AN IMPROVED EIGENVALUE ESTIMATE FOR EMBEDDED MINIMAL HYPERSURFACES IN THE SPHERE

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Abstract. Suppose that $\Sigma^n \subset \mathbb{S}^{n+1}$ is a closed embedded minimal hypersurface. We prove that the first non-zero eigenvalue λ_1 of the induced Laplace-Beltrami operator on Σ satisfies $\lambda_1 \geq \frac{n}{2} + a_n(\Lambda^6 + b_n)^{-1}$, where a_n and b_n are explicit dimensional constants and Λ is an upper bound for the length of the second fundamental form of Σ. This provides the first explicitly computable improvement on Choi and Wang's lower bound $\lambda_1 \geq \frac{n}{2}$ without any further assumptions on Σ.

keywords: eigenvalue estimate, minimal hypersurfaces

1. Introduction

An important problem in geometric analysis is to understand the spectrum of the Laplace-Beltrami operator on a Riemannian manifold, and to study its relation to the underlying intrinsic and/or ambient geometry. From the geometric perspective, it is of particular interest to address such questions for manifolds embedded in spaces of constant curvature. In this paper, we obtain a new lower bound for the first non-zero eigenvalue $\lambda_1(\Sigma)$ of the induced Laplace-Beltrami operator $-\Delta^\Sigma$ on a smooth closed hypersurface Σ^n minimally embedded in the unit sphere $\mathbb{S}^{n+1}$ (which we always assume to be equipped with the round metric g).

In this direction, an argument of Choi and Wang [**6**], later refined by Brendle [**1**], gives the lower bound

$$\lambda_1(\Sigma) > \frac{n}{2}. \tag{1.1}$$

An important application of (1.1) and the Yang-Yau inequality [**16**] is an area bound for embedded minimal surfaces in $\mathbb{S}^3$ in terms of their genus; this plays a crucial role in the compactness theory of Choi and Schoen [**5**]. Moreover, (1.1) provides evidence towards a famous conjecture of Yau [**17**], which predicts that $\lambda_1(\Sigma)$ is equal to n. Note that the restriction to $\mathbb{S}^{n+1}$ of each coordinate function on $\mathbb{R}^{n+2}$ is an eigenfunction for $-\Delta^\Sigma$ with eigenvalue n, and thus the upper bound $\lambda_1(\Sigma) \leq n$ is clear.

Despite an extensive literature relating to the study of $\lambda_1(\Sigma)$ under additional assumptions on Σ since the work of Choi and Wang (see e.g. [**4, 15**] and the references therein), (1.1) has remained the strongest *explicit* lower bound that is known to hold for a general embedded minimal hypersurface in $\mathbb{S}^{n+1}$. In this paper, we obtain an explicit improvement on (1.1) which depends only on the dimension n and an upper bound for $\|A\| := \sqrt{\mathrm{trace}(g^{-1}A)^2}$, where for a fixed choice of orientation N on Σ, $A(X,Y) := -g(\nabla_X N, Y)$ is the second fundamental form of Σ and $(g^{-1}A)X := -\nabla_X N$ is the corresponding shape operator, obtained by raising an index of A using the inverse metric g^{-1}. We recall that the *mean curvature* of Σ is defined to be the trace of the shape operator, and that Σ is called *minimal* if its mean curvature is identically zero. Our main result is as follows:

Theorem 1.1. *Let $\Sigma^n \subset \mathbb{S}^{n+1}$ be a closed embedded minimal hypersurface and denote $\Lambda = \max_\Sigma \|A\|$. Then there exist constants*

$$a_n \geq \frac{(n-1)n^2}{32000} \quad \text{and} \quad b_n \leq \frac{5n^2}{216} \tag{1.2}$$

such that

$$\lambda_1(\Sigma) \geq \frac{n}{2} + \frac{a_n}{\Lambda^6 + b_n}. \tag{1.3}$$

Remark 1.2. In the proof of Theorem 1.1, we will actually show that one can take

$$a_n \geq \frac{3(n-1)n^{7/2}}{3200} \arctan^3\left(\frac{1}{3\sqrt{n}}\right) \quad \text{and} \quad b_n \leq \frac{5n^{7/2}}{8} \arctan^3\left(\frac{1}{3\sqrt{n}}\right).$$

The inequalities in (1.2) follow since for $n \geq 2$ we have $\frac{7}{200} \leq n^{3/2} \arctan^3(\frac{1}{3\sqrt{n}}) \leq \frac{1}{27}$.

Remark 1.3. In recent related work [**18**], Zhao obtained an estimate of the form (1.3) when $n = 2$, although his constants are not explicit.

Whilst we are only interested in explicitly computable lower bounds for $\lambda_1(\Sigma)$ in this paper, we note that upper bounds for either $\lambda_1(M^n, g)$ or $\lambda_1(M^n, g) \cdot \mathrm{Vol}(M^n, g)^{2/n}$ on Riemannian manifolds (M^n, g) have also been studied extensively – see for instance the classical works of Cheng [**3**], Li and Yau [**10, 11**], Yang and Yau [**16**] and Korevaar [**9**]. In particular, recall that for a closed orientable Riemannian surface (Σ^2, g) of genus γ, the Yang-Yau inequality [**7, 16**] states that $\lambda_1(\Sigma, g)\,\mathrm{Area}(\Sigma, g) \leq 8\pi\lfloor\frac{\gamma+3}{2}\rfloor$, where $\lfloor x \rfloor$ denotes the integer part of x. The following result is then an immediate corollary of the Yang-Yau inequality and Theorem 1.1:

Corollary 1.4. *Let $\Sigma^2 \subset \mathbb{S}^3$ be a closed embedded minimal surface of genus γ and denote $\Lambda = \max_\Sigma \|A\|$. Then there exist constants a_n and b_n satisfying* (1.2) *such that*

$$\mathrm{Area}(\Sigma) \leq \left(\frac{n}{2} + \frac{a_n}{\Lambda^6 + b_n}\right)^{-1} 8\pi \left\lfloor \frac{\gamma+3}{2} \right\rfloor.$$

Remark 1.5. As a consequence of our *method* for proving Theorem 1.1, we will also obtain an explicit volume bound for closed embedded mean-convex hypersurfaces in $\mathbb{S}^{n+1}$ in terms of n and Λ – see Proposition 2.2. We note that our proof of Proposition 2.2 does not invoke any lower bound for λ_1.

