:

- The control input $U=u$ is restricted to be a deterministic function of time. In particular, it does not depend upon the observations. For such a control input, the solution $Y=y$ of the BSDE is a deterministic function of time, and $V=0$. The BSDE becomes a PDE:

$$-\frac{\partial y_t}{\partial t}(x)=(\mathcal{A}y_t)(x)+x^{\intercal}Hu_t,\quad y_T=c1 \tag{13}$$

  where the lower-case notation is used to stress the fact that u and y are now deterministic functions of time.
- Consider a finite (d-) dimensional space of linear functions:

$$\mathsf{L}:=\{f:\mathbb{R}^d\to\mathbb{R}\ :\ f(x)=x^{\intercal}\tilde{f}\text{ where }\tilde{f}\in\mathbb{R}^d\}$$

  Then L is an invariant subspace for the dynamics (13). On L, expressing $y_t(x)=x^{\intercal}\tilde{y}_t$, the PDE reduces to an ODE:

$$-\frac{\mathrm{d}y_t}{\mathrm{d}t}=Ay_t+Hu_t,\quad y_T=0 \tag{14}$$

  where we have dropped the tilde for notational ease. The terminal condition is set to 0 because it is the only constant function which is also linear.

In this manner, we have recovered the dual control system (2) familiar from the linear systems theory (and discussed in Sec. I). The *only* assumption in going from BSDE (8) to the ODE (14) is that the control U is deterministic. Evidently, such a choice suffices for the purposes of linear Gaussian estimation. A detailed explanation of why this is the case appears in the companion paper [8, Sec. III-C].

Remark 12: In the stability analysis of the Kalman filter, detectability of the deterministic state-output model (1) is a standard condition used in proofs: The condition is both necessary and sufficient to show that the solution of the DRE converges [50, Thm. 3.7]. Because the two models are different (compare (1) and (12)), the fact that detectability of the deterministic model is also the appropriate condition for the stochastic model is somewhat curious. The above provides an explanation by showing that the dual control systems for the two models are the same (compare (2) and (14)).

G. Comparison between linear and nonlinear systems

For pedagogical reasons, it is useful to draw parallels between the linear deterministic and the nonlinear stochastic cases. In both cases, controllability (resp. observability) is a property of the range space (resp. null space) of a certain

TABLE I
LINEAR OPERATORS AND THEIR ADJOINTS FOR DETERMINISTIC (TOP) AND STOCHASTIC (BOTTOM) SYSTEMS

(state-output system Eq. (1))	(initial condition at $t=0$)	$\xi \in \mathbb{R}^d$	$\xrightarrow{\mathcal{L}^\dagger}$	$\{z_t : 0 \le t \le T\} \in L^2([0,T];\mathbb{R}^m)$	(output)
(state-input system Eq. (2))	(initial condition at $t=0$)	$y_0 \in \mathbb{R}^d$	$\xleftarrow{\mathcal{L}}$	$\{u_t : 0 \le t \le T\} \in L^2([0,T];\mathbb{R}^m)$	(control input)
(Zakai Eq. (3))	(measure at $t=0$)	$\mu \in \mathcal{M}(\mathbb{S})$	$\xrightarrow{\mathcal{L}^\dagger}$	$\{\sigma_t^\mu(h) : 0 \le t \le T\} \in L^2_{\mathcal{Z}}([0,T];\mathbb{R}^m)$	(un-norm. filter)
(BSDE Eq. (8))	(function at $t=0$)	$Y_0 \in C_b(\mathbb{S})$	$\xleftarrow{\mathcal{L}}$	$\{U_t : 0 \le t \le T\} \in L^2_{\mathcal{Z}}([0,T];\mathbb{R}^m)$	(control input)

linear operator (resp. the adjoint operator). Table I provides a comparison of the linear operators together with the domain and the co-domain spaces. Based on these definitions, Table II provides a side-by-side comparison of the controllability-observability duality in the two cases.

