MASSES AT NULL INFINITY FOR EINSTEIN'S EQUATIONS IN HARMONIC COORDINATES

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Dedicated to Demetrios Christodoulou on the occasion of his 70th birthday

Abstract. In this work we give a complete picture of how to in a direct simple way define the mass at null infinity in harmonic coordinates in three different ways that we show satisfy the Bondi mass loss law. The first and second way involve only the limit of metric (Trautman mass) respectively the null second fundamental forms along asymptotically characteristic surfaces (asymptotic Hawking mass) that only depend on the ADM mass. The last involves construction of special characteristic coordinates at null infinity (Bondi mass). The results here rely on asymptotics of the metric derived in [27].

1. Introduction

The first definition of mass at null infinity was given by Trautman [36]. In Trautman's definition, the mass is defined as the integral of the so called superpotential which is expressed in terms of the metric and its first order derivatives over the spheres receding to null infinity, see §1.4 for precise definition. We refer the readers to [37, 38, 3, 16] and [5, §3.1] and references therein for more details.

In 1960, Bondi [6] introduced a new approach which was based on the outgoing null rays to study the gravitational waves. Later, Bondi, Metzner and van der Burg [7] considered the axisymmetric spacetimes. Soon after, Sachs [33] generalized the formalism to non axisymmetric spacetimes. In the Bondi-Sachs formalism, the coordinates which are called Bondi-Sachs coordinates, are adapted to the null geodesics of the space time. With respect to such a coordinate system, only 6 metric quantities are needed to describe the spacetime, and the Bondi mass and radiated energy at null infinity are defined in terms of certain lower order terms of these metric components. Hintz-Vasy [18] showed the existence of the Bondi-Sachs coordinates for a specific class of initial data and identified the Bondi mass in a generalized wave coordinates.

Christodoulou [11] introduced an alternative approach to defining the mass at null infinity without the need to use the Bondi-Sachs coordinates. The definition was given as the limit of the Hawking mass of a family of spheres that converge to a round metric sphere along the outgoing null hypersurfaces towards null infinity. Christodoulou proved that the limit of the Hawking mass exists and satisfies a mass loss law for the initial data used in [14] by analyzing the null structure Einstein equations. Later on, the limit of Hawking mass of suitable spheres was analyzed in the settings of the work [4, 24].

These three masses are defined completely differently and each has been analyzed in several settings. As summarized in [5], in the setting where the Bondi-Sachs formalism can be carried out, the limit of the Hawking mass along suitable family of spheres recovers the Trautman mass and Bondi mass. However, the notion of the mass and radiated energy at null infinity in harmonic coordinates remains to be clarified. In this work we give a complete picture of how to define the mass at null infinity in harmonic coordinates in the different ways that we show satisfy the Bondi mass loss law and therefore coincide.

1.1. Einstein vacuum equations in harmonic coordinates.

Einstein's equations in harmonic are a system of nonlinear wave equations

$$\widetilde{\square}_g\, g_{\mu\nu} = F_{\mu\nu}(g)(\partial g, \partial g), \qquad \text{where} \quad \widetilde{\square}_g = \sum g^{\alpha\beta}\partial_\alpha\partial_\beta, \tag{1.1}$$

for a Lorentzian metric $g_{\alpha\beta}$, that in addition satisfy the preserved wave coordinate condition

$$\partial_\alpha\big(\sqrt{|g|}g^{\alpha\beta}\big) = 0, \qquad \text{where} \quad |g| = |\det{(g)}|. \tag{1.2}$$

Choquet-Bruhat [9] proved local existence in these coordinates. Christodoulou-Klainerman [14] proved global existence for Einstein's vacuum equations $R_{\mu\nu}=0$ for small asymptotically flat initial data:

$$g_{ij}\big|_{t=0}=(1+Mr^{-1})\,\delta_{ij}+o(r^{-1-\gamma}),\quad \partial_t g_{ij}\big|_{t=0}=o(r^{-2-\gamma}),\quad r=|x|,\qquad 0<\gamma<1, \tag{1.3}$$

where $M>0$ by the positive mass theorem [35, 40]. The proof avoids using coordinates since it was believed the metric in harmonic coordinates would blow up for large times. John [19, 20] had noticed that solutions to some nonlinear wave equations blow up for small data, whereas Klainerman [22, 23], see also Christodoulou [10], came up with the "null condition", that guaranteed global existence for small data. However Einstein's equations do not satisfy the null condition. The null condition provide a cancellation of the nonlinear terms so that solutions decay like solutions of linear equations. Hörmander introduced a simplified asymptotic system, by neglecting angular derivatives which we expect decay faster due to the rotational invariance, to study blowup. Lindblad [26] showed that the asymptotic system corresponding to the quasilinear part of Einstein's equations does not blow up and gave an example of a nonlinear equation of this form that have global solutions that do not decay as much. Lindblad-Rodnianski [28] introduced the *weak null condition* requiring that the corresponding asymptotic system have global solutions and showed that Einstein's equations in *wave coordinates* satisfy the weak null condition which was used in [29, 30] to prove global existence. Starting from the L^2 estimates in [29, 30], Lindblad [27] derived more detailed asymptotics that we will rely on. We expect the result of this manuscript to be true in the presence of matter since these asymptotics can also be derived by directly using a change of coordinates which is equivalent to generalized wave coordinates as in Kauffmann-Lindblad [21] and [8, 17].

1.2. **The characteristic surfaces.** In order to unravel the effect of the quasilinear terms in (1.1) one can change to characteristic coordinates as in [14], but this loses regularity and is not explicit. Instead Lindblad [27] used the asymptotics of the metric to determine the characteristic surfaces asymptotically and used this to construct coordinates. Due to the wave coordinate condition (1.2) the outgoing light cones of a solution with asymptotically flat data (1.3) approach those of the Schwarzschild metric with the same mass M. In [27] it was shown that there is a solution to the eikonal equation that approaches the one for Schwarzschild

$$g^{\alpha\beta}\partial_\alpha u\,\partial_\beta u=0,\qquad u\to u^{*}\!=t-r^{*}_{,}\quad \text{when}\qquad r>t/2\to\infty,\qquad \text{where } r^{*}\!=r+M\ln r+O(M/r). \tag{1.4}$$

1.3. **The asymptotics of the metric.** In [27] the precise asymptotics of the metric was given. Asymptotically the metric is Minkowski metric $m_{\mu\nu}$ plus

$$h_{\mu\nu}(t,r\omega)\sim H_{\mu\nu}(\widetilde{r}-t,\omega)/(t+r)+K_{\mu\nu}\big(\tfrac{t+\widetilde{r}}{|\widetilde{r}-t|+1},\omega\big)/(t+r),\qquad \widetilde{r}=r+M\ln r,\qquad \omega=x/|x|.$$

Here H is concentrated close to the outgoing light cones $\widetilde{q}=\widetilde{r}-t$ constant, $|H(\widetilde{q},\omega)|\le\varepsilon(1+|\widetilde{q}|)^{-\gamma'}$ where $\gamma'=\gamma-C\varepsilon$ for some constant C and small constant ε, and K is homogeneous of degree 0 with a log singularity at the light cone $|K(s,\omega)|\le\varepsilon\ln|s|$ for the nontangential components. H is the radiation field of the free curved wave operator, the left of (1.1), and K is the backscattering of the wave operator with a source term $F_{\mu\nu}\sim P_{\mathcal{S}}(\partial_\mu h,\partial_\nu h)$ in the right of (1.1), where $P_{\mathcal{S}}$ is the norm of the components tangential to the spheres. In the wave zone

$$K_{\mu\nu}\big(\tfrac{t+\widetilde{r}}{|\widetilde{r}-t|+1},\omega\big)\sim L_\mu(\omega)L_\nu(\omega)\int_{\widetilde{r}-t}^{\infty}\frac12\ln\Big(\frac{t+\widetilde{r}+\widetilde{q}}{t-\widetilde{r}+\widetilde{q}}\Big)n\big(\widetilde{q},\omega\big)\,d\widetilde{q},\quad \text{when}\quad |t-\widetilde{r}|<<t+\widetilde{r},$$

where $L_\mu\!=\!m_{\mu\nu}L^\nu$, in a null frame $L\!=\!(1,\omega)$, $\underline{L}\!=\!(1,-\omega)$ and orthonormal $S_1,S_2\!\in\! T(\mathbb{S}^2)$, we have

$$n(\widetilde{q},\omega)=-P_{\mathcal{S}}\big(\partial_{\widetilde{q}}H,\partial_{\widetilde{q}}H\big)(\widetilde{q},\omega),\quad \text{where}\quad P_{\mathcal{S}}(D,E)=-D_{AB}\,E^{AB}/\,2,\quad A,B\in\{S_1,S_2\}. \tag{1.5}$$

Remark. In this manuscript $\widetilde{\bullet},\widehat{\bullet},\overline{\bullet}$ refer to the quantities expressed in the coordinates $(\widetilde{t}=t,\widetilde{r}\omega)$ where $\widetilde{r}=r+M\ln r$, modified asymptotically Schwarzschild null coordinates $\widehat{y}^p$ as defined in 2.1 and the Bondi-Sachs coordinates respectively, unless otherwise specified.

1.4. **The Trautman mass and radiated energy.** We will use the surface $\widetilde{u}=\widetilde{t}-\widetilde{r}$ constant instead of the null cones to define the Trautman mass and radiated energy at null infinity in terms of the asymptotics of the metric components in wave coordinates.

We let $\partial_\mu = A_\mu^\nu \widetilde{\partial}_\nu$ and $\tilde{g}^{\mu\nu} = A_\alpha^\mu A_\beta^\nu g^{\alpha\beta}$, where $A_\mu^\nu = \partial \widetilde{x}^\nu / \partial x^\mu$ and $\widetilde{x} = \widetilde{r}\omega$ where $\widetilde{r} = r + M \ln r$. Let $S_{\widetilde{u},\widetilde{r}} = \{(\widetilde{t}, \widetilde{x}); \widetilde{t} = \widetilde{u} + \widetilde{r}\}$ be a sphere, following [36, 37, 38, 5] we define the *Trautman four-momentum* as

$$M_T^\alpha(\widetilde{u}) = \lim_{r\to\infty} \frac{1}{4\pi} \int_{S_{\widetilde{u},\widetilde{r}}} \widetilde{\mathbb{U}}^{\alpha\beta\gamma}\, dS_{\beta\gamma}.$$

Here $dS_{\beta\gamma} = n_{[\beta}k_{\gamma]}\widetilde{r}^2 dS(\omega)$ with $n_\gamma = (d\widetilde{r})_\gamma = (0, \omega_i)$, $k_\beta = (d\widetilde{t})_\beta = (1, 0, 0, 0)$, and the superpotential $\widetilde{\mathbb{U}}^{\alpha\beta\gamma}$ is

$$\widetilde{\mathbb{U}}^{\alpha\beta\gamma} = \sqrt{|\tilde{g}|}\tilde{g}^{\alpha\mu}\widetilde{\mathbb{U}}_\mu^{\beta\gamma} \quad \text{where} \quad \widetilde{\mathbb{U}}_\mu^{\beta\gamma} = \sqrt{|\tilde{g}|}\tilde{g}^{\alpha\mu}\tilde{g}^{\sigma[\rho}\delta_\mu^\gamma \tilde{g}^{\beta]\tau}\widetilde{\partial}_\tau \tilde{g}_{\rho\sigma}.$$

Here the square brackets denote the antisymmetric part of a tensor, i.e., $T^{[a_1\cdots a_l]} = \sum_\sigma (-1)^\sigma T^{a_{\sigma(1)}\cdots a_{\sigma(l)}}$ where the sum is taken over all permutations σ of $1, \dots, l$ and $(-1)^\sigma$ is 1 for even permutations and -1 for odd permutations. A direct computation implies

$$\widetilde{\mathbb{U}}^{\alpha\beta\gamma} = -\tilde{\lambda}^{\alpha\beta\mu}, \quad \text{where} \quad \tilde{\lambda}^{\alpha\beta\mu} = \widetilde{\partial}_\nu\big(|\tilde{g}|(\tilde{g}^{\alpha\beta}\tilde{g}^{\mu\nu} - \tilde{g}^{\alpha\mu}\tilde{g}^{\beta\nu})\big).$$

Therefore with $L_\alpha = (-1, \omega_i)$ and $\underline{L}_\alpha = (-1, -\omega_i)$ we can write

$$M_T^\alpha(\widetilde{u}) = \frac{1}{4\pi}\int_{\mathbb{S}^2} m_T^\alpha(\widetilde{u}, \omega) dS(\omega), \qquad \text{where} \quad m_T^\alpha(\widetilde{u}, \omega) = \lim_{\widetilde{r}\to\infty} (\widetilde{r})^2 \big(\widetilde{\lambda}^{\alpha\beta\gamma}\underline{L}_\gamma L_\beta\big)(\widetilde{u} - \widetilde{r}, \widetilde{r}\omega).$$

The *Trautman radiated four-momentum*[1] is defined as

$$E_T^\alpha(\widetilde{u}) = \lim_{\widetilde{r}\to\infty} \frac{1}{2\pi} \int_{S_{\widetilde{u},r}} |\tilde{g}|\tilde{\pi}^{\alpha\beta}\, dS_\beta.$$

Here $dS_\beta = n_\beta \widetilde{r}^2 dS(\omega)$ with $n_\beta = (0, \omega_i)$ and $\tilde{\pi}^{\alpha\beta}$ is *Landau-Lifshitz* pseudotensor [25, §101],

$$\tilde{\pi}^{\alpha\beta} = -2\tilde{G}^{\alpha\beta} + \frac{1}{|\tilde{g}|}\widetilde{\partial}_\mu \tilde{\lambda}^{\alpha\beta\mu} \quad \text{where} \quad \tilde{G}^{\alpha\beta} = \tilde{R}^{\alpha\beta} - \frac{1}{2}\tilde{g}^{\alpha\beta}\tilde{R}.,$$

which is a symmetric. We write

$$E_T^\alpha(\widetilde{u}) = \frac{1}{2\pi}\int_{\mathbb{S}^2} \Delta m_T^\alpha(\widetilde{u}, \omega)\, d\omega \quad \text{where} \quad \Delta m_T^\alpha(\widetilde{u}, \omega) = \lim_{\widetilde{r}\to\infty} \widetilde{r}^2 |\tilde{g}|\tilde{\pi}^{\alpha i}\frac{\widetilde{x}_i}{\widetilde{r}}.$$

Remark. Many known gravitational pseudotensors can be derived from the above superpotentials, including the mixed Einstein pseudotensor of energy and momentum and the symmetric Landau-Lifshitz pseudotensor. We refer the readers to [3, 16, 37] for a more detailed discussion of different pseusotensors and their relations.

We will refer to $M_T^0(\widetilde{u})$ as the *Trautman mass* and to $E_T^0(\widetilde{u})$ as the radiated energy at null infinity. Using the asymptotics in [27] we prove in subsection 4.4

Theorem 1.1. *The Trautman four-momentum $M_T^\alpha(\widetilde{u})$ and Trautman radiated four momentum $E_T^\alpha(\widetilde{u})$ are well defined and satisfy the mass loss law*

$$M_T^\alpha(\widetilde{u}_2) - M_T^\alpha(\widetilde{u}_1) = -\int_{\widetilde{u}_1}^{\widetilde{u}_2} E_T^\alpha(\widetilde{u})\, d\widetilde{u}.$$

Moreover the radiated energy E_T^0 at null infinity can be expressed in terms of n in (1.5) *as*

$$E_T^0(\widetilde{u}) = \int_{\mathbb{S}^2} n(-\widetilde{u}, \omega) dS(\omega)/8\pi,$$

and the Trautman mass $M_T^0(\widetilde{u}) \to M$, the ADM mass, as $\widetilde{u} \to -\infty$ and $M_T^0(\widetilde{u}) \to 0$ as $\widetilde{u} \to \infty$.

[1]For the original definition of the radiated four-momentum [36, 37, 38], Trautman uses the mixed Einstein pseudotensor of energy and momentum. In [16] it is noted that the symmetric Landau-Lifshitz pseudotensor has the same total energy and momentum as the mixed Einstein pseudotensor.

1.5. **The asymptotic Hawking mass and radiated energy.** We will use the asymptotically null surfaces $u^* = t - r^*$ constant where $r^* = r + M\ln r + O(M/r)$ instead of null cones to define the asymptotic Hawking mass and the radiated energy at null infinity.

Following [11, 13] define the radius of a surface S by $r(S) = \sqrt{\text{Area}(S)/4\pi}$. Let $\hat{L}$ and $\underline{\hat{L}}$ be the outgoing respectively incoming null normals to S satisfying $g(\hat{L}, \underline{\hat{L}}) = -2$. $\hat{L}$ and $\underline{\hat{L}}$ are unique up to the transformation $\hat{L} \to a\hat{L}$ and $\underline{\hat{L}} \to a^{-1}\underline{\hat{L}}$. The null second fundamental form and the conjugate null second fundamental form are defined by $\chi(X,Y) = g(\nabla_X \hat{L}, Y)$ respectively $\underline{\chi}(X,Y) = g(\nabla_X \underline{\hat{L}}, Y)$, for any vectors X, Y tangent to S at a point, where ∇_X is covariant differentiation. The Hawking mass

$$M_{\mathcal{H}}(S) = r(S)\big(1 + \int_S \text{tr}\chi\, \text{tr}\underline{\chi}\, dS/16\pi\big),$$

is invariant under the transformation since $\chi \to a\chi$ and $\underline{\chi} \to \underline{\chi}/a$. If $\text{tr}\chi\, \text{tr}\underline{\chi} < 0$ we can fix $\hat{L}$ and $\underline{\hat{L}}$ by $\text{tr}\chi + \text{tr}\underline{\chi} = 0$. Let $\hat{\chi}$ and $\underline{\hat{\chi}}$ be the traceless parts. The incoming and outgoing energy fluxes are

$$E(S) = \int_S \hat{\chi}^2 dS/16\pi, \qquad \text{and} \qquad \underline{E}(S) = \int_S \underline{\hat{\chi}}^2 dS/16\pi.$$

We use the family of spheres $S_{u^*,r} = \{(t,x); t = u^* + r^*(r),\ |x| = r\}$ to define the *asymptotic Hawking mass* and the radiated energy at null infinity as follows

$$M_{AH}(u^*) = \lim_{r\to\infty} M_{\mathcal{H}}(S_{u^*,r}) \quad \text{and} \quad E_{AH}(u^*) = \lim_{r\to\infty} \underline{E}(S_{u^*,r}),$$

with $r(S)^2 \not{g}$ converging to a round metric where $\not{g}$ is the restriction of g on the spheres $S_{u^*,r}$.

Remark 1.2. As pointed out in [34], it is absolutely essential in the limit process that the spheres $S_{u^*,r}$converge to a round metric sphere. Otherwise the limit of the Hawking mass has nothing to do with the Bondi mass, in general. This is somehow related to the undesirable fact that the Hawking mass of any spherical surface in Euclidean space is negative unless it is a metric sphere where it is zero.

With the asymptotics results in [27] we prove in subsection 5.4:

Theorem 1.3. *The asymptotic Hawking mass $M_{AH}(u^*)$ and the radiated energy $E_{AH}(u^*)$ are well defined and in fact with n in* (1.5)

$$M_{AH}(u^*) = M - \frac{1}{8\pi}\int_{-u^*}^{\infty}\int_{\mathbb{S}^2} n(\eta,\omega)\, dS(\omega) d\eta, \quad \textit{and} \quad E_{AH}(u^*) = \frac{1}{8\pi}\int_{\mathbb{S}^2} n(-u^*,\omega) dS(\omega).$$

Therefore, they satisfy the mass loss law

$$\frac{d}{du^*} M_{AH}(u^*) = -E_{AH}(u^*).$$

Moreover, $M_{AH}(u^*) \to M$, the ADM mass, as $u^* \to -\infty$ and $M_{AH}(u^*) =\!\to 0$ as $u^* \to \infty$.

1.6. **The Bondi-Sachs coordinates.** The definition of Bondi mass introduced in 1962 in [7, 33] requires the existence of the so called Bondi-Sachs coordinates. In this manuscript we will construct the Bondi-Sachs coordinates $\overline{y}^p = (u, \overline{r}, \overline{y}^3, \overline{y}^4)$ under which we denote the solution to (1.1) by $\overline{g}$. The Bondi-Sachs coordinates $\overline{y}^p = (u, \overline{r}, \overline{y}^3, \overline{y}^4)$ are based on a family of outgoing null hypersurfaces $\overline{y}^1 = u = const$. The two angular coordinates $\overline{y}^a$, $(a, b, c, ... = 3, 4)$, are constant along the null rays, i.e. $g^{\alpha\beta}\partial_\beta u \partial_\alpha \overline{y}^a = 0$. The coordinate $\overline{y}^2 = \overline{r}$, which varies along the null rays, is chosen to be an areal coordinate such that $\det[\overline{g}_{ab}] = \overline{r}^4 \mathfrak{q}$, where $\mathfrak{q}(\overline{y}^a)$ is the determinant of the unit sphere metric $\overline{q}_{ab}$ associated with the angular coordinates $\overline{y}^a$. In these coordinates, the metric takes the Bondi-Sachs form (see Proposition 7.1)

$$\overline{g}_{pq} d\overline{y}^p d\overline{y}^q = -\frac{V}{\overline{r}} e^{2\beta} du^2 - 2e^{2\beta} du d\overline{r} + \overline{r}^2 h_{ab}\Big(d\overline{y}^a - U^a du\Big)\Big(d\overline{y}^b - U^b du\Big).$$

1.7. **The Bondi mass and radiated energy.** Once we write the metric in the Bondi-Sachs form as above, following [31] we define the mass aspect M_A and news tensor N_{ab} as follows

$$M_A(u, \overline{y}^a) := -\lim_{\overline{r}\to\infty}\Big(V(u, \overline{r}, \overline{y}^a) - \overline{r}\Big),$$

$$N_{ab}(u, \overline{y}^c) := \frac{1}{2}\partial_u C_{ab}(u, \overline{y}^c) \quad \text{where} \quad C_{ab}(u, \overline{y}^c) := \lim_{\overline{r}\to\infty} \overline{r}(h_{ab}(u, \overline{r}, \overline{y}^c) - \overline{q}_{ab}(\overline{y}^c)).$$

The Bondi mass M_B[2] radiated energy E_B are defined by

$$M_B(u) = \frac{1}{4\pi}\int_{\mathbb{S}^2} M_A(u,\overline{y}^a)d\overline{S}(\overline{y}^a) \quad \text{and} \quad E_B(u) = \frac{1}{4\pi}\int_{\mathbb{S}^2} |N|^2\, d\overline{S}(\overline{y}^a).$$

where $d\overline{S}(\overline{y}^a) = \sqrt{\mathfrak{q}(\overline{y}^a)}d\overline{y}^3 d\overline{y}^4$ is the volume form associated to the unit sphere metric $\overline{q}_{ab}$ and $|N|^2 = \overline{q}^{ac}\overline{q}^{bd}N_{ab}N_{cd}$. We will prove the existence of $M_B(u)$ and $E_B(u)$ and the Bondi mass loss law in subsection 7.2

Theorem 1.4. *Let M_A, N_{ab}, M_B, E be defined as above, then we have*

$$M_B(u) = M - \frac{1}{8\pi}\int_{\mathbb{S}^2}\int_{-u}^{\infty} n(\eta,\overline{y}^a)\, d\eta d\overline{S}(\overline{y}^a).$$

The radiated energy is expressed as

$$E_B(u) = \frac{1}{8\pi}\int_{\mathbb{S}^2} n(-u,\overline{y}^a)\, d\overline{S}(\overline{y}^a).$$

They satisfy the Bondi mass loss law

$$\frac{d}{du}M_B(u) = -E_B(u).$$

Moreover, $M_B(u)\to M$ as $u\to-\infty$ where M is the ADM mass and $M_B(u)\to 0$ as $u\to\infty$.

Remark 1.5. According to Theorem 1.1, 1.3 and 1.4, we see that the masses and radiated energies at null infinity defined in the above three ways are equivalent. We also note that these three different ways rely on the same collections of the asymptotics in [27]. In [27] it was shown that $\int_{-\infty}^{\infty}\int_{\mathbb{S}^2} n(\eta,\omega)\, dS(\omega)d\eta/8\pi = M$, and we can conclude that the total radiated energy is equal to the ADM mass. In particular, this implies that if $n = 0$ then $M = 0$, and then by the positive mass theorem [35, 40] the spacetime is Minkowski space.

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2. The metric in modified asymptotically Schwarzschild null coordinates

In this section we introduce the modified asymptotically Schwarzschild null coordinates and review some results concerning the asymptotics of the metric established in [27].

2.1. Modified asymptotically Schwarzschild null coordinates.

Suppose $g_{\alpha\beta} = m_{\alpha\beta} + h^0_{\alpha\beta} + h^1_{\alpha\beta}$ where $h^0_{\alpha\beta} = \frac{M}{r}\delta_{\alpha\beta}\chi(\frac{r}{1+t})$ and $\chi(s) = 1$ when $s \geq 1/2$ and 0 when $s \leq 1/4$. Then the inverse metric $g^{\alpha\beta} = m^{\alpha\beta} + h_0^{\alpha\beta} + h_1^{\alpha\beta}$ where $h_0^{\alpha\beta} = -\frac{M}{r}\delta^{\alpha\beta}\chi(\frac{r}{1+t})$ and $h_1^{\alpha\beta} = -m^{\alpha\mu}h^1_{\mu\nu}m^{\nu\beta} + O(h^2)$.

We introduce the modified asymptotically Schwarzschild null coordinates $\widehat{y}^p = (v^* = t + r^*, u^* = t - r^*, \widehat{y}^3, \widehat{y}^4)$. Here we let $r = |x|$, $\omega = \frac{x}{r} \in \mathbb{S}^2$ and $\widehat{y}^a = (\widehat{y}^3, \widehat{y}^4)$ be local coordinates on $\mathbb{S}^2$ and define $r^* = r + M\ln r + O(M/r)$ which is slightly different from $\widetilde{r} = r + M\ln r$ by solving

$$\frac{dr^*}{dr} = \rho'(r) = \Big(\frac{1+M/r}{1-M/r}\Big)^{1/2} = 1 + \frac{M}{r} + O(M^2/r^2).$$

In what follows indices $\widehat{y}^p, \widehat{y}^q, \dots$ will stand for all the modified asymptotically Schwarzschild null coordinates whereas $\widehat{y}^a, \widehat{y}^b, \dots$ stand for the coordinates on the sphere only. We will now calculate the changes of variables

[2]The Bondi energy-momentum vector for the outgoing null hypersurfaces $u = const$ is defined as [7, 33, 15, 1] the average of the Bondi mass aspect M_A over the unit round sphere weighted by a vector $N^\alpha = (1, N^i)$ where $N^i (1 \leq i \leq 3)$ are the $l = 1$ spherical harmonics. That is, $N^i = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$ in the natural spherical coordinates (θ,φ) for a unit round sphere. More specifically, the Bondi energy-momentum vector is defined by

$$M_B^\alpha(u) = \frac{1}{4\pi}\int_{\mathbb{S}^2} M_A(u,\overline{y}^a)N^\alpha d\overline{S}(\overline{y}^a).$$

The time component M_B^0 is referred as the Bondi energy in [2, 34, 1] and the Bondi mass in [7, 33, 15] respectively. In this manuscript we adopt the latter definition.

$\partial_{\widehat{y}^p} = \widehat{A}_p^\mu \partial_\mu$ and $\partial_\mu = \widehat{A}_\mu^p \partial_{\widehat{y}^p}$. We define $L^* = L^{*\mu}\partial_\mu = \partial_t + \partial_{r^*}$, $\underline{L}^* = \underline{L}^{*\mu}\partial_\mu = \partial_t - \partial_{r^*}$, $L^*_\mu = -\partial_\mu u^*$ and $\underline{L}^*_\mu = -\partial_\mu v^*$, then we have

$$\partial_{v^*} = \frac{1}{2}L^* = \frac{1}{2}(\partial_t + (\omega^i/\rho')\partial_i), \quad \partial_{u^*} = \frac{1}{2}\underline{L}^* = \frac{1}{2}(\partial_t - (\omega^i/\rho')\partial_i), \quad \partial_{\widehat{y}^a} = \widehat{A}_a^\mu \partial_\mu,$$

where $\omega = x/r$, $x = r\omega$, and

$$\partial_\mu = -\frac{1}{2}L^*_\mu \partial_{\underline{L}^*} - \frac{1}{2}\underline{L}^*_\mu \partial_{L^*} + \widehat{X}^a_\mu \partial_{\widehat{x}^a} = -L^*_\mu \partial_{u^*} - \underline{L}^*_\mu \partial_{v^*} + \widehat{A}^a_\mu \partial_{\widehat{y}^a}.$$

Here we have $\widehat{A}_a^\mu \widehat{A}_\mu^b = \delta_a^b$ and $\widehat{A}_a^\mu = O(r), \widehat{A}^a_\mu = O(1/r)$.

