

RESEARCH ARTICLE

Global existence of weak solutions to the two-dimensional nematic liquid crystal flow with partially free boundary

Yannick Sire¹ | Yantao Wu¹ | Yifu Zhou²

¹Department of Mathematics, Johns Hopkins University, Baltimore, Maryland, USA

²School of Mathematics and Statistics, Wuhan University, Wuhan, China

Correspondence

Yannick Sire, Department of Mathematics, Johns Hopkins University, 3400 N. Charles Street, Baltimore, MD 21218, USA.
Email: ysire1@jhu.edu

Funding information

NSF, Grant/Award Number: DMS Grant 2154219; Fundamental Research Funds for the Central Universities

Abstract

We consider a nematic liquid crystal flow with partially free boundary in a smooth bounded domain in $\mathbb{R}^2$. We prove regularity estimates and the global existence of weak solutions enjoying partial regularity properties, and a uniqueness result.

MSC 2020

35A01 (primary), 35A02, 35A21

Contents

1. INTRODUCTION AND MAIN RESULTS	2
2. NOTATIONS AND PRELIMINARIES	7
3. HÖLDER REGULARITY OF SOLUTION	13
Proof of Theorem 1.2	19
4. EXISTENCE OF SHORT-TIME SMOOTH SOLUTIONS	19
5. ENERGY ESTIMATION	23
6. GLOBAL WEAK SOLUTION AND PROOF OF THEOREM 1.3	25
Proof of Theorem 1.3	28
7. ETERNAL BEHAVIOR: PROOF OF THEOREMS 1.5 AND 1.6	37
APPENDIX A: BOUNDARY CONDITION AND THE BASIC ENERGY LAW	39
ACKNOWLEDGMENTS	41
REFERENCES	41

1 | INTRODUCTION AND MAIN RESULTS

We consider the following hydrodynamic system modeling the flow of liquid crystal materials in dimension 2:

$$\begin{cases} u_t + u \cdot \nabla u - \nu \Delta u + \nabla P = -\lambda \nabla \cdot (\nabla d \odot \nabla d - \frac{1}{2} |\nabla d|^2 \mathbb{I}_2) \\ \nabla \cdot u = 0 \\ d_t + u \cdot \nabla d = \gamma (\Delta d + |\nabla d|^2 d) \end{cases} \quad \text{in } \Omega \times (0, T), \quad (1.1)$$

where domain $\Omega \subseteq \mathbb{R}^2$ is assumed to be a connected bounded domain with boundary $\Gamma := \partial\Omega$ of class C^∞ . Function $u(x, t) : \Omega \times (0, +\infty) \rightarrow \mathbb{R}^2$ represents the velocity field of the flow, $d(x, t) : \Omega \times (0, +\infty) \rightarrow \mathbb{S}^2 \subseteq \mathbb{R}^3$ is a unit vector that represents the director field of macroscopic molecular orientation of the liquid crystal material, and $P(x, t) : \Omega \times (0, +\infty) \rightarrow \mathbb{R}$ represents the pressure function. Three positive constants ν , λ , and γ , respectively, quantify the viscosity, the competition between kinetic energy and elastic energy, and the microscopic elastic relaxation time for the director field. Here, we assume that $\nu = \lambda = \gamma = 1$ since the exact values of these constants play no role in our qualitative results. $\nabla \cdot$ denotes the divergence operator, and $\nabla d \odot \nabla d$ denotes the 2×2 matrix whose (i, j) th entry is given by $\partial_{x_i} d \cdot \partial_{x_j} d$ for $i, j \in \{1, 2\}$.

The system (1.1) was first proposed by Lin in [40], and it is a simplified version of the Ericksen–Leslie system [14, 31]. In fact, these models both share the same type of energy law, coupling structure, and dissipative properties. There has been growing interest concerning the global existence of weak solutions, partial regularity results, singularity formation, and others. Lin and Liu [41] established the global existence of weak and classical solutions in two and three dimensions. A partial regularity result of Caffarelli–Kohn–Nirenberg type ([9]) for suitable weak solutions was shown in [42]. In two dimensions, the global existence of Leray–Hopf–Struwe-type weak solutions of (1.1) was proved in Lin–Lin–Wang [34], and the uniqueness of such weak solution was later shown in [37]. See also [30] for the construction of finite-time singularities. Much less is known in three dimensions due to the supercriticality. In [39], Lin and Wang proved the global existence of suitable weak solutions that satisfy the global energy inequality, under the assumption that the initial orientation field $d_0(\Omega) \subset \mathbb{S}_+^2$. There are also blow-up examples and criteria for finite-time singularities, for instance, [21, 23]. We refer to a comprehensive survey by Lin and Wang [38] for recent vital developments in the mathematical analysis of nematic liquid crystals.

In this paper, our concern is a free boundary model of (1.1) recently introduced in [35]. We consider the system (1.1) with partially free boundary conditions

$$\begin{cases} u \cdot \hat{n} = 0 \\ (\mathbb{D}u \cdot \hat{n})_\tau = 0 \\ d(x, t) \in \Sigma \\ \nabla_{\hat{n}} d(x, t) \perp T_{d(x, t)} \Sigma \end{cases} \quad \text{on } \partial\Omega \times (0, T),$$

where $\hat{n}$ and τ are the unit outer normal vector and tangential vector of $\partial\Omega$, and $\mathbb{D}u$ is deformation tensor associated with the velocity field u

$$\mathbb{D}u = \frac{1}{2}(\nabla u + (\nabla u)^T),$$

and $\Sigma \subset \mathbb{S}^2$ is a simple, closed, and smooth curve. The case that Σ is a circle in $\mathbb{S}^2$ is physically relevant. The first two boundary conditions are the usual *Navier perfect-slip boundary conditions* for the velocity field, indicating a zero friction along the boundary, and the last two are referred to as *partially free boundary conditions* for the map. This boundary condition is physically natural in that it agrees with *the basic energy law* (A.1). Let us assume that Σ is the equator for simplicity, then the partially free boundary condition can be simplified to be

$$\begin{cases} (\mathbb{D}u \cdot \hat{n})_\tau = u \cdot \hat{n} = 0 \\ \nabla_{\hat{n}} d_1 = \nabla_{\hat{n}} d_2 = d_3 = 0 \end{cases} \quad \text{on } \partial\Omega \times (0, T). \quad (1.2)$$

The free boundary system (1.1)–(1.2) is a nonlinearly coupled system between the incompressible Navier–Stokes equations and the harmonic map heat flow with a partially free boundary condition. The latter is a geometric flow with the Plateau and Neumann boundary conditions. The motivation for studying this model stems from a recent surge of interest in geometric variational problems and the Navier–Stokes equation with Navier boundary condition. The former includes an interesting and classical topic of the harmonic map heat flow with free boundary. We refer to Hamilton [19], Struwe [56], Chen–Lin [11], Ma [43], Sire–Wei–Zheng [52], Hyder–Segatti–Sire–Wang [24], and the references therein for both seminal and more recent results. On the other hand, the Navier–Stokes equation with Navier boundary condition is more physical in some contexts. See, for instance, [1, 4–6, 10, 12, 26, 32, 44, 57] and their references. As derived and discussed in [35], the system (1.1) turns out to be physically natural and compatible with the free boundary condition (1.2) imposed, and it enjoys the same dissipative properties as those for the case of Dirichlet boundary. Moreover, (1.1)–(1.2) may trigger new boundary behaviors of solutions, such as the finite-time blow-up via bubbling on the boundary (and in the interior) constructed in [35]. This new boundary behavior was already observed by Chen and Lin [11] in the context of harmonic map heat flow with free boundary.

In this paper, we are interested in the global existence of weak solutions, partial regularity, and uniqueness of (1.1)–(1.2). The well-posedness under consideration is motivated by the interesting work of Lin, Lin, and Wang [34], which considers the no-slip boundary condition for velocity field v and Dirichlet boundary condition for director field d .

Let us first define

$$\begin{aligned} \mathbf{L} &= \{u \in L^2(\Omega, \mathbb{R}^2) : \nabla \cdot u = 0 \text{ in } \Omega \text{ and } v \text{ satisfies (1.2)}_1\}, \\ \mathbf{H} &= \{u \in H^1(\Omega, \mathbb{R}^2) : \nabla \cdot u = 0 \text{ in } \Omega \text{ and } v \text{ satisfies (1.2)}_1\}, \\ \mathbf{J} &= \{d \in H^1(\Omega, \mathbb{R}^3) : d \in \mathbb{S}^2 \text{ a.e. in } \Omega \text{ and } d \text{ satisfies (1.2)}_2\}. \end{aligned}$$

We assume that the initial data

$$(u(x, 0), d(x, 0)) = (u_0(x), d_0(x)), \quad x \in \Omega \quad (1.3)$$

for (1.1)–(1.2) satisfy

$$u_0 \in \mathbf{L} \text{ and } d_0 \in \mathbf{J}. \quad (1.4)$$

Since $\Delta u = \nabla \cdot (\nabla u + (\nabla u)^T) = 2\nabla \cdot \mathbb{D}u$ and partially free boundary condition gives $\hat{n}^T(\mathbb{D}u)\tau = \hat{n}(\nabla d \odot \nabla d)\tau = 0$, the following weak formulation can help us get rid of boundary terms.

Definition 1.1 (Weak solution). For $0 < T \leq +\infty$, $u \in L^\infty([0, T], \mathbf{L}) \cap L^2([0, T], \mathbf{H})$ and $d \in L^2([0, T], \mathbf{J})$ is a weak solution of (1.1)–(1.3), if

$$\begin{aligned} & - \int_{\Omega \times [0, T]} \langle u, \psi' \varphi \rangle + \int_{\Omega \times [0, T]} [\langle u \cdot \nabla u, \psi \varphi \rangle + \langle 2\mathbb{D}u, \psi \nabla \varphi \rangle] \\ & = -\psi(0) \int_{\Omega} \langle u_0, \varphi \rangle + \int_{\Omega \times [0, T]} \langle \nabla d \odot \nabla d - \frac{1}{2} |\nabla d|^2 \mathbb{I}_2, \psi \nabla \varphi \rangle, \\ & - \int_{\Omega \times [0, T]} \langle d, \psi' \phi \rangle + \int_{\Omega \times [0, T]} [\langle u \cdot \nabla d, \psi \phi \rangle + \langle \nabla d, \psi \nabla \phi \rangle] \\ & = -\psi(0) \int_{\Omega} \langle d_0, \phi \rangle + \int_{\Omega \times [0, T]} |\nabla d|^2 \langle d, \psi \phi \rangle \end{aligned}$$

for any $\psi \in C^\infty([0, T])$ with $\psi(T) = 0$, $\varphi \in H^1(\Omega, \mathbb{R}^2)$ with $\varphi \cdot \hat{n} = 0$ on $\partial\Omega$, and $\phi \in H^1(\Omega, \mathbb{R}^3)$ with $\phi_3 = 0$ on $\partial\Omega$. Moreover, (u, d) satisfies (1.4) in the sense of trace.

Our first regularity theorem for the weak solution is stated as follows.

Theorem 1.2. For $0 < T < +\infty$, assume $u \in L^\infty([0, T], \mathbf{L}) \cap L^2([0, T], \mathbf{H})$ and $d \in L^2([0, T], \mathbf{J})$ is a weak solution of (1.1)–(1.3) with initial data satisfying (1.4). If d further belongs to $L^2([0, T], H^2(\Omega))$, then $(u, d) \in C^\infty(\Omega \times (0, T]) \cap C_{\alpha}^{2,1}(\overline{\Omega} \times (0, T])$ for some $\alpha \in (0, 1)$.

Our second theorem concerns the existence of global weak solutions that enjoy the partial smoothness property and a result of uniqueness.

Theorem 1.3. There exists a global weak solution $u \in L^\infty([0, \infty), \mathbf{L}) \cap L^2([0, +\infty), \mathbf{H})$ and $d \in L^\infty([0, \infty), H^1(\Omega, \mathbb{S}^2))$ of (1.1)–(1.3) with (1.4), such that the following properties hold.

(1) There exists $L \in \mathbb{N}$ depending only on (u_0, d_0) and $0 < T_1 < \dots < T_L$, $1 \leq i \leq L$, such that

$$(u, d) \in C^\infty(\Omega \times ((0, \infty) \setminus \{T_i\}_{i=1}^L)) \cap C_{\alpha}^{2,1}(\overline{\Omega} \times ((0, +\infty) \setminus \{T_i\}_{i=1}^L)).$$

(2) Global weak solution (u, d) is unique in the class of functions that

$$d \in L^\infty([0, \infty), H^1(\Omega)) \bigcap_{i=0}^{K-1} \bigcap_{\epsilon > 0} L^2([T_i, T_{i+1} - \epsilon], H^2(\Omega)) \cap L^2([T_K, +\infty), H^2(\Omega))$$

for some $0 < T_1 < \dots < T_K < +\infty$.

(3) Each singular time T_i ($1 \leq i \leq L$) can be characterized by

$$\liminf_{t \nearrow T_i} \max_{x \in \overline{\Omega}} \int_{\Omega \cap B_r(x)} (|u|^2 + |\nabla d|^2)(y, t) dy \geq 4\pi, \quad \forall r > 0. \quad (1.5)$$

Moreover, there exist $x_m^i \rightarrow x_0^i \in \overline{\Omega}$, $t_m^i \nearrow T_i$, $r_m^i \searrow 0$ such that

- (a) if $x_0^i \in \Omega$ or $x_0^i \in \partial\Omega$ with $\lim_{m \rightarrow \infty} \frac{|x_m^i - x_0^i|}{r_m} = \infty$, then there exists a nonconstant smooth harmonic map $\omega_i : \mathbb{R}^2 \rightarrow \mathbb{S}^2$ with finite energy, such that as $m \rightarrow \infty$,

$$(u_m^i, d_m^i) \rightarrow (0, \omega_i) \text{ in } C_{\text{loc}}^2(\mathbb{R}^2 \times [-\infty, 0]);$$

- (b) if $x_0^i \in \partial\Omega$ and $\lim_{m \rightarrow \infty} \frac{|x_m^i - x_0^i|}{r_m} < \infty$, then there exists a nonconstant smooth harmonic map $\omega'_i : \mathbb{R}_a^2 \rightarrow \mathbb{S}^2$ with finite energy for some half plane $\mathbb{R}_a^2$, such that as $m \rightarrow \infty$,

$$(u_m^i, d_m^i) \rightarrow (0, \omega'_i) \text{ in } C_{\text{loc}}^2(\mathbb{R}_a^2 \times [-\infty, 0]),$$

where

$$u_m^i(x, t) = r_m^i u(x_m^i + r_m^i x, t_m^i + (r_m^i)^2 t), \quad d_m^i(x, t) = d(x_m^i + r_m^i x, t_m^i + (r_m^i)^2 t).$$

- (4) Set $T_0 = 0$. Then, for $0 \leq i \leq L-1$,

$$|d_t| + |\nabla^2 d| \in L^2(\Omega \times [T_i, T_{i+1} - \epsilon]), \quad |u_t| + |\nabla^2 u| \in L^{\frac{4}{3}}(\Omega \times [T_i, T_{i+1} - \epsilon])$$

for any $\epsilon > 0$, and for any $0 < T_L < T < +\infty$,

$$|d_t| + |\nabla^2 d| \in L^2(\Omega \times [T_L, T]), \quad |u_t| + |\nabla^2 u| \in L^{\frac{4}{3}}(\Omega \times [T_L, T]).$$

Remark 1.4.

- (1) Theorem 1.3 is established by using Theorem 1.2, the global and local energy inequalities, and estimates of the pressure function P in Section 3 below.
- (2) Compared to [34], the interior estimates in the proof are similar. However, extra care shall be taken for the estimates near the boundary.
- (3) The global existence of weak solutions for the nematic liquid crystal flow in three dimensions is a notoriously hard problem. In the case of the Dirichlet boundary, it was solved by Lin and Wang [39] with the restriction $d_0(\Omega) \subset \mathbb{S}_+^2$. The significant difficulties stem from the energy supercritical harmonic map heat flow and the three-dimensional Navier–Stokes equation with a supercritical forcing term. The situation might be better for the free boundary system (1.1)–(1.2) since the vorticity $\omega = \nabla \times u$ may have better estimates with the Navier boundary condition.