IV. EXPLICIT FORMULAE FOR FINITE STATE-SPACE

A. Notation

The state-space $\mathbb{S}$ is finite, namely $\mathbb{S} = \{1, 2, \ldots, d\}$. In this case, the space $C_b(\mathbb{S})$ and $\mathcal{M}(\mathbb{S})$ are both isomorphic to $\mathbb{R}^d$: a real-valued function f (resp. a measure μ) are both identified with a column vector in $\mathbb{R}^d$ where the i^{th} element of the vector represents $f(i)$ (resp. $\mu(i)$), and $\mu(f) = \mu^{\intercal} f$. In this manner, the observation function h is also identified with a matrix $H \in \mathbb{R}^{d\times m}$. We denote the j^{th} column and the i^{th} row of the matrix H by H^j and H_i, respectively.

The generator $\mathcal{A}$ of the Markov process is identified with a row-stochastic rate matrix $A \in \mathbb{R}^{d\times d}$ (the non-diagonal elements of A are non-negative and the row sum is zero). It acts on a function through: $\mathcal{A} : f \mapsto Af$. Its adjoint $\mathcal{A}^\dagger$ acts on measures through: $\mathcal{A}^\dagger : \mu \mapsto A^{\intercal}\mu$ where $A^{\intercal}$ is the matrix transpose.

B. Dual control system

The dual processes Y and V are $\mathbb{R}^d$ and $\mathbb{R}^{d\times m}$-valued, respectively. The BSDE (8) is finite-dimensional as follows:

$$
\begin{aligned}
-\,\mathrm{d}Y_t &= \Big(AY_t + HU_t + \sum_{j=1}^{m} \mathrm{diag}(H^j)V_t^j\Big)\,\mathrm{d}t - V_t\,\mathrm{d}Z_t,\\
Y_T &= c1
\end{aligned}
\tag{15}
$$

where 1 is now the d-dimensional column vector of all ones, $\mathrm{diag}(H^j)$ is a diagonal matrix formed from the j^{th} column of the matrix H, and V_t^j denotes the j^{th} column of the matrix V_t. The solution pair is $(Y, V) \in L^2_{\mathcal{Z}}([0,T];\mathbb{R}^d) \times L^2_{\mathcal{Z}}([0,T];\mathbb{R}^{d\times m})$.

We refer to the BSDE (15) as the nonlinear model (A, H) and the ODE (2) as the linear model (A, H).

C. Controllable subspace

The controllable space $\mathcal{C}$ is a subspace of $\mathbb{R}^d$. Note that if U is deterministic then $V = 0$ and the BSDE (15) reduces to the ODE (2). This means that the controllable subspace for the linear model $\mathrm{span}\{H,\, AH,\, A^2H, \ldots\} \subset \mathcal{C}$. Directly by applying Prop. 1,

$$
\begin{aligned}
\mathcal{C} = \mathrm{span}\,\big\{1, H,\, AH,\, A^2H,\, A^3H,\, \ldots,&\\
H\cdot H,\, A(H\cdot H),\, H\cdot(AH),\, A^2(H\cdot H), \ldots,&\\
H\cdot(H\cdot H),\, (AH)\cdot(H\cdot H),\, \ldots\big\}&
\end{aligned}
\tag{16}
$$

where the dot notation denotes the element-wise (Hadamard) product between two matrices. Again, one observes that the first row $\{H, AH, A^2H, \ldots\}$ is the same as the controllability matrix of the linear model (A, H). Therefore, if (A, H) is controllable in the sense of linear systems theory then the nonlinear model is also controllable (from duality the HMM is then observable.) The presence of additional entries in (16) means that the nonlinear model (A, H) may be controllable even though the linear model (A, H) is not. To highlight the difference, the following proposition gives a sufficient condition for controllability which does not depend upon A.

Proposition 3: The nonlinear model (A, H) is controllable if H is an injective map from $\mathbb{S}$ into $\mathbb{R}^m$ (the map is injective if and only if $H_i \neq H_j$ for all $i \neq j$). If $A = 0$ then the injective property is also necessary for controllability.