To put Theorem 1.1 into context, we now briefly discuss some related results. We first observe that, in light of the strictness of the inequality in (1.1), non-explicit improved lower bounds depending only on quantities such as dimension, index, topological type and curvature bounds follow from suitable compactness results. For example, if $\mathcal{A}(\Lambda, n)$ denotes the class of closed embedded minimal hypersurfaces in $\mathbb{S}^{n+1}$ satisfying $\max_\Sigma \|A\| \leq \Lambda$, then it is well-known that $\mathcal{A}(\Lambda, n)$ is compact in the C^k topology for any $k \geq 2$. Combined with (1.1), it follows that there exists a constant $\alpha(\Lambda, n) > 0$ such that

$$\lambda_1(\Sigma) \geq \frac{n}{2} + \alpha(\Lambda, n) \quad \text{for all } \Sigma \in \mathcal{A}(\Lambda, n). \tag{1.4}$$

We stress that, in contrast with (1.4), the estimate (1.3) obtained in Theorem 1.1 provides an *explicitly computable* improvement on (1.1). Moreover, our lower bound (1.3) is obtained by arguing more directly in the spirit of [**6**], rather than appealing to any compactness theory.

C^k compactness results have also been established in other classes. For example, Choi and Schoen showed in [**5**] that the class $\mathcal{B}(\gamma)$ of closed embedded minimal surfaces in $\mathbb{S}^3$ with genus less than γ is compact in the C^k topology for any $k \geq 2$. In combination with (1.1), this implies the existence of a constant $\beta(\gamma) > 0$ such that

$$\lambda_1(\Sigma) \geq 1 + \beta(\gamma) \quad \text{for all } \Sigma \in \mathcal{B}(\gamma).$$

A more recent compactness of result of Sharp [**13, Corollary 2.6**] shows that the class $\mathcal{C}(V, I, n)$ of closed embedded minimal hypersurfaces in $\mathbb{S}^{n+1}$ with volume bounded from above by V and index bounded from above by I is compact in the C^k topology for $k \geq 2$ when $2 \leq n \leq 6$. Combined with (1.1), this implies the existence of a constant $\delta(V, I, n) > 0$ such that

$$\lambda_1(\Sigma) \geq \frac{n}{2} + \delta(V, I, n) \quad \text{for all } \Sigma \in \mathcal{C}(V, I, n) \text{ when } 2 \leq n \leq 6.$$

In a similar vein to Theorem 1.1, it would be interesting to derive improved lower bounds for $\lambda_1(\Sigma)$ with explicit dependence on quantities such as genus (when $n = 2$), volume and/or index. Such results could provide a step towards proving Yau's conjecture within certain classes of minimal hypersurfaces in $\mathbb{S}^{n+1}$. Recently, Yau's conjecture was established for the class of embedded isoparametric minimal hypersurfaces in $\mathbb{S}^{n+1}$ – see Tang and Yan [**15**] and the references therein. We refer also to the work of Choe and Soret [**4**], where Yau's conjecture was established for a class of symmetric minimal surfaces in $\mathbb{S}^3$.

Remark 1.6. The aforementioned results of [**5, 6, 13**] apply more generally when $\mathbb{S}^{n+1}$ is replaced by a closed manifold (M^{n+1}, g) whose Ricci curvature satisfies $\mathrm{Ric}_g \geq kg$ for some constant $k > 0$. The bound (1.1) is then replaced by $\lambda_1(\Sigma) > \frac{k}{2}$, and our subsequent discussion generalises in the obvious way. In attempting to generalise Theorem 1.1 to this more general context, it seems that our method introduces constants that depend on *sectional curvature bounds*. To keep the exposition simple, and since the case of the sphere is the one of most interest, we will not discuss such generalisations in this paper.

The plan of the paper is as follows. In Section 2 we prove a preliminary result on the embeddedness of parallel hypersurfaces in $\mathbb{S}^{n+1}$. As a corollary, we obtain an explicit volume bound for closed embedded mean-convex hypersurfaces in $\mathbb{S}^{n+1}$ in terms of an upper bound for $\|A\|$. In Section 3 we prove Theorem 1.1. The key here is to estimate a positive term which is dropped in the estimate of Choi and Wang in [**6**]. Here, our integral estimates require working in a neighbourhood of Σ whose thickness is controlled away from zero; this control is provided by our results in Section 2.

2. Embeddedness of parallel hypersurfaces

Suppose that Σ^n is a smooth, closed and embedded hypersurface in $\mathbb{S}^{n+1}$. As observed in [**6**], Σ divides the sphere into two components $\mathbb{S}^{n+1} = M_1 \cup M_2$, where $\partial M_1 = \partial M_2 = \Sigma$. Let $N\Sigma$ denote the normal bundle of $\Sigma \subset \mathbb{S}^{n+1}$ and $\exp^{N\Sigma}$ the corresponding exponential map. We fix the orientation on Σ determined by the unit normal vector field X on Σ pointing into M_1, and for $t \in \mathbb{R}$ we define

$$\Sigma^t = \{\exp^{N\Sigma}(p, tX_p) \in \mathbb{S}^{n+1} : p \in \Sigma\}.$$

Geometrically, Σ^t is the hypersurface parallel to Σ and of signed distance t to Σ. It is well-known (see e.g. Theorems 2.1 and 2.2 in [**2**]) that if $\kappa_1(p), \ldots, \kappa_n(p)$ are the principal curvatures of Σ at p and $\kappa_{\max} = \max_{p\in\Sigma, i\in\{1,\ldots,n\}} |\kappa_i(p)|$, then Σ^t is a smooth immersed hypersurface in $\mathbb{S}^{n+1}$ for

$$|t| < \arctan(\kappa_{\max}^{-1}) =: T_\Sigma.$$

Moreover, we may consider n continuous functions $\kappa_i(\cdot,\cdot) : \Sigma \times (-T_\Sigma, T_\Sigma) \to \mathbb{R}$ defined by

$$\kappa_i(\cdot, t) = \frac{\kappa_i(\cdot, 0) + \tan t}{1 - \kappa_i(\cdot, 0)\tan t}, \quad \kappa_i(\cdot, 0) := \kappa_i(\cdot). \tag{2.1}$$

