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Smooth Connes–Thom isomorphism, cyclic homology, and equivariant quantization

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Dedicated to Professor Alain Connes on the Occasion of his 75th Birthday

Abstract: Using a smooth version of the Connes–Thom isomorphism in Grensing's bivariant K-theory for locally convex algebras, we prove an equivariant version of the Connes–Thom isomorphism in periodic cyclic homology. As an application, we prove that periodic cyclic homology is invariant with respect to equivariant strict deformation quantizations.

1. Introduction

In this paper, we are interested in the following C^*-dynamical systems. Let $\mathbb{R}^n$ act strongly continuously on a C^*-algebra A by an action α. Also let G be a finite group acting on A by β. Let $\rho : G \to GL_n(\mathbb{R})$ be a representation of G. As G is finite, there is always a G-invariant inner product on $\mathbb{R}^n$. Without loss of generality, we assume in this paper that the G representation ρ preserves the standard metric on $\mathbb{R}^n$. We assume also that the actions α and β are compatible, i.e.,

$$\beta_g \alpha_x = \alpha_{\rho_g(x)} \beta_g, \quad \text{for all } g \in G,\ x \in \mathbb{R}^n. \tag{1.1}$$

When ρ is the trivial group homomorphism, the actions α and β commute. The above compatibility condition states in fact that there is a strongly continuous action of the semi-direct product $\mathbb{R}^n \rtimes G$ on A. With the above data, we consider the crossed product algebra $A \rtimes_\alpha \mathbb{R}^n$, $A \rtimes_\beta G$, and $(A \rtimes_\alpha \mathbb{R}^n) \rtimes_{\beta \rtimes \rho} G$. We expect that our results naturally generalize to compact groups. We have decided to focus our presentation on finite groups motivated by the applications, cf. [ELPW10, TY14].

Connes established in [Con81] a far-reaching generalization of the Thom isomorphism theorem in K-theory to noncommutative geometry,

$$\mathrm{K}_\bullet(A) \cong \mathrm{K}_{\bullet+n}(A \rtimes_\alpha \mathbb{R}^n),$$

with the G-equivariant version proved by Kasparov [Kas95]

$$\mathrm{K}_\bullet(A \rtimes G) \cong \mathrm{K}_\bullet\Big(\big((A \otimes \mathbb{C}_n) \rtimes \mathbb{R}^n\big) \rtimes G\Big),$$

where $\mathbb{C}_n$ is the complexified Clifford algebra associated to $\mathbb{R}^n$. This result quickly becomes one of the key tools in the study of noncommutative geometry, e.g. [Con86, Con94, Kas95].

In cyclic cohomology, Elliott, Natsume, and Nest in [ENN88] proved a cyclic version of the Connes–Thom isomorphism. Let A^∞ be the smooth subalgebra of A of smooth vectors for the action α. Then A^∞ is a Fréchet algebra with respect to a system of seminorms (p_i), which is induced from the norm of A. Elliott, Natsume, and Nest in [ENN88] proved the following identity in periodic cyclic cohomology.

$$\mathrm{HP}^\bullet(A^\infty) \cong \mathrm{HP}^{\bullet+1}(A^\infty \rtimes \mathbb{R}), \qquad \mathrm{HP}^\bullet(A^\infty) \cong \mathrm{HP}^{\bullet+2}(A^\infty \rtimes \mathbb{R}^2).$$

Accordingly, they conclude by induction that

$$\mathrm{HP}^\bullet(A^\infty) \cong \mathrm{HP}^{\bullet+n}(A^\infty \rtimes \mathbb{R}^n).$$

In the setting with a G-representation $\rho : G \to GL_n(\mathbb{R})$ and a G-action β on A, G also acts on A^∞. So we consider the crossed product algebras $A^\infty \rtimes_\beta G$ and $(A^\infty \rtimes_\alpha \mathbb{R}^n) \rtimes_{\beta\rtimes\rho} G$. Elliott, Natsume, and Nest [ENN88] studied the Thom isomorphism in cyclic cohomology. The ideas of [ENN88] and our paper both go back to the work of [Kas95] in the study of the Baum-Connes conjecture. In this paper we work on the cyclic homology side, and provide a dual map of the construction in [ENN88] with emphasis on the connection to K-theory and Dirac operator.

In this article, we prove an equivariant version of the cyclic Connes–Thom isomorphism in periodic cyclic homology for locally convex algebras with a new approach.

Theorem 1.1 (Corollary 4.12). *With the above notations and assumptions, we have*

$$\mathrm{HP}_\bullet\Big(\big((A^\infty \otimes \mathbb{C}_n) \rtimes_\alpha \mathbb{R}^n\big) \rtimes_{\beta\rtimes\rho} G\Big) \cong \mathrm{HP}_\bullet(A^\infty \rtimes_\beta G). \tag{1.2}$$

To prove Theorem 1.1, we present the following smooth version of the equivariant Connes–Thom isomorphism using Grensing's bivariant K-theory [Gre12] for locally convex algebras, from which Theorem 1.1 follows as a corollary.

Theorem 1.2 (Theorem 4.1). *With the above notations and assumptions, we have the following equation,*

$$\mathrm{H}\big(((A^\infty \otimes \mathbb{C}_n) \rtimes_\alpha \mathbb{R}^n\big) \rtimes_{\beta\rtimes\rho} G) \cong \mathrm{H}(A^\infty \rtimes_\beta G),$$

where H *is a split-exact, diffotopy invariant,* $\mathcal{K}^\infty$*-stable functor on the category of locally convex algebras, and* $\mathbb{C}_n$ *is the complexified Clifford algebra associated with* $\mathbb{R}^n$*, carrying the trivial action of* $\mathbb{R}^n$ *and the action of* G *which is induced from the* G*-action on* $\mathbb{R}^n$*.*

Our proof of Theorem 1.2 is inspired by the KK-theory proof of the Connes-Thom isomorphism [FS81]. Our strategy is to develop a smooth version of the Connes-Thom isomorphism by working with smooth subalgebras of C^*-algebras. The key ingredient in our proof of Theorem 1.2 is the pseudodifferential calculus for strongly continuous $\mathbb{R}^n$-actions which was introduced by Connes in [Con80,Baa88a,Baa88b]. Using this pseudodifferential calculus with noncommutative symbols, we introduce a smooth version of the Dirac and dual Dirac operator for $\mathbb{R}^n$-actions, which leads us to the desired isomorphism in Grensing's theory. Another important ingredient in our proof is an equivariant Takesaki–Takai duality that there is a G-equivariant isomorphism between $(A^\infty \rtimes_\alpha \mathbb{R}^n) \rtimes_{\widehat{\alpha}} \widehat{\mathbb{R}^n}$ and $A^\infty \otimes \mathcal{K}^\infty$, where $\widehat{\mathbb{R}^n}$ is the Pontryagin dual group with a natural action $A^\infty \rtimes_\alpha \mathbb{R}^n$ and $\mathcal{K}^\infty$ is the algebra of smoothing operators on $\mathbb{R}^n$.

As an application of Theorem 1.1, we study the periodic cyclic homology of equivariant strict deformation quantization introduced by Rieffel [Rie93].

Let J be a real antisymmetric $n \times n$ matrix. Define $GL_n(J)$ to be the group of invertible matrices g such that $g^t Jg = J$, where $SL_n(\mathbb{R}, J) := SL_n(\mathbb{R}) \bigcap GL_n(J)$. Assume that $\rho : G \to GL_n(\mathbb{R})$ takes value in $SL_n(\mathbb{R}, J)$. Rieffel [Rie93] constructed a strict deformation quantization A_J^∞. It was checked in [TY14] that the actions α and β of $\mathbb{R}^n$ and G extend to A_J^∞ with the same properties. Using Theorem 1.1, we prove the following result about the cyclic homology of $A_J^\infty \rtimes G$, which is the cyclic homology analog of the K-theory result obtained by the last two authors in [TY14] (see also [ELPW10]). This result can also be viewed as an equivariant generalization of the homotopy invariance of periodic cyclic homology to locally convex algebras, c.f. [Con85,Get93,Goo85,Yas17].

Theorem 1.3 (Corollary 5.7). *For a general deformation quantization A_J^∞ defined by Equation (5.1),* $\mathrm{HP}_\bullet(A_J^\infty \rtimes G)$ *is independent of the J parameter.*

In this paper, we focus on cyclic homology, e.g., Theorem 1.1 and Theorem 1.3. In [CTY], we will discuss the cyclic cohomology version of the results in this paper and construct explicit cocycles to detect the K-theory elements constructed in [DL13]. Theorem 1.2 also applies to the bivariant kk-theory, e.g. [CT06], as an example of the functor H.

This article is organized as follows. In Sect. 2, we recall the material about Grensing's bivariant K-theory of locally convex algebras. In Sect. 3, we prove an equivariant Takesaki–Takai duality theorem. Theorem 1.1 and 1.2 are proved in Sect. 4. Theorem 1.3 together with examples are presented in Sect. 5. In Appendix A, we develop an equivariant version of Grensing's results [Gre12], which is needed in our proofs. In Appendix B, we briefly recall Connes' pseudodifferential calculus for $\mathbb{R}^n$-actions, and some analytic properties about the symbol of the dual Dirac operator introduced in the proof of Theorem 1.2 is discussed in Appendix C.

2. Preliminaries

In this section, we briefly recall some preliminary material about the bivariant K-theory of (graded) locally convex algebras. We remark that analogous to the equivariant KK-theory [Kas95], it may seem natural to introduce and work with a G-equivariant version of Grensing's bivariant K-theory. To avoid the technical difficulty in defining an equivariant Kasparov module for locally convex algebras, we have chosen to work with the nonequivariant version of bivariant K-theory [Gre12] in this paper.

2.1. Locally convex algebra. Recall that a locally convex algebra $\mathcal{A}$ is a complete locally convex space endowed with a continuous multiplication $\mathcal{A} \times \mathcal{A} \to \mathcal{A}$ such that for every continuous seminorm $\mathfrak{p}$ on $\mathcal{A}$ there is a continuous seminorm $\mathfrak{q}$ on $\mathcal{A}$ for which the inequality

$$\mathfrak{p}(ab) \leq \mathfrak{q}(a)\mathfrak{q}(b)$$

holds for all $a, b \in \mathcal{A}$. Throughout the paper, we will always work with projective tensor product of locally convex algebras. By a slight abuse of notation, we will use the symbol $\mathcal{A} \otimes \mathcal{B}$ to denote the *projective tensor product* of two locally convex algebras $\mathcal{A}$ and $\mathcal{B}$. When a locally convex algebra $\mathcal{A}$ is equipped with a grading operator $\epsilon \in \mathrm{Aut}(\mathcal{A})$ such that $\epsilon^2 = 1$, we call $(\mathcal{A}, \epsilon)$ a graded locally convex algebra. For the following definitions, we will restrict ourselves to Fréchet algebras only.

Let $\mathcal{A}$ be a (graded) Fréchet algebra. We will use $\mathcal{S}(\mathbb{R}^n, \mathcal{A})$ to denote the locally convex space of Schwartz functions on $\mathbb{R}^n$ with value in $\mathcal{A}$. Suppose that $\mathbb{R}^n$, as an abelian group, acts smoothly isometrically on the algebra $\mathcal{A}$ by the group homomorphism $\alpha : \mathbb{R}^n \to \mathrm{Aut}(\mathcal{A})$. The following twisted convolution product $*$ makes $\mathcal{S}(\mathbb{R}^n, \mathcal{A})$ into a Fréchet algebra, i.e.,

$$(f * f')(x) = \int_{\mathbb{R}^n} f(y)\alpha_y(f'(x-y))dy,$$

where dy is the Lebesgue measure on $\mathbb{R}^n$. This algebra will be denoted by $\mathcal{A} \rtimes \mathbb{R}^n$.

An automorphism $u \colon \mathcal{A} \to \mathcal{A}$ is said to be *almost isometric* if for all seminorms $\|\cdot\|_\nu$, there exists a positive constant C_ν such that

$$\|u(a)\|_\nu \leq C_\nu \|a\|_\nu, \quad \forall a \in \mathcal{A}.$$

We assume that G is a discrete group that acts on $\mathcal{A}$ from the left by almost isometric automorphisms. And by a G-Fréchet algebra, we mean a Fréchet algebra equipped with an almost isometric G-action.

Let $\mathcal{A}$ be a G-Fréchet algebra.

Definition 2.1 (cf. [Per08, Sec. 7]). We define the *smooth crossed product* $\mathcal{A} \rtimes G$ as the completion of the vector space spanned by the elements $(a, g) \in \mathcal{A} \oplus G$ in the system of seminorms

$$\left\| \sum_{g \in G} (a_g, g) \right\|_\nu = C_\alpha \sum_{g \in G} \left\| g^{-1}(a_g) \right\|_\nu,$$

with multiplication given by

$$(a, g) \cdot (a', g') = (ag(a'), gg'), \quad a, a' \in \mathcal{A}, \; g, g' \in G.$$

It can be checked directly that the seminorms $\|\cdot\|_\nu$ make $\mathcal{A} \rtimes G$ into a Fréchet algebra. Furthermore, if $\mathcal{A}$ is graded, then $\mathcal{A} \rtimes G$ is also graded. Note that, for notational conveniences, we do not distinguish between seminorms of $\mathcal{A} \rtimes G$ and seminorms of $\mathcal{A}$.

2.2. KK-theory for locally convex algebras. In this subsection we briefly recall the smooth bivariant K-theory introduced by Grensing [Gre12].

2.2.1. Abstract Kasparov module Let $\mathcal{A}$ and $\mathcal{B}$ be (graded) locally convex algebras. An abstract Kasparov module from $\mathcal{A}$ to $\mathcal{B}$ is given by a quadruple $(\alpha, \bar{\alpha}, U, \hat{\mathcal{B}})$, where $\hat{\mathcal{B}}$ is a (graded) algebra containing $\mathcal{B}$, and $U \in \hat{\mathcal{B}}$ is an invertible element, and $\alpha, \bar{\alpha} : \mathcal{A} \to \hat{\mathcal{B}}$ are two (degree 0) morphisms such that

$$\mathcal{A} \to \hat{\mathcal{B}} : \ a \mapsto \alpha(a) - U^{-1}\bar{\alpha}(a)U$$

is a $\mathcal{B}$-valued continuous map (with respect to the topology on $\mathcal{A}$ and $\mathcal{B}$).