Under the coordinates $(v^*, u^*, \widehat{y}^3, \widehat{y}^4)$, we denote the corresponding metric by $\widehat{g}$. The inverse of the metric then satisfies $\widehat{g}^{pq} = g^{\alpha\beta}\widehat{A}^p_\alpha \widehat{A}^q_\beta$. We have in the region $r > t/2$

$$\widehat{g}^{v^*v^*} = \widehat{h}_1^{v^*v^*}, \quad \widehat{g}^{v^*u^*} = -2(1+\frac{M}{r}) + \widehat{h}_1^{v^*u^*}, \quad \widehat{g}^{v^*a} = \widehat{h}_1^{v^*a}, \quad \widehat{g}^{u^*u^*} = \widehat{h}_1^{u^*u^*}, \quad \widehat{g}^{u^*a} = \widehat{h}_1^{u^*a}, \quad \widehat{g}^{ab} = (1-\frac{M}{r})\frac{1}{r^2}\widehat{q}^{ab} + \widehat{h}_1^{ab}.$$

where $\widehat{q}_{ab}$ is the unit sphere metric on $\mathbb{S}^2$ associated with the angular coordinates $(\widehat{y}^3, \widehat{y}^4)$ and $\widehat{q}^{ac}\widehat{q}_{cb} = \delta^a_b$. In fact this follows from decomposing the leading part $m+h_0$ into a time part, a radial part and a sphere part

$$\big(m^{\alpha\beta} + h_0^{\alpha\beta}\big)\xi_\alpha\eta_\beta = -(1+\frac{M}{r})\xi_0\eta_0 + (1-\frac{M}{r})\omega^i\omega^j\xi_i\eta_j + (1-\frac{M}{r})\big(\delta^{ij} - \omega^i\omega^j\big)\xi_i\eta_j, \tag{2.1}$$

and composing with the change of variables in the radial part.

2.2. **Asymptotics result.** Let $(\widetilde{t}, \widetilde{x})$ be the asymptotically Schwarzschild coordinates with

$$\widetilde{t} = t, \qquad \widetilde{x}^i = \widetilde{r}\omega^i, \qquad \text{where} \quad \omega^i = x^i/r, \qquad \widetilde{r} = r + M\ln r, \quad r = |x|.$$

We write $\widetilde{h}^0_{\alpha\beta} + \widetilde{h}^1_{\alpha\beta} = h^0_{\alpha\beta} + h^1_{\alpha\beta}$ where $\widetilde{h}^0_{\alpha\beta} = \chi(\frac{\widetilde{r}}{1+t})\frac{M}{\widetilde{r}}\delta_{\alpha\beta}$ and $\widetilde{q} = \widetilde{r} - t$. We now restate several propositions from [27], which will be used frequently in this manuscript. The first proposition is the sharp decay estimates in asymtotically Schwarzschild coordinates

Proposition 2.1 ([27, Proposition 17])**.** *For $|I| \le N-6$ with $\gamma' = \gamma - C\varepsilon$, $\widetilde{q} = \widetilde{r} - t$ and $\langle \widetilde{q}\rangle = \sqrt{1+\widetilde{q}^2}$ we have*

$$|\widetilde{Z}^I\widetilde{h}^1| \lesssim \frac{\varepsilon^2 S^0(t,\widetilde{r})}{(1+t+\widetilde{r})(1+\widetilde{q}_+)^{1-C\varepsilon}} + \frac{\varepsilon}{1+t+\widetilde{r}}\frac{1}{(1+|\widetilde{q}|)^{\gamma'}}, \quad \text{where} \quad S^0(t,\widetilde{r}) = \frac{t}{\widetilde{r}}\ln\Big(\frac{\langle t+\widetilde{r}\rangle}{\langle t-\widetilde{r}\rangle}\Big) \lesssim \frac{1}{\varepsilon}\Big(\frac{\langle t+\widetilde{r}\rangle}{\langle t-\widetilde{r}\rangle}\Big)^\varepsilon.$$

For $\widetilde{r} \ge t/2$ we have

$$|\widetilde{Z}^I\widetilde{h}^1_{TU}| \lesssim \frac{\varepsilon}{(1+t+\widetilde{r})(1+\widetilde{q}_+)^{\gamma'}}, \tag{2.2}$$

$$|\partial\widetilde{Z}^I\widetilde{h}^1_{LT}| + |\partial\widetilde{Z}^I\delta^{AB}\widetilde{h}^1_{AB}| \lesssim \frac{\varepsilon}{(1+t+\widetilde{r})^{2-\varepsilon}(1+|\widetilde{q}|)^\varepsilon(1+\widetilde{q}_+)^{\gamma'}}, \tag{2.3}$$

$$|\widetilde{Z}^I\widetilde{h}^1_{LT}| + |\widetilde{Z}^I\delta^{AB}\widetilde{h}^1_{AB}| \lesssim \frac{\varepsilon}{(1+t+\widetilde{r})^{1+\gamma'}} + \frac{\varepsilon}{1+t+\widetilde{r}}\Big(\frac{1+\widetilde{q}_-}{1+t+\widetilde{r}}\Big)^{1-\varepsilon}, \tag{2.4}$$

where $\widetilde{q}_+ = \max\{0,\widetilde{q}\}$, $\widetilde{q}_- = \max\{0,-\widetilde{q}\}$. Here $\widetilde{h}^1_{UV} = \widetilde{h}^1_{\alpha\beta}U_\alpha V_\beta$ where $U_\alpha = m_{\alpha\beta}U^\beta$ and $U,V \in \{L, \underline{L}, A, B\}$, the null frame associated to the coordinates (t,x), and $\widetilde{Z}^I$ stands for a product of $|I|$ of the vector fields

$$\{\partial_{\widetilde{x}^\alpha},\ \widetilde{x}^i\partial_{\widetilde{x}^j} - \widetilde{x}^i\partial_{\widetilde{x}^j},\ \widetilde{x}^i\partial_{\widetilde{t}} + \widetilde{t}\partial_{\widetilde{x}^i},\ \widetilde{S} = \widetilde{t}\partial_{\widetilde{t}} + \widetilde{x}^i\partial_{\widetilde{x}^i}\}, \quad \text{where} \quad \widetilde{x}^\alpha = (\widetilde{t} = t, \widetilde{x}^i) = (t, \widetilde{r}\omega^i). \tag{2.5}$$

Here and in what follows we focus on the region $\{\widetilde{r} > t/2\} \cap \{r^* > t/2\}$.

Remark 2.2. In view of Proposition 2.1 we have the sharp decay estimates for the metric components in modified asymptotically Schwarzschild null coordinates. For $r^* > t/2$, with $q^* = -u^*, 0 < \gamma < 1$ and $\gamma' = \gamma - C\varepsilon$ we see that for $|I| \le N-6$

$$|Z^{*I}\hat{h}^{v^*v^*}| \lesssim \frac{\varepsilon^2}{(1+t+r^*)(1+q^*_+)^{1-C\varepsilon}}\ln\Big(\frac{\langle t+r^*\rangle}{\langle t-r^*\rangle}\Big), \tag{2.6}$$

$$|Z^{*I}\hat{h}_1^{v^*u^*}| + |Z^{*I}(r\hat{h}_1^{v^*a})| \lesssim \frac{\varepsilon}{(1+t+r^*)(1+q^*_+)^{\gamma'}}, \tag{2.7}$$

$$|Z^{*I}\hat{h}_1^{u^*u^*}| + |Z^{*I}(r\hat{h}_1^{u^*a})| + |Z^{*I}(r^2\hat{q}_{ab}\hat{h}_1^{ab})| \lesssim \frac{\varepsilon}{(1+t+r^*)^{1+\gamma'}} + \frac{\varepsilon}{1+t+r^*}\Big(\frac{1+q^*_-}{1+t+r^*}\Big)^{1-\varepsilon}. \tag{2.8}$$

Here Z^{*I} stands for a product of $|I|$ of the vector fields

$$\{\partial_{x^{*\alpha}},\ x^{*i}\partial_{x^{*j}} - x^{*i}\partial_{x^{*j}},\ x^{*i}\partial_{t^*} + t^*\partial_{x^{*i}},\ t^*\partial_{t^*} + x^{*i}\partial_{x^{*i}}\}, \quad \text{where} \quad x^{*\alpha} = (t^*, x^{*i}) = (t, r^*\omega^i). \tag{2.9}$$

The second one establishes the estimates for L derivatives

Proposition 2.3 ([27, Proposition 18]). *With $\delta_{UV}^{\underline{LL}}=1$ if $U=V=\underline{L}$ and 0 otherwise and $|I| \leq N-6$ we have*

$$\frac{1}{r}\,|(\partial_t + \partial_{\widetilde{r}})(\widetilde{r}\widetilde{Z}^I\widetilde{h}^1_{UV})| \lesssim \frac{\varepsilon(1+\widetilde{q}_-)^{\gamma - C\varepsilon}}{(1+t+\widetilde{r})^{2+\gamma-C\varepsilon}} + \delta_{UV}^{\underline{LL}}\,\frac{\varepsilon(1+\widetilde{q}_+)^{-\gamma}}{(1+t+\widetilde{r})^2}. \tag{2.10}$$

Remark 2.4. Correspondingly in modified asymptotically Schwarzschild null coordinates we have

$$\frac{1}{r}\,|\partial_{L^*}(r^*Z^{*I}\widehat{h}_1^{v^*u^*})| \lesssim \frac{\varepsilon(1+q^*_-)^{\gamma-C\varepsilon}}{(1+t+r^*)^{2+\gamma-C\varepsilon}} \quad \text{for} \quad |I| \leq N-6. \tag{2.11}$$

The third one provides us with the asymtotics for the metric in asymptotically Scwarzschild coordinates

Proposition 2.5 ([27, Proposition 20, 22]). *Let $H^1_{TU}(\widetilde{q},\omega,\widetilde{r}) = \widetilde{r}h^1_{TU}(\widetilde{r}-\widetilde{q},\widetilde{r}\omega)$,then the limit*

$$H^{1\infty}_{TU}(\widetilde{q},\omega) = \lim_{\widetilde{r}\to\infty} H^1_{TU}(\widetilde{q},\omega,\widetilde{r}),$$

exists and satisfies $H^{1\infty}_{TU} = H^{1\infty}_{UT}$, and $H^{1\infty}_{LT}(\widetilde{q},\omega) = \delta^{AB}H^{1\infty}_{AB}(\widetilde{q},\omega) = 0$. Moreover, for $|\alpha| + k \leq N-6$ and $|J|+|K| = k$ and $\widetilde{r} > t/2$

$$\big|\,\partial^\alpha_\omega\big((1+|\,\widetilde{q}|)\partial_{\widetilde{q}}\big)^k H^{1\infty}_{TU}(\widetilde{q},\omega)\big| \lesssim \varepsilon(1+\widetilde{q}_+)^{-\gamma'},$$

$$\big|\partial^\alpha_\omega \widetilde{S}^J\partial^K_t H^1_{TU}(\widetilde{q},\omega,\widetilde{r}) - \partial^\alpha_\omega(\widetilde{q}\,\partial_{\widetilde{q}})^J(-\partial_{\widetilde{q}})^K H^{1\infty}_{TU}(\widetilde{q},\omega)\big| \lesssim \varepsilon\Big(\frac{1+\widetilde{q}_-}{1+t+\widetilde{r}}\Big)^{\gamma'}.$$

Let

$$n(\widetilde{q},\omega) = \tfrac12\delta^{CD}\delta^{C'D'}V^\infty_{CC'}(\widetilde{q},\omega)V^\infty_{DD'}(\widetilde{q},\omega) \quad \textit{where} \quad V^\infty_{TU}(\widetilde{q},\omega) = \partial_{\widetilde{q}}H^{1\infty}_{TU}(\widetilde{q},\omega), \tag{2.12}$$

for the component $h^1_{\underline{LL}}(t,\widetilde{r}\omega)$ we have when $\widetilde{r} \gg 1$

$$h^1_{\underline{LL}}(t,\widetilde{r}\omega) = 2\frac{M}{\widetilde{r}}(\chi^e(\widetilde{q})-1) + \int_{\widetilde{r}-t}^\infty \frac{2}{\widetilde{r}}\ln\Big(\frac{t+\widetilde{r}+\eta}{t-\widetilde{r}+\eta}\Big)n\big(\eta,\omega\big)\,d\eta + \frac{H^1_{\underline{LL}}(\widetilde{q},\omega)}{\widetilde{r}} + \widetilde{\mathcal{R}}.$$

Here $\chi^e(s) = 1$ when $s \geq 2$ and $\chi^e(s) = 0$ when $s \leq 1$, and for $|\alpha| + k = |I| \leq N-7$ we see that

$$\big|\,\partial^\alpha_\omega\big((1+|\,\widetilde{q}|)\partial_{\widetilde{q}}\big)^k H^{1\infty}_{\underline{LL}}(\widetilde{q},\omega)\big| \lesssim \varepsilon(1+\widetilde{q}_+)^{-\gamma'}, \qquad |\widetilde{Z}^I\widetilde{\mathcal{R}}| \lesssim \varepsilon\frac{(1+\widetilde{q}_-)^{\gamma'}}{(1+t+\widetilde{r})^{1+\gamma'}}.$$

Remark 2.6. In modified asymptotically Schwarzschild null coordinates, since $|q^* - \widetilde{q}| \lesssim M/r$ it follows from Proposition 2.5 that the following limits

$$\widehat{H}^{v^*u^*}_{1\infty}(q^*,\widehat{y}^a) = \lim_{r^*\to\infty} r^*\widehat{h}_1^{v^*u^*}(v^*,-q^*,\widehat{y}^a), \quad \widehat{H}^{u^*u^*}_{1\infty}(q^*,\widehat{y}^a) = \lim_{r^*\to\infty} r^*\widehat{h}_1^{u^*u^*}(v^*,-q^*,\widehat{y}^a),$$
$$\widehat{H}^{u^*a}_{1\infty}(q^*,\widehat{y}^a) = \lim_{r^*\to\infty} r^{*2}\widehat{h}_1^{u^*a}(v^*,-q^*,\widehat{y}^a), \quad \widehat{H}^{v^*a}_{1\infty}(q^*,\widehat{y}^a) = \lim_{r^*\to\infty} r^{*2}\widehat{h}_1^{v^*a}(v^*,-q^*,\widehat{y}^a),$$
$$\widehat{H}^{ab}_{1\infty}(q^*,\widehat{y}^a) = \lim_{r^*\to\infty} r^{*3}\widehat{h}_1^{ab}(v^*,-q^*,\widehat{y}^a)$$

exist and satisfy $\widehat{H}^{u^*u^*}_{1\infty}(q^*,\widehat{y}^a) = \widehat{H}^{u^*a}_{1\infty}(q^*,\widehat{y}^a) = \widehat{q}_{ab}\widehat{H}^{ab}_{1\infty}(q^*,\widehat{y}^a) = 0$. Moreover for $|\alpha| + k \leq N-6$ we have

$$\big|\,\partial^\alpha_{\widehat{y}^a}\big((1+|\,q^*)\partial_{q^*}\big)^k\widehat{H}^{pq}_{1\infty}(q^*,\widehat{y}^a)\big| \lesssim \varepsilon(1+q^*_+)^{-\gamma'}, \qquad (p,q) \neq v^*,v^*),$$

and when $r^* \gg 1$

$$\widehat{h}_1^{v^*u^*}(v^*,-q^*,\widehat{y}^a) = \frac{\widehat{H}^{v^*u^*}_{1\infty}(q^*,\widehat{y}^a)}{r^*} + \widehat{\mathcal{R}}^{u^*v^*}, \qquad \widehat{h}_1^{v^*u^*}(v^*,-q^*,\widehat{y}^a) = \frac{\widehat{H}^{u^*u^*}_{1\infty}(q^*,\widehat{y}^a)}{r^*} + \widehat{\mathcal{R}}^{u^*u^*}$$
$$\widehat{h}_1^{v^*u^*}(v^*,-q^*,\widehat{y}^a) = \frac{\widehat{H}^{u^*a}_{1\infty}(q^*,\widehat{y}^a)}{r^{*2}} + \widehat{\mathcal{R}}^{u^*a}, \qquad \widehat{h}_1^{v^*a}(v^*,-q^*,\widehat{y}^a) = \frac{\widehat{H}^{v^*a}_{1\infty}(q^*,\widehat{y}^a)}{r^{*2}} + \widehat{\mathcal{R}}^{v^*a},$$
$$\widehat{h}_1^{ab}(v^*,-q^*,\widehat{y}^a) = \frac{\widehat{H}^{ab}_{1\infty}(q^*,\widehat{y}^a)}{r^{*3}} + \widehat{\mathcal{R}}^{ab},$$

where the remainders $\widehat{\mathcal{R}}$ satisfy

$$|Z^{*I}\widehat{\mathcal{R}}^{v^*u^*}|+|Z^{*I}\widehat{\mathcal{R}}^{u^*u^*}|\lesssim\frac{\varepsilon(1+q^*_-)^{\gamma'}}{(1+t+r^*)^{1+\gamma'}},\qquad |Z^{*I}\widehat{\mathcal{R}}^{ab}|\lesssim\frac{\varepsilon(1+q^*_-)^{\gamma'}}{(1+t+r^*)^{3+\gamma'}},$$

$$|Z^{*I}\widehat{\mathcal{R}}^{v^*a}|+|Z^{*I}\widehat{\mathcal{R}}^{u^*a}|\lesssim\frac{\varepsilon(1+q^*_-)^{\gamma'}}{(1+t+r^*)^{2+\gamma'}}.$$

Let $n(q^*,\widehat{y}^a)=n(q^*,\omega(\widehat{y}^a))$, then we have

$$n(q^*,\widehat{y}^a)=\tfrac12\widehat{q}_{ab}\widehat{q}_{a'b'}\widehat{V}^{aa'}(q^*,\widehat{y}^a)\widehat{V}^{bb'}(q^*,\widehat{y}^a)\quad\text{with}\quad \widehat{V}^{ab}=\partial_{q^*}\widehat{H}^{ab}(q^*,\widehat{y}^a). \tag{2.13}$$

As for the component $\widehat{h}_1^{v^*v^*}$, we have when $r^*\gg1$

$$\widehat{h}_1^{v^*v^*}(v^*,u^*,\widehat{y}^a)=-\frac{2M}{r^*}(\chi^e(q^*)-1)-\int_{q^*}^{\infty}\frac{2}{r^*}\ln\Big(\frac{v^*+\eta}{u^*+\eta}\Big)n(q^*,\omega)\,d\eta+\frac{\widehat{H}_{1\infty}^{v^*v^*}(q^*,\omega)}{r^*}+\mathcal{R}^{v^*v^*}.$$

Here for $|\alpha|+k=|I|\le N-7$, $\widehat{H}_{1\infty}^{v^*v^*}$ and the remainder $\mathcal{R}^{v^*v^*}$ satisfy

$$\big|\,\partial_\omega^\alpha\big((1+|\,q^*)\partial_{q^*}\big)^k\widehat{H}_{1\infty}^{v^*v^*}(q^*,\omega)\big|\lesssim\varepsilon(1+q^*_+)^{-\gamma'},\qquad |Z^{*I}\mathcal{R}^{v^*v^*}|\lesssim\varepsilon\frac{(1+q^*_-)^{\gamma'}}{(1+t+r^*)^{1+\gamma'}}.$$

The last proposition gives a relation between M and n.

Proposition 2.7 ([27, Proposition 28])**.** *We have*

$$\frac12\int_{-\infty}^{+\infty}\int_{\mathbb{S}^2}n(\widetilde{q},\omega)\frac{dS(\omega)}{4\pi}d\widetilde{q}=M. \tag{2.14}$$

In what follows we write $A=O(B)$ if $A\le CB$ and $A=O_k(B)$ if $\sum_{|I|\le k}|\widetilde{Z}^IA|+|Z^{*I}A|\le CB$ with $\widetilde{Z}$ and Z^* defined in (2.5) and (2.9) respectively for some universal constant C. We define $\sigma=\min\{\gamma',1-3\varepsilon\}>0$.

2.3. **Wave coordinate condition in modified asymptotically Schwarzschild null coordinates.** Let N be some fixed large integer ($N=9$ works). We express the wave coordinate condition in modified asymptotically Schwarzschild null coordinates.

$$\partial_\alpha\Big(g^{\alpha\beta}\sqrt{|g|}\Big)=\Big(-\frac12L^*_\alpha\partial_{\underline{L}^*}-\frac12\underline{L}^*_\alpha\partial_{L^*}+\widehat{A}^a_\alpha\partial_{\widehat{y}^a}\Big)\Big(g^{\alpha\beta}\sqrt{|g|}\Big)=0. \tag{2.15}$$

2.3.1. *Contraction with L^*.*

Proposition 2.8. *We have*

$$\frac12\partial_{\underline{L}^*}(\widehat{h}_1^{u^*u^*})+\frac{\widehat{h}_1^{v^*u^*}}{r}=O_2(\frac{\varepsilon}{r^{2+\sigma}}). \tag{2.16}$$

Proof. Contracting (2.15) with L^*_β we obtain

$$L^*_\beta\partial_\alpha\big(g^{\alpha\beta}\sqrt{|g|}\big)=-\frac12L^*_\beta L^*_\alpha\partial_{\underline{L}^*}\big(g^{\alpha\beta}\sqrt{|g|}\big)-\frac12L^*_\beta\underline{L}^*_\alpha\partial_{L^*}\big(g^{\alpha\beta}\sqrt{|g|}\big)+L^*_\beta\widehat{A}^c_\alpha\partial_{\widehat{y}^c}\big(g^{\alpha\beta}\sqrt{|g|}\big)=0.$$

Here and in what follows the repeated indices c,d are summed over a,b. We first analyze the first term

$$\begin{aligned}-\frac12L^*_\beta L^*_\alpha\partial_{\underline{L}^*}\Big(g^{\alpha\beta}\sqrt{|g|}\Big)&=-\frac12\partial_{\underline{L}^*}\Big(L^*_\beta L^*_\alpha g^{\alpha\beta}\sqrt{|g|}\Big)+\partial_{\underline{L}^*}(L^*_\alpha)L^*_\beta g^{\alpha\beta}\sqrt{|g|}\\&=-\frac12\partial_{\underline{L}^*}(\widehat{h}_1^{u^*u^*})+\omega^i\delta_{i\alpha}\frac{M}{r^2}\frac{1}{\rho'}L^*_\beta g^{\alpha\beta}\sqrt{|g|}+O_2(\frac{\varepsilon}{r^{2+\sigma}})=-\frac12\partial_{\underline{L}^*}(\widehat{h}_1^{u^*u^*})+\frac{M}{r^2}+O_2(\frac{\varepsilon}{r^{2+\sigma}})\end{aligned}$$

where we used the estimate $\partial_{\underline{L}^*}(\sqrt{|g|})=O_2(\varepsilon r^{-1})$. Here and in what follows the repeated indices i,j are summed over $1,2,3$. We note that the error term is of order $O_2(\varepsilon r^{-2-\sigma})$ because it only depends on the metric. For the second term, we have

$$\begin{aligned}-\frac12L^*_\beta L^*_\alpha\partial_{L^*}\Big(g^{\alpha\beta}\sqrt{|g|}\Big)&=-\frac12\partial_{L^*}\Big(L^*_\beta\underline{L}^*_\alpha g^{\alpha\beta}\sqrt{|g|}\Big)+\frac12\partial_{L^*}(\underline{L}^*_\alpha)L^*_\beta g^{\alpha\beta}\sqrt{|g|}+\frac12\partial_{L^*}(L^*_\beta)\underline{L}^*_\alpha g^{\alpha\beta}\sqrt{|g|}\\&=-\frac{M}{r^2}-\frac12\partial_{L^*}(\widehat{h}_1^{v^*u^*})-\frac{M}{r^2}+\frac12\partial_{L^*}(\widehat{h}_1^{v^*u^*})+\frac{M}{2r^2}+\frac{M}{2r^2}+O_2(\frac{\varepsilon}{r^{2+\sigma}})=-\frac{M}{r^2}+O_2(\frac{\varepsilon}{r^{2+\sigma}}).\end{aligned}$$

Before analyzing the last term, we calculate the Christoffle symbols $\widehat{\Gamma}^c_{ab}$ on sphere under the coordinates $\widehat{y}^a$:

$$\widehat{\Gamma}^c_{ab} = -\frac{\partial x^\alpha}{\partial \widehat{y}^a}\frac{\partial x^\beta}{\partial \widehat{y}^b}\frac{\partial^2 \widehat{y}^c}{\partial x^\alpha \partial x^\beta} = -\widehat{A}^\alpha_a \partial_{\widehat{y}^b}\widehat{A}^c_\alpha.$$

As we have $\widehat{A}^c_\alpha \widehat{A}^\beta_c = (\delta^j_i - \omega_i\omega^j)\delta_{j\alpha}\delta^{i\beta}$, we see that

$$\begin{aligned}\widehat{A}^c_\beta\widehat{\Gamma}^a_{cb} = -\widehat{A}^c_\beta\widehat{A}^\alpha_c\partial_{\widehat{y}^b}\widehat{A}^a_\alpha = -\partial_{\widehat{y}^b}\widehat{A}^a_\beta - \widehat{A}^a_\alpha\omega^j\delta_{j\beta}\partial_{\widehat{y}^b}(\omega_i\delta^{i\alpha}) = -\partial_{\widehat{y}^b}\widehat{A}^a_\beta - \widehat{A}^a_\alpha\omega^j\delta_{j\beta}\widehat{A}^\mu_b\partial_\mu(\omega_i\delta^{i\alpha})\\ = -\partial_{\widehat{y}^b}\widehat{A}^a_\beta - \widehat{A}^a_\alpha\omega^j\delta_{j\beta}\widehat{A}^\mu_b\frac{\delta^l_k - \omega_k\omega^l}{r}\delta_{l\mu}\delta^{k\alpha} = -\partial_{\widehat{y}^b}\widehat{A}^a_\beta - \frac{\delta^a_b\omega^j\delta_{j\beta}}{r}.\end{aligned}$$

Finally, we compute

$$\begin{aligned}L^*_\beta\widehat{A}^c_\alpha\partial_{\widehat{y}^c}\left(g^{\alpha\beta}\sqrt{|g|}\right) &= \partial_{\widehat{y}^c}\left(L^*_\beta\widehat{A}^c_\alpha g^{\alpha\beta}\sqrt{|g|}\right) - \partial_{\widehat{y}^c}(L^*_\beta)\widehat{A}^c_\alpha g^{\alpha\beta}\sqrt{|g|} - \partial_{\widehat{y}^c}(\widehat{A}^c_\alpha)L^*_\beta g^{\alpha\beta}\sqrt{|g|}\\ &= -\rho'\frac{\delta^{ij}-\omega^i\omega^j}{r}\delta_{i\alpha}\delta_{j\beta}g^{\alpha\beta}\sqrt{|g|} + (\widehat{A}^d_\alpha\widehat{\Gamma}^c_{dc} + \frac{\omega^i\delta_{i\alpha}\delta^c_c}{r})L^*_\beta g^{\alpha\beta}\sqrt{|g|}\\ &= -\frac{2}{r}(1+\frac{M}{r})(1-\frac{M}{r})\sqrt{|g|} + \frac{\rho'}{r}\not{t}r h^1 + \frac{2}{r}\frac{L^*_\alpha - \underline{L}^*_\alpha}{2\rho'}L^*_\beta g^{\alpha\beta}\sqrt{|g|} + O_2(\frac{\varepsilon}{r^{2+\sigma}}) = -\frac{\widehat{h}_1^{v^*u^*}}{r} + O_2(\frac{\varepsilon}{r^{2+\sigma}}).\end{aligned}$$

Gathering our estimates yield the lemma. □

2.3.2. *Contraction with* $\widehat{A}^a$.