Then for each $t \in (-T_\Sigma, T_\Sigma)$, the quantities $\kappa_1(p,t), \ldots, \kappa_n(p,t)$ are the principal curvatures of Σ^t at $\exp^{N\Sigma}(p, tX_p)$, with respect to the orientation on Σ^t determined by parallel transporting X along geodesics normal to Σ by a signed distance t. The formula (2.1) can be derived somewhat directly following the proof of Theorem 2.2 in [**2**]. Alternatively, it can be derived from the more general fact that the principal curvatures of parallel hypersurfaces in a Riemannian manifold satisfy a certain Riccati equation, which in the case of the sphere can be integrated directly – see Corollary 3.5 in [**8**].

Whilst it is well-known that Σ^t remains embedded for t sufficiently small, in general the range of t for which Σ^t is embedded is *not* controlled by the curvature of Σ, since Σ may be arbitrarily close to 'self-touching'. We show that in the case that Σ is mean-convex (that is, the mean curvature H_Σ of Σ is nonnegative), we do in fact have such control:

Proposition 2.1. *Suppose $\Sigma^n \subset \mathbb{S}^{n+1}$ is a smooth, closed and embedded mean-convex hypersurface. Then Σ^t is a smooth, closed and embedded strictly mean-convex hypersurface in $\mathbb{S}^{n+1}$ for $|t| \in (0, T_\Sigma)$.*

Proof. We consider the case $t > 0$; the case $t < 0$ is similar. Let

$$t_* = \sup\{t > 0 : \Sigma^t \text{ is smooth and embedded}\},$$

and suppose for a contradiction that $t_* = \arctan(\varepsilon\kappa_{\max}^{-1})$ for some $0 < \varepsilon < 1$. Since $t_* < T_\Sigma$, by the discussion above Σ^{t_*} is a smooth, immersed, non-embedded hypersurface. Therefore, for some point $x \in \Sigma^{t_*}$, there exist distinct points $p, q \in \Sigma$ such that $x = \exp^{N\Sigma}(p, t_*X_p) = \exp^{N\Sigma}(q, t_*X_q)$. Now, locally near p (resp. q), Σ^t is a smooth graph

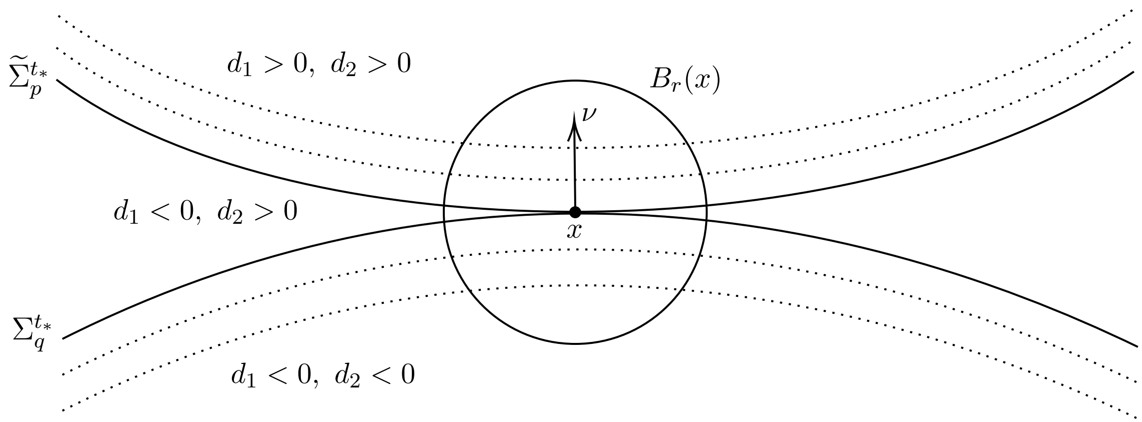

Figure 1.

over a neighbourhood of the origin in $T_p\Sigma$ (resp. $T_q\Sigma$) for $t \leq t_*$. Denote these graphs by Σ_p^t and Σ_q^t, respectively. Then by (2.1), on Σ_p^t we have

$$\begin{aligned}\kappa_i(p, t_*) = \frac{\kappa_i(p,0) + \tan t_*}{1 - \kappa_i(p,0)\tan t_*} &= \kappa_i(p,0) + \frac{(1+\kappa_i(p,0)^2)\tan t_*}{1-\kappa_i(p,0)\tan t_*} \\ &> \kappa_i(p,0) + \frac{(1+\kappa_i(p,0)^2)\tan t_*}{1+\varepsilon},\end{aligned} \tag{2.2}$$

and likewise on Σ_q^t we have

$$\kappa_i(q,t_*) > \kappa_i(q,0) + \frac{(1+\kappa_i(q,0)^2)\tan t_*}{1+\varepsilon}. \tag{2.3}$$

Summing over i in (2.2) and (2.3), and using mean-convexity of Σ to assert $\sum_i \kappa_i(p,0) \geq 0$ and $\sum_i \kappa_i(q,0) \geq 0$, we see that the mean curvature $H_{\Sigma_p^{t_*}}$ of $\Sigma_p^{t_*}$ at the point x satisfies

$$H_{\Sigma_p^{t_*}}(x) > \frac{(n + \|A(p)\|^2)\tan t_*}{1+\varepsilon} > 0, \tag{2.4}$$

and likewise

$$H_{\Sigma_q^{t_*}}(x) > \frac{(n + \|A(q)\|^2)\tan t_*}{1+\varepsilon} > 0. \tag{2.5}$$