When we consider the eternal behavior of solutions to (1.1) with (1.2) as $t \rightarrow +\infty$, it turns out that there are two distinct situations depending on whether the domain is axisymmetric or not. To clarify the definition of axisymmetry, here we adopt the convention from Desvillettes and Villani [13]: A domain in $\mathbb{R}^2$ is axisymmetric if it has a circular symmetry around some point; a domain in $\mathbb{R}^3$ is axisymmetric if it admits an axis of symmetry (which means that it is preserved by a rotation of arbitrary angle around this axis).

In particular, because we only consider open, bounded, and connected domains in $\mathbb{R}^2$, throughout this paper, we mean an axisymmetric domain by either a disk or an annulus, and a nonaxisymmetric domain by any other open, bounded, and connected domain. We use this

axisymmetry and nonaxisymmetry in Korn's inequality 2.12 to study whether a domain permits a nontrivial fluid flow in the kernel of the symmetrized gradient $\mathbb{D}$. For nonaxisymmetric domains, one can utilize the stronger Korn's inequality in Lemma 2.12 to show that the velocity field finally decays to $u_\infty \equiv 0$. For axisymmetric domains, (1.1)–(1.2), in fact, permit a stationary and circular velocity field. See [17] for related discussions in three dimensions. It remains a problem what properties of initial data (u_0, d_0) can explain the behavior of eternal weak limit (u_∞, d_∞) .

Theorem 1.5 (Eternal behavior).

- (1) *There exist $t_k \nearrow +\infty$, $u_\infty \in H^1(\Omega)$ with $\mathbb{D}u_\infty = 0$, and a harmonic map $d_\infty \in C^\infty(\Omega, \mathbb{S}^2) \cap C^{2,\alpha}(\overline{\Omega}, \mathbb{S}^2)$ with d_∞ satisfies (1.2)₂ on $\partial\Omega$ such that $u(\cdot, t_k) \rightharpoonup u_\infty$ weakly in $H^1(\Omega)$, $d(\cdot, t_k) \rightharpoonup d_\infty$ weakly in $H^1(\Omega)$, and there exist $l, l' \in \mathbb{N}$, points $\{x_i\}_{i=1}^l \subseteq \Omega$, $\{y_i\}_{i=1}^{l'} \subseteq \partial\Omega$ and $\{m_i\}_{i=1}^l, \{m'_i\}_{i=1}^{l'} \subseteq \mathbb{N}$ such that*

$$|\nabla d(\cdot, t_k)|^2 dx \rightharpoonup |\nabla d_\infty|^2 dx + \sum_{i=1}^l 8\pi m_i \delta_{x_i} + \sum_{i=1}^{l'} 4\pi m'_i \delta_{y_i} \text{ in Radon measure.}$$

Moreover, if Ω is nonaxisymmetric, we can further conclude that $u_\infty \equiv 0$.

- (2) *Suppose that (u_0, d_0) satisfies*

$$\int_{\Omega} |u_0|^2 + |\nabla d_0|^2 \leq 4\pi,$$

then $(u, d) \in C^\infty(\Omega, (0, +\infty)) \cap C^{2,1}_\alpha(\overline{\Omega} \times (0, +\infty))$. Moreover, there exist $t_k \nearrow +\infty$, $u_\infty \in H^1(\Omega)$ with $\mathbb{D}u_\infty = 0$, and a harmonic map $d_\infty \in C^\infty(\Omega, \mathbb{S}^2) \cap C^{2,\alpha}(\overline{\Omega}, \mathbb{S}^2)$ with d_∞ satisfies (1.2)₂ on $\partial\Omega$ such that $(u(\cdot, t_k), d(\cdot, t_k)) \rightarrow (u_\infty, d_\infty)$ in $C^2(\overline{\Omega})$.

Moreover, if Ω is nonaxisymmetric, we can further conclude that $u_\infty \equiv 0$.

As a complement of the above theorem, we further describe the eternal weak solution (u_∞, d_∞) in the situation of axisymmetric domains.

Theorem 1.6 (Potential profiles of eternal weak solution in axisymmetric domain). *Suppose that the domain Ω is axisymmetric. There exist $t_k \nearrow +\infty$, a nontrivial u_∞ and a harmonic map $d_\infty \in C^\infty(\Omega, \mathbb{S}^2) \cap C^{2,\alpha}(\overline{\Omega}, \mathbb{S}^2)$ with d_∞ satisfies (1.2)₂ on $\partial\Omega$ such that $(u(\cdot, t_k), d(\cdot, t_k)) \rightharpoonup (u_\infty, d_\infty)$ weakly in $H^1(\Omega)$. Moreover, (u_∞, d_∞) can be classified in the following two cases.*

- (1) *If Ω is a disk B_r , then*

$$\begin{cases} u_\infty = c(x_2, -x_1), & c \neq 0 \\ d_\infty \equiv (K_1, K_2, K_3), & K_1^2 + K_2^2 + K_3^2 = 1 \end{cases} \quad \text{or} \quad \begin{cases} u_\infty \equiv 0 \\ d_\infty \text{ is a harmonic map.} \end{cases} \quad (1.6)$$

- (2) *If Ω is an annulus $B_{r_2} \setminus B_{r_1}$, then*

$$\begin{cases} u_\infty = c(x_2, -x_1) \\ d_1 = K_1 \cos\left(\alpha \ln\left(\frac{r}{r_1}\right)\right) \\ d_2 = K_2 \cos\left(\alpha \ln\left(\frac{r}{r_1}\right)\right) \\ d_3 = K_3 \sin\left(\alpha \ln\left(\frac{r}{r_1}\right)\right) \end{cases} \quad \text{or} \quad \begin{cases} u_\infty \equiv 0 \\ d_\infty \text{ is a harmonic map,} \end{cases} \quad (1.7)$$

where

$$|K_3| = 1, \quad K_1^2 + K_2^2 = 1, \quad \alpha \ln \left(\frac{r_2}{r_1} \right) = k\pi, \quad k \in \mathbb{Z}, \quad c \in \mathbb{R}. \quad (1.8)$$

Remark 1.7.

- (1) We emphasize that such nontrivial weak limit for u_∞ is a new feature triggered by the partially free boundary condition (1.2), which is not possible in the case of Dirichlet boundary.
- (2) The stability of (u_∞, d_∞) in this free boundary model might be an interesting and challenging problem.

The rest of this paper is devoted to the proof of Theorem 1.2, Theorem 1.3, Theorem 1.5, and Theorem 1.6.

2 | NOTATIONS AND PRELIMINARIES

This section introduces some notations and estimates that will be used throughout this paper.

We use $A \lesssim B$ to denote $A \leq CB$ for some universal constant $C > 0$. For $x_0 \in \mathbb{R}^2$, $t_0 \in \mathbb{R}$, $z_0 = (x_0, t_0)$, denote

$$B_r(x_0) = \{x \in \mathbb{R}^2 : |x - x_0| \leq r\},$$

$$P_r(z_0) = B_r(x_0) \times [t_0 - r^2, t_0]$$

to be spatial neighborhood and parabolic cylinder, respectively. We use Ω_t to denote $\Omega \times [0, t]$. For $x_0 \in \partial\Omega$, we use $Q_r(z_0)^+$ to denote $P_r(z_0) \cap \Omega_t$ to denote the parabolic cylinder at boundary. Denote the boundary of parabolic cylinder $\partial_p P_r(z_0)$ to be

$$\partial_p P_r(z_0) = (B_r(x_0) \times \{t_0 - r^2\}) \cup (\partial B_r(x_0) \times [t_0 - r^2, t_0]).$$

For $1 < p, q < \infty$, denote $L^{p,q}(P_r(z_0)) = L^q([t_0 - r^2, t_0], L^p(B_r(x_0)))$ with norm

$$\|f\|_{L^{p,q}(P_r(z_0))} = \left(\int_{t_0 - r^2}^{t_0} \|f(\cdot, t)\|_{L^p(B_r(x_0))}^q dt \right)^{\frac{1}{q}}.$$

Further, denote $W_{p,q}^{1,0}(P_r(x_0)) = L^q([t_0 - r^2, t_0], W^{1,p}(B_r(x_0)))$, with norm

$$\|f\|_{W_{p,q}^{1,0}(P_r(x_0))} = \|f\|_{L^{p,q}(P_r(z_0))} + \|\nabla f\|_{L^{p,q}(P_r(z_0))}.$$

Denote $W_{p,q}^{2,1}(P_r(x_0)) = \{f \in W_{p,q}^{1,0}(P_r(x_0)) : \nabla^2 f, \partial_t f \in L^{p,q}(P_r(x_0))\}$, with norm

$$\|f\|_{W_{p,q}^{2,1}(P_r(x_0))} = \|f\|_{W_{p,q}^{1,0}(P_r(x_0))} + \|\nabla^2 f\|_{W_{p,q}^{1,0}(P_r(x_0))} + \|\partial_t f\|_{W_{p,q}^{1,0}(P_r(x_0))}.$$

If $p = q$, then the above notation can be simplified as $L^{p,p} = L^p$, $W_{p,p}^{1,0} = W_p^{1,0}$, $W_{p,p}^{2,1} = W_p^{2,1}$.

Here are some techniques that we are going to utilize.

Lemma 2.1 (Gradient estimates for the heat equation). *If $d : \Omega \rightarrow \mathbb{R}^d$ solves heat equation $\partial_t d - \Delta d = 0$ with Dirichlet boundary condition ($d = 0$ on $\partial\Omega$) or Neumann boundary condition ($\hat{n} \cdot \nabla d = 0$ on $\partial\Omega$), then we have the following integral estimates for the gradient ∇d :*

$$\begin{aligned} \text{interior: } \int_{P_{\theta R}(z_1)} |\nabla d|^4 &\lesssim \theta^4 \int_{P_R(z_1)} |\nabla d|^4, \\ \text{boundary: } \int_{P_{\theta R}^+(z_1)} |\nabla d|^4 &\lesssim \theta^4 \int_{P_R^+(z_1)} |\nabla d|^4. \end{aligned}$$

Proof. See [33, Lemma 4.5, Lemma 4.13, Lemma 4.20, Theorem 7.35]. $\square$

Lemma 2.2 (Parabolic Morrey's decay lemma). *Suppose for any $z \in \bar{\Omega}$ and any $0 < r < \min(\text{diam}(\Omega), \sqrt{T})$, we have*

$$\int_{P_r(z) \cap \Omega_T} |\nabla d|^p + r^p |d_t|^p \lesssim r^{n+2+(\alpha-1)p},$$

then $d \in C^\alpha(P_{\frac{1}{2}}(z_1) \cap \Omega_T)$.

Proof. We use the definition of Campanato space and apply Poincaré inequality

$$r^{-(n+2+\alpha p)} \int_{P_r(z) \cap \Omega_T} |d - d_{z,r}|^p \lesssim r^{-(n+2+\alpha p)} \int_{P_r(z) \cap \Omega_T} r^p |\nabla d|^p + r^{2p} |d_t|^p \lesssim 1. \quad \square$$

Lemma 2.3 (A variant of Ladyzhenskaya's inequality). *There exists C_0 and R_0 depending only on Ω such that for any $T > 0$, if $u \in L^{2,\infty}(\Omega_T) \cap W_2^{1,0}(\Omega_T)$, then for $R \in (0, R_0)$,*

$$\int_{\Omega_T} |u|^4 \leq C_0 \sup_{(x,t) \in \bar{\Omega}_T} \int_{\Omega \cap B_R(x)} |u|^2(\cdot, t) \left(\int_{\Omega_T} |\nabla u|^2 + \frac{1}{R^2} \int_{\Omega_T} |u|^2 \right).$$

Proof. See [54, Lemma 3.1]. $\square$

Lemma 2.4 (Refined embedding theorem). *For $u \in W_0^{1,p}(\Omega)$ with $1 \leq p < n$ and $1 \leq r \leq p^* = \frac{np}{n-p}$, we have for any $q \in [r, p^*]$,*

$$\|u\|_{L^q(\Omega)} \lesssim \|\nabla u\|_{L^p(\Omega)}^\alpha \|u\|_{L^r(\Omega)}^{1-\alpha},$$

where $\alpha = (\frac{1}{r} - \frac{1}{q})(\frac{1}{r} - \frac{1}{p^})^{-1}$.*

Lemma 2.5 ($L^p - L^q$ regularity for Neumann heat equation). *Let $1 \leq p \leq q \leq \infty$, ($q \neq 1, p \neq \infty$). Let $(e^{t\Delta})_{t \geq 0}$ be the Neumann heat semigroup in Ω , and let $\lambda_1 > 0$ denote the first nonzero eigenvalue of $-\Delta$ in Ω under Neumann boundary conditions. Then, there exist constant $C(\Omega, t_0)$ such that*

$$\begin{aligned} \|e^{t\Delta} f\|_{L^q(\Omega)} &\leq C(\Omega, t_0) t^{-\frac{n}{2}(\frac{1}{p} - \frac{1}{q})} e^{-\lambda_1 t} \|f\|_{L^p(\Omega)} \\ \|\nabla e^{t\Delta} f\|_{L^q(\Omega)} &\leq C(\Omega, t_0) t^{-\frac{1}{2} - \frac{n}{2}(\frac{1}{p} - \frac{1}{q})} e^{-\lambda_1 t} \|f\|_{L^p(\Omega)} \end{aligned}$$

for $t \leq t_0$.

Proof. See [58, Lemma 1.3]. $\square$

Lemma 2.6 ($L^p - L^q$ regularity for Stokes operator with Navier boundary condition). *Let $\Omega \subseteq \mathbb{R}^n$, $n \geq 2$ be a bounded C^3 -smooth domain. Let $1 < p \leq q < \infty$, $\delta > 0$, let $\mathbb{P}_p : L^p(\Omega) \rightarrow L^p_\sigma(\Omega)$ denote the Helmholtz projection, and let $\mathbb{A}_p = \mathbb{P}_p \Delta$ be the Stokes operator with Navier boundary condition. Suppose either one of the following two conditions holds.*

- (1) *If $p < \frac{n}{2}$ and $p \leq q \leq \frac{np}{n-2p}$ where $0 \leq \gamma = \frac{n}{2}(\frac{1}{p} - \frac{1}{q}) \leq 1$.*
- (2) *If $p \geq \frac{n}{2}$ and $p \leq q$ where $1 \geq \gamma \geq 1 - \frac{p}{q} \geq 0$,*

then

$$\|e^{-t\mathbb{A}_p} u\|_{L^q(\Omega)} \leq C(\delta, p, \Omega) \left(\frac{t+1}{t} \right)^\gamma e^{\delta t} \|u\|_{L^p(\Omega)}$$

for $u \in L^p_\sigma(\Omega)$.

Proof. See [15, Corollary 1.4]. Based on this inequality, one can also utilize interpolation of Sobolev space to derive

$$\|e^{-t\mathbb{A}_p} \nabla u\|_{L^q(\Omega)} \leq C(\delta, p, \Omega, t_0) t^{-\frac{1}{2}-\gamma} \|u\|_{L^p(\Omega)}$$

for $t \leq t_0$.

□

Remark 2.7.