2.2.2. Locally convex Kasparov module Let $\mathcal{A}$ and $\mathcal{B}$ be (graded) locally convex algebras. A locally convex Kasparov module from $\mathcal{A}$ to $\mathcal{B}$ is given by a (graded) locally convex algebra $\hat{\mathcal{B}}$, whose grading is defined by an element $\epsilon \in \hat{\mathcal{B}}$ with $\epsilon^2 = 1$, containing $\mathcal{B}$ as a graded subalgebra, an odd element $F \in \hat{\mathcal{B}}$ and a (degree 0) map $\phi : \mathcal{A} \to \hat{\mathcal{B}}$ such that the maps

- $\mathcal{B} \to \hat{\mathcal{B}} : \ b \mapsto bF, Fb, \epsilon b, b\epsilon,$
- $\mathcal{A} \otimes \mathcal{B} \to \hat{\mathcal{B}} : \ (a, b) \mapsto \phi(a)b, b\phi(a),$
- $\mathcal{A} \to \hat{\mathcal{B}} : \ a \mapsto \phi(a)(1 - F^2), (1 - F^2)\phi(a),$
- $\mathcal{A} \to \hat{\mathcal{B}} : \ a \mapsto [\phi(a), F]$

are all $\mathcal{B}$-valued and continuous (with respect to the topology on $\mathcal{A}$ and $\mathcal{B}$). We denote this locally convex Kasparov $(\mathcal{A}, \mathcal{B})$-module by $(\hat{\mathcal{B}}, \phi, F)$.

2.2.3. Quasihomomorphism Let $\mathcal{A}$ and $\mathcal{B}$ be (graded) locally convex algebras. Let $\hat{\mathcal{B}}$ be a (graded) locally convex algebra containing $\mathcal{B}$. A quasihomomorphism from $\mathcal{A}$ to $\mathcal{B}$ is given by a pair of (degree 0) maps $(\alpha, \bar{\alpha})$ from $\mathcal{A}$ to $\hat{\mathcal{B}}$ such that

- $\mathcal{A} \to \hat{\mathcal{B}} : \ a \mapsto \alpha(a) - \bar{\alpha}(a),$
- $\mathcal{A} \otimes \mathcal{B} \to \hat{\mathcal{B}} : \ (a, b) \mapsto \alpha(a)b, b\alpha(a),$

are all $\mathcal{B}$-valued maps and continuous (with respect to the topology on $\mathcal{A}$ and $\mathcal{B}$).

Let $\mathcal{A}, \mathcal{B}$ be (graded) locally convex algebras. The *cylinder algebra* is the subalgebra $\mathscr{Z}\mathcal{A} \subseteq C^\infty([0, 1], \mathcal{A})$ consisting of $\mathcal{A}$-valued functions whose derivatives of order ≥ 1 vanish at the endpoints. Since $\mathscr{Z} := \mathscr{Z}\mathbb{C}$ is nuclear, therefore $\mathscr{Z}\mathcal{A} = \mathscr{Z} \otimes \mathcal{A}$.

Two morphisms $\phi_0, \phi_1 : \mathcal{A} \to \mathcal{B}$ are called *diffotopic* if there exists a (degree 0) morphism $\phi : \mathcal{A} \to \mathscr{Z}\mathcal{B}$ called *diffotopy*, such that $\mathrm{ev}_0 \circ \phi = \phi_0$, $\mathrm{ev}_1 \circ \phi = \phi_1$. The relation of diffotopy is clearly an equivalence relation.

If $\mathcal{A}$ carries an action α of G, then we define the action of G on $\mathscr{Z}\mathcal{A} = \mathscr{Z} \otimes \mathcal{A}$ to be $\mathrm{id} \otimes \alpha$.

According to [Gre12, Definition 18], if H is a split-exact functor on the category of locally convex algebras, a quasihomomorphism $(\alpha, \bar{\alpha})$ from $\mathcal{A}$ to $\mathcal{B}$ gives rise to a map from $\mathrm{H}(\mathcal{A})$ to $\mathrm{H}(\mathcal{B})$.

An abstract Kasparov module from $\mathcal{A}$ to $\mathcal{B}$ gives rise to a quasihomomorphism. Suppose that we have an abstract Kasparov module $x := (\alpha, \bar{\alpha}, U, \hat{\mathcal{B}})$ from $\mathcal{A}$ to $\mathcal{B}$. This immediately gives a quasihomomorphism from $\mathcal{A}$ to $\mathcal{B}$ given by $Qh(x) := (\alpha, U^{-1}\bar{\alpha}U)$. Also if we have a locally convex Kasparov $(\mathcal{A}, \mathcal{B})$-module by $(\hat{\mathcal{B}}, \phi, F)$, we can set elements $P_\epsilon := \frac{1}{2}(1 + \epsilon)$, $P_\epsilon^\perp := 1 - P_\epsilon$ and define an abstract Kasparov module from $\mathcal{A}$ to $\mathcal{B}$ which is given by the quadruple $(P_\epsilon \phi, P_\epsilon^\perp \phi, W_F, \hat{\mathcal{B}})$, where

$$W_F := P_\epsilon F + P_\epsilon^\perp F(2 - F^2) + \epsilon(1 - F^2).$$

See [Gre12, Proposition 39] for the details of the previous discussion.

We will use the following results of Grensing from [Gre12]. We remark that Grensing [Gre12] worked with trivially graded locally convex algebras. However, it is not hard to see that all the arguments [Gre12] naturally extend to graded locally convex algebras with the above definitions with essentially identical proofs. So we will use these results for graded locally convex algebras without detailed proofs. In the following, by a morphism between graded algebras we always mean a graded morphism.

Proposition 2.2 ([Gre12, Proposition 22]). *Let* $(\alpha, \bar{\alpha}) : \mathcal{A} \to \hat{\mathcal{B}} \trianglerighteq \mathcal{B}$ *be a quasihomomorphism of (graded) locally convex algebras,* H *a split-exact functor on the category of (graded) locally convex algebras with value in abelian groups. Then the following properties hold:*

(1) $\mathrm{H}((\alpha, \bar{\alpha}) \circ \phi) = \mathrm{H}(\alpha, \bar{\alpha}) \circ \mathrm{H}(\phi)$ *for every morphism* $\phi : \mathcal{C} \to \mathcal{A}$,
(2) $\mathrm{H}(\psi \circ (\alpha, \bar{\alpha})) = \mathrm{H}(\psi) \circ \mathrm{H}(\alpha, \bar{\alpha})$ *for every homomorphism* $\psi : \hat{\mathcal{B}} \to \hat{\mathcal{C}}$ *of locally convex algebras, such that there is a locally convex subalgebra* $\mathcal{C} \subseteq \hat{\mathcal{C}}$ *such that* $(\psi \circ \alpha, \psi \circ \bar{\alpha}) : \mathcal{A} \to \hat{\mathcal{C}} \trianglerighteq \mathcal{C}$ *is a quasihomomorphism,*
(3) *if* $(\beta, \bar{\beta}) : \mathcal{A}' \to \hat{\mathcal{B}}' \trianglerighteq \mathcal{B}'$ *is a quasihomomorphism, and* $\phi : \mathcal{A} \to \mathcal{A}'$, $\psi : \mathcal{B} \to \mathcal{B}'$ *are homomorphisms of locally convex algebras such that* $\psi \circ (\alpha - \bar{\alpha}) = (\beta - \bar{\beta}) \circ \phi$, *then*

$$\mathrm{H}(\psi) \circ \mathrm{H}(\alpha, \bar{\alpha}) = \mathrm{H}(\beta, \bar{\beta}) \circ \mathrm{H}(\phi),$$

(4) $\mathrm{H}(\alpha, \bar{\alpha}) = -\mathrm{H}(\bar{\alpha}, \alpha)$,
(5) *if* $\alpha - \bar{\alpha}$ *is a homomorphism orthogonal to* $\bar{\alpha}$, *then* $\mathrm{H}(\alpha, \bar{\alpha}) = \mathrm{H}(\alpha - \bar{\alpha})$.

If $(\alpha, \bar{\alpha}) : \mathcal{A} \to \hat{\mathcal{B}} \trianglerighteq \mathcal{B}$ is a quasihomomorphism of (graded) locally convex algebras, the sum $\mathcal{D} := \mathcal{A} \oplus \mathcal{B}$, equipped with the multiplication

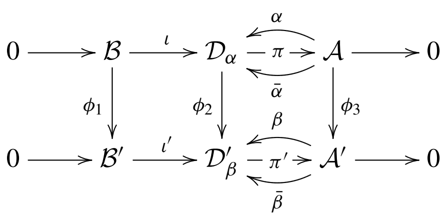
$$(a, b)(a', b') := (aa', \alpha(a)b' + b\alpha(a') + bb')$$

is a (graded) locally convex algebra $\mathcal{D}_\alpha$. $\mathcal{B}$ sits in $\mathcal{D}_\alpha$ by the inclusion $\iota_{\mathcal{B}} : \mathcal{B} \to \mathcal{D}_\alpha,\ b \mapsto (0, b)$, and $\mathcal{A}$ is a quotient of $\mathcal{D}_\alpha$. Also we have the following split exact sequence

$$0 \longrightarrow \mathcal{B} \longrightarrow \mathcal{D}_\alpha \longrightarrow \mathcal{A} \longrightarrow 0,$$

with split $\alpha' := \mathrm{id}_{\mathcal{A}} \oplus 0$. Note that there is another split $\bar{\alpha}' := \mathrm{id}_{\mathcal{A}} \oplus (\bar{\alpha} - \alpha)$. By abuse of notations, we denote them by α and $\bar{\alpha}$ again, respectively.

Proposition 2.3. ([Gre12, Lemma 19])*d If* (ϕ_1, ϕ_2, ϕ_3) *is a morphism of double split extensions, i.e.:*



(2.1)

which commutes in the usual sense satisfying $\phi_2 \circ \alpha = \beta \circ \phi_3$, $\phi_2 \circ \bar{\alpha} = \bar{\beta} \circ \phi_3$, *then for every split exact functor* H

$$\mathrm{H}(\phi_1) \circ \mathrm{H}(\alpha, \bar{\alpha}) = \mathrm{H}(\beta, \bar{\beta}) \circ \mathrm{H}(\phi_3).$$

2.3. Notations. In this paper, we shall denote (graded) C^*-algebras by $A, B, \cdots$, and (graded) locally convex algebras by $\mathcal{A}, \mathcal{B}, \cdots$. If A is equipped with a strongly continuous $\mathbb{R}^n$-action, we denote the subalgebra of A of smooth vectors with respect to an $\mathbb{R}^n$-action by A^∞, which is a Fréchet algebra (see Appendix B). And G will always be a finite group. All the tensor products we use are projective tensor products of (graded) locally convex algebras. For the convenience to readers, we give a short list of notations:

- $e(s)$ denotes the number $e^{2\pi\sqrt{-1}s}$.
- $\mathcal{K}^\infty$ denotes the (trivially graded) locally convex algebra of smooth compact operators (smoothing operators on $\mathbb{R}^n$ with rapid decay Schwartz kernels), and $\mathcal{K}$, the C^*-algebra of compact operators.
- H is a split-exact, diffotopy invariant, $\mathcal{K}^\infty$-stable functor on the category of (graded) locally convex algebras.
- $AKM(\mathcal{A}, \mathcal{B})$ denotes the set of locally convex Kasparov modules between (graded) locally convex algebras $\mathcal{A}$ and $\mathcal{B}$.[1]
- $\mathrm{KK}^G(A, B)$ denotes the usual equivariant Kasparov group for the C^*-algebras A and B. We denote the non-equivariant one by $\mathrm{KK}(A, B)$.
- $QH(\mathcal{A}, \mathcal{B})$ denotes the quasihomomorphim group between $\mathcal{A}$ and $\mathcal{B}$. $Qh(x)$ denotes the quasi-homomorphism group associated to the locally convex Kasparov module x.
- If $x \in QH(\mathcal{A}, \mathcal{B})$, then $\mathrm{H}(x)$ denotes the map from $\mathrm{H}(\mathcal{A})$ to $\mathrm{H}(\mathcal{B})$.

Let $R(G)$ be the representation ring of the group G. We will introduce an $R(G)$-module structure on $\mathrm{H}(\mathcal{K}^\infty \rtimes_\rho G)$ in Appendix A for a diffotopy invariant, split-exact, $\mathcal{K}^\infty$-stable functor H. We will need the following G-equivariant variation of Grensing's result in the study of the Connes–Thom isomorphism.

Recall that G acts on $\mathbb{R}^n$ and therefore acts on $\mathcal{K}^\infty$. Let $(\mathcal{H}, \varphi, \tilde{F})$ be a G-equivariant Kasparov $(\mathbb{C}, \mathcal{K})$-module for C^*-algebras. We assume that $(B(\mathcal{H}), \varphi, \tilde{F})$ defines a locally convex Kasparov $(\mathbb{C}, \mathcal{K}^\infty)$-module. Through out the paper, we use $B(\mathcal{H})$ to denote the algebra of bounded linear operators on $\mathcal{H}$. By the descent construction in equivariant KK-theory, $(\mathcal{H}, \varphi, \tilde{F})$ gives rise to a Kasparov $(\mathbb{C}G, \mathcal{K} \rtimes_\rho G)$-module for C^*-algebras in the following way. Endow $C(G, \mathcal{H})$, which we denote by $\mathcal{H} \rtimes G$, with the following operations:

$$\langle x, y\rangle_{\mathcal{K}\rtimes G}(t) := \sum_G \beta_{s^{-1}}\big(\langle x(s), y(st)\rangle_{\mathcal{K}}\big),$$

$$(x \cdot \lambda)(t) := \sum_G x(s)\beta_s\big(\lambda(s^{-1}t)\big),$$

for $x, y \in C(G, \mathcal{H}) := \mathcal{H} \rtimes G$, and $\lambda \in \mathcal{K} \rtimes_\rho G$. Define F on $C(G, \mathcal{H})$ by $F(x)(t) = \tilde{F}(x(t))$, and $\phi : \mathbb{C}G \to B(\mathcal{H} \rtimes G)$ by

$$\big(\phi(f)x\big)(t) := \sum_s \varphi\big(f(s)\big)\gamma_s\big(x(s^{-1}t)\big),$$

for $f \in \mathbb{C}G, x \in \mathcal{H} \rtimes G$. With the assumption that $(B(\mathcal{H}), \varphi, \tilde{F})$ is a locally convex Kasparov $(\mathbb{C}, \mathcal{K}^\infty)$-module, it is straightforward to check that the triple $\big(B(\mathcal{H} \rtimes G), \phi, F\big)$ defines a locally convex Kasparov $(\mathbb{C}G, \mathcal{K}^\infty \rtimes_\rho G)$-module. We have the following property generalizing [Gre12, Proposition 45].