Proof: See Appendix E. ∎

Remark 13: In [25], a test for stochastic observability is given for the white noise observation model in the finite state-space settings. The test is given in terms of the dimension of the space of observable functions (Def. 2). Since the notation is somewhat different, we recall that [25] denotes the set of distinct possible values of noise-free observations by $h(\mathbb{S}) := \{h_1, \ldots, h_r\}$ (the value h_i should not to be confused with $h(i)$). Note $r \le d$ with equality holds when $h(i) \neq h(j)$ for all $i \neq j$. Next for each $h_k \in h(\mathbb{S})$, a diagonal projection matrix $P_{h_k} \in \mathbb{R}^{d\times d}$ is defined whose non-zero elements are $P_{h_k}(i,j) = 1$ if $h(i) = h_k$ and $i = j$. In terms of these matrices, the space of observable functions is shown to be [25, Lemma 9]

$$
\mathcal{O} = \mathrm{span}\,\big\{P_{n_0}AP_{n_1}AP_{n_2}\cdots AP_{n_k}1 : k \ge 0, n_i \in h(\mathbb{S})\big\}
$$

It is shown in Appendix F that $\mathcal{O} = \mathcal{C}$ (formula in (16)).

D. Controllability gramian

For the nonlinear model (A, H), the controllability gramian W is most directly expressed in terms of the solution operator

TABLE II
COMPARISON OF THE CONTROLLABILITY–OBSERVABILITY DUALITY FOR LINEAR AND NONLINEAR SYSTEMS

	Linear deterministic systems	**Nonlinear stochastic systems**
Function space for inputs and outputs	$\mathcal{U} = L^2([0,T];\mathbb{R}^m)$ $\langle u, v\rangle = \int_0^T u_t^{\intercal} v_t \,\mathrm{d}t$	$\mathcal{U} = L^2_{\mathcal{Z}}(\Omega \times [0,T];\mathbb{R}^m)$ $\langle U, V\rangle = \tilde{\mathsf{E}}\Big(\int_0^T U_t^{\intercal} V_t \,\mathrm{d}t\Big)$
Function space for the dual state	$\mathcal{Y} = \mathbb{R}^d$ $\langle x, y\rangle = x^{\intercal} y$	$\mathcal{Y} = C_b(\mathbb{S})$, $\mathcal{Y}^\dagger = \mathcal{M}(\mathbb{S})$ $\langle \mu, y\rangle = \mu(y)$
Linear operators	$\mathcal{L}: \mathcal{U} \to \mathcal{Y}$, $u \mapsto y_0$ by ODE (2) $\mathcal{L}^\dagger: \mathcal{Y} \to \mathcal{U}$, $x_0 \mapsto \{z_t : 0 \le t \le T\}$ by ODE (1)	$\mathcal{L}: \mathcal{U} \times \mathbb{R} \to \mathcal{Y}$, $(U, c) \mapsto Y_0$ by BSDE (8) $\mathcal{L}^\dagger: \mathcal{Y}^\dagger \to \mathcal{U} \times \mathbb{R}$, $\mu \mapsto (\{\sigma_t(h) : 0 \le t \le T\}, \mu(1))$ by Zakai Eq. (3)
Controllability	$\mathsf{R}(\mathcal{L}) = \mathbb{R}^d$	$\overline{\mathsf{R}(\mathcal{L})} = \mathcal{Y}$
Observability	$\mathsf{N}(\mathcal{L}^\dagger) = \{0\}$	$\mathsf{N}(\mathcal{L}^\dagger) = \{0\}$
Duality	$\mathsf{R}(\mathcal{L})^\perp = \mathsf{N}(\mathcal{L}^\dagger) \quad \Longrightarrow$	A state-output system is observable iff the dual control system is controllable.

of the Zakai equation defined as follows:

$$\mathrm{d}\Psi_t = A^{\intercal}\Psi_t \,\mathrm{d}t + \sum_{j=1}^m \mathrm{diag}(H^j)\Psi_t \,\mathrm{d}Z_t^j, \quad \Psi_0 = I_d$$

where I_d is the $d \times d$ identity matrix. In Appendix G, it is shown that

$$\mathsf{W} = 11^{\intercal} + \tilde{\mathsf{E}}\Big(\int_0^T \Psi_t^{\intercal} H H^{\intercal} \Psi_t \,\mathrm{d}t\Big) \tag{17}$$