- In the setup of the Stokes operator with zero Dirichlet boundary condition, the $L^p - L^q$ regularity is in the form of Lemma 2.5. This result is proved by showing that the resolvent operator $R(\lambda, \mathbb{A}_p) = (\lambda I + \mathbb{A}_p)^{-1}$ is sectorial, and then study the fractional and purely imaginary power of $\mathbb{A}_p$, and finally conclude the $L^p - L^q$ regularity by using Komatsu semigroup decaying inequality (cf. [27, Theorem 12.1]). In dimension 3, such scheme also works in the setup of Stokes operator with Navier-type boundary condition (i.e., $u \cdot \hat{n} = \text{curl } u \times \hat{n} = 0$, see [2]), as well as in the setup Navier slip boundary condition (i.e., $2[(\mathbb{D}u)n]_\tau + \alpha u \cdot \tau = 0$, see [3]). In the latter situation, either a nontrivial friction α is required, or the nonaxisymmetric property of the domain Ω is required. This requirement suggests that in the current setup of two-dimensional Navier perfect-slip boundary condition, generally, we have no $\mathcal{H}^\infty$ -calculus (consider the counterexample of stationary vortex flow in disk). Indeed, Lemma 2.6 has no long-time decaying property, and its proof in [15] is based on equivalent norms on $D(\mathbb{A}_q)$ and interpolation of Sobolev space, without using fractional semigroup.
- See also [28, Theorem 3.10] for related semigroup estimates.

Lemma 2.8 (Parabolic Sobolev embedding theorem). *We have continuous embedding $W_q^{2l,l}(Q_T) \subseteq W_p^{s,r}(Q_T)$ if $2l - 2r - s - (\frac{1}{q} - \frac{1}{p})(n+2) \geq 0$.*

Proof. See [29, Lemma 3.3].

□

Lemma 2.9 (Boundary $W_{p,q}^{2,1}$ -estimate for Stokes equation). *For a homogeneous, nonstationary Stokes equation $\partial_t u - \Delta u + \nabla P = 0$ in Ω with Navier boundary condition on $\partial\Omega$. For $s \leq p$ and*

arbitrary $p_0(t)$, we have

$$\|u\|_{W_{p,q}^{2,1}(Q_1^+)} + \|\nabla P\|_{L^{p,q}(Q_1^+)} \lesssim \|\nabla u\|_{L^{s,q}(Q_1^+)} + \|P - p_0\|_{L^{s,q}(Q_1^+)}.$$

Proof. For a homogeneous, nonstationary Stokes equation with zero Dirichlet boundary condition, this boundary $W_{p,q}^{2,1}$ -estimate has been proved in [48, Lemma 3.2], using the property of maximal $L^p - L^q$ regularity for Stokes system with nonzero divergence. Such property is intensively studied by Shibata–Shimizu in [50, 51] and [49]. For maximal $L^p - L^q$ regularity in the setting of compressible fluid, see [25]. However, in the setting of the Stokes system with Navier boundary condition and nonzero divergence, such a maximal regularity is false because there is no control of the Neumann term as in [50]. Instead, we follow Seregin's argument in [47] and use maximal $L^p - L^q$ regularity for the Stokes system with Navier boundary and zero divergence in [16].

Let $v_1(x, t) = \phi(x)u(x, t)$ and $P_1 = \phi(x)(P(x, t) - p_0(t))$ for some cutoff function $\phi \equiv 1$ on $B_{\frac{1}{2}}^+$, $\text{supp } \phi \subseteq B_1^+$, and $\hat{n} \cdot \nabla \phi = 0$ on $\partial\Omega$. Then, (v_1, P_1) solves

$$\begin{cases} \partial_t v_1 - \Delta v_1 + \nabla P_1 = g_1 & \text{in } \Omega \\ \nabla \cdot v_1 = u \cdot \nabla \phi & \text{in } \Omega \\ v_1 \text{ satisfies Navier slip boundary condition (1.2)}_1 & \text{on } \partial\Omega, \end{cases}$$

where $g_1 = -2\nabla u \nabla \phi - u \Delta \phi + (P - p_0) \nabla \phi$.

Further, we let $(v_2(x), P_2(x))$ solves the following stationary Stokes system:

$$\begin{cases} -\Delta v_2 + \nabla P_2 = 0 & \text{in } \Omega \\ \nabla \cdot v_2 = u \cdot \nabla \phi & \text{in } \Omega \\ v_2 \text{ satisfies (1.2)}_1 & \text{on } \partial\Omega, \end{cases}$$

and $(\partial_t v_2, \partial_t P_2)$ solves

$$\begin{cases} -\Delta(\partial_t v_2) + \nabla(\partial_t P_2) = 0 & \text{in } \Omega \\ \nabla \cdot (\partial_t v_2) = \partial_t u \cdot \nabla \phi & \text{in } \Omega \\ \partial_t v_2 \text{ satisfies (1.2)}_1 & \text{on } \partial\Omega. \end{cases}$$

Thus, $v_3 = v_1 - v_2$ and $P_3 = P_1 - P_2$ solve

$$\begin{cases} \partial_t v_3 - \Delta v_3 + \nabla P_3 = g_3 = g_1 - \partial_t v_2 & \text{in } \Omega \\ \nabla \cdot v_3 = 0 & \text{in } \Omega \\ v_3 \text{ satisfies (1.2)}_1 & \text{on } \partial\Omega. \end{cases}$$

Let $\chi(t) : \mathbb{R} \rightarrow [0, 1]$ be the smooth function such that $\chi \equiv 0$ on $(-\infty, -1)$ and $\chi \equiv 1$ on $(-\frac{1}{4}, \infty)$. We see that $v_4 = v_3 \chi$ and $P_4 = P_3 \chi$ solve

$$\begin{cases} \partial_t v_4 - \Delta v_4 + \nabla P_4 = g_4 = \chi g_3 + v_3 \partial_t \chi & \text{in } \Omega \\ \nabla \cdot v_4 = 0 & \text{in } \Omega \\ v = 0 & \text{at } t = -1 \\ v \text{ satisfies (1.2)}_1 & \text{on } \partial\Omega. \end{cases}$$

By the maximal $L^p - L^q$ regularity to the Stokes system with Navier perfect slip boundary condition (see [16, Theorem 2.9]), we have

$$\|v_4\|_{W_{p,q}^{2,1}(\Omega \times (-1,0))} + \|\nabla P_4\|_{L^{p,q}(\Omega \times (-1,0))} \lesssim \|g_4\|_{L^{p,q}(\Omega \times (-1,0))}.$$

First, we can directly estimate g_1 that

$$\|g_1(\cdot, t)\|_{L^p(\Omega)} \lesssim \|u\|_{W_p^1(B_1^+)} + \|P - p_0\|_{L^p(B_1^+)}.$$

To estimate $\partial_t v_2$ in g_3 , we notice that $\partial_t u \cdot \nabla \phi = \nabla a + b$ where $a = -(P - p_0)\nabla \phi + \nabla u \nabla \phi$ and $b = -\nabla u : \nabla^2 \phi + (P - p_0)\Delta \phi$, we can check that the duality argument of [53, Theorem 2.4] also works with Navier boundary condition, and it gives

$$\|\partial_t v_2\|_{L^p(\Omega)} \lesssim \|a\|_{L^p(\Omega)} + \|b\|_{L^p(\Omega)} + \|a \cdot \hat{n}\|_{L^p(\partial\Omega)},$$

while the last term can be estimated by

$$\|a \cdot \hat{n}\|_{L^p(\partial\Omega)} \lesssim \delta^{\frac{1}{q}} \left(\|\nabla^2 u\|_{L^p(B_1^+)} + \|\nabla P\|_{L^p(B_1^+)} \right) + \left(\frac{1}{\delta} \right)^{\frac{p'}{pq}} \left(\|u\|_{W_p^1(B_1^+)} + \|P - p_0\|_{L^p(B_1^+)} \right).$$

We can also use test function $\psi \in C^\infty(\Omega)$ with $\hat{n} \cdot \nabla \psi = 0$ on $\partial\Omega$ to estimate that

$$\|v_2\|_{W_p^2(\Omega)} + \|\nabla P_2\|_{L^p(\Omega)} \lesssim \|\nabla(u \cdot \nabla \phi)\|_{L^p(\Omega)} \lesssim \|u\|_{W^{1,p}(B_1^+)}.$$

Combining the above estimates, it follows that

$$\|u\|_{W_{p,q}^{2,1}(Q_{\frac{1}{2}}^+)} + \|\nabla P\|_{L^{p,q}(Q_{\frac{1}{2}}^+)} \lesssim \|\nabla u\|_{L^{p,q}(Q_1^+)} + \|P - p_0\|_{L^{p,q}(Q_1^+)}.$$

Moreover, for $s \leq p$, one can use Sobolev embedding to lift above RHS to terms with $W_{s,q}^{2,1}$ norm and $W_{s,q}^{1,0}$ norm, then apply the above inequality again to deduce that

$$\|u\|_{W_{p,q}^{2,1}(Q_{\frac{1}{2}}^+)} + \|\nabla P\|_{L^{p,q}(Q_{\frac{1}{2}}^+)} \lesssim \|\nabla u\|_{L^{s,q}(Q_1^+)} + \|P - p_0\|_{L^{s,q}(Q_1^+)}. \quad (2.1)$$

□

Lemma 2.10 (Riesz potential estimates between parabolic Morrey space). *Let $\tilde{M}^{p,\lambda}(\Omega_T)$ and $\tilde{M}^{p,\lambda}(\Omega_T)_*$, respectively, denote the parabolic Morrey space, and the weak parabolic Morrey space, where $1 \leq p < \infty$, $0 \leq \lambda \leq n + 2$, and*

$$\begin{aligned} \|f\|_{\tilde{M}^{p,\lambda}(\Omega_T)}^p &:= \sup_{r>0, z \in \Omega_T} r^{\lambda-(n+2)} \|f\|_{L^p(\Omega_T \cap P_r(z))}^p, \\ \|f\|_{\tilde{M}_*^{p,\lambda}(\Omega_T)}^p &:= \sup_{r>0, z \in \Omega_T} r^{\lambda-(n+2)} \|f\|_{L^{p,*}(\Omega_T \cap P_r(z))}^p. \end{aligned}$$

Let $\tilde{I}_\beta(f)$ denote the parabolic Riesz potential of order $\beta \in [0, n + 2]$,

$$\tilde{I}_\beta(f) = \int_{\mathbb{R}^{n+1}} \frac{f(y, s)}{\delta((x, t), (y, s))^{n+2-\beta}} dy ds,$$

where $f \in L^p(\mathbb{R}^{n+1})$ and parabolic distance $\delta((x, t), (y, s)) = \max(|x - y|, \sqrt{|t - s|})$. Then, for any $\beta > 0, 0 < \lambda \leq n + 2, 1 < p < \frac{\lambda}{\beta}$, if $f \in L^p(\mathbb{R}^{n+1}) \cap \tilde{M}^{p, \lambda}(\mathbb{R}^{n+1})$, then $\tilde{I}_\beta(f) \in L^{\tilde{p}}(\mathbb{R}^{n+1}) \cap \tilde{M}^{\tilde{p}, \lambda}(\mathbb{R}^{n+1})$ with $\tilde{p} = \frac{p\lambda}{\lambda - p\beta}$. Further, for any $0 < \beta < \lambda \leq n + 2$, if $f \in L^1(\mathbb{R}^{n+1}) \cap \tilde{M}^{1, \lambda}(\mathbb{R}^{n+1})$, then $\tilde{I}_\beta(f) \in L^{\frac{\lambda}{\lambda - \beta}, * }(\mathbb{R}^{n+1}) \cap \tilde{M}_*^{\frac{\lambda}{\lambda - \beta}, \lambda}(\mathbb{R}^{n+1})$.

Proof. See [22, Theorem 3.1]. $\square$

Lemma 2.11 (Density of smooth maps in $\mathbf{L}$ and $\mathbf{J}$). *For $n = 2$, and any given map $v \in \mathbf{L}$ and $f \in \mathbf{J}$, there exists sequence $\{v_k\} \subseteq C^\infty(\Omega, \mathbb{S}^2) \cap C^{2, \alpha}(\bar{\Omega}, \mathbb{S}^2) \cap \mathbf{L}$ and $\{f_k\} \subseteq C^\infty(\Omega, \mathbb{R}^2) \cap C^{2, \alpha}(\bar{\Omega}, \mathbb{R}^2) \cap \mathbf{J}$ such that*

$$\lim_{k \rightarrow \infty} \|v_k - v\|_{L^2(\Omega)} = \lim_{k \rightarrow \infty} \|f_k - f\|_{H^1(\Omega)} = 0.$$

Proof. For the classical result that smooth maps are dense in Sobolev maps, see [7]. Since we need lots of other properties such as Hölder continuity, Navier boundary condition for v_k , and free boundary condition for f_k , our method is to consider the solution of an evolution equation with initial data v and f with free boundary condition (1.2)₁; then we take backward slices $v_k = v(\cdot, t = 2^{-k})$ and $f_k = f(\cdot, 2^{-k})$ as smooth approximations. For director field $f(x, t) : \Omega \times (0, T) \rightarrow \mathbb{S}^2$, we consider the heat flow of harmonic map

$$\begin{cases} f_t = \Delta f + |\nabla f|^2 f & \text{in } \Omega \times (0, t_f) \\ f(\cdot, 0) = f_0 \in \mathbf{J} \\ \nabla_{\hat{n}} f_1 = \nabla_{\hat{n}} f_2 = f_3 = 0 & \text{on } \partial\Omega \times (0, t_f). \end{cases}$$

A classical result in short-time existence and regularity shows that if t_f is smaller than the first singular time, then $f(x, t) \in C^\infty(\Omega \times (0, t_f))$, see [55]. That argument is the same as we shall use to prove Theorem 1.3. Smoothness of solution of equation gives $f \in C([0, T], H^1(\Omega))$ and thus $\lim_{k \rightarrow \infty} \|f_k - f\|_{H^1(\Omega)} = 0$. Hölder regularity follows easily from an estimation of Morrey–Campanato norm, and the argument is the same as what we shall do in proving Theorem 1.2.

Similarly, consider the solution of the Navier–Stokes equation with initial data v_0 with Navier–slip boundary condition (1.2)₂.

$$\begin{cases} v_t + v \cdot \nabla v - \Delta v + \nabla P = 0 & \text{in } \Omega \times (0, t_v) \\ \nabla \cdot v = 0 & \text{in } \Omega \times (0, t_v) \\ v(\cdot, 0) = v_0 \\ (\mathbb{D}u \cdot \hat{n})_\tau = u \cdot \hat{n} = 0 & \text{on } \partial\Omega \times (0, t_v). \end{cases}$$

We similarly use existence and regularity for two-dimensional Navier–Stokes to show that $v \in C([0, T], L^2(\Omega))$ and thus $\lim_{k \rightarrow \infty} \|v_k - v\|_{L^2(\Omega)} = 0$, see [26]. Also, Hölder regularity follows from an estimation of Morrey–Campanato norm, which is the argument we shall use in Theorem 1.2. Alternatively, since the proof of Theorem 1.2 does not require this lemma, we can also use the result of Theorem 1.2 with $u \equiv 0$ or $d \equiv C$ to obtain the Hölder regularity. $\square$

Lemma 2.12 (Korn's inequality). *Let $\Omega \subseteq \mathbb{R}^n$ with $n \geq 2$ be an open, bounded, and C^1 -smooth domain. Let v be a vector field on Ω with $\nabla v \in L^2(\Omega)$. Then, there exists $C(\Omega) > 0$ such that*

$$\|v\|_{L^2(\Omega)} + \|\mathbb{D}v\|_{L^2(\Omega)} \geq C(\Omega) \|\nabla v\|_{L^2(\Omega)}.$$

Assume further that Ω is nonaxisymmetric and v has tangential boundary condition (i.e., $v \cdot \hat{n} = 0$ on $\partial\Omega$), then

$$\|\mathbb{D}v\|_{L^2(\Omega)} \geq K(\Omega) \|\nabla v\|_{L^2(\Omega)}$$

for some constant $K(\Omega) > 0$ measuring the deviation of domain Ω from being axisymmetric.