[1] Here we follow Grensing's definition, so we do not deal with unbounded Fredholm modules in this paper.

Proposition 2.4. ([Gre12, Proposition 45]) *Let* $x = (B(\mathcal{H} \rtimes G), \phi, F)$ *be a locally convex Kasparov* $(\mathbb{C}G, \mathcal{K}^\infty \rtimes_\rho G)$ *-module defined above from a G-equivariant Kasparov module* $(\mathcal{H}, \varphi, \tilde{F})$ *for* C^**-algebras* $(\mathbb{C}, \mathcal{K})$*, and* H *be a diffotopy invariant, split-exact,* $\mathcal{K}^\infty$*-stable functor. Then* $\mathrm{H}(Qh(x)) = \mathrm{index}_G \tilde{F} \circ \theta_*$*, where* $\theta_* : \mathrm{H}(\mathbb{C}G) \to \mathrm{H}(\mathcal{K}^\infty \rtimes_\rho G)$ *denotes the G-stabilization map, and* $\mathrm{index}_G \tilde{F}$ *is the G-index of the operator* $\tilde{F}$ *which is an element of* $R(G)$*.*

The proof of the above proposition is presented in Appendix A, Proposition A.3.

For our applications (Corollary 4.12, Corollary 5.6), the main example of H will be the periodic cyclic homology functor, HP, for locally convex algebras. We refer to [Con85, Cun97, CST04, Cun05], and the references therein for the basic properties of the functor HP . In particular, from [Cun97, the proof of Korollar 6.8] and [Cun05, Section 9], we have the following lemma.

Lemma 2.5. *The periodic cyclic homology,* $\mathrm{HP}(-)$*, is a split-exact, diffotopy invariant, and* $\mathcal{K}^\infty$*-stable functor on the category of (graded) locally convex algebras..*

3. The Equivariant Takesaki–Takai Duality Theorem

We consider a C^*-dynamical system $(A, \mathbb{R}^n, \alpha)$. Let $\widehat{\mathbb{R}^n}$ be the Pontryagin dual group of $\mathbb{R}^n$ as an abelian group. We observe that there is a dual action $\hat{\alpha}$ of $\widehat{\mathbb{R}^n}$ on $A^\infty \rtimes_\alpha \mathbb{R}^n$ given by

$$\hat{\alpha}_x(f)(s) = e(\langle x, s\rangle) f(s).$$

The action of G on a smoothing operator T' (viewed as an operator on $L^2(\mathbb{R}^n)$) is $g(T')(f)(x) = T'(g^{-1} \cdot f)(g^{-1}x)$. If we realize a compact operator by a kernel function $k(r, s)$ on $\mathbb{R}^n \times \mathbb{R}^n$ then the G-action is the diagonal action, i.e., $g \cdot k(x, y) = k(g^{-1}x, g^{-1}y)$.

Following [Wil07, Page 190], we prove a G-equivariant version of the Takesaki–Takai duality theorem.

Theorem 3.1. *The algebra* $(A^\infty \rtimes_\alpha \mathbb{R}^n) \rtimes_{\hat{\alpha}} \widehat{\mathbb{R}^n}$ *is isomorphic to* $A^\infty \otimes \mathcal{K}^\infty$*. And the isomorphism can be made G-equivariant.*

Proof. Let γ be the action of $\mathbb{R}^n$ on $\mathcal{S}(\mathbb{R}^n, A^\infty)$ given by

$$(\gamma_t f)(s) = f(s - t).$$

From the proof of [Wil07, Theorem 7.1], we have an isomorphism of $(A^\infty \rtimes_\alpha \mathbb{R}^n) \rtimes_{\hat{\alpha}} \widehat{\mathbb{R}^n}$ and $\mathcal{S}(\mathbb{R}^n, A^\infty) \rtimes_\gamma \mathbb{R}^n$ given by Φ, where

$$\Phi(F)(s, r) = \int_{\widehat{\mathbb{R}^n}} \alpha_r^{-1}(F(t, s)) e(\langle r - s, t\rangle) dt, \quad F \in \mathcal{S}(\widehat{\mathbb{R}^n} \times \mathbb{R}^n, A).$$

In the following, we show that the above isomorphism ([Wil07, Lemma 7.6]) between $\mathcal{S}(\mathbb{R}^n, A^\infty) \rtimes_\gamma \mathbb{R}^n$ and $A^\infty \otimes \mathcal{K}^\infty$ is G-equivariant, i.e., $g \cdot \Phi(F) = \Phi(g \cdot F)$,

$$\begin{aligned}
\Phi(g\cdot F)(s,r) &= \int_{\widehat{\mathbb{R}^n}} \alpha_r^{-1}(g\cdot F(x,s))e(\langle r-s,x\rangle)dx\\
&= \int_{\widehat{\mathbb{R}^n}} \alpha_r^{-1}(\beta_g(F(g^t x, g^{-1}s)))e(\langle r-s,x\rangle)dx,\\
\Phi(g\cdot F)(gs,r) &= \int_{\widehat{\mathbb{R}^n}} \alpha_r^{-1}(\beta_g(F(g^t x, s)))e(\langle r-gs,x\rangle)dx\\
&= \int_{\widehat{\mathbb{R}^n}} \alpha_r^{-1}(\beta_g(F(x, s)))e(\langle r-gs,(g^t)^{-1}x\rangle)dx\\
&= \int_{\widehat{\mathbb{R}^n}} \alpha_r^{-1}(\beta_g(F(x, s)))e(\langle r-gs,(g^t)^{-1}x\rangle)dx\\
&= \int_{\widehat{\mathbb{R}^n}} \alpha_r^{-1}(\beta_g(F(x, s)))e(\langle g^{-1}r-s,x\rangle)dx,\\
\Phi(g\cdot F)(gs,gr) &= \int_{\widehat{\mathbb{R}^n}} \alpha_{gr}^{-1}(\beta_g(F(x, s)))e(\langle r-s,x\rangle)dx\\
&= \int_{\widehat{\mathbb{R}^n}} \beta_g(\alpha_r^{-1}(F(x, s)))e(\langle r-s,x\rangle)dx,\\
\Phi(g\cdot F)(s,r) &= \int_{\widehat{\mathbb{R}^n}} \beta_g(\alpha_{g^{-1}r}^{-1}(F(x, g^{-1}s)))e(\langle g^{-1}r-g^{-1}s,x\rangle)dx\\
&= \beta_g(\Phi(F)(g^{-1}s, g^{-1}r))\\
&= (g\cdot\Phi(F))(s,r).
\end{aligned}$$

□

Corollary 3.2. *The above isomorphism gives a G-equivariant isomorphism between the C^*-algebras $(A \rtimes_\alpha \mathbb{R}^n) \rtimes_{\hat{\alpha}} \widehat{\mathbb{R}^n}$ and $A \otimes \mathcal{K}$. And we have the following isomorphism of algebras*

$$\left((A \rtimes_\alpha \mathbb{R}^n) \rtimes_{\hat{\alpha}} \widehat{\mathbb{R}^n}\right) \rtimes_{\beta\rtimes\rho} G \cong \left(A \otimes \mathcal{K}\right) \rtimes_\beta G.$$

4. Smooth Connes–Thom Isomorphism

In [Con81], Connes proves a Thom isomorphism theorem in K-theory, which offers a fundamental tool in noncommutative geometry. More precisely, let A be a (trivially graded) C^*-algebra with a strongly continuous $\mathbb{R}^n$-action. Then

$$\mathrm{K}_\bullet(A) \cong \mathrm{K}_{\bullet+n}(A \rtimes \mathbb{R}^n).$$

In this paper, we present a smooth version of the above Connes–Thom isomorphism for locally convex algebras.

Theorem 4.1 (Equivariant Connes–Thom isomorphism). *Let $\mathbb{R}^n$, G, and A be defined as in Sect. 1. Then*

$$\mathrm{H}\left(((A^\infty \otimes \mathbb{C}_n) \rtimes_\alpha \mathbb{R}^n\right) \rtimes_{\beta\rtimes\rho} G) \cong \mathrm{H}(A^\infty \rtimes_\beta G),$$

where H *is a split-exact, diffotopy invariant, $\mathcal{K}^\infty$-stable functor on the category of (graded) locally convex algebras, and $\mathbb{C}_n$ is the complexified Clifford algebra associated with $\mathbb{R}^n$, carrying the trivial action of $\mathbb{R}^n$ and the action of G which is induced from the G-action on $\mathbb{R}^n$.*

Let $B^\infty := A^\infty \otimes \mathbb{C}_n$. We extend the action α of $\mathbb{R}^n$ on A (and A^∞) to B^∞ by a trivial action on the component $\mathbb{C}_n$ and denote the action again by α. We construct a pair of Dirac and dual Dirac elements $x^\infty_{n,\alpha}$ and $y^\infty_{n,\alpha}$ as a locally convex Hilbert module from $(B^\infty \rtimes_\alpha \mathbb{R}^n) \rtimes_{\beta \rtimes \rho} G$ to $(A^\infty \otimes \mathcal{K}^\infty) \rtimes_{\beta \rtimes \rho} G$ and from $A^\infty \rtimes_\beta G$ to $(B^\infty \rtimes_\alpha \mathbb{R}^n) \rtimes_{\beta \rtimes \rho} G$ respectively, whose images under the functor H are inverses to each other. When the algebra A is $\mathbb{C}$, the two elements

$$x_n^\infty \in AKM\Big((\mathcal{S}(\mathbb{R}^n) \otimes \mathbb{C}_n) \rtimes_\rho G, \mathcal{K}^\infty \rtimes_\rho G\Big), \text{ and } y_n^\infty \in AKM\Big(\mathbb{C}G, (\mathcal{S}(\mathbb{R}^n) \otimes \mathbb{C}_n) \rtimes_\rho G\Big),$$

modify Grensing's construction [Gre12] to allow the generalization to general A.

We recall the standard Hodge-de Rham operator as $d + d^*$ on $\mathbb{R}^n$, where we equip $\mathbb{R}^n$ with the standard metric and d^* is the adjoint of d. Our key observation is that the principal symbol of $d + d^*$ is the the Clifford multiplication of ξ on $\wedge^\bullet \mathbb{C}^{n*}$, which is independent of x and only a function of ξ. Using the symbol calculus by Connes [Con80], we can generalize the Hodge-de Rham operator to strongly continuous $\mathbb{R}^n$-actions on C^*-algebras, and therefore leads to generalizations of x_n and y_n.

4.1. A (dual) Dirac element. Following [HR00, Section 10.6], we use a normalizing function to construct the Dirac element.

Recall that a normalizing function is a smooth function $\chi : \mathbb{R} \to [-1, 1]$ satisfying the following properties

(1) χ is odd,
(2) $\chi(\lambda) > 0$ for all $\lambda > 0$,
(3) $\chi(\lambda) \to \pm 1$ as $\lambda \to \pm\infty$.

An explicit construction of a normalizing function χ is given in [HR00, Ex. 10.9.3] with an extra nice property that the Fourier transform $\widehat{\chi}$ has compact support and $s\widehat{\chi}(s)$ is a smooth function on $\mathbb{R}$.

Given $\xi \in \mathbb{R}^n$, we use $c(\xi)$ to denote the Clifford multiplication with respect to ξ. We use $e^{isc(\xi)}$ to denote the wave operator on $\mathbb{C}_n$ defined by the wave equation

$$\frac{d}{ds} f = \sqrt{-1} sc(\xi)(f),$$

on $C^\infty(\mathbb{R}, \mathbb{C}_n)$.

Let χ be a normalizing function. Define an endomorphism $\chi\big(c(\xi)\big)$ on $\mathbb{C}_n$ by

$$\chi\big(c(\xi)\big) := \int_{\mathbb{R}} \widehat{\chi}(s) e^{\sqrt{-1} sc(\xi)} ds.$$

Consider an order zero symbol $\Sigma(\xi)$ as follows,

$$\Sigma(\xi) := 1 \otimes \chi\big(c(\xi)\big).$$

A crucial property here is that $\Sigma(x, \xi)$ is independent of x and only a function of ξ. This allows us to apply Connes' pseudo-differential calculus, which is reviewed in Appendix B.

For a C^*-dynamical system $(A, \mathbb{R}^n, \alpha)$, let $t \mapsto V_t$ be the canonical representation of $\mathbb{R}^n$ in $M(A \rtimes_\alpha \mathbb{R}^n)$, the multiplier algebra, with $V_x a V_x^* = \alpha_x(a)$ $(a \in A)$. Consider $B = A \otimes \mathbb{C}_n$ with the extended action α by $\mathbb{R}^n$, which acts trivially on the component $\mathbb{C}_n$. The smooth subalgebra B^∞ of α is identified with $A^\infty \otimes \mathbb{C}_n$. As $\Sigma \in S^0(\mathbb{R}^n, B^\infty)$ (as a symbol independent of A^∞),

$$\widehat{\Sigma}(x) = \int_{\widehat{\mathbb{R}^n}} \Sigma(\xi) e(-\langle x, \xi \rangle) d\xi$$

is a well-defined distribution on $\mathbb{R}^n$ with values in B^∞. Using Connes' pseudo-differential calculus [Con80] (c.f. Appendix B) for $(A, \mathbb{R}^n, \alpha)$, we define $X_\alpha^\infty \in M(B \rtimes_\alpha \mathbb{R}^n)$ by

$$X_\alpha^\infty := \int \widehat{\Sigma}(x) V_x dx.$$

Lemma 4.2. *The symbol function $\frac{\partial \Sigma}{\partial \xi_j}$, for $j = 1, ..., n$, is a Schwartz function on $\mathbb{R}^n$.*

Proof. The proof of this property is presented in Appendix C. □

The grading operator on $M(B \rtimes_\alpha \mathbb{R}^n)$ is defined by the inner automorphism with respect to the element $\epsilon \in M(B \rtimes_\alpha \mathbb{R}^n)$, where ϵ is the grading operator on the spinors associated to $\mathbb{C}_n$. It is straightforward to check that ϵ is invariant with respect to the G-action.