Since W is a deterministic matrix, controllability admits to a rank condition test:

$$(A, H) \text{ is controllable} \quad \Longleftrightarrow \quad \mathsf{W} \text{ is full rank}$$

E. Stabilizability

Because A is a stochastic matrix, a simple application of the Geršgorin circle theorem shows that the eigenvalues of A are either in the open left-half-plane or at zero, and

$$S_s^\perp = S_0 := \{f \in \mathbb{R}^d : \; Af = 0\}$$

Therefore, stabilizability is equivalent to an inclusion property:

$$(A, H) \text{ is stabilizable} \quad \Longleftrightarrow \quad S_0 \subset \mathcal{C}$$

A meaningful characterization is possible by partitioning the finite state-space $\mathbb{S}$ into r *ergodic classes* $\{\mathbb{S}_k : k = 1, 2, \ldots, r\}$ as follows:

1) $\mathbb{S} = \bigcup_{k=1}^r \mathbb{S}_k$ where $\mathsf{P}([X_t \in \mathbb{S}_l] \mid [X_0 \in \mathbb{S}_k]) = 0$ for all $t \ge 0$ and $l \ne k$.
2) By choosing an appropriate coordinate, the matrix

$$A = \begin{pmatrix} A_1 & 0 & \cdots & 0 \\ 0 & A_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_r \end{pmatrix}$$

where A_k is a row stochastic matrix on $\mathbb{S}_k$ for $k = 1, 2, \ldots, r$.

The Markov process is said to be *ergodic* if it has a single ergodic class ($r = 1$). The following proposition provides an explicit characterization of stabilizability:

Proposition 4: Consider the nonlinear model (A, H) and an associated ergodic partition $\mathbb{S} = \cup_{k=1}^r \mathbb{S}_k$. Then

- If $\mathbb{S}$ has a single ergodic class ($r = 1$) then (A, H) is stabilizable.
- If $r > 1$ then (A, H) is stabilizable if and only if the indicator functions $1_{\mathbb{S}_k} \in \mathcal{C}$ for $k = 1, 2, \ldots, r$.

Proof: The matrix A has exactly r eigenvalues at zero with an r-dimensional eigenspace $S_0 = \mathrm{span}\{1_{\mathbb{S}_k} : k = 1, 2, \ldots, r\}$. ■

Remark 14: The two bullets in Prop. 4 correspond to the ergodic signal case and the non-ergodic signal case as discussed in Sec. I. The first bullet says that if the Markov process is ergodic then the dual control system is stabilizable irrespective of H. For the non-ergodic case, the second bullet provides a simple condition for stabilizability. It can be shown that the condition is both necessary and sufficient for the optimal filter to asymptotically detect the correct ergodic class [7, Ch. 8] (see also [14]).

V. CONCLUSIONS AND DIRECTIONS FOR FUTURE WORK

In the study of deterministic linear and nonlinear systems, duality has played a central role in defining, interpreting, and using the property of observability. In this paper, we presented the first such dual control system for studying stochastic observability of HMMs with white noise observations. We related the dual control system to both nonlinear filtering (the Zakai equation) and the linear Gaussian model. The latter relationship is shown to recover the classical duality between controllability and observability of linear systems. In fact, the development is entirely parallel in the nonlinear and the linear systems. This is shown with the aid of the Table II with a side-by-side comparison.

Because the stress in this paper was on the duality between controllability and observability, we did not explicitly relate the process Y and the hidden process X. In fact, it is possible as well and yields the following attractive formula

$$\mathsf{E}\big(Y_T(X_T)\big) = \mu(Y_0) - \mathsf{E}\Big(\int_0^T U_t^{\mathsf{T}}\,\mathrm{d}Z_t\Big)$$

The formula is the starting point for the companion paper (part II) [8] which is concerned with the use of duality to express the nonlinear filtering problem as a dual optimal control problem. The connection to optimal control theory opens up several avenues of research, namely study of asymptotic stability of the optimal filter, a definition of suitable supply rates that yields useful dissipative characterizations of the HMM, and opportunities for algorithm design. These are also discussed at length as part of the conclusions of the companion paper.