Now, by minimality of t_*, the graphs $\Sigma_p^{t_*}$ and $\Sigma_q^{t_*}$ meet tangentially at x and thus have opposite orientations at x. Let us denote by $\widetilde{\Sigma}_p^{t_*}$ the hypersurface $\Sigma_p^{t_*}$ but with the opposite orientation. Then by (2.4) and (2.5), we have

$$H_{\widetilde{\Sigma}_p^{t_*}}(x) < 0 < H_{\Sigma_q^{t_*}}(x). \tag{2.6}$$

Now denote by d_1 the signed distance to $\widetilde{\Sigma}_p^{t_*}$ and d_2 the signed distance to $\Sigma_q^{t_*}$, so that the functions d_i are positive in the direction ν of the common orientation of $\widetilde{\Sigma}_p^{t_*}$ and $\Sigma_q^{t_*}$; note that the functions d_i are well defined in a sufficiently small geodesic ball $B_r(x)$ (see Figure 1). Since the orientation on $\widetilde{\Sigma}_p^{t_*}$ is given by ∇d_1 and the orientation on $\Sigma_q^{t_*}$ is given by ∇d_2, the definition of the mean curvature as the trace of the shape operator implies that $H_{\widetilde{\Sigma}_p^{t_*}}(x) = -\Delta d_1(x)$ and $H_{\Sigma_q^{t_*}}(x) = -\Delta d_2(x)$. Therefore, (2.6) can be rewritten as

$$-\Delta d_1(x) < 0 < -\Delta d_2(x),$$

and by continuity it follows that $-\Delta d_1 < 0 < -\Delta d_2$ in $B_r(x) \cap \{d_1 > 0\}$ for sufficiently small r, i.e. $\Delta(d_2 - d_1) < 0$ in $B_r(x) \cap \{d_1 > 0\}$. But $d_2 - d_1 \geq 0$ in $B_r(x) \cap \{d_1 > 0\}$ and $(d_2 - d_1)(x) = 0$. By the Hopf lemma, either $d_2 - d_1$ is constant in $B_r(x) \cap \{d_1 > 0\}$, or $\nabla_\nu(d_2 - d_1)(x) > 0$. In either case we obtain a contradiction: the first possibility contradicts the strict inequality $\Delta(d_2 - d_1) < 0$ in $B_r(x) \cap \{d_1 > 0\}$, and the second possibility contradicts the fact that $\nabla_\nu d_1(x) = \nabla_\nu d_2(x) = 1$.

We have therefore shown that $t_* = T_\Sigma$. It is also clear from the computations (2.2)–(2.5) that Σ^t is strictly mean convex for $t \in (0, T_\Sigma)$, which completes the proof of the proposition. $\square$

As a corollary of Proposition 2.1 we obtain an explicit volume bound for closed embedded mean-convex hypersurfaces in $\mathbb{S}^{n+1}$:

Proposition 2.2. *Suppose $\Sigma^n \subset \mathbb{S}^{n+1}$ is a smooth, closed and embedded mean-convex hypersurface with $\max_\Sigma \|A\| \leq \Lambda$, and define*

$$\mathrm{I}_\Lambda = \int_0^{\arctan(\Lambda^{-1})} (\cos t)^n (1 - \Lambda \tan t)^n \, dt.$$

Then

$$\mathrm{Vol}(\Sigma^n) \leq \frac{1}{2\,\mathrm{I}_\Lambda} \mathrm{Vol}(\mathbb{S}^{n+1}). \tag{2.7}$$

In particular, there exists a dimensional constant $c_n \leq \frac{25}{3}\left(\frac{5}{4}\right)^{n-2}$ such that

$$\mathrm{Vol}(\Sigma^n) \leq c_n \Lambda \, \mathrm{Vol}(\mathbb{S}^{n+1}) \tag{2.8}$$

whenever $\Lambda \geq \frac{1}{4}$.

Remark 2.3. Suppose in addition to the hypotheses of Proposition 2.2 that Σ is minimal and not totally geodesic. Then by the inequality $\int_\Sigma \|A\|^2(\|A\|^2 - n)\, dS_g \geq 0$ of Simons [**14**], $\Lambda \geq \sqrt{n}$ and thus the assumption $\Lambda \geq \frac{1}{4}$ is automatically satisfied. Note that the restriction $\Lambda \geq \frac{1}{4}$ is somewhat arbitrary, allowing us to make a crude estimation of the quantity I_Λ in the proof below.

Proof. Let $V^\pm(R)$ denote the volume of region swept out by the parallel hypersurfaces $\Sigma^{\pm t}$ for $0 \leq t \leq R$. Then by [**8, Exercise 3.5**] and Proposition 2.1, the following formula is valid for $R \leq \arctan(\Lambda^{-1})$:

$$V^\pm(R) = \int_\Sigma \left(\int_0^R (\cos t)^n \prod_{i=1}^n (1 \mp \kappa_i \tan t)\, dt \right) dS.$$

Taking $R = \arctan(\Lambda^{-1})$, we therefore see that

$$\begin{aligned}\mathrm{Vol}(\mathbb{S}^{n+1}) &\geq V^+(\arctan(\Lambda^{-1})) + V^-(\arctan(\Lambda^{-1}))\\ &\geq 2\int_\Sigma \left(\int_0^{\arctan(\Lambda^{-1})} (\cos t)^n (1-\Lambda\tan t)^n\, dt\right) dS = 2\,\mathrm{I}_\Lambda \,\mathrm{Vol}(\Sigma^n), \qquad (2.9)\end{aligned}$$

which proves (2.7).

Now suppose that $\Lambda \geq \frac{1}{4}$. Then on the interval $[0, \frac{5}{54\Lambda}]$ it is easy to verify that $\cos t \geq \frac{9}{10}$ and $\tan t \leq \frac{27t}{25} \leq \frac{1}{10\Lambda}$, and moreover $\frac{5}{54\Lambda} \leq \arctan(\Lambda^{-1})$. Therefore, by (2.9) we obtain

$$\mathrm{Vol}(\mathbb{S}^{n+1}) \geq 2\,\mathrm{Vol}(\Sigma^n)\int_0^{\frac{5}{54\Lambda}} (\cos t)^n(1-\Lambda\tan t)^n\, dt \geq \frac{5}{27}\left(\frac{9}{10}\right)^{2n}\frac{\mathrm{Vol}(\Sigma^n)}{\Lambda},$$

from which the estimate (2.8) easily follows with $c_n \leq \frac{25}{3}\left(\frac{5}{4}\right)^{n-2}$. □

3. The improved estimate

In this section we prove Theorem 1.1. We begin in Section 3.1 by recalling the argument of Choi and Wang [**6**]. In Section 3.2 we give the proof of Theorem 1.1 assuming the validity of two propositions. In Sections 3.3 and 3.4 we give the proofs of these two propositions.