Lemma 4.3. *The Fourier transform which sends Σ to $\widehat{\Sigma}$ is G-equivariant, i.e., $\widehat{g \cdot \Sigma} = g \cdot \widehat{\Sigma}$.*

Proof. We have the following computation.

$$\begin{aligned}\widehat{g \cdot \Sigma}(x) &= \int_{\widehat{\mathbb{R}^n}} (g \cdot \Sigma)(\xi) e(-\langle x, \xi \rangle) d\xi = \int_{\widehat{\mathbb{R}^n}} \beta_g(\Sigma(g^t \xi)) e(-\langle x, \xi \rangle) d\xi \\ &= \int_{\widehat{\mathbb{R}^n}} \beta_g(\Sigma(\xi)) e(-\langle x, (g^t)^{-1} \xi \rangle) d\xi = \int_{\widehat{\mathbb{R}^n}} \beta_g(\Sigma(\xi)) e(-\langle g^{-1} x, \xi \rangle) d\xi \\ &= \beta_g(\widehat{\Sigma}(g^{-1} x)) = (g \cdot \widehat{\Sigma})(x).\end{aligned}$$

□

Remark 4.4. The integrands of the above give rise to divergent integrals: we took the Fourier transform of an order zero symbol (as tempered distribution). To regularize divergent oscillatory integrals, one does the following: since we realize $\widehat{\Sigma}$ as a distribution, for $u \in \mathcal{S}(\mathbb{R}^n, A^\infty)$, $\langle \widehat{\Sigma}, u \rangle$ is given by

$$\lim_{\epsilon \to 0} \int_{\mathbb{R}^n} \int_{\widehat{\mathbb{R}^n}} \Sigma(\xi) u(x) \exp\left(-\frac{\epsilon \|\xi\|^2}{2}\right) e(-\langle x, \xi \rangle) d\xi dx.$$

Now since the expression $\exp\left(-\frac{\epsilon \|\xi\|^2}{2}\right)$ is G-invariant, the change of variables in the above proof makes sense. From now on we will use change of variables in oscillatory integrals for the G-action without any further explanation.

Lemma 4.5. *For any $a \in A^\infty$, we have $[a, X_\alpha^\infty] \in B^\infty \rtimes_\alpha \mathbb{R}^n$.*

Proof. Following the definition of X_α^∞, we compute the commutator $[a, X_\alpha^\infty]$ as follows.

$$\begin{aligned}[a, X_\alpha^\infty] &= \left[a, \int_{\mathbb{R}^n} \widehat{\Sigma}(x) V_x dx\right] \\ &= \int_{\mathbb{R}^n} \int_{\widehat{\mathbb{R}^n}} [a, \chi\big(c(\xi)\big) e(-\langle x, \xi\rangle) V_x d\xi dx] \\ &= \int_{\mathbb{R}^n} \int_{\widehat{\mathbb{R}^n}} \chi\big(c(\xi)\big) e(-\langle x, \xi\rangle)(a - \alpha_x(a)) V_x d\xi dx \\ &= \int_{\mathbb{R}^n} \int_{\widehat{\mathbb{R}^n}} \int_{\mathbb{R}} \widehat{\chi}(s) e^{isc(\xi)} e(-\langle x, \xi\rangle)(a - \alpha_x(a)) V_x ds d\xi dx.\end{aligned}$$

Define a function $Y : \mathbb{R}^n \to B^\infty$ by $Y(x) := \alpha_x(a)$. As a belongs to B^∞, $Y(x)$ is a smooth function on $\mathbb{R}^n$. Therefore, we can write

$$Y(x) - Y(0) = \int_0^1 \frac{d}{ds} Y(sx) ds = \int_0^1 x^i \frac{\partial}{\partial x^i} Y(sx) ds.$$

Define $Z_i : \mathbb{R}^n \to B^\infty$ by

$$Z_i(x) := \int_0^1 \frac{\partial}{\partial x^i} Y(sx) ds.$$

It is not hard to check that Z_i is again a smooth function on $\mathbb{R}^n$ with values in B^∞.

In summary, there are smooth functions Z_i, $i = 1, \ldots, n$, such that the following equation holds,

$$\widehat{\Sigma}(x)\big(\alpha_x(a) - a\big) = \widehat{\Sigma}(x) \sum_i x^i Z_i(x) = \sum_i \widehat{\Sigma}(x) x^i Z_i(x).$$

By the properties of the Fourier transform, we have

$$\widehat{\Sigma}(x) x^i = \frac{1}{\sqrt{-1}} \widehat{\frac{\partial \Sigma}{\partial \xi_i}}(x).$$

Recall that $\Sigma(\xi)$ is defined as

$$\Sigma(\xi) := 1 \otimes \chi\big(c(\xi)\big).$$

By Lemma 4.2, $\frac{\partial \Sigma}{\partial \xi_i}$ is a Schwartz function, and therefore $\widehat{\Sigma}(x) x^i$ is a Schwartz function. From the above discussion, we conclude that as a sum of elements in $B^\infty \rtimes_\alpha \mathbb{R}^n$,

$$\widehat{\Sigma}(x)\big(\alpha_x(a) - a\big) = \sum_i \widehat{\Sigma}(x) x^i G_i(x)$$

belongs to $B^\infty \rtimes_\alpha \mathbb{R}^n$. □

Lemma 4.6. *For $a \in A^\infty \hookrightarrow M(B \rtimes_\alpha \mathbb{R}^n)$, which is the multiplier algebra of the C^*-algebra $B \rtimes_\alpha \mathbb{R}^n$, we have $a(1 - (X_\alpha^\infty)^2) \in B^\infty \rtimes_\alpha \mathbb{R}^n$.*

Proof. We observe that $\hat{\Sigma} * \hat{\Sigma}$ is well-defined as a distribution, and

$$(1 - (X_\alpha^\infty)^2) = \int_{\mathbb{R}^n} \widehat{(1 - \Sigma^2}(x)) V_x dx.$$

Now

$$(1 - \Sigma^2)(\xi) = 1 \otimes \big(1 - \chi(c(\xi))^2\big).$$

As the function $s\widehat{\chi}(s)$ is smooth with compact support, the corresponding Fourier transform, $\frac{d\chi}{d\lambda}$, is a Schwartz function. Furthermore, $\chi^2(\lambda) \to 1$ as $\lambda \to \pm\infty$. It follows that $\chi^2(\lambda) - 1$ is a Schwartz function. Therefore, $(1 - \Sigma^2)$ is a Schwartz function and hence a symbol of order $-\infty$. So the element $\int_{\mathbb{R}^n} \widehat{(1 - \Sigma^2}(x)) V_x dz$ is in $B^\infty \rtimes_\alpha \mathbb{R}^n$, which proves our claim. □

Lemma 4.7. *We have $g \cdot X_\alpha^\infty = X_\alpha^\infty$.*

Proof. We start by checking $\widehat{\Sigma}(gx) = g \cdot \widehat{\Sigma}(x)$. Indeed,

$$\begin{aligned}
\widehat{\Sigma}(gx) &= \int_{\widehat{\mathbb{R}^n}} \Sigma(\xi) e(-\langle gx, \xi\rangle) d\xi = \int_{\widehat{\mathbb{R}^n}} \Sigma(\xi) e(-\langle x, g^t\xi\rangle) d\xi \\
&= \int_{\widehat{\mathbb{R}^n}} \Sigma((g^t)^{-1}\xi) e(-\langle x, \xi\rangle) d\xi = \int_{\widehat{\mathbb{R}^n}} \Sigma(g \cdot \xi) e(-\langle x, \xi\rangle) d\xi \\
&= \int_{\widehat{\mathbb{R}^n}} \beta_g(\Sigma(\xi)) e(-\langle x, \xi\rangle) d\xi = g \cdot \widehat{\Sigma}(x).
\end{aligned}$$

Using the above property, we verify the following equality.

$$g \cdot X_\alpha^\infty = \int_{\mathbb{R}^n} g \cdot \widehat{\Sigma}(x) V_{gx} dx = \int_{\mathbb{R}^n} \widehat{\Sigma}(gx) V_{gx} dx = \int_{\mathbb{R}^n} \widehat{\Sigma}(x) V_x dx = X_\alpha^\infty.$$

□

As the above locally convex Kasparov module $\big(M(B \rtimes_\alpha \mathbb{R}^n), \iota, X_\alpha^\infty\big)$ is G-invariant, we introduce the following Kasparov module for the corresponding crossed product algebras. Consider the multiplier algebra $M\big((B \rtimes_\alpha \mathbb{R}^n) \rtimes_{\beta\rtimes\rho} G\big)$ of the C^*-algebra $(B \rtimes_\alpha \mathbb{R}^n) \rtimes_{\beta\rtimes\rho} G$ with the natural inclusion map $\iota : A^\infty \rtimes_\beta G \to M\big((B \rtimes_\alpha \mathbb{R}^n) \rtimes_{\beta\rtimes\rho} G\big)$. The grading operator ϵ on $M(B \rtimes_\alpha \mathbb{R}^n)$ extends to a grading operator on $M\big((B \rtimes_\alpha \mathbb{R}^n) \rtimes_{\beta\rtimes\rho} G\big)$ defined by

$$\tilde{\epsilon}(f)(g) = \epsilon\big(f(g)\big), \ \forall g \in G.$$

The operator X_α^∞ extends to an element $\widetilde{X}_\alpha^\infty$ in $M\big((B \rtimes_\alpha \mathbb{R}^n) \rtimes_{\beta\rtimes\rho} G\big)$ by

$$\widetilde{X}_\alpha^\infty(f)(g) = X_\alpha^\infty(f(g)), \ \forall g \in G.$$

With the above preparation, we directly conclude the following proposition by Lemma 4.5, 4.6 and 4.7.

Proposition 4.8. *The multiplier $\widetilde{X}_\alpha^\infty$ of the algebra $(B^\infty \rtimes_\alpha \mathbb{R}^n) \rtimes_{\beta\rtimes\rho} G$ defines a locally convex Kasparov module $(M\big((B \rtimes_\alpha \mathbb{R}^n) \rtimes_{\beta\rtimes\rho} G\big), \iota, \widetilde{X}_\alpha^\infty)$ in $AKM\big(A^\infty \rtimes_\beta G, (B^\infty \rtimes_\alpha \mathbb{R}^n) \rtimes_{\beta\rtimes\rho} G\big)$, which we will denote by $y_{n,\alpha}^\infty$.*

Applying Proposition 4.8 to the dual action $\hat{\alpha}$ of $\widehat{\mathbb{R}^n}$ on $(B^\infty \otimes \mathbb{C}_n) \rtimes_\alpha \mathbb{R}^n$, we obtain an element

$$\Big(M\Big(\big(((B \otimes \mathbb{C}_n) \rtimes_\alpha \mathbb{R}^n) \rtimes_{\hat{\alpha}} \widehat{\mathbb{R}^n}\big) \rtimes_{\beta \rtimes \rho} G\Big), \iota, X^\infty_{\hat{\alpha}}\Big)$$
$$\in AKM\Big((B^\infty \rtimes_\alpha \mathbb{R}^n) \rtimes_{\beta\rtimes\rho} G, \big(((B^\infty \otimes \mathbb{C}_n) \rtimes_\alpha \mathbb{R}^n) \rtimes_{\hat{\alpha}} \widehat{\mathbb{R}^n}\big) \rtimes_{\beta\rtimes\rho} G\Big).$$

Using the G-equivariant Takesaki–Takai duality (Theorem 3.1), we have the isomorphism

$$\big(((B^\infty \rtimes \mathbb{C}_n) \rtimes_\alpha \mathbb{R}^n) \rtimes_{\hat{\alpha}} \widehat{\mathbb{R}^n}\big) \rtimes_{\beta \rtimes \rho} G \cong (A \otimes \mathbb{C}_{2n} \otimes \mathcal{K}^\infty) \rtimes_{\beta \rtimes \rho} G.$$

With the above isomorphism, we get a Dirac element

$$x^\infty_{n,\alpha} \in AKM\big((B^\infty \rtimes_\alpha \mathbb{R}^n) \rtimes_{\beta \rtimes \rho} G, (A^\infty \otimes \mathbb{C}_{2n} \otimes \mathcal{K}^\infty) \rtimes_{\beta \rtimes \rho} G\big).$$

In summary, we have constructed in this section a (dual) Dirac element for an $\mathbb{R}^n$-action, i.e.,

$$x^\infty_{n,\alpha} \in AKM\big((B^\infty \rtimes_\alpha \mathbb{R}^n) \rtimes_{\beta \rtimes \rho} G, (A^\infty \otimes \mathbb{C}_{2n} \otimes \mathcal{K}^\infty) \rtimes_{\beta \rtimes \rho} G\big),$$
$$y^\infty_{n,\alpha} \in AKM\big(A^\infty \rtimes_{\beta \rtimes \rho} G, (B^\infty \rtimes_\alpha \mathbb{R}^n) \rtimes_{\beta \rtimes \rho} G\big).$$

We next prove that $\mathrm{H}(y^\infty_{n,\alpha})$ and $\mathrm{H}(x^\infty_{n,\alpha})$ are inverses to each other for a split-exact, diffotopy invariant, $\mathcal{K}^\infty$-stable functor H.

4.2. Equivariant Bott periodicity. When α is trivial, we call the elements $y^\infty_{n,\alpha}$ and $x^\infty_{n,\alpha}$ by y^∞_n and x^∞_n respectively. If we take the extension of the elements y^∞_n and x^∞_n to the corresponding C^*-algebras to get elements in $\mathrm{KK}\Big(\big(B \otimes C_0(\mathbb{R}^n)\big) \rtimes_{\beta \rtimes \rho} G, A \rtimes_\beta G\Big)$ and $\mathrm{KK}\Big(A \rtimes_\beta G, \big(B \otimes C_0(\mathbb{R}^n)\big) \rtimes_{\beta \rtimes \rho} G\Big)$, we get the usual class of Kasparov Dirac-dual Dirac element, since the usual class and the extended class are compact perturbation to each other.

Remark 4.9. Grensing [Gre12] defined similar elements. Our $y^\infty_{n,\alpha}$ and $x^\infty_{n,\alpha}$ are slightly different from his. But as we want to use $\mathcal{K}^\infty$-stability rather than stability with the Schatten ideals, we prefer to work with this modified class. Grensing's class and our class match up to compact perturbation if we view the classes as KK elements (with the obvious extensions) in the corresponding C^*-algebras.