Proof. See [13, Theorem 3]. □

3 | HÖLDER REGULARITY OF SOLUTION

This section uses the standard hole-filling argument to obtain the regularity of solution v to a parabolic system (P) . This argument has mainly two parts: first, we cut a hole (e.g., $P_r(x_0)$ or Ω_T) and fill it with the solution v' of a homogeneous system (P') (e.g., heat equation, Stokes system, harmonic map heat flow) with boundary condition inherited from the original solution v . Then, $v - v'$ solves the parabolic system $(P - P')$, which is usually inhomogeneous but has a good boundary condition on the hole. This good boundary condition gives us more freedom to perform integration by parts and allows us to handle inhomogeneity only. We leave the boundary condition to good system (P') and deal with it independently. Ultimately, we achieve a gradient estimate and obtain C^α -regularity of v . The next step is to use C^α -regularity to obtain $C^{1,\beta}$ -regularity for some $\beta \in (0, 1)$. One approach given by [22] is to analyze the Riesz potential between parabolic Morrey space. We apply the first part of the hole-filling argument in the following three lemmas and apply the second part in the proof of Theorem 1.2.

We start with the $L^{\frac{4}{3}}$ estimate of $|\nabla P|$ to the solution (u, d, P) of the free boundary system (1.1)–(1.4).

Lemma 3.1 (Estimation of pressure). *For $0 < T < \infty$, suppose that $u \in L^{2,\infty} \cap W_2^{1,0}(\Omega_T)$ and $d \in L^\infty([0, T], H^1(\Omega)) \cap L^2([0, T], H^2(\Omega))$ is a weak solution to (1.1)–(1.4). Then, $\nabla P \in L^{\frac{4}{3}}(\Omega_T)$ and we have the following estimate:*

$$\|P - P_\Omega\|_{L^{\frac{4}{3}}(\Omega_T)} \lesssim \|\nabla P\|_{L^{\frac{4}{3}}(\Omega_T)} \lesssim \|u\|_{L^4(\Omega_T)} \|\nabla u\|_{L^2(\Omega_T)} + \|\nabla d\|_{L^4(\Omega_T)} \|\nabla^2 d\|_{L^2(\Omega_T)}.$$

Proof. We write $u = v^1 + v^2$ where $v^1 : \Omega_T \rightarrow \mathbb{R}^2$ solves homogeneous heat equation $\partial_t v^1 - \Delta v^1 = 0$ with initial condition $v^1 = u_0$ on $\Omega \times \{t = 0\}$ and partially free boundary condition (v^1 satisfies (1.2)₁ on $\partial\Omega$). Then $v^2 = v - v^1$ solves nonhomogeneous, nonstationary Stokes equation $\partial_t v^2 - \Delta v^2 + \nabla P = -u \cdot u - \nabla \cdot (\nabla d \odot \nabla d - \frac{1}{2} |\nabla d|^2 \mathbb{I}_2)$ with initial condition $v^2 = 0$ on $\Omega \times \{t = 0\}$ and partially free boundary condition v^2 satisfies (1.2)₁ on $\partial\Omega$. Then, Sobolev inequality

and $W_{\frac{4}{3}}^{2,1}$ -estimate give

$$\begin{aligned} \|P - P_\Omega\|_{L^{\frac{4}{3}}(\Omega_T)} &\lesssim \|\nabla P\|_{L^{\frac{4}{3}}(\Omega_T)} \lesssim \|u\|_{L^{\frac{4}{3}}(\Omega_T)} \|\nabla u\|_{L^{\frac{4}{3}}(\Omega_T)} + \|\nabla d\|_{L^{\frac{4}{3}}(\Omega_T)} \|\nabla^2 d\|_{L^{\frac{4}{3}}(\Omega_T)} \\ &\lesssim \|u\|_{L^4(\Omega_T)} \|\nabla u\|_{L^2(\Omega_T)} + \|\nabla d\|_{L^4(\Omega_T)} \|\nabla^2 d\|_{L^2(\Omega_T)}. \end{aligned} \quad \square$$

Lemma 3.2 (Local smallness). *For any $\alpha \in (0, 1)$, there exists $\epsilon_0 > 0$ such that for $z_0 = (x_0, t_0) \in \mathbb{R}^3$ and $r > 0$, if $(u, d) \in W_2^{1,0}(P_{2r}(z_0))$, $P \in W^{1,\frac{4}{3}}(P_{2r}(z_0))$ is a weak solution to (1.1) and*

$$\int_{P_r(z_0)} |u|^4 + |\nabla d|^4 \leq \epsilon_0^4, \quad (3.1)$$

then $(u, d) \in C^\alpha(P_r(z_0), \mathbb{R}^2 \times \mathbb{S}^2)$. Moreover,

$$[d]_{C^\alpha(P_r(z_0))} \leq C(\|u\|_{L^4(P_r(z_0))} + \|\nabla d\|_{L^4(P_r(z_0))}), \quad (3.2)$$

$$[u]_{C^\alpha(P_r(z_0))} \leq C(\|u\|_{L^4(P_r(z_0))} + \|\nabla d\|_{L^4(P_r(z_0))} + \|\nabla P\|_{L^{\frac{4}{3}}(P_r(z_0))}). \quad (3.3)$$

Proof. The proof is similar to that of [34, Lemma 2.2]. Here, we give a sketch of proof for self-containedness.

Step 1: First, we want to have a local growth control of $|\nabla d|$ in the following sense:

$$\int_{P_r(z_1)} |\nabla d|^4 \leq \left(\frac{r}{R}\right)^{4\alpha} \int_{P_R(z_1)} |\nabla d|^4 \text{ for } 0 < r \leq R. \quad (3.4)$$

To do so, we decompose $d = d^1 + d^2$ where $d^1 : P_R(z_1) \rightarrow \mathbb{R}^3$ satisfies the homogeneous heat equation $\partial_t d^1 - \Delta d^1 = 0$ with Dirichlet boundary condition $d^1 = d$ on $\partial P_R(z_1)$, and $d^2 : P_R(z_1) \rightarrow \mathbb{R}^3$ satisfies the inhomogeneous heat equation $\partial_t d^2 - \Delta d^2 = -u \cdot d + |\nabla d|^2 d$ with zero boundary condition $d^2 = 0$ on $\partial P_R(z_1)$. We test d^2 equation with Δd^2 and apply Hölder inequality to obtain

$$\sup_{t_1 - R^2 \leq t \leq t_1} \int_{B_R(x_1)} |\nabla d^2|^2 + \int_{P_R(z_1)} |\Delta d^2|^2 \lesssim (\|u\|_{L^4(P_R(z_1))}^2 + \|\nabla d\|_{L^4(P_R(z_1))}^2) \|\nabla d\|_{L^4(P_R(z_1))}^2.$$

This, together with Ladyzhenskaya inequality in Lemma 2.3, yields

$$\int_{P_R(z_1)} |\nabla d^2|^4 \lesssim \left(\int_{P_R(z_1)} |u|^4 + \int_{P_R(z_1)} |\nabla d|^4 \right) \int_{P_R(z_1)} |\nabla d|^4.$$

We use Lemma 2.1 for d^1 and obtain $\|\nabla d^1\|_{L^4(P_{\theta R})} \lesssim \theta \|\nabla d^1\|_{L^4(P_R)}$, and thus

$$\int_{P_{\theta R}(z_1)} |\nabla d^1|^4 \leq C\theta^4 \int_{P_R(z_1)} |\nabla d|^4 + C \left(\int_{P_R(z_1)} |u|^4 + \int_{P_R(z_1)} |\nabla d|^4 \right) \int_{P_R(z_1)} |\nabla d|^4,$$

which gives inequality (3.4) if we choose $\theta = r/R < \theta_0$ and $\epsilon_0 < \theta_0$ such that $2C\theta_0^4 \leq \theta_0^{4\alpha}$.

Step 2: Next, we want to control the local growth of $|d_t|$. We test (1.1)₃ with $d_t\phi^2$ with cut-off function $\phi \in C_0^\infty(B_r(x_0))$, $0 \leq \phi \leq 1$, $\phi \equiv 1$ on $B_{\frac{r}{2}}(x_0)$, and $|\nabla\phi| \leq 2/r$. It follows that

$$\int_{B_{\frac{r}{2}}(x_0)} |d_t|^2 \phi^2 + \frac{d}{dt} \int_{B_r(x_0)} |\nabla\phi|^2 \phi^2 \lesssim r^{-2} \int_{B_r(x_0)} |\nabla d|^2 + \int_{B_r(x_0)} |u|^2 |\nabla d|^2.$$

We select $s_0 \in (t_0 - r^2 - t_0 - \frac{r^2}{4})$ such that $\int_{B_r(x_0)} |\nabla(\cdot, s_0)|^2 \lesssim r^{-2} \int_{P_r(z_0)} |\nabla d|^2$. As a consequence,

$$\int_{P_{\frac{r}{2}}(z_0)} |d_t|^2 \lesssim (1 + \|u\|_{L^4(P_r(z_0))}^2) \|\nabla d\|_{L^4(P_r(z_0))}^2 \lesssim \left(\frac{r}{R}\right)^{2\alpha} \left(\int_{P_R(z_0)} |\nabla d|^4\right)^{\frac{1}{2}}. \quad (3.5)$$

Estimations (3.4) and (3.5) imply the Hölder continuity by Lemma 2.2.

Step 3: We will estimate the local growth of $|\nabla u|$ by separating solution $u = u^1 + u^2$ of (1.1)₁ into two functions, where $u^1 : P_R(z_1) \rightarrow \mathbb{R}^2$ satisfies homogeneous heat equation $\partial_t u^1 - \Delta u^1 = 0$ with Dirichlet boundary condition $u^1 = u$ on $\partial P_R(z_1)$ and $u^2 = u - u^1 : P_R(z_1) \rightarrow \mathbb{R}^2$ satisfies inhomogeneous heat equation $\partial_t u^2 - \Delta u^2 + \nabla P = -\nabla \cdot [u \otimes (u - u_{z_1, R}) + \nabla d \odot \nabla d - \frac{1}{2} |\nabla d|^2 \mathbb{I}_2]$ with zero boundary condition $u^2 = 0$ on $\partial P_R(z_1)$. We test it with u^2 and use Hölder inequality to estimate

$$\frac{1}{2} \frac{d}{dt} \int_{B_R(x_1)} |u^2|^2 + \int_{B_R(x_1)} |\nabla u^2|^2 \lesssim \int_{B_R(x_1)} |\nabla d|^4 + |u|^2 |u - u_{z_1, R}|^2 + |u^2| |\nabla P|.$$

We integrate over $[t_1 - R^2, t_1]$ and use Hölder inequality to obtain

$$\sup_{t_1 - R^2 \leq t \leq t_1} \int_{B_R(x_1)} |u^2|^2 + \int_{P_R} |\nabla u^2|^2 \lesssim \int_{P_R} |\nabla d|^4 + \|u\|_{L^4(P_R)}^2 \|u - u_{z_1, R}\|_{L^4}^2 + \|u^2\|_{L^4(P_R)} \|\nabla P\|_{L^{\frac{4}{3}}(P_R)},$$

which, together with Ladyzhenskaya inequality, yields

$$\int_{P_R(z_1)} |u^2|^4 \lesssim \left(\int_{P_R(z_1)} |\nabla d|^4\right)^2 + \int_{P_R(z_1)} |u - u_{z_1, R}|^4 \cdot \int_{P_R(z_1)} |u|^4 + \left(\int_{P_R(z_1)} |\nabla P|^{\frac{4}{3}}\right)^3.$$

Collecting the above two estimates gives

$$\left(\int_{P_R(z_1)} |\nabla u^2|^2\right)^2 \lesssim \left(\int_{P_R(z_1)} |\nabla d|^4\right)^2 + \int_{P_R(z_1)} |u - u_{z_1, R}|^4 \cdot \int_{P_R(z_1)} |u|^4 + \left(\int_{P_R(z_1)} |\nabla P|^{\frac{4}{3}}\right)^3.$$

Applying Lemma 2.1 to u^1 , we have $\|u^1 - u_{z_1, \partial R}^1\|_{L^4(P_{\partial R}(z_1))} \lesssim \theta^2 \|u^1 - u_{z_1, R}^1\|_{L^4(P_R(z_1))} \lesssim \theta^2 \|u - u_{z_1, R}\|_{L^4(P_R(z_1))} + \|u^2\|_{L^4(P_R(z_1))}$ and $\|\nabla u^1\|_{L^2(P_{\partial R}(z_1))} \lesssim \theta^2 \|\nabla u\|_{L^2(P_R(z_1))} + \|\nabla u^2\|_{L^2(P_R(z_1))}$. Thus, we obtain the following estimates for local growth of $|u - u_{z_1, R}|$ and $|\nabla u|$:

$$\begin{aligned} \int_{P_{\partial R}(z_1)} |u - u_{z_1, \partial R}|^4 &\lesssim (\theta^8 + \|u\|_{L^4(P_R(z_1))}^4) \int_{P_R(z_1)} |u - u_{z_1, R}|^4 + \|\nabla d\|_{L^4(P_R)}^8 + \|\nabla P\|_{L^{\frac{4}{3}}(P_R)}^4, \\ \left(\int_{P_{\partial R}(z_1)} |\nabla u|^2\right)^2 &\lesssim \theta^8 \left(\int_{P_R(z_1)} |\nabla u|^2\right)^2 + \|u\|_{L^4(P_R)}^4 \int_{P_R(z_1)} |u - u_{z_1, R}|^4 \\ &\quad + \|\nabla d\|_{L^4(P_R)}^8 + \|\nabla P\|_{L^{\frac{4}{3}}(P_R)}^4. \end{aligned} \quad (3.6)$$

Step 4: We need to estimate $\|\nabla P\|_{L^{\frac{4}{3}}}$. Again, we separate $P = P^1 + P^2$ where $P^1 : B_R(x_1) \rightarrow \mathbb{R}$ satisfies Poisson's equation $\Delta P^1 = -\nabla \cdot (u \cdot \nabla u + \nabla \cdot (\nabla d \odot \nabla d - \frac{1}{2}|\nabla d|^2 \mathbb{I}_2))$ with zero boundary condition $P^1 = 0$ on $\partial B_R(x_1)$, and harmonic function P^2 satisfies Laplace equation with Dirichlet boundary condition $P^2 = P$ on $\partial B_R(x_1)$. The Calderon-Zygmund theory, together with interior $W^{2,1}_2$ -estimate $\|\nabla^2 d\|_{L^2(P_{\frac{3}{4}})} \lesssim \|u\|_{L^4(P_1)} + \|\nabla d\|_{L^4(P_1)}$ for (1.1)₃, gives

$$\|\nabla P^1\|_{L^{\frac{4}{3}}(P_{\frac{3}{2}}(z_1))} \lesssim \|u\|_{L^4(P_R(z_1))} \|\nabla u\|_{L^2(P_R(z_1))} + (\|u\|_{L^4(P_R(z_1))} + \|\nabla d\|_{L^4(P_R(z_1))}) \|\nabla d\|_{L^4(P_R(z_1))}.$$

We apply Harnack inequality for ∇P^2 and use the above estimate to get

$$\int_{P_{\theta R}(z_1)} |\nabla P|^{\frac{4}{3}} \lesssim \theta^2 \int_{P_R} |\nabla P|^{\frac{4}{3}} + \|u\|_{L^4(P_R)}^{\frac{4}{3}} \|\nabla u\|_{L^2(P_R)}^{\frac{4}{3}} + \left(\|u\|_{L^4(P_R)}^{\frac{4}{3}} + \|\nabla d\|_{L^4(P_R)}^{\frac{4}{3}} \right) \|\nabla d\|_{L^4(P_R)}^{\frac{4}{3}}. \quad (3.7)$$

Step 5: Adding the above estimates (3.4)–(3.7) for $\nabla d, d_t, \nabla u, \nabla P$ together, we have

$$A_{\theta R} \lesssim (\theta^6 + \|u\|_{L^4(P_R)}^4) A_R + (\|u\|_{L^4(P_R)}^4 + \|\nabla d\|_{L^4(P_R)}^4) \|\nabla d\|_{L^4(P_R)}^4 \lesssim \theta^{4+4\alpha_1} A_R + R^{4+4\alpha_1},$$

where $A_r = \|u - u_{z_r}\|_{L^4(P_r)}^4 + \|\nabla u\|_{L^2(P_r)}^4 + \|\nabla P\|_{L^{\frac{4}{3}}(P_r)}^4$, and we take $\alpha_1 = \frac{1+\alpha}{4}, \epsilon_0 \leq \theta^{\frac{3}{2}} \leq R \leq \theta$. The characterization of Campanato spaces states that $u \in C^{\alpha_1}(P_{\frac{1}{2}})$. $\square$