Proposition 2.9. *We have*

$$\frac{1}{2}\partial_{\underline{L}^*}(\widehat{h}_1^{u^*a}) + \frac{1}{2}\partial_{L^*}(\widehat{h}_1^{v^*a}) + \frac{2\widehat{h}_1^{v^*a}}{r} + \frac{1}{2r^2}\widehat{q}^{ac}\partial_{\widehat{y}^c}(\widehat{h}_1^{v^*u^*}) + \widehat{\nabla}_c\widehat{h}_1^{ac} = O_2(\frac{\varepsilon}{r^{3+\sigma}}). \tag{2.17}$$

Proof. Contracting (2.15) with $\widehat{A}^a_\beta$ yields

$$\widehat{A}^a_\beta\partial_\alpha\left(g^{\alpha\beta}\sqrt{|g|}\right) = -\frac{1}{2}\widehat{A}^a_\beta L^*_\alpha\partial_{\underline{L}^*}\left(g^{\alpha\beta}\sqrt{|g|}\right) - \frac{1}{2}\widehat{A}^a_\beta\underline{L}^*_\alpha\partial_{L^*}\left(g^{\alpha\beta}\sqrt{|g|}\right) + \widehat{A}^a_\beta\widehat{A}^c_\alpha\partial_{\widehat{y}^c}\left(g^{\alpha\beta}\sqrt{|g|}\right) = 0.$$

As for the first term, since we have $\partial_r\widehat{A}^a_\alpha = -\widehat{A}^a_\alpha/r$, we see that

$$\begin{aligned}-\frac{1}{2}\widehat{A}^a_\beta L^*_\alpha\partial_{\underline{L}^*}\Big(g^{\alpha\beta}\sqrt{|g|}\Big) &= -\frac{1}{2}\partial_{\underline{L}^*}\Big(\widehat{A}^a_\beta L^*_\alpha g^{\alpha\beta}\sqrt{|g|}\Big) + \frac{1}{2}\partial_{\underline{L}^*}(L^*_\alpha)\widehat{A}^a_\beta g^{\alpha\beta}\sqrt{|g|} + \frac{1}{2}\partial_{\underline{L}^*}(L^*_\alpha)\widehat{A}^a_\beta g^{\alpha\beta}\sqrt{|g|}\\ &= \frac{1}{2}\partial_{\underline{L}^*}(\widehat{h}_1^{u^*a}) + O_2(\frac{\varepsilon}{r^{3+\sigma}}).\end{aligned}$$

Next we calculate the second term

$$\begin{aligned}-\frac{1}{2}\widehat{A}^a_\beta\underline{L}^*_\alpha\partial_{L^*}\Big(g^{\alpha\beta}\sqrt{|g|}\Big) &= -\frac{1}{2}\partial_{L^*}\Big(\widehat{A}^a_\beta\underline{L}^*_\alpha g^{\alpha\beta}\sqrt{|g|}\Big) + \frac{1}{2}\partial_{L^*}(\underline{L}^*_\alpha)\widehat{A}^a_\beta g^{\alpha\beta}\sqrt{|g|} + \frac{1}{2}\partial_{L^*}(\underline{L}^*_\alpha)\widehat{A}^a_\beta g^{\alpha\beta}\sqrt{|g|}\\ &= \frac{1}{2}\partial_{L^*}(\widehat{h}_1^{v^*a}) + \frac{\widehat{h}_1^{v^*a}}{2r} + O_2(\frac{\varepsilon}{r^{3+\sigma}}).\end{aligned}$$

Finally,

$$\begin{aligned}\widehat{A}^a_\beta\widehat{A}^c_\alpha\partial_{\widehat{y}^c}\left(g^{\alpha\beta}\sqrt{|g|}\right) &= \widehat{A}^a_\beta\widehat{A}^c_\alpha g_0^{\alpha\beta}\partial_{\widehat{y}^c}\left(\sqrt{|g|}\right) + \widehat{A}^a_\beta\widehat{A}^c_\alpha\partial_{\widehat{y}^c}\left(h_1^{\alpha\beta}\sqrt{|g|}\right)\\ &= \frac{1}{2r^2}\widehat{q}^{ac}\partial_{\widehat{y}^c}(\widehat{h}_1^{v^*u^*}) + \widehat{A}^a_\beta\widehat{A}^c_\alpha\partial_{\widehat{y}^c}\left(h_1^{\alpha\beta}\sqrt{|g|}\right) + O_2(\frac{\varepsilon}{r^{3+\sigma}}).\end{aligned}$$

We write

$$\begin{aligned}\widehat{A}^a_\beta\widehat{A}^c_\alpha\partial_{\widehat{y}^c}\left(h_1^{\alpha\beta}\sqrt{|g|}\right) &= \partial_{\widehat{y}^c}\left(\widehat{A}^a_\beta\widehat{A}^c_\alpha h_1^{\alpha\beta}\sqrt{|g|}\right) - \widehat{A}^c_\alpha\partial_{\widehat{y}^c}(\widehat{A}^a_\beta)h_1^{\alpha\beta}\sqrt{|g|} - \widehat{A}^a_\beta\partial_{\widehat{y}^c}(\widehat{A}^c_\alpha)h_1^{\alpha\beta}\sqrt{|g|}\\ &= \partial_{\widehat{y}^c}(\widehat{h}_1^{ac}) + \Big(\widehat{A}^c_\alpha\widehat{A}^d_\beta\widehat{\Gamma}^a_{dc} + \frac{\delta^a_c\omega^j\delta_{j\beta}}{r} + \widehat{A}^a_\beta\widehat{A}^d_\alpha\widehat{\Gamma}^c_{dc} + \frac{\delta^c_c\omega^j\delta_{j\alpha}}{r}\Big)h_1^{\alpha\beta}\sqrt{|g|} + O_2(\frac{1}{r^{3+\sigma}})\\ &= \widehat{\nabla}_c\widehat{h}_1^{ac} + \frac{\widehat{h}_1^{v^*a}}{2r} + \frac{\widehat{h}_1^{v^*a}}{r} + O_2(\frac{\varepsilon}{r^{3+\sigma}}) = \widehat{\nabla}_c\widehat{h}_1^{ac} + \frac{3\widehat{h}_1^{v^*a}}{2r} + O_2(\frac{\varepsilon}{r^{3+\sigma}})\end{aligned}$$

where $\widehat{\nabla}$ is the covariant derivative on sphere. Putting all together yields the conclusion. □

2.3.3. *Contraction with $\underline{L}^*$.*

Proposition 2.10. *We have*

$$\frac{1}{2}\partial_{\underline{L}^*}(\not{\mathrm{tr}}h^1)+\frac{\widehat{h}_1^{v^*u^*}}{r}-\frac{1}{2}\partial_{L^*}(\widehat{h}_1^{v^*v^*})-\frac{\widehat{h}_1^{v^*v^*}}{r}-\widehat{\not\nabla}_c(\widehat{h}_1^{v^*c})-\frac{1}{4}\partial_{\underline{L}^*}(\widehat{h}_1^{ac}r^2\widehat{q}_{cb}\widehat{h}_1^{bd}r^2\widehat{q}_{da})=O_2(\frac{\varepsilon}{r^{2+\sigma}}). \qquad (2.18)$$

In order to prove this proposition, we now need the following lemmas.

Lemma 2.11. *We have*

$$\partial_{\underline{L}^*}(\sqrt{|g|})=\frac{M}{r^2}-\frac{1}{2}\partial_{\underline{L}^*}(h^1_{L^*\underline{L}^*})+\frac{M}{r}\partial_{\underline{L}^*}(\widehat{h}_1^{v^*u^*})+\frac{1}{2}\partial_{\underline{L}^*}(\not{\mathrm{tr}}h^1)-\frac{1}{4}\partial_{\underline{L}^*}(\widehat{h}_1^{ac}r^2\widehat{q}_{cb}\widehat{h}_1^{bd}r^2\widehat{q}_{da})+O_2(\frac{\varepsilon}{r^{2+\sigma}}).$$

Proof. We first notice that $h^1_{UV}=h^1_{\alpha\beta}U^\alpha V^\beta$. Given the facts that $2\partial_\alpha(\sqrt{|g|})=\sqrt{|g|}g^{\mu\nu}\partial_\alpha g_{\mu\nu}$, $\sqrt{|g|}=1+\frac{M}{r}-\frac{1}{2}h^1_{L\underline{L}}+O_2(\varepsilon r^{-2-\sigma})$ and $h^1_{L\underline{L}}=h^1_{L^*\underline{L}^*}+O_3(\varepsilon r^{-1-\sigma})=-\widehat{h}_1^{v^*u^*}+O_3(\varepsilon r^{-1-\sigma})$, we obtain

$$\begin{aligned}\partial_{\underline{L}^*}(\sqrt{|g|})&=\sqrt{|g|}\Big(\frac{M}{r^2}+\frac{1}{2}g^{\alpha\beta}\partial_{\underline{L}^*}(h^1_{\alpha\beta})\Big)=\sqrt{|g|}\Big(\frac{M}{r^2}-\frac{1}{2}\partial_{\underline{L}^*}(h^1_{L^*\underline{L}^*})-\frac{M}{2r}\partial_{\underline{L}^*}(h^1_{L^*\underline{L}^*})+\frac{1}{2}\partial_{\underline{L}^*}(\not{\mathrm{tr}}h^1)\Big)\\&\quad+\sqrt{|g|}\Big(-\frac{1}{4}h^1_{L^*\underline{L}^*}\partial_{\underline{L}^*}(h^1_{L^*\underline{L}^*})-\frac{1}{4}\partial_{\underline{L}^*}(\widehat{h}_1^{ac}r^2\widehat{q}_{cb}\widehat{h}_1^{bd}r^2\widehat{q}_{da})\Big)+O_2(\frac{\varepsilon}{r^{2+\sigma}})\\&=\frac{M}{r^2}-\frac{1}{2}\partial_{\underline{L}^*}(h^1_{L^*\underline{L}^*})+\frac{M}{r}\partial_{\underline{L}^*}(\widehat{h}_1^{v^*u^*})+\frac{1}{2}\partial_{\underline{L}^*}(\not{\mathrm{tr}}h^1)-\frac{1}{4}\partial_{\underline{L}^*}(\widehat{h}_1^{ac}r^2\widehat{q}_{cb}\widehat{h}_1^{bd}r^2\widehat{q}_{da})+O_2(\frac{\varepsilon}{r^{2+\sigma}}).\end{aligned}$$

□

Lemma 2.12. *We have*

$$\partial_{\underline{L}^*}(h^1_{L^*\underline{L}^*})+\partial_{\underline{L}^*}(\widehat{h}_1^{v^*u^*})=-\widehat{h}_1^{v^*u^*}\partial_{\underline{L}^*}(\widehat{h}_1^{v^*u^*})+\frac{2M}{r}\partial_{\underline{L}^*}(\widehat{h}_1^{v^*u^*})+O_2(\frac{\varepsilon}{r^{2+\sigma}}).$$

Proof. Since we express $g_{\alpha\beta}=g^0_{\alpha\beta}+h^1_{\alpha\beta}$ and $g^{\alpha\beta}=g_0^{\alpha\beta}+h_1^{\alpha\beta}$, we see that

$$h_1^{\alpha\beta}=-m^{\alpha\mu}h^1_{\mu\nu}m^{\nu\beta}+m^{\alpha\alpha'}(\frac{M}{r}\delta_{\alpha'\mu}+h^1_{\alpha'\mu})m^{\mu\nu}(\frac{M}{r}\delta_{\nu\beta'}+h^1_{\nu\beta'})m^{\beta'\beta}+O_3(\frac{\varepsilon}{r^{2+\sigma}}).$$

Therefore

$$\widehat{h}_1^{v^*u^*}=h_1^{\alpha\beta}\underline{L}^*_\alpha L^*_\beta=-m^{\alpha\mu}h^1_{\mu\nu}m^{\nu\beta}\underline{L}^*_\alpha L^*_\beta+\underline{L}^{\alpha'}L^{\beta'}m^{\mu\nu}(\frac{M}{r}\delta_{\alpha'\mu}+h^1_{\alpha'\mu})(\frac{M}{r}\delta_{\nu\beta'}+h^1_{\nu\beta'})+O_3(\frac{\varepsilon}{r^{2+\sigma}}).$$

We analyze the first term

$$\begin{aligned}-m^{\alpha\mu}h^1_{\mu\nu}m^{\nu\beta}\underline{L}^*_\alpha L^*_\beta&=-h^1_{\mu\nu}(\underline{L}^{*\mu}+(\frac{1}{\rho'}-\rho')\omega_j\delta^{j\mu})(L^{*\nu}+(\rho'-\frac{1}{\rho'})\omega_j\delta^{j\nu})\\&=-h^1_{L^*\underline{L}^*}+\frac{M}{r}(h^1_{L^*L^*}+h^1_{\underline{L}^*\underline{L}^*}-2h^1_{L^*\underline{L}^*})+O_3(\frac{\varepsilon}{r^{2+\sigma}}).\end{aligned}$$

Now we turn to the second term. Using the null frame ($L=\partial_t+\partial_r,\underline{L}=\partial_t-\partial_r,A,B$), we rewrite $\underline{L}^{\alpha'}\delta_{\alpha'\mu}$, $L^{\beta'}\delta_{\nu\beta'}$, $\underline{L}^{\alpha'}h^1_{\alpha'\mu}$, $L^{\beta'}h^1_{\nu\beta'}$ as follows

$$\begin{aligned}\underline{L}^{\alpha'}h^1_{\alpha'\mu}&=-\frac{1}{2}\underline{L}_\mu h^1_{L\underline{L}}-\frac{1}{2}L_\mu h^1_{\underline{L}\underline{L}}+A_\mu h^1_{A\underline{L}}, &\qquad \underline{L}^{\alpha'}\delta_{\alpha'\mu}&=-\frac{1}{2}L_\mu\delta_{\underline{L}\underline{L}}=-L_\mu,\\L^{\beta'}h^1_{\nu\beta'}&=-\frac{1}{2}\underline{L}_\nu h^1_{LL}-\frac{1}{2}L_\nu h^1_{\underline{L}L}+A_\nu h^1_{AL}, &\qquad L^{\beta'}\delta_{\nu\beta'}&=-\frac{1}{2}\underline{L}_\nu\delta_{LL}=-\underline{L}_\nu.\end{aligned}$$

We see that

$$\underline{L}^{\alpha'}L^{\beta'}m^{\mu\nu}(\frac{M}{r}\delta_{\alpha'\mu}+h^1_{\alpha'\mu})(\frac{M}{r}\delta_{\nu\beta'}+h^1_{\nu\beta'})=-\frac{1}{2}h^1_{L\underline{L}}h^1_{L\underline{L}}-\frac{M}{r}h^1_{LL}-\frac{M}{r}h^1_{\underline{L}\underline{L}}-\frac{2M}{r^2}+O_3(\frac{\varepsilon}{r^{2+\sigma}}).$$

Thus we find

$$\widehat{h}_1^{v^*u^*}+h^1_{L^*\underline{L}^*}=-\frac{1}{2}h^1_{L\underline{L}}h^1_{L\underline{L}}-\frac{2M}{r}h^1_{L\underline{L}}-\frac{2M}{r^2}+O_3(\frac{\varepsilon}{r^{2+\sigma}})$$

and

$$\partial_{\underline{L}^*}(\widehat{h}_1^{v^*u^*}+h^1_{L^*\underline{L}^*})=-h^1_{L\underline{L}}\partial_{\underline{L}^*}(h^1_{L\underline{L}})-\frac{2M}{r}\partial_{\underline{L}^*}(h^1_{L\underline{L}})+O_2(\frac{\varepsilon}{r^{2+\sigma}})=-\widehat{h}_1^{v^*u^*}\partial_{\underline{L}^*}(\widehat{h}_1^{v^*u^*})+\frac{2M}{r}\partial_{\underline{L}^*}(\hat{h}_1^{v^*u^*})+O_2(\frac{\varepsilon}{r^{2+\sigma}}).$$

□

Proof of Proposition 2.10. Contracting with $\underline{L}^*_\beta$ we see that

$$\underline{L}^*_\beta\partial_\alpha\left(g^{\alpha\beta}\sqrt{|g|}\right)=-\frac{1}{2}\underline{L}^*_\beta L^*_\alpha\partial_{\underline{L}^*}\left(g^{\alpha\beta}\sqrt{|g|}\right)-\frac{1}{2}\underline{L}^*_\beta\underline{L}^*_\alpha\partial_{L^*}\left(g^{\alpha\beta}\sqrt{|g|}\right)+\underline{L}^*_\beta\widehat{A}^c_\alpha\partial_{\widehat{y}^c}\left(g^{\alpha\beta}\sqrt{|g|}\right)=0.$$

We first analyze the last two expressions

$$-\frac{1}{2}\underline{L}^*_\beta\underline{L}^*_\alpha\partial_{L^*}\left(g^{\alpha\beta}\sqrt{|g|}\right)=-\frac{1}{2}\partial_{L^*}\left(\underline{L}^*_\beta\underline{L}^*_\alpha g^{\alpha\beta}\sqrt{|g|}\right)+\partial_{L^*}(\underline{L}^*_\alpha)\underline{L}^*_\beta g^{\alpha\beta}\sqrt{|g|}=-\frac{1}{2}\partial_{L^*}(\widehat{h}_1^{v^*v^*})-\frac{M}{r^2}+O_2(\frac{\varepsilon}{r^{2+\sigma}})$$

and

$$\begin{aligned}\underline{L}^*_\beta\widehat{A}^c_\alpha\partial_{\widehat{y}^c}\left(g^{\alpha\beta}\sqrt{|g|}\right)&=\partial_{\widehat{y}^c}\left(\underline{L}^*_\beta\widehat{A}^c_\alpha g^{\alpha\beta}\sqrt{|g|}\right)-\partial_{\widehat{y}^c}(\widehat{A}^c_\alpha)\underline{L}^*_\beta g^{\alpha\beta}\sqrt{|g|}-\partial_{\widehat{y}^c}(\underline{L}^*_\beta)\widehat{A}^c_\alpha g^{\alpha\beta}\sqrt{|g|}\\&=-\partial_{\widehat{y}^c}(\widehat{h}_1^{v^*c})+(\widehat{A}^d_\alpha\widehat{\Gamma}^c_{dc}+\frac{\delta^c_c\omega^j\delta_{j\beta}}{r})\underline{L}^*_\beta g^{\alpha\beta}\sqrt{|g|}+\rho'\frac{\delta^{ij}-\omega^i\omega^j}{r}\delta_{i\alpha}\delta_{j\beta}g^{\alpha\beta}+O_2(\frac{\varepsilon}{r^{2+\sigma}})\\&=-\widehat{\nabla\!\!\!/}_c(\widehat{h}_1^{v^*c})+\frac{\widehat{h}_1^{v^*u^*}}{r}-\frac{\widehat{h}_1^{v^*v^*}}{r}+O_2(\frac{\varepsilon}{r^{2+\sigma}}).\end{aligned}$$

Then it remains to compute the term $-\frac{1}{2}\underline{L}^*_\beta L^*_\alpha\partial_{\underline{L}^*}\left(g^{\alpha\beta}\sqrt{|g|}\right)$. Now we compute

$$-\frac{1}{2}\underline{L}^*_\beta L^*_\alpha\partial_{\underline{L}^*}\left(g^{\alpha\beta}\sqrt{|g|}\right)=-\frac{1}{2}\partial_{\underline{L}^*}\left(\underline{L}^*_\beta L^*_\alpha g^{\alpha\beta}\sqrt{|g|}\right)-\frac{M}{r^2}+O_2(\frac{\varepsilon}{r^{2+\sigma}}).$$

Using Lemma 2.11 and Lemma 2.12, we further calculate

$$\begin{aligned}-\frac{1}{2}\partial_{\underline{L}^*}\left(\underline{L}^*_\beta L^*_\alpha g^{\alpha\beta}\sqrt{|g|}\right)&=\frac{M}{r^2}-\frac{1}{2}\partial_{\underline{L}^*}(\widehat{h}_1^{v^*u^*})-\frac{M}{2r}\partial_{\underline{L}^*}(\widehat{h}_1^{v^*u^*})-\frac{1}{4}\widehat{h}_1^{v^*u^*}\partial_{\underline{L}^*}(\widehat{h}_1^{v^*u^*})-\frac{1}{2}\widehat{g}^{v^*u^*}\partial_{\underline{L}^*}(\sqrt{|g|})+O_2(\frac{\varepsilon}{r^{2+\sigma}})\\&=\frac{2M}{r^2}+\frac{1}{2}\partial_{\underline{L}^*}(\not{t}\!r h^1)-\frac{1}{4}\partial_{\underline{L}^*}(\widehat{h}_1^{ac}r^2\widehat{q}_{cb}\widehat{h}_1^{bd}r^2\widehat{q}_{da})+O_2(\frac{\varepsilon}{r^{2+\sigma}}).\end{aligned}$$

Putting all together finishes the proof. □

3. Construction of outgoing characteristic surfaces

In [27] we show that the eikonal equation

$$g^{\alpha\beta}\partial_\alpha u\,\partial_\beta u=0,\qquad\text{in}\quad r>|t|/2,\tag{3.1}$$

has a unique solution with asymptotic data at infinity $u\sim u^*=t-r^*$, as $t\to\infty$.

Remark 3.1. In the construction of u coordinate, we may change the asymptotic data imposed at infinity, i.e., we may require $u\sim u^*+f(\widehat{y}^3,\widehat{y}^4)$. This leads to the transformation at future null infinity $\mathscr{I}^+:u\to u+f(\overline{y}^3,\overline{y}^4)$ where $(\overline{y}^3,\overline{y}^4)$ are the angular coordinates in the Bondi-Sachs coordinate system associated to u whose construction will be given in Subsection 6.1. So the changes of asymptotic data for u of this type generate the *supertranslations* which is an infinite dimensional subgroup of the asymptotic symmetry group at null infinity—Bondi-Metzner-Sachs group [32, 39, 31].

Proposition 3.2 ([27, Proposition 26])**.** *The eikonal equation* (3.1) *has a solution* $u=\mathring{u}+u^*$ *satisfying*

$$\sum\nolimits_{|I|\le 2}|Z^{*I}\mathring{u}|\le C_1\varepsilon\Big(\frac{1+(r^*-|t|)_-}{1+t+|q^*|}\Big)^{\gamma'},\quad r>|t|/2.\tag{3.2}$$

Remark 3.3. Following the proof of Proposition 26 in [27], we can prove Proposition 3.2 for $|I|\le 3$. We commutate the vector fields $X\in\mathcal{X}=\{S^*=t\partial_t+x^{*i}\partial_{x^{*i}},\ \Omega_{ij},\ \partial_t\}$ through the equation (3.1). Let $\widetilde{X}=X-\delta_{XS^*}$ and $\widetilde{\mathcal{L}}_X=\mathcal{L}_X+2\delta_{XS^*}$, where $\delta_{XS^*}=1$ if $X=S^*$, and $=0$ otherwise. In fact, when $|I|=3$ we consider the equation $\partial_{\widetilde{L}}\widetilde{X}\widetilde{Y}\widetilde{Z}\tilde{u}=-H(g,u)/2$ where $\widetilde{L}^\alpha=g^{\alpha\beta}\partial_\beta u$ and

$$\begin{aligned}H(g,u)=&\,\widetilde{\mathcal{L}}_X\widetilde{\mathcal{L}}_Y\widetilde{\mathcal{L}}_Z g(\partial u,\partial u)\\&+2\widetilde{\mathcal{L}}_X\widetilde{\mathcal{L}}_Y g(\partial\widetilde{Z}u,\partial u)+2\widetilde{\mathcal{L}}_X\widetilde{\mathcal{L}}_Z g(\partial\widetilde{Y}u,\partial u)+2\widetilde{\mathcal{L}}_Y\widetilde{\mathcal{L}}_Z g(\partial\widetilde{X}u,\partial u)+2\widetilde{\mathcal{L}}_X g(\partial\widetilde{Y}\widetilde{Z}u,\partial u)\\&+2\widetilde{\mathcal{L}}_Y g(\partial\widetilde{X}\widetilde{Z}u,\partial u)+2\widetilde{\mathcal{L}}_Z g(\partial\widetilde{X}\widetilde{Y}u,\partial u)+2\widetilde{\mathcal{L}}_X g(\partial\widetilde{Z}u,\partial\widetilde{Y}u)+2\widetilde{\mathcal{L}}_Y g(\partial\widetilde{Z}u,\partial\widetilde{X}u)\\&+2\widetilde{\mathcal{L}}_Z g(\partial\widetilde{X}u,\partial\widetilde{Y}u)+2g(\partial\widetilde{X}\widetilde{Y}u,\partial\widetilde{Z}u)+2g(\partial\widetilde{X}\widetilde{Z}u,\partial\widetilde{Y}u)+2g(\partial\widetilde{Y}\widetilde{Z}u,\partial\widetilde{X}u)\end{aligned}$$

We notice that only the new term $\widetilde{\mathcal{L}}_X\widetilde{\mathcal{L}}_Y\widetilde{\mathcal{L}}_Z g_0(\partial u,\partial u)$ needs additional analysis. Since we already know that $\partial\widetilde{X}u^*=0$ and $\widetilde{\mathcal{L}}_X\widetilde{\mathcal{L}}_Y g_0(\partial u^*,\partial u^*)=0$, we obtain

$$\widetilde{\mathcal{L}}_X\widetilde{\mathcal{L}}_Y\widetilde{\mathcal{L}}_Z g_0(\partial u^*,\partial u^*)=X\Big(\widetilde{\mathcal{L}}_Y\widetilde{\mathcal{L}}_Z g_0(\partial u^*,\partial u^*)\Big)-2\widetilde{\mathcal{L}}_Y\widetilde{\mathcal{L}}_Z g_0(\partial\widetilde{X}u^*,\partial u^*)=0.$$

Moreover, we have $\widetilde{\mathcal{L}}_{\partial_t}g_0=\widetilde{\mathcal{L}}_\Omega g_0=0$ and $\widetilde{\mathcal{L}}_{S^*}g_0=\kappa_3 g_0-2(\kappa_1-\kappa_2)\overline{g}_0$ where $\kappa_1\sim M\ln r/r$, $\kappa_2\sim\kappa_3\sim M/r$ and $\overline{g}_0(\partial u,\partial v)=g_0^{ij}\bar{\partial}_i u\,\bar{\partial}_j v$. Using $\overline{g}_0(\partial\widetilde{X}^I u^*,\partial w)=0$ for $|I|\le 2$ we obtain $\widetilde{\mathcal{L}}_Y\overline{g}_0(\partial\widetilde{X}^I u^*,\partial w)=-\overline{g}_0(\partial\widetilde{Y}\widetilde{X}^I u^*,\partial w)-\overline{g}_0(\partial\widetilde{X}^I u^*,\partial\widetilde{Y}w)=0$ for $|I|\le 1$ and then $\widetilde{\mathcal{L}}_X\widetilde{\mathcal{L}}_Y\overline{g}_0(\partial u^*,\partial w)=-\widetilde{\mathcal{L}}_Y\overline{g}_0(\partial\widetilde{X}u^*,\partial w)-\widetilde{\mathcal{L}}_Y\overline{g}_0(\partial u^*,\partial\widetilde{X}w)=0$. Hence

$$|\widetilde{\mathcal{L}}_X\widetilde{\mathcal{L}}_Y\widetilde{\mathcal{L}}_Z g_0(\partial u^*,\partial\mathring{u})|\lesssim|\partial_{L^*}\mathring{u}|.$$

Using $\widetilde{\mathcal{L}}_X\overline{g}_0(\partial v,\partial w)=X(\overline{g}_0(\partial v,\partial w))-\overline{g}_0(\partial\widetilde{X}v,\partial w)-\overline{g}_0(\partial v,\partial\widetilde{X}w)$ and the expression for $\widetilde{\mathcal{L}}_{S^*}g_0$ we have

$$|\widetilde{\mathcal{L}}_X\widetilde{\mathcal{L}}_Y\widetilde{\mathcal{L}}_Z g_0(\partial\mathring{u},\partial\mathring{u})|\lesssim|\partial_{L^*}\mathring{u}||\partial_{\underline{L}^*}\mathring{u}|+(|\bar{\partial}\mathring{u}|+|\bar{\partial}\widetilde{X}\mathring{u}|)(|\bar{\partial}\mathring{u}|+|\bar{\partial}\widetilde{Y}\mathring{u}|)+|\bar{\partial}\mathring{u}||\bar{\partial}\widetilde{X}\widetilde{Y}\mathring{u}|.$$

Finally the estimates for the remaining terms in $H(g,u)$ and thus the bounds of $Z^{*I}\mathring{u}$ with $|I|=3$ follow as in Proposition 26 in [27]. Then we have

$$|\partial_{L^*}Z^{*I}\mathring{u}|=O(\frac{\varepsilon}{r^{1+\sigma}}),\quad\text{and}\quad|\partial_{\underline{L}^*}Z^{*I}\mathring{u}|+|\partial_{\widehat{y}^a}Z^{*I}\mathring{u}|=O(\frac{\varepsilon}{r^{\sigma}}),\quad\text{for}\quad|I|\le 2.$$

The estimate for $\partial_{\underline{L}^*}Z^{*I}\mathring{u}$ is not precise enough and we need to sharpen it. We first record three lemmas in [27], which will be of use when refining the expressions for $\partial_{\underline{L}^*}Z^{*I}\mathring{u}$.

Lemma 3.4 ([27, Lemma 21]). *If $Z=\partial_t$ then with $h_1^{\alpha\beta}=g^{\alpha\beta}-g_0^{\alpha\beta}$ and $\widetilde{L}^\alpha=g^{\alpha\beta}\partial_\beta u$ we have*

$$\partial_{\widetilde{L}}Z\mathring{u}=-\tfrac{1}{2}h_{1Z}(\partial u,\partial u)\tag{3.3}$$

with the notation $h_{1Z}(U,V)=h_{1Z}^{\alpha\beta}U_\alpha V_\beta$ where the Lie derivative $h_{1Z}^{\alpha\beta}=\mathcal{L}_Z h_1^{\alpha\beta}$ is given by

$$h_{1Z}^{\alpha\beta}\partial_\alpha u\,\partial_\beta w=(Zh_1^{\alpha\beta})\partial_\alpha u\,\partial_\beta w+h_1^{\alpha\beta}\partial_\alpha u\,[Z,\partial_\beta]w+h_1^{\alpha\beta}[Z,\partial_\alpha]u\,\partial_\beta w.\tag{3.4}$$

Lemma 3.5 ([27, Lemma 25]). *We have*

$$h_{1\partial_t}(\partial u,\partial u)=\partial_t(\widehat{h}_1^{u^*u^*})+2\partial_t(\widehat{h}_1^{u^*u^*})\partial_t\mathring{u}+\partial_t(\widehat{h}_1^{u^*u^*})(\partial_t\mathring{u})^2+O(\frac{\varepsilon}{r^{2+\sigma}}).\tag{3.5}$$

Lemma 3.6 ([27, Lemma 24]). *If $\Omega=x^i\partial_j-x^j\partial_i$ then with $k^{\alpha\Omega/r}=k^{\alpha i}\omega_j-k^{\alpha j}\omega_i$ we have*

$$(\mathcal{L}_\Omega k)(\partial u,\partial v)=(\Omega k)(\partial u,\partial v)+k([\Omega,\partial]u,\partial v)+k(\partial u,[\Omega,\partial]v),\tag{3.6}$$

$$k^{\alpha\beta}[\partial_\beta,\Omega]u=k^{\alpha\Omega/r}\partial_r u+\big(k^{\alpha i}\overline{\partial}_j-k^{\alpha j}\overline{\partial}_i\big)u.\tag{3.7}$$

Now we are ready to refine $\partial_{\underline{L}^*}Z^{*I}\mathring{u}$.