3.1. The estimate of Choi and Wang.

Our proof of Theorem 1.1 initially proceeds in the same way as in [**6**]; we derive the relevant estimate of [**6**] here for the convenience of the reader. The starting point is the following identity due to Reilly [**12**], which is an integral version of Bochner's formula:

Lemma 3.1 (Reilly's formula)**.** *Let (X^{n+1}, g) be a smooth orientable Riemannian manifold with boundary $\Sigma^n := \partial X^{n+1}$. Denote by dv_g the volume element on (X^{n+1}, g), dS_g the volume element of the induced metric on Σ, u_ν the outward normal derivative of u on Σ, ∇^Σ the gradient operator of the induced metric on Σ, A the second fundamental form of Σ defined with respect to the inward unit normal, and H the mean curvature of Σ with respect to the inward unit normal. Then for $u \in C^2(\overline{X})$,*

$$\begin{aligned}\int_X \left((\Delta u)^2 - |\nabla^2 u|^2\right) dv_g = \int_X \mathrm{Ric}_X(\nabla u, \nabla u)\, dv_g + \int_\Sigma (\Delta^\Sigma u + H u_\nu) u_\nu\, dS_g\\ - \int_\Sigma \langle \nabla^\Sigma u, \nabla^\Sigma u_\nu\rangle\, dS_g + \int_\Sigma A(\nabla^\Sigma u, \nabla^\Sigma u)\, dS_g.\end{aligned}$$

Remark 3.2. Our convention that A and H be defined with respect to the inward unit normal on Σ is opposite to the convention used in [**6**].

Recall that under the setup of Theorem 1.1, we may write $\mathbb{S}^{n+1} = M_1 \cup M_2$, where $\partial M_1 = \partial M_2 = \Sigma$. Denote by Ψ an L^2-normalised eigenfunction corresponding to the first non-zero eigenvalue λ_1 of $-\Delta^\Sigma$, so that $-\Delta^\Sigma\Psi = \lambda_1\Psi$ and $\|\Psi\|_{L^2(\Sigma)} = 1$, and let u

be the unique solution to

$$\begin{cases} \Delta u = 0 & \text{in } M_1 \\ u = \Psi & \text{on } \Sigma. \end{cases} \tag{3.1}$$

In what follows, we fix the orientation on Σ pointing into M_1, and we denote by g the round metric on $\mathbb{S}^{n+1}$. We may assume that $\int_\Sigma A(\nabla^\Sigma u, \nabla^\Sigma u)\, dS_g \geq 0$, otherwise we work on M_2 instead. Then by Reilly's formula and minimality of Σ, the solution u to (3.1) satisfies

$$\begin{aligned} -\int_{M_1} |\nabla^2 u|^2\, dv_g &\geq n \int_{M_1} |\nabla u|^2\, dv_g + \int_\Sigma u_\nu \Delta^\Sigma u\, dS_g - \int_\Sigma \langle \nabla^\Sigma u, \nabla^\Sigma u_\nu \rangle\, dS_g \\ &= n \int_{M_1} |\nabla u|^2\, dv_g + 2\int_\Sigma u_\nu \Delta^\Sigma u\, dS_g \\ &= n \int_{M_1} |\nabla u|^2\, dv_g - 2\lambda_1 \int_\Sigma u_\nu u\, dS_g. \end{aligned} \tag{3.2}$$

On the other hand, integration by parts and the fact that $\Delta u = 0$ on M_1 gives

$$\int_\Sigma u_\nu u\, dS_g = \int_{M_1} \big(|\nabla u|^2 + u\Delta u\big)\, dv_g = \int_{M_1} |\nabla u|^2\, dv_g, \tag{3.3}$$

and substituting (3.3) back into (3.2) yields

$$2\bigg(\lambda_1 - \frac{n}{2}\bigg) \int_{M_1} |\nabla u|^2\, dv_g \geq \int_{M_1} |\nabla^2 u|^2\, dv_g \geq 0. \tag{3.4}$$

This is precisely the estimate derived in [**6**]; the lower bound $\lambda_1 \geq \frac{n}{2}$ follows immediately from (3.4), since $|\nabla u| \not\equiv 0$. We note that in [**1**], Brendle gave a refinement of the above argument to show that $\lambda_1 > \frac{n}{2}$, although we will not need to use this strict inequality in our subsequent arguments.

3.2. **Proof of Theorem 1.1.** As seen above, the term $\int_{M_1} |\nabla^2 u|^2\, dv_g$ in (3.4) is simply dropped in the argument of Choi and Wang. In order to prove Theorem 1.1, we obtain a lower bound for $\int_{M_1} |\nabla^2 u|^2\, dv_g$ in terms of $\int_{M_1} |\nabla u|^2\, dv_g$.

