THE WEAK NULL CONDITION ON KERR BACKGROUNDS

HANS LINDBLAD AND MIHAI TOHANEANU

Abstract. We study a system of semilinear wave equations on Kerr backgrounds that satisfies the weak null condition. Under the assumption of small initial data, we prove global existence and pointwise decay estimates.

Contents

1. Introduction

The semilinear system of wave equations in $\mathbb{R}^{1+3}$

$$\Box\phi = Q[\partial\phi, \partial\phi], \qquad \phi|_{t=0} = \phi_0, \qquad \partial_t\phi|_{t=0} = \phi_1,$$

where Q is a quadratic form, for small initial data has been studied extensively. For the scalar equation, it is known that the solution can blows up in finite time for $\Box\phi = (\partial_t\phi)^2$, see [28]. On the other hand, if the nonlinearity satisfies the null condition by Klainerman [30], e.g. $\Box\phi = (\partial_t\phi)^2 - |\partial_x\phi|^2$, it was shown independently in [10] and [31] that the solution exists globally. This result was extended to quasilinear systems with multiple speeds, as well as the case of exterior domains; see, for instance, [54], [55], [56], [24], [41], [33], [1], [38, 39], [62], [18]. There have also been many works for small data in the variable coefficient case. Almost global existence for nontrapping metrics was shown in [9], [60]. Global existence for stationary, small perturbations of Minkowski was shown in [66], for nonstationary, compactly supported perturbations in [67], and for large, asymptotically flat perturbations that satisfy the strong local energy decay estimates in [50]. In the context of black holes, global existence was shown in [51] for Kerr space-times with small angular momentum, and in [4] for the Reissner-Nordström backgrounds.

Written in harmonic coordinates, the Einstein Equations take the form

$$\Box_g g_{\mu\nu} = P[\partial_\mu g, \partial_\nu g] + Q_{\mu\nu}[\partial g, \partial g],$$

where $\Box_g$ is the wave operator on the background of the Lorentzian metric g, and P and $Q_{\mu\nu}$ are quadratic forms with coefficients depending on the metric. Unfortunately the nonlinear terms do not satisfy the null condition. Yet Christodoulou-Klainerman[11] were able to prove global existence for Einstein vacuum equations $R_{\mu\nu} \equiv 0$ for small asymptotically flat initial data. Their proof avoids using coordinates since it was believed the metric in harmonic coordinates would blow up for large times. However, later Lindblad-Rodnianski [42] noticed that Einstein's equations in harmonic coordinates satisfy a weak null condition, and subsequently used it to prove stability of Minkowski in harmonic coordinates [43], [44]. Whereas it is still unknown if general equations satisfying the weak null condition have global existence for small initial data, there has been a number of results in that direction, including detailed asymptotics of the solution, see for example [1], [38, 39, 40], [29], [17], [68], [69].

There has recently been a lot of activity in proving asymptotic stability of black holes. As a first step people have proved decay of solutions to wave equations on Schwarzschild and Kerr background [6, 7, 8, 52, 13] and [64, 16, 3]. People have also studied semilinear perturbations [51, 27] satisfying the null condition, but apart from our recent papers [46, 47], little is known about quasilinear perturbations or semilinear perturbations satisfying the weak null condition. There has more recently been progress on the nonlinear stability of Schwarzschild and Kerr [34, 35, 36, 12, 22]. These proofs are very long, using sophisticated geometric constructions. We hope that studying models of Einstein's equations in wave coordinates will simplify the proofs and lead to a better understanding and extensions as it did for the stability of Minkowski space.

Finally we remark that there are several recent works on the cosmological case. Hintz-Vasy proved the stability of Kerr de Sitter with small angular momentum [25], see also [19, 20] for an alternative proof. More recently there have been works on the wave equation on Kerr-deSitter background for large angular momentum assuming there are no growing modes [48, 53].

1.0.1. *The semilinear Einstein model.* An example of a simple semilinear systems satisfying the weak null condition, but not the classical null condition, is the system

$$\Box\phi_1 = (\partial_t\phi_2)^2, \qquad \Box\phi_2 = 0$$

It is trivial to see that this has global solutions, and moreover that ϕ_1 decays slower than $1/t$. A less trivial example is the semilinear system

$$\Box\phi_1 = (\partial_t\phi_2)^2 + Q_1[\partial\phi,\partial\phi], \qquad \Box\phi_2 = Q_2[\partial\phi,\partial\phi]$$

where Q_j are null forms. These systems have the advantage the components ϕ_1 and ϕ_2 decouple to highest order. For Einstein's equations there is the additional difficulty that this decoupling can only be seen in a null frame, and contractions with the frame do not commute with the wave operator as far as the L^2 estimate. Hence a more realistic model is the system is

$$\Box\phi_{\mu\nu} = P[\partial_\mu\phi,\partial_\nu\phi] + Q_{\mu\nu}[\partial\phi,\partial\phi],$$

where P is assumed to have a certain weak null structure. Contracting with a nullframe this resembles the decoupled systems with $\phi_{\underline{LL}}$ in place of ϕ_1, where $\underline{L}^\mu\partial_\mu = \partial_t - \partial_r$, and ϕ_2 replaced by the other components ϕ_{TU} where T is tangential to the outgoing light cones. The only really bad component is $\partial\phi_{\underline{LL}}$ but this one does not show up quadratically in P for Einstein's Equations. It shows up linearly but multiplied with a component $\partial\phi_{LL}$ that has vanishing radiation field due to the wave coordinate condition.

With the goal of understanding Einstein's Equations in (generalized) harmonic coordinates close to Kerr with small angular momentum, we will focus on the following system, which resembles the semilinear part of Einstein's equations:

$$\Box_K\phi_{\mu\nu} = P[\partial_\mu\phi,\partial_\nu\phi] + Q_{\mu\nu}[\partial\phi,\partial\phi], \qquad \tilde{t}\geq 0, \qquad \phi|_{\tilde{t}=0} = \phi_0, \qquad \tilde{T}\phi|_{\tilde{t}=0} = \phi_1. \tag{1.1}$$

Here $\Box_K$ denotes the d'Alembertian with respect to the Kerr metric, and $\tilde{T}$ is a smooth, everywhere timelike vector field that equals ∂_t away from the black hole. The coordinate $\tilde{t}$ is chosen so that the slice $\tilde{t} = const$ are space-like and $\tilde{t} = t$ away from the black hole. For simplicity we will consider compactly supported smooth initial data, but suitably weighted Sobolev spaces of large enough order would suffice. Moreover, $Q_{\mu\nu}$ are null forms and P is a symmetric quadratic form:

$$P[\phi,\psi] = P^{\alpha\beta\gamma\delta}(x/\tilde{t})\,\phi_{\alpha\beta}\psi_{\gamma\delta},$$

with coefficients with a certain weak null structure. We remove the component $\partial\phi_{\underline{LL}}$ by imposing the condition

$$P^{\underline{LL}\alpha\beta}(x/\tilde{t}) = P^{\alpha\beta\underline{LL}}(x/\tilde{t}) = 0.$$

For this system we cannot have different energy estimates for different components because the null structure is only seen in a null frame and contractions with the frame do not commute with the wave operator. Because of this one can not get the decay estimates directly from the L^2 estimates but one has to use the equations again to get improved decay estimates. As a result, the proof is more involved. The method we develop avoids boosts vector fields and combines local energy decay at the origin with estimates in characteristic coordinates at the light cone. It gives an essentially optimal decay of almost $\tilde{t}^{-1}$, which is an improvement over $\tilde{t}^{-1/2}$ which can be obtained more easily from energy estimates. The method in particular works close to Minkowski where it gives the optimal decay without using boosts.

Finally we remark that this system can be combined with the quasilinear system that we previously studied [46], [47] (see also [49] for improved pointwise bounds) to resemble also the quasilinear part of Einstein's equations

$$\Box_{g[\phi]}\phi_{\mu\nu} = P[\partial_\mu\phi,\partial_\nu\phi] + Q_{\mu\nu}[\partial\phi,\partial\phi],$$

where

$$g^{\alpha\beta}[\phi] = K^{\alpha\beta} + H^{\alpha\beta}[\phi], \qquad \text{where} \quad H^{\alpha\beta}[\phi] = H^{\alpha\beta\mu\nu}(x/\tilde{t})\,\phi_{\mu\nu}, \quad \text{and} \quad H^{\underline{LL}\mu\nu}(x/\tilde{t}) = 0.$$

1.0.2. *Statement of the results.* We are now ready to state the our main result. We define $\tilde{r}$ to be some function that equals r near the event horizon, and r_K^* away from it, see Section 2 for more details.

Theorem 1.1. *Fix $R_0 > r_e$, and assume that ϕ_0, ϕ_1 are smooth and compactly supported in $\tilde{r} \le R_0$. Then there exists a global classical solution to (1.1), provided that, for a certain $\epsilon_0 \ll 1$ and large enough N, we have*

$$\mathcal{E}_N(0) = \|(\phi_0, \phi_1)\|_{H^{N+1}\times H^N} \le \epsilon_0.$$

Moreover, for some fixed positive integer m, independent of N, we have for any $\delta > 0$

$$|\phi_{\le N-m}| \lesssim \frac{\mathcal{E}_N(0)}{\langle \tilde{t}-\tilde{r}\rangle^{\delta}\langle \tilde{t}\rangle^{1-\delta}}, \qquad |\partial\phi_{\le N-m}| \lesssim \frac{\tilde{t}^{\delta}\mathcal{E}_N(0)}{r\langle \tilde{t}-\tilde{r}\rangle^{1+\delta}},$$

$$|(\partial\phi_{TU})_{\le N-m}| \lesssim \frac{\mathcal{E}_N(0)}{r\langle \tilde{t}-\tilde{r}\rangle^{1-\delta}}.$$

Note that is an improvement of the decay estimates we previously proved essentially by a factor of $\tilde{t}^{-1/2}$. Note also the structure here, that a derivative decreases the homogeneity, but because the homogeneous vector fields we can use together with the wave operator do not span the tangent space at the origin or at the light cone a derivative only improves by a power of r close to the origin and a power of $\tilde{t}-\tilde{r}$ close to the light cone. Note also that close to the light cone we have a better estimate for the good components which is due to the weak null structure.