Proposition 3.7. *The eikonal equation* (3.1) *has a solution $u=\mathring{u}+u^*$ satisfying*

$$\partial_{\underline{L}^*}Z^I\mathring{u}=Z^I(\widehat{h}_1^{v^*u^*})+O(\frac{\varepsilon}{r^{1+\sigma}})\quad\textit{for}\quad Z\in\{\partial_t,\Omega_{ij}\}\quad\textit{and}\quad|I|\le 2.\tag{3.8}$$

Proof. Putting (2.16), (3.3) and (3.5) together, we obtain

$$\partial_{\widetilde{L}}\partial_t\mathring{u}=\frac{1}{2}\frac{\widehat{h}_1^{v^*u^*}}{r}+O(\frac{\varepsilon}{r^{2+\sigma}}).$$

It follows from Remark 2.4 that

$$\partial_{\widetilde{L}}\partial_t\mathring{u}=-\frac{1}{2}\partial_{L^*}(\widehat{h}_1^{v^*u^*})+O(\frac{\varepsilon}{r^{2+\sigma}}).$$

Since

$$\widetilde{L}=g^{\alpha\beta}\partial_\alpha u\partial_\beta=(-2+O(\frac{\varepsilon}{r^{\sigma}}))\partial_{v^*}+O(\frac{\varepsilon}{r^{1+\sigma}})\partial_{u^*}+O(\frac{\varepsilon}{r^{2+\sigma}})\partial_{\widehat{x}^a},\tag{3.9}$$

we find

$$\partial_{\widetilde{L}}\partial_t\mathring{u}=\frac{1}{2}\partial_{\widetilde{L}}(\widehat{h}_1^{v^*u^*})+O(\frac{\varepsilon}{r^{2+\sigma}}).$$

Therefore we can conclude that

$$\partial_t\mathring{u}=\frac{1}{2}\widehat{h}_1^{v^*u^*}+O(\frac{\varepsilon}{r^{1+\sigma}}).$$

If we commutate the vector fields $Z \in \{\partial_t, \Omega_{ij} = x^i\partial_j - x^j\partial_i\}$ through the equation $\partial_{\widetilde{L}}\partial_t\mathring{u} = -\frac{1}{2}h_{1\partial_t}(\partial u, \partial u)$, using proposition 3.2, Remark 3.3 and Lemma 3.6 and then integrating along the integral curves of $\widetilde{L}$ yield

$$\partial_t Z^I \mathring{u} = \frac{1}{2} Z^I(\widehat{h}_1^{v^*u^*}) + O(\frac{\varepsilon}{r^{1+\sigma}}).$$

Then (3.8) follows from the fact that $\partial_{\underline{L}^*} = 2\partial_t + \partial_{L^*}$. □

4. The Trautman mass

4.1. The asymtotically Schwarzschild coordinates. In this section we will use the asymptotically Schwarzschild coordinates $(\widetilde{t}, \widetilde{x})$ with

$$\widetilde{t} = t, \qquad \widetilde{x}^i = \widetilde{r}\omega^i, \qquad \text{where} \quad \omega^i = x^i/r, \qquad \widetilde{r} = r + M\ln r, \quad r = |\,x|.$$

Then

$$\partial_t = \partial_{\widetilde{t}}, \qquad \partial_{x^i} = \Big(\omega^i\omega^j(1+\frac{M}{r}) + \frac{\widetilde{r}}{r}(\delta^{ij} - \omega^i\omega^j)\Big)\partial_{\widetilde{x}^j} = \Big(\delta^{ij} + \frac{\widetilde{r}-r}{r}(\delta^{ij} - \omega^i\omega^j) + \frac{M}{r}\omega^i\omega^j\Big)\partial_{\widetilde{x}^j}.$$

In particular,

$$\frac{x^i}{|x|}\partial_{x^i} = (1+\frac{M}{r})\frac{\widetilde{x}^i}{|\widetilde{x}|}\partial_{\widetilde{x}^i}, \qquad \frac{x^i}{|x|}\partial_{x^k} - \frac{x^k}{|x|}\partial_{x^i} = \frac{\widetilde{r}}{r}\Big(\frac{\widetilde{x}^i}{|\widetilde{x}|}\partial_{\widetilde{x}^k} - \frac{\widetilde{x}^k}{|\widetilde{x}|}\partial_{\widetilde{x}^i}\Big).$$

Denote the corresponding metric components by $\tilde{g}_{\alpha\beta}$. We see that

$$\partial_{x^\alpha} = A_{\alpha\beta}\partial_{\widetilde{x}^\beta},$$

where the matrix A has the form

$$A_{\alpha\beta} = \delta_{\alpha\beta} + \frac{M\ln r}{r}(\delta^{ij} - \omega^i\omega^j)\delta_{\alpha i}\delta_{\beta j} + \frac{M}{r}\omega^i\omega^j\delta_{\alpha i}\delta_{\beta j},$$

where the sums are over $i, j = 1, 2, 3$ only. As a consequence, we have that

$$\tilde{g}^{\alpha\beta} = A_{\alpha\mu}A_{\beta\nu}g^{\mu\nu}.$$

Expanding the metric

$$g^{\mu\nu} = m^{\mu\nu} - M\delta^{\mu\nu}/r + h_1^{\mu\nu}$$

and using that $|h_1| \le r^{-1}\ln r$ we obtain

$$\begin{aligned}\tilde{g}^{\alpha\beta} &= m^{\alpha\beta} - \frac{M}{r}\delta^{\alpha\beta} + h_1^{\alpha\beta} + \frac{2M\ln r}{r}(\delta^{ij} - \omega^i\omega^j)\delta_{\alpha i}\delta_{\beta j} + \frac{2M}{r}\omega^i\omega^j\delta_{\alpha i}\delta_{\beta j} + O(\frac{\ln^2 r}{r^2}) + O(\frac{\ln r}{r}h_1)\\ &= (1+\frac{M}{r})m^{\alpha\beta} + h_1^{\alpha\beta} + \frac{2M(\ln r - 1)}{r}(\delta^{ij} - \omega^i\omega^j)\delta_{\alpha i}\delta_{\beta j} + O(\frac{\ln^2 r}{r^2}).\end{aligned}$$

Lemma 4.1. *Relative to the asymptotically Schwarzschild coordinates the metric components $\tilde{g}^{\alpha\beta}$ verify*

$$\tilde{g}^{\alpha\beta} = (1+\frac{M}{r})m^{\alpha\beta} + \frac{2M(\ln r - 1)}{r}(\delta^{ij} - \omega^i\omega^j)\delta_{\alpha i}\delta_{\beta j} + h_1^{\alpha\beta} + O(\frac{\ln^2 r}{r^2}),$$

in the region $r > t/2$. Here the sum is over $i, j = 1, 2, 3$ only.

Next we examine the wave coordinate expression $\partial_{\widetilde{x}^\alpha}\big(\tilde{g}^{\alpha\beta}\sqrt{|\tilde{g}|}\big)$ evaluated in $\widetilde{x}$-coordinates.

Lemma 4.2. *Relative to the asymptotically Schwarzschild coordinates the wave coordinate expression satisfy*

$$\partial_{\widetilde{x}^\alpha}\Big(\tilde{g}^{\alpha\beta}\sqrt{|\tilde{g}|}\Big) = -2\frac{M\ln r}{r^2}\omega^j\delta_{j\beta} + O(\frac{1}{r^2}).$$

Proof. We have

$$\partial_{\widetilde{x}^\alpha}\big(\tilde{g}^{\alpha\beta}\sqrt{|\tilde{g}|}\big) = (A^{-1})^{\alpha\mu}\partial_{x^\mu}\big(A_{\alpha\nu}A_{\beta\delta}|A|^{-1}g^{\nu\delta}\sqrt{|g|}\big) = (A^{-1})^{\alpha\mu}\partial_{x^\mu}\big(A_{\alpha\nu}A_{\beta\delta}|A|^{-1}\big)g^{\nu\delta}\sqrt{|g|},$$

where we used that the wave coordinate condition is satisfied in the x coordinates. Here the expression $\partial_{x^\mu}\left(A_{\alpha\nu}A_{\beta\delta}|A|^{-1}\right)$ is already at most of the order of $M\ln r/r^2$. Therefore, ignoring the terms of the order of $M^2\ln^2 r/r^3$ allows us to replace the above expression by

$$\partial_{x^\alpha}\left(m^{\nu\delta}A_{\alpha\nu}A_{\beta\delta}|A|^{-1}\right).$$

Replacing the matrix A by its expansion

$$A_{\alpha\beta} = \delta_{\alpha\beta} + \frac{M\ln r}{r}(\delta^{ij} - \omega^i\omega^j)\delta_{\alpha i}\delta_{\beta j} + O(\frac{1}{r}), \qquad |A| = 1 + 2\frac{M\ln r}{r} + \frac{M}{r} + O(\frac{\ln^2 r}{r^2}),$$

and ignoring the terms of the order of M/r^2 we obtain

$$2\partial_{x^i}\Big(\frac{M\ln r}{r}(\delta^{ij} - \omega^i\omega^j)\delta_{j\beta}\Big) - 2m^{\nu\beta}\partial_\nu\Big(\frac{M\ln r}{r}\Big).$$

The proof follows since again up to the terms of order M/r^2 we have

$$-2\Big(\frac{M\omega^i\ln r}{r^2}(\delta^{ij} - \omega^i\omega^j)\delta_{j\beta}\Big) - 2\frac{M\ln r}{r^2}2\omega^j\delta_{j\beta} + 2\delta_{j\beta}\frac{M\omega^j\ln r}{r^2} = -2\frac{M\ln r}{r^2}\omega^j\delta_{j\beta}. \qquad \square$$

The same calculation also gives

Lemma 4.3. *We have*

$$\partial_{\widetilde{x}^\alpha}\left(\tilde{g}^{\alpha\beta}|\tilde{g}|\right) = A_{\beta\delta}|A|^{-2}\partial_{x^\nu}\left(g^{\nu\delta}|g|\right) + O(\frac{1}{r^2}) = \frac{\tilde{g}^{\alpha\beta}}{\sqrt{|g|}}\partial_{\widetilde{x}^\alpha}\left(\sqrt{|g|}\right) + O(\frac{1}{r^2}),$$

$$\partial_{\widetilde{x}^\alpha}\left(\tilde{g}^{\alpha\beta}\sqrt{|g|}\right) = -4\frac{M\ln r}{r^2}\omega^j\delta_{j\beta} + O(\frac{1}{r^2}), \qquad \partial_{\widetilde{x}^\alpha}\left(\tilde{g}^{\alpha\beta}|\tilde{g}|/\sqrt{|g|}\right) = O(\frac{1}{r^2}).$$

4.2. **The Landau-Lifshitz pseudotensor.** In view of Proposition 3.2 the characteristic hypersurfaces of the metric g become asymptotic to the null cones of the Schwarzschild metric, we recast the Einstein equations in the form explicitly involving the asymptotically Schwarzschild coordinates $(\widetilde{t} = t, \widetilde{x})$ as opposed to the original Minkowski (t, x) harmonic coordinates.

Let $S_{\widetilde{u},\widetilde{r}} = \{(\widetilde{t},\widetilde{x}); \widetilde{t} = \widetilde{u} + \widetilde{r}\}$ be a sphere, following [36, 37, 38, 5] we define the *Trautman four-momentum*

$$M_T^\alpha(\widetilde{u}) = \lim_{r\to\infty}\frac{1}{4\pi}\int_{S_{\widetilde{u},\widetilde{r}}}\tilde{\mathbb{U}}^{\alpha\beta\gamma}\, dS_{\beta\gamma}.$$

Here $dS_{\beta\gamma} = n_{[\beta}k_{\gamma]}\widetilde{r}^2 dS(\omega)$ with $n_\gamma = (d\widetilde{r})_\gamma = (0,\omega_i)$, $k_\beta = (d\widetilde{t})_\beta = (1,0,0,0)$, and the superpotential $\tilde{\mathbb{U}}^{\alpha\beta\gamma}$ is

$$\tilde{\mathbb{U}}^{\alpha\beta\gamma} = \sqrt{|\tilde{g}|}\tilde{g}^{\alpha\mu}\tilde{\mathbb{U}}_\mu^{\beta\gamma} \quad \text{where} \quad \tilde{\mathbb{U}}_\mu^{\beta\gamma} = \sqrt{|\tilde{g}|}\tilde{g}^{\alpha\mu}\tilde{g}^{\sigma[\rho}\delta_\mu^\gamma\tilde{g}^{\beta]\tau}\widetilde{\partial}_\tau\tilde{g}_{\rho\sigma}.$$

Here the square brackets denote the antisymmetric part of a tensor, i.e., $T^{[a_1\cdots a_l]} = \sum_\sigma(-1)^\sigma T^{a_{\sigma(1)}\cdots a_{\sigma(l)}}$ where the sum is taken over all permutations σ of $1,\dots,l$ and $(-1)^\sigma$ is 1 for even permutations and -1 for odd permutations. A direct computation implies

$$\tilde{\mathbb{U}}^{\alpha\beta\gamma} = -\tilde{\lambda}^{\alpha\beta\mu}, \quad \text{where} \quad \tilde{\lambda}^{\alpha\beta\mu} = \widetilde{\partial}_\nu\big(|\tilde{g}|(\tilde{g}^{\alpha\beta}\tilde{g}^{\mu\nu} - \tilde{g}^{\alpha\mu}\tilde{g}^{\beta\nu})\big).$$

Therefore we can write

$$M_T^\alpha(\widetilde{u}) = \frac{1}{4\pi}\int_{\mathbb{S}^2} m_T^\alpha(\widetilde{u},\omega)dS(\omega),$$

where with $L_\alpha = (-1,\omega_i$ and $\underline{L}_\alpha = (-1,-\omega_i)$

$$m_T^\alpha(\widetilde{u},\omega) = \lim_{\widetilde{r}\to\infty}(\widetilde{r})^2\big(\tilde{\lambda}^{\alpha\beta\gamma}\underline{L}_\gamma L_\beta\big)(\widetilde{u} - \widetilde{r}, \widetilde{r}\omega).$$

The *Trautman radiated four-momentum* is defined as

$$E_T^\alpha(\widetilde{u}) = \lim_{\widetilde{r}\to\infty}\frac{1}{2\pi}\int_{S_{\widetilde{u},r}}|\tilde{g}|\tilde{\pi}^{\alpha\beta}\, dS_\beta.$$

Here $dS_\beta = n_\beta\widetilde{r}^2 dS(\omega)$ with $n_\beta = (0,\omega_i)$ and $\tilde{\pi}^{\alpha\beta}$ is *Landau-Lifshitz* pseudotensor [25, §101], which is a symmetric pseudotensor satisfying

$$\tilde{\pi}^{\alpha\beta} = -2\tilde{G}^{\alpha\beta} + \frac{1}{|\tilde{g}|}\widetilde{\partial}_\mu\tilde{\lambda}^{\alpha\beta\mu} \quad \text{where} \quad \tilde{G}^{\alpha\beta} = \tilde{R}^{\alpha\beta} - \frac{1}{2}\tilde{g}^{\alpha\beta}\tilde{R}.$$

We write

$$E_T^\alpha(\widetilde{u}) = \frac{1}{2\pi}\int_{\mathbb{S}^2}\Delta m_T^\alpha(\widetilde{u},\omega)\, dS(\omega) \quad \text{where} \quad \Delta m_T^\alpha(\widetilde{u},\omega) = \lim_{\widetilde{r}\to\infty}\widetilde{r}^2|\tilde{g}|\tilde{\pi}^{\alpha i}\frac{\widetilde{x}_i}{\widetilde{r}}.$$

It can be shown that the Einstein-vacuum equations $R_{\alpha\beta}(g)=0$ can be written in the form

$$|\tilde g|\tilde\pi^{\alpha\beta}=\frac{\partial\tilde\lambda^{\alpha\beta\mu}}{\partial\widetilde{x}^{\mu}},$$

where

$$\tilde\lambda^{\alpha\beta\mu}=\frac{\partial}{\partial\widetilde{x}^{\nu}}\left(|\tilde g|(\tilde g^{\alpha\beta}\tilde g^{\mu\nu}-\tilde g^{\alpha\mu}\tilde g^{\beta\nu})\right),\qquad \tilde\lambda^{\alpha\beta\mu}=-\tilde\lambda^{\alpha\mu\beta},$$

and $\tilde\pi^{\alpha\beta}$ is the Landau-Lifshitz pseudo tensor

$$\begin{aligned}\tilde\pi^{\alpha\beta}=(2\tilde\Gamma^{\gamma}_{\mu\nu}\tilde\Gamma^{\delta}_{\gamma\delta}-\tilde\Gamma^{\gamma}_{\mu\delta}\tilde\Gamma^{\delta}_{\nu\gamma}-\tilde\Gamma^{\gamma}_{\mu\gamma}\tilde\Gamma^{\delta}_{\nu\delta})(\tilde g^{\alpha\nu}\tilde g^{\beta\mu}-\tilde g^{\alpha\beta}\tilde g^{\mu\nu})+\tilde g^{\alpha\gamma}\tilde g^{\mu\nu}(\tilde\Gamma^{\beta}_{\gamma\delta}\tilde\Gamma^{\delta}_{\mu\nu}+\tilde\Gamma^{\beta}_{\mu\nu}\tilde\Gamma^{\delta}_{\gamma\delta}-\tilde\Gamma^{\beta}_{\mu\delta}\tilde\Gamma^{\delta}_{\gamma\nu}-\tilde\Gamma^{\beta}_{\gamma\nu}\tilde\Gamma^{\delta}_{\mu\delta})\\+\tilde g^{\beta\gamma}\tilde g^{\mu\nu}(\tilde\Gamma^{\alpha}_{\gamma\delta}\tilde\Gamma^{\delta}_{\mu\nu}+\tilde\Gamma^{\alpha}_{\mu\nu}\tilde\Gamma^{\delta}_{\gamma\delta}-\tilde\Gamma^{\alpha}_{\mu\delta}\tilde\Gamma^{\delta}_{\gamma\nu}-\tilde\Gamma^{\alpha}_{\gamma\nu}\tilde\Gamma^{\delta}_{\mu\delta})+\tilde g^{\mu\gamma}\tilde g^{\nu\delta}(\tilde\Gamma^{\alpha}_{\mu\nu}\tilde\Gamma^{\beta}_{\gamma\delta}-\tilde\Gamma^{\alpha}_{\mu\gamma}\tilde\Gamma^{\beta}_{\nu\delta}),\end{aligned}$$

where $\tilde\Gamma^{\alpha}_{\mu\nu}$ are the Christoffel symbols of $\tilde g$. Alternatively, with $\tilde{\mathcal G}^{\alpha\beta}=\sqrt{|\tilde g|}\tilde g^{\alpha\beta}$,

$$\begin{aligned}|\tilde g|\tilde\pi^{\alpha\beta}=\partial_{\widetilde{x}^{\mu}}\tilde{\mathcal G}^{\alpha\beta}\partial_{\widetilde{x}^{\nu}}\tilde{\mathcal G}^{\mu\nu}-\partial_{\widetilde{x}^{\mu}}\tilde{\mathcal G}^{\alpha\mu}\partial_{\widetilde{x}^{\nu}}\tilde{\mathcal G}^{\beta\nu}+\tfrac12\tilde g^{\alpha\beta}\tilde g_{\mu\nu}\partial_{\widetilde{x}^{\delta}}\tilde{\mathcal G}^{\gamma\mu}\partial_{\widetilde{x}^{\gamma}}\tilde{\mathcal G}^{\delta\nu}-\big(\tilde g^{\alpha\gamma}\tilde g_{\mu\nu}\partial_{\widetilde{x}^{\delta}}\tilde{\mathcal G}^{\beta\mu}\partial_{\widetilde{x}^{\gamma}}\tilde{\mathcal G}^{\delta\nu}+\tilde g^{\beta\gamma}\tilde g_{\mu\nu}\partial_{\widetilde{x}^{\delta}}\tilde{\mathcal G}^{\alpha\mu}\partial_{\widetilde{x}^{\gamma}}\tilde{\mathcal G}^{\delta\nu}\big)\\+\tilde g_{\mu\nu}\tilde g^{\gamma\delta}\partial_{\widetilde{x}^{\gamma}}\tilde{\mathcal G}^{\alpha\mu}\partial_{\widetilde{x}^{\delta}}\tilde{\mathcal G}^{\beta\nu}+\frac18(2\tilde g^{\alpha\mu}\tilde g^{\beta\nu}-\tilde g^{\alpha\beta}\tilde g^{\mu\nu})(2\tilde g_{\gamma\delta}\tilde g_{\rho\sigma}-\tilde g_{\gamma\rho}\tilde g_{\delta\sigma})\partial_{\widetilde{x}^{\mu}}\tilde{\mathcal G}^{\gamma\rho}\partial_{\widetilde{x}^{\nu}}\tilde{\mathcal G}^{\delta\sigma}.\end{aligned}$$

The tensor $\tilde\pi^{\alpha\beta}=\tilde\pi^{\beta\alpha}$ is symmetric and due to the anti-symmetry of $\tilde\lambda^{\alpha\beta\gamma}$ is divergence free

$$\partial_{\widetilde{x}^{\beta}}\left(|\tilde g|\tilde\pi^{\alpha\beta}\right)=0.$$

Integrating the above identity in the region $\{(\widetilde{t},\widetilde{x}):\ \widetilde{q}_1\le\widetilde{q}=\widetilde{r}-\widetilde{t}\le\widetilde{q}_2,\ |\widetilde{x}|\le R\}$ we obtain

$$\int_{|\widetilde{x}|\le R,\,\widetilde{q}=\widetilde{q}_1}|\tilde g|\tilde\pi^{\alpha\beta}L_\beta=\int_{|\widetilde{x}|\le R,\,\widetilde{q}=\widetilde{q}_2}|\tilde g|\tilde\pi^{\alpha\beta}L_\beta+\frac1R\int_{|\widetilde{x}|=R,\,\widetilde{q}_1\le\widetilde{q}\le\widetilde{q}_2}|\tilde g|\tilde\pi^{\alpha i}\widetilde{x}_i.\tag{4.1}$$

Using that $|\,\tilde g|\tilde\pi^{\alpha\beta}=\partial\tilde\lambda^{\alpha\beta\gamma}/\partial\widetilde{x}^{\gamma}$ we have

$$\int_{|\widetilde{x}|\le R,\,\widetilde{q}=\widetilde{q}_1}|\tilde g|\tilde\pi^{\alpha\beta}L_\beta=\int_{|\widetilde{x}|\le R,\,\widetilde{q}=\widetilde{q}_1}\partial_{\widetilde{x}^{\gamma}}\tilde\lambda^{\alpha\beta\gamma}L_\beta.$$

As usual we use the decomposition

$$\partial_{\widetilde{x}^{\gamma}}\tilde\lambda^{\alpha\beta\gamma}=-\frac12L_\gamma\partial_{\underline{L}}\tilde\lambda^{\alpha\beta\gamma}-\frac12\underline{L}_\gamma\partial_L\tilde\lambda^{\alpha\beta\gamma}+A_\gamma\partial_A\tilde\lambda^{\alpha\beta\gamma}+B_\gamma\partial_B\tilde\lambda^{\alpha\beta\gamma}.$$

where $(L,\underline{L},A,B)$ is the null frame associated to the asymptotically Schwarzschild coordinates $(\widetilde{t},\widetilde{x})$. By anti-symmetry of $\tilde\lambda^{\alpha\beta\gamma}$

$$L_\gamma\partial_{\underline{L}}\tilde\lambda^{\alpha\beta\gamma}L_\beta=0.$$

Therefore,

$$\int_{|\widetilde{x}|\le R,\,\widetilde{q}=\widetilde{q}_1}\partial_{\widetilde{x}^{\gamma}}\tilde\lambda^{\alpha\beta\gamma}L_\beta=-\frac12\int_{|\widetilde{x}|\le R,\,\widetilde{q}=\widetilde{q}_1}\partial_L\left(\tilde\lambda^{\alpha\beta\gamma}\underline{L}_\gamma L_\beta\right)+\int_{|\widetilde{x}|\le R,\,\widetilde{q}=\widetilde{q}_1}A_\gamma L_\beta\partial_A\tilde\lambda^{\alpha\beta\gamma}+B_\gamma L_\beta\partial_B\tilde\lambda^{\alpha\beta\gamma}.$$

On the surface $\widetilde{q}=\widetilde{q}_1$ we introduce coordinates (s,ω) with $s=\frac12(t+\widetilde{r}+\widetilde{q}_1)$ so that $\partial_L=\partial_s$ and the volume form is $s^2d\omega$. Then

$$\begin{aligned}\frac12\int_{|\widetilde{x}|\le R,\,\widetilde{q}=\widetilde{q}_1}\partial_L\left(\tilde\lambda^{\alpha\beta\gamma}\underline{L}_\gamma L_\beta\right)&=\frac12\int_0^R\int_{\mathbb S^2}\frac{d}{ds}\left(\tilde\lambda^{\alpha\beta\gamma}\underline{L}_\gamma L_\beta\right)s^2dsd\omega\\&=\frac12\int_{\mathbb S^2}\left(\tilde\lambda^{\alpha\beta\gamma}\underline{L}_\gamma L_\beta\right)(R,\omega)R^2d\omega-\int_0^R\int_{\mathbb S^2}\left(\tilde\lambda^{\alpha\beta\gamma}\underline{L}_\gamma L_\beta\right)sdsd\omega.\end{aligned}\tag{4.2}$$

On the other hand, using that $\partial_AL_\beta=A^k\partial_{\widetilde{x}^k}\widetilde{x}_\beta/\widetilde{r}=A^\beta/\widetilde{r}$ and that $\tilde\lambda^{\alpha\beta\gamma}$ is anti symmetric,

$$A_\gamma L_\beta\partial_A\tilde\lambda^{\alpha\beta\gamma}=A_\gamma\partial_A\big(L_\beta\tilde\lambda^{\alpha\beta\gamma}\big)-\frac1sA_\gamma A_\beta\tilde\lambda^{\alpha\beta\gamma}=A_\gamma\partial_A\big(L_\beta\tilde\lambda^{\alpha\beta\gamma}\big).\tag{4.3}$$

We now need the following lemma

Lemma 4.4. *Let A, B be orthonormal vector fields on $\mathbb{S}^2$, independent of $\widetilde{r}$; $\partial_{\widetilde{r}} A = \partial_{\widetilde{r}} B = 0$. Then*

$$|\partial A| + |\partial B| \lesssim 1/\widetilde{r}, \tag{4.4}$$

and with $\partial_A = A^k \partial_{\widetilde{x}^k} = A^k \overline{\partial}_{\widetilde{x}^k}$ we have

$$\partial_A(A^\ell) = -\omega^\ell/\widetilde{r} + \langle \partial_A A, B\rangle B^\ell, \qquad \partial_A(B^\ell) = \langle \partial_A B, A\rangle A^\ell. \tag{4.5}$$

Moreover, if $\overline{F}^k$ is tangential to $\mathbb{S}^2$ then

$$\operatorname{d\!\!/iv} \overline{F} = \overline{\partial}_k \overline{F}^k = \partial_k \overline{F}^k = A^k \partial_A \overline{F}_k + B^k \partial_B \overline{F}_k$$

satisfies

$$\int_{\mathbb{S}^2} \operatorname{d\!\!/iv} \overline{F}\, d\omega = 0. \tag{4.6}$$

On the other hand if F is not tangential then

$$\int_{\mathbb{S}^2} (A^\ell \partial_A + B^\ell \partial_B) F_\ell \, d\omega = \frac{2}{\widetilde{r}} \int_{\mathbb{S}^2} \omega^\ell F_\ell \, d\omega. \tag{4.7}$$

Using that $\omega_\gamma = \frac{1}{2}(L_\gamma - \underline{L}_\gamma)$ and by anti-symmetry of $\tilde{\lambda}^{\alpha\beta\gamma} L_\beta L_\gamma = 0$ we obtain

$$\int_{\mathbb{S}^2} A_\gamma L_\beta \partial_A \tilde{\lambda}^{\alpha\beta\gamma} + B_\gamma L_\beta \partial_B \tilde{\lambda}^{\alpha\beta\gamma} \, d\omega = -\int_{\mathbb{S}^2} \frac{1}{s} \underline{L}_\gamma L_\beta \tilde{\lambda}^{\alpha\beta\gamma} \, d\omega.$$

Combining this with (4.2) we finally obtain

$$\int_{|\widetilde{x}| \le R,\, \widetilde{q} = \widetilde{q}_1} \partial_{\widetilde{x}^\gamma} \tilde{\lambda}^{\alpha\beta\gamma} L_\beta = -\frac{1}{2} \int_{\mathbb{S}^2} \left(\tilde{\lambda}^{\alpha\beta\gamma} \underline{L}_\gamma L_\beta \right)(R, \widetilde{q}_1, \omega) R^2 d\omega.$$

Substituting this into (4.1) we obtain

$$\int_{\mathbb{S}^2} \left(\tilde{\lambda}^{\alpha\beta\gamma} \underline{L}_\gamma L_\beta \right)(R, \widetilde{q}_1, \omega) R^2 d\omega = \int_{\mathbb{S}^2} \left(\tilde{\lambda}^{\alpha\beta\gamma} \underline{L}_\gamma L_\beta \right)(R, \widetilde{q}_2, \omega) R^2 d\omega - 2 \int_{|\widetilde{x}| = R,\, \widetilde{q}_1 \le \widetilde{q} \le \widetilde{q}_2} |\tilde{g}| \tilde{\pi}^{\alpha i} \frac{\widetilde{x}_i}{R}.$$

Assume for a moment that the following limits exist

$$m_T^\alpha(\widetilde{q}, \omega) = \lim_{R \to \infty} R^2 \left(\tilde{\lambda}^{\alpha\beta\gamma} \underline{L}_\gamma L_\beta \right)(R, \widetilde{q}, \omega), \qquad \Delta m_T^\alpha(\widetilde{q}, \omega) = \lim_{R \to \infty} R^2 |\tilde{g}| \tilde{\pi}^{\alpha i} \widetilde{x}_i / R,$$

then we have the following analog of the *Bondi mass loss formula*

$$M_T^\alpha(\widetilde{q}_1) = M_T^\alpha(\widetilde{q}_2) - \int_{\widetilde{q}_1}^{\widetilde{q}_2} E_T^\alpha(\widetilde{q}) d\widetilde{q}. \tag{4.8}$$

In what follows we will establish existence of the above limits together with non-positivity of Δm_T.