Our proof of Theorem 1.1 can be decomposed into two main propositions, which we describe now. We introduce parameters $0 < \varepsilon \leq \frac{\Lambda}{2}$ and $\beta > 0$, which are to be fixed later in the proof of Theorem 1.1 but assumed sufficiently small for now so that $\gamma := \sqrt{2n} - \frac{\Lambda\varepsilon}{\Lambda - \varepsilon}(\frac{n}{\Lambda^2} + 1) - \beta > 0$. We also define $\delta = n \arctan(\frac{\varepsilon}{n})$ and $T = \frac{\delta}{2\Lambda^2}$, and for $t \geq 0$ we denote $M_1^t = \{x \in M_1 : d(x) > t\}$, where d is the distance to Σ in M_1. Note $\partial M_1^t = \Sigma^t$ is a smooth embedded hypersurface for $0 \leq t < \arctan(\Lambda^{-1})$ by Proposition 2.1, and in particular this holds for $0 \leq t < 2T$. Our two main propositions are then as follows:

Proposition 3.3. *Suppose $\varepsilon, \beta, \gamma, \delta$ and T are as above, and $\Lambda \geq \sqrt{n}$. Then*

$$\int_{M_1} |\nabla u|^2\, dv_g \leq \frac{2\Lambda^2}{\delta\gamma} \int_{M_1^T \setminus M_1^{2T}} |\nabla u|^2\, dv_g + \frac{1}{\beta\gamma} \int_{M_1} |\nabla^2 u|^2\, dv_g. \tag{3.5}$$

Proposition 3.4. *Suppose $\varepsilon, \beta, \gamma, \delta$ and T are as above. Then*

$$\int_{M_1^T} |\nabla u|^2 \, dv_g \leq \frac{16\Lambda^4}{(n-1)\delta^2} \int_{M_1} |\nabla^2 u|^2 \, dv_g. \tag{3.6}$$

Assuming the validity of Propositions 3.3 and 3.4 for now, we proceed to give the proof of Theorem 1.1:

Proof of Theorem 1.1. By Simons' inequality $\int_\Sigma \|A\|^2(\|A\|^2 - n)\, dS_g \geq 0$ for minimal hypersurfaces in $\mathbb{S}^{n+1}$ [**14**], if $\Lambda < \sqrt{n}$ then $A \equiv 0$ and thus Σ is a totally geodesic n-sphere. In this case, it is well-known that $\lambda_1(\Sigma) = n$, and so (1.3) clearly holds. For the remainder of the proof, we may therefore assume that $\Lambda \geq \sqrt{n}$.

Substituting the estimate (3.6) of Proposition 3.4 back into the estimate (3.5) of Proposition 3.3, we obtain

$$\int_{M_1} |\nabla u|^2 \, dv_g \leq \left(\frac{32\Lambda^6}{(n-1)\delta^3\gamma} + \frac{1}{\beta\gamma} \right) \int_{M_1} |\nabla^2 u|^2 \, dv_g.$$

Therefore

$$\int_{M_1} |\nabla^2 u|^2 \, dv_g \geq \frac{a_n}{\Lambda^6 + b_n} \int_{M_1} |\nabla u|^2 \, dv_g$$

where

$$a_n = \frac{(n-1)\delta^3\gamma}{32} \quad \text{and} \quad b_n = \frac{(n-1)\delta^3}{32\beta}.$$

Now, since we assume $\varepsilon \leq \frac{\Lambda}{2}$ we have $\frac{\Lambda}{\Lambda - \varepsilon} \leq 2$, and since we assume $\Lambda \geq \sqrt{n}$ we have $\frac{n}{\Lambda^2} \leq 1$. Substituting these inequalities back into the definition of γ, we see

$$\gamma = \sqrt{2n} - \frac{\Lambda\varepsilon}{\Lambda - \varepsilon}\left(\frac{n}{\Lambda^2} + 1\right) - \beta \geq \sqrt{n}\left(\sqrt{2} - \frac{4\varepsilon}{\sqrt{n}} - \frac{\beta}{\sqrt{n}}\right).$$

Choosing $\beta = \frac{\sqrt{n}}{20}$ and $\varepsilon = \frac{\sqrt{n}}{3}$, we then see that $\gamma \geq \sqrt{n}(\sqrt{2} - \frac{4}{3} - \frac{1}{20}) \geq \frac{3\sqrt{n}}{100}$ and $\delta = n \arctan(\frac{\varepsilon}{n}) = n \arctan(\frac{1}{3\sqrt{n}})$. Therefore

$$a_n \geq \frac{3(n-1)n^{7/2}}{3200} \arctan^3\left(\frac{1}{3\sqrt{n}}\right) \quad \text{and} \quad b_n \leq \frac{5n^{7/2}}{8} \arctan^3\left(\frac{1}{3\sqrt{n}}\right).$$

As explained in Remark 1.2, this completes the proof of the theorem. □

The rest of the paper is devoted to the proofs of Propositions 3.3 and 3.4.

3.3. **Proof of Proposition 3.3.** To describe our setup for the proof of Proposition 3.3, let d be the signed distance to Σ in $\mathbb{S}^{n+1}$:

$$d(x) = \begin{cases} -\operatorname{dist}(x, \Sigma) & \text{if } x \in \overline{M_2} \\ \operatorname{dist}(x, \Sigma) & \text{if } x \in M_1. \end{cases}$$

As before, we equip the surfaces Σ^d with the orientation induced by Σ, i.e. the orientation given by the normal vector field ∇d on Σ^d. Then the mean curvature of Σ^d is given by $H_{\Sigma^d} = -\operatorname{div} \nabla d = -\Delta d$.

By Proposition 2.1, the parallel hypersurfaces Σ^d are smooth and embedded for

$$|d| \in \big[0, \arctan(\Lambda^{-1})\big). \tag{3.7}$$

However, to gain control on the mean curvature of the hypersurfaces parallel to Σ, in the proof of Proposition 3.3 we will need to work in a neighbourhood around Σ of thickness smaller than that determined by (3.7). To this end, for $0 < \varepsilon \leq \frac{\Lambda}{2}$ we define

$$D_\varepsilon = \arctan(\varepsilon\Lambda^{-2}).$$

Since $\frac{\varepsilon}{\Lambda} < 1$, clearly $D_\varepsilon < \arctan(\Lambda^{-1})$ and thus Σ^t is a smooth embedded hypersurface for $|t| \in [0, D_\varepsilon]$. Our first estimate towards the proof of Proposition 3.3 is an upper bound on the mean curvature of the hypersurfaces Σ^t parallel to Σ when $t \in [0, D_\varepsilon]$:

Lemma 3.5. *Let $0 < \varepsilon \leq \frac{\Lambda}{2}$. Then for $t \in [0, D_\varepsilon]$,*

$$H_{\Sigma^t} \leq \widetilde{\varepsilon} := \frac{\Lambda\varepsilon}{\Lambda - \varepsilon}\left(\frac{n}{\Lambda^2} + 1\right).$$