1.0.3. *Structure of the proof.* The starting point is the local energy estimate in Section 2. The local energy scales like the energy which is consistent with a decay $\tilde{t}^{-1/2}$ of order $-1/2$ for ϕ and $-3/2$ for the derivatives, and this is also the decay we were able to obtain in our previous paper from a bound of the local energy applied to scaling and rotation vector fields, see Section 3. Assuming these decay estimates one can go back into the equation and get improved decay estimates. In fact from these decay estimates the total decay of the inhomogeneous term would be -3 which would be consistent with a solution of the wave equation with decay of order -1. We prove this using L^∞ estimates for the wave operator from Section 5. However the first improved estimates we obtain have the improved decay in r or $\tilde{t}-\tilde{r}$ and we need improved decay in $\tilde{t}$. For this we have other estimates turn decay in r or $\tilde{t}-\tilde{r}$ into decay in $\tilde{t}$, see Section 4. The whole argument is put together in the last section.

The paper is structured as follows. In Section 2 we introduce the Kerr metric, the vector fields we will use, and the local energy estimates which will play a key role in the proof. Sections 3, 4 and 5, and 6 contain various estimates that will allow us to extract the necessary pointwise bounds for (vector fields applied to) the solution. Finally, Section 7 contains the bootstrap argument.

2. The Kerr metric and local energy estimates

2.1. The Kerr metric.

The Kerr geometry in Boyer-Lindquist coordinates is given by

$$ds^2 = g_{tt}^K dt^2 + g_{t\phi}dtd\phi + g_{rr}^K dr^2 + g_{\phi\phi}^K d\phi^2 + g_{\theta\theta}^K d\theta^2,$$

where $t \in \mathbb{R}$, $r > 0$, (ϕ, θ) are the spherical coordinates on $\mathbb{S}^2$ and

$$g_{tt}^K = -\frac{\Delta - a^2\sin^2\theta}{\rho^2}, \qquad g_{t\phi}^K = -2a\frac{2Mr\sin^2\theta}{\rho^2}, \qquad g_{rr}^K = \frac{\rho^2}{\Delta},$$

$$g_{\phi\phi}^K = \frac{(r^2+a^2)^2 - a^2\Delta\sin^2\theta}{\rho^2}\sin^2\theta, \qquad g_{\theta\theta}^K = \rho^2,$$

with

$$\Delta = r^2 - 2Mr + a^2, \qquad \rho^2 = r^2 + a^2\cos^2\theta.$$

Here M represents the mass of the black hole, and aM its angular momentum.

A straightforward computation gives us the inverse of the metric:

$$g_K^{tt} = -\frac{(r^2+a^2)^2 - a^2\Delta\sin^2\theta}{\rho^2\Delta}, \qquad g_K^{t\phi} = -a\frac{2Mr}{\rho^2\Delta}, \qquad g_K^{rr} = \frac{\Delta}{\rho^2},$$

$$g_K^{\phi\phi} = \frac{\Delta - a^2\sin^2\theta}{\rho^2\Delta\sin^2\theta}, \qquad g_K^{\theta\theta} = \frac{1}{\rho^2}.$$

The case $a = 0$ corresponds to the Schwarzschild space-time. We shall subsequently assume that a is small $0 < a \ll M$, so that the Kerr metric is a small perturbation of the Schwarzschild metric. Note also that the coefficients depend only r and θ but are independent of ϕ and t. We denote the d'Alembertian associated to the Kerr metric by $\Box_K$.

In the above coordinates the Kerr metric has singularities at $r = 0$, on the equator $\theta = \pi/2$, and at the roots of Δ, namely $r_\pm = M \pm \sqrt{M^2 - a^2}$. To remove the singularities at $r = r_\pm$ we introduce functions $r_K^* = r_K^*(r)$, $v_+ = t + r_K^*$ and $\phi_+ = \phi_+(\phi, r)$ so that (see [23])

$$dr_K^* = (r^2 + a^2)\Delta^{-1} dr, \qquad dv_+ = dt + dr_K^*, \qquad d\phi_+ = d\phi + a\Delta^{-1} dr.$$

Note that when $a = 0$ the r_K^* coordinate becomes the Schwarzschild Regge-Wheeler coordinate

$$r^* = r + 2M \log(r - 2M)$$

The Kerr metric can be written in the new coordinates (v_+, r, ϕ_+, θ)

$$\begin{aligned} ds^2 = & -(1 - \frac{2Mr}{\rho^2}) dv_+^2 + 2 dr dv_+ - 4a\rho^{-2} Mr \sin^2\theta dv_+ d\phi_+ - 2a \sin^2\theta dr d\phi_+ + \rho^2 d\theta^2 \\ & + \rho^{-2}[(r^2 + a^2)^2 - \Delta a^2 \sin^2\theta] \sin^2\theta \, d\phi_+^2 \end{aligned}$$

which is smooth and nondegenerate across the event horizon up to but not including $r = 0$. We introduce the function

$$\tilde{t} = v_+ - \mu(r),$$

where μ is a smooth function of r. In the $(\tilde{t}, r, \phi_+, \theta)$ coordinates the metric has the form

$$\begin{aligned} ds^2 = & (1 - \frac{2Mr}{\rho^2}) d\tilde{t}^2 + 2\left(1 - (1 - \frac{2Mr}{\rho^2})\mu'(r)\right) d\tilde{t} dr \\ & - 4a\rho^{-2} Mr \sin^2\theta d\tilde{t} d\phi_+ + \left(2\mu'(r) - (1 - \frac{2Mr}{\rho^2})(\mu'(r))^2\right) dr^2 \\ & - 2a(1 + 2\rho^{-2} Mr\mu'(r)) \sin^2\theta dr d\phi_+ + \rho^2 d\theta^2 \\ & + \rho^{-2}[(r^2 + a^2)^2 - \Delta a^2 \sin^2\theta] \sin^2\theta d\phi_+^2. \end{aligned}$$

On the function μ we impose the following two conditions:
(i) $\mu(r) \geq r_K^*$ for $r > 2M$, with equality for $r > 5M/2$.
(ii) The surfaces $\tilde{t} = const$ are space-like, i.e.

$$\mu'(r) > 0, \qquad 2 - (1 - \frac{2Mr}{\rho^2})\mu'(r) > 0.$$

As long as a is small, we can use the same function μ as in the case of the Schwarzschild space-time in [52].

We also introduce

$$\tilde{\phi} = \zeta(r)\phi_+ + (1 - \zeta(r))\phi,$$

where ζ is a cutoff function supported near the event horizon.

We fix r_e satisfying $r_- < r_e < r_+$. The choice of r_e is unimportant, and for convenience we may simply use $r_e = M$ for all Kerr metrics with $a/M \ll 1$. Let $\mathcal{M} = \{\tilde{t} \geq 0, \ r \geq r_e\}$, $\Sigma(T) = \mathcal{M} \cap \{\tilde{t} = T\}$, and $d\Sigma_K$ be the induced volume element on $\Sigma(T)$.

Let $\tilde{r}$ denote a smooth strictly increasing function (of r) that equals r for $r \leq R$ and r_K^* for $r \geq 2R$ for some large R. We will use the coordinates $(\tilde{t}, x^i)$, where $x^i = \tilde{r}\omega$. Note that, since $r \approx \tilde{r}$, we can use r^k and $\tilde{r}^k$ interchangeably when defining our spaces of functions in what follows.

2.2. **Vector fields and spaces of functions.**

Our favorite sets of vector fields will be

$$\partial = \{\partial_{\tilde{t}}, \partial_i\}, \qquad \Omega = \{x^i \partial_j - x^j \partial_i\}, \qquad S = \tilde{t}\partial_{\tilde{t}} + \tilde{r}\partial_{\tilde{r}},$$

namely the generators of translations, rotations and scaling. We set $Z = \{\partial, \Omega, S\}$.

We also denote by $\partial\!\!\!/$ the angular derivatives,

$$\partial_j = \frac{x^i}{\tilde{r}}\partial_{\tilde{r}} + \partial\!\!\!/_i$$

and let

$$\overline{\partial} := (\partial_v, \partial\!\!\!/), \quad \partial_v = \partial_{\tilde{t}} + \partial_{\tilde{r}}$$

denote the tangential derivatives. We also let $\underline{L} = \partial_{\tilde{t}} - \partial_{\tilde{r}}$.

For a triplet $\alpha = (i, j, k)$ we define $|\alpha| = i + 3j + 3k$ and

$$u_\alpha = \partial^i \Omega^j S^k u, \quad u_{\leq m} = (u_\Lambda)_{|\Lambda| \leq m}$$

Given a norm $\|\cdot\|_X$, we write

$$\|u_{\le m}\|_X = \sum_{|\Lambda|\le m} \|u_\Lambda\|_X$$

We define the classes $S^Z(r^k)$ of functions in $\mathbb{R}^+ \times \mathbb{R}^3$ by

$$f \in S^Z(r^k) \iff |Z^j f(t,x)| \le c_j \langle r\rangle^k, \quad j \ge 0.$$

Given a family of functions $\mathcal{G}$, we will also use the notation

$$f \in S^Z(r^k)\mathcal{G}$$

to mean that

$$f = \sum h_i g_i, \quad h_i \in S^Z(r^k), \quad g_i \in \mathcal{G}.$$

We will also use the notation U for an element of $S^Z(1)Z$, and T for an element of $S^Z(1)\overline{\partial}$.

An important observation is that, since

$$\partial_v = \frac{\tilde t - \tilde r}{\tilde t}\partial_{\tilde r} + \frac{1}{\tilde t}S, \quad \not\partial\phi \in S^Z(r^{-1})\Omega\phi$$

we have

$$|\overline{\partial} w| \lesssim \frac{\tilde t - \tilde r}{r}|\partial w| + \frac{1}{r}|\Omega w|. \tag{2.2}$$

Moreover, an easy computation gives

$$[\Box_K, \partial]\phi \in S^Z(r^{-2})\partial\partial^{\le 1}\phi, \quad [\Box_K, \Omega]\phi \in S^Z(r^{-2})\partial\partial^{\le 1}\phi,$$
$$[\Box_K, S]\phi \in S^Z(1)\Box_K\phi + S^Z(r^{-2+})\partial\phi + S^Z(r^{-2+})\partial\Omega\phi + S^Z(r^{-2})\partial\partial^{\le 1}\phi,$$

and thus by induction we obtain that

$$[\Box_K, Z^\alpha]\phi = F_1 + F_2, \quad F_1 \in S^Z(1)(\Box_K\phi)_{\le|\alpha|}, \quad F_2 \in S^Z(r^{-2+})\partial\phi_{\le|\alpha|}. \tag{2.3}$$

We now claim that

$$[Z, \overline{\partial}] \in S^Z(1)\overline{\partial} + S^Z(r^{-1})\partial \tag{2.4}$$

Indeed, we compute

$$[\partial_{\tilde t}, \overline{\partial}] = 0$$
$$[\partial_i, \partial_v] = [\not\partial_i, \partial_{\tilde r}] \in S^Z(r^{-1})\not\partial$$
$$[\partial_i, \not\partial] \in S^Z(r^{-1})\partial$$
$$[\Omega, \partial_v] = 0$$
$$[\Omega, \not\partial] \in S^Z(1)\not\partial$$
$$[S, \partial_v] = \partial_v$$
$$[S, \not\partial] \in S^Z(1)\not\partial$$

This proves (2.4).