Theorem 4.10. *Let* H *be a split-exact, diffotopy invariant, $\mathcal{K}^\infty$-stable functor. Then for the trivial action α, $\mathrm{H}(x^\infty_{n,\alpha})$ and $\mathrm{H}(y^\infty_{n,\alpha})$ are inverse to one another.*

Proof. We notice that the elements in

$$\mathrm{KK}\Big(\big(B \otimes C_0(\mathbb{R}^n)\big) \rtimes_{\beta \rtimes \rho} G, A \rtimes_\beta G\Big) \text{ and } \mathrm{KK}\Big(A \rtimes_\beta G, \big(B \otimes C_0(\mathbb{R}^n)\big) \rtimes_{\beta \rtimes \rho} G\Big)$$

are the usual Kasparov Dirac and dual Dirac elements. Now let y and x be the quasihomomorphisms associated to y^∞_n and x^∞_n respectively. We wish to calculate $\mathrm{H}(y) \circ \mathrm{H}(x)$. Now let γ be the product of these two elements whose index is 1 (using Kasparov's Bott periodicity [Kas95]). And therefore the element γ has a trivial G-index. And the equality $\mathrm{H}(y^\infty_n) \circ \mathrm{H}(x^\infty_n) = 1$ follows from the second part of the proof of [Gre12, Theorem 57] using the standard (G-equivariant) rotation argument, Proposition 2.4, and $\mathcal{L}^p$ replaced by $\mathcal{K}^\infty$.

Now using the arguments similar to Grensing's (the first part of [Gre12, Theorem 57]) and Proposition 2.4, we can similarly prove $\mathrm{H}(x_n^\infty) \circ \mathrm{H}(y_n^\infty) = 1$. □

As a corollary of Theorem 4.10, when $A = \mathbb{C}$, we have the following version of Bott-periodicity.

Corollary 4.11. *Let* H *be a split-exact, diffotopy invariant, $\mathcal{K}^\infty$-stable functor. Then for the trivial α,* $\mathrm{H}(x_{n,\alpha}^\infty)$ *and* $\mathrm{H}(y_{n,\alpha}^\infty)$ *establish an isomorphism between* $\mathrm{H}(\mathbb{C}G)$ *and* $\mathrm{H}\Big(\big(\mathbb{C}_n \otimes \mathcal{S}(\mathbb{R}^n)\big) \rtimes_\rho G\Big)$.

4.3. Proof of the equivariant Connes–Thom isomorphism. With the development in Sect. 4.1 and 4.2, we are ready to prove Theorem 4.1.

Proof. Define a deformation α^s of the action α for $s \in [0,1]$. Define $\alpha^s : \mathbb{R}^n \times A^\infty \to A^\infty$ by

$$\alpha_x^s(a) := \alpha_{sx}(a), \qquad a \in A, x \in \mathbb{R}^n.$$

Then the same formulas as above define

$$y_{n,\alpha^s}^\infty \in AKM\Big(A^\infty \rtimes_\beta G, \big(B^\infty \rtimes_{\alpha^s} \mathbb{R}^n\big) \rtimes_{\beta\rtimes\rho} G\Big),$$
$$x_{n,\alpha^s}^\infty \in AKM\Big(\big(B^\infty \rtimes_{\alpha^s} \mathbb{R}^n\big) \rtimes_{\beta\rtimes\rho} G, \big(A^\infty \otimes \mathbb{C}_{2n} \otimes \mathcal{K}^\infty\big) \rtimes_{\beta\rtimes\rho} G\Big).$$

Now let $y_s := y_{n,\alpha^s}^\infty = (\beta_s, \bar{\beta}_s)$ and $x_s := x_{n,\alpha^s}^\infty = (\hat{\beta}_s, \bar{\hat{\beta}}_s)$. Denote the corresponding quasihomomorphisms also by y_s and x_s, respectively.

Let us now consider the Fréchet algebra $D^\infty = \mathscr{Z}A^\infty$, and the $\mathbb{R}^n$-action on the algebra D^∞ by $\mu_x(f)(s) = \alpha_{sx}(f(s))$. Now we have the following elements:

$$\mathrm{H}(\mathrm{ev}_s) \in QH\big(D^\infty \rtimes_\beta G, A^\infty \rtimes_\beta G\big) \quad (\text{since}\, \mathrm{ev}_s \text{ is } G\text{-equivariant}),$$
$$\mathrm{H}(\widehat{\mathrm{ev}_s}) \in QH\Big(\big((D^\infty \otimes \mathbb{C}_n) \rtimes_\mu \mathbb{R}^n\big) \rtimes_{\beta\rtimes\rho} G, \big((A^\infty \otimes \mathbb{C}_n) \rtimes_{\alpha^s} \mathbb{R}^n\big) \rtimes_{\beta\rtimes\rho} G\Big),$$
$$(\widehat{\mathrm{ev}_s} \text{ is extended from } \mathrm{ev}_s)$$
$$\mathrm{H}(\widehat{\widehat{\mathrm{ev}_s}}) = \mathrm{H}(\mathrm{ev}_s \otimes \mathrm{id} \otimes \mathrm{id}) \in QH((D^\infty \otimes \mathbb{C}_{2n} \otimes \mathcal{K}^\infty) \rtimes_{\beta\rtimes\rho} G, (A^\infty \otimes \mathbb{C}_{2n} \otimes \mathcal{K}^\infty) \rtimes_{\beta\rtimes\rho} G),$$

which is defined using the Takesaki–Takai isomorphism (Theorem 3.1) and viewing $\mathrm{ev}_s \otimes \mathrm{id} \otimes \mathrm{id}$ as a G-equivariant map from $D^\infty \otimes \mathbb{C}_{2n} \otimes \mathcal{K}^\infty$ to $A^\infty \otimes \mathbb{C}_{2n} \otimes \mathcal{K}^\infty$. Together with

$$y^\infty := y_{n,\gamma}^\infty = (\gamma, \overline{\gamma}) \in QH\Big(D^\infty \rtimes_{\beta\rtimes\rho} G, \big((D^\infty \otimes \mathbb{C}_n) \rtimes_\mu \mathbb{R}^n\big) \rtimes_{\beta\rtimes\rho} G\Big),$$
$$x^\infty := x_{n,\gamma}^\infty = (\hat{\gamma}, \overline{\hat{\gamma}}) \in QH\Big(\big((D^\infty \otimes \mathbb{C}_n) \rtimes_\mu \mathbb{R}^n\big) \rtimes_{\beta\rtimes\rho} G, (D^\infty \otimes \mathbb{C}_{2n} \otimes \mathcal{K}^\infty) \rtimes_{\beta\rtimes\rho} G\Big),$$

they satisfy the following identities,

$$\mathrm{H}(\mathrm{ev}_s) \circ \mathrm{H}(y_s) = \mathrm{H}(y^\infty) \circ \mathrm{H}(\widehat{\mathrm{ev}_s}), \text{ and } \mathrm{H}(x^\infty) \circ \mathrm{H}(\mathrm{ev}_s) = \mathrm{H}(\widehat{\mathrm{ev}_s}) \circ \mathrm{H}(x_s).$$

The above two identities come from the following diagrams and Proposition 2.3:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \big((A^\infty\otimes\mathbb{C}_n)\rtimes_{\alpha^s}\mathbb{R}^n\big)\rtimes_{\beta\rtimes\rho}G & \longrightarrow & \mathcal{D}_{\hat{\beta}_s}\rtimes_\beta G & \longrightarrow & (A^\infty\otimes\mathbb{C}_{2n}\otimes\mathcal{K}^\infty)\rtimes_{\beta\rtimes\rho}G & \longrightarrow & 0\\
 & & \uparrow{\scriptstyle\widehat{\mathrm{ev}_s}} & & \uparrow{\scriptstyle\overline{\mathrm{ev}_s}} & & \uparrow{\scriptstyle\mathrm{ev}_s\otimes\mathrm{id}\otimes\mathrm{id}} & & \\
0 & \longrightarrow & \big((D^\infty\otimes\mathbb{C}_n)\rtimes_{\mu}\mathbb{R}^n\big)\rtimes_{\beta\rtimes\rho}G & \longrightarrow & \mathcal{D}_{\hat{\gamma}}\rtimes_\beta G & \longrightarrow & (D^\infty\otimes\mathbb{C}_{2n}\otimes\mathcal{K}^\infty)\rtimes_{\beta\rtimes\rho}G & \longrightarrow &
\end{array} \tag{4.1}$$

and

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A^\infty\rtimes_\beta G & \longrightarrow & \mathcal{D}_{\beta_s}\rtimes_\beta G & \longrightarrow & \big((A^\infty\otimes\mathbb{C}_n)\rtimes_{\alpha^s}\mathbb{R}^n\big)\rtimes_{\beta\rtimes\rho}G & \longrightarrow & 0\\
 & & \uparrow{\scriptstyle\mathrm{ev}_s} & & \uparrow{\scriptstyle\overline{\mathrm{ev}_s}} & & \uparrow{\scriptstyle\widehat{\mathrm{ev}_s}} & & \\
0 & \longrightarrow & D^\infty\rtimes_\beta G & \longrightarrow & \mathcal{D}_{\gamma}\rtimes_\beta G & \longrightarrow & \big((D^\infty\otimes\mathbb{C}_n)\rtimes_{\mu}\mathbb{R}^n\big)\rtimes_{\beta\rtimes\rho}G & \longrightarrow & 0
\end{array} \tag{4.2}$$

The above two diagrams are reinterpretations of the naturality of the Thom elements. The diagrams commute and also commute with the double-splits (as in Proposition 2.3). Note that using $\mathcal{K}^\infty$-stability of H, $\mathrm{H}(\mathrm{ev}_s\otimes\mathrm{id}\otimes\mathrm{id})=\mathrm{H}(\mathrm{ev}_s)$.

Now

$$\begin{aligned}
(\mathrm{H}(y^\infty)\circ\mathrm{H}(x^\infty))\circ\mathrm{H}(\mathrm{ev}_s) &= \mathrm{H}(y^\infty)\circ(\mathrm{H}(x^\infty)\circ\mathrm{H}(\mathrm{ev}_s)) = \mathrm{H}(y^\infty)\circ(\mathrm{H}(\widehat{\mathrm{ev}_s})\circ\mathrm{H}(x_s))\\
&= (\mathrm{H}(y^\infty)\circ\mathrm{H}(\widehat{\mathrm{ev}_s}))\circ\mathrm{H}(x_s) = \mathrm{H}(\mathrm{ev}_s)\circ\mathrm{H}(y_s)\circ\mathrm{H}(x_s).
\end{aligned}$$

But using diffotopy invariance of the H functor, we have $\mathrm{H}(\mathrm{ev}_s)=\mathrm{H}(\mathrm{ev}_0)$. So

$$\mathrm{H}(y_s)\circ\mathrm{H}(x_s)=\mathrm{H}(\mathrm{ev}_0)^{-1}\circ(\mathrm{H}(y^\infty)\circ\mathrm{H}(x^\infty))\circ\mathrm{H}(\mathrm{ev}_0)$$

shows that $\mathrm{H}(y_s)\circ\mathrm{H}(x_s)$ is independent of s. Similarly starting with $\mathrm{H}(x^\infty)\circ\mathrm{H}(y^\infty)\circ\mathrm{H}(\widehat{\mathrm{ev}_s})$, we conclude that $\mathrm{H}(x_s)\circ\mathrm{H}(y_s)$ is independent of s. This implies that

$$\mathrm{H}(x^\infty_{n,\alpha})\circ\mathrm{H}(y^\infty_{n,\alpha})=\mathrm{H}(x^\infty_{n,\alpha^0})\circ\mathrm{H}(y^\infty_{n,\alpha^0}).$$

The product $\mathrm{H}(x^\infty_{n,\alpha})\circ\mathrm{H}(y^\infty_{n,\alpha})$ is reduced to the computation to the Bott-periodicity result, Theorem 4.10:

$$\mathrm{H}(x^\infty_{n,\alpha^0})\circ\mathrm{H}(y^\infty_{n,\alpha^0})=1,$$

where the action of $\mathbb{R}^n$ on A^∞ is trivial. We conclude that $\mathrm{H}(x^\infty_{n,\alpha})\circ\mathrm{H}(y^\infty_{n,\alpha})$ is equal to 1 in $QH\big(A^\infty\rtimes_\beta G,(A^\infty\otimes\mathcal{K}^\infty)\rtimes_{\beta\rtimes\rho}G\big)$. And a similar computation shows that $\mathrm{H}(y^\infty_{n,\alpha})\circ\mathrm{H}(x^\infty_{n,\alpha})$ is 1 in $QH\big((B^\infty\rtimes_\alpha\mathbb{R}^n)\rtimes_{\beta\rtimes\rho}G,(B^\infty\rtimes_\alpha\mathbb{R}^n)\rtimes_{\beta\rtimes\rho}G\big)$. □

We have the following corollary of Theorem 4.1.

Corollary 4.12. *With all the notations and conditions as in Theorem 4.1, when* H *is the periodic cyclic homology functor* $\mathrm{HP}_\bullet$*, we have*

$$\mathrm{HP}_\bullet(((A^\infty\otimes\mathbb{C}_n)\rtimes_\alpha\mathbb{R}^n)\rtimes_{\beta\rtimes\rho}G)\cong\mathrm{HP}_\bullet(A^\infty\rtimes_\beta G).$$

Proof. By Lemma 2.5, periodic cyclic homology, $\mathrm{HP}(-)$, has all the properties which are assumed for the functor H in Theorem 4.1. As a corollary of Theorem 4.1, we obtain the desired formula. □

5. Application to the Rieffel Strict Deformation Quantization

In this section, we apply the equivariant Connes–Thom isomorphism (Theorem 4.1) for locally convex algebras to study the cyclic homology of Rieffel's strict deformation quantization. Recall that given a strongly continuous action α of $\mathbb{R}^n$ on a C^*-algebra A, and a skew-symmetric form J on $\mathbb{R}^n$, Rieffel [Rie93] constructed a strict deformation quantization A_J of A via oscillatory integrals,

$$a \times_J b := \int_{\mathbb{R}^n \times \mathbb{R}^n} \alpha_{Jx}(a)\alpha_y(b)e(x \cdot y)dxdy, \tag{5.1}$$

for $x, y \in \mathbb{R}^n$, and $a, b \in A^\infty$. The first copy of $\mathbb{R}^n$ in $\mathbb{R}^n \times \mathbb{R}^n$ is basically $\widehat{\mathbb{R}^n}$ after the identification of $\widehat{\mathbb{R}^n}$ and $\mathbb{R}^n$ (see the discussion at Page 11 of [Rie93, Chapter 2]). $\mathbb{R}^n$ acts on A_J by the same action α ([Rie93, Proposition 2.5]), and we denote the smooth vectors for this action by A_J^∞.