A natural question also is to relate this work to the study of deterministic models. Unfortunately, our work crucially relies on additive Gaussian form of the measurement noise. This effectively precludes the zero measurement noise case. Even worse, there are well known counter-examples in stochastic filtering theory that show that the observability property in deterministic and stochastic models is fundamentally dissimilar [51]. On the other hand, considering the limit as the covariance of the measurement noise goes to zero may be meaningful from robustness perspective, and may yield useful insights also for the deterministic models. It is hoped that duality helps bring together the two communities of researchers studying deterministic and stochastic models.

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Appendix

A. Proof of Thm. 1

- ($1 \iff 2$) It directly follows from the relative entropy formula (5).
- ($2 \implies 3$) The Zakai equation (3) with $f=1$ gives
$$\sigma_t(1) = 1 + \int_0^t \sigma_s(h)^{\mathsf{T}}\,\mathrm{d}Z_s = 1 + \int_0^t \sigma_s(1)\pi_s(h)^{\mathsf{T}}\,\mathrm{d}Z_s \tag{18}$$
where the formula (4) is used to obtain the second equality. It follows that if $\pi_s^\mu(h) = \pi_s^\nu(h)$ for all $0 \le s \le t$ then $\sigma_t^\mu(1) = \sigma_t^\nu(1)$. Using (4), $\sigma_t^\mu(h) = \sigma_t^\nu(h)$.
- ($3 \implies 2$) From the first equality in (18), it follows that if $\sigma_s^\mu(h) = \sigma_s^\nu(h)$ for all $0 \le s \le t$ then $\sigma_t^\mu(1) = \sigma_t^\nu(1)$. Using (4), $\pi_t^\mu(h) = \pi_t^\nu(h)$.

B. Proof of Thm. 2

The space $\mathcal{U} = L^2_{\mathcal{Z}}([0,T];\mathbb{R}^m)$ is a Hilbert space with the inner product $\langle U,V\rangle_{\mathcal{U}} := \tilde{\mathsf{E}}\Big(\int_0^T U_t^{\mathsf{T}} V_t\,\mathrm{d}t\Big)$. Therefore, $\mathcal{U}\times\mathbb{R}$ is a Hilbert space with a natural inner product $\langle (U,c),(V,d)\rangle_{\mathcal{U}\times\mathbb{R}} := \langle U,V\rangle_{\mathcal{U}} + cd$. For a function $f\in\mathcal{Y}$ and a measure $\mu\in\mathcal{Y}^\dagger$, the duality pairing is denoted by $\langle \mu, f\rangle_{\mathcal{Y}} := \mu(f)$. Let $U\in\mathcal{U}$, $c\in\mathbb{R}$, and $\mu\in\mathcal{Y}^\dagger$. By linearity, $\mathcal{L}(U,c) = \mathcal{L}(U,0) + c1$ and therefore

$$\langle \mu, \mathcal{L}(U,c)\rangle_{\mathcal{Y}} = \langle \mu, \mathcal{L}(U,0)\rangle_{\mathcal{Y}} + c\mu(1)$$

Thus, the main calculation is to show $\langle \mu, \mathcal{L}(U,0)\rangle_{\mathcal{Y}} = \langle \sigma(h), U\rangle_{\mathcal{U}}$ where $\sigma(h) = \{\sigma_t(h)\in\mathbb{R}^m : 0\le t\le T\}$ solves the Zakai equation (3) with $\sigma_0 = \mu$. This is done by using the Itô-Wentzell formula for measure valued processes [52, Thm. 1.1] (note here that all stochastic processes are forward adapted),

$$\begin{aligned}\mathrm{d}\big(\sigma_t(Y_t)\big) &= \big(\sigma_t(\mathcal{A}Y_t)\,\mathrm{d}t + \sigma_t(h^{\mathsf{T}}Y_t)\,\mathrm{d}Z_t\big) + \sigma_t(h^{\mathsf{T}}V_t)\,\mathrm{d}t \\ &\quad + \big(\sigma_t(-\mathcal{A}Y_t - h^{\mathsf{T}}U_t - h^{\mathsf{T}}V_t)\,\mathrm{d}t + \sigma_t(V_t^{\mathsf{T}})\,\mathrm{d}Z_t\big) \\ &= -U_t^{\mathsf{T}}\sigma_t(h)\,\mathrm{d}t + \sigma_t(h^{\mathsf{T}}Y_t + V_t^{\mathsf{T}})\,\mathrm{d}Z_t\end{aligned}$$