Given vector fields X and Y, we define

$$\phi_{XY} = X^\alpha Y^\beta \phi_{\alpha\beta}$$

Similarly, we can write the coefficients P with respect to the vector frame $\{\underline{L}, \overline{\partial}\}$ as

$$P^{\alpha\beta\gamma\delta} = P^{\underline{L}\underline{L}\gamma\delta}\underline{L}^\alpha \underline{L}^\beta + \sum P^{TU\gamma\delta}T^\alpha U^\beta$$

$$P^{\alpha\beta\gamma\delta} = P^{\alpha\beta\underline{L}\underline{L}}\underline{L}^\gamma \underline{L}^\delta + \sum P^{\alpha\beta TU}T^\gamma U^\delta$$

The assumptions on the coefficients $P^{\alpha\beta\gamma\delta}$ are the following:

$$P^{\alpha\beta\gamma\delta} \in S^Z(1), \tag{2.5}$$

$$P^{\underline{L}\underline{L}\alpha\beta} = P^{\alpha\beta\underline{L}\underline{L}} = 0. \tag{2.6}$$

(2.6) means that terms like $\underline{L}\phi_{\underline{L}\underline{L}}\partial\phi$ do not appear on the right hand side of (1.1).

The assumption on the null forms $Q_{\mu\nu}$ is that

$$Q_{\mu\nu}[\partial\phi, \partial\phi] \in S^Z(1)\partial\phi\overline{\partial}\phi. \tag{2.7}$$

2.3. **Local energy estimates.**

We consider a partition of $\mathbb{R}^3$ into the dyadic sets $A_R = \{R \le \langle \tilde{r} \rangle \le 2R\}$ for $R \ge 1$.

We now introduce the local energy norm LE

$$\|u\|_{LE} = \sup_R \|\langle r\rangle^{-\frac{1}{2}} u\|_{L^2(\mathcal{M}\cap\mathbb{R}\times A_R)}$$
$$\|u\|_{LE[\tilde{t}_0,\tilde{t}_1]} = \sup_R \|\langle r\rangle^{-\frac{1}{2}} u\|_{L^2(\mathcal{M}\cap[\tilde{t}_0,\tilde{t}_1]\times A_R)},$$

its H^1 counterpart

$$\|u\|_{LE^1} = \|\partial u\|_{LE} + \|\langle r\rangle^{-1} u\|_{LE}$$
$$\|u\|_{LE^1[\tilde{t}_0,\tilde{t}_1]} = \|\partial u\|_{LE[\tilde{t}_0,\tilde{t}_1]} + \|\langle r\rangle^{-1} u\|_{LE[\tilde{t}_0,\tilde{t}_1]},$$

as well as the dual norm

$$\|f\|_{LE^*} = \sum_R \|\langle r\rangle^{\frac{1}{2}} f\|_{L^2(\mathcal{M}\cap\mathbb{R}\times A_R)}$$
$$\|f\|_{LE^*[\tilde{t}_0,\tilde{t}_1]} = \sum_R \|\langle r\rangle^{\frac{1}{2}} f\|_{L^2(\mathcal{M}\cap[\tilde{t}_0,\tilde{t}_1]\times A_R)}.$$

We also define similar norms for higher Sobolev regularity

$$\|u_{\le m}\|_{LE^1} = \sum_{|\alpha|\le m} \|u_\alpha\|_{LE^1}$$
$$\|u_{\le m}\|_{LE^1[\tilde{t}_0,\tilde{t}_1]} = \sum_{|\alpha|\le m} \|u_\alpha\|_{LE^1[\tilde{t}_0,\tilde{t}_1]}$$
$$\|u_{\le m}\|_{LE[\tilde{t}_0,\tilde{t}_1]} = \sum_{|\alpha|\le m} \|u_\alpha\|_{LE[\tilde{t}_0,\tilde{t}_1]},$$

respectively

$$\|f\|_{LE^{*,k}} = \sum_{|\alpha|\le k} \|\partial^\alpha f\|_{LE^*}$$
$$\|f\|_{LE^{*,k}[\tilde{t}_0,\tilde{t}_1]} = \sum_{|\alpha|\le k} \|\partial^\alpha f\|_{LE^*[\tilde{t}_0,\tilde{t}_1]}.$$

Finally, we introduce a weaker version of the local energy decay norm

$$\|u\|_{LE^1_w} = \|(1-\chi_{ps})\partial u\|_{LE} + \|\partial_r u\|_{LE} + \|\langle r\rangle^{-1} u\|_{LE}$$
$$\|u\|_{LE^1_w[\tilde{t}_0,\tilde{t}_1]} = \|(1-\chi_{ps})\partial u\|_{LE[\tilde{t}_0,\tilde{t}_1]} + \|\partial_r u\|_{LE[\tilde{t}_0,\tilde{t}_1]} + \|\langle r\rangle^{-1} u\|_{LE[\tilde{t}_0,\tilde{t}_1]},$$

To measure the inhomogeneous term, we define

$$\|f\|_{LE^*_w} = \inf_{f_1+f_2=f} \|f_1\|_{L^1L^2} + \|(1-\chi_{ps})f_2\|_{LE^*}$$
$$\|f\|_{LE^*_w[\tilde{t}_0,\tilde{t}_1]} = \inf_{f_1+f_2=f} \|f_1\|_{L^1[\tilde{t}_0,\tilde{t}_1]L^2} + \|(1-\chi_{ps})f_2\|_{LE^*[\tilde{t}_0,\tilde{t}_1]}.$$

Here χ_{ps} is a smooth, compactly supported spatial cutoff function that equals 1 in a neighborhood of the trapped set. We also define the higher order weak norms as above.

We define the (nondegenerate) energy

$$E[u](\tilde{t}) = \left(\int_{\Sigma(\tilde{t})} |\partial u|^2 d\Sigma_K\right)^{1/2}.$$

We now fix some $\delta_1 \ll 1$, and define

$$\mathcal{E}_N(T) = \sup_{0\le\tilde{t}\le T} E[\phi_{\le N}](\tilde{t}) + \|\phi_{\le N}\|_{LE^1_w[0,T]} + \|\langle \tilde{t}-\tilde{r}\rangle^{\frac{-1-\delta_1}{2}} \overline{\partial}\phi_{\le N}\|_{L^2[0,T]L^2(r\ge R_1)}. \tag{2.8}$$

We will need the following local energy estimates for the linear problem:

Lemma 2.2. *Assume that $\Box_K \phi = F$, and N is any nonnegative integer. We then have for any $T \ge 0$ that*

$$\mathcal{E}_N(T) \lesssim \mathcal{E}_N(0) + \|F_{\le N}\|_{L^1[0,T]L^2 + LE^*_w[0,T]} \tag{2.9}$$

where the implicit constant is independent of T.

Proof. Indeed, Theorem 4.5 from [64] gives the desired bound for the first two terms. On the other hand, Lemma 4.3 in [46] and Cauchy Schwarz yield

$$\|\langle \tilde t - \tilde r\rangle^{\frac{-1-\delta_1}{2}}\bar\partial\phi\|_{L^2[0,T]L^2(r\geq R_1)} \lesssim \|\phi\|_{LE^1_w[0,T]} + \|F\|_{LE^*_w[0,T]}, \tag{2.10}$$

which is the desired bound when $N = 0$. Moreover, for any multiindex α we have from applying (2.10) to ϕ_α that

$$\|\langle \tilde t - \tilde r\rangle^{\frac{-1-\delta_1}{2}}\bar\partial\phi_\alpha\|_{L^2[0,T]L^2(r\geq R_1)} \lesssim \|\phi_\alpha\|_{LE^1_w[0,T]} + \|F_\alpha\|_{LE^*_w[0,T]} + \|[\Box_K, Z^\alpha]\phi\|_{LE^*_w[0,T]}.$$

We are left with bounding the last term on RHS. By (2.3) we have

$$\|[\Box_K, Z^\alpha]\phi\|_{LE^*_w[0,T]} \lesssim \|F_{\leq|\alpha|}\|_{LE^*_w[0,T]} + \|r^{-2+}\partial\phi_{\leq|\alpha|}\|_{LE^*_w[0,T]} \lesssim \|F_{\leq|\alpha|}\|_{LE^*_w[0,T]} + \|\phi_{\leq|\alpha|}\|_{LE^1_w[0,T]}.$$

This finishes the proof of the lemma. □

The first estimate of this kind was obtained by Morawetz for the Klein-Gordon equation [59]. In the Schwarzschild case, similar estimates were shown in [6, 7], [8], [13], [14], [52]. The estimate for Kerr with small angular momentum was proven in [64] (see also [3] and [15] for related works). For large angular momentum see [16] ($|a| < M$), and [5] ($|a| = M$).

3. Pointwise estimates from local energy decay estimates

The goal of this section is to show how to extract (weak) pointwise estimates from local energy norms. These bounds will serve as the starting point in an iteration that will yield strong enough pointwise bounds to close the bootstrap argument in Section 7.