4.3. **Existence of the Trautman mass.** Here we are concerned with establishing existence of the limit

$$m_T^\alpha(\widetilde{q}, \omega) = \lim_{\widetilde{r} \to \infty} (\widetilde{r})^2 \left(\tilde{\lambda}^{\alpha\beta\gamma} \underline{L}_\gamma L_\beta \right)(\widetilde{r}, \widetilde{q}, \omega)$$

and thus the Trautman mass M_T^0.

Proposition 4.5. *The Trautman four-momentum*

$$M_T^\alpha(\widetilde{u}) = \frac{1}{4\pi} \int_{\mathbb{S}^2} m_T^\alpha(\widetilde{u}, \omega) dS(\omega)$$

is well defined.

Proof. We consider the quantity $\tilde{\lambda}^{\alpha\beta\gamma} \underline{L}_\gamma L_\beta$ and show that for sufficiently large $r > t/2$

$$|\tilde{\lambda}^{\alpha\beta\gamma} \underline{L}_\gamma L_\beta| \le C r^{-2}.$$

Moreover, from our discussion and analysis in [27] it will be clear that the quantities defining the r^{-2} behavior $\tilde{\lambda}$ all have well defined limits as $r \to \infty$. To estimate $\tilde{\lambda}^{\alpha\beta\gamma} \underline{L}_\gamma L_\beta$ we first note that by Lemma 4.3 we have

$$\tilde{\lambda}^{\alpha\beta\gamma} \underline{L}_\gamma L_\beta = \underline{L}_\gamma L_\beta \frac{\partial}{\partial \widetilde{x}^\mu} \left(|\tilde{g}| (\tilde{g}^{\alpha\beta} \tilde{g}^{\gamma\mu} - \tilde{g}^{\alpha\gamma} \tilde{g}^{\beta\mu}) \right) = \frac{|\tilde{g}|}{\sqrt{|g|}} \underline{L}_\gamma L_\beta \left(\tilde{g}^{\gamma\mu} \partial_{\widetilde{x}^\mu} (\sqrt{|g|} \tilde{g}^{\alpha\beta}) - \tilde{g}^{\beta\mu} \partial_{\widetilde{x}^\mu} (\sqrt{|g|} \tilde{g}^{\alpha\gamma}) \right) + O(\frac{1}{r^2}).$$

According to Lemma 4.1

$$\tilde{g}^{\alpha\beta} = (1+\frac{M}{r})m^{\mu\nu} + \frac{2M(\ln r-1)}{r}(\delta^{ij}-\omega^i\omega^j)\delta_{\alpha i}\delta_{\beta j} + h_1^{\mu\nu} + O(\frac{\ln^2 r}{r^2}). \tag{4.9}$$

Using that

$$(\delta^{ij}-\omega^i\omega^j)\delta_{\alpha i}\delta_{\beta j}L^\beta = (\delta^{ij}-\omega^i\omega^j)\delta_{\alpha i}\delta_{\beta j}\underline{L}^\beta = 0$$

and the crude estimates $|\partial g| + |\partial\tilde{g}| \le \varepsilon r^{-1+\varepsilon}$, we obtain

$$\begin{aligned}\tilde{\lambda}^{\alpha\beta\gamma}\underline{L}_\gamma L_\beta &= (1+\frac{M}{r})\frac{|\tilde{g}|}{\sqrt{|g|}}\,\underline{L}_\gamma L_\beta\left(m^{\gamma\mu}\partial_{\tilde{x}^\mu}(\sqrt{|g|}\tilde{g}^{\alpha\beta}) - m^{\beta\mu}\partial_{\tilde{x}^\mu}(\sqrt{|g|}\tilde{g}^{\alpha\gamma})\right)\\ &\quad + \frac{|\tilde{g}|}{\sqrt{|g|}}\,\underline{L}_\gamma L_\beta\left(h_1^{\gamma\mu}\partial_{\tilde{x}^\mu}(\sqrt{|g|}\tilde{g}^{\alpha\beta}) - h_1^{\beta\mu}\partial_{\tilde{x}^\mu}(\sqrt{|g|}\tilde{g}^{\alpha\gamma})\right) + O(\frac{1}{r^2}).\end{aligned}$$

We first analyze the expression

$$\underline{L}_\gamma L_\beta\left(m^{\gamma\mu}\partial_{\tilde{x}^\mu}(\sqrt{|g|}\tilde{g}^{\alpha\beta}) - m^{\beta\mu}\partial_{\tilde{x}^\mu}(\sqrt{|g|}\tilde{g}^{\alpha\gamma})\right) = L_\beta\,\partial_{\underline{L}}(\sqrt{|g|}\tilde{g}^{\alpha\beta}) - \underline{L}_\beta\partial_L(\sqrt{|g|}\tilde{g}^{\alpha\beta}).$$

We write

$$L_\beta\,\partial_{\underline{L}}(\sqrt{|g|}\tilde{g}^{\alpha\beta}) = -\underline{L}_\beta\,\partial_L(\sqrt{|g|}\tilde{g}^{\alpha\beta}) + 2C_\beta\partial_C(\sqrt{|g|}\tilde{g}^{\alpha\beta}) - 2\partial_{\tilde{x}^\beta}(\sqrt{|g|}\tilde{g}^{\alpha\beta}).$$

Here and in what follows repeated index C is summed over $C = A, B$. Therefore

$$\underline{L}_\gamma L_\beta\left(m^{\gamma\mu}\partial_{\tilde{x}^\mu}(\sqrt{|g|}\tilde{g}^{\alpha\beta}) - m^{\beta\mu}\partial_{\tilde{x}^\mu}(\sqrt{|g|}\tilde{g}^{\alpha\gamma})\right) = -2\underline{L}_\beta\,\partial_L(\sqrt{|g|}\tilde{g}^{\alpha\beta}) + 2C_\beta\partial_C(\sqrt{|g|}\tilde{g}^{\alpha\beta}) - 2\partial_{\tilde{x}^\beta}(\sqrt{|g|}\tilde{g}^{\alpha\beta}).$$

We analyze the expression

$$\begin{aligned}2C_\beta\,\partial_C(\sqrt{|g|}\tilde{g}^{\alpha\beta}) &= 2(1+\frac{M}{r})C_\beta\partial_C(\sqrt{|g|}\,m^{\alpha\beta}) + 4\frac{M(\ln r-1)}{r}C_\beta\partial_C(\sqrt{|g|}\,(\delta_{ij}-\omega^i\omega^j)\delta^{\alpha j}\delta^{\beta i})\\ &\quad + 2C_\beta\partial_C(\sqrt{|g|}\,h_1^{\alpha\beta}) + O(\frac{1}{r^{3-\varepsilon}})\\ &= 2(1+\frac{M}{r})C^\alpha\partial_C(\sqrt{|g|}) + 4\frac{M(\ln r-1)}{r}\left(\partial_C(\sqrt{|g|}\,C^\alpha) - \sqrt{|g|}\,(\delta_{ij}-\omega^i\omega^j)\delta^{\alpha j}\delta^{\beta i}\partial_C C_\beta\right)\\ &\quad + 2\partial_C(\sqrt{|g|}\,h_1^{\alpha\beta}C_\beta) - 2\sqrt{|g|}\,h_1^{\alpha\beta}\partial_C C_\beta + O(\frac{1}{r^{3-\varepsilon}}).\end{aligned}$$

Here we used that $(\delta_{ij}-\omega^i\omega^j)\delta^{\alpha j}\delta^{\beta i}$ is the orthogonal projection on $\mathbb{S}^2$. In particular, $(\delta_{ij}-\omega^i\omega^j)\delta^{\alpha j}\delta^{\beta i}C_\beta = C^\alpha$ and by Lemma 4.4

$$\partial_A(A^k) = -\omega^k/\tilde{r} + \langle\partial_A A, B\rangle B^k. \tag{4.10}$$

Thus,

$$\begin{aligned}&\partial_A(\sqrt{|g|}A^k) - \sqrt{|g|}\,(\delta_{ij}-\omega^i\omega^j)\delta^{kj}\delta^{\beta i}\partial_A A_\beta\\ &\quad = A^k\partial_A(\sqrt{|g|}) + \sqrt{|g|}\big(\partial_A A^k - (\delta_{ij}-\omega^i\omega^j)\delta^{kj}\delta^{\beta i}\partial_A A_\beta\big) = A^k\partial_A(\sqrt{|g|}) - \frac{\omega^k}{\tilde{r}}.\end{aligned} \tag{4.11}$$

Furthermore by Proposition 2.1 we have the following estimates

$$|\,h_1^{\alpha\beta}A_\beta| \le \frac{C\varepsilon}{\tilde{r}}, \qquad |\,\partial_A\sqrt{|g|}| \le \frac{C\varepsilon}{\tilde{r}^2}, \qquad |\,\partial_A(h_1^{\alpha\beta}A_\beta)| \le \frac{C\varepsilon}{\tilde{r}^2}.$$

We remark that the last estimate above is very sensitive since its not true for each term in $(\partial_A h_1^{\alpha\beta})A_\beta + h_1^{\alpha\beta}\partial_A A_\beta$. As a result,

$$2\sum_{C=A,B} C_\beta\partial_C(\sqrt{|g|}\tilde{g}^{\alpha\beta}) = -8\frac{M\ln r}{r\tilde{r}}\omega^j\delta^{j\alpha} + \frac{4}{\tilde{r}}h_1^{\alpha\beta}\omega^j\delta_{j\beta} + O(\frac{1}{r^2}) = -8\frac{M\ln r}{r^2}\omega^j\delta^{j\alpha} + \frac{4}{r}h_1^{\alpha\beta}\omega^j\delta_{j\beta} + O(\frac{1}{r^2}).$$

Using Lemma 4.3 we further compute

$$-2\partial_{\tilde{x}^\alpha}\left(\tilde{g}^{\alpha\beta}\sqrt{|g|}\right) = 8\frac{M\ln r}{r^2}\omega^j\delta_{j\beta} + O(\frac{1}{r^2}).$$

Finally, with the help of the estimate $|\partial_L(\sqrt{|g|})| \le \frac{C\varepsilon}{r^{-2}}$ we obtain

$$\begin{aligned}
-2\underline{L}_\beta\, \partial_L(\sqrt{|g|}\tilde{g}^{\alpha\beta}) &= -2\underline{L}_\beta\, \partial_L(\tilde{g}^{\alpha\beta}) + O(\frac{1}{r^2}) \\
&= -2\underline{L}_\beta\, \partial_L((1+\frac{M}{r})m^{\alpha\beta}) - 2\partial_L\Big(\frac{2M\ln r}{r}(\delta^{ij}-\omega^i\omega^j)\delta^{\alpha i}\delta^{\beta j}\underline{L}_\beta\Big) - 2\,\partial_L(h_1^{\alpha\beta}\underline{L}_\beta) + O(\frac{1}{r^2}) \\
&= -2\,\partial_L(h_1^{\alpha\beta}\underline{L}_\beta) + O(\frac{1}{r^2}).
\end{aligned}$$

Gathering our estimates we obtain

$$\begin{aligned}
\underline{L}_\gamma L_\beta\left(m^{\gamma\mu}\partial_{\widetilde{x}^\mu}(\sqrt{|g|}\tilde{g}^{\alpha\beta}) - m^{\beta\mu}\partial_{\widetilde{x}^\mu}(\sqrt{|g|}\tilde{g}^{\alpha\gamma})\right) &= \frac{4}{r}h_1^{\alpha\beta}\omega^j\delta_{j\beta} - 2\,\partial_L(h_1^{\alpha\beta}\underline{L}_\beta) + O(\frac{1}{r^2}) \\
&= -\frac{2}{r}h_1^{\alpha\beta}\underline{L}_\beta - 2\,\partial_L(h_1^{\alpha\beta}\underline{L}_\beta) + O(\frac{1}{r^2}),
\end{aligned}$$

where in the last line we used that $\omega^j\delta_{j\beta} = (L_\beta - \underline{L}_\beta)/2$ and $|h_1^{\alpha\beta}L_\beta| \le C\varepsilon r^{-1}$. We now note that

$$\frac{2}{r}h_1^{\alpha\beta}\underline{L}_\beta + 2\,\partial_L(h_1^{\alpha\beta}\underline{L}_\beta) = \frac{2}{r}\partial_L(rh_1^{\alpha\beta})\underline{L}_\beta = O(\frac{1}{r^2}),$$

by the results in Proposition 2.3. Therefore,

$$\underline{L}_\gamma L_\beta\left(m^{\gamma\mu}\partial_{\widetilde{x}^\mu}(\sqrt{|\,g|}\,\tilde{g}^{\alpha\beta}) - m^{\beta\mu}\partial_{\widetilde{x}^\mu}(\sqrt{|\,g|}\,\tilde{g}^{\alpha\gamma})\right) = O(\frac{1}{r^2}).$$

To achieve the desired result for $\tilde{\lambda}^{\alpha\beta\gamma}\underline{L}_\gamma L_\beta$ it remains to show that the expression

$$\underline{L}_\gamma L_\beta\left(h_1^{\gamma\mu}\partial_{\widetilde{x}^\mu}(\sqrt{|\,g|}\,\tilde{g}^{\alpha\beta}) - h_1^{\beta\mu}\partial_{\widetilde{x}^\mu}(\sqrt{|\,g|}\,\tilde{g}^{\alpha\gamma})\right) = O(\frac{1}{r^2}).$$

Given the fact that $|(h_1)_{LT}| \le C\varepsilon r^{-1-\sigma}$, the term

$$h_1^{\beta\mu}\partial_{\widetilde{x}^\mu}(\sqrt{|\,g|}\,\tilde{g}^{\alpha\gamma})\underline{L}_\gamma L_\beta = O(\frac{1}{r^{2+\sigma}}).$$

It is clear that we only need to analyze the term $h^1_{L\underline{L}}\partial_{\underline{L}}\left(\sqrt{|\,g|}\,\tilde{g}^{\alpha\beta}L_\beta\right)$. Since

$$|h^1_{L\underline{L}}| \le \frac{C\varepsilon}{r}, \qquad |\partial\sqrt{|\,g|}| \le \frac{C\varepsilon}{r}, \qquad |\partial\tilde{g}^{\alpha\beta}L_\beta| \le \frac{C\varepsilon}{r},$$

we see that

$$h^1_{L\underline{L}}\partial_{\underline{L}}\left(\sqrt{|\,g|}\,\tilde{g}^{\alpha\beta}L_\beta\right) = O(\frac{1}{r^2}).$$ □

4.4. **Existence of the news function Δm_T.** We now establish existence of the limit

$$\Delta m_T^\alpha(\widetilde{q},\omega) = 2\lim_{\widetilde{r}\to\infty}\widetilde{r}^2|\tilde{g}|\,\tilde{\pi}^{\alpha i}\omega^i.$$

Proposition 4.6.

$$\Delta m_T^\alpha(\widetilde{q},\omega) = \frac{1}{4}L^\alpha\lim_{\widetilde{r}\to\infty}\widetilde{r}^2|\partial_{\widetilde{q}}\hat{\gamma}|^2(\widetilde{r},\widetilde{q},\omega).$$

Here $\hat{\gamma}_{CD} = h^1_{CD} - \frac{1}{2}\delta_{CD}(h^1_{AA} + h^1_{BB})$ is the traceless part of of the angular part of the metric g. Using Proposition 2.5 we can identify Δm_T^α with the expression.

$$\Delta m_T^\alpha(\widetilde{q},\omega) = \frac{1}{4}L^\alpha\delta^{CC'}\delta^{DD'}\partial_{\widetilde{q}}H^{1\infty}_{CD}\partial_{\widetilde{q}}H^{1\infty}_{C'D'} = \frac{1}{2}L^\alpha n(\widetilde{q},\omega).$$

Combining Propositions 4.5, 4.6 with the analog of the Bondi mass formula stated in (4.8) we obtain

Theorem 4.7.

$$M_T^\alpha(\widetilde{q}_1) = M_T^\alpha(\widetilde{q}_2) - \frac{1}{8\pi}\int_{\widetilde{q}_1}^{\widetilde{q}_2}\int_{\mathbb{S}^2}L^\alpha n(\widetilde{q},\omega)\,d\omega\,d\widetilde{q}.$$

Remark 4.8. Note that since

$$|\,\partial_{\widetilde{q}}H^{1\infty}_{AB}| \le \frac{C\varepsilon}{1+|\,\widetilde{q}|},$$

the News function is easily integrable with respect to the variable $\widetilde{q}$.

Proof of Proposition 4.6. Recall that

$$|\tilde g|\tilde\pi^{\alpha\beta} = \partial_{\widetilde{x}^\nu}\tilde{\mathcal G}^{\alpha\beta}\partial_{\widetilde{x}^\lambda}\tilde{\mathcal G}^{\nu\lambda} - \partial_{\widetilde{x}^\nu}\tilde{\mathcal G}^{\alpha\nu}\partial_{\widetilde{x}^\lambda}\tilde{\mathcal G}^{\beta\lambda} + \frac12 \tilde g^{\alpha\beta}\tilde g_{\nu\lambda}\partial_{\widetilde{x}^\delta}\tilde{\mathcal G}^{\nu\mu}\partial_{\widetilde{x}^\mu}\tilde{\mathcal G}^{\delta\lambda} - \tilde g^{\alpha\nu}\tilde g_{\lambda\mu}\partial_{\widetilde{x}^\delta}\tilde{\mathcal G}^{\beta\mu}\partial_{\widetilde{x}^\nu}\tilde{\mathcal G}^{\lambda\delta}$$
$$- \tilde g^{\beta\nu}\tilde g_{\lambda\mu}\partial_{\widetilde{x}^\delta}\tilde{\mathcal G}^{\alpha\mu}\partial_{\widetilde{x}^\nu}\tilde{\mathcal G}^{\lambda\delta} + \tilde g_{\nu\lambda}\tilde g^{\mu\delta}\partial_{\widetilde{x}^\mu}\tilde{\mathcal G}^{\alpha\nu}\partial_{\widetilde{x}^\delta}\tilde{\mathcal G}^{\beta\lambda} + \frac18(2\tilde g^{\alpha\nu}\tilde g^{\beta\lambda} - \tilde g^{\alpha\beta}\tilde g^{\nu\lambda})(2\tilde g_{\mu\delta}\tilde g_{\gamma\rho} - \tilde g_{\delta\gamma}\tilde g_{\mu\rho})\partial_{\widetilde{x}^\nu}\tilde{\mathcal G}^{\mu\rho}\partial_{\widetilde{x}^\lambda}\tilde{\mathcal G}^{\delta\gamma},$$

with $\tilde{\mathcal G}^{\alpha\beta} = \sqrt{|\tilde g|}\tilde g^{\alpha\beta}$. Recall that by Lemma 4.2

$$\partial_{\widetilde{x}^\alpha}\tilde{\mathcal G}^{\alpha\beta} = O(\frac{\ln r}{r^2}). \tag{4.12}$$

We also use the crude estimate

$$|\partial_{\widetilde{x}}\tilde{\mathcal G}| \le \frac{C\varepsilon}{r^{1-\varepsilon}(1+|\widetilde{q}|)^{1+\varepsilon}}.$$

Based on this we can replace

$$|\tilde g|\,\tilde\pi^{\alpha\beta} = \frac12 m^{\alpha\beta}m_{\nu\lambda}\partial_{\widetilde{x}^\delta}\tilde{\mathcal G}^{\nu\mu}\partial_{\widetilde{x}^\mu}\tilde{\mathcal G}^{\delta\lambda} - (m^{\alpha\nu}m_{\lambda\mu}\partial_{\widetilde{x}^\delta}\tilde{\mathcal G}^{\beta\mu}\partial_{\widetilde{x}^\nu}\tilde{\mathcal G}^{\lambda\delta} + m^{\beta\nu}m_{\lambda\mu}\partial_{\widetilde{x}^\delta}\tilde{\mathcal G}^{\alpha\mu}\partial_{\widetilde{x}^\nu}\tilde{\mathcal G}^{\lambda\delta})$$
$$+ m_{\nu\lambda}m^{\mu\delta}\partial_{\widetilde{x}^\mu}\tilde{\mathcal G}^{\alpha\nu}\partial_{\widetilde{x}^\delta}\tilde{\mathcal G}^{\beta\lambda} + \frac18(2m^{\alpha\nu}m^{\beta\lambda} - m^{\alpha\beta}m^{\nu\lambda})(2m_{\mu\delta}m_{\gamma\rho} - m_{\delta\gamma}m_{\mu\rho})\partial_{\widetilde{x}^\nu}\tilde{\mathcal G}^{\mu\rho}\partial_{\widetilde{x}^\lambda}\tilde{\mathcal G}^{\delta\gamma} + O(\frac{1}{r^{3-2\varepsilon}(1+|\widetilde{q}|)}).$$

Taking into account that

$$|\partial_A\tilde{\mathcal G}| + |\partial_L\tilde{\mathcal G}| \le \frac{C}{r^{2-\varepsilon}(1+|\,\widetilde{q}|)^{\varepsilon}}, \tag{4.13}$$

and using that modulo tangential derivatives $\partial_{\widetilde{x}^\alpha}$ is $L_\alpha\partial_{\widetilde{q}}$ we can further write

$$|\tilde g|\,\tilde\pi^{\alpha\beta} = \frac12 m^{\alpha\beta}m_{\nu\lambda}L_\delta L_\mu\partial_{\widetilde{q}}\tilde{\mathcal G}^{\nu\mu}\partial_{\widetilde{q}}\tilde{\mathcal G}^{\delta\lambda} - (m_{\lambda\mu}L^\alpha L_\delta\partial_{\widetilde{q}}\tilde{\mathcal G}^{\beta\mu}\partial_{\widetilde{q}}\tilde{\mathcal G}^{\lambda\delta} + m_{\lambda\mu}L^\beta L_\delta\partial_{\widetilde{q}}\tilde{\mathcal G}^{\alpha\mu}\partial_{\widetilde{q}}\tilde{\mathcal G}^{\lambda\delta})$$
$$+ \frac14 L^\alpha L^\beta(2m_{\mu\delta}m_{\gamma\rho} - m_{\delta\gamma}m_{\mu\rho})\partial_{\widetilde{q}}\tilde{\mathcal G}^{\mu\rho}\partial_{\widetilde{q}}\tilde{\mathcal G}^{\delta\gamma} + O(\frac{1}{r^{3-2\varepsilon}(1+|\,\widetilde{q}|)}).$$

Using the expression (4.9) for the metric g

$$\tilde g^{\alpha\beta} = (1+\frac{M}{r})m^{\mu\nu} + \frac{2M(\ln r - 1)}{r}(\delta^{ij} - \omega^i\omega^j)\delta^{\alpha i}\delta^{\beta j} + h_1^{\mu\nu} + O(\frac{\ln^2 r}{r^2}).$$

We easily see that[3]

$$|\,\partial_{\widetilde{q}}\tilde{\mathcal G}^{\lambda\mu}T_\mu| \le \frac{C\varepsilon}{r(1+|\,\widetilde{q}|)^{1-\varepsilon}}.$$

Moreover, using the wave coordinate condition

$$|\,\partial_{\widetilde{q}}\tilde g_{LT}| \le \frac{C\varepsilon}{r^{2-\varepsilon}(1+|\,\widetilde{q}|)^{\varepsilon}},$$

we conclude that

$$|\,\partial_{\widetilde{q}}\tilde{\mathcal G}_{LT}| \le \frac{C\varepsilon}{r^{2-\varepsilon}(1+|\,\widetilde{q}|)^{\varepsilon}}.$$

In the above we used the fact that $\tilde g_{LT} = O(1/r)$. Using that $m_{\mu\nu} = -(L_\mu\underline{L}_\nu + \underline{L}_\mu L_\nu)/2 + A_\mu A_\nu + B_\mu B_\nu$ this allows us to conclude that

$$|\tilde g|\,\tilde\pi^{\alpha\beta} = -\frac12\big(-L_\mu L^\alpha\partial_{\widetilde{q}}\tilde{\mathcal G}^{\beta\mu}\partial_{\widetilde{q}}\tilde{\mathcal G}_{\underline{L}L} - L_\mu L^\beta L_\delta\partial_{\widetilde{q}}\tilde{\mathcal G}^{\alpha\mu}\partial_{\widetilde{q}}\tilde{\mathcal G}_{\underline{L}L}\big) +$$
$$+ \frac14 L^\alpha L^\beta(2m_{\mu\delta}m_{\gamma\rho} - m_{\delta\gamma}m_{\mu\rho})\partial_{\widetilde{q}}\tilde{\mathcal G}^{\mu\rho}\partial_{\widetilde{q}}\tilde{\mathcal G}^{\delta\gamma} + O(\frac{1}{r^{3-2\varepsilon}(1+|\,\widetilde{q}|)})$$
$$= -\frac12 L^\alpha L^\beta(\partial_{\widetilde{q}}\tilde{\mathcal G}_{L\underline{L}})^2 + \frac14 L^\alpha L^\beta\left(2\partial_{\widetilde{q}}\tilde{\mathcal G}_{\mu\nu}\partial_{\widetilde{q}}\tilde{\mathcal G}^{\mu\nu} - (m^{\mu\nu}\partial_{\widetilde{q}}\tilde{\mathcal G}_{\mu\nu})^2\right) + O(\frac{1}{r^{3-2\varepsilon}(1+|\,\widetilde{q}|)}).$$

Here we used that we can expand any vector in the null frame $V^\alpha = -(L^\alpha F_{\underline{L}} + \underline{L}^\alpha F_L)/2 + A^\alpha F_A + B^\alpha F_B$. We can write

$$-2\partial_{\widetilde{q}}\tilde{\mathcal G}_{L\underline{L}} = -\partial_L\tilde{\mathcal G}_{\underline{LL}} + 2\partial_C\tilde{\mathcal G}_{\mu\nu}C^\mu\underline{L}^\nu - 2m^{\mu\delta}\partial_{\widetilde{x}^\delta}\tilde{\mathcal G}_{\mu\nu}\underline{L}^\nu.$$

[3]The ϵ loss occurs only due to the presence of the logarithmic terms.

Using (4.12) and (4.13) we obtain that

$$|\,\partial_{\widetilde{q}}\tilde{\mathcal{G}}_{L\underline{L}}| \leq \frac{C\varepsilon}{r^{2-\varepsilon}}.$$

Based on this we obtain

$$|\tilde{g}|\,\tilde{\pi}^{\alpha\beta} = \frac{1}{4}L^\alpha L^\beta\left(2\partial_{\widetilde{q}}\tilde{\mathcal{G}}_{CD}\partial_{\widetilde{q}}\tilde{\mathcal{G}}^{CD} - (\delta^{CD}\partial_{\widetilde{q}}\tilde{\mathcal{G}}_{CD})^2\right) + O(\frac{1}{r^{3-2\varepsilon}(1+|\widetilde{q}|)}) = \frac{1}{2}L^\alpha L^\beta|\partial_{\widetilde{q}}\hat{\mathcal{G}}|^2 + O(\frac{1}{r^{3-2\varepsilon}(1+|\widetilde{q}|)}).$$

Here $\hat{\tilde{\mathcal{G}}}_{CD}$ is a tensor

$$\hat{\tilde{\mathcal{G}}}_{CD} = \tilde{\mathcal{G}}_{CD} - \frac{1}{2}\delta_{CD}(\tilde{\mathcal{G}}_{AA} + \tilde{\mathcal{G}}_{BB}).$$

Remembering that $\tilde{\mathcal{G}}_{\alpha\beta} = \sqrt{|\tilde{g}|}\tilde{g}_{\alpha\beta}$ we obtain

$$\partial_{\widetilde{q}}\hat{\tilde{\mathcal{G}}}_{AB} = \partial_{\widetilde{q}}(\sqrt{|\tilde{g}|})\hat{\tilde{\gamma}}_{AB} + \sqrt{|\tilde{g}|}\,\partial_{\widetilde{q}}(\hat{\tilde{\gamma}}).$$

Here

$$\hat{\tilde{\gamma}}_{CD} = \tilde{g}_{CD} - \frac{1}{2}\delta_{CD}(\tilde{g}_{AA} + \tilde{g}_{BB}) = \hat{\gamma} + O(\frac{\ln^2 r}{r^2}), \qquad \hat{\gamma}_{CD} = h^1_{CD} - \frac{1}{2}\delta_{CD}(h^1_{AA} + h^1_{BB}).$$

In the above we used (4.9). Therefore we obtain

$$|\,\tilde{g}|\tilde{\pi}^{\alpha\beta} = \frac{1}{2}L^\alpha L^\beta|\partial_{\widetilde{q}}\hat{\gamma}|^2 + O(\frac{1}{r^{3-2\varepsilon}(1+|\,\widetilde{q}|)}). \qquad \square$$

4.5. **The ADM mass.** It remains to show that the Trautman mass as $\widetilde{q} \to \infty$ tends to the ADM mass of initial data which in our setting is M. It follows from the results in the previous section that all components of h^1 tend to 0 faster than r^{-1} as $r \to \infty$ so the limit of the mass as $r \to \infty$ only depend on $h^0_{\mu\nu} = M\delta_{\mu\nu}/r$. In fact a direct calculation implies the limit is exactly M. That this constant is positive would follow from proving that the Bondi mass tend to 0 as $\widetilde{q} \to -\infty$. This in turn would follow from Proposition 2.7.