Proof. Summing over i in (2.1) and appealing to minimality of Σ, we see that for $t \in [0, D_\varepsilon]$ we have

$$H_{\Sigma^t} = \sum_{i=1}^n \left(\kappa_i(\cdot, 0) + \frac{(1 + \kappa_i(\cdot, 0)^2)\tan t}{1 - \kappa_i(\cdot, 0)\tan t}\right) = \sum_{i=1}^n \frac{(1 + \kappa_i(\cdot, 0)^2)\tan t}{1 - \kappa_i(\cdot, 0)\tan t}. \tag{3.8}$$

Now, by definition of D_ε, we have $1 - \kappa_i(\cdot, 0)\tan t \geq \frac{\Lambda - \varepsilon}{\Lambda}$ on $[0, D_\varepsilon]$ for each i, and it therefore follows from (3.8) that for $t \in [0, D_\varepsilon]$,

$$H_{\Sigma^t} \leq \frac{\Lambda}{\Lambda - \varepsilon}(n + \Lambda^2)\tan t = \frac{\Lambda}{\Lambda - \varepsilon}\left(\frac{n}{\Lambda^2} + 1\right)\Lambda^2 \tan t \leq \frac{\Lambda\varepsilon}{\Lambda - \varepsilon}\left(\frac{n}{\Lambda^2} + 1\right),$$

as claimed. □

We now use Lemma 3.5 to show:

Lemma 3.6. *Let $0 < \varepsilon \leq \frac{\Lambda}{2}$ and suppose v is a smooth function defined on $\overline{M_1}$. Then for $t \in [0, D_\varepsilon]$ and any $\beta > 0$,*

$$\int_\Sigma |\nabla v|^2 \, dS_g \leq \int_{\Sigma^t} |\nabla v|^2 \, dS_g + (\widetilde{\varepsilon} + \beta)\int_{M_1 \setminus M_1^t} |\nabla v|^2 \, dv_g + \beta^{-1}\int_{M_1 \setminus M_1^t} |\nabla^2 v|^2 \, dv_g. \tag{3.9}$$

Proof. Recall that if $x \in \mathbb{S}^{n+1}$ is a signed distance s from Σ, then $H_{\Sigma^s}(x) = -\Delta d(x)$. By Lemma 3.5, we therefore have

$$-\int_{M_1 \setminus M_1^t} |\nabla v|^2 \Delta d \, dv_g \leq \widetilde{\varepsilon}\int_{M_1 \setminus M_1^t} |\nabla v|^2 \, dv_g. \tag{3.10}$$

On the other hand, by the divergence theorem

$$-\int_{M_1 \setminus M_1^t} |\nabla v|^2 \Delta d \, dv_g = \int_{M_1 \setminus M_1^t} \langle \nabla d, \nabla |\nabla v|^2 \rangle \, dv_g - \int_{\Sigma \cup \Sigma^t} |\nabla v|^2 \langle \nabla d, \nu \rangle \, dS_g, \tag{3.11}$$

where ν is the outward pointing unit normal to the region $M_1 \backslash M_1^t$. By definition of d, we have $\langle \nabla d, \nu \rangle = -1$ on Σ and $\langle \nabla d, \nu \rangle = 1$ on Σ^t. Therefore, by (3.11), for any $\beta > 0$ we have

$$\begin{aligned} -\int_{M_1\backslash M_1^t} |\nabla v|^2 \Delta d \, dv_g &\geq -2\int_{M_1\backslash M_1^t} |\nabla v||\nabla^2 v| \, dv_g + \int_\Sigma |\nabla v|^2 \, dS_g - \int_{\Sigma^t} |\nabla v|^2 \, dS_g \\ &\geq -\beta \int_{M_1\backslash M_1^t} |\nabla v|^2 \, dv_g - \beta^{-1}\int_{M_1\backslash M_1^t} |\nabla^2 v|^2 \, dv_g + \int_\Sigma |\nabla v|^2 \, dS_g \\ &\quad - \int_{\Sigma^t} |\nabla v|^2 \, dS_g. \end{aligned} \tag{3.12}$$

Substituting (3.12) into (3.10) and rearranging, we arrive at (3.9). □

Whilst the desired estimate in Proposition 3.3 involves $\int_{M_1} |\nabla u|^2 \, dv_g$ on the LHS, the estimate in Lemma 3.6 (therein taking $v = u$) involves $\int_\Sigma |\nabla u|^2 \, dS_g$ on the LHS. These two quantities are related by the following lemma:

Lemma 3.7. *The solution u to* (3.1) *satisfies*

$$\int_\Sigma |\nabla u|^2 \, dS_g \geq \sqrt{2n}\int_{M_1} |\nabla u|^2 \, dv_g. \tag{3.13}$$

Proof. Integrating by parts, using $\Delta u = 0$ in M_1 and the fact that $\|u|_\Sigma\|_{L^2(\Sigma)} = \|\Psi\|_{L^2(\Sigma)} = 1$, we have

$$\left(\int_{M_1} |\nabla u|^2 \, dv_g\right)^2 = \left(\int_\Sigma u_\nu u \, dS_g\right)^2 \leq \int_\Sigma u_\nu^2 \, dS_g \int_\Sigma u^2 \, dS_g = \int_\Sigma u_\nu^2 \, dS_g. \tag{3.14}$$

On the other hand,

$$\int_\Sigma u_\nu^2 \, dS_g = \int_\Sigma |\nabla u|^2 \, dS_g - \int_\Sigma |\nabla^\Sigma u|^2 \, dS_g = \int_\Sigma |\nabla u|^2 \, dS_g - \lambda_1, \tag{3.15}$$

with the second identity in (3.15) following from the variational characterisation of λ_1 and the fact that $u|_\Sigma = \Psi$. Substituting (3.15) into (3.14) and applying Young's inequality, we obtain

$$\int_\Sigma |\nabla u|^2 \, dS_g \geq \lambda_1 + \left(\int_{M_1} |\nabla u|^2 \, dv_g\right)^2 \geq 2\lambda_1^{1/2}\int_{M_1} |\nabla u|^2 \, dv_g. \tag{3.16}$$