We recall the following result about Rieffel's strict deformation quantization from [Nes14]:

Proposition 5.1. *The map Θ_J from $A_J^\infty \rtimes \mathbb{R}^n$ to $A^\infty \rtimes \mathbb{R}^n$ defined by*

$$\Theta_J(f)(x) = \int_{\mathbb{R}^n} \alpha_{Jy}(\hat{f}(y))e(x \cdot y)dy$$

is an isomorphism, where $\hat{f}$ is the Fourier transform of $f \in \mathcal{S}(\mathbb{R}^n, A)$ and $e(t) := e^{2\pi\sqrt{-1}t}$.

Proof. See [Nes14, Theorem 1.1]. □

Let G be a finite group acting on A as in Sect. 1. This means that the G-action $\beta : G \to \mathrm{Aut}(A)$ on A satisfies

$$\beta_g \alpha_x = \alpha_{\rho_g(x)}\beta_g, \quad \text{for any } g \in G, x \in \mathbb{R}^n.$$

We compute

$$\begin{aligned}
\beta_g(a) \times_J \beta_g(b) &= \int_{\mathbb{R}^n \times \mathbb{R}^n} \alpha_{Jx}(\beta_g(a))\alpha_y(\beta_g(b))e(x \cdot y)dxdy \\
&= \int_{\mathbb{R}^n \times \mathbb{R}^n} \alpha_{(g^t)^{-1}Jg^{-1}x}\beta_g(a)\alpha_{gy}\beta_g(b)e(x \cdot gy)dxdy \\
&= \int_{\mathbb{R}^n \times \mathbb{R}^n} \alpha_{(g^t)^{-1}Jx}\beta_g(a)\alpha_{gy}\beta_g(b)e(gx \cdot gy)dxdy \\
&= \int_{\mathbb{R}^n \times \mathbb{R}^n} \beta_g\alpha_{Jx}(a)\beta_g\alpha_y(b)e(x \cdot y)dxdy \\
&= \beta_g(a \times_J b).
\end{aligned}$$

The above computation shows that the action of G on A_J by β is well-defined. So we get a G-action on $A_J^\infty \rtimes \mathbb{R}^n$ and $A^\infty \rtimes \mathbb{R}^n$. Abusively we call both actions by β again.

The following property ensures that we can make the isomorphism in Proposition 5.1 G-equivariant.

Proposition 5.2. *With the notations introduced in Proposition 5.1,*

$$\beta_g(\Theta_J(f)) = \Theta_J(\beta_g(f))$$

Proof. We have

$$\begin{aligned}
\Theta_J(\beta_g(f))(x) &= \int_{\mathbb{R}^n} \alpha_{Jy}(\widehat{\beta_g f}(y))e(x\cdot y)dy \\
&= \int_{\mathbb{R}^n} \alpha_{Jy}\left(\int_{\mathbb{R}^n} \beta_g f(t)e(-y\cdot t)dt\right)e(x\cdot y)dy \\
&= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \alpha_{Jy}\beta_g(f(g^{-1}t))e(-y\cdot t)e(x\cdot y)dtdy \\
&= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \beta_g\alpha_{g^{-1}Jy}(f(g^{-1}t))e(-y\cdot t)e(x\cdot y)dtdy
\end{aligned}$$

$$\begin{aligned}
&= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \beta_g\alpha_{g^{-1}Jy}(f(t))e(-y\cdot gt)e(x\cdot y)dtdy \\
&= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \beta_g\alpha_{g^{-1}J(g^t)^{-1}y}(f(t))e(-y\cdot t)e(g^{-1}x\cdot y)dtdy \\
&= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \beta_g\alpha_{Jy}(f(t))e(-y\cdot t)e(g^{-1}x\cdot y)dtdy \\
&= \beta_g(\Theta_J(f))(x),
\end{aligned}$$

where in the above formulas we have used gx to denote $\rho_g(x)$. □

Remark 5.3. In the above argument, since $y \in \widehat{\mathbb{R}^n}$, abusively we have written gy for $(g^t)^{-1}y$ and we have used the fact that $g^{-1}J(g^t)^{-1} = J$. In general one should be careful with $\widehat{\mathbb{R}^n}$ and $\mathbb{R}^n$.

Example 5.4. Recall that an n-dimensional noncommutative torus A_θ is the universal C^*-algebra generated by unitaries $U_1, U_2, U_3, \cdots, U_n$ subject to the relations

$$U_kU_j = e(\theta_{jk})U_jU_k$$

for $j, k = 1, 2, 3, \cdots, n$ and $\theta := (\theta_{jk})$ being a skew symmetric real $n \times n$ matrix. If we look at the holomorphically closed smooth subalgebra A_θ^∞ of A_θ, the algebra A_θ^∞ can also be viewed as the Rieffel strict deformation quantization [Rie93] of $C^\infty(\mathbb{T}^n)$ by the translation action of $\mathbb{R}^n$ on $C^\infty(\mathbb{T}^n)$ and θ a skew symmetric form on $\mathbb{R}^n$.

Example 5.5. Let G be $\mathbb{Z}_2$ or $\mathbb{Z}_3$ or $\mathbb{Z}_4$ or $\mathbb{Z}_6$ as finite cyclic groups (can be viewed as matrices in $SL_2(\mathbb{Z})$) acting on $\mathbb{R}^2$. Since the action is $\mathbb{Z}^2$-preserving, the 2-torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ inherits an action of G from the G-action on $\mathbb{R}^2$. Let A be the C^*-algebra of continuous functions on $\mathbb{T}^2$. The group $\mathbb{R}^2$ acts on $\mathbb{T}^2$ by translation. For $\theta \in \mathbb{R}$, we consider the symplectic form $\theta dx_1 \wedge dx_2$, also denoted by θ. So A_θ is just like the previous example of a 2-dimensional noncommutative torus. The action α (and β) of $\mathbb{R}^2$ (and G) on A satisfy Eq. (1.1). Now the G-action on A_θ is well-defined.

Recall that we can also consider the twisted group C^*-algebra $C^*(\mathbb{Z}^2 \rtimes G, \omega_\theta)$, where ω_θ is a 2-cocycle of $\mathbb{Z}^2$ ($\omega_\theta(x, y) := e^{2\pi i\langle\theta x, y\rangle}$, θ being a real number) extended trivially to the semi-direct product. These algebras are considered in [ELPW10]. Now it is not hard to see that $C^*(\mathbb{Z}^2 \rtimes G, \omega_\theta) = A_\theta \rtimes G$, where the latter is defined as in the previous paragraph (see [ELPW10, Lemma 2.10]). In general, with the above 2-cocycles, the twisted group algebras of groups like $\mathbb{Z}^n \rtimes G$ are basically coming from equivariant strict deformation quantization of $\mathbb{R}^n$ action on $A = C(\mathbb{T}^n)$.

Corollary 5.6. *The cyclic homology groups* $\mathrm{HP}_\bullet(A_\theta^\infty \rtimes G)$ *is independent of* θ.

Proof. Here we have $A = C(\mathbb{T}^2)$ and $J = \theta$. From the above discussion and Proposition 5.2 we get,

$$(A_\theta^\infty \rtimes \mathbb{R}^2) \rtimes G \simeq (A^\infty \rtimes \mathbb{R}^2) \rtimes G.$$

Now applying the HP functor on both sides we get,

$$\mathrm{HP}_\bullet\left((A_\theta^\infty \rtimes \mathbb{R}^2) \rtimes G\right) = \mathrm{HP}_\bullet\left((A^\infty \rtimes \mathbb{R}^2) \rtimes G\right). \tag{5.2}$$

Now since in this particular case G is a finite cyclic group, the G-action on $\mathbb{R}^2$ is $spin^c$ preserving. Indeed, the diagram

$$\begin{array}{ccccccccc} & & & & & & G & & \\ & & & & & & \downarrow & & \\ 0 & \longrightarrow & \mathbb{S}^1 & \longrightarrow & Spin^c(n) & \longrightarrow & SO(n) & \longrightarrow & 0 \end{array}$$

determines a group 2-cocycle on $SO(n)$. And since the restriction of this cocycle to G is trivial (as G is cyclic and preserving the symplectic structure J), the lift

$$\begin{array}{ccccccccc} & & & & & \swarrow\!\!\cdots & G & & \\ & & & & & & \downarrow & & \\ 0 & \longrightarrow & \mathbb{S}^1 & \longrightarrow & Spin^c(n) & \longrightarrow & SO(n) & \longrightarrow & 0, \end{array}$$

is always possible. Hence

$$\begin{aligned} \mathrm{HP}_\bullet\left((A_\theta^\infty \rtimes \mathbb{R}^2) \rtimes G\right) &= \mathrm{HP}_\bullet\left(\left((A_\theta^\infty \otimes \mathbb{C}_2) \rtimes \mathbb{R}^2\right) \rtimes G\right) \\ &= \mathrm{HP}_\bullet(A_\theta^\infty \rtimes G) \text{ (by Corollary 4.12).} \end{aligned}$$

Similar computation gives $\mathrm{HP}_\bullet((A^\infty \rtimes \mathbb{R}^2) \rtimes G) = \mathrm{HP}_\bullet(A^\infty \rtimes G)$. So the claim follows from the isomorphism given by Equation (1.2). □

Now, from [ELPW10, Theorem 0.1], we know that $\mathrm{K}_0(A_\theta \rtimes \mathbb{Z}_2) = \mathbb{Z}^6$, $\mathrm{K}_0(A_\theta \rtimes \mathbb{Z}_3) = \mathbb{Z}^8$, $\mathrm{K}_0(A_\theta \rtimes \mathbb{Z}_4) = \mathbb{Z}^9$, $\mathrm{K}_0(A_\theta \rtimes \mathbb{Z}_6) = \mathbb{Z}^{10}$ and $\mathrm{K}_1 = 0$ for all the cases. It is also well-known that $A_\theta^\infty \rtimes G$ is holomorphically closed inside $A_\theta \rtimes G$, for such G (see [CY19, Proposition 6.6]). Since the Chern character from $\mathrm{K}_\bullet(C^\infty(\mathbb{T}^2) \rtimes G) \otimes \mathbb{C}$ to $\mathrm{HP}_\bullet(C^\infty(\mathbb{T}^2) \rtimes G)$ is an isomorphism (by a result of Baum and Connes, see [Sol05, Page 279–80, equation 11 and 13]), we conclude that $\mathrm{HP}_0(A_\theta^\infty \rtimes \mathbb{Z}_2) = \mathbb{C}^6$, $\mathrm{HP}_0(A_\theta^\infty \rtimes \mathbb{Z}_3) = \mathbb{C}^8$, $\mathrm{HP}_0(A_\theta^\infty \rtimes \mathbb{Z}_4) = \mathbb{C}^9$, $\mathrm{HP}_0(A_\theta^\infty \rtimes \mathbb{Z}_6) = \mathbb{C}^{10}$ and $\mathrm{HP}_1 = 0$ for all these algebras.

Corollary 5.7. *For a general deformation* A_J^∞ *defined by Equation* (5.1), $\mathrm{HP}_\bullet(A_J^\infty \rtimes G)$ *is independent of the* J *parameter.*

Proof. We first prove the case when J is nondegenerate. From Proposition 5.1 we get,

$$A_J^\infty \rtimes \mathbb{R}^n \simeq A^\infty \rtimes \mathbb{R}^n.$$

Again, the G-action on $\mathbb{R}^n$ preserves the matrix J and therefore preserves a $spin^c$ structure. Now applying the functor $HP_\bullet$ on the both sides we get the desired conclusion for A_J^∞ with a skew symmetric $n \times n$ matrix J by a similar computation as in the proof of Corollary 5.6.

For a general J, we decompose $\mathbb{R}^n$ into a direct sum of $V \oplus W$ for two G-invariant subspaces of $\mathbb{R}^n$ where the restriction of J on V is zero, and the restriction of J on W is nondegenerate. One directly checks that the algebra A_J^∞ is isomorphic to $A_{J|_W}^\infty$ with the W-action on A and $J|_W$. We can then deduce the desired result for general $A_J^\infty = A_{J|_W}^\infty$ from the above nondegenerate case. □

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Appendix A. An equivariant version of Grensing's results

Let G be a finite group. We develop an equivariant version of Grensing's results [Gre12] and prove Proposition 2.4.

The following result is a straightforward generalization of Grensing's work [Gre12, Lemma 34]. Since G is a finite group, $\mathbb{C}G$ is isomorphic to $\oplus_{i=1}^d M_{n_i}(\mathbb{C})$ with a trivial grading, where d is the number of conjugacy classes of G, and for $i = 1, ..., d$, $\mathbb{C}^{n_i}$ is an irreducible G-representation. And for $i = 1, ...,$ $\mathbb{C}^{n_i}$ is an irreducible G-representation.

Lemma A.1. *Let* $(\alpha, \bar{\alpha}) : \mathbb{C}G \to \hat{\mathcal{B}} \trianglerighteq \mathcal{B} \rtimes G$ *be a quasihomomorphism. Then there is a quasihomomorphism* $(\alpha', \bar{\alpha}') : \mathbb{C}G \to M_{2d}((\mathcal{B} \rtimes G)^+) \trianglerighteq M_{2d}(\mathcal{B} \rtimes G)$ *such that*

$$\mathrm{H}(\theta_{\mathcal{B}}) \circ \mathrm{H}(\alpha, \bar{\alpha}) \circ \mathrm{H}(\kappa)^{-1} = \mathrm{H}(\alpha', \bar{\alpha}')$$

for every split-exact $\mathcal{K}^\infty$*-stable functor, where* $\theta_{\mathcal{B}} : \mathcal{B} \to M_{2d}(\mathcal{B})$ *denotes the stabilization, and* κ *is the stabilization map from* $\mathbb{C}^d$ *to* $\oplus_{i=1}^d M_{n_i}(\mathbb{C})$.