Let

$$C_T = \{T \leq \tilde t \leq 2T, \quad \tilde r \leq \tilde t\}.$$

We use a double dyadic decomposition of C_T with respect to either the size of $\tilde t - \tilde r$ or the size of r, depending on whether we are close or far from the cone,

$$C_T = \bigcup_{1\leq R\leq T/4} C_T^R \ \bigcup\ \bigcup_{1\leq U < T/4} C_T^U,$$

where for $R, U > 1$ we set

$$C_T^R = C_T \cap \{R < r < 2R\}, \qquad C_T^U = C_T \cap \{U < \tilde t - \tilde r < 2U\},$$

while for $R = 1$ and $U = 1$ we have

$$C_T^{R=1} = C_T \cap \{0 < r < 2\}, \qquad C_T^{U=1} = C_T \cap \{0 < \tilde t - \tilde r < 2\}.$$

The sets C_T^R and C_T^U represent the setting in which we apply Sobolev embeddings, which allow us to obtain pointwise bounds from L^2 bounds. Precisely, we have (see Lemma 3.8 from [58] and Lemma 6.2 in [46]):

Lemma 3.3. *For any function w and all $T \geq 1$ and $1 \leq R, U \leq T/4$ we have*

$$\|w\|_{L^\infty(C_T^R)} \lesssim \frac{1}{T^{\frac12}R^{\frac32}}\sum_{i\leq 1, j\leq 2}\|S^i\Omega^j w\|_{L^2(C_T^R)} + \frac{1}{T^{\frac12}R^{\frac12}}\sum_{i\leq 1, j\leq 2}\|S^i\Omega^j \partial w\|_{L^2(C_T^R)}, \tag{3.11}$$

respectively

$$\|w\|_{L^\infty(C_T^U)} \lesssim \frac{1}{T^{\frac32}U^{\frac12}}\sum_{i\leq 1, j\leq 2}\|S^i\Omega^j w\|_{L^2(C_T^U)} + \frac{U^{\frac12}}{T^{\frac32}}\sum_{i\leq 1, j\leq 2}\|S^i\Omega^j \partial w\|_{L^2(C_T^U)}. \tag{3.12}$$

Using the lemma above, we prove the following pointwise bound:

$$\|w\|_{L^\infty(C_T)} \lesssim \langle \tilde t\rangle^{-1}\langle \tilde t - \tilde r\rangle^{1/2}\|w_{\leq 12}\|_{LE^1[T,2T]}. \tag{3.13}$$

Indeed, in the region C_T^R, this is an immediate application of (3.11). On the other hand, in the region C_T^U this follows from (3.12) and Hardy's inequality, see (6.7) in [46].

We also need an L^∞ bound on the derivative that is better than (3.13) for large r. This is the content of the following, which is essentially Proposition 3.5 in [50]

Proposition 3.4. *Let*

$$\mu := \min(\langle \tilde t\rangle, \langle \tilde t - \tilde r\rangle)^{1/2}.$$

Assume that ϕ solve (1.1) for $t \in [T, 2T]$. Then for any dyadic region $C \in \{C_T^R, C_U^R\}$ and $m \geq 0$ we have

$$\|\partial\phi_{\leq m}\|_{L^\infty(C)} \leq \bar C_m \frac{1}{\mu}\left(\frac{1}{\langle r\rangle} + \|\partial\phi_{\leq\frac{m+10}{2}}\|_{L^\infty(C)}\right)\|\phi_{\leq m+5}\|_{LE^1[T,2T]} \tag{3.14}$$

Here the crucial estimate was the following Klainerman-Sideris type estimate, see Lemma 6.3 in [46] (for Schwarzschild) combined with the remarks after (5.13) in [47]:

Lemma 3.5. *For any w we have in the region $r \geq 2R_1$ that*

$$|\partial^2 w| \lesssim \frac{\tilde{t}}{r\langle t-\tilde{r}\rangle} \sum_{i+j\leq 1} |\partial S^i \Omega^j w| + \frac{t}{\langle t-\tilde{r}\rangle} |\Box_K w|$$

We now apply (3.12) to $\partial\phi_\Lambda$. We obtain

$$\begin{aligned}\|\partial\phi_\Lambda\|_{L^\infty(C_T^U)} \lesssim & \frac{1}{T^{\frac32}U^{\frac12}} \sum_{i\leq 1, j\leq 2} \|S^i\Omega^j \partial\phi_\Lambda\|_{L^2(C_T^U)} + \frac{U^{\frac12}}{T^{\frac32}} \sum_{i\leq 1, j\leq 2} \|S^i\Omega^j\partial^2\phi_\Lambda\|_{L^2(C_T^U)} \\ \lesssim & \frac{1}{TU^{\frac12}} \|\phi_{\leq|\Lambda|+13}\|_{LE^1[T,2T]} + \frac{1}{(TU)^{\frac12}} \|(\Box_K\phi)_{\leq|\Lambda|+10}\|_{L^2(C_T^U)}\end{aligned}$$

Since

$$|(\Box_K\phi)_{\leq|\Lambda|+10}| \lesssim |\partial\phi_{\leq\frac{|\Lambda|}{2}+5}||\partial\phi_{\leq|\Lambda|+10}|$$

the conclusion follows in the region C_T^U. A similar computation yields the result in C_T^R.

4. Improved pointwise bounds

We will use three lemmas that will help us improve our pointwise bounds. The first one is Proposition 3.14 from [58], which will allow us to turn r-decay into t-decay in the region $r \leq t/2$.

Lemma 4.6. *The the following estimate holds for all $m \geq 0$ and some fixed (m-independent) n:*

$$\|u_{\leq m}\|_{LE^1(C_T^{<T/2})} \lesssim T^{-1}\|\langle r\rangle u_{\leq m+n}\|_{LE^1(C_T^{<T/2})} + \|(\Box_K u)_{\leq m+n}\|_{LE^*(C_T^{<T/2})}.$$

The second lemma is a slight modification of Lemma 3.11 from [58], the difference being that we may not enlarge our regions in time. The role of the lemma is to gain a factor of $\frac{\tilde{t}}{r\langle\tilde{t}-\tilde{r}\rangle}$ for the derivative.

We let $\tilde{C}_T^R$ and $\tilde{C}_T^U$ denote enlargements of C_T^R and C_T^U in space (but not in time) that contain all the integral curves of the scaling vector field S (i.e. if $(t,x) \in C_T^R$ then $(st, sx) \in \tilde{C}_T^R$ as long as $T \leq st \leq 2T$ and similarly for C_T^U). More precisely, let

$$\tilde{C}_T^R = \{T \leq \tilde{t} \leq 2T, \quad \frac{8}{10}\frac{T}{2R} \leq \frac{\tilde{t}}{\tilde{r}} \leq \frac{12}{10}\frac{2T}{R}\}, \quad \tilde{C}_T^R(\tau) = \tilde{C}_T^R \cap \{\tilde{t} = \tau\}.$$

$$\tilde{C}_T^U = \{T \leq \tilde{t} \leq 2T, \quad \frac{8}{10}\frac{T}{T-2U} \leq \frac{\tilde{t}}{\tilde{r}} \leq \frac{12}{10}\frac{2T}{2T-U}\}, \quad \tilde{C}_T^U(\tau) = \tilde{C}_T^U \cap \{\tilde{t} = \tau\}$$

An important observation here is that $\tilde{r} \approx R$ and $\tilde{t}-\tilde{r} \approx U$ in $\tilde{C}_T^R$ and $\tilde{C}_T^U$ respectively.

Lemma 4.7. *For $1 \ll U, R \leq T/4$ we have*

$$\|\partial w\|_{L^2(C_T^R)} \lesssim R^{-1}\|w\|_{L^2(\tilde{C}_T^R)} + T^{-1}\big(\|Sw\|_{L^2(\tilde{C}_T^R)} + \|S^2 w\|_{L^2(\tilde{C}_T^R)}\big) + R\|\Box_K w\|_{L^2(\tilde{C}_T^R)} \tag{4.15}$$

respectively

$$\|\partial w\|_{L^2(C_T^U)} \lesssim U^{-1}\big(\|w\|_{L^2(\tilde{C}_T^U)} + \|Sw\|_{L^2(\tilde{C}_T^U)} + \|S^2 w\|_{L^2(\tilde{C}_T^U)}\big) + T\|\Box_K w\|_{L^2(\tilde{C}_T^U)} \tag{4.16}$$

Proof. The proof is similar to the one in Lemma 3.11 from [58], except that we need to estimate the boundary terms at $\tilde{t} = T$ and $\tilde{t} = 2T$.

To keep the ideas clear we first prove the lemma with $\Box_K$ replaced by $\Box$. We consider a cutoff function χ supported in $[8/20, 22/10]$ which equals 1 on $[9/20, 21/20]$. Let

$$\beta(\tilde{t},\tilde{r}) = \chi\Big(\frac{\tilde{r}}{\tilde{t}}\frac{T}{R}\Big)$$

Note that $\beta \equiv 1$ on C_T^R, and that β is supported in $\tilde{C}_T^R$.

Integrating $\beta\,\Box w^2/2 = \beta\big(w\Box w + m^{\alpha\beta}\partial_\alpha w\,\partial_\beta w\big)$ by parts twice gives

$$\int_T^{2T}\int \beta(|\partial_x w|^2 - |\partial_t w|^2)dx\,dt = \int_T^{2T}\int \Box w\cdot\beta w dx\,dt - \frac12\int_T^{2T}\int(\Box\beta)w^2 dx\,dt - \int\big(\beta w\partial_t w - \beta_t w^2/2\big)\,dx\Big|_T^{2T}.$$

Since we can write $w_t = (Sw - x^i\partial_i w)/t$ it follows after integration by parts that

$$\int \beta w\partial_t w\,dx = \frac{1}{\tilde{t}}\int \beta w S w\,dx + \frac{1}{2\tilde{t}}\int w^2\partial_i(x^i\beta)\,dx.$$

Since $|\partial_i(x^i\beta)| + \tilde{t}|\partial_t\beta| \le C$ on the support of β, it follows that the boundary terms are bounded by

$$CT^{-1}\Big(\|w(2T,\cdot)\|^2_{L^2(\tilde{C}^R_T(2T))} + \|Sw(2T,\cdot)\|^2_{L^2(\tilde{C}^R_T(2T))} + \|w(T,\cdot)\|^2_{L^2(\tilde{C}^R_T(T))} + \|Sw(T,\cdot)\|^2_{L^2(\tilde{C}^R_T(T))}\Big).$$