Integrating both sides,

$$\sigma_T(Y_T) - \mu(Y_0) = -\int_0^T U_t^{\mathsf{T}}\sigma_t(h)\,\mathrm{d}t + \int_0^T \sigma_t(h^{\mathsf{T}}Y_t + V_t^{\mathsf{T}})\,\mathrm{d}Z_t$$

With $Y_T = 0$, because Z is a $\tilde{\mathsf{P}}$-B.M.,

$$\langle \mu, \mathcal{L}(U,0)\rangle_{\mathcal{Y}} = \mu(Y_0) = \tilde{\mathsf{E}}\Big(\int_0^T U_t^{\mathsf{T}}\sigma_t(h)\,\mathrm{d}t\Big) = \langle \sigma(h), U\rangle_{\mathcal{U}}$$

In summary,

$$\langle \mu, \mathcal{L}(U,c)\rangle_{\mathcal{Y}} = \langle \sigma(h), U\rangle_{\mathcal{U}} + c\mu(1) = \big\langle \mathcal{L}^\dagger\mu, (U,c)\big\rangle_{\mathcal{U}\times\mathbb{R}}$$

C. Proof of Prop. 1

The proof is adapted from [53, Thm. 3.2]. For notational ease, we assume $m=1$. For $m>1$, the procedure is repeated for each component of h. The definition of $\mathsf{N}(\mathcal{L}^\dagger)$ is:

$$\mu\in\mathsf{N}(\mathcal{L}^\dagger) \Leftrightarrow \mu(1) = 0 \text{ and } \sigma_t(h)\equiv 0 \quad \forall\, t\in[0,T]$$

Since $\mathsf{N}(\mathcal{L}^\dagger)$ is the annihilator of $\mathsf{R}(\mathcal{L})$, we have $1, h\in\mathsf{R}(\mathcal{L})$. Consider next the Zakai equation (3) with the initial condition $\mu\in\mathsf{N}(\mathcal{L}^\dagger)$ and $f=h$:

$$\sigma_t(h) = \mu(h) + \int_0^t \sigma_s(\mathcal{A}h)\,\mathrm{d}s + \int_0^t \sigma_s(h^2)\,\mathrm{d}Z_s$$

Since t is arbitrary, the left-hand side is identically zero for all $t\in[0,T]$ if and only if

$$\mu(h) = 0, \quad \sigma_t(\mathcal{A}h)\equiv 0, \quad \sigma_t(h^2)\equiv 0 \quad \forall\, t\in[0,T]$$

and in particular, this implies $\mathcal{A}h, h^2\in\mathsf{R}(\mathcal{L})$.

The subspace $\mathcal{C}$ is obtained by continuing to repeat the steps ad infinitum: If at the conclusion of the k^{th} step, we find a function $g\in\mathcal{C}$ such that $\sigma_t(g)\equiv 0$ for all $t\in[0,T]$. Then through the use of the Zakai equation,

$$\mu(g) = 0, \quad \sigma_t(\mathcal{A}g)\equiv 0, \quad \sigma_t(hg)\equiv 0 \quad \forall\, t\in[0,T]$$

so $\mathcal{A}g, hg\in\mathcal{C}$. By construction, because $\mu\in\mathsf{N}(\mathcal{L}^\dagger)$, $\mathcal{C} = \mathsf{R}(\mathcal{L})$.