Proof. For each summand of $\mathbb{C}G$, $M_{n_i}(\mathbb{C})$, consider the quasihomomorphism κ from $\mathbb{C}$ to $M_{n_i}(\mathbb{C})$ given by stabilization. Now the composition of κ and $(\alpha, \bar{\alpha})$ is given by a quasihomomorphism (using [Gre12, Part (i) of Corollary 56]) from $\mathbb{C}$ to $\mathcal{B} \rtimes G$. So by [Gre12, Lemma 34], this quasihomomorphism gives rise to a quasihomomorphism $(\alpha', \bar{\alpha}') : \mathbb{C} \to M_2((\mathcal{B} \rtimes G)^+) \trianglerighteq M_2(\mathcal{B} \rtimes G)$ such that

$$\mathrm{H}(\theta) \circ \mathrm{H}(\alpha, \bar{\alpha}) \circ \mathrm{H}(\kappa)^{-1} = \mathrm{H}(\alpha', \bar{\alpha}'),$$

where θ is the M_2-stabilization. Now the result follows from taking the direct sum of each summand. □

As an immediate corollary we get the following generalization of [Gre12, Part (i) of Corollary 56].

Corollary A.2. *For two quasihomomorphisms* $(\alpha, \bar{\alpha}) : \mathbb{C}G \to \hat{\mathcal{B}} \trianglerighteq \mathcal{B} \rtimes G$ *and* $(\beta, \bar{\beta}) : \mathcal{B} \rtimes G \to \hat{C} \trianglerighteq C$*, there is a product between them (up to* M_{2d}*-stabilization) with respect to split-exact* $\mathcal{K}^{\infty}$*-stable functors.*

Proof. $\mathcal{K}^{\infty}$-stability implies M_{2d}-stability. Now the same arguments as in [Gre12, Part (i), Corollary 56] and Lemma A.1 give the result. □

Now we define an $R(G)$-module structure of $\mathrm{H}(\mathcal{A} \rtimes G)$ following the construction in [Phi87, Section 2.7, Definition 2.7.8] for a split-exact, $\mathcal{K}^{\infty}$-stable functor H on the category of (graded) locally convex algebras. For any element ρ_0 of $R(G)$, we define a morphism $[\rho_0]$ from $\mathrm{H}(\mathcal{A} \rtimes G)$ to itself in the following way.

Suppose that ρ is a G-representation on a finite dimensional vector space V of dimension l, and so it induces a G-action on $L(V)$. Let p_V and $p_{\mathbb{C}}$ be the projections from $V \oplus \mathbb{C}$ onto V and $\mathbb{C}$. We consider the algebra homomorphisms $\Phi_{V\oplus\mathbb{C}}, \Phi_{\mathbb{C}} : \mathcal{A} \rtimes G \to (L(V \oplus \mathbb{C}) \otimes \mathcal{A}) \rtimes G$ defined by

$$\Phi_{V\oplus\mathbb{C}}(a \otimes f) = (p_V \otimes a) \otimes f, \;\; \Phi_{\mathbb{C}}(a \otimes f) = (p_{\mathbb{C}} \otimes a) \otimes f.$$

Observe that $\Phi_{\mathbb{C}} : \mathcal{A} \rtimes G \to (End(V \oplus \mathbb{C}) \otimes \mathcal{A}) \rtimes G$ is a Morita equivalence, so the results of Grensing shows that $\mathrm{H}(\Phi_{\mathbb{C}})$ is invertible. We define $[\rho_0] : \mathrm{H}(\mathcal{A} \rtimes G) \to \mathrm{H}(\mathcal{A} \rtimes G)$ to be

$$\mathrm{H}(\Phi_{\mathbb{C}})^{-1} \circ \mathrm{H}(\Phi_{V\oplus\mathbb{C}}) : \mathrm{H}(\mathcal{A} \rtimes G) \to \mathrm{H}(\mathcal{A} \rtimes G).$$

To check that the above definition defines an $R(G)$-module structure on $\mathrm{H}(\mathcal{A} \rtimes G)$ for K-theory, the author of [Phi87, Section 2.7, Definition 2.7.8] used the Green-Julg descent theorem from the G-equivariant K-theory to the K-theory of the crossed product algebra. We can generalize this approach to $\mathrm{H}(\mathcal{A} \rtimes G)$. On the other hand, since G is finite, we can also directly check the $R(G)$-module structure on $\mathrm{H}(\mathcal{A} \rtimes G)$. For example, for a pair $[(\rho_1, V_1)], [(\rho_2, V_2)] \in R(G)$, we have the following algebra homomorphisms,

$$\Phi_{V_1\oplus V_2\oplus\mathbb{C}} : \mathcal{A} \rtimes G \to \big(L(V_1 \oplus V_2 \oplus \mathbb{C}) \otimes \mathcal{A}\big) \rtimes G, \;\; \Phi_{\mathbb{C}} : \mathcal{A} \rtimes G \to \big(L(V_1 \oplus V_2 \oplus \mathbb{C}) \otimes \mathcal{A}\big) \rtimes G.$$

The above algebra homomorphisms give the action of $[\rho_1 \oplus \rho_2]$ on $H(\mathcal{A} \rtimes G)$. Likewise, to define $[\rho_1] \oplus [\rho_2] : H(\mathcal{A} \rtimes G) \to H(\mathcal{A} \rtimes G)$, we consider the map

$$\mathcal{A} \rtimes G \xrightarrow{\Phi_{V_2\oplus\mathbb{C}} \oplus \Phi_{V_1\oplus\mathbb{C}}} \Big(\big(L(V_2 \oplus \mathbb{C}) \oplus L(V_1 \oplus \mathbb{C})\big) \otimes \mathcal{A}\Big) \rtimes G,$$

and

$$\mathcal{A} \rtimes G \xrightarrow{(\Phi_{\mathbb{C}}, \Phi_{\mathbb{C}})} \Big(\big(L(V_2 \oplus \mathbb{C}) \oplus L(V_1 \oplus \mathbb{C})\big) \otimes \mathcal{A}\Big) \rtimes G.$$

To show the additive property of the module structure, i.e. $[\rho_1 \oplus \rho_2] = [\rho_1] \oplus [\rho_2]$, we can map both $\big(L(V_1 \oplus V_2 \oplus \mathbb{C}) \otimes \mathcal{A}\big) \rtimes G$ and $\Big(\big(L(V_2 \oplus \mathbb{C}) \oplus L(V_1 \oplus \mathbb{C})\big) \otimes \mathcal{A}\Big) \rtimes G$ into

$\big(L(V_1 \oplus V_2 \oplus \mathbb{C} \oplus \mathbb{C}) \otimes \mathcal{A}\big) \rtimes G$, and prove that $[\rho_1 \oplus \rho_2] = [\rho_1] \oplus [\rho_2]$ by identifying the lifted maps to $\big(L(V_1 \oplus V_2 \oplus \mathbb{C} \oplus \mathbb{C}) \otimes \mathcal{A}\big) \rtimes G$. We can use a similar construction for $(V_1 \oplus \mathbb{C}) \otimes (V_2 \oplus \mathbb{C})$ to obtain the multiplicative property, i.e., $[\rho_1] \cdot [\rho_2] = [\rho_1 \circ \rho_2]$, by mapping $\big(L(V_1 \otimes V_2 \oplus \mathbb{C}) \otimes \mathcal{A}\big) \rtimes G$ in $\Big(L\big((V_1 \oplus \mathbb{C}) \otimes (V_2 \oplus \mathbb{C})\big) \otimes A\Big) \rtimes G$, the detail of which is left to the reader.

Now we have the following generalization of [Gre12, Proposition 45].

Proposition A.3. *(Proposition 2.4) Let* $x = (B(\mathcal{H} \rtimes G), \phi, F)$ *be a locally convex Kasparov* $(\mathbb{C}G, \mathcal{K}^\infty \rtimes G)$*-module defined above from a* G*-equivariant Kasparov module* $(\mathcal{H}, \varphi, \tilde{F})$ *for* C^**-algebras* $(\mathbb{C}, \mathcal{K})$*, and* H *be a diffotopy invariant, split-exact,* $\mathcal{K}^\infty$*-stable functor. Then* $\mathrm{H}(Qh(x)) = \mathrm{index}_G\, \tilde{F} \circ \theta_*$*, where* $\theta_* : \mathrm{H}(\mathbb{C}G) \to \mathrm{H}(\mathcal{K}^\infty \rtimes_\rho G)$ *denotes the stabilization map, and* $\mathrm{index}_G\, \tilde{F}$ *is the* G*-index of the operator* $\tilde{F}$ *which is an element of* $R(G)$*.*

Proof. The proof is essentially same as the proof of [Gre12, Proposition 45]. Using Grensing's notation, we assume $\tilde{F} = \begin{pmatrix} 0 & S \\ T & 0 \end{pmatrix}$. By the hypothesis, T is Fredholm and hence has closed co-kernel. The co-kernel is a G-space, and we assume further that $1 - TT'$ (using Grensing's notation) be a map from $\mathbb{C}^l \rtimes G$ to $(\mathbb{C}^l \otimes \mathcal{K}^\infty) \rtimes_\rho G$ (as a map of Hilbert modules), where $\mathbb{C}^l$ carries the G-action via an irreducible representation χ of G. If the action were trivial we would get $\mathrm{H}(1 - TT') = -l\theta_*$ (using Grensing's non-equivariant version). We observe that since G is finite, there always exists a G-invariant rank-one projection in $\mathcal{K}^\infty$. Then we can reduce the proof to the trivial case by applying the element $[\chi]$ on $\mathrm{H}(\mathcal{K}^\infty \rtimes_\rho G)$. □

Appendix B. Connes' pseudo-differential calculus

We review briefly in this appendix the key results of Connes' psuedodifferential calculus for $\mathbb{R}^n$-actions, [Con80]. We assume that the reader is familiar with the definition of classical pseudo-differential calculus on $\mathbb{R}^n$, i.e., Hörmander classes of symbols, and refer to [Hör65,Hör07] for a thorough discussion of the classical theory. Connes in [Con80] introduced an anisotropic version of Hörmander classes of symbols, which was studied later by Baaj in [Baa88a,Baa88b] in detail.

Suppose that $\mathbb{R}^n$, an abelian Lie group, acts (with the action denoted by α) strongly continuously on a unital (adjoining a unit if necessary) C^*-algebra A (possibly graded). Let A^∞ be the subspace of A of smooth vectors for the action. For a multi-index k, and $a \in A^\infty$, let us denote the k-th derivative of a (with respect to the action of $\mathbb{R}^n$) by $\delta^k(a)$. We will equip A^∞ with the semi-norms (p_i),

$$p_i(a) := \left\| \delta^i(a) \right\|,$$

for multi-indices i. With this family of seminorms A^∞ is a Fréchet algebra, and α is isometric on A^∞. We refer to Chapter 1 and Chapter 2 of [Rie93] for more details about A^∞. For the C^*-dynamical system $(A, \mathbb{R}^n, \alpha)$, let $x \mapsto V_x$ be the canonical representation of $\mathbb{R}^n$ in $M(A \rtimes_\alpha \mathbb{R}^n)$, the multiplier algebra, with $V_x a V_x^* = \alpha_x(a)$ $(a \in A)$. Let $\widehat{\mathbb{R}^n}$ be the Fourier dual of $\mathbb{R}^n$ as before. We shall say that ρ, a C^∞ map from $\widehat{\mathbb{R}^n}$ to A^∞, is a symbol of order m, $\rho \in S^m(\widehat{\mathbb{R}^n}, A^\infty)$ if the following properties hold:

(1) for all multi-indices i, j, there exists $C_{ij} < \infty$ such that

$$p_i\left(\left(\frac{\partial}{\partial \xi}\right)^j \rho(\xi)\right) \leq C_{ij}(1+|\xi|)^{m-|j|};$$

(2) there exists $s \in C^\infty(\widehat{\mathbb{R}^n}\backslash\{0\}, A^\infty)$ such that when $\lambda \to +\infty$ one has

$$\lambda^{-m}\rho(\lambda\xi) \to s(\xi)$$

(for the topology of $C^\infty(\widehat{\mathbb{R}^n}\backslash\{0\}, A^\infty)$).

When A is $C_0(\mathbb{R}^n)$ and $\mathbb{R}^n$ is acting on A by the translation action, we may think of ρ as a function of two variables. In this case we get back to the classical symbols ([LMR10, Lemma 2.7]).

Connes proved that an order zero symbol gives rise to an element of the multiplier algebra of the crossed product $A \rtimes_\alpha \mathbb{R}^n$. Indeed, if ρ is a symbol of order zero then we can take the Fourier transform (in the sense of distribution):

$$\widehat{\rho}(x) = \int_{\widehat{\mathbb{R}^n}} \rho(\xi)e(-\langle x, \xi\rangle)d\xi,$$

which is a well-defined distribution on $\mathbb{R}^n$ with value in A^∞ (it will be clear later what a distribution means).

Recall that the following inner product

$$(\xi, \eta) := \int \xi^*(x)\eta(x)dx, \qquad \forall \xi, \eta \in \mathcal{S}(\mathbb{R}^n, A^\infty).$$

makes $\mathcal{S}(\mathbb{R}^n, A^\infty)$ into a pre-C^*-module. Following [Con80, Prop. 8] and [Baa88a, Théorème 4.1.], we define a continuous operator D_ρ on $\mathcal{S}(\mathbb{R}^n, A^\infty)$ with respect to the pre-C^*-module norm, which extends to an element $D_\rho \in M(A \rtimes_\alpha \mathbb{R}^n)$ by

$$D_\rho := \int_{\mathbb{R}^n} \widehat{\rho}(x)V_x dx.$$

It follows from [Baa88a, Definition 3.1] that D_ρ acts on the smooth sub-algebra $\mathcal{S}(\mathbb{R}^n, A^\infty)$ of $A \rtimes_\alpha \mathbb{R}^n$ by the oscillatory integral (see [Abe12, Section 3.3])

$$D_\rho(u)(t) := \int_{\mathbb{R}^n}\int_{\widehat{\mathbb{R}^n}} \alpha_{-t}(\rho(\xi))u(s)e(-\langle(t-s), \xi\rangle)dsd\xi.$$

To get the motivation of the above equation, let us take $\rho \in \mathcal{S}(\mathbb{R}^n, A^\infty)$. Then

$$\begin{aligned} D_\rho(u)(t) &= \Big(\int_{\mathbb{R}^n} \widehat{\rho}(s)V_s u ds\Big)(t) = \int_{\mathbb{R}^n} \alpha_{-t}(\widehat{\rho}(s))V_s(u(t))ds \\ &= \int_{\mathbb{R}^n} \alpha_{-t}(\widehat{\rho}(s))u(t-s)ds = \int_{\mathbb{R}^n} \alpha_{-t}(\widehat{\rho}(t-s))u(s)ds \\ &= \int_{\mathbb{R}^n}\int_{\widehat{\mathbb{R}^n}} \alpha_{-t}(\rho(\xi))u(s)e(-\langle(t-s), \xi\rangle)dsd\xi. \end{aligned}$$

Note that the above integrals exist in the usual sense.