Let $\chi(t/T)$ be another smooth cutoff such that $\chi(2) = 1$ and $\chi(1) = 0$. We write

$$w(2T,x)^2 = \int_{1/2}^1 \frac{d}{ds}(\chi w^2)(s2T, sx)\,ds = \int_{1/2}^1 S(\chi w^2)(s2T,sx)\,\frac{ds}{s} = \int_T^{2T} S(\chi w^2)(t, tx/2T)\,\frac{dt}{t}$$

and thus

$$\|w(2T,\cdot)\|^2_{L^2(\tilde{C}^R_T(2T))} \lesssim \frac{1}{T}\|S(\chi w^2)(t,x)\|^2_{L^2(\tilde{C}^R_T)} \lesssim \frac{1}{T}\Big(\|w\|^2_{L^2(\tilde{C}^R_T)} + \|Sw\|^2_{L^2(\tilde{C}^R_T)}\Big).$$

A similar argument holds for $2T$ replaced by T, and for w replaced by Sw. Hence the boundary term can be estimated by

$$\frac{1}{T^2}\sum_{j=0}^2 \|S^j w\|^2_{L^2(\tilde{C}^R_T)}.$$

To estimate ∂w we use the pointwise inequality

$$|\partial w|^2 \le M\frac{1}{(\tilde{t}-\tilde{r})^2}|Sw|^2 + \frac{\tilde{t}}{\tilde{t}-\tilde{r}}(|\partial_x w|^2 - |\partial_t w|^2) \tag{4.17}$$

which is valid inside the cone C for a fixed large M. Hence

$$\int \beta|\partial w|^2 dxdt \lesssim \int \frac{1}{(\tilde{t}-\tilde{r})^2}\beta|Sw|^2 + \frac{\tilde{t}}{\tilde{t}-\tilde{r}}|\Box\beta|w^2 + \frac{\tilde{t}}{\tilde{t}-\tilde{r}}\beta|\Box w||w|dxdt \tag{4.18}$$

where all weights have a fixed size in the support of β. The function β also satisfies $|\Box\beta| \lesssim R^{-2}$. Then the conclusion of the lemma follows by applying Cauchy-Schwarz to the last term.

The argument for C^U_T is similar. We now consider

$$\beta(\tilde{t},\tilde{r}) = \chi\Big(\frac{\tilde{t}-\tilde{r}}{\tilde{t}}\frac{T}{U}\Big)$$

We multiply by βw and integrate by parts as above. The boundary terms are now controlled by

$$CU^{-1}\Big(\|w(2T,\cdot)\|^2_{L^2(\tilde{C}^U_T(2T))} + \|Sw(2T,\cdot)\|^2_{L^2(\tilde{C}^U_T(2T))} + \|w(T,\cdot)\|^2_{L^2(\tilde{C}^U_T(T))} + \|Sw(T,\cdot)\|^2_{L^2(\tilde{C}^U_T(T))}\Big).$$

which in turn is controlled, by using the scaling S as above, by

$$\frac{1}{TU}\sum_{j=0}^2 \|S^j w\|^2_{L^2(\tilde{C}^R_T)}.$$

The estimate now follows from (4.18), using the fact that $|\Box\beta| \lesssim T^{-1}U^{-1}$.

Now consider the above proof but with $\Box$ replaced by $\Box_K$. Integrating $\beta\,\Box_K w^2/2 = \beta\big(w\Box_K w + g_K^{\alpha\beta}\partial_\alpha w\,\partial_\beta w\big)$ by parts twice gives

$$-\int_T^{2T}\!\!\int \beta g_K^{\alpha\beta}\partial_\alpha w\,\partial_\beta w\sqrt{|g_K|}dx\,dt = \int_T^{2T}\!\!\int \big(\beta w\Box_K w - \frac{1}{2}(\Box_K\beta)w^2\big)\sqrt{|g_K|}dx\,dt - \frac{1}{2}\int \big(\beta g_K^{0\alpha}\partial_\alpha w^2 - g_K^{\alpha 0}w^2\partial_\alpha\beta\big)\sqrt{|g_K|}dx\Big|_T^{2T}.$$

First we estimate the boundary term. The terms with $\alpha = 0$ are handled as before and so is the second term with $\alpha > 0$. For the first term with $\alpha > 0$ we integrate by parts and see that it is bounded by a term of the same form as the second term plus a term of the form

$$\frac{1}{2}\int \beta\partial_\alpha(g_K^{0\alpha}\sqrt{|g_K|})\,w^2\,dx \lesssim \int \beta r^{-2}w^2\,dx,$$

which can be estimated as above. To estimate the interior term we just note that

$$\sqrt{|g_K|}g_K^{\alpha\beta}\partial_\alpha w\,\partial_\beta w = |\partial_x w|^2 - |\partial_t w|^2 + O(r^{-1})|\partial w|^2,$$

where the error term can be absorbed in the left of (4.17) for large enough R.

This finishes the proof of (4.15). (4.16) follows in a similar manner. ∎

Applying Lemma 4.7 to w_α for some multiindex α, and using (2.3) we obtain the higher order version of the estimates:

$$\|\partial w_\alpha\|_{L^2(C^R_T)} \lesssim R^{-1}\|w_{|\alpha|+n}\|_{L^2(\tilde C^R_T)} + R\|(\Box_K w)_{|\alpha|+n}\|_{L^2(\tilde C^R_T)} \tag{4.19}$$

$$\|\partial w_\alpha\|_{L^2(C^U_T)} \lesssim U^{-1}\|w_{|\alpha|+n}\|_{L^2(\tilde C^U_T)} + T\|(\Box_K w)_{|\alpha|+n}\|_{L^2(\tilde C^U_T)} \tag{4.20}$$

Combining the two estimates above (4.19) and (4.20) with the Sobolev embeddings from Lemma 3.3 and the pointwise estimate for second order derivatives in Lemma 3.5 we obtain

Corollary 4.8. *For all $T \geq 1$ and $1 \leq R, U \leq T/4$ we have for some n independent of α:*

$$\|\partial w_\alpha\|_{L^\infty(C^R_T)} \lesssim \frac{1}{R}\|w_{\leq|\alpha|+n}\|_{L^\infty(\tilde C^R_T)} + R\|(\Box_K w)_{|\alpha|+n}\|_{L^\infty(\tilde C^R_T)},$$

respectively

$$\|\partial w_\alpha\|_{L^\infty(C^U_T)} \lesssim \frac{1}{U}\|w_{\leq|\alpha|+n}\|_{L^\infty(\tilde C^U_T)} + T\|(\Box_K w)_{|\alpha|+n}\|_{L^\infty(\tilde C^U_T)}.$$

Finally, we will derive a sharp estimate for the bad first order derivative, following [37].

Lemma 4.9. *Let $D_t = \{x;\, 0 \leq t-|x| \leq t/4\}$, $C^q_t = \{x;\, t-|x| = q\}$, and let $\overline{w}(q)$ be any positive continuous function, where $q = t-r$. Suppose that $\Box\phi = F$. Then the following holds in D_t, $t \geq 1$:*

$$t\,|\partial\phi(t,x)\,\overline{w}(q)| \lesssim \sup_{4q\leq\tau\leq t}\Big(\|\,q\,\partial\phi(\tau,\cdot)\,\overline{w}\|_{L^\infty(C^q_\tau)} + \sum\nolimits_{|I|\leq 1}\|Z^I\phi(\tau,\cdot)\,\overline{w}\|_{L^\infty(C^q_\tau)}\Big)$$
$$+ \int_{4q}^t \Big(\langle\tau\rangle\|\,F(\tau,\cdot)\,\overline{w}\|_{L^\infty(C^q_\tau)} + \sum\nolimits_{|I|+|J|\leq 2}\langle\tau\rangle^{-1}\|\partial^I\Omega^J\phi(\tau,\cdot)\,\overline{w}\|_{L^\infty(C^q_\tau)}\Big)\,d\tau.$$

Proof. We write

$$\Box\phi = -\frac{1}{r}\,\partial_v\partial_u(r\phi) + \frac{1}{r^2}\triangle_\omega\phi,$$

where $\partial_u = \partial_t - \partial_r$ and $\partial_v = \partial_t + \partial_r$. Hence in D_t

$$\Big|\partial_v\partial_u(r\phi)\Big| \lesssim \Big|r\Box\phi\Big| + \langle r\rangle^{-1}\sum\nolimits_{|I|+|J|\leq 2}|\partial^I\Omega^J\phi| \lesssim \Big|\langle t\rangle\Box\phi\Big| + \langle t\rangle^{-1}\sum\nolimits_{|I|+|J|\leq 2}|\partial^I\Omega^J\phi| \tag{4.21}$$

Integrating this along the flow lines of the vector field ∂_v from the boundary of $D = \cup_{\tau\geq 0}D_\tau$ to any point inside D_t for $t \geq 1$. Using that $\overline{w}$ is constant along the flow lines, and (4.21), we obtain

$$|\partial_u(r\phi(t,x))\,\overline{w}(q)| \lesssim |\partial_u(r\phi)(4q,3q)\overline{w}(q)| + \int_{4q}^t \Big(\langle\tau\rangle\|\,F(\tau,\cdot)\,\overline{w}\|_{L^\infty(C^q_\tau)} + \sum\nolimits_{|I|+|J|\leq 2}\langle\tau\rangle^{-1}\|\partial^I\Omega^J\phi(\tau,\cdot)\,\overline{w}\|_{L^\infty(C^q_\tau)}\Big)\,d\tau.$$

Moreover

$$t|\partial_u\phi(t,x)\overline{w}(q)| \lesssim |\partial_u(r\phi(t,x))\,\overline{w}(q)| + |\phi(t,x)\overline{w}(q)|,$$

and

$$|\partial_u(r\phi)(4q,3q)\overline{w}(q)| \lesssim |q\partial_u\phi(4q,3q)\overline{w}(q)| + |\phi(4q,3q)\overline{w}(q)|.$$

The last three inequalities yield

$$t|\partial_u\phi(t,x)\overline{w}(q)| \lesssim \sup_{4q\leq\tau\leq t}\Big(\|\,q\,\partial\phi(\tau,\cdot)\,\overline{w}\|_{L^\infty(C^q_\tau)} + \|\phi(\tau,\cdot)\,\overline{w}\|_{L^\infty(C^q_\tau)}\Big)$$
$$+ \int_{4q}^t \Big(\langle\tau\rangle\|\,F(\tau,\cdot)\,\overline{w}\|_{L^\infty(C^q_\tau)} + \sum\nolimits_{|I|+|J|\leq 2}\langle\tau\rangle^{-1}\|\partial^I\Omega^J\phi(\tau,\cdot)\,\overline{w}\|_{L^\infty(C^q_\tau)}\Big)\,d\tau.$$

The lemma follows from also using that $r|\partial\phi| \lesssim |r\partial_q\phi| + |S\phi| + |\Omega\phi|$. □

5. Pointwise estimates from the Minkowski fundamental solution

In this section, we translate pointwise bounds on the inhomogeneous terms into pointwise bounds for the solution by using the fundamental solution of the Minkowski metric.