Lemma 3.3 (Boundary smallness). *For any $\alpha \in (0, 1)$, there exists ϵ_0 such that if for $z_0 = (x_0, t_0) \in \partial\Omega_+ \times \mathbb{R}$ and $r > 0$, $(u, d) \in W^{1,0}_{2r}(P^+_{2r}(z_0)), P \in W^{1,\frac{4}{3}}(P^+_{2r}(z_0))$ is a weak solution of (1.1), with (u, d) satisfies (1.2) on $\Gamma^+_{2r}(x_0) \times [t_0 - r^2, t_0]$ and*

$$\int_{P^+_{r}(z_0)} |u|^4 + |\nabla d|^4 \leq \epsilon_0^4, \quad (3.8)$$

then $(u, d) \in C^{\alpha}(P^+_{\frac{r}{2}}(z_0))$. Moreover,

$$[d]_{C^{\alpha}(P^+_{\frac{r}{2}}(z_0))} \lesssim \|u\|_{L^4(P^+_{r}(z_0))} + \|\nabla d\|_{L^4(P^+_{r}(z_0))}, \quad (3.9)$$

$$[u]_{C^{\alpha}(P^+_{\frac{r}{2}}(z_0))} \lesssim \|u\|_{L^4(P^+_{r}(z_0))} + \|\nabla d\|_{L^4(P^+_{r}(z_0))} + \|\nabla P\|_{L^{\frac{4}{3}}(P^+_{r}(z_0))}. \quad (3.10)$$

Proof. *Step 1:* First, we estimate boundary growth for ∇d . Consider $z_1 = (x_1, t_1) \in \Gamma_{\frac{1}{2}} \times [-\frac{1}{4}, 0]$ and $0 < R \leq \frac{1}{4}$. We separate $d = d^1 + d^2$, where $d^1 : P^+_{\frac{1}{2}}(z_1) \rightarrow \mathbb{R}^3$ solves homogeneous heat equation $\partial_t d^1 - \Delta d^1 = 0$ in $P^+_{\frac{1}{2}}(z_1)$ with mixed boundary condition $d^1 = d$ on $S_R(x_1) \times [t_1 - R^2, t_1]$, $d^1 = d$ on $B^+_{\frac{1}{2}}(x_1) \times \{t = t_1 - R^2\}$, and d^1 satisfies (1.2) on $\Gamma_R(x_1) \times [t_1 - R^2, t_1]$. And $d^2 = d - d^1$ solves inhomogeneous heat equation $\partial_t d^2 - \Delta d^2 = -u \cdot \nabla d + |\nabla d|^2 d$ in $P^+_{\frac{1}{2}}(z_1)$ with mixed boundary condition $d^2 = 0$ on $B^+_{\frac{1}{2}}(x_1) \times \{t = t_1 - R^2\}$, $d^2 = 0$ on $S_R(x_1) \times [t_1 - R^2, t_1]$, and d^2 satisfies (1.2) on $\Gamma_R(x_1) \times [t_1 - R^2, t_1]$. Same as in Lemma 3.2, we test equation with Δd^2 , and in the end, we obtain

$$\int_{P^+_{\frac{1}{2}}(z_1)} |\nabla d^2|^4 \lesssim \left(\int_{P^+_{\frac{1}{2}}(z_1)} |u|^4 + |\nabla d|^4 \right) \int_{P^+_{\frac{1}{2}}(z_1)} |\nabla d|^4.$$

We apply Lemma 2.1 to components of d^1 and follow the same argument in Lemma 3.2 to obtain

$$\int_{P_{\theta R}^+(z_1)} |\nabla d|^4 \leq C(\theta^4 + \epsilon_0^4) \int_{P_R^+(z_1)} |\nabla d|^4 \leq \theta_0^{4\alpha} \int_{P_R^+(z_1)} |\nabla d|^4, \quad (3.11)$$

given that $\theta \leq \theta_0$ with $2C\theta_0^4 \leq \theta_0^{4\alpha}$ and $\epsilon_0 \leq \theta_0$.

Step 2: Then, we estimate boundary growth for $|d_t|$. Because components of d_t satisfy either Neumann boundary condition or zero Dirichlet boundary condition, we test (1.1)₃ by $d_t \phi^2$ where the cut-off function $\phi \in C_0^\infty(B_R(x_1))$ is such that $0 \leq \phi \leq 1$, $\phi \equiv 1$ on $B_{\frac{r}{2}}(x_1)$, and $|\nabla \phi| \lesssim \frac{1}{r}$. The same computation as in Lemma 3.2 yields

$$\int_{P_{\frac{r}{2}}^+(z_1)} |d_t|^2 \lesssim r^2 + \left(\frac{r}{R}\right)^{2\alpha} \|\nabla d\|_{L^4(P_R^+(z_1))}^2. \quad (3.12)$$

Estimations (3.11) and (3.12), together with parabolic Morrey's lemma, imply that $d \in C^\alpha(P_{\frac{1}{2}}^+)$.

Step 3: Now we estimate the boundary local growth for $|\nabla u|$. Let $H \subseteq \mathbb{R}^2$ be a bent half-plane such that $P_R^+(z_1) \subseteq H$ and $\Gamma_R(z_1) \subseteq \partial H$. Let $u^1 : H \times [-1, 0] \rightarrow \mathbb{R}^2$ solve nonhomogeneous, non-stationary Stokes equation $\partial_t u^1 - \Delta u^1 + \nabla P^1 = -(u \cdot \nabla u + \nabla \cdot (\nabla d \odot \nabla d - \frac{1}{2} |\nabla d|^2 \mathbb{I}_2)) \chi_{P_R^+}$ with Navier perfect-slip boundary condition (u^1 satisfies (1.2)₁ on ∂H) and zero initial condition. Then, $u^2 = u - u^1 : P_R^+(z_1) \rightarrow \mathbb{R}^2$ solves homogeneous, nonstationary Stokes equation $\partial_t u^2 - \Delta u^2 + \nabla P^2 = 0$ with partially free boundary condition (u^2 satisfies (1.2)₁ on $\Gamma_R(z_1)$) and initial condition $u^2 = u$ at $t = t_1 - R^2$.

We apply parabolic Sobolev inequality Lemma 2.8 to u^1 and use $W_{\frac{4}{3}}^{2,1}$ -estimates for u^1 (cf. [8, Theorem 2.3]), together with Hölder inequality to obtain

$$\|\nabla u^1\|_{L^2(P_R^+)} + \|u^1\|_{L^4(P_R^+)} + \|\nabla P^1\|_{L^{\frac{4}{3}}(P_R^+)} \lesssim \|u\|_{L^4(P_R^+)} \|\nabla u\|_{L^2(P_R^+)} + \|\nabla^2 d\|_{L^2(P_R^+)} \|\nabla d\|_{L^4(P_R^+)}.$$

We apply Lemma 2.9 to u^2 and have, for any $q > 4$,

$$\begin{aligned} R^{\frac{3}{2}-\frac{2}{q}} \left(\|u^2\|_{W_{q, \frac{4}{3}}^{2,1}(P_{\frac{R}{2}}^+(z_1))} + \|\nabla P^2\|_{L^{\frac{4}{3}}(P_R^+(z_1))} \right) &\lesssim \|\nabla u^2\|_{L^2(P_R^+(z_1))} + \|\nabla P^2\|_{L^{\frac{4}{3}}(P_R^+(z_1))} \\ &\lesssim \|\nabla u\|_{L^2(P_R^+(z_1))} + \|\nabla P\|_{L^{\frac{4}{3}}(P_R^+(z_1))} + \|u\|_{L^4(P_R^+(z_1))} \|\nabla u\|_{L^2(P_R^+(z_1))} + \|\nabla^2 d\|_{L^2(P_R^+(z_1))} \|\nabla d\|_{L^4(P_R^+(z_1))}. \end{aligned}$$

Moreover, we use the Sobolev inequality and Hölder inequality to see that for any $\theta \in (0, \frac{1}{4})$,

$$\begin{aligned} \|u^2 - u_{z_1, \theta R}^2\|_{L^4(P_{\theta R}^+(z_1))} + \|\nabla P^2\|_{L^{\frac{4}{3}}(P_{\theta R}^+(z_1))} &\lesssim (\theta R)^{\frac{3}{2}-\frac{2}{q}} \left(\|u^2\|_{W_{q, \frac{4}{3}}^{2,1}(P_{\frac{R}{2}}^+(z_1))} + \|\nabla P^2\|_{L^{\frac{4}{3}}(P_R^+(z_1))} \right) \\ &\lesssim \theta^{\frac{3}{2}-\frac{2}{q}} \left(\|\nabla u\|_{L^2(P_R^+)} + \|\nabla P\|_{L^{\frac{4}{3}}(P_R^+)} + \|u\|_{L^4(P_R^+)} \|\nabla u\|_{L^2(P_R^+)} + \|\nabla^2 d\|_{L^2(P_R^+)} \|\nabla d\|_{L^4(P_R^+)} \right). \end{aligned}$$

To estimate $\|\nabla u^2\|_{L^2}$, let $\phi \in C_0^1(B_{\theta R}(x_1))$ be such that $0 \leq \phi \leq 1$, $\phi \equiv 1$ on $B_{\frac{\theta R}{2}}(x_1)$, and $|\nabla \phi| \leq \frac{2}{\theta R}$. Multiplying the equation of u^2 by $(u^2 - u_{z_1, \theta R}^2)\phi^2$ and integrating over $B_{\theta R}^+(x_1)$, we obtain

$$\begin{aligned} \frac{d}{dt} \int_{B_{\theta R}^+(x_1)} |u^2 - u_{z_1, \theta R}^2|^2 \phi^2 + \int_{B_{\theta R}^+(x_1)} |\nabla u^2| \phi^2 &\lesssim \frac{1}{(\theta R)^2} \int_{B_{\theta R}^+} |u^2 - u_{z_1, \theta R}^2|^2 \\ &+ \int_{B_{\theta R}^+} |\nabla P^2| |u^2 - u_{z_1, \theta R}^2|. \end{aligned}$$

Integrating over $[s_0, t_1]$, where $s_0 \in [t_1 - (\theta R)^2, t_1 - \frac{1}{2}(\theta R)^2]$ is such that

$$\int_{B_{\theta R}^+(x_1) \times \{s_0\}} |u^2 - u_{z_1, \theta R}^2|^2 \lesssim (\theta R)^{-2} \int_{P_{\theta R}^+(z_1)} |u^2 - u_{z_1, \theta R}^2|^2,$$

we obtain

$$\begin{aligned} \int_{P_{\frac{\theta R}{2}}^+(z_1)} |\nabla u^2|^2 &\lesssim (\theta R)^{-2} \int_{P_{\theta R}^+(x_1)} |u^2 - u_{z_1, \theta R}^2|^2 + \int_{P_{\theta R}^+(x_1)} |\nabla P^2| |u^2 - u_{z_1, \theta R}^2| \\ &\lesssim \|u^2 - u_{z_1, \theta R}^2\|_{L^4(P_{\theta R}^+(z_1))}^2 + \|\nabla P^2\|_{L^{\frac{4}{3}}(P_{\theta R}^+(z_1))} \|u^2 - u_{z_1, \theta R}^2\|_{L^4(P_{\theta R}^+(z_1))} \\ &\lesssim \|u^2 - u_{z_1, \theta R}^2\|_{L^4(P_{\theta R}^+(z_1))}^2 + \|\nabla P^2\|_{L^{\frac{4}{3}}(P_{\theta R}^+(z_1))}^2. \end{aligned}$$

Collecting the above estimates and we set $q = 8$, we arrive

$$\begin{aligned} &\|u - u_{z_1, \theta R}\|_{L^4(P_{\theta R}^+(z_1))} + \|\nabla u\|_{L^2(P_{\theta R}^+(z_1))} + \|\nabla P\|_{L^{\frac{4}{3}}(P_{\theta R}^+(z_1))} \\ &\lesssim \left(\theta^{\frac{5}{4}} + \|u\|_{L^4(P_R^+(z_1))} \right) \left(\|u - u_{z_1, R}\|_{L^4(P_R^+(z_1))} + \|\nabla u\|_{L^2(P_R^+(z_1))} + \|\nabla P\|_{L^{\frac{4}{3}}(P_R^+(z_1))} \right) \\ &\quad + \|\nabla^2 d\|_{L^2(P_{\frac{R}{2}}^+(z_1))} \|\nabla d\|_{L^4(P_R^+(z_1))}, \end{aligned}$$

and we have $W_2^{2,1}$ -estimate for d that

$$\|\nabla^2 d\|_{L^2(P_{\frac{R}{2}}^+(z_1))} \lesssim \|u\|_{L^4(P_R^+(z_1))} + \|\nabla d\|_{L^4(P_R^+(z_1))}.$$

Step 4: By choosing $\theta = \theta_0$ sufficiently small and $\varepsilon_0 \leq \theta_0$, we obtain for $0 < r \leq \frac{1}{4}$,

$$\Theta^+(z_1, \theta R) \leq \theta \Theta^+(z_1, R) + C(\theta + R^\alpha)R^\alpha,$$

where

$$\Theta^+(z_1, r) := \|u - u_{z_1, r}\|_{L^4(P_r^+(z_1))} + \|\nabla u\|_{L^2(P_r^+(z_1))} + \|\nabla P\|_{L^{\frac{4}{3}}(P_r^+(z_1))}.$$

We repeat the same argument as in Lemma 3.2 combined with the characterization of Campanato space and conclude that $u \in C^\alpha(P_{\frac{1}{2}}^+)$. $\square$

Proof of Theorem 1.2

Proof. Assumption $u \in L^\infty([0, T], L^2(\Omega)) \cap L^2([0, T], H^1(\Omega))$ and Ladyzhenskaya inequality give $u \in L^4(\Omega_T)$. Equation (1.1)₃ and the fact that $|d| = 1$ give $d\Delta d + |\nabla d|^2 = 0$, which, together with assumption $\nabla d \in L^2([0, T], H^1(\Omega))$, gives $|\nabla d| \in L^4(\Omega_T)$ and hence $u \cdot \nabla u, \nabla \cdot (\nabla d \odot \nabla d - \frac{1}{2}|\nabla d|^2 \mathbb{I}_2) \in L^{\frac{4}{3}}(\Omega_T)$. It follows from absolute continuity of $\int |u|^4 + |\nabla d|^4$ that if $z_0 = (x_0, t_0) \in \bar{\Omega} \times (0, T]$, there exists $r_0 > 0$ such that for $r < r_0$,

$$\int_{P_r(x_0) \cap \Omega_T} |u|^4 + |\nabla d|^4 \leq \epsilon_0^4,$$

where ϵ_0 is given in Lemma 3.2 and Lemma 3.3. Hence, we deduce that $(u, d) \in C^\alpha(P_{r_0/2}(z_0) \cap \Omega \times (0, T])$. Consequently, we have $(u, d) \in C^\alpha(\bar{\Omega} \times (0, T])$.