The norm closure of all multipliers, which come from order zero symbols, is denoted by $\mathcal{D}(A \rtimes_\alpha \mathbb{R}^n)$. From [Con80, Prop. 8] and [Baa88a] there is an exact sequence

$$0 \longrightarrow A \rtimes_\alpha \mathbb{R}^n \xrightarrow{\phi} \mathcal{D}(A \rtimes_\alpha \mathbb{R}^n) \xrightarrow{\psi} A \otimes C(S^{n-1}) \longrightarrow 0\cdot$$

This exact sequence is often called the pseudo-differential extension. It is well-known that there is a non-degenerate morphism from $C^*(\mathbb{R}^n)$ to $M(A \rtimes_\alpha \mathbb{R}^n)$. So this morphism extends to the multiplier algebra of $C^*(\mathbb{R}^n)$ and in particular to the sub-algebra $\mathcal{D}(C^*(\mathbb{R}^n))$. So if we say $D \in \mathcal{D}(C^*(\mathbb{R}^n))$, we view D inside $M(A \rtimes_\alpha \mathbb{R}^n)$.

Theorem B.1. *For $a \in A \hookrightarrow M(A \rtimes_\alpha \mathbb{R}^n)$ and $D \in \mathcal{D}(C^*(\mathbb{R}^n))$, we have $[D, a] \in A \rtimes_\alpha \mathbb{R}^n$.*

Proof. See [Baa88a, Section 4], also [DS15, Proposition 4.3] for a more general version of this property. □

Let us recall the definition of asymptotic expansion of a symbol. For a decreasing divergent sequence $(m_j)_{j\in\{0,1,2,\cdots\}}$, and $\rho_j \in S^{m_j}(\widehat{\mathbb{R}^n}, A^\infty)$, we say $\rho \in S^{m_0}(\widehat{\mathbb{R}^n}, A^\infty)$ admits an asymptotic expansion $\sum \rho_j$ (written as $\rho \sim \sum \rho_j$), if for all integers $k \geq 1$,

$$\rho - \sum_{j\leq k} \rho_j \in S^{m_k}(\widehat{\mathbb{R}^n}, A^\infty).$$

Theorem B.2. *For $\rho_1 \in S^{m_1}(\widehat{\mathbb{R}^n}, A^\infty)$ and $\rho_2 \in S^{m_2}(\widehat{\mathbb{R}^n}, A^\infty)$, there exists a unique $\rho \in S^{m_1+m_2}(\widehat{\mathbb{R}^n}, A^\infty)$ such that $D_\rho = D_{\rho_1} D_{\rho_2}$. Also ρ admits an asymptotic expansion:*

$$\rho(\xi) \sim \sum_k \frac{\sqrt{-1}^{|k|}}{k!} \rho_1^{(k)}(\xi)\delta^k(\rho_2(\xi)).$$

Proof. See [Baa88a, Proposition 3.2]. Also [LM16, Theorem 2.2], for twisted dynamical systems. □

Theorem B.3. *For $\rho \in S^0(\widehat{\mathbb{R}^n}, A^\infty)$, the adjoint $(D_\rho)^*$ of the operator D_ρ exists and $(D_\rho)^* = D_{\rho'}$, where ρ' admits an asymptotic expansion:*

$$\rho'(\xi) \sim \sum_k \frac{\sqrt{-1}^{|k|}}{k!} \delta^k((\rho')^{(k)}(\xi)^*).$$

Proof. See [Baa88a, Proposition 3.3]. Also [LM16, Theorem 2.2], for twisted dynamical systems. □

Remark B.4. Since the unitization of A (and A') sits inside $M(A)$ (and $M(A')$) nondegenerately, we get ([DS15, Proposition 3.2]) a non-degenerate morphism from $A' \rtimes \mathbb{R}^n$ to $M(A \rtimes \mathbb{R}^n)$ giving a morphism from $\mathcal{D}(A' \rtimes_\alpha \mathbb{R}^n)$ to $M(A \rtimes \mathbb{R}^n)$. Hence, though we adjoin a unit for the non-unital A, ultimately we end up with getting an element in $M(A \rtimes \mathbb{R}^n)$.

Appendix C. Proof of Lemma 4.2

In the following, we outline a proof of Lemma 4.2 that $\frac{\partial \Sigma}{\partial \xi_j}$, $j = 1, ..., n$, is a Schwartz function.

Lemma C.1.

$$e^{\sqrt{-1}sc(\xi)} = \cos(s|\xi|) + \sqrt{-1}c(\xi)\frac{\sin(s|\xi|)}{|\xi|}$$
$$\frac{\partial}{\partial \xi_j}e^{\sqrt{-1}sc(\xi)} = s\sin(s|\xi|)\frac{\xi_j}{|\xi|} + \sqrt{-1}\Big(c^j\frac{\sin(s|\xi|)}{|\xi|} + c(\xi)\frac{\partial}{\partial \xi_j}\frac{\sin(s|\xi|)}{|\xi|}\Big).$$

Proof. Using the power series, we have

$$\begin{aligned} e^{\sqrt{-1}sc(\xi)} &= \sum_{n=0}^{\infty}\frac{\left(\sqrt{-1}sc(\xi)\right)^n}{n!} \\ &= \sum\frac{\left(\sqrt{-1}sc(\xi)\right)^{2n}}{(2n)!} + \sum\frac{\left(\sqrt{-1}sc(\xi)\right)^{2n+1}}{(2n+1)!} \\ &= \sum\frac{(-1)^n s^{2n}|\xi|^{2n}}{(2n)!} + \sqrt{-1}\sum\frac{(-1)^n s^{2n+1}c(\xi)|\xi|^{2n}}{(2n+1)!} \\ &= \cos(s|\xi|) + \sqrt{-1}\frac{c(\xi)}{|\xi|}\sin(s|\xi|). \end{aligned}$$

Let $\{e^j\}$ be an orthonormal basis of $\mathbb{R}^n$, and denote $c^j = c(e^j)$. Then $c(\xi) = \sum_j \xi_j c^j$ for $\xi = \sum_j \xi_j e^j$. Differentiate $e^{isc(\xi)}$ with respect to ξ_j.

$$\frac{\partial}{\partial \xi_j}e^{\sqrt{-1}sc(\xi)} = s\sin(s|\xi|)\frac{\xi_j}{|\xi|} + \sqrt{-1}\Big(c^j\frac{\sin(s|\xi|)}{|\xi|} + c(\xi)\frac{\partial}{\partial \xi_j}\frac{\sin(s|\xi|)}{|\xi|}\Big).$$

□

Notice that $\frac{\sin(y)}{y}$ is a smooth even function of the variable y. Therefore, the function

$$\frac{\sin(s|\xi|)}{s|\xi|}$$

is a smooth function with respect to s and $|\xi|^2$, so is subsequently smooth with respect to ξ_j, for $j = 1, \cdots, n$.

Set $H(s,\xi) = \frac{\sin(s|\xi|)}{s|\xi|}$. Then

$$\frac{\partial}{\partial \xi_j}e^{\sqrt{-1}sc(\xi)} = s^2 H(s,\xi)\xi_j + \sqrt{-1}s\Big(c^j H(s,\xi) + c(\xi)\frac{\partial}{\partial \xi_j}H(s,\xi)\Big).$$

Recall that the function Σ is defined as follows

$$1 \otimes \chi(c(\xi)), \text{ where } \chi(c(\xi)) = \int_{\mathbb{R}} \widehat{\chi}(s)e^{\sqrt{-1}sc(\xi)}ds.$$

The derivative $\frac{\partial}{\partial \xi_j}\chi(c(\xi))$ can be expressed as follows,

$$\frac{\partial}{\partial \xi_j}\chi(c(\xi)) = \int_{\mathbb{R}} \widehat{\chi}(s)s\Big(sH(s,\xi)\xi_j + \sqrt{-1}s\big(c^j H(s,\xi) + c(\xi)\frac{\partial}{\partial \xi_j}H(s,\xi)\big)\Big)ds.$$

We remark that as $\widehat{\chi}(s)s$ is a compactly supported smooth function, and $H(s,\xi)$ is a smooth function in both s and ξ, the above integral formula for $\frac{\partial}{\partial \xi_j}\chi(c(\xi))$ holds true. We can even conclude that $\frac{\partial}{\partial \xi_j}\chi(c(\xi))$ is a smooth function with respect to the variable ξ.

In the following, we show that $\frac{\partial}{\partial \xi_j}\chi(c(\xi))$ is a Schwartz function.

Lemma C.2. *Let* $h(y) = \frac{\sin(y)}{y}$. $h(y)$ *is a smooth function on* $\mathbb{R}$. *Furthermore, for any* n, *there are polynomials* $\phi_n(y)$ *and* $\psi_n(y)$ *of degree less than or equal to* n *such that*

$$\frac{d^n}{dy^n}h(y) = \frac{\sin(y)\phi_n(y) + \cos(y)\psi_n(y)}{y^{n+1}}.$$

Proof. This can be proved by induction with direct computation. □

Lemma C.3. *For every* $J = (j_1, \cdots, j_m) \in \mathbb{N} \times \cdots \mathbb{N}$, *there are polynomials* $\Phi_{J,k}$ *and* $\Psi_{J,k}$ *of* n*-variables with* $2(j_1 + \cdots + j_m)) \geq deg(\Phi_{J,k})$ *and* $(deg(\Psi_{J,k}) + 1)$ *satisfying*

$$\frac{\partial^{j_1+\cdots+j_m}}{\partial \xi_J}H(s,\xi) = \sum_{k=0}^{j_1+\cdots+j_m} s^k \frac{d^k}{dy^k}h(s|\xi|)\frac{\Phi_{J,k} + |\xi|\Psi_{J,k}}{|\xi|^{2(j_1+\cdots+j_m)+1}}.$$

Proof. This can be proved directly by induction on the total order of derivatives $\nu = j_1 + \cdots + j_m$. □

We now look at the function $\frac{\partial}{\partial \xi_j}\chi(c(\xi))$. It is the sum of three terms

$$\int_{\mathbb{R}} \widehat{\chi}(s)s^2 H(s,\xi)\xi_j ds,\ ic^j \int_{\mathbb{R}} \widehat{\chi}(s)s^2 H(s,\xi)ds,\ i\int_{\mathbb{R}} \widehat{\chi}(s)sc(\xi)\frac{\partial}{\partial \xi_j}H(s,\xi)ds.$$

To prove that $\frac{\partial}{\partial \xi_j}\chi(c(\xi))$ is a Schwartz function, it suffices to prove that each of them is a Schwartz function. As they are all similar, it is enough to prove that for a Schwartz function $\kappa(s)$, the following function

$$|\xi|^l \int_{\mathbb{R}} \kappa(s)\frac{\partial^{j_1+\cdots+j_m}}{\partial \xi_J}H(s,\xi)ds \tag{C.1}$$

is bounded for every fixed $j_1, \cdots, j_m$ and l.

By Lemma C.3, we are reduced to prove for each k, the following function

$$\begin{aligned} &|\xi|^l \int_{\mathbb{R}} \kappa(s)s^k \frac{d^k}{dy^k}h(s|\xi|)\frac{\Phi_{J,k} + |\xi|\Psi_{J,k}}{|\xi|^{2(j_1+\cdots+j_m)+1}}ds \\ &= \frac{\Phi_{J,k} + |\xi|\Psi_{J,k}}{|\xi|^{2(j_1+\cdots+j_m)+1}}|\xi|^l \int_{\mathbb{R}} \kappa(s)s^k \frac{d^k}{dy^k}h(s|\xi|)ds. \end{aligned} \tag{C.2}$$

Notice that $\frac{\partial^l}{\partial s^l}\frac{d^k}{dy^k}h(s|\xi|) = |\xi|^l \frac{d^k}{dy^k}h(s|\xi|)$. We have the following equation

$$\frac{\Phi_{J,k} + |\xi|\Psi_{J,k}}{|\xi|^{2(j_1+\cdots+j_m)+1}}|\xi|^l \int_{\mathbb{R}} \kappa(s)s^k \frac{d^k}{dy^k}h(s|\xi|)ds = \frac{\Phi_{J,k} + |\xi|\Psi_{J,k}}{|\xi|^{2(j_1+\cdots+j_m)+1}} \int_{\mathbb{R}} \kappa(s)s^k \frac{\partial^l}{\partial s^l}\frac{d^k}{dy^k}h(s|\xi|)ds.$$

Integration by parts gives that the right hand side of the equation can be written as

$$\frac{\Phi_{J,k} + |\xi|\Psi_{J,k}}{|\xi|^{2(j_1+\cdots+j_m)+1}}(-1)^l \int_{\mathbb{R}} \frac{d^l}{ds^l}\left(\kappa(s)s^k\right)\frac{d^k}{dy^k}h(s|\xi|)ds. \tag{C.3}$$

By the degree counting, when $|\xi|$ is sufficiently large,

$$\frac{\Phi_{J,k} + |\xi|\Psi_{J,k}}{|\xi|^{2(j_1+\cdots+j_m)+1}}$$

is uniformly bounded.

By Lemma C.2, the function $\frac{d^k}{dy^k}h$ is uniformly bounded again for all k. Therefore, $\frac{d^k}{dy^k}h(s|\xi|)$ is uniformly bounded. Finally, as κ is assumed to be a Schwartz functions, $\frac{d^l}{ds^l}\left(\kappa s^k\right)$ is again a Schwartz function. Therefore, the integral

$$\int_{\mathbb{R}} \frac{d^l}{ds^l}\left(\kappa(s)s^k\right)\frac{d^k}{dy^k}h(s|\xi|)ds$$

is uniformly bounded. Hence, we summarize from the above discussion that the whole function

$$\frac{\Phi_{J,k} + |\xi|\Psi_{J,k}}{|\xi|^{2(j_1+\cdots+j_m)+1}}(-1)^l \int_{\mathbb{R}} \frac{d^l}{ds^l}\left(\kappa(s)s^k\right)\frac{d^k}{dy^k}h(s|\xi|)ds$$

introduced in Equation (C.3) is bounded, and therefore the function introduced in Equation (C.1) is bounded via Equation (C.2). From this property, we can conclude that the function $\frac{\partial}{\partial \xi_j}\chi(c(\xi))$ is a Schwartz function. And it follows that $\frac{\partial \Sigma}{\partial \xi_j}$ is a Schwartz function.

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