For any $\beta, \gamma, \eta \in \mathbb{R}$, we define the weighted L^∞ norms

$$\|G\|_{L^\infty_{\beta,\gamma,\eta}} = \|\langle r\rangle^\beta\langle t\rangle^\gamma\langle t-r\rangle^\eta H(t,r)\|_{L^\infty_{t,r}}, \quad H(t,r) = \sum_0^2 \|\Omega^i G(t,r\omega)\|_{L^2(\mathbb{S}^2)}.$$

We use the following lemma (see Section 6 of [65]).

Lemma 5.10. *Let ψ solve*

$$\Box\psi = G, \qquad \psi(0) = 0, \quad \partial_t\psi(0) = 0,$$

where G is supported in $\{|x| \le t + R_0\}$. Assume also that $2 \le \beta \le 3$ and $\eta \ge -1/2$. We define, for any arbitrary $\delta > 0$,

$$\tilde{\eta} = \begin{cases} \eta - \delta - 2 & \eta < 1, \\ -1 & \eta > 1 \end{cases} .$$

i) If $\gamma \ge 0$, we have

$$r\psi(t,x) \lesssim \frac{1}{\langle t-r\rangle^{\beta+\gamma+\tilde{\eta}-1}} \|G\|_{L^\infty_{\beta,\gamma,\eta}}, \tag{5.22}$$

ii) If $\gamma < 0$, we have

$$r\psi(t,x) \lesssim \frac{t^{-\gamma}}{\langle t-r\rangle^{\beta+\tilde{\eta}-1}} \|G\|_{L^\infty_{\beta,\gamma,\eta}}, \tag{5.23}$$

iii) If $|x| \le t-1$, and $\eta > 1$, we have

$$r\psi(t,x) \lesssim \ln\frac{t}{t-r} \|G\|_{L^\infty_{2,0,\eta}}, \tag{5.24}$$

Proof. Note first that, after a translation in time, we may assume that $R_0 = 0$.

We use the ideas from [58]. Define

$$H(t,r) = \sum_0^2 \|\Omega^i G(t, r\omega)\|_{L^2(\mathbb{S}^2)}.$$

By Sobolev embeddings on the sphere, we have $|G| \lesssim H$. Let v be the radial solution to

$$\Box v = H, \qquad v[0] = 0.$$

By the positivity of the fundamental solution, we have that $|\psi| \lesssim |v|$. On the other hand, we can write v explicitly:

$$rv(t,r) = \frac{1}{2}\int_{D_{tr}} \rho H(s,\rho) ds d\rho,$$

where D_{tr} is the rectangle

$$D_{tr} = \{0 \le s - \rho \le t - r, \quad t - r \le s + \rho \le t + r\}.$$

We partition the set D_{tr} into a double dyadic manner as

$$D_{tr} = \bigcup_{R \le t} D^R_{tr}, \quad D^R_{tr} = D_{tr} \cap \{R < r < 2R\}$$

and estimate the corresponding parts of the above integral.

We clearly have

$$\int_{D^R_{tr}} \rho H ds d\rho \lesssim \|G\|_{L^\infty_{\beta,\gamma,\eta}} \int_{D^R_{tr}} \rho^{1-\beta}\langle s\rangle^{-\gamma}\langle s-\rho\rangle^{-\eta} d\rho ds.$$

We now consider two cases:

(i) $R < (t-r)/8$. Here we have $\rho \sim R$ and $s \approx s - \rho \approx \langle t-r\rangle$; therefore we obtain

$$\int_{D^R_{tr}} \rho^{1-\beta}\langle s\rangle^{-\gamma}\langle s-\rho\rangle^{-\eta} d\rho ds \lesssim R^{3-\beta}\langle t-r\rangle^{-\gamma-\eta},$$

and after summation, using that $\beta \le 3$, we obtain

$$\sum_{R<(t-r)/8} \int_{D^R_{tr}} \rho H ds d\rho \lesssim \frac{\ln\langle t-r\rangle\langle t-r\rangle^{3-\beta}}{\langle t-r\rangle^{\gamma+\eta}} \lesssim \frac{1}{\langle t-r\rangle^{\beta+\tilde{\eta}}},$$

which is the desired bound in all cases.

(ii) $(t-r)/8 < R < t$. Here we have $\rho \sim R$ and $t \ge s \gtrsim R$. Denote $u = s - \rho$.

Assume first that $\gamma \ge 0$; then

$$\int_{D^R_{tr}} \rho^{1-\beta}\langle s\rangle^{-\gamma}\langle s-\rho\rangle^{-\eta} d\rho ds \lesssim R^{2-\beta-\gamma}\int_0^{t-r}\langle u\rangle^{-\eta} du \lesssim R^{2-\beta-\gamma}\langle t-r\rangle^{\mu(\eta)},$$

where

$$\mu(\eta) = \begin{cases} 1-\eta & \eta < 1, \\ 0 & \eta > 1 \end{cases} .$$

If $\beta+\gamma>2$, we obtain after summation

$$\sum_{R>(t-r)/8}\int_{D^R_{tr}} \rho H ds d\rho \lesssim \langle t-r\rangle^{2-\beta-\gamma+\mu(\eta)},$$

which is (5.22).

On the other hand, if $\beta = 2$ and $\gamma = 0$, and taking into account that there are $\ln\frac{t}{t-r}$ dyadic regions when $(t-r)/8 < R < t$, we obtain (5.24) after summation.

Finally, if $\gamma < 0$ we obtain

$$\int_{D^R_{tr}} \rho^{1-\beta}\langle s\rangle^{-\gamma}\langle s-\rho\rangle^{-\eta} d\rho ds \lesssim R^{2-\beta}t^{-\gamma}\int_0^{t-r}\langle u\rangle^{-\eta}du \lesssim R^{2-\beta}t^{-\gamma}\langle t-r\rangle^{\mu(\eta)},$$

Since $\beta\geq 2$, we obtain after summation

$$\sum_{R>(t-r)/8}\int_{D^R_{tr}} \rho H ds d\rho \lesssim t^{-\gamma}\langle t-r\rangle^{2-\beta+\mu(\eta)},$$

which is (5.23). □

6. Setup for pointwise estimates

In this section, we will slightly adjust $\Box_K$ to an operator closer to $\Box$ (with respect to the $(\tilde t, x)$ coordinates). Indeed, we let

$$P = |g_K|^{1/4}(-g_K^{\tilde t\tilde t})^{-1/2}\Box_K(-g_K^{\tilde t\tilde t})^{-1/2}|g_K|^{-1/4}.$$

P is self-adjoint with respect to $d\tilde t dx$. More importantly, a quick computation yields that

$$P = \partial_\alpha\left(g_K^{\alpha\beta}(-g_K^{\tilde t\tilde t})\partial_\beta\right) + V, \quad V = |g_K|^{1/4}(-g_K^{\tilde t\tilde t})^{-1/2}\Box_K\left((-g_K^{\tilde t\tilde t})^{-1/2}|g_K|^{-1/4}\right).$$

It is easy to see that $V\in S^Z(r^{-3})$.

Let us first consider the Schwarzschild metric. In this case we have that for large r, $-g_S^{\tilde t\tilde t} = g_S^{r^*r^*}$ and $g_S^{\tilde t r^*} = 0$. We thus have

$$P = \Box + P_{lr},$$

where the long range spherically symmetric part P_{lr} has the form

$$P_{lr} = g_{lr}(r)\Delta_\omega + V, \qquad g_{lr}\in S^Z(r^{-3}), \qquad V\in S^Z(r^{-3}). \tag{6.25}$$

For the Kerr metric, we use the fact that the metric coefficients have the following properties:

$$g_K^{\alpha\beta} - g_S^{\alpha\beta} \in S^Z(r^{-2}), \tag{6.26}$$

$$\partial g_K \in S^Z(r^{-2}), \quad \partial^2 g_K\in S^Z(r^{-3}) \tag{6.27}$$

Using (6.25) and (6.26) we see that we can write

$$P = \Box + P_{lr} + P_{sr}, \tag{6.28}$$

where the short-range part P_{sr} has the form

$$P_{sr} = \partial_\alpha g_{sr}^{\alpha\beta}\partial_\beta, \quad g_{sr}^{\alpha\beta}\in S^Z(r^{-2}). \tag{6.29}$$

Using (6.27) we see that for any function ϕ we have

$$P\phi = (-g_K^{\tilde t\tilde t})\Box_K\phi + h_1\phi + h_2\partial\phi, \quad h_1\in S^Z(r^{-3}), \quad h_2\in S^Z(r^{-2}). \tag{6.30}$$

Now pick any multiindex α. After commuting with vector fields, using (6.28), (6.25), and (6.29), we obtain

$$P\phi_\alpha \in S^Z(1)(\Box_K\phi)_{\leq|\alpha|} + S^Z(r^{-3})\phi_{\leq|\alpha|+6} + S^Z(r^{-2})\partial\phi_{\leq|\alpha|+5},$$

which in turn implies, using (6.28)

$$\Box\phi_\alpha \in S^Z(1)(\Box_K\phi)_{\leq|\alpha|} + S^Z(r^{-3})\phi_{\leq|\alpha|+6} + S^Z(r^{-2})\partial\phi_{\leq|\alpha|+5}. \tag{6.31}$$

Moreover, by finite speed of propagation, and the assumption on the support of the initial data, the right hand side is supported in the forward light cone $\{|x| < \tilde t + R_0\}$.

We will use (6.31) in the next section to extract more decay for the solution.

7. The bootstrap argument for the Einstein model

We now prove Theorem 1.1 by using a bootstrap argument. We first write

$$\mathcal{E}_N(0) = \mu_N \epsilon$$

where $\mu_N > 0$ is a fixed, small N-dependent constant to be determined below (see (7.36), (7.37)).