D. Proof of Prop. 2

Suppose $f \in \mathsf{R}(\mathsf{W})$. Then there exists $\mu \in \mathcal{M}(\mathbb{S})$ such that $\mathsf{W}\mu = f$. Let $\big(U, \mu(1)\big) = \mathcal{L}^\dagger \mu$, and apply the control U to the BSDE with terminal condition $Y_T = \mu(1)1$. Then

$$Y_0 = \mathcal{L}(U, \mu(1)) = \mathcal{L}\mathcal{L}^\dagger \mu = \mathsf{W}\mu = f$$

Suppose another $(\tilde{U}, c) \in \mathcal{U} \times \mathbb{R}$ gives $\mathcal{L}(\tilde{U}, c) = f$. Then

$$\begin{aligned} 0 &= \big\langle \mu, \mathcal{L}\big(U - \tilde{U}, \mu(1) - c\big)\big\rangle_{\mathcal{Y}} \\ &= \big\langle \mathcal{L}^\dagger \mu, \big(U - \tilde{U}, \mu(1) - c\big)\big\rangle_{\mathcal{U}\times\mathbb{R}} \\ &= \big\langle \big(U, \mu(1)\big), \big(U - \tilde{U}, \mu(1) - c\big)\big\rangle_{\mathcal{U}\times\mathbb{R}} \end{aligned}$$

The minimum norm formula (11) follows because of this orthogonality property.

E. Proof of Proposition 3

Step 1: We first provide the proof for the case when $m = 1$. In this case, H is a column vector and H_i denotes its i^{th} element. We claim that if $H_i \neq H_j$ for all $i \neq j$, then

$$\text{span}\{1,\, H,\, H \cdot H,\, \ldots,\, \underbrace{H \cdot H \cdots H}_{(d-1)\text{ times}}\} = \mathbb{R}^d \qquad (19)$$

where (as before) the dot denotes the element-wise product. Assuming that the claim is true, the result easily follows because the vectors on left-hand side are contained in $\mathsf{R}(\mathcal{L})$ (see (16)). It remains to prove the claim. For this purpose, express the left-hand side of (19) as the column space of the following matrix:

$$\begin{pmatrix} 1 & H_1 & H_1^2 & \cdots & H_1^{d-1} \\ 1 & H_2 & H_2^2 & \cdots & H_2^{d-1} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 1 & H_d & H_d^2 & \cdots & H_d^{d-1} \end{pmatrix}$$

This matrix is easily seen to be full rank by using the Gaussian elimination:

$$\begin{pmatrix} 1 & H_1 & H_1^2 & \cdots & H_1^{d-1} \\ 0 & H_2 - H_1 & H_2^2 - H_1^2 & \cdots & H_2^{d-1} - H_1^{d-1} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & \prod_{i=1}^{d-1}(H_d - H_i) \end{pmatrix}$$

The diagonal elements are non-zero because $H_i \neq H_j$.

Step 2: In the general case, H is a $d \times m$ matrix and H_i denotes its i^{th} row. We claim that if $H_i \neq H_j$ for all $i \neq j$ then there exists a vector $\tilde{H}$ in the column span of H such that $\tilde{H}_i \neq \tilde{H}_j$ for all $i \neq j$. Assuming that the claim is true, the result follows from the $m = 1$ case by considering (19) with $\tilde{H}$. It remains to prove the claim. Let $\{e_1, \ldots, e_d\}$ denote the canonical basis in $\mathbb{R}^d$. The assumption means $(e_i - e_j)^{\mathsf{T}} H$ is a non-zero row-vector in $\mathbb{R}^m$ for all $i \neq j$. Therefore, the null-space of $(e_i - e_j)^{\mathsf{T}} H$ is a $(m-1)$-dimensional hyperplane in $\mathbb{R}^m$. Since there are only finite such hyperplanes, there must exist a vector $a \in \mathbb{R}^m$ such that $(e_i - e_j)^{\mathsf{T}} H a \neq 0$ for all $i \neq j$. Pick such an a and define $\tilde{H} := Ha$.

Step 3: To show the necessity of the injective property when $A = 0$, assume $H_i = H_j$ for some $i \neq j$. Then the corresponding rows are identical, so the rank is less than d.