For the higher order regularity, we use Lemma 2.10: for interior point $(x_0, t_0) \in \Omega \times (0, T)$, pick $r_0 > 0$ sufficiently small as above and also $P_{r_0}(x_0) \subseteq \Omega \times (0, T)$. Previous estimates give $\nabla d \in M^{2, 2-2\alpha}(P_{r_0/2}(x))$ for any $\alpha \in (0, 1)$. Consider $\tilde{d} = d\phi : P_{r_0}(x_0) \rightarrow \mathbb{R}^3$ where the cutoff $\phi \in C_0^\infty(P_{r_0}(x_0))$, $0 \leq \phi \leq 1$, $\phi \equiv 1$ on $P_{r_0/2}(x_0)$, and $|\phi_t|, |\nabla \phi|, |\nabla^2 \phi| \lesssim 1/r_0^2$. Then, $\tilde{d}$ solves $\partial_t \tilde{d} - \Delta \tilde{d} = F$ where

$$F = (\phi|\nabla d|^2 d - d(\phi_t - \Delta \phi) - 2\nabla \phi \cdot \nabla d - u \cdot \nabla d \phi) \chi_{P_{r_0}(x_0)}.$$

Since $\nabla d \in L^4(\Omega_T)$, $u \in C^\alpha(\bar{\Omega} \times (0, T])$, and (3.4), we can check that $F \in L^1(\mathbb{R}^{n+1})$ and $F \in \tilde{M}^{1, 2-2\alpha}(\mathbb{R}^{n+1})$. Because we have $\tilde{d} = G * F$ and thus $\nabla \tilde{d} = \nabla G * F$ where G is the fundamental solution of the heat equation on $\mathbb{R}^n$, and $|\nabla G(x, t)| \lesssim \frac{1}{\delta((0, 0), (x, t))^{1+n}}$, direct computation together with Lemma 2.10 yields

$$|\nabla \tilde{d}(z)| \lesssim \tilde{I}_1(|F|)(z) \in L^{\frac{2-2\alpha}{1-2\alpha}, *}(\mathbb{R}^{n+1}).$$

Notice that $\frac{2-2\alpha}{1-2\alpha} \rightarrow +\infty$ as $\alpha \nearrow \frac{1}{2}$. Thus, we use interpolation of (weak) L^p spaces to get $\nabla \tilde{d} \in L^q(\mathbb{R}^{n+1})$ for any $q > 1$. Hence, $W_q^{2,1}$ -estimate for the heat equation gives $\tilde{d} \in W_q^{2,1}(P_{r_0}(x_0))$ and thus $d \in W_q^{2,1}(P_{r_0/2}(x_0))$. Sobolev inequality then gives $\nabla d \in C^\alpha(P_{r_0/2}(x_0))$ for any $\alpha \in (0, 1)$.

Consequently, we apply Schauder estimate to system (1.1)₃ to obtain $d \in C_{\alpha}^{2,1}(\Omega \times (0, T))$. This gives the Hölder regularity of external forces in (1.1)₁, and thus, the standard $C_{\alpha}^{2,1}$ -regularity theory implies that $u \in C_{\alpha}^{2,1}(\Omega \times (0, T))$. Starting from $C_{\alpha}^{2,1}$ -regularity for pair (u, d) , we use the standard boot-strap argument to conclude that $(u, d) \in C^\infty(\Omega \times (0, T))$. Boundary $C_{\alpha}^{2,1}$ -regularity for (u, d) can be obtained similarly, by taking instead $\tilde{d} = (d - d')\phi$ where $d' : P_r(x_0)^+ \rightarrow \mathbb{R}^3$ solves heat equation and $d' = d$ on $\Gamma_r(x_0) \times (t_0 - t^2, t_0) \cup S_r(x_0) \times (t_0 - t^2, t_0)$. $\square$

4 | EXISTENCE OF SHORT-TIME SMOOTH SOLUTIONS

We would like to show that short-time smooth solutions to (1.1)–(1.3) exist for smooth initial and boundary data. Later, in the proof of Theorem 1.3, we shall use an approximation of smooth data and then show the existence. The argument to show the short-time existence in this section also applies to the case of three dimensions.

Theorem 4.1. For any $\alpha > 0$, if $u_0 \in C^{2,\alpha}(\overline{\Omega}, \mathbb{R}^2) \cap \mathbf{H}$ and $d_0 \in C^{2,\alpha}(\overline{\Omega}, \mathbb{S}^2) \cap \mathbf{J}$, then there exists $T > 0$ depending on $\|u_0\|_{C^{2,\alpha}(\Omega)}, \|d_0\|_{C^{2,\alpha}(\Omega)}$ such that there is a unique smooth solution $(u, d) \in C^{2,1}_\alpha(\overline{\Omega} \times [0, T], \mathbb{R}^2 \times \mathbb{S}^2)$ to the initial-boundary value problem (1.1)–(1.3).

Proof. We will use the contraction mapping principle in a similar spirit as [34]. For $T > 0$ and $K > 0$ to be chosen later, denote

$$\begin{aligned} X &= \{(v, f) \in C^{2,1}_\alpha(\overline{\Omega}_T, \mathbb{R}^2 \times \mathbb{R}^3) \mid \nabla \cdot v = 0, (v, f)|_{t=0} \\ &= (u_0, d_0), \|v - u_0\|_{C^{2,1}_\alpha(\Omega_T)} + \|f - d_0\|_{C^{2,1}_\alpha(\Omega_T)} \leq K\}. \end{aligned}$$

Equip X with the norm

$$\|(v, f)\|_X := \|v\|_{C^{2,1}_\alpha(\Omega_T)} + \|f\|_{C^{2,1}_\alpha(\Omega_T)}.$$

Then, we can show that $(X, \|\cdot\|_X)$ is a Banach space. For any $(v, f) \in X$, let (u, d) be the unique solution to the following mixed system (inhomogeneous Stokes system with initial condition and Navier-slip boundary condition; inhomogeneous heat equation with initial condition and Neumann boundary condition 1.2):

$$\begin{aligned} \partial_t u - \Delta u + \nabla P &= -v \cdot \nabla v - \nabla \cdot \left(\nabla d \odot \nabla d - \frac{1}{2} |\nabla d|^2 \mathbb{I}_2 \right) \text{ in } \Omega_T \\ \nabla \cdot u &= 0 \text{ in } \Omega_T \\ \partial_t d - \Delta d &= |\nabla f|^2 f - v \cdot \nabla f \text{ in } \Omega_T \\ (u, d) &= (u_0, d_0) \text{ on } \Omega \times \{t = 0\} \\ (u, d) &\text{ satisfies 1.2 on } \partial\Omega \times (0, T). \end{aligned} \tag{4.1}$$

We define the operator $L : X \rightarrow C^{2,1}_\alpha(\overline{\Omega}_T, \mathbb{R}^2 \times \mathbb{R}^3)$ to be $L(v, f) = (u, d)$. In the following two lemmas, we will show that for $T > 0$ sufficiently small and $K > 0$ sufficiently large, $L : X \rightarrow X$ and L is a contraction map. Then, there exists a unique solution $(u, d) \in C^{2,1}_\alpha(\overline{\Omega} \times [0, T], \mathbb{R}^2 \times \mathbb{R}^3)$ to (1.1)–(1.4). It remains to show that $|d| = 1$. In fact, this can be done by applying the maximum principle to the equation for $|d|^2$. $\square$

Lemma 4.2. There exist $T > 0$ and $K > 0$ such that $L : X \rightarrow X$.

Proof. For any $(v, f) \in X$. Let $(u, d) = L(v, f)$ be the unique solution to system (4.1). We use $C_0 > 0$ to denote constants depending only on $\|u_0\|_{C^{2,\alpha}}$ and $\|d_0\|_{C^{2,\alpha}}$.

Step 1: Estimation of $\|d - d_0\|_{C^{2,1}_\alpha(\Omega_T)}$. Assume $K \geq C_0$. By the Schauder theory of parabolic systems, we have

$$\|d - d_0\|_{C^{2,1}_\alpha(\Omega_T)} \lesssim \|v \cdot \nabla f\|_{C^\alpha(\Omega_T)} + \| |\nabla f|^2 f \|_{C^\alpha(\Omega_T)}. \tag{4.2}$$

And we can estimate the first term in (4.2) as

$$\begin{aligned} \|v \cdot \nabla f\|_{C^\alpha(\Omega_T)} &\leq \|v \cdot \nabla f - u_0 \cdot \nabla d_0\|_{C^\alpha(\Omega_T)} + \|u_0 \cdot \nabla d_0\|_{C^\alpha(\Omega)} \\ &\leq \|(v - u_0) \cdot \nabla f\|_{C^\alpha(\Omega_T)} + \|u_0 \cdot \nabla(f - d_0)\|_{C^\alpha(\Omega_T)} + C_0 \\ &\leq 2K(\|v - u_0\|_{C^0(\Omega_T)} + \|v - u_0\|_{C^\alpha(\Omega_T)}) + C_0(1 + \|\nabla(f - d_0)\|_{C^0(\Omega_T)} + \|\nabla(f - d_0)\|_{C^\alpha(\Omega_T)}). \end{aligned}$$

Since $v - u_0 = f - d_0 = 0$ at $t = 0$, it is easy to see

$$\|v - u_0\|_{C^0(\Omega_T)} \leq KT, \quad \|\nabla(f - d_0)\|_{C^0(\Omega_T)} \leq KT.$$

Meanwhile, we use interpolation inequality (see [18, Lemma 6.32]) to control terms with C^α -norm:

$$\begin{aligned} \|v - u_0\|_{C^\alpha(\Omega_T)} &\lesssim \frac{1}{\delta} \|v - u_0\|_{C^0(\Omega_T)} + \delta \|v - u_0\|_{C_\alpha^{2,1}(\Omega_T)} \lesssim \left(\delta + \frac{1}{\delta}\right) K, \\ \|\nabla(d - f)\|_{C^\alpha(\Omega_T)} &\lesssim \frac{1}{\delta} \|\nabla(d - f)\|_{C^0(\Omega_T)} + \delta \|d - f\|_{C_\alpha^{2,1}(\Omega_T)} \lesssim \left(\delta + \frac{1}{\delta}\right) K. \end{aligned}$$

Collecting the above estimates together, we obtain

$$\|v \cdot \nabla f\|_{C^\alpha(\Omega_T)} (C_0 K + CK^2) \left(T + \delta + \frac{T}{\delta}\right) + C_0 \leq \frac{\sqrt{K}}{4}$$

with $K = 16C_0^2$, $\delta \lesssim ((C_0 + C^2K)\sqrt{K})^{-1}$, and $T = \delta^2$.

Then, we will estimate the second term in (4.2). Notice that

$$\begin{aligned} \|\nabla f\|^2 f\|_{C^\alpha(\Omega_T)} &\leq \| |\nabla f|^2 f - |\nabla d_0|^2 d_0 \|_{C^\alpha(\Omega_T)} + \| |\nabla d_0|^2 d_0 \|_{C^\alpha(\Omega_T)} \\ &\leq \| |\nabla f|^2 (f - d_0) \|_{C^\alpha(\Omega_T)} + \| d_0 (|\nabla f|^2 - |\nabla d_0|^2) \|_{C^\alpha(\Omega_T)} + C_0 := I_1 + I_2 + C_0, \end{aligned}$$

where I_1 can be estimated by

$$\begin{aligned} I_1 &\leq \|f - d_0\|_{C^\alpha(\Omega_T)} \|\nabla f\|_{C^0(\Omega_T)}^2 + \|f - d_0\|_{C^0(\Omega_T)} \|\nabla f\|_{C^\alpha(\Omega_T)}^2 \\ &\lesssim K^2 (\|f - d_0\|_{C^0(\Omega_T)} + \|f - d_0\|_{C^\alpha(\Omega_T)}) \\ &\lesssim K^2 \left(\left(1 + \frac{1}{\delta}\right) \|f - d_0\|_{C^0(\Omega_T)} + \delta \|f - d_0\|_{C_\alpha^{2,1}(\Omega_T)} \right) \lesssim K^3 \left(\frac{T}{\delta} + \delta \right), \end{aligned}$$

and I_2 can be estimated by

$$\begin{aligned} I_2 &\leq \| |\nabla f|^2 - |\nabla d_0|^2 \|_{C^\alpha(\Omega_T)} + \| |\nabla f|^2 - |\nabla d_0|^2 \|_{C^0(\Omega_T)} \|d_0\|_{C^\alpha(\Omega_T)} \\ &\leq \| (|\nabla f| + |\nabla d_0|) |\nabla(f - d_0)| \|_{C^\alpha(\Omega_T)} + C_0 \| (|\nabla f| + |\nabla d_0|) |\nabla(f - d_0)| \|_{C^0(\Omega_T)} \\ &\lesssim (1 + C_0) K (\|\nabla(f - d_0)\|_{C^0(\Omega_T)} + \|\nabla(f - d_0)\|_{C^\alpha(\Omega_T)}) \lesssim (1 + C_0) K^2 \left(T + \delta + \frac{T}{\delta}\right). \end{aligned}$$

Therefore, we collect the above estimates together and conclude

$$\|\nabla f\|^2 f\|_{C^\alpha(\Omega_T)} \lesssim K^3 \left(\frac{T}{\delta} + \delta \right) + (1 + C_0) K^2 \left(T + \delta + \frac{T}{\delta}\right)$$

with $K = 16C_0^2$, $\delta \lesssim ((1 + C_0)K^{\frac{5}{2}})^{-1}$, and $T = \delta^2$.

As a consequence, we can simplify (4.2) to be

$$\|d - d_0\|_{C_\alpha^{2,1}(\Omega_T)} \leq \frac{\sqrt{K}}{2}. \quad (4.3)$$

Step 2: We estimate $\|u - u_0\|_{C_{\alpha}^{2,1}(\Omega_T)}$ in a similar way. By the Schauder theory for nonhomogeneous, nonstationary Stokes equations, we have

$$\|u - u_0\|_{C_{\alpha}^{2,1}(\Omega_T)} \lesssim \|v \cdot \nabla v\|_{C^{\alpha}(\Omega_T)} + \|\nabla \cdot (\nabla d \odot \nabla d - \frac{1}{2}|\nabla d|^2 \mathbb{I}_2)\|_{C^{\alpha}(\Omega_T)} \quad (4.4)$$

and estimate the first term in (4.4) as

$$\begin{aligned} \|v \cdot \nabla v\|_{C^{\alpha}(\Omega_T)} &\leq \|(v - u_0) \cdot \nabla v\|_{C^{\alpha}(\Omega_T)} + \|u_0 \cdot \nabla(v - u_0)\|_{C^{\alpha}(\Omega_T)} + \|u_0 \cdot \nabla u_0\|_{C^{\alpha}(\Omega_T)} \\ &\leq K(\|v - u_0\|_{C^0(\Omega_T)} + \|v - u_0\|_{C^{\alpha}(\Omega_T)}) + C_0(\|\nabla(v - u_0)\|_{C^0(\Omega_T)} + \|\nabla(v - u_0)\|_{C^{\alpha}(\Omega_T)} + C_0) \\ &\lesssim (C_0 K + K^2) \left(T + \delta + \frac{T}{\delta} \right) + C_0 \leq \frac{K}{4} \end{aligned}$$

with $K = 8C_0$, $\delta \lesssim ((1 + C_0)K)^{-1}$, and $T = \delta^2$.