5. The asymptotic Hawking mass

In this section we will use the modified asymptotically Schwarzschild null coordinates $\widehat{y}^p = (v^*, u^*, \widehat{y}^3, \widehat{y}^4)$ as defined in subsection 2.1.

5.1. **The definition of the asymptotic Hawking mass and radiated energy.** We define the radius of a surface S to be $r(S) = \sqrt{\text{Area}(S)/4\pi}$. Let $\hat{L}$ and $\hat{\underline{L}}$ be the outgoing respectively incoming null normals to S satisfying $g(\hat{L}, \hat{\underline{L}}) = -2$. $\hat{L}$ and $\hat{\underline{L}}$ are unique up to the transformation $\hat{L} \to a\hat{L}$ and $\hat{\underline{L}} \to a^{-1}\hat{\underline{L}}$. The null second fundamental form and the conjugate null second fundamental form are defined to be the tensors

$$\chi(X,Y) = g(\nabla_X\hat{L}, Y), \qquad \text{respectively} \qquad \underline{\chi}(X,Y) = g(\nabla_X\hat{\underline{L}}, Y),$$

for any vectors X, Y tangent to S at a point, where ∇_X is covariant differentiation. Under the transformation above $\chi \to a\chi$ and $\underline{\chi} \to a^{-1}\underline{\chi}$ so the Hawking mass of S,

$$M_{\mathcal{H}}(S) = r(S)\big(1 + \int_S \text{tr}\chi\,\text{tr}\underline{\chi}\,dS/16\pi\big),$$

is invariant. If $\text{tr}\chi\,\text{tr}\underline{\chi} < 0$ we can fix $\hat{L}$ and $\hat{\underline{L}}$ by $\text{tr}\chi + \text{tr}\underline{\chi} = 0$. Let $\hat{\chi}$ and $\hat{\underline{\chi}}$ be the traceless parts. The incoming respectively outgoing energy flux through S are

$$E(S) = \int_S \hat{\chi}^2 dS/16\pi, \qquad \text{and} \qquad \underline{E}(S) = \int_S \hat{\underline{\chi}}^2 dS/16\pi.$$

Owing to Proposition 3.2 the outgoing characteristic surfaces of g is asymptotic the null cones $u^* = t - r^*$ constant, we use the family of spheres $S_{u^*,r} = \{(t,x); t = u^* + r^*(r),\ |x| = r\}$ to define the asymptotic Hawking mass and the radiated energy at null infinity as follows

$$M_{AH}(u^*) = \lim_{r\to\infty} M_{\mathcal{H}}(S_{u^*,r}) \qquad \text{and} \qquad E_{AH}(u^*) = \lim_{r\to\infty} \underline{E}(S_{u^*,r}),$$

with $r(S)^2 \not{g}$ converging to a round metric where $\not{g}$ is the restriction of g on the spheres $S_{u^*,r}$.

In order for the limit of the Hawking mass to exist as well as the energy to be well defined we must have that $r(S)\,\text{tr}\chi \sim 2$ and $r(S)\,\text{tr}\underline{\chi} \sim -2$, as $r(S) \to \infty$. In order for the mass to be defined we also require that the rescaled spheres S^1 (scaled by $r(S)$ so that the radius is 1) converge to a round sphere, i.e. that the Gaussian curvature $r(S)^2 K \sim 1$, as $r(S) \to \infty$.

5.2. **The radiated energy at null infinity.** Assume that $r(S)^2 \not{g}$ converges to a round metric which we will prove in next subsection. We now prove the radiated energy at infinity is well defined.

Proposition 5.1. *The rdiated energy at null infinity is given by*

$$E_{AH}(u^*) = \frac{1}{8\pi}\int_{\mathbb{S}^2} n(-u^*,\omega)dS(\omega). \tag{5.1}$$

Proof. Since $\hat{\underline{L}}\, g(X,Y) = g(\nabla_{\hat{\underline{L}}} X,Y) + g(X,\nabla_{\hat{\underline{L}}} Y)$ and $\nabla_{\hat{\underline{L}}} X = \nabla_X \hat{\underline{L}} - [X,\hat{\underline{L}}]$ and since χ is symmetric we have

$$2\chi(X,Y) = \hat{L}\, g(X,Y) + g(X,[Y,\hat{L}]) + g([X,\hat{L}],Y), \tag{5.2}$$

$$2\underline{\chi}(X,Y) = \hat{\underline{L}}\, g(X,Y) + g(X,[Y,\hat{\underline{L}}]) + g([X,\hat{\underline{L}}],Y). \tag{5.3}$$

Since $g = m + h^0 + h^1$ this is true for the surface measures

$$g_{AB} = d^2\delta_{AB} + h^1_{AB}, \quad \text{and} \quad \det(g_{AB}) = d^4 + d^2\delta^{AB}h^1_{AB} + O\big((h^1)^2\big), \tag{5.4}$$

where $d = (1+M/r)^{1/2}$ and $\{A,B\}$ are orthonormal vector fields on $\mathbb{S}^2$ associated to the coordinates x, i.e. $A = A^k\partial_{x^k}$. Since $r\,\delta^{AB}h^1_{AB} \to 0$ it follows from our estimates that $\sqrt{\det(g_{AB})} = d^2 + O(r^{-1-\gamma'})$. Hence $dS_{u^*,r} = \sqrt{\det(g_{AB})}\, r^2 dS(\omega) \sim r^2 dS(\omega)$ and $r(S_{u^*,r}) \sim r$.

Let us define $\hat{L}$ and $\hat{\underline{L}}$ by $\hat{L}^\alpha = g^{\alpha\beta}\hat{L}_\beta$ and $\hat{\underline{L}}^\alpha = g^{\alpha\beta}\hat{\underline{L}}_\beta$, where $\hat{L}_i = -\hat{\underline{L}}_i = a\omega_i$, for $i = 1,2,3$, and $\hat{L}_0 = a\tau$, $\hat{\underline{L}}_0 = a\underline{\tau}$. The condition that they are outgoing respectively incoming null normal is then equivalent to $g^{00}\tau^2 + 2g^{0i}\tau\omega_i + g^{ij}\omega_i\omega_j = 0$ respectively $g^{00}\underline{\tau}^2 - 2g^{0i}\underline{\tau}\omega_i + g^{ij}\omega_i\omega_j = 0$. Completion of the squares gives $(\tau+\tau_1)^2 = (\underline{\tau}-\tau_1)^2 = \tau_0^2$, where $\tau_0 = \big(\tau_1^2 - g^{ij}\omega_i\omega_j/g^{00}\big)^{1/2}$ and $\tau_1 = g^{0i}\omega_i/g^{00}$. Hence $\tau + \underline{\tau} = -2\tau_0$ and $\tau - \underline{\tau} = -2\tau_1$. If we set $a = 2b/(\tau+\underline{\tau})$ we get $g(\hat{\underline{L}},\hat{L}) = a^2\big(g^{00}\tau\underline{\tau} - g^{0i}(\tau-\underline{\tau})\omega_i - g^{ij}\omega_i\omega_j\big) = 2b^2 g^{00}$. $\hat{L} + \hat{\underline{L}}$ is normal to the hyperplanes t constant and $\hat{L} - \hat{\underline{L}}$ is the normal to the spheres $S_{u^*,r}$ in these hyperplanes. If we set $a = (-g^{00}\tau_0^2)^{-1/2}$ we get $g(\hat{L},\hat{\underline{L}}) = -2$.

It follows that $\hat{L} = dL^* + O(h^1)$ and $\hat{\underline{L}} = d\underline{L}^* + O(h^1)$, where $L^* = \partial_t + \partial_{r^*}$, $\underline{L}^* = \partial_t - \partial_{r^*}$ and $d = (1+M/r)^{1/2}$. Moreover, this is true also for the derivatives. We therefore have $\chi \sim \chi^*$ and $\underline{\chi} \sim \underline{\chi}^*$, where $\chi^*(X,Y) = dg(\nabla_X L^*, Y)$ and $\underline{\chi}^*(X,Y) = dg(\nabla_X \underline{L}^*, Y)$. Since $[A,\underline{L}^*] = -r^{-1}(dr/dr^*)A = -r^{-1}A + O(r^{-2})$, we obtain

$$\underline{\chi}^*(A,B) = d\underline{L}^* g(A,B)/2 - dg(A,B)/r + O(r^{-2}), \quad \text{and} \quad \chi^*(A,B) = dL^* g(A,B)/2 + dg(A,B)/r + O(r^{-2}),$$

by (5.2)-(5.3). With $\mathrm{tr}k = \not{g}^{AB}k_{AB}$, where $\not{g}^{AB}$ is the inverse of g_{AB}, we have

$$\mathrm{tr}\underline{\chi}^*(A,B) = d\,\det(g(A,B))^{-1}\underline{L}^*\,\det(g(A,B))/2 - 2d/r + O(r^{-2}).$$

Here we used (5.4) and the identity $Z\det A = \det A\,\mathrm{tr}(A^{-1}ZA)$. Hence

$$\mathrm{tr}\underline{\chi}^*(A,B) = d^{-1}\underline{L}^*\,\delta^{AB}h^1_{AB}/2 - 2d/r + O\big(h^1\partial h^1\big) + O(r^{-2}),$$

$$\mathrm{tr}\chi^*(A,B) = d^{-1}L^*\,\delta^{AB}h^1_{AB}/2 + 2d/r + O\big(h^1\partial h^1\big) + O(r^{-2}).$$

Hence with $\hat{\underline{\chi}}^*(A,B) = \underline{\chi}^*(A,B) - \mathrm{tr}\underline{\chi}^*\, g(A,B)/2$ we have

$$\hat{\underline{\chi}}^*(A,B) = \underline{L}^*\hat{h}^1(A,B)/2 + O\big(h^1\partial h^1\big) + O(r^{-2}), \quad \text{and} \quad \hat{\chi}^*(A,B) = L^*\hat{h}^1(A,B)/2 + O\big(h^1\partial h^1\big) + O(r^{-2}).$$

where $\hat{h}^1_{AB} = h^1_{AB} - \delta_{AB}\,\delta^{CD}h^1_{CD}/2$. Since we shown that $r\,\delta^{CD}\partial_{q^*}h^1_{CD} \to 0$ and that $r\hat{h}^1$ has a limit as $r \to \infty$ along the curves $(u^* + r^*(r), r\omega)$ in Remark 2.6 it follows that

$$r^2|\hat{\underline{\chi}}^*|^2 \sim r^2(\partial_{q^*}\hat{h}^1)_{AB}\,(\partial_{q^*}\hat{h}^1)^{AB} \to 2n(-u^*,\omega). \qquad \square$$

5.3. **The convergence to a round metric sphere.**

Proposition 5.2. $r(S)^2\not{g}$ *converging to a round metric where* $\not{g}$ *is the restriction of* g *on the spheres* $S_{u^*,r}$

Proof. By Gauss equation ([13]) $K + \mathrm{tr}\chi\mathrm{tr}\underline{\chi}/2 - (\chi,\underline{\chi})/2 = \not{g}^{AC}\not{g}^{BD}R_{ABCD}$, where $\not{g}^{AC}$ is the inverse of the restriction of the metric to the sphere S. That $r(S)^2K \sim 1$ follows if we show that $r(S)^2\not{g}^{AC}\not{g}^{BD}R_{ABCD} \sim 0$.

We now calculate the curvature components in modified asymptotically Schwarzschild null coordinates:

$$\widehat{R}(g)_{abc}{}^p = \frac{\partial\widehat{\Gamma}(g)^p{}_{ac}}{\partial\widehat{x}^b} - \frac{\partial\widehat{\Gamma}(g)^p{}_{bc}}{\partial\widehat{x}^a} + \widehat{\Gamma}(g)^q{}_{ac}\widehat{\Gamma}(g)^p{}_{bq} - \widehat{\Gamma}(g)^q{}_{bc}\widehat{\Gamma}(g)^p{}_{aq},$$

and

$$\widehat{R}(g)_{abcd} = \widehat{g}_{dp}\frac{\partial\widehat{\Gamma}(g)^p{}_{ac}}{\partial\widehat{x}^b} - \widehat{g}_{dp}\frac{\partial\widehat{\Gamma}(g)^p{}_{bc}}{\partial\widehat{x}^a} + \widehat{\Gamma}(g)^q{}_{ac}\widehat{\Gamma}(g)_{dbq} - \widehat{\Gamma}(g)^q{}_{bc}\widehat{\Gamma}(g)_{daq}.$$

Since $g = g^0 + h^1$ with $g^0_{\alpha\beta} = m_{\alpha\beta} + M\delta_{\alpha\beta}/r$ and the inverse metric $g^{-1} = (g^0)^{-1} + h_1 + O(M^2/r^2)$, we calculate

$$-\widehat{\Gamma}(g^0)^{v^*}{}_{ab},\, \widehat{\Gamma}(g^0)^{u^*}{}_{ab} = f(r)\widehat{q}_{ab}, \qquad -\widehat{\Gamma}(g^0)_{v^*ab},\, \widehat{\Gamma}(g^0)_{u^*ab} = g(r)\widehat{q}_{ab}/2 \text{ where } f(r), g(r) = r + O_1(1)$$
$$\widehat{\Gamma}(g^0)_{abc} = (r^2 + Mr)\widehat{\Gamma}(m)_{abc}, \qquad \widehat{\Gamma}(g^0)^c{}_{ab} = \widehat{\Gamma}(m)^c{}_{ab}, \qquad \widehat{R}_{abcd}(g^0) = O(r).$$

Next we compute

$$\widehat{\Gamma}(g)_{abv^*} = \widehat{\Gamma}(g^0)_{abv^*} + \widehat{\Gamma}(h^1)_{abv^*} = \widehat{\Gamma}(g^0)_{abv^*} + \frac{1}{2}\Big(\frac{\partial\widehat{h}^1_{ab}}{\partial v^*} + \frac{\partial\widehat{h}^1_{av^*}}{\partial\widehat{x}^b} - \frac{\partial\widehat{h}^1_{bv^*}}{\partial\widehat{x}^a}\Big) = \widehat{\Gamma}(g^0)_{abv^*} + O_1(1),$$
$$\widehat{\Gamma}(g)_{abu^*} = \widehat{\Gamma}(g^0)_{abu^*} + \partial_{u^*}\widehat{h}^1_{ab}/2 + O_1(1), \qquad \widehat{\Gamma}(g)_{abc} = \widehat{\Gamma}(g^0)_{abc} + O_1(r).$$

and

$$\widehat{\Gamma}(g)^{v^*}{}_{ab} = \widehat{g}^{v^*p}\widehat{\Gamma}(g)_{pab} = \widehat{\Gamma}(g^0)^{v^*}{}_{ab} + \partial_{u^*}\widehat{h}^1_{ab} + O_1(r^{1-\sigma}),$$
$$\widehat{\Gamma}(g)^{u^*}{}_{ab} = \widehat{g}^{u^*p}\widehat{\Gamma}(g)_{pab} = \widehat{\Gamma}(g^0)^{u^*}{}_{ab} + O_1(1), \qquad \widehat{\Gamma}(g)^c{}_{ab} = \widehat{g}^{cp}\widehat{\Gamma}(g^0)_{pab} = \widehat{\Gamma}(g^0)^c{}_{ab} + O_1(r^{-1}).$$

Therefore

$$\partial_{\widehat{x}^d}\big(\widehat{\Gamma}(g)^{v^*}{}_{ab}\big) = \partial_{\widehat{x}^d}\big(\widehat{\Gamma}(g^0)^{v^*}{}_{ab}\big) + \partial_{\widehat{x}^d}\partial_{u^*}\widehat{h}^1_{ab}/2 + O_1(r^{1-\sigma}),$$
$$\partial_{\widehat{x}^d}\big(\widehat{\Gamma}(g)^{u^*}{}_{ab}\big) = \partial_{\widehat{x}^d}\big(\widehat{\Gamma}(g^0)^{u^*}{}_{ab}\big) + O_1(1), \qquad \partial_{\widehat{x}^d}\big(\widehat{\Gamma}(g)^c{}_{ab}\big) = \partial_{\widehat{x}^d}\big(\widehat{\Gamma}(g^0)^c{}_{ab}\big) + O_1(r^{-1}).$$

Finally we conclude that

$$\begin{aligned}\widehat{R}(g)_{3434} &= \widehat{R}(g^0)_{3434} + \big(r\widehat{q}_{44}\partial_{u^*}\widehat{h}^1_{33} + r\widehat{q}_{33}\partial_{u^*}\widehat{h}^1_{44}\big)/2 - \big(r\widehat{q}_{34}\partial_{u^*}\widehat{h}^1_{34} + r\widehat{q}_{34}\partial_{u^*}\widehat{h}^1_{34}\big)/2 + O(r^{2-\sigma})\\ &= r^3\det(\widehat{q}_{ab})\partial_{u^*}(\not{t}\!r h^1)/2 + O(r^{2-\sigma}) = O(r^{2-\sigma}).\end{aligned}$$

where we used $\widehat{q}^{33} = \det(\widehat{q}_{ab})^{-1}\widehat{q}_{44}$, $\widehat{q}^{44} = \det(\widehat{q}_{ab})^{-1}\widehat{q}_{33}$, $\widehat{q}^{34} = -\det(\widehat{q}_{ab})^{-1}\widehat{q}_{34}$ and $r^{-2}\widehat{q}^{ab}\widehat{h}^1_{ab} = \not{t}\!r h^1$ in the second step and Remark 2.2 in the last step. Due to the symmetry of the Riemann curvature tensor we see that $\widehat{R}(g)_{abcd} = O(r^{2-\sigma})$. Since $\not{g}^{ab}$ is the inverse of $\widehat{g}_{ab} = (1+M/r)r^2\widehat{q}_{ab} + \widehat{h}^1_{ab}$ we have $\not{g}^{ab} = r^{-2}\widehat{q}^{ab} + O(r^{-3})$ and thus $\not{g}^{ac}\not{g}^{bd}\widehat{R}(g)_{abcd} = O(r^{-2-\sigma})$. This proves that the rescaled spheres converge to a round sphere. □

5.4. **The limit of the Hawking mass along the asymptotically null hypersurfaces.** We now establish the existence of the asymptotic Hawking mass

$$M_{AH}(u^*) = \lim_{r\to\infty} M_{\mathcal{H}}(S_{u^*,r}).$$

Proposition 5.3.

$$M_{AH}(u^*) = \lim_{r\to\infty} M_{\mathcal{H}}(S_{u^*,r}) = M - \frac{1}{8\pi}\int_{-u^*}^{\infty}\int_{\mathbb{S}^2} n(\eta,\omega)\, dS(\omega)d\eta. \tag{5.5}$$

In view of (5.1) and the above proposition, we establish the following *Bondi mass loss formula*

Theorem 5.4.

$$\frac{d}{du^*}M_{AH}(u^*) = -E_{AH}(u^*). \tag{5.6}$$

Moreover we see that $M(u^*) = M$ *as* $u^* = t - r^* \to -\infty$ *and by Proposition 2.7* $M(u^*) = 0$ *as* $u^* = t - r^* \to \infty$.

Proof of Proposition 5.3. In order to obtain the explicit expression for $M_{AH}(u^*)$, we need to refine the expressions for $\mathrm{tr}\chi$, $\mathrm{tr}\underline{\chi}$. Recall that the outgoing and incoming null normals $\hat{L}, \hat{\underline{L}}$ to the surfaces $S_{u^*,r}$ are expressed in terms of $\tau, \underline{\tau}, a$ as defined in the proof of proposition 5.1. A direct computation implies

$$\tau = -1 + \frac{M}{r} - \frac{1}{2}(h_1^{00} + h_1^{ij}\omega_i\omega_j) + h_1^{0i}\omega_i + O_1(\frac{\varepsilon}{r^{1+\sigma}}) = -1 + \frac{M}{r} + O_1(\frac{\varepsilon}{r^{1+\sigma}}),$$
$$\underline{\tau} = -1 + \frac{M}{r} - \frac{1}{2}(h_1^{00} + h_1^{ij}\omega_i\omega_j) - h_1^{0i}\omega_i + O_1(\frac{\varepsilon}{r^{1+\sigma}}) = -1 + \frac{M}{r} - \frac{\widehat{h}_1^{v^*v^*}}{2} + O_1(\frac{\varepsilon}{r^{1+\sigma}}),$$
$$a = 1 + \frac{M}{2r} - \frac{1}{2}h_1^{ij}\omega_i\omega_j + O_1(\frac{\varepsilon}{r^{1+\sigma}}) = 1 + \frac{M}{2r} + \frac{1}{8}(2\widehat{h}_1^{v^*u^*} - \widehat{h}_1^{v^*v^*}) + O_1(\frac{\varepsilon}{r^{1+\sigma}}).$$

Suppose $\{A, B\}$ are orthonormal vector fields on $\mathbb{S}^2$ associated to the coordinates x, i.e. $A = A^k\partial_{x^k}$, we have

$$\hat{L} = \Big(1 + \frac{M}{2r} - \frac{\widehat{h}_1^{v^*v^*}}{8} - \frac{\widehat{h}_1^{v^*u^*}}{4} + O_1(\frac{\varepsilon}{r^{1+\sigma}})\Big)L^* + O_1(\frac{\varepsilon}{r^{1+\sigma}})\underline{L}^* + O_1(\frac{\varepsilon}{r^{1+\sigma}})\partial_A,$$
$$\underline{\hat{L}} = \Big(-\frac{\widehat{h}_1^{v^*v^*}}{4} + O_1(\frac{\varepsilon}{r^{1+\sigma}})\Big)L^* + \Big(1 + \frac{M}{2r} + \frac{\widehat{h}_1^{v^*v^*}}{8} - \frac{\widehat{h}_1^{u^*v^*}}{4} + O_1(\frac{\varepsilon}{r^{1+\sigma}})\Big)\underline{L}^* + \Big(-\widehat{h}_1^{v^*A} + O_1(\frac{\varepsilon}{r^{1+\sigma}})\Big)\partial_A.$$

Recall that

$$2\chi(A,B) = \hat{L}\, g(A,B) + g(A,[B,\hat{L}]) + g([A,\hat{L}],B), \quad \text{and} \quad 2\underline{\chi}(A,B) = \underline{\hat{L}}\, g(A,B) + g(A,[B,\underline{\hat{L}}]) + g([A,\underline{\hat{L}}],B).$$

With $\mathrm{tr}k = \not{g}^{AB}k_{AB}$, where $\not{g}^{AB}$ is the inverse of $g_{AB} = g(A,B)$, we have

$$\mathrm{tr}\chi = \det(g(A,B))^{-1}\hat{L}\,\det(g(A,B))/2 + \not{g}^{AB}\big(g(A,[B,\hat{L}]) + g([A,\hat{L}],B)\big)/2,$$
$$\mathrm{tr}\underline{\chi} = \det(g(A,B))^{-1}\underline{\hat{L}}\,\det(g(A,B))/2 + \not{g}^{AB}\big(g(A,[B,\underline{\hat{L}}]) + g([A,\underline{\hat{L}}],B)\big)/2.$$

Here we used the identity $Z\det A = \det A\,\mathrm{tr}(A^{-1}ZA)$. A direct calculation yields

$$L^*\det(g_{AB}) = -2Mr^{-2} + O(\varepsilon r^{-2-\sigma}), \quad \partial_A\det(g_{AB}) = O(\varepsilon r^{-2-\sigma}),$$
$$\underline{L}^*\det(g_{AB}) = 2Mr^{-2} + \underline{L}^*\not{\mathrm{tr}}h^1 + \underline{L}^*\det(h^1_{AB}) + O(\varepsilon r^{-2-\sigma}),$$

Therefore

$$\hat{L}\det(g_{AB}) = -2Mr^{-2} + O(\varepsilon r^{-2-\sigma}), \qquad \text{and} \qquad \underline{\hat{L}}\det(g_{AB}) = 2Mr^{-2} + \underline{L}^*\not{\mathrm{tr}}h^1 + \underline{L}^*\det h^1 + O(\varepsilon r^{-2-\sigma}).$$

and

$$\mathrm{tr}\chi = -Mr^{-2} + \not{g}^{AB}\big(g(A,[B,\hat{L}]) + g([A,\hat{L}],B)\big)/2 + O(\varepsilon r^{-2-\sigma}),$$
$$\mathrm{tr}\underline{\chi} = Mr^{-2} + \underline{L}^*\not{\mathrm{tr}}h^1/2 + \underline{L}^*\det(h^1_{AB})/2 + \not{g}^{AB}\big(g(A,[B,\underline{\hat{L}}]) + g([A,\underline{\hat{L}}].B)\big)/2 + O(\varepsilon r^{-2-\sigma}).$$

where we used $\det(g_{AB})^{-1} = 1 + O(\varepsilon r^{-1})$. It remains to control the commutators terms. We compute

$$g(A,[B,L^*]) = g(A, \frac{dr}{dr^*}\frac{\partial_B}{r}) = \Big(\frac{1}{r} - \frac{M}{r^2} + O(\frac{M^2}{r^3})\Big)g_{AB},$$
$$g(A,[B,\underline{L}^*]) = g(A, -\frac{dr}{dr^*}\frac{\partial_B}{r}) = -\Big(\frac{1}{r} - \frac{M}{r^2} + O(\frac{M^2}{r^3})\Big)g_{AB},$$
$$g(A,[B,C]) = g(A,(\not{\Gamma}^D_{BC} - \not{\Gamma}^D_{CB})D) = (\not{\Gamma}^D_{BC} - \not{\Gamma}^D_{CB})g_{AD}.$$

Here $\not{\Gamma}^C_{AB} = m(\not{\nabla}_{\partial_A}\partial_B, D)$ where $\not{\nabla}$ is the covariant differentiation on sphere and then $\not{\Gamma}$ are the associated frame-Christoffel symbol and homogeneous functions of degree -1 with respect to the radial variable r. Then

$$\frac{1}{2}\not{g}^{AB}\big(g(A,[B,\hat{L}]) + g([A,\hat{L}],B)\big) = \frac{2}{r} - \frac{M}{r^2} - \frac{\widehat{h}_1^{v^*v^*}}{4r} - \frac{\widehat{h}_1^{v^*u^*}}{2r} + O(\frac{\varepsilon}{r^{2+\sigma}}),$$
$$\frac{1}{2}\not{g}^{AB}\big(g(A,[B,\hat{L}]) + g([A,\hat{L}],B)\big) = -\frac{2}{r} + \frac{M}{r^2} - \frac{3\widehat{h}_1^{v^*v^*}}{4r} + \frac{\widehat{h}_1^{v^*u^*}}{2r} - \partial_C\widehat{h}_1^{v^*C} - \widehat{h}_1^{v^*C}(\not{\Gamma}^D_{DC} - \not{\Gamma}^D_{CD}) + O(\frac{\varepsilon}{r^{2+\sigma}})$$
$$= -\frac{2}{r} + \frac{M}{r^2} - \frac{3\widehat{h}_1^{v^*v^*}}{4r} + \frac{\widehat{h}_1^{v^*u^*}}{2r} - \not{\nabla}_C\widehat{h}_1^{v^*C} + O(\frac{\varepsilon}{r^{2+\sigma}}).$$

Here we used the fact that $\not{\Gamma}^D_{CD} = m(\not{\nabla}_{\partial_C}\partial_D, D) = \partial_C(m(D,D))/2 = 0$. Hence

$$\mathrm{tr}\chi = \frac{2}{r} - \frac{2M}{r^2} - \frac{\widehat{h}_1^{v^*v^*}}{4r} - \frac{\widehat{h}_1^{v^*u^*}}{2r} + O(\frac{\varepsilon}{r^{2+\sigma}}),$$
$$\mathrm{tr}\underline{\chi} = -\frac{2}{r} + \frac{2M}{r^2} - \frac{3\widehat{h}_1^{v^*v^*}}{4r} + \frac{\widehat{h}_1^{v^*u^*}}{2r} - \not{\nabla}_C\widehat{h}_1^{v^*C} + \frac{\underline{L}^*\not{\mathrm{tr}}h^1}{2} + \frac{\underline{L}^*\det(h^1_{AB})}{2} + O(\frac{\varepsilon}{r^{2+\sigma}}).$$

Using (2.18) we obtain

$$\mathrm{tr}\underline{\chi} = -\frac{2}{r} + \frac{2M}{r^2} + \frac{\partial_{L^*}\widehat{h}_1^{v^*v^*}}{2} + \frac{\widehat{h}_1^{v^*v^*}}{4r} - \frac{\widehat{h}_1^{v^*u^*}}{2r} + O(\frac{\varepsilon}{r^{2+\sigma}}).$$

Here we used these facts $\not\nabla_C \widehat{h}_1^{v^*C} = \widehat{\nabla}_c \widehat{h}_1^{v^*c}$, $\partial_{\underline{L}^*}(\widehat{h}_1^{ac} r^2 \widehat{q}_{cb} \widehat{h}_1^{bd} r^2 \widehat{q}_{da}) = \partial_{\underline{L}^*}(h^1_{AB} h^{1AB}) + O_2(\varepsilon r^{-2-\sigma})$ and $h^1_{AB} h^{1AB} + 2\det h^1 = (\not{tr} h^1)^2$. Hence

$$\mathrm{tr}\chi \mathrm{tr}\underline{\chi} = -\frac{4}{r^2} + \frac{8M}{r^3} + \frac{\partial_{L^*}\widehat{h}_1^{v^*v^*}}{r} + \frac{\widehat{h}_1^{v^*v^*}}{r^2} + O(\frac{\varepsilon}{r^{3+\sigma}}).$$

Since $dS_{u^*,r} = \sqrt{\det(g_{AB})} r^2 dS(\omega) = (1 + M/r + O(\varepsilon r^{-1-\sigma}))r^2 dS(\omega)$ and $r(S_{u^*,r}) = r + O(\varepsilon)$, It follows that

$$\begin{aligned} M_{\mathcal{H}}(S_{u^*,r}) &= r(S_{u^*,r})\big(1 + \int_{S_{u^*,r}} \mathrm{tr}\chi \,\mathrm{tr}\underline{\chi}\, dS_{u^*,r}/16\pi\big) \\ &= \big(r + O(\varepsilon)\big)\Big(\frac{M}{r} + \frac{1}{16\pi}\int_{S_{u^*,r}} r\partial_{L^*}\widehat{h}_1^{v^*v^*} + \widehat{h}_1^{v^*v^*} dS(\omega) + O(\frac{\varepsilon}{r^{1+\sigma}})\Big) = M - \frac{1}{8\pi}\int_{-u^*}^{\infty} n(\eta,\omega)\, d\eta + O(\frac{\varepsilon}{r^{\sigma}}), \end{aligned}$$

where we used the asymptotics result for the metric component $\widehat{h}_1^{v^*v^*}$ in Remark 2.6. Therefore

$$M_{AH}(u^*) = \lim_{r\to\infty} M_{\mathcal{H}}(S_{u^*,r}) = M - \frac{1}{8\pi}\int_{-u^*}^{\infty}\int_{\mathbb{S}^2} n(\eta,\omega)\, dS(\omega) d\eta. \qquad \square$$

Remark 5.5. According to the proof of Proposition 5.3, we find that the existence of the limit $M_{AH}(u^*)$ of the Hawking mass along the asymptotically null hypersurfaces does not require the null infinity to extend all the way back to the spatial infinity. Suppose the null infinity could be extended back to the spatial infinity, the past limit $\lim_{u^*\to-\infty} M_{AH}(u^*)$ equals to the ADM mass.