The desired estimate (3.13) then follows from (3.16) and the fact that $\lambda_1 \geq \frac{n}{2}$. □

We are now in a position to give the proof of Proposition 3.3:

Proof of Proposition 3.3. We first take $v = u$ in the estimate (3.9) of Lemma 3.6, where u is the solution to (3.1). Substituting (3.13) back into (3.9), we therefore arrive at

$$\underbrace{(\sqrt{2n} - \widetilde{\varepsilon} - \beta)}_{=:\gamma}\int_{M_1} |\nabla u|^2 \, dv_g \leq \int_{\Sigma^t} |\nabla u|^2 \, dS_g + \beta^{-1}\int_{M_1} |\nabla^2 u|^2 \, dv_g. \tag{3.17}$$

Now recall that we define $\delta = n\arctan(\frac{\varepsilon}{n})$. Noting that $\frac{\delta}{x^2} \leq \arctan(\frac{\varepsilon}{x^2})$ for $x \geq \sqrt{n}$, we see that $\frac{\delta}{\Lambda^2} \leq \arctan(\frac{\varepsilon}{\Lambda^2}) = D_\varepsilon$ for $\Lambda \geq \sqrt{n}$. In particular, we are justified in integrating both sides of (3.17) with respect to t over the interval $[T, 2T]$, where $T = \frac{\delta}{2\Lambda^2}$. This yields (3.5), completing the proof of Proposition 3.3. $\square$

3.4. **Proof of Proposition 3.4.** The proof of Proposition 3.4 is a consequence of two lemmas. The first of these is as follows:

Lemma 3.8. *Let $\Omega \subset \mathbb{S}^{n+1}$ be a domain and v a smooth function defined on Ω satisfying $\Delta v = 0$ in Ω. Then*

$$\Delta|\nabla v|^2 = 2|\nabla^2 v|^2 + 2n|\nabla v|^2 \quad \text{in } \Omega. \tag{3.18}$$

Proof. This is an immediate consequence of the Bochner formula

$$\Delta|\nabla w|^2 = 2\langle \nabla \Delta w, \nabla w\rangle + 2|\nabla^2 w|^2 + 2\operatorname{Ric}_g(\nabla w, \nabla w)$$

for a smooth function w defined on a Riemannian manifold (N, g), and the fact that $\operatorname{Ric}_g = ng$ on $\mathbb{S}^{n+1}$ equipped with the round metric g. $\square$

For a domain $\Omega \subset \mathbb{S}^{n+1}$ with smooth boundary, we denote by Ω^s the set of points in Ω whose distance to $\partial\Omega$ is greater than s. We now use Lemma 3.8 to show:

Lemma 3.9. *Let $\Omega \subset \mathbb{S}^{n+1}$ be a domain with smooth boundary $\partial\Omega$, and v a smooth function defined on Ω satisfying $\Delta v = 0$ in Ω. Suppose that $t > 0$ is sufficiently small so that $\partial(\Omega^{2t})$ is a smooth embedded hypersurface in $\mathbb{S}^{n+1}$. Then*

$$\int_{\Omega^{2t}} |\nabla v|^2 \, dv_g \leq \frac{1}{n-1} t^{-2} \int_\Omega |\nabla^2 v|^2 \, dv_g. \tag{3.19}$$

Proof. Let $\zeta \in C_c^\infty(\Omega)$ be a cutoff function whose properties will be specified later in the proof. Multiplying the inequality (3.18) by ζ^2 and integrating over Ω, we see

$$\begin{aligned}
\int_\Omega \zeta^2(n|\nabla v|^2 + |\nabla^2 v|^2)\, dv_g &= \frac{1}{2}\int_\Omega \zeta^2 \Delta |\nabla v|^2 \, dv_g \\
&= -\int_\Omega \zeta \langle \nabla\zeta, \nabla|\nabla v|^2\rangle \, dv_g \\
&= -2\int_\Omega \zeta \nabla^2 v(\nabla v, \nabla \zeta)\, dv_g \\
&\leq \int_\Omega \zeta^2 |\nabla v|^2 \, dv_g + \int_\Omega |\nabla\zeta|^2 |\nabla^2 v|^2 \, dv_g.
\end{aligned}$$

Therefore

$$\begin{aligned}
0 &\geq (n-1)\int_\Omega \zeta^2|\nabla v|^2\, dv_g + \int_\Omega (\zeta^2 - |\nabla\zeta|^2)|\nabla^2 v|^2 \, dv_g \\
&\geq (n-1)\int_\Omega \zeta^2 |\nabla v|^2 \, dv_g - \int_\Omega |\nabla \zeta|^2 |\nabla^2 v|^2 \, dv_g,
\end{aligned}$$

which yields

$$\int_\Omega \zeta^2 |\nabla v|^2 \, dv_g \leq \frac{1}{n-1}\int_\Omega |\nabla\zeta|^2 |\nabla^2 v|^2 \, dv_g. \tag{3.20}$$

Now, for each $\varepsilon > 0$, one can choose a smooth cutoff function ζ such that $\zeta \equiv 0$ in $\Omega \backslash \Omega^t$, $\zeta \equiv 1$ in Ω^{2t} and $|\nabla \zeta| \leq (1+\varepsilon)t^{-1}$. Thus (3.20) implies

$$\int_{\Omega^{2t}} |\nabla v|^2 \, dv_g \leq \frac{(1+\varepsilon)^2}{n-1} t^{-2} \int_{\Omega} |\nabla^2 v|^2 \, dv_g, \tag{3.21}$$

and the estimate (3.19) then follows after taking $\varepsilon \to 0$ in (3.21). □

Proof of Proposition 3.4. In the statement of Lemma 3.9, let $\Omega = M_1$ and let $v = u$, where u is the solution to (3.1). Following the reasoning given in the proof Proposition 3.3, we are then justified in taking $t = T/2$ in Lemma 3.9, where $T = \frac{\delta}{2\Lambda^2}$ as before. The desired estimate (3.6) then follows. □

Having established Propositions 3.3 and 3.4, the proof of Theorem 1.1 is complete, as explained in Section 3.2.

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