For the second term in (4.4), we have

$$\begin{aligned} \|\nabla \cdot (\nabla d \odot \nabla d - \frac{1}{2}|\nabla d|^2 \mathbb{I}_2)\|_{C^{\alpha}(\Omega_T)} &\leq \| |\nabla^2(d - d_0)| |\nabla d| \|_{C^{\alpha}(\Omega_T)} + \| |\nabla^2 d_0| |\nabla(d - d_0)| \|_{C^{\alpha}(\Omega_T)} \\ &\quad + \| |\nabla^2 d_0| |\nabla d_0| \|_{C^{\alpha}(\Omega_T)} \leq C_0 + \|d - d_0\|_{C_{\alpha}^{2,1}(\Omega_T)} \|d\|_{C_{\alpha}^{2,1}(\Omega_T)} + C_0 \|d - d_0\|_{C_{\alpha}^{2,1}(\Omega_T)} \\ &\leq C_0 + \frac{\sqrt{K}}{2} \left(C_0 + \frac{\sqrt{K}}{2} \right) + C_0 \sqrt{K} \leq \frac{K}{2}. \end{aligned}$$

Combining the above estimates, we have $\|u - u_0\|_{C_{\alpha}^{2,1}(\Omega_T)} \leq \frac{3}{4}K$, which together with (4.4), yields

$$\|u - u_0\|_{C_{\alpha}^{2,1}(\Omega_T)} + \|d - d_0\|_{C_{\alpha}^{2,1}(\Omega_T)} \leq K.$$

Consequently, $L : X \rightarrow X$. □

Lemma 4.3. *There exist sufficiently large $K > 0$ and sufficiently small $T > 0$ such that $L : X \rightarrow X$ is a contraction map.*

Proof. For any $(v_i, f_i) \in X$, $i = 1, 2$, let $(u_i, d_i) = L(v_i, f_i) \in X$. Denote $u = u_1 - u_2$, $d = d_1 - d_2$, $P = P_1 - P_2$, $v = v_1 - v_2$, $f = f_1 - f_2$. Then, (u, d) solves the following mixed system:

$$\begin{aligned} \partial_t u - \Delta u + \nabla P &= G \quad \text{in } \Omega_T \\ \nabla \cdot u &= 0 \quad \text{in } \Omega_T \\ \partial_t d - \Delta d &= H \quad \text{in } \Omega_T \\ (u, d) &= (0, 0) \quad \text{on } \Omega \times \{t = 0\} \\ (u, d) &\text{ satisfies (1.2) on } \partial\Omega \times (0, T), \end{aligned} \quad (4.5)$$

where

$$G = -(v \cdot \nabla v_1 + v_2 \cdot \nabla v) - \nabla \cdot (\nabla d \odot \nabla d_1 + \nabla d_2 \odot \nabla d),$$

and

$$H = |\nabla f_1|^2 f + \nabla(f_1 + f_2) \cdot \nabla f f_2 - (v \cdot \nabla f_1 - v_2 \cdot \nabla f).$$

By the previous lemma, we know that for $i = 1, 2$,

$$\|u_i - u_0\|_{C_\alpha^{2,1}(\Omega_T)} + \|d_i - d_0\|_{C_\alpha^{2,1}(\Omega_T)} \leq K.$$

We apply the Schauder theory of parabolic systems for (4.5)₃, and we have

$$\begin{aligned} \|d\|_{C_\alpha^{2,1}(\Omega_T)} &\lesssim \|H\|_{C^\alpha(\Omega_T)} \lesssim \|v \cdot \nabla f_1 + |v_2 \cdot \nabla f| + |\nabla f_1|^2 |f| + |f_2|(|\nabla f_1| + |\nabla f_2|)|\nabla f|\|_{C^\alpha(\Omega_T)} \\ &\lesssim K^2(\|v\|_{C^\alpha(\Omega_T)} + \|f\|_{C^\alpha(\Omega_T)} + \|\nabla f\|_{C^\alpha(\Omega_T)}) \\ &\lesssim K^2\left(\delta(\|v\|_{C_\alpha^{2,1}(\Omega_T)} + \|f\|_{C_\alpha^{2,1}(\Omega_T)}) + \frac{1}{\delta}(\|v\|_{C^\alpha(\Omega_T)} + \|f\|_{C^\alpha(\Omega_T)})\right) \\ &\lesssim K^2\left(\delta + \frac{T}{\delta}\right)(\|v\|_{C_\alpha^{2,1}(\Omega_T)} + \|f\|_{C_\alpha^{2,1}(\Omega_T)}), \end{aligned} \quad (4.6)$$

where we have used $\|v\|_{C^\alpha(\Omega_T)} + \|f\|_{C^\alpha(\Omega_T)} \lesssim (\|v\|_{C_\alpha^{2,1}(\Omega_T)} + \|f\|_{C_\alpha^{2,1}(\Omega_T)})T$ that originates from the fact that $v = d = 0$ at $t = 0$.

We apply the Schauder theory of nonhomogeneous, nonstationary Stokes system for (4.5)₁–(4.5)₂, and we have

$$\begin{aligned} \|u\|_{C_\alpha^{2,1}(\Omega_T)} &\lesssim \|G\|_{C^\alpha(\Omega_T)} \lesssim \| |v| |\nabla v_1| + |v_2| |\nabla v| + |\nabla^2 d| |\nabla d_1| + |\nabla^2 d_2| |\nabla d| \|_{C^\alpha(\Omega_T)} \\ &\lesssim K \|d\|_{C_\alpha^{2,1}(\Omega_T)} + K(\|v\|_{C^\alpha(\Omega_T)} + \|\nabla v\|_{C^\alpha(\Omega_T)}) \\ &\lesssim K^3\left(\delta + \frac{T}{\delta}\right)(\|v\|_{C_\alpha^{2,1}(\Omega_T)} + \|f\|_{C_\alpha^{2,1}(\Omega_T)}) + K^2\left(\delta + \frac{T}{\delta}\right)\|v\|_{C_\alpha^{2,1}(\Omega_T)} \\ &\lesssim K^3\left(\delta + \frac{T}{\delta}\right)(\|v\|_{C_\alpha^{2,1}(\Omega_T)} + \|f\|_{C_\alpha^{2,1}(\Omega_T)}). \end{aligned} \quad (4.7)$$

It follows from (4.6) and (4.7) that

$$\|L(v_1, f_1) - L(v_2, f_2)\|_X \lesssim K^3\left(\delta + \frac{T}{\delta}\right)(\|v\|_{C_\alpha^{2,1}(\Omega_T)} + \|f\|_{C_\alpha^{2,1}(\Omega_T)})\frac{1}{2}\|(v_1, f_1) - (v_2, f_2)\|_X,$$

provided that δ and T are sufficiently small. Then, $L : X \rightarrow X$ is a contraction map as desired. $\square$

5 | ENERGY ESTIMATION

Lemma 5.1 (Global energy). *For $0 < t < +\infty$, suppose $u \in L^{2,\infty}(\Omega \times [0, T]) \cap W_2^{1,0}(\Omega_T)$, $d \in L^\infty([0, T], H^1(\Omega)) \cap L^2([0, T], H^2(\Omega))$, and $\nabla p \in L^{\frac{4}{3}}(\Omega_T)$ is a weak solution to (1.1)–(1.4). Then, for any $0 < t \leq T$, we have*

$$\int_\Omega (|u|^2 + |\nabla d|^2)(\cdot, t) + \int_{\Omega_t} (4|\mathbb{D}u|^2 + 2|\Delta d + |\nabla d|^2 d|^2) = \int_\Omega (|u_0|^2 + |\nabla d_0|^2). \quad (5.1)$$

Proof. Partially free boundary condition (1.2) permits

$$\begin{aligned} \int_{\Omega} (u \cdot \nabla u) \cdot u &= 0, \quad \int_{\Omega} \nabla P \cdot u = 0, \quad \int_{\Omega} d_t \cdot \Delta d = -\frac{d}{dt} \int_{\Omega} \frac{1}{2} |\nabla d|^2 \\ \int_{\partial\Omega} (\hat{n} \cdot \nabla) d \cdot (u \cdot \nabla) d &= 0, \quad \int_{\Omega} (u \cdot \nabla) (|\nabla d|^2) = 0. \end{aligned}$$

Assumption $u \in L^\infty([0, T], L^2(\Omega)) \cap L^2([0, T], H^1(\Omega))$ and Ladyzhenskaya inequality yield $u \in L^4(\Omega_T)$. Equation (1.1)₃ and the fact that $|d| = 1$ gives $d\Delta d + |\nabla d|^2 = 0$, which, together with assumption $\nabla \in L^2([0, T], H^1(\Omega))$, gives $|\nabla d| \in L^4(\Omega_T)$. Now we test (1.1)₁ with u and get

$$\frac{1}{2} \frac{d}{dt} \int_{\Omega} |u|^2 + 2 \int_{\Omega} |\mathbb{D}u|^2 = \int_{\Omega} \nabla d \odot \nabla d : \nabla u,$$

where the operator: stands for the inner product of two matrices.

Testing (1.1)₃ with $\Delta d + |\nabla d|^2 d$, we obtain

$$\int_{\Omega} (d_t + u \cdot \nabla d) \cdot \Delta d = \int_{\Omega} |\Delta d + |\nabla d|^2 d|^2.$$

Further, we can compute that

$$\begin{aligned} \int_{\Omega} \Delta d \cdot (u \cdot \nabla d) &= - \int_{\Omega} \nabla d : \nabla((u \cdot \nabla) d) = - \int_{\Omega} \nabla d \odot \nabla d : \nabla u - (u \cdot \nabla) |\nabla d|^2 \\ &= - \int_{\Omega} \nabla d \odot \nabla d : \nabla u. \end{aligned}$$

Adding the above equations together implies the global energy equality (5.1). $\square$

Lemma 5.2 (Interior and boundary energy inequalities). *For $0 < t < +\infty$, suppose $u \in L^{2,\infty}(\Omega \times [0, T]) \cap W_2^{1,0}(\Omega_T)$, $d \in L^\infty([0, T], H^1(\Omega)) \cap L^2([0, T], H^2(\Omega))$, and $\nabla P \in L^{\frac{4}{3}}(\Omega_T)$ is a weak solution to (1.1)-(1.4). Then, for any nonnegative $\phi \in C^\infty(\Omega)$ and $0 < s < t \leq T$, it holds that*

$$\begin{aligned} \int_{\Omega} \phi(|u|^2 + |\nabla d|^2)(t) + \int_s^t \int_{\Omega} \phi(4|\mathbb{D}u|^2 + 2|\Delta d + |\nabla d|^2 d|^2) &\leq \\ \int_{\Omega} \phi(|u|^2 + |\nabla d|^2)(s) + C \int_s^t \int_{\Omega} |\nabla \phi|(|u|^3 + |P - P_\Omega||u| + |\mathbb{D}u||u| + |\nabla d|^2|u| + |d_t||\nabla d|), \end{aligned} \quad (5.2)$$

where P_Ω is the average of P over Ω .

Proof. Testing (1.1)₁ by $u\phi$, we have

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \int_{\Omega} |u|^2 \phi + \int_{\Omega} 2|\mathbb{D}u|^2 \phi &= \int_{\Omega} -u \cdot \nabla u \cdot u \phi - 2\mathbb{D}u \cdot u \cdot \nabla \phi + (P - P_\Omega) \cdot u \cdot \nabla \phi \\ &\quad + (\nabla d \odot \nabla d - \frac{1}{2} |\nabla d|^2 \mathbb{I}_2) : \nabla(u\phi), \end{aligned}$$

and notice that

$$\int_{\Omega} -u \cdot \nabla u \cdot u \phi = \frac{1}{2} \int_{\Omega} |u|^2 u \cdot \nabla \phi$$

$$\int_{\Omega} (\nabla d \odot \nabla d - \frac{1}{2} |\nabla d|^2 \mathbb{I}_2) : \nabla(u\phi) = \int_{\Omega} (\nabla d \odot \nabla d) : \nabla u \phi + \int_{\Omega} u \cdot \nabla d \cdot \nabla d \nabla \phi - \frac{1}{2} |\nabla d|^2 u \cdot \nabla \phi.$$

Hence, (1.1)₁ gives energy estimates for u :

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \int_{\Omega} |u|^2 \phi + \int_{\Omega} 2|\mathbb{D}u|^2 \phi &\leq \int_{\Omega} (\nabla d \odot \nabla d) : \nabla u \phi \\ &+ \left(\frac{1}{2} |u|^3 + |u| |2\mathbb{D}u| + |P - P_{\Omega}| |u| + \frac{3}{2} |\nabla d|^2 |u| \right) |\nabla \phi|. \end{aligned} \quad (5.3)$$

We test (1.1)₃ with $(\Delta d + |\nabla d|^2 d)\phi$ to get

$$\int_{\Omega} (d_t + u \cdot \nabla d) \cdot \Delta d \phi = \int_{\Omega} |\Delta d + |\nabla d|^2 d|^2 \phi,$$

and notice that

$$\begin{aligned} \int_{\Omega} d_t \cdot \Delta d \phi &= -\frac{1}{2} \frac{d}{dt} \int_{\Omega} |\nabla d|^2 \phi - \int_{\Omega} d_t \cdot \nabla d \cdot \nabla \phi \\ \int_{\Omega} u \cdot \nabla d \cdot \Delta d \phi &= -\int_{\Omega} \frac{1}{2} u^i \partial_i (|\nabla d|^2) \phi + \nabla d \odot \nabla d : \nabla u \phi + (u \cdot \nabla) d (\nabla \phi \cdot \nabla d) \\ &= \int_{\Omega} \frac{1}{2} |\nabla d|^2 u \cdot \nabla \phi - \nabla d \odot \nabla d : \nabla u \phi - (u \cdot \nabla) d (\nabla \phi \cdot \nabla d). \end{aligned}$$

Therefore, (1.1)₃ gives energy estimates for ∇d :

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \int_{\Omega} |\nabla d|^2 \phi + \int_{\Omega} |\Delta d + |\nabla d|^2 d|^2 \phi &= \int_{\Omega} -d_t \cdot \nabla d \cdot \nabla \phi + \frac{1}{2} |\nabla d|^2 u \cdot \nabla \phi \\ &- \nabla d \odot \nabla d : \nabla u \phi - (u \cdot \nabla) d (\nabla \phi \cdot \nabla d). \end{aligned} \quad (5.4)$$

Adding energy estimates (5.3) and (5.4) yields (5.2). $\square$

Remark 5.3. By virtue of Lemma 2.12 for nonaxisymmetric domain, $\|\nabla u\|_{L^2(\Omega)}$ and $\|\mathbb{D}u\|_{L^2(\Omega)}$ are equivalent norms; thus, Lemma 5.1 and Lemma 5.2 can have corresponding inequalities with $|\mathbb{D}u|^2$ replaced by $|\nabla u|^2$. We shall use this variant in the proof of Theorem 1.3 because we need to use the convergence of L^2 -norm of ∇u to zero in some domain in order to conclude that $u \rightarrow 0$ in H^1 . The original version of energy estimates in terms of $\mathbb{D}u$ is insufficient.

6 | GLOBAL WEAK SOLUTION AND PROOF OF THEOREM 1.3

We now derive the life span estimate for smooth solutions in terms of Sobolev space norms of initial data.