Let $N_1 = \frac{N}{2}$. We will assume that the following a-priori bounds hold for some large constant $\tilde{C}$ independent of ϵ and $\tilde{t}$, and a fixed small $\delta > 0$

$$\mathcal{E}_N(\tilde{t}) \leq \tilde{C}\mu_N \epsilon \langle \tilde{t}\rangle^{\delta}, \tag{7.32}$$

$$|\phi_{\leq N_1+2}| \leq \frac{\epsilon \tilde{r}^{\delta}}{\langle \tilde{t}\rangle}, \qquad |\partial\phi_{\leq N_1+2}| \leq \frac{\epsilon}{\tilde{r}^{1-\delta}\langle \tilde{t}-\tilde{r}\rangle} \tag{7.33}$$

$$|(\partial\phi_{TU})_{\leq N_1+2}| \leq \frac{\epsilon}{\langle \tilde{t}\rangle} \tag{7.34}$$

Clearly (7.32), (7.33) and (7.34) hold for small times. We assume now that the bounds hold on some time interval $0 \leq \tilde{t} \leq T$, and we improve the constants by 1/2. By the continuity method this implies that the solution exists globally, and that the bounds also hold globally.

In order to improve (7.32), we show that, for small enough ϵ, there is C_N independent of T so that

$$\mathcal{E}_N(\tilde{t}) \leq C_N \langle \tilde{t}\rangle^{C_N \epsilon}\mathcal{E}_N(0), \quad 0 \leq \tilde{t} \leq T \tag{7.35}$$

If we now additionally take $\tilde{C} = 2C_N$ and $\epsilon < \frac{\delta}{C_N}$ we thus improve the a-priori bound for $\mathcal{E}_N(\tilde{t})$ to

$$\mathcal{E}_N(\tilde{t}) \leq \frac{1}{2}\tilde{C}\mu_N\epsilon\langle \tilde{t}\rangle^{\delta}.$$

In order to improve the pointwise bounds, we will show that, for some fixed positive integer m, independent of N, we have

$$|\phi_{\leq N-m}| \lesssim \frac{\mathcal{E}_N(0)}{\langle \tilde{t}-\tilde{r}\rangle^{\delta}\langle \tilde{t}\rangle^{1-\delta}}, \qquad |\partial\phi_{\leq N-m}| \lesssim \frac{\tilde{t}^{\delta}\mathcal{E}_N(0)}{r\langle \tilde{t}-\tilde{r}\rangle^{1+\delta}} \tag{7.36}$$

$$|(\partial\phi_{TU})_{\leq N-m}| \lesssim \frac{\mathcal{E}_N(0)}{r\langle \tilde{t}-\tilde{r}\rangle^{1-\delta}} \tag{7.37}$$

We can now pick a small μ_N to improve (7.33) and (7.34).

7.1. **The energy estimates.** We will now use assumptions (7.33) and (7.34) to show (7.35) for small enough ϵ.

By Gronwall's inequality and (2.9), it is enough to show that

$$\|(\Box_K\phi)_{\leq N}\|_{LE^*_w[0,\tilde{t}]} \lesssim \int_0^{\tilde{t}} \frac{\epsilon}{\tau}\mathcal{E}_N(\tau)d\tau + \epsilon\mathcal{E}_N(\tilde{t}) \tag{7.38}$$

We can write, using (2.5), (2.6) and (2.7):

$$\Box_K\phi \in S^Z(1)(\partial\phi_{TU})^2 + S^Z(1)\partial\phi\overline{\partial}\phi$$

After commuting with vector fields, and using (2.4), we also get that

$$(\Box_K\phi)_{\leq N} \lesssim (\partial\phi_{TU})_{\leq N_1}(\partial\phi_{TU})_{\leq N} + \partial\phi_{\leq N_1}\overline{\partial}\phi_{\leq N} + \overline{\partial}\phi_{\leq N_1}\partial\phi_{\leq N} + r^{-1}\partial\phi_{\leq N_1}\partial\phi_{\leq N-1} \tag{7.39}$$

The first term is easy. By (7.34) we have

$$\|(\partial\phi_{TU})_{\leq N_1}(\partial\phi_{TU})_{\leq N}\|_{L^1[0,\tilde{t}]L^2} \lesssim \int_0^{\tilde{t}} \frac{\epsilon}{\tau}\mathcal{E}_N(\tau)d\tau.$$

Similarly, the last term can be estimated in L^1L^2. Indeed, we note that (7.33) implies that

$$|r^{-1}\partial\phi_{\leq N_1}| \lesssim \frac{\epsilon}{\tilde{t}}$$

and thus

$$\|r^{-1}\partial\phi_{\leq N_1}\partial\phi_{\leq N-1}\|_{L^1[0,\tilde{t}]L^2} \lesssim \int_0^{\tilde{t}} \frac{\epsilon}{\tau}\mathcal{E}_N(\tau)d\tau.$$

For the second term, we divide it into two parts. When $r < R_1$ we have by (7.33):

$$\|\partial\phi_{\leq N_1}\overline{\partial}\phi_{\leq N}\|_{L^1[0,\tilde{t}]L^2(r<R_1)} \lesssim \int_0^{\tilde{t}} \frac{\epsilon}{\tau}\mathcal{E}_N(\tau)d\tau$$

When $r > R_1$, we use (7.33) and the last term in (2.8):

$$\|\partial\phi_{\le N_1}\overline{\partial}\phi_{\le N}\|^2_{LE^*[0,t]} \lesssim \int_0^{\tilde{t}}\int_{r>R_1}\frac{\epsilon^2\tau^{2\delta}}{r\langle\tau-\tilde{r}\rangle^{2+2\delta}}|\overline{\partial}\phi_{\le N}|^2 dV \lesssim \|\epsilon\langle\tau-\tilde{r}\rangle^{\frac{-1-\delta_1}{2}}\overline{\partial}\phi_{\le N}\|^2_{L^2[0,t]L^2(r\ge R_1)} \lesssim \left(\epsilon\mathcal{E}_N(\tilde{t})\right)^2$$

For the third term, note that (2.2) and (7.33) imply that

$$|\overline{\partial}\phi_{\le N_1}| \lesssim \frac{\epsilon}{\langle\tilde{t}\rangle} \tag{7.40}$$

Using (7.40) gives

$$\|\overline{\partial}\phi_{\le N_1}\partial\phi_{\le N}\|_{L^1[0,\tilde{t}]L^2} \lesssim \int_0^{\tilde{t}}\frac{\epsilon}{\tau}\mathcal{E}_N(\tau)d\tau$$

Putting all these together we obtain (7.38).

7.2. **The decay estimates.** We now show that (7.36) and (7.37) hold.

The proof uses an iteration procedure. The most important part here is to obtain pointwise decay rates of $\tilde{t}^{-1}$ near the trapped set for all components. We start with a weak decay rate of $\tilde{t}^{-1/2+C\epsilon}$ given by the slow growth $\tilde{t}^{C\epsilon}$ combined with the results of Section 3. We then use Lemma 5.10 to improve decay in r, followed by Corollary 4.8 to improve the decay of derivatives. Lemma 4.6 then allows us to turn the r-decay into $\tilde{t}$ decay. This yields an improved global decay rate of $\tilde{t}^{-1+C\epsilon}$, which is barely not enough. We then use Lemma 4.9 to improve the decay of the derivative of the good components $\partial\phi_{TU}$ to $\tilde{t}^{-1}$ near the cone. We can now go back to the iteration procedure, and use the improved bounds combined with Lemma 5.10, Corollary 4.8 and Lemma 4.6 to improve the decay rate of all components to $\tilde{t}^{-1}$ away from the cone. This finishes the proof.

Let $N_2 = N - 13$. We first note that (3.13) and (3.14), combined with the energy bounds (7.35), yield the weak pointwise bounds

$$|\partial\phi_{\le N_2}| \lesssim \frac{\langle\tilde{t}\rangle^{C\epsilon}\mathcal{E}_N(0)}{r\langle\tilde{t}-\tilde{r}\rangle^{1/2}}, \qquad |\phi_{\le N_2}| \lesssim \frac{\langle\tilde{t}-\tilde{r}\rangle^{1/2}\mathcal{E}_N(0)}{\langle\tilde{t}\rangle^{1-C\epsilon}} \tag{7.41}$$

We now need to improve the decay of $\phi_{\le N-m}$ and $\partial\phi_{\le N-m}$. To that extent, we will use Lemma 5.10, followed by Lemma 4.6 and Corollary 4.8.

We cannot apply Lemma 5.10 directly. On one hand, we have no control on the solution for $r \ll 2M$, and on the other hand, the initial data is not trivial. Instead, let

$$\chi = \chi_1(\tilde{r})\chi_2(\tilde{t})$$

Here $\chi_1 \equiv 1$ for $\tilde{r} \ge R \gg M$ and supported in $\tilde{r} \ge R/2$, while $\chi_2 \equiv 1$ for $\tilde{t} \ge 1$ and supported in $\tilde{t} \ge 1/2$.

We now consider $\psi_{\alpha\beta} = \chi\phi_{\alpha\beta}$. Using (6.31), we see that ψ satisfies the system

$$\Box(\psi_{\le n}) = G_n, \quad G_n \in S^Z(r^{-2})\partial\phi_{\le n+5} + S^Z(r^{-3})\phi_{\le n+6} + S^Z(1)(\partial\phi_{\le n})^2$$

with trivial initial data, and G_n supported in the region $r \ge R/2$. Using (7.41), we see that, for all $n \le N_3 := N_2 - 12$, we have

$$G_{n+6} \lesssim \mathcal{E}_N(0)\left(\frac{\langle\tilde{t}\rangle^{C\epsilon}}{r^3\langle\tilde{t}-\tilde{r}\rangle^{1/2}} + \frac{\langle\tilde{t}-\tilde{r}\rangle^{1/2}}{r^3\langle\tilde{t}\rangle^{1-C\epsilon}} + \frac{\langle\tilde{t}\rangle^{C\epsilon}}{r^2\langle\tilde{t}-\tilde{r}\rangle}\right)$$

We now apply Lemma 5.10. The first term on the right hand side is controlled by the other two terms. For the second term we use (5.22) with $\beta = 3$, $\gamma = 1 - C\epsilon$ and $\eta = -1/2$. For the third term, we use (5.22) with $\beta = 2$, $\gamma = -C\epsilon$ and $\eta = 1 - C\epsilon$. We obtain

$$|\phi_{\le N_3}| \lesssim \frac{\tilde{t}^{C\epsilon}}{r}\mathcal{E}_N(0). \tag{7.42}$$