6. Bondi-Sachs coordinates

In this section, our goal is to construct the Bondi-Sachs coordinates $\overline{y}^p = (u, \overline{r}, \overline{y}^3, \overline{y}^4)$ under which we denote the metric by $\overline{g}$. The Bondi-Sachs coordinates $\overline{y}^p = (u, \overline{r}, \overline{y}^3, \overline{y}^4)$ are based on a family of outgoing null hypersurfaces $\overline{y}^1 = u = const$. The two angular coordinates $\overline{y}^a$, $(a, b, c, ... = 3, 4)$, are constant along the null rays, i.e. $g^{\alpha\beta}\partial_\beta u \partial_\alpha \overline{y}^a = 0$. The coordinate $\overline{y}^2 = \overline{r}$, which varies along the null rays, is chosen to be an areal coordinate such that $\det[\overline{g}_{ab}] = \overline{r}^4 \mathfrak{q}$, where $\mathfrak{q}(\overline{y}^a)$ is the determinant of the unit sphere metric $\overline{q}_{ab}$ associated with the angular coordinates $\overline{y}^a$. In these coordinates, the metric takes the Bondi-Sachs form

$$\overline{g}_{pq} d\overline{y}^p d\overline{y}^q = -\frac{V}{\overline{r}} e^{2\beta} du^2 - 2e^{2\beta} du d\overline{r} + \overline{r}^2 h_{ab}\Big(d\overline{y}^a - U^a du\Big)\Big(d\overline{y}^b - U^b du\Big).$$

We have already constructed u coordinate in section 3, it remains to construct the angular coordinates $\overline{y}^a$ and areal coordinate $\overline{r}$.

6.1. **Construction of angular coordinates.** Since $\{u = const\}$ are null hypersurfaces, for any point P, it must be at some null geodesic. After reparametrization, we see that P must be at some $X(s) = (s, u^*(s), \widehat{y}^3(s), \widehat{y}^4(s)) = X(s; u, \overline{y}^3, \overline{y}^4)$ where we use the notation $X(s; u, \overline{y}^3, \overline{y}^4)$ to emphasize that the integral curve $X(s)$ of the vector field $\widehat{g}^{pq}\partial_{\widehat{y}^q} u \partial_{\widehat{y}^p} / \widehat{g}^{v^*q}\partial_{\widehat{y}^q} u$ satisfies that $(u^*(s), \widehat{y}^3(s), \widehat{y}^4(s)) \to (u, \overline{y}^3, \overline{y}^4)$ as $s \to \infty$. Therefore, for any point $P \in X(s; u, \overline{y}^3, \overline{y}^4)$, we define $(\overline{y}^3, \overline{y}^4)$ to be the angular coordinates in Bondi-Sachs coordinates. Using the estimate

$$\frac{\widehat{g}^{aq}\partial_{\widehat{y}^q} u}{\widehat{g}^{v^*q}\partial_{\widehat{y}^q} u} = \frac{\widehat{h}^{u^*a} + O(\varepsilon r^{-3-\sigma})}{-1 + O(\varepsilon r^{-1})} = -\widehat{h}^{u^*a} + O(\frac{\varepsilon}{r^{3+\sigma}}),$$

and integrating along the curve $X(s; u, \overline{y}^3, \overline{y}^4)$ yield

$$\mathring{y}^a := \overline{y}^a - \widehat{y}^a = \int_{v^*}^{\infty} \frac{\widehat{g}^{aq}\partial_{\widehat{y}^q} u}{\widehat{g}^{v^*q}\partial_{\widehat{y}^q} u}\, ds = O(\frac{\varepsilon}{r^{1+\sigma}}).$$

We now have new angular coordinates $\overline{y}^a(v^*, u^*, \widehat{y}^3, \widehat{y}^4) = \widehat{y}^a + \mathring{y}^a(v^*, u^*, \widehat{y}^3, \widehat{y}^4)$ which we use in Bondi-Sachs coordinate system and then we derive the system for the derivatives of $\overline{y}^a$. According to the construction,

$$g^{\alpha\beta}\partial_\alpha u \partial_\beta \overline{y}^a = 0. \tag{6.1}$$

Differentiating (6.1) gives

$$g^{\alpha\beta}\partial_\alpha u\, \partial_\beta Z\overline{y}^a = -g_Z^{\alpha\beta}\partial_\alpha u\, \partial_\beta \overline{y}^a - g^{\alpha\beta}\partial_\alpha Z u\, \partial_\beta \overline{y}^a, \tag{6.2}$$

where the Lie derivative $g_Z^{\alpha\beta} = \mathcal{L}_Z g^{\alpha\beta}$ is given by

$$g_Z^{\alpha\beta}\partial_\alpha u\,\partial_\beta w = (Zg^{\alpha\beta})\partial_\alpha u\,\partial_\beta w + g^{\alpha\beta}\partial_\alpha u\,[Z,\partial_\beta]w + g^{\alpha\beta}[Z,\partial_\alpha]u\,\partial_\beta w. \tag{6.3}$$

Hence with the notation $g_Z(U,V) = g_Z^{\alpha\beta}U_\alpha V_\beta$ and using the facts $\partial Z u^* = 0,\ g_{0Z}(U,V) = 0$ for $Z \in \mathcal{Z} = \{\Omega_{ij}, \partial_t\}$, (6.1) respectively (6.2) become

$$\partial_{\widetilde{L}}\overline{y}^a = 0, \tag{6.4}$$

$$\partial_{\widetilde{L}} Z\mathring{y}^a = -h_{1Z}(\partial u, \partial\overline{y}^a) - g^{\alpha\beta}\partial_\alpha Z\mathring{u}\,\partial_\beta\overline{y}^a - \partial_{\widetilde{L}} Z\widehat{y}^a. \tag{6.5}$$

In order to estimate this system we need the following lemmas.

Lemma 6.1. *If* $|\Omega\mathring{y}^a| \le 1/20$ *we have*

$$|\partial_{v^*}\mathring{y}^a| \le \frac{C_0\varepsilon}{r^{1+\sigma}}|\partial_t\mathring{y}^a| + \frac{C_0\varepsilon}{r^{2+\sigma}}. \tag{6.6}$$

Proof. Using (3.2) and (3.8) we know that

$$\partial_{\widehat{y}^p} u = \Big(O(\frac{\varepsilon}{r^{1+\sigma}}),\ 1 + \frac{\widehat{h}_1^{v^*u^*}}{2} + O(\frac{\varepsilon}{r^{1+\sigma}}),\ O(\frac{\varepsilon}{r^\sigma}),\ O(\frac{\varepsilon}{r^\sigma})\Big), \tag{6.7}$$

and

$$\partial_{\widehat{y}^p}\overline{y}^a = \Big(\partial_{v^*}\mathring{y}^a,\ \partial_{u^*}\mathring{y}^a,\ \delta_b^a + \partial_{\widehat{y}^b}\mathring{y}^a\Big). \tag{6.8}$$

Since $g^{\alpha\beta}\partial_\alpha u\partial_\beta\overline{y}^a = \widehat{g}^{pq}\partial_{\widehat{y}^p}u\partial_{\widehat{y}^q}\overline{y}^a = 0$, with the estimates of $\widehat{g}$ and the assumption $|\Omega\mathring{y}^a| \le 1/20$ we obtain

$$\big(-2 + O(\frac{\varepsilon}{r})\big)\partial_{v^*}\mathring{y}^a + O(\frac{\varepsilon}{r^{1+\sigma}})\partial_t\mathring{y}^a + O(\frac{\varepsilon}{r^{2+\sigma}}) = 0.$$

This finishes the proof of the lemma. □

Lemma 6.2. *If* $|\Omega\mathring{y}^a| \le 1/20$ *we have*

$$|\partial_{\widetilde{L}}\partial_t\mathring{y}^a| \le \frac{C_0\varepsilon}{r}||\partial_{v^*}\mathring{y}^a| + \frac{C_0\varepsilon}{r^{1+\sigma}}|\partial_t\mathring{y}^a| + \frac{C_0\varepsilon}{r^{2+\sigma}}. \tag{6.9}$$

Proof. If $Z = \partial_t$, we have $\partial_{\widetilde{L}}\partial_t\widehat{y}^a = 0$ and by (6.7) and (6.8)

$$h_{1Z}(\partial u, \partial\overline{y}^a) = (\partial_t h_1^{\alpha\beta})\partial_\alpha u\partial_\beta\overline{y}^a = (\partial_t\widehat{h}_1^{pq})\partial_{\widehat{y}^p}u\partial_{\widehat{y}^q}\overline{y}^a = O(\frac{\varepsilon}{r})\partial_{v^*}\mathring{y}^a + O(\frac{\varepsilon}{r^{1+\sigma}})\partial_t\mathring{y}^a + O(\frac{\varepsilon}{r^{2+\sigma}}).$$

By Proposition 3.2 and (3.8)

$$\partial_{\widehat{y}^p}\partial_t\mathring{u} = \Big(O(\frac{\varepsilon}{r^{1+\sigma}}),\ \frac{\partial_t(\widehat{h}_1^{v^*u^*})}{2} + O(\frac{\varepsilon}{r^{1+\sigma}}),\ O(\frac{\varepsilon}{r^\sigma}),\ O(\frac{\varepsilon}{r^\sigma})\Big),$$

Hence by (6.8)

$$g^{\alpha\beta}\partial_\alpha\partial_t\mathring{u}\,\partial_\beta\overline{y}^a = \widehat{g}^{pq}\partial_{\widehat{y}^p}\partial_t\mathring{u}\partial_{\widehat{y}^q}\overline{y}^a = O(\frac{\varepsilon}{r})\partial_{v^*}\mathring{y}^a + O(\frac{\varepsilon}{r^{1+\sigma}})\partial_t\mathring{y}^a + O(\frac{\varepsilon}{r^{2+\sigma}}).$$

Then this lemma follows from (6.5) with $Z = \partial_t$ and the assumption $|\Omega\mathring{y}^a| \le 1/20$. □

Lemma 6.3. *If* $|\Omega\mathring{y}^a| \le 1/20$ *we have*

$$|\partial_{\widetilde{L}}\Omega\mathring{y}^a| \le \frac{C_0\varepsilon}{r}||\partial_{v^*}\mathring{y}^a| + \frac{C_0\varepsilon}{r^{1+\sigma}}|\partial_t\mathring{y}^a| + \frac{C_0\varepsilon|\Omega\mathring{y}^a|}{r^2} + \frac{C_0\varepsilon}{r^{2+\sigma}}. \tag{6.10}$$

Proof. The estimates for$(\Omega h_1^{\alpha\beta})\partial_\alpha u\partial_\beta\overline{y}^a$ and $g^{\alpha\beta}\partial_\alpha\Omega\mathring{u}\,\partial_\beta\overline{y}^a$ are similar to those in the proof of Lemma 6.2. Therefore we are left with $h_1^{\alpha\beta}\partial_\alpha u\,[Z,\partial_\beta]\overline{y}^a$, $h_1^{\alpha\beta}[Z,\partial_\alpha]u\,\partial_\beta\overline{y}^a$ and $\partial_{\widetilde{L}}\Omega\widehat{y}^a$. By Lemma 3.6, if $\Omega = x^i\partial_j - x^j\partial_i$

$$h_1^{\alpha\beta}[\partial_\beta, \Omega]u = k^{\alpha\Omega/r}\partial_r u + \big(k^{\alpha i}\overline{\partial}_j - k^{\alpha j}\overline{\partial}_i\big)u,$$

with $h_1^{\alpha\Omega/r} = k^{\alpha i}\omega_j - k^{\alpha j}\omega_i$. By (6.7) and (6.8) we conclude that

$$|h_1^{\alpha\beta}\partial_\alpha u\,[Z,\partial_\beta]\overline{y}^a| + |h_1^{\alpha\beta}[Z,\partial_\alpha]u\,\partial_\beta\overline{y}^a| = O(\frac{\varepsilon}{r})\partial_{v^*}\mathring{y}^a + O(\frac{\varepsilon}{r^{1+\sigma}})\partial_t\mathring{y}^a + O(\frac{\varepsilon}{r^2})|\Omega\mathring{y}^a|.$$

In view of (3.9), we have $\partial_{\widetilde{L}}\Omega\widehat{y}^a = O(\varepsilon r^{-2-\sigma})\partial_{\widehat{y}^c}\Omega\widehat{y}^a = O(\varepsilon r^{-2-\sigma})$. □

Proposition 6.4. *If $\varepsilon > 0$ is sufficiently small we have for $r > t/2$ and $r > 2$ with constants $C_1 = 2C_0C_\sigma$ for some universal constant C_σ*

$$|\partial_{v^*}\mathring{y}^a| \le \frac{2C_0\varepsilon}{r^{2+\sigma}}, \tag{6.11}$$

$$|\partial_t\mathring{y}^a| \le \frac{C_1\varepsilon}{r^{1+\sigma}}, \tag{6.12}$$

$$|\Omega\mathring{y}^a| \le \frac{C_1\varepsilon}{r^{1+\sigma}}. \tag{6.13}$$

Proof. We can prove this by assuming these estimates are true and show that they imply better estimates if ε is sufficiently small. First from (6.6) and the assumed bound (6.12), we prove (6.11) with $2C_0$ replaced by $3C_0/2$ if ε is sufficiently small such that $2C_0^2C_\sigma\varepsilon \le 1/2$. From the construction of $\overline{y}^a$ we see that $\partial_t\mathring{y}^a, \Omega\mathring{y}^a \to 0$ as $v^* \to \infty$. If we integrate (6.5) with $Z = \partial_t$ and use the assumed bound (6.11) and (6.12) we obtain

$$|\partial_t\mathring{y}^a| \le \frac{\varepsilon C_\sigma(2\varepsilon C_0^2C_0 + \varepsilon C_0C_1 + C_0)}{r^{1+\sigma}} \le \frac{3C_1\varepsilon/4}{r^{1+\sigma}},$$

if ε is sufficiently small such that $2C_0^2\varepsilon + \varepsilon C_1 \le 1/2$ which proves (6.12). If we integrate (6.5) with $Z = \Omega$ and use the assumed bound (6.11), (6.12) and (6.13) we obtain

$$|\Omega\mathring{y}^a| \le \frac{\varepsilon C_\sigma(2\varepsilon C_0^2C_0 + 2\varepsilon C_0C_1 + C_0)}{r^{1+\sigma}} \le \frac{3C_1\varepsilon/4}{r^{1+\sigma}},$$

if ε is sufficiently small such that $2C_0^2\varepsilon + 2\varepsilon C_1 \le 1/2$ which proves (6.13). □

We now turn to higher order derivatives of $\mathring{y}^a$.

Proposition 6.5. *We have*

$$\sum\nolimits_{|I|\le 2}|Z^{*I}\mathring{y}^a| = O(\frac{\varepsilon}{r^{1+\sigma}}). \tag{6.14}$$

Proof. Following the proof of Proposition 3.2 we commutate the vector fields $X \in \mathcal{X} = \{S^* = t\partial_t + x^{*i}\partial_{x^{*i}},\ \Omega_{ij},\ \partial_t\}$ through the equation (6.4). Let $\widetilde{X} = X - \delta_{XS^*}$ and $\widetilde{\mathcal{L}}_X = \mathcal{L}_X + 2\delta_{XS^*}$, where $\delta_{XS^*} = 1$ if $X = S^*$, and $= 0$ otherwise. Then $X\big(k(\partial u, \partial v\big) = (\widetilde{\mathcal{L}}_X k)(\partial u, \partial v) + k(\partial\widetilde{X}u, \partial v) + k(\partial u, \partial\widetilde{X}v)$ and $\partial\widetilde{X}u^* = 0$. Since $g(\partial u, \partial\overline{y}^a) = 0$ we get $\partial_{\widetilde{L}}\widetilde{X}\widetilde{Z}\overline{y}^a = -H(g, u, \overline{y}^a)$ where

$$\begin{aligned} H(g,u,\overline{y}^a) &= \widetilde{\mathcal{L}}_X\widetilde{\mathcal{L}}_Z g(\partial u, \partial\overline{y}^a) + \widetilde{\mathcal{L}}_X g(\partial u, \partial\widetilde{Z}\overline{y}^a) + \widetilde{\mathcal{L}}_X g(\partial\widetilde{Z}u, \partial\overline{y}^a) + \widetilde{\mathcal{L}}_Z g(\partial u, \partial\widetilde{X}\overline{y}^a) \\ &\quad + \widetilde{\mathcal{L}}_Z g(\partial\widetilde{X}u, \partial\overline{y}^a) + g(\partial\widetilde{X}\widetilde{Z}u, \partial\overline{y}^a) + g(\partial\widetilde{Z}u, \partial\widetilde{X}\overline{y}^a) + g(\partial\widetilde{X}u, \partial\widetilde{Z}\overline{y}^a). \end{aligned}$$

Here $\widetilde{\mathcal{L}}_X g = \widetilde{\mathcal{L}}_X g_0 + \widetilde{\mathcal{L}}_X h_1$, where $\widetilde{\mathcal{L}}_{\partial_t} g_0 = \widetilde{\mathcal{L}}_\Omega g_0 = 0$ and $\widetilde{\mathcal{L}}_{S^*} g_0 = \kappa_3 g_0 - 2(\kappa_1 - \kappa_2)\overline{g}_0$. Here $\kappa_1 \sim M\ln r/r$, $\kappa_2 \sim \kappa_3 \sim M/r$ and $\overline{g}_0(\partial u, \partial v) = g_0^{ij}\not\partial_i u\,\not\partial_j v$. Since $g_0(\partial\widetilde{X}^I u, \partial\overline{y}^a) = O(\varepsilon r^{-2-\sigma})$ and $\overline{g}_0(\partial\widetilde{X}^I u, \partial\overline{y}^a) = O(\varepsilon r^{-2-\sigma})$ for $|I| \le 1$, we have $\widetilde{\mathcal{L}}_X g_0(\partial\widetilde{X}^I u, \partial\overline{y}^a) = O(\varepsilon r^{-2-\sigma})$ for $|I| \le 1$. Moreover, $\widetilde{\mathcal{L}}_X \overline{g}_0(\partial u, \partial\overline{y}^a) = X(\overline{g}_0(\partial u, \partial\overline{y}^a)) - \overline{g}_0(\partial\widetilde{X}u, \partial\overline{y}^a) - \overline{g}_0(\partial u, \partial\widetilde{X}\overline{y}^a)$. It follows that $|\widetilde{\mathcal{L}}_X^I g_0(\partial u, \partial\overline{y}^a)| = O(\varepsilon r^{-2-\sigma})|\Omega\widetilde{X}\overline{y}^a| + O(\varepsilon r^{-2-\sigma})$, for $|I| \le 2$. Hence

$$|\widetilde{\mathcal{L}}_X\widetilde{\mathcal{L}}_Z g(\partial u, \partial\overline{y}^a) + \widetilde{\mathcal{L}}_X g(\partial\widetilde{Z}u, \partial\overline{y}^a) + \widetilde{\mathcal{L}}_Z g(\partial\widetilde{X}u, \partial\overline{y}^a) + g(\partial\widetilde{X}\widetilde{Z}u, \partial\overline{y}^a)| = O(\frac{\varepsilon}{r^{2+\sigma}})\Big(|\Omega\widetilde{X}\overline{y}^a| + |\Omega\widetilde{Z}\overline{y}^a|\Big) + O(\frac{\varepsilon}{r^{2+\sigma}}).$$

Then it remains to control the terms containing second order derivatives of $\overline{y}^a$

$$\begin{aligned} &|\widetilde{\mathcal{L}}_X g(\partial u, \partial\widetilde{Z}\overline{y}^a) + \widetilde{\mathcal{L}}_Z g(\partial u, \partial\widetilde{X}\overline{y}^a) + g(\partial\widetilde{Z}u, \partial\widetilde{X}\overline{y}^a) + g(\partial\widetilde{X}u, \partial\widetilde{Z}\overline{y}^a)| \\ &= O(\frac{\varepsilon}{r})\Big(|\partial_{L^*}\widetilde{X}\overline{y}^a| + |\partial_{L^*}\widetilde{Z}\overline{y}^a|\Big) + O(\frac{\varepsilon}{r^{1+\sigma}})\Big(|\partial_t\widetilde{X}\overline{y}^a| + |\partial_t\widetilde{Z}\overline{y}^a|\Big) + O(\frac{\varepsilon}{r^{2+\sigma}})\Big(|\Omega\widetilde{X}\overline{y}^a| + |\Omega\widetilde{Z}\overline{y}^a|\Big) + O(\frac{\varepsilon}{r^{2+\sigma}}). \end{aligned}$$

We have $\partial_{\widetilde{L}}\widetilde{X}\overline{y}^a = -\widetilde{\mathcal{L}}_X g(\partial u, \partial\overline{y}^a) - g(\partial\widetilde{X}u, \partial\overline{y}^a)$, so $|\partial_{\widetilde{L}}\widetilde{X}\overline{y}^a| = O(\varepsilon r^{-2-\sigma})$. By (3.9)

$$|\partial_{L^*}\widetilde{X}\overline{y}^a| \lesssim |\partial_{\widetilde{L}}\widetilde{X}\overline{y}^a| + O(\frac{\varepsilon}{r^{1+\sigma}})|\partial_t\widetilde{X}\overline{y}^a| + O(\frac{\varepsilon}{r^{2+\sigma}})|\Omega\widetilde{X}\overline{y}^a|.$$

Therefore we conclude

$$|\partial_{\widetilde{L}}\widetilde{X}\widetilde{Z}\mathring{y}^a| = O(\frac{\varepsilon}{r^{1+\sigma}})\Big(|\partial_t\widetilde{X}\tilde{y}^a| + |\partial_t\widetilde{Z}\tilde{y}^a|\Big) + O(\frac{\varepsilon}{r^{2+\sigma}})\Big(|\Omega\widetilde{X}\tilde{y}^a| + |\Omega\widetilde{Z}\tilde{y}^a|\Big) + O(\frac{\varepsilon}{r^{2+\sigma}}).$$

where we used the facts that $\Omega\widetilde{X}\widehat{y}^a = O(1)$, $\partial_t\widetilde{X}\widehat{y}^a = 0$ and $\partial_{\widetilde{L}}\widetilde{X}\widetilde{Z}\widehat{y}^a = O(\varepsilon r^{-2-\sigma})$. We repeat the proof of Proposition 6.4 and the conclusion follows. □

We now refine $\partial_t Z^I \mathring{y}^a$ with $|I| \le 1$.