F. Proof that $\mathcal{O} = \mathcal{C}$ in Rem. 13

We begin with $A = 0$ case so

$$\mathcal{C} = \text{span}\{1, H, \text{diag}(H)H, \text{diag}(H)^2 H, \ldots\}$$

Since $\text{diag}(H)^n H = [h^{n+1}(1), \ldots, h^{n+1}(d)]^{\mathsf{T}}$, an element of $f \in \mathcal{C}$ can be expressed by

$$f = \sum_{j=0}^{\infty} a_j [h^j(1), \ldots, h^j(d)]^{\mathsf{T}} = \sum_{k=1}^{r} \sum_{j=0}^{\infty} a_j h_k^j P_{h_k} 1 \in \mathcal{O}$$

Therefore, $\mathcal{C} \subset \mathcal{O}$. To show $\mathcal{O} \subset \mathcal{C}$, let

$$f = \sum_{k=1}^{r} b_k P_{h_k} 1$$

It suffices to show that there exists $\{a_j : j = 0, 1, \ldots\}$ such that $b_k = \sum_{j=0}^{\infty} a_j h_k^j$ for all $k = 1, \ldots, r$. In fact, such a_j can be found explicitly by setting $a_j = 0$ for $j \geq r$ and inverting the following matrix:

$$\begin{pmatrix} 1 & h_1 & h_1^2 & \cdots & h_1^{r-1} \\ 1 & h_2 & h_2^2 & \cdots & h_2^{r-1} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 1 & h_r & h_r^2 & \cdots & h_r^{r-1} \end{pmatrix}$$

It is invertible because h_i are distinct (the proof is by Gaussian elimination as before).

For general $A \neq 0$ case, we repeat the same procedure as above for arbitrary matrices M_1 and M_2 which are multiples of A and $\text{diag}(H)$, to claim that

$$\begin{aligned} &\text{span}\{M_1 M_2 H, M_1 \,\text{diag}(H) M_2 H, M_1 \,\text{diag}(H)^2 M_2 H, \ldots\} \\ &\qquad = \text{span}\{M_1 P_{h_k} M_2 H : k = 1, \ldots, r\} \end{aligned}$$

The proposition is proved by repeating this for countable times.

G. Formula (17) for the gramian

For a given measure $\mu \in \mathbb{R}^d$, $\mathsf{W}\mu = Y_0$ is obtained by solving the BSDE

$$\begin{aligned} -\,\mathrm{d}Y_t &= \big(AY_t + HH^{\mathsf{T}}\sigma_t + \sum_{j=1}^{m} H^j \cdot V_t^j\big)\,\mathrm{d}t - V_t\,\mathrm{d}Z_t \\ Y_T &= (\mu^{\mathsf{T}}1)1 \end{aligned}$$

Consider the process

$$\Theta_t := \Psi_t^{\mathsf{T}} Y_t + \int_0^t \Psi_s^{\mathsf{T}} HH^{\mathsf{T}} \sigma_s \,\mathrm{d}s, \quad 0 \leq t \leq T$$

Then by the Itô product formula,

$$\mathrm{d}\Theta_t = \sum_{j=1}^{m} \Psi_t^{\mathsf{T}} \big(H^j Y_t + V_t^j\big)\,\mathrm{d}Z_t^j$$

Therefore, $\{\Theta_t : 0 \leq t \leq T\}$ is a $\tilde{\mathsf{P}}$-martingale. In particular,

$$Y_0 = \Theta_0 = \tilde{\mathsf{E}}(\Theta_T) = \tilde{\mathsf{E}}\big(\Psi_T^{\mathsf{T}} 11^{\mathsf{T}} \mu + \int_0^T \Psi_t^{\mathsf{T}} HH^{\mathsf{T}} \sigma_t \,\mathrm{d}t\big)$$

Since the un-normalized filter is given by $\sigma_t = \Psi_t \mu$,

$$\mathsf{W}\mu = \tilde{\mathsf{E}}\Big(\Psi_T^{\mathsf{T}} 11^{\mathsf{T}} + \int_0^T \Psi_t^{\mathsf{T}} HH^{\mathsf{T}} \Psi_t \,\mathrm{d}t\Big)\mu$$

Finally, $\tilde{\mathsf{E}}(\Psi_T^{\mathsf{T}} 11^{\mathsf{T}}) = 11^{\mathsf{T}}$ because $\frac{\mathrm{d}}{\mathrm{d}t}\tilde{\mathsf{E}}(\Psi_t^{\mathsf{T}} 1) = 0$.

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