We now plug in the bounds (7.42) and (7.41) into Corollary 4.8. We thus obtain for $N_4 = N_3 - n$ with n from Corollary 4.8:

$$\|\partial\phi_{N_4}\|_{L^\infty(C_T^R)} \lesssim \frac{1}{R}\frac{T^{C\epsilon}}{R}\mathcal{E}_N(0) + R\left(\frac{T^{C\epsilon}}{RT^{1/2}}\mathcal{E}_N(0)\right)^2 \lesssim \frac{T^{C\epsilon}}{R^2}\mathcal{E}_N(0)$$

$$\|\partial\phi_{N_4}\|_{L^\infty(C_T^U)} \lesssim \frac{1}{U}\frac{T^{C\epsilon}}{R}\mathcal{E}_N(0) + T\left(\frac{T^{C\epsilon}}{RU^{1/2}}\mathcal{E}_N(0)\right)^2 \lesssim \frac{T^{C\epsilon}}{RU}\mathcal{E}_N(0)$$

The last two inequalities can be written as

$$|\partial\phi_{\le N_4}| \lesssim \frac{\tilde{t}^{1+C\epsilon}}{r^2\langle\tilde{t}-\tilde{r}\rangle}\mathcal{E}_N(0) \tag{7.43}$$

We now use Lemma 4.6. Note that (7.42) and (7.43) yield

$$\|\langle r\rangle\phi_{\leq N_4}\|_{LE^1(C_T^{<T/2})} \lesssim T^{1/2+C\epsilon}\mathcal{E}_N(0)$$

Moreover, (7.41) implies that

$$\|(\Box_K\phi)_{\leq N_4}\|_{LE^*(C_T^{<T/2})} \lesssim T^{-1/2+C\epsilon}\mathcal{E}_N(0)$$

The two inequalities above and Lemma 4.6 with $N_5 = N_4 - n$ give us

$$\|\phi_{\leq N_5}\|_{LE^1(C_T^{<T/2})} \lesssim T^{-1/2+C\epsilon}\mathcal{E}_N(0)$$

which combined with the Sobolev embeddings from Lemma 3.3 give for $N_6 = N_5 - 13$:

$$|\phi_{\leq N_6}| \lesssim \tilde{t}^{-1+C\epsilon}\mathcal{E}_N(0) \tag{7.44}$$

We now plug in the bounds (7.44) and (7.41) into Corollary 4.8. We thus obtain for $N_7 = N_6 - n$

$$\|\partial\phi_{\leq N_7}\|_{L^\infty(C_T^R)} \lesssim \frac{1}{R}\frac{T^{C\epsilon}}{T}\mathcal{E}_N(0) + R\left(\frac{T^{C\epsilon}}{RT^{1/2}}\mathcal{E}_N(0)\right)^2 \lesssim \frac{T^{C\epsilon}}{RT}\mathcal{E}_N(0)$$

which combined with (7.43) gives

$$|\partial\phi_{\leq N_7}| \lesssim \frac{\tilde{t}^{C\epsilon}}{r\langle\tilde{t}-\tilde{r}\rangle}\mathcal{E}_N(0) \tag{7.45}$$

This finishes the proof of (7.36) when $\tilde{r} > \tilde{t}/2$.

Note also that (2.2), (7.44) and (7.45) give

$$|\overline{\partial}\phi_{\leq N_7-2}| \lesssim \frac{\tilde{t}^{C\epsilon}}{r\langle\tilde{t}\rangle}\mathcal{E}_N(0) \tag{7.46}$$

We now use the fact that ψ_{TU} actually satisfies better decay estimates. Indeed, note first that

$$\Box(T^\alpha U^\beta\phi_{\alpha\beta}) - T^\alpha U^\beta\Box\phi_{\alpha\beta} \in S^Z(r^{-2})\phi_{\leq 1}$$

Using (6.28) and (6.30) we obtain

$$\Box\phi_{TU} \in S^Z(1)(\Box_K\phi)_{TU} + S^Z(r^{-2})\phi_{\leq 6}$$

and since

$$(\Box_K\phi)_{TU} \in S^Z(1)\partial\phi\overline{\partial}\phi$$

we thus have

$$\Box\phi_{TU} \in S^Z(1)\partial\phi\overline{\partial}\phi + S^Z(r^{-2})\phi_{\leq 6}$$

After commuting with vector fields (in particular using (2.4)) and applying the cutoff we thus obtain

$$\Box(\psi_{TU})_{\leq m} = H_m, \quad H_m \in S^Z(r^{-2})\phi_{\leq m+6} + S^Z(1)\partial\phi_{\leq m}\overline{\partial}\phi_{\leq m} + S^Z(r^{-1})(\partial\phi_{\leq m})^2$$

Using (7.44), (7.45) and (7.46), we see that

$$H_m \lesssim \frac{\mathcal{E}_N(0)}{r^2\langle\tilde{t}\rangle^{1-C\epsilon}}, \quad m \leq N_7 - 2 \tag{7.47}$$

Let $N_8 = N_7 - 6$. We now apply Lemma 4.9 with $\overline{w}(q) = \langle q\rangle^{1-\delta}$ to $(\psi_{TU})_{\leq N_8}$. Note first that, due to (7.45) and (7.44) we have

$$\sup_{4q\leq\tau\leq\tilde{t}}\left(\|\,q\,\partial\phi_{\leq N_8}(\tau,\cdot)\,\overline{w}\|_{L^\infty(C_\tau^q)} + \sum\nolimits_{|I|\leq 1}\|Z^I\phi_{\leq N_8}(\tau,\cdot)\,\overline{w}\|_{L^\infty(C_\tau^q)}\right) \lesssim \mathcal{E}_N(0)$$

Moreover, (7.44) implies that

$$\int_{4q}^{\tilde{t}}\sum\nolimits_{|I|\leq 2}\langle\tau\rangle^{-1}\|\Omega^I\phi_{\leq N_8}(\tau,\cdot)\,\overline{w}\|_{L^\infty(C_\tau^q)}\,d\tau \lesssim \int_{4q}^{\tilde{t}}\langle\tau\rangle^{-1}\frac{\langle q\rangle^{1-\delta}\mathcal{E}_N(0)}{\langle\tau\rangle^{1-C\epsilon}}d\tau \lesssim \mathcal{E}_N(0).$$

Finally, we obtain by (7.47) that

$$\int_{4q}^{\tilde{t}}\langle\tau\rangle\|\,H_m(\tau,\cdot)\,\overline{w}\|_{L^\infty(C_\tau^q)} \lesssim \int_{4q}^{\tilde{t}}\langle\tau\rangle\frac{\langle q\rangle^{1-\delta}\mathcal{E}_N(0)}{\langle\tau\rangle^{3-C\epsilon}}d\tau \lesssim \mathcal{E}_N(0).$$

Lemma 4.9 thus implies, in conjunction with (7.45), that

$$|\partial(\psi_{TU})_{\leq N_8}| \lesssim \frac{\mathcal{E}_N(0)}{r\langle\tilde{t}-\tilde{r}\rangle^{1-\delta}}. \tag{7.48}$$

This finishes the proof of (7.37).

Finally, to obtain a decay rate of $1/\tilde{t}$ in the interior, we see that, using (6.31) and (7.39), we can write our system as

$$\Box(\psi_{\le m}) = J_m, \quad J_m \in S^Z(r^{-2})\partial\phi_{\le m+5} + S^Z(r^{-3})\phi_{\le m+6} + S^Z(1)(\partial\phi_{TU})^2_{\le m} + S^Z(1)\partial\phi_{\le m}\overline{\partial}\phi_{\le m} + S^Z(r^{-1})(\partial\phi_{\le m})^2$$

and J_m is supported in the region $\{\tilde{t} \ge 1/2, \tilde{r} \ge R/2\}$. Due to the improved bounds (7.44), (7.45) and (7.48) we obtain

$$J_{m+6} \lesssim \mathcal{E}_N(0)\left(\frac{t^{C\epsilon}}{r^3\langle\tilde{t}-\tilde{r}\rangle} + \frac{1}{r^2\langle\tilde{t}-\tilde{r}\rangle^{2-2\delta}}\right), \quad m \le N_9 := N_8 - 8.$$

We now apply Lemma 5.10 and in particular (5.23) to control the last term. We obtain

$$\phi_{\le N_9} \lesssim \frac{1}{r}\mathcal{E}_N(0), \quad \tilde{r} < 3\tilde{t}/4. \tag{7.49}$$

Corollary 4.8 thus implies, with $N_{10} = N_9 - n$:

$$\partial\phi_{\le N_{10}} \lesssim \frac{1}{r^2}\mathcal{E}_N(0), \quad \tilde{r} < 3\tilde{t}/4. \tag{7.50}$$

We now use Lemma 4.6. Note that (7.49) and (7.50) yield

$$\|\langle r\rangle\phi_{\le N_{10}}\|_{LE^1(C_T^{<T/2})} \lesssim T^{1/2}\mathcal{E}_N(0)$$

Moreover, (7.45) implies that

$$\|(\Box_K\phi)_{\le N_{10}}\|_{LE^*(C_T^{<T/2})} \lesssim T^{-1/2}\mathcal{E}_N(0)$$

The two inequalities above and Lemma 4.6 give us for $N_{11} = N_{10} - n$:

$$\|\phi_{\le N_{11}}\|_{LE^1(C_T^{<T/2})} \lesssim T^{-1/2}\mathcal{E}_N(0)$$

which combined with the Sobolev embeddings from Lemma 3.3 give with $N_{12} = N_{11} - 13$

$$|\phi_{\le N_{12}}| \lesssim \frac{\mathcal{E}_N(0)}{\langle\tilde{t}\rangle}, \quad \tilde{r} \le \tilde{t}/2$$

Finally, one last application of Corollary 4.8 with $N_{13} = N_{12} - n$ gives

$$|\partial\phi_{\le N_{13}}| \lesssim \frac{\mathcal{E}_N(0)}{r\langle\tilde{t}\rangle}, \quad \tilde{r} \le \tilde{t}/2$$

This finishes the proof of (7.36) if we pick N large enough so that $N_{13} \ge N_1$.

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Department of Mathematics, Johns Hopkins University, Baltimore, MD 21218
Email address: lindblad@math.jhu.edu

Department of Mathematics, University of Kentucky, Lexington, KY 40506
Email address: mihai.tohaneanu@uky.edu