Proposition 6.6. *If $Z \in \{\partial_t, \Omega_{ij}\}$ and $|I| \le 1$ we have*

$$\partial_t Z^I \mathring{y}^a = \frac{1}{2} Z^I(\widehat{h}_1^{v^* a}) + \frac{r^*}{2} Z^I(\widehat{\nabla}_c \widehat{h}_1^{ac}) + O(\frac{\varepsilon}{r^{2+\sigma}}). \tag{6.15}$$

Proof. We analyze (6.5) with $Z = \partial_t$ to find that

$$\partial_{\widetilde{L}} \partial_t \mathring{y}^a = 2\partial_{v^*}\partial_t \mathring{u} \partial_t \mathring{y}^a - \partial_t(\widehat{h}_1^{u^* a}) - \frac{1}{r^2}\widehat{q}^{ab}\partial_{\widehat{y}^b}\partial_t \mathring{u} + O(\frac{\varepsilon}{r^{3+\sigma}}).$$

Using (2.17) and (3.8) we obtain

$$\partial_{\widetilde{L}} \partial_t \mathring{y}^a - 2\partial_{v^*}\partial_t \mathring{u}\partial_t \mathring{y}^a = \frac{1}{2}\partial_{L^*}(\widehat{h}_1^{v^* a}) + \frac{2\widehat{h}_1^{v^* a}}{r} + \widehat{\nabla}_c \widehat{h}_1^{ac} + O(\frac{\varepsilon}{r^{3+\sigma}}).$$

Along an integral curve $(v^*(s), u^*(s), \widehat{y}^a(s))$ of the vector field $\widetilde{L}$, we have the following equation with $H = \int_s^\infty 2\partial_{v^*}\partial_t \mathring{u}\, d\eta = O(\frac{1}{r^\sigma})$

$$\frac{d}{ds}\Big(e^H \partial_t \mathring{y}^a\Big) = e^H\Big(\frac{1}{2}\partial_{L^*}(\widehat{h}_1^{v^* a}) + \frac{2\widehat{h}_1^{v^* a}}{r} + \widehat{\nabla}_c\widehat{h}_1^{ac}\Big) O(\frac{\varepsilon}{r^{3+\sigma}}) = \frac{1}{2}\partial_{L^*}(\widehat{h}_1^{v^* a}) + \frac{2\widehat{h}_1^{v^* a}}{r} + \widehat{\nabla}_c\widehat{h}_1^{ac} + O(\frac{\varepsilon}{r^{3+\sigma}}).$$

Using the asymptotics results in Remark 2.6 and integrating backward along the integral curve we conclude

$$\partial_t \mathring{y}^a = \frac{1}{2}\widehat{h}_1^{v^* a} + \frac{r^*}{2}\widehat{\nabla}_c \widehat{h}_1^{ac} + O(\frac{\varepsilon}{r^{2+\sigma}}). \tag{6.16}$$

Once we have (6.16), Proposition 3.2, Remark 3.3 and Proposition 6.5 at our disposal, we are able to express $\partial_t Z \mathring{y}^a$ with $Z \in \{\partial_t, \Omega_{ij}\}$ more precisely. In fact we have the following equations

$$\begin{aligned}
\partial_{\widetilde{L}}\partial_t^2 \mathring{y}^a &= -h_{1\partial_t}(\partial u, \partial\partial_t \mathring{y}^a) - g(\partial\partial_t \mathring{u}, \partial\partial_t \mathring{y}^a) + \partial_t\big(\partial_{\widetilde{L}}\partial_t \mathring{y}^a\big) \\
&= 4\partial_{v^*}\partial_t \mathring{u}\partial_t^2 \mathring{y}^a + \frac{1}{2}\partial_{L^*}\partial_t(\widehat{h}_1^{v^* a}) + \frac{2\partial_t(\widehat{h}_1^{v^* a})}{r} + \partial_t(\widehat{\nabla}_c \widehat{h}_1^{ac}) + O(\frac{\varepsilon}{r^{3+\sigma}}),
\end{aligned}$$

$$\partial_{\widetilde{L}}\partial_t \Omega \mathring{y}^a = -h_{1\Omega}(\partial u, \partial\partial_t \mathring{y}^a) - g(\partial\Omega\tilde{u}, \partial\partial_t \mathring{y}^a) + \Omega\big(\partial_{\widetilde{L}}\partial_t \mathring{y}^a\big) + \frac{1}{2}\partial_{L^*}\Omega(\widehat{h}_1^{v^* a}) + \frac{2\Omega(\widehat{h}_1^{v^* a})}{r} + \Omega(\widehat{\nabla}_c \widehat{h}_1^{ac}) + O(\frac{\varepsilon}{r^{3+\sigma}}).$$

where we used (3.8), (2.16) and (2.17). Then we repeat the proof of (6.16). □

6.2. **Construction of areal coordinate.** We now construct the areal coordinate such that $\det[\overline{g}_{ab}] = \overline{r}^4 \mathfrak{q}$, where $\mathfrak{q}(\overline{y}^a)$ is the determinant of the unit sphere metric $\overline{q}_{ab}$ associated with the angular coordinates $\overline{y}^a$. Since $\overline{g}$ takes the Bondi-Sachs form, we have $\overline{g}_{ac}\overline{g}^{cb} = \delta_a^b$ and thus $\det[\overline{g}_{ab}] = 1/\det[\overline{g}^{ab}]$. Then we define

$$\overline{r} = \big(\det[\overline{g}^{ab}]\det[\overline{q}_{ab}]\big)^{-1/4} = \big(\det[\overline{q}_{ac}\overline{g}^{cb}]\big)^{-1/4}. \tag{6.17}$$

By Proposition 6.5 and 6.6 we have $\partial_{\widehat{y}^p}\overline{y}^a = \big(O_1(r^{-2-\sigma}), (\widehat{h}_1^{v^* a} + r^*\widehat{\nabla}_c \widehat{h}_1^{ca} + O_1(r^{-2-\sigma}))/2, \delta_c^a + \partial_{\widehat{y}^c}\mathring{y}^a\big)$ and

$$\begin{aligned}
\overline{g}^{ab} &= \widehat{g}^{pq}\frac{\partial \overline{y}^a}{\partial \widehat{y}^p}\frac{\partial \overline{y}^b}{\partial \widehat{y}^q} = \frac{1}{r^2}\Big((1-\frac{M}{r})\widehat{q}^{ab} + r^2\widehat{h}_1^{ab} + \widehat{q}^{ac}\frac{\partial \mathring{y}^b}{\partial \widehat{y}^c} + \widehat{q}^{bc}\frac{\partial \mathring{y}^a}{\partial \widehat{y}^c} + O_1(\frac{\varepsilon}{r^{2+\sigma}})\Big), \\
\overline{q}_{ab} &= \widehat{q}_{ab} + \mathring{y}^c\partial_{\widehat{y}^c}(\widehat{q}_{ab}) + O_2(\frac{\varepsilon}{r^{2+2\sigma}}).
\end{aligned}$$

Then we find

$$\begin{aligned}
\overline{q}_{ac}\overline{g}^{cb} &= \frac{1}{r^2}\Big((1-\frac{M}{r})I_2 + r^2\widehat{q}_{ac}\widehat{h}_1^{cb} + \widehat{q}_{ac}\widehat{q}^{cd}\frac{\partial \mathring{y}^b}{\partial \widehat{y}^d} + \widehat{q}_{ac}\widehat{q}^{bd}\frac{\partial \mathring{y}^c}{\partial \widehat{y}^d} + \widehat{q}^{cb}\mathring{y}^d\partial_{\widehat{y}^d}(\widehat{q}_{ac}) + O_1(\frac{\varepsilon}{r^{2+\sigma}})\Big) \\
&= \frac{1-M/r}{r^2}\Big(I_2 + r^2\widehat{q}_{ac}\widehat{h}_1^{cb} + \widehat{q}_{ac}\widehat{q}^{cd}\frac{\partial \mathring{y}^b}{\partial \widehat{y}^d} + \widehat{q}_{ac}\widehat{q}^{bd}\frac{\partial \mathring{y}^c}{\partial \widehat{y}^d} + \widehat{q}^{cb}\mathring{y}^d\partial_{\widehat{y}^d}(\widehat{q}_{ac}) + O_1(\frac{\varepsilon}{r^{2+\sigma}})\Big).
\end{aligned}$$

Therefore

$$\det[\overline{q}_{ac}\overline{g}^{cb}] = \frac{(1-M/r)^2}{r^4}\Big(1 + r^2\widehat{q}_{ab}\widehat{h}_1^{ab} + 2\widehat{q}_{ac}\widehat{q}^{cd}\frac{\partial \mathring{y}^a}{\partial \widehat{y}^d} + \widehat{q}^{ab}\mathring{y}^d\partial_{\widehat{y}^d}(\widehat{q}_{ab}) - \frac{1}{2}r^2\widehat{q}_{ac}\widehat{h}_1^{cb}r^2\widehat{q}_{bd}\widehat{h}_1^{da} + O_1(\frac{\varepsilon}{r^{2+\sigma}})\Big),$$

and

$$\overline{r} = r + \frac{M}{2} + \frac{3M^2}{8r} + r\Big(-\frac{1}{4}r^2\widehat{q}_{ab}\widehat{h}_1^{ab} - \frac{1}{2}\widehat{q}_{ac}\widehat{q}^{cd}\frac{\partial \mathring{y}^a}{\partial \widehat{y}^d} - \frac{1}{4}\widehat{q}^{ab}\mathring{y}^d\partial_{\widehat{y}^d}(\widehat{q}_{ab})\Big) + r\Big(\frac{1}{8}r^2\widehat{q}_{ac}\widehat{h}_1^{cb}r^2\widehat{q}_{bd}\widehat{h}_1^{da} + O_1(\frac{\varepsilon}{r^{2+\sigma}})\Big). \tag{6.18}$$

From the expression of $\overline{r}$ (6.18), it is clear that

$$\partial_{L^*}\overline{r} = 1 - \frac{M}{r} + O(\frac{\varepsilon}{r^{1+\sigma}}) \qquad \text{and} \qquad \partial_{\widehat{y}^a}\overline{r} = O(\frac{\varepsilon}{r^{\sigma}}). \tag{6.19}$$

As for $\partial_{\underline{L}^*}\overline{r}$, we need more delicate analysis. First given that $\widehat{q}^{ab}\partial_{\widehat{y}^d}(\widehat{q}_{ab}) = 2\widehat{\Gamma}^c_{dc}$ and using (6.15) we get

$$\begin{aligned} -\frac{1}{2}\widehat{q}_{ac}\widehat{q}^{cd}\partial_{\underline{L}^*}\partial_{\widehat{y}^d}\mathring{y}^a - \frac{1}{4}\widehat{q}^{ab}\partial_{\underline{L}^*}(\mathring{y}^d)\partial_{\widehat{y}^d}(\widehat{q}_{ab}) &= -\frac{1}{2}(\partial_{\underline{L}^*}\partial_{\widehat{y}^d}\mathring{y}^d + \widehat{\Gamma}^c_{dc}\partial_{\underline{L}^*}\mathring{y}^d) \\ = -\frac{1}{2}\partial_{\underline{L}^*}(\widehat{\nabla}_d\mathring{y}^d) &= -\frac{1}{2}\Big(\widehat{\nabla}_a\widehat{h}_1^{v^*a} + r^*\widehat{\nabla}_a\widehat{\nabla}_b\widehat{h}_1^{ab}\Big) + O(\frac{\varepsilon}{r^{2+\sigma}}). \end{aligned}$$

In order to handle the term $\partial_{\underline{L}^*}(r^2\widehat{q}_{ab}\widehat{h}_1^{ab})$, we need the following lemma

Lemma 6.7. *We have*

$$\partial_{\underline{L}^*}(r^2\widehat{q}_{ab}\widehat{h}_1^{ab}) = -\partial_{\underline{L}^*}(\not{t}rh^1) + \partial_{\underline{L}^*}(r^2\widehat{q}_{ac}\widehat{h}_1^{cb}r^2\widehat{q}_{bd}\widehat{h}_1^{da}) + O(\frac{\varepsilon}{r^{2+\sigma}}). \tag{6.20}$$

Proof. As we can see in the proof of Lemma 2.12, we have

$$h_1^{\alpha\beta} = -m^{\alpha\mu}h^1_{\mu\nu}m^{\nu\beta} + m^{\alpha\alpha'}(\frac{M}{r}\delta_{\alpha'\mu} + h^1_{\alpha'\mu})m^{\mu\nu}(\frac{M}{r}\delta_{\nu\beta'} + h^1_{\nu\beta'})m^{\beta'\beta} + O_1(\frac{\varepsilon}{r^{2+\sigma}}).$$

Then we repeat the calculation in the proof of Lemma 2.12 and obtain

$$\begin{aligned} \mathrm{tr}h^1 = m^{\alpha\beta}h^1_{\alpha\beta} &= -(1+\frac{2M}{r})m_{\alpha\beta}h_1^{\alpha\beta} + \frac{4M}{r^2} + \frac{4M}{r}h^1_{00} + m_{\alpha\beta}m_{\mu\nu}h_1^{\alpha\mu}h_1^{\beta\nu} + O(\frac{\varepsilon}{r^{2+\sigma}}) \\ &= -\big(1+\frac{2M}{r}\big)\big(-h_{1L\underline{L}} + r^2\widehat{q}_{ab}\widehat{h}_1^{ab}\big) + \frac{4M}{r^2} + \frac{M}{r}h^1_{\underline{LL}} + \frac{2M}{r}h^1_{L\underline{L}} + r^2\widehat{q}_{ac}\widehat{h}_1^{cb}r^2\widehat{q}_{bd}\widehat{h}_1^{da} + \frac{1}{2}(h_{1L^*\underline{L}^*})^2 + O_1(\frac{\varepsilon}{r^{2+\sigma}}), \\ h^1_{L\underline{L}} &= -h_{1L\underline{L}} - \frac{1}{2}(h^1_{L\underline{L}})^2 - \frac{2M}{r^2} - \frac{M}{r}h^1_{\underline{LL}} + O_1(\frac{\varepsilon}{r^{2+\sigma}}). \end{aligned}$$

where we used the notations $h^1_{UV} = h^1_{\alpha\beta}U^\alpha V\beta$ and $h_{1UV} = h_1^{\alpha\beta}U_\alpha V_\beta$. In view of the facts that $h^1_{UV} = -h_{1UV} + O_1(r^{-1-\sigma})$ if $U_\alpha = m_{\alpha\beta}U^\beta$ and $V_\alpha = m_{\alpha\beta}V^\beta$ and $h_{1U^*V^*} = h_{1UV} + O_1(r^{-1-\sigma})$, we conclude that

$$\not{t}rh^1 = \mathrm{tr}h^1 + h^1_{L\underline{L}} = -r^2\hat{q}_{ab}\hat{h}_1^{ab} + \frac{2M}{r^2} + r^2\widehat{q}_{ac}\widehat{h}_1^{cb}r^2\widehat{q}_{bd}\widehat{h}_1^{da} + O_1(\frac{\varepsilon}{r^{2+\sigma}}).$$

Applying $\underline{L}^*$ proves the lemma. □

Therefore by (2.18) we conclude that

$$\partial_{\underline{L}^*}\overline{r} = -1 + \frac{M}{r} + \frac{r\partial_{L^*}(\widehat{h}_1^{v^*v^*})}{4} + \frac{\widehat{h}_1^{v^*v^*}}{2} - \frac{\widehat{h}_1^{v^*u^*}}{2} - \frac{r^2\widehat{\nabla}_a\widehat{\nabla}_b\widehat{h}_1^{ab}}{2} + O(\frac{\varepsilon}{r^{1+\sigma}}). \tag{6.21}$$

6.3. **The Jacobian.** We now give the Jacobian of the mapping from the coordinates $\widehat{y}^p$ to the Bondi-Sachs coordinates $\overline{y}^p$. According to Propositions 3.2, 3.7, 6.5, 6.6, and identities (6.19), (6.21)

$$\begin{aligned} \partial_{\widehat{y}^p}u &= \Big(O(\frac{\varepsilon}{r^{1+\sigma}}),\ 1 + \frac{\widehat{h}_1^{v^*u^*}}{2} + O(\frac{\varepsilon}{r^{1+\sigma}}),\ O(\frac{\varepsilon}{r^{\sigma}}),\ O(\frac{\varepsilon}{r^{\sigma}})\Big), \\ \partial_{\widehat{y}^p}\overline{r} &= \Big(\frac{1}{2} - \frac{M}{2r} + O(\frac{\varepsilon}{r^{1+\sigma}}),\ \frac{1}{2}\partial_{\underline{L}^*}\overline{r},\ O(\frac{\varepsilon}{r^{\sigma}}),\ O(\frac{\varepsilon}{r^{\sigma}})\Big), \\ \partial_{\widehat{y}^p}\overline{y}^3 &= \Big(O(\frac{\varepsilon}{r^{2+\sigma}}),\ \frac{1}{2}\widehat{h}_1^{v^*2} + \frac{r^*}{2}\widehat{\nabla}_c\widehat{h}_1^{c2} + O(\frac{\varepsilon}{r^{2+\sigma}}),\ 1 + O(\frac{\varepsilon}{r^{1+\sigma}}),\ O(\frac{\varepsilon}{r^{1+\sigma}})\Big), \\ \partial_{\widehat{y}^p}\overline{y}^4 &= \Big(O(\frac{\varepsilon}{r^{2+\sigma}}),\ \frac{1}{2}\widehat{h}_1^{v^*3} + \frac{r^*}{2}\widehat{\nabla}_c\widehat{h}_1^{c3} + O(\frac{\varepsilon}{r^{2+\sigma}}),\ O(\frac{\varepsilon}{r^{1+\sigma}}),\ 1 + O(\frac{\varepsilon}{r^{1+\sigma}})\Big), \end{aligned} \tag{6.22}$$

where

$$\partial_{\underline{L}^*}\overline{r} = -1 + \frac{M}{r} + \frac{r\partial_{L^*}(\widehat{h}_1^{v^*v^*})}{4} + \frac{\widehat{h}_1^{v^*v^*}}{2} - \frac{\widehat{h}_1^{v^*u^*}}{2} - \frac{r^2\widehat{\nabla}_a\widehat{\nabla}_b\widehat{h}_1^{ab}}{2} + O(\frac{\varepsilon}{r^{1+\sigma}}).$$

Then we have

$$\begin{aligned} \overline{g}^{pq}\partial_{\overline{y}^p}\partial_{\overline{y}^q} = &-2\Big(1+O(\frac{\varepsilon}{\overline{r}^{1+\sigma}})\Big)\partial_u\partial_{\overline{r}} + \Big(1-\frac{M}{r} - \frac{r\partial_{L^*}(\widehat{h}_1^{v^*v^*})}{4} - \frac{\widehat{h}_1^{v^*v^*}}{4} + \frac{r^2\widehat{\nabla}_c\widehat{\nabla}_d\widehat{h}_1^{cd}}{2} + O(\frac{\varepsilon}{\overline{r}^{1+\sigma}})\Big)\partial^2_{\overline{r}} \\ &+ 2\Big(-\frac{1}{2}r\widehat{\nabla}_c\widehat{h}_1^{ca} + O(\frac{\varepsilon}{\overline{r}^{2+\sigma}})\Big)\partial_{\overline{r}}\partial_{\overline{y}^a} + \frac{1}{r^2}\Big((1-\frac{M}{r})\widehat{q}^{ab} + r^2\widehat{h}_1^{ab} + O(\frac{\varepsilon}{\overline{r}^{1+\sigma}})\Big)\partial_{\overline{y}^a}\partial_{\overline{y}^b}. \end{aligned} \tag{6.23}$$

7. The Bondi Mass

7.1. **The Bondi-Sachs metric.** Now we can establish the following expression for the Bondi-Sachs metric

Proposition 7.1. *We have*

$$\begin{aligned}\overline{g}_{pq}d\overline{y}^p d\overline{y}^q = -\Big(1-\frac{M}{r}-\frac{r\partial_{L^*}(\widehat{h}_1^{v^*v^*})}{4}-\frac{\widehat{h}_1^{v^*v^*}}{4}+\frac{r^2\widehat{\nabla}_c\widehat{\nabla}_d\widehat{h}_1^{cd}}{2}+O(\frac{\varepsilon}{\overline{r}^{1+\sigma}})\Big)du^2-2\Big(1+O(\frac{\varepsilon}{\overline{r}^{1+\sigma}})\Big)dud\overline{r} \\ +r^2\Big((1+\frac{M}{r})\widehat{q}_{ab}-r^2\widehat{q}_{ac}\widehat{h}_1^{cd}\widehat{q}_{db}+O(\frac{\varepsilon}{\overline{r}^{1+\sigma}})\Big)\Big(d\overline{y}^a-U^a du\Big)\Big(d\overline{y}^b-U^b du\Big),\end{aligned} \tag{7.1}$$

where $U^a = -\frac{r}{2}\widehat{\nabla}_c\widehat{h}_1^{ca}+O(\frac{\varepsilon}{\overline{r}^{2+\sigma}})$.

Proof. Taking matrix inversion to (6.23) yields (7.1). □

Remark 7.2. The existence of a compactification with smooth future null infinity $\mathscr{I}^+$, i.e., the conformally rescaled metric can extend smoothly across $\mathscr{I}^+$, is a delicate issue as it is sensitive to the choice of conformal factor and the smooth structure near $\mathscr{I}^+$ [39]. The Bondi-Sachs coordinates constructed above allows us to obtain $C^{1,\delta}$ regularity of $\mathscr{I}^+$ where $0<\delta<1$, which is consistent with the result in [18]. More specifically, we let coordinates $(u,\overline{\rho}=\overline{r}^{-1},\overline{y}^3,\overline{y}^4)$ be a smooth coordinate system near $\mathscr{I}^+$ and choose $\overline{\rho}^2$ as the conformal factor. Then we find that

$$\overline{\rho}^2\overline{g}(\partial_u,\partial_u)\in C^{3,\delta},\quad \overline{\rho}^2\overline{g}(\partial_u,\partial_{\overline{y}^a})\in C^{2,\delta}\quad\text{and}\quad \overline{\rho}^2\overline{g}(\partial_u,\partial_{\overline{\rho}}),\ \overline{\rho}^2\overline{g}(\partial_{\overline{y}^a},\partial_{\overline{y}^b})\in C^{1,\delta}$$

which implies $\overline{\rho}^2\overline{g}\in C^{1,\delta}$ and thus $\mathscr{I}^+$ is of the class $C^{1,\delta}$. We also note that the work by Christodoulou [12] strongly suggests that the conformally compactification is generically at most of class $C^{1,\alpha}$ with $\alpha<1$.

Then by Proposition 7.1 we see that in the Bondi-Sachs coordinates $\overline{y}^\alpha=(u,\overline{r},\overline{y}^3,\overline{y}^4)$, the metric takes the following Bondi-Sachs form

$$\overline{g}_{pq}d\overline{y}^p d\overline{y}^q = -V\overline{r}^{-1}e^{2\beta}du^2-2e^{2\beta}dud\overline{r}+\overline{r}^2h_{ab}(d\overline{y}^a-U^a du)(d\overline{y}^b-U^b du),$$

where

$$V=\overline{r}-M-\frac{\overline{r}^2\partial_{L^*}(\widehat{h}_1^{v^*v^*})}{4}-\frac{\overline{r}\widehat{h}_1^{v^*v^*}}{4}+\frac{\overline{r}^3\widehat{\nabla}_c\widehat{\nabla}_d\widehat{h}_1^{cd}}{2}+O(\frac{1}{\overline{r}^{\sigma}}),\quad\text{and}\quad h_{ab}=\overline{q}_{ab}-r^2\overline{q}_{ac}\widehat{h}_1^{cd}\overline{q}_{db}+O(\frac{1}{\overline{r}^{1+\sigma}}). \tag{7.2}$$

Here we used the fact that $r=\overline{r}-M/2+O(\overline{r}^{-\sigma})$ which is implied by the definition of $\overline{r}$ (6.18).

The mass aspect M_A and news tensor N_{ab} are defined as follows

$$M_A(u,\overline{y}^a):=-\lim_{\overline{r}\to\infty}\big(V(u,\overline{r},\overline{y}^a)-\overline{r}\big),$$

$$N_{ab}(u,\overline{y}^c):=\frac{1}{2}\partial_u C_{ab}(u,\overline{y}^c)\quad\text{where}\quad C_{ab}(u,\overline{y}^c):=\lim_{\overline{r}\to\infty}\overline{r}(h_{ab}(u,\overline{r},\overline{y}^c)-\overline{q}_{ab}(\overline{y}^c)).$$

The Bondi mass M_B and radiated energy E_B are defined by

$$M_B(u)=\frac{1}{4\pi}\int_{\mathbb{S}^2}M_A(u,\overline{y}^a)d\overline{S}(\overline{y}^a)\quad\text{and}\quad E_B(u)=\frac{1}{4\pi}\int_{\mathbb{S}^2}|N|^2\,d\overline{S}(\overline{y}^a),$$

where $d\overline{S}(\overline{y}^a)=\sqrt{\mathfrak{q}(\overline{y}^a)}d\overline{y}^3d\overline{y}^4$ is the volume form of the unit sphere metric $\overline{q}_{ab}$ and $|N|^2=\overline{q}^{ac}\overline{q}^{bd}N_{ab}N_{cd}$.

7.2. **Bondi mass loss law.** We will prove the existence of $M_B(u)$ and $E_B(u)$ and the Bondi mass loss law.

Theorem 7.3. *Let M_A, N_{ab}, M_B, E_B be defined as above, then the Bondi mass is given by*

$$M_B(u)=\frac{1}{4\pi}\int_{\mathbb{S}^2}M_A(u,\overline{y}^a)d\overline{q}=M-\frac{1}{16\pi}\int_{\mathbb{S}^2}\int_{-u}^{\infty}n(\eta,\overline{y}^a)\,d\eta d\overline{S}(\overline{y}^a), \tag{7.3}$$

and the radiated energy is equal to

$$E_B(u)=\frac{1}{8\pi}\int_{\mathbb{S}^2}n(-u,\overline{y}^a)\,d\overline{S}(\overline{y}^a). \tag{7.4}$$

where $n(q^*,\widehat{y}^a)=\widehat{q}_{ac}\widehat{q}_{bd}V^{ab}\widehat{V}^{cd}/2$ with $\widehat{V}^{ab}=\partial_{q^*}\widehat{H}_{1\infty}^{ab}$ defined in (2.13)*.They satisfy the Bondi mass loss law*

$$\frac{d}{du}M_B(u)=-E(u). \tag{7.5}$$

Moreover, $M_B(u)\to M$ as $u\to-\infty$ where M is the ADM mass and $M_B(u)\to 0$ as $u\to\infty$.

Proof. By Remark 2.6 we write with $q^* = -u^* = r^* - t$

$$\widehat{h}_1^{ab}(v^*_, -q^*, \widehat{y}^a) = \frac{\widehat{H}_{1\infty}^{ab}(q^*, \widehat{y}^a)}{r^{*3}} + O(\frac{1}{r^{*3+\sigma}}),$$

$$\widehat{h}_1^{v^*v^*}(v^*_, -q^*, \widehat{y}^a) = -\frac{2}{r^*}\int_{q^*}^{\infty} \ln\Big(\frac{v^* + \eta}{u^* + \eta}\Big) n(\eta, \widehat{y}^a)\, d\eta + \frac{\widehat{H}_{1\infty}^{v^*v^*}(q^*, \widehat{y}^a)}{r^*} + \frac{2M}{r^*}\Big(1 - \chi^e(q^*)\Big) + O(\frac{1}{r^{*1+\sigma}}),$$

where $\chi^e(s) = 1$ when $s \geq 2$, $\chi^e(s) = 0$ when $s \leq 1$. Plugging these expressions for $\widehat{h}_1^{v^*v^*}$ and $\widehat{h}_1^{ab}$ into (7.2) gives

$$V = \overline{r} - M + \frac{1}{2}\int_{q^*}^{\infty} n(\eta, \widehat{y}^a)\, d\eta + \frac{\widehat{\nabla\!\!\!/}_c \widehat{\nabla\!\!\!/}_d \widehat{H}^{cd}(q^*, \widehat{y}^a)}{2} + O(\frac{1}{\overline{r}^{\sigma}}), \qquad \text{and} \qquad h_{ab} = \overline{q}_{ab} - \frac{\overline{q}_{ac}\widehat{H}^{cd}\overline{q}_{db}}{\overline{r}} + O(\frac{1}{\overline{r}^{1+\sigma}}).$$

Therefore we conclude

$$M_A(u, \overline{y}^a) = M - \frac{1}{2}\lim_{\overline{r}\to\infty}\Big(\int_{\mathring{u}-u}^{\infty} n(\eta, \overline{y}^a - \mathring{y}^a)\, d\eta + \widehat{\nabla\!\!\!/}_c \widehat{\nabla\!\!\!/}_d \widehat{H}^{cd}(\mathring{u} - u, \overline{y}^a - \mathring{y}^a)\Big)$$

$$= M - \frac{1}{2}\lim_{\overline{r}\to\infty}\Big(\int_{\mathring{u}-u}^{\infty} n(\eta, \overline{y}^a - \mathring{y}^a)\, d\eta + \overline{\nabla\!\!\!/}_c \overline{\nabla\!\!\!/}_d \widehat{H}^{cd}(\mathring{u} - u, \overline{y}^a - \mathring{y}^a)\Big) = M - \frac{1}{2}\int_{-u}^{\infty} n(\eta, \overline{y}^a)\, d\eta - \frac{\overline{\nabla\!\!\!/}_c \overline{\nabla\!\!\!/}_d \widehat{H}^{cd}(-u, \overline{y}^a)}{2}.$$

where we used the fact that $\overline{\nabla\!\!\!/}_c \overline{\nabla\!\!\!/}_d H^{cd} = \widehat{\nabla\!\!\!/}_c \widehat{\nabla\!\!\!/}_d \widehat{H}^{cd} + O(\overline{r}^{-\sigma})$. Then the Bondi mass $M_B(u)$ is given by

$$M_B(u) = \frac{1}{4\pi}\int_{\mathbb{S}^2} M_A(u, \overline{y}^a) d\overline{q} = M - \frac{1}{16\pi}\int_{\mathbb{S}^2}\int_{-u}^{\infty} n(\eta, \overline{y}^a)\, d\eta d\overline{S}(\overline{y}^a), \tag{7.6}$$

where we used the fact that the integral of the spherical divergence $\overline{\nabla\!\!\!/}_c \overline{\nabla\!\!\!/}_d \widehat{H}^{cd}$ over the sphere is 0. Since

$$C_{ab}(u, \overline{y}^a) = -\lim_{\overline{r}\to\infty} \overline{q}_{ac}\widehat{H}^{cd}(\mathring{u} - u, \overline{y}^a - \mathring{y}^a)\overline{q}_{db} = -\overline{q}_{ac}\widehat{H}^{cd}(-u, \overline{y}^a)\overline{q}_{db},$$

with $\widehat{V}^{ab}(q^*, \widehat{y}^a) = \partial_{q^*}\widehat{H}^{ab}(q^*, \widehat{y}^a)$ defined in (2.13) we obtain

$$N_{ab}(u, \overline{y}^a) = \frac{1}{2}\overline{q}_{ac}\overline{q}_{db}\widehat{V}^{cd}(-u, \overline{y}^a).$$

Therefore with (2.13) the radiated energy flux is equal to

$$\begin{aligned} E_B(u) &= \frac{1}{4\pi}\int_{\mathbb{S}^2} |N|^2\, d\overline{S}(\overline{y}^a) = \frac{1}{4\pi}\int_{\mathbb{S}^2} \overline{q}^{ac}\overline{q}^{bd} N_{cd} N_{ab}\, d\overline{S}(\overline{y}^a) \\ &= \frac{1}{16\pi}\int_{\mathbb{S}^2} \overline{q}_{ac}\overline{q}_{bd}\widehat{V}^{ab}\widehat{V}^{cd}(-u, \overline{y}^a)\, d\overline{S}(\overline{y}^a) = \frac{1}{8\pi}\int_{\mathbb{S}^2} n(-u, \overline{y}^a)\, d\overline{S}(\overline{y}^a). \end{aligned} \tag{7.7}$$

By (7.6) and (7.7) we establish the mass loss formula

$$\frac{d}{du}M_B(u) = -E_B(u). \tag{7.8}$$

Moreover, since $n(\eta, \overline{y}^a)$ is integrable in η, we see that $M_B(u) \to M$ as $u \to -\infty$. By Proposition 2.7 we know that $\int_{-\infty}^{\infty} E_B(u)\, du = M$ and thus $M_B(u) \to 0$ as $u \to \infty$. □

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