Lemma 6.1. Let $\epsilon_0 > 0$ be given in Lemmas 3.2 and 3.3. There exist $0 < \epsilon_1 \ll \epsilon_0$, $0 < R_0 \leq 1$, and $\theta_0 = \theta_0(\epsilon_1, E_0) \in (0, \frac{1}{4})$ with $E_0 = \int_{\Omega} |u_0|^2 + |\nabla d_0|^2$, such that if $u_0 \in C^{2,\beta}(\overline{\Omega}, \mathbb{R}^2) \cap \mathbf{H}$ and $d_0 \in C^{2,\beta}(\overline{\Omega}, \mathbb{S}^2) \cap \mathbf{J}$ satisfy

$$\sup_{x \in \overline{\Omega}} \int_{\Omega \cap B_{2R_0}(x)} |u_0|^2 + |\nabla d_0|^2 \leq \epsilon_1^2, \quad (6.1)$$

then there exist $T_0 \geq \theta_0 R_0^2$ and a unique solution $(u, d) \in C^\infty(\Omega \times (0, T_0), \mathbb{R}^2 \times \mathbb{S}^2) \cap C_{\beta}^{2,1}(\overline{\Omega} \times [0, T_0], \mathbb{R}^2 \times \mathbb{S}^2)$ to (1.1)–(1.4) satisfying

$$\sup_{(x,t) \in \overline{\Omega}_{T_0}} \int_{\Omega \cap B_{R_0}(x)} (|u|^2 + |\nabla d|^2)(\cdot, t) \leq 2\epsilon_1^2. \quad (6.2)$$

Proof. Theorem 4.1 states the existence of $T_0 > 0$ such that there exists a unique smooth solution $(u, d) \in C^\infty(\Omega \times (0, T_0), \mathbb{R}^2 \times \mathbb{S}^2) \cap C_{\alpha}^{2,1}(\overline{\Omega} \times [0, T_0], \mathbb{R}^2 \times \mathbb{S}^2)$ to (1.1)–(1.2). Let $0 < t_0 \leq T_0$ be the maximal times such that

$$\sup_{0 \leq t \leq t_0} \sup_{x \in \overline{\Omega}} \int_{\Omega \cap B_{R_0}(x)} (|u|^2 + |\nabla d|^2)(\cdot, t) \leq 2\epsilon_1^2. \quad (6.3)$$

Since t_0 is defined to be the maximal time, we have

$$\sup_{x \in \overline{\Omega}} \int_{\Omega \cap B_{R_0}(x)} (|u|^2 + |\nabla d|^2)(\cdot, t_0) = 2\epsilon_1^2.$$

Now, we estimate the lower bound of t_0 as follows. Assume $t_0 \leq R_0^2 \leq 1$ (otherwise, we have finished the proof). Set

$$\begin{aligned} E(t) &= \int_{\Omega} (|u|^2 + |\nabla d|^2)(\cdot, t), \\ E_0 &= \int_{\Omega} (|u_0|^2 + |\nabla d_0|^2). \end{aligned}$$

Observe the energy inequality in Lemma 5.1: for $0 < t \leq t_0$,

$$E(t) + \int_{\Omega_t} (|\mathbb{D}u|^2 + |\Delta d + |\nabla d|^2 d|^2) \leq E_0.$$

Lemma 2.12 allows us to deduce that

$$\int_{\Omega_t} |\nabla u|^2 \lesssim \int_{\Omega_t} |u|^2 + \int_{\Omega_t} |\mathbb{D}u|^2 \leq (t_0 + 1)E_0 \leq 2E_0.$$

Also, we use the Ladyzhenskaya inequality in Lemma 2.3, and it follows that

$$\int_{\Omega_t} |\nabla d|^4 \lesssim \mathcal{E}_{R_0}^2(t) \left(\int_{\Omega_t} |\Delta d|^2 + \frac{tE_0}{R_0^2} \right),$$

where

$$\begin{aligned}\mathcal{E}_{R_0}^2(t) &= \sup_{(x,s) \in \Omega_t} \int_{\Omega \cap B_{R_0}(x)} |\nabla d|^2(\cdot, s) \\ \mathcal{E}_{R_0}^1(t) &= \sup_{(x,s) \in \Omega_t} \int_{\Omega \cap B_{R_0}(x)} |u|^2(\cdot, s) \\ \mathcal{E}_{R_0}(t) &= \mathcal{E}_{R_0}^1(t) + \mathcal{E}_{R_0}^2(t).\end{aligned}$$

By (6.3), we have $\mathcal{E}_{R_0}(t) \leq 4\epsilon_1^2$ for all $0 \leq t \leq t_0$. As a consequence,

$$\int_{\Omega_{t_0}} |\nabla d|^4 \lesssim \epsilon_1^2 \left(\int_{\Omega_{t_0}} |\Delta d|^2 + \frac{t_0 E_0}{R_0^2} \right).$$

The above inequality, together with the energy inequality and the fact that $|\Delta d|^2 \leq 2(|\Delta d + |\nabla d|^2 d| + |\nabla d|^4)$, implies

$$\int_{\Omega_{t_0}} |\Delta d|^2 \leq E_0 + C_0 \epsilon_1^2 \left(\int_{\Omega_{t_0}} |\Delta d|^2 + \frac{t_0 E_0}{R_0^2} \right).$$

Therefore, by taking $0 < \epsilon_1^2 \leq \min(1, \frac{1}{2C_0})$, we have

$$\int_{\Omega_{t_0}} |\Delta d|^2 \leq C_0 E_0,$$

and hence,

$$\int_{\Omega_{t_0}} |\nabla d|^4 \leq C_0 \epsilon_1^2 E_0. \quad (6.4)$$

Similarly, we apply Lemma 2.3 to $\int_{\Omega_{t_0}} |u|^4$ and get

$$\begin{aligned}\int_{\Omega_{t_0}} |u|^4 &\lesssim \mathcal{E}_{R_0}^1(t_0) \left(\int_{\Omega_{t_0}} |\nabla u|^2 + \frac{1}{R_0^2} \int_{\Omega_{t_0}} |u|^2 \right) \lesssim \mathcal{E}_{R_0}^1(t_0) \left(\int_{\Omega_{t_0}} |\nabla u|^2 + \frac{t_0 E_0}{R_0^2} \right) \\ &\lesssim \epsilon_1^2 \left(E_0 + \frac{t_0}{R_0^2} E_0 \right) \leq C_0 \epsilon_1^2 E_0.\end{aligned} \quad (6.5)$$

We would like to estimate $\mathcal{E}_{R_0}(t)$. For any $x \in \overline{\Omega}$, let cut-off function $\phi \in C_0^\infty(B_{2R_0}(x))$ such that $0 \leq \phi \leq 1$, $\phi \equiv 1$ on $B_{R_0}(x)$, and $|\nabla \phi| \lesssim 1/R_0$. Then, by interior and boundary energy inequalities (5.2), we have

$$\begin{aligned}\sup_{0 \leq t \leq t_0} \int_{\Omega \cap B_{R_0}(x)} (|u|^2 + |\nabla d|^2) - \mathcal{E}_{2R_0}(0) &\leq \sup_{0 \leq t \leq t_0} \int_{\Omega \cap B_{2R_0}(x)} (|u|^2 + |\nabla d|^2) \phi - \mathcal{E}_{2R_0}(0) \\ &\lesssim \int_{\Omega_{t_0}} |\nabla \phi| (|u|^3 + |P - P_\Omega||u| + |\nabla u||u| + |d_t||\nabla d|)\end{aligned}$$

$$\begin{aligned} & \lesssim \left(\frac{t_0}{R_0^2} \right)^{\frac{1}{4}} \left(\|u\|_{L^4(\Omega_{t_0})}^3 + \|\nabla u\|_{L^2(\Omega_{t_0})} \|u\|_{L^4(\Omega_{t_0})} + \|\nabla d\|_{L^4(\Omega_{t_0})}^2 \|u\|_{L^4(\Omega_{t_0})} \right. \\ & \quad \left. + \|d_t\|_{L^2(\Omega_{t_0})} \|\nabla d\|_{L^4(\Omega_{t_0})} \right) + \|P - P_\Omega\|_{L^{\frac{4}{3}}(\Omega_{t_0})} \|u\|_{L^4(B_{2R_0}(x) \times (0, t_0) \cap \Omega_{t_0})}. \end{aligned} \quad (6.6)$$

Notice that $\|u\|_{L^2(\Omega_{t_0})} \leq (t_0 E_0)^{\frac{1}{2}} \leq E_0^{\frac{1}{2}}$. To estimate d_t , we test (1.1)₃ with d_t and use (6.4)–(6.5) to obtain

$$\int_{\Omega_{t_0}} |d_t|^2 \lesssim \int_{\Omega} |\nabla d_0|^2 + \int_{\Omega_{t_0}} |u|^2 |\nabla d|^2 \lesssim E_0 + \|u\|_{L^4(\Omega_{t_0})}^2 \|\nabla d\|_{L^4(\Omega_{t_0})}^2 \leq C_0 E_0.$$

The above inequality, together with Lemma 3.1, (6.4), (6.5), into (6.6), yields

$$2\epsilon_1^2 = \sup_{0 \leq t \leq t_0} \int_{\Omega \cap B_{R_0}(x)} (|u|^2 + |\nabla d|^2) \leq \epsilon_1^2 + C_0 \left(\frac{t_0}{R_0^2} \right)^{\frac{1}{4}} \epsilon_1^{\frac{1}{2}} E_0^{\frac{3}{4}} + \mathcal{O}(R_0) \epsilon_1^{\frac{1}{2}} E_0^{\frac{3}{4}},$$

where term $\|u\|_{L^4(B_{R_0}(x) \times (0, t_0) \cap \Omega_{t_0})}$ in (6.6) gives higher order term of R_0 that becomes negligible if we take $R_0 \leq \epsilon_1^2$. This implies

$$t_0 \geq \frac{\epsilon_1^6}{C_0^4 E_0^3} R_0^2 := \theta_0 R_0^2.$$

The proof is thus complete by taking $T_0 = t_0$. □

Proof of Theorem 1.3

Proof. We follow the argument in [34].

Step 1: Approximation of initial data. Because we only assume initial condition $u_0 \in \mathbf{L}$ and $d_0 \in \mathbf{J}$, we need to approximate them by smooth functions so that we can utilize Lemma 6.1. By density of smooth maps $L^2(\Omega, \mathbb{R}^2)$ and by Lemma 2.11, we assume that there exist $\{u_0^k\}_{k=1}^\infty \subseteq C^\infty(\Omega, \mathbb{R}^2) \cap C^{2,\alpha}(\overline{\Omega}, \mathbb{R}^2) \cap \mathbf{L}$ and $\{d_0^k\} \subseteq C^\infty(\Omega, \mathbb{S}^2) \cap C^{2,\alpha}(\overline{\Omega}, \mathbb{S}^2) \cap \mathbf{J}$ such that

$$\begin{aligned} \lim_{k \rightarrow \infty} \|u_0^k - u_0\|_{L^2(\Omega)} &= 0, \\ \lim_{k \rightarrow \infty} \|d_0^k - d_0\|_{H^1(\Omega)} &= 0. \end{aligned}$$

By the absolute continuity of $\int |u_0|^2 + |\nabla d_0|^2$, there exists $R_0 > 0$ such that

$$\sup_{x \in \overline{\Omega}} \int_{\Omega \cap B_{2R_0}(x)} |u_0|^2 + |\nabla d_0|^2 \leq \frac{\epsilon_1^2}{2},$$

where $\epsilon_1 > 0$ is given in Lemma 6.1. By the strong convergence of $(u_0^k, \nabla d_0^k) \rightarrow (u_0, \nabla d_0)$ in $L^2(\Omega)$, it follows that

$$\sup_{x \in \bar{\Omega}} \int_{\Omega \cap B_{2R_0}(x)} |u_0^k|^2 + |\nabla d_0^k|^2 \leq \epsilon_1^2 \text{ for } k \text{ sufficiently large.}$$

By discarding finitely many k 's and taking the rest as a subsequence, we may assume without loss of generality that the above holds for $k \geq 1$. As a consequence, Lemma 6.1 states that there exists $\theta_0 = (\epsilon_1, E_0) \in (0, 1)$ and life spans $T_0^k \geq \theta_0 R_0^2$ such that each initial condition (u_0^k, d_0^k) admits a $(u^k, d^k) \in C^\infty(\Omega \times (0, T_0^k), \mathbb{R}^2 \times \mathbb{S}^2) \cap C_\alpha^{2,1}(\bar{\Omega} \times (0, T_0^k), \mathbb{R}^2 \times \mathbb{S}^2)$ as a solution to (1.1)–(1.4).

Moreover, we have

$$\sup_{(x,t) \in \bar{\Omega}_{T_0^k}} \int_{\Omega \cap B_{R_0}(x)} (|u^k|^2 + |\nabla d^k|^2)(\cdot, t) \leq 2\epsilon_1^2, \quad (6.7)$$

and Lemma 5.1 gives energy estimate

$$\sup_{0 \leq t \leq T_0^k} \int_{\Omega} (|u^k|^2 + |\nabla d^k|^2)(\cdot, t) + \int_{\Omega_{T_0^k}} (4|\mathbb{D}u^k|^2 + 2|\Delta d^k + |\nabla d^k|^2 d^k|^2) \lesssim \int_{\Omega} |u_0^k|^2 + |\nabla d_0^k|^2 \lesssim E_0. \quad (6.8)$$

The above two estimates, together with Lemma 2.3, imply that

$$\int_{\Omega_{T_0^k}} |u^k|^4 + |\nabla d^k|^4 \lesssim \epsilon_1^2 E_0, \quad (6.9)$$

$$\int_{\Omega_{T_0^k}} |d_t^k|^2 + |\nabla^2 d^k|^2 \lesssim E_0. \quad (6.10)$$

We use Lemma 3.1 and estimates (6.7), (6.9), and (6.10) to conclude that

$$\|\nabla P^k\|_{L^{\frac{4}{3}}(\Omega_{T_0^k})} \lesssim \|\nabla u^k\|_{L^2(\Omega_{T_0^k})} \|u^k\|_{L^4(\Omega_{T_0^k})} + \|\nabla^2 d^k\|_{L^2(\Omega_{T_0^k})} \|\nabla d^k\|_{L^4(\Omega_{T_0^k})} \lesssim \epsilon_1^{\frac{1}{2}} E_0^{\frac{3}{4}}. \quad (6.11)$$

We collect estimates (6.9)–(6.11) and apply Theorem 1.2 to obtain that for any $\delta > 0$,

$$\begin{aligned} \|(u^k, d^k)\|_{C_\alpha^{2,1}(\bar{\Omega} \times [\delta, T_0^k])} &\leq C \left(\delta, E_0, \|u^k\|_{L^4(\Omega_{T_0^k})}, \|\nabla d^k\|_{L^4(\Omega_{T_0^k})}, \|\nabla P^k\|_{L^{\frac{4}{3}}(\Omega_{T_0^k})} \right) \\ &\leq C(\delta, E_0, \epsilon_1). \end{aligned} \quad (6.12)$$

Furthermore, for any compact subdomain $K \Subset \Omega$ and $\delta > 0$,

$$\|(u^k, d^k)\|_{C^l(K \times [\delta, T_0^k])} \leq C(\text{dist}(K, \partial\Omega), \delta, l, E_0).$$

Hence, after possibly passing to subsequences, there exist $T_0 \geq \theta_0 R^2$, $u \in W_2^{1,0}(\Omega_{T_0}, \mathbb{R}^2)$, and $d \in W_2^{2,1}(\Omega_{T_0}, \mathbb{S}^2)$ such that

$$u^k \rightharpoonup u \text{ weakly in } W_2^{1,0}(\Omega_{T_0}, \mathbb{R}^2), \quad d^k \rightharpoonup d \text{ weakly in } W_2^{2,1}(\Omega_{T_0}, \mathbb{S}^2),$$

$$\lim_{k \rightarrow \infty} \|u^k - u\|_{L^4(\Omega_{T_0})} = 0,$$

$$\lim_{k \rightarrow \infty} \|d^k - d\|_{L^4(\Omega_{T_0})} + \|\nabla d^k - \nabla d\|_{L^2(\Omega_{T_0})} = 0,$$

and for any $l \geq 2$, $\delta > 0$, $\gamma < \beta$, and compact $K \Subset \Omega$,

$$\lim_{k \rightarrow \infty} \|(u^k, d^k) - (u, d)\|_{C^l(K \times [\delta, T_0])} = 0,$$

$$\lim_{k \rightarrow \infty} \|(u^k, d^k) - (u, d)\|_{C_\gamma^{2,1}(\overline{\Omega} \times [\delta, T_0])} = 0.$$

As a result, $(u, d) \in C^\infty(\Omega \times (0, T_0], \mathbb{R}^2 \times \mathbb{S}^2) \cap C_\beta^{2,1}(\overline{\Omega} \times (0, T_0], \mathbb{R}^2 \times \mathbb{S}^2)$ solves (1.1)–(1.3) in $\Omega \times (0, T_0]$ and satisfies the boundary condition.

By (1.1)₁, $u_t^k \in L^2([0, T_0^k], H^{-1}(\Omega))$ and $\|u_t^k\|_{L^2([0, T_0^k], H^{-1}(\Omega))} \leq CE_0$, this together with (6.10) states that after possibly passing to a subsequence, $(u, \nabla d)(\cdot, t) \rightharpoonup (u_0, \nabla d_0)$ weakly in $L^2(\Omega)$ as $t \rightarrow 0$. Consequently,

$$E(0) \leq \liminf_{t \rightarrow 0} E(t).$$

On the other hand, steps in energy estimates (6.8) give

$$E(0) \geq \limsup_{t \rightarrow 0} E(t).$$

This implies that $(u, \nabla d)(\cdot, t)$ converges to $(u_0, \nabla d_0)$ strongly in $L^2(\Omega)$. Hence, (u, d) satisfies initial condition (1.3).

Step 2: weak extension beyond singular time. Let $T_1 \in (T_0, \infty)$ be the first singular time of (u, d) , that is,

$$(u, d) \in C^\infty(\Omega \times (0, T_1), \mathbb{R}^2 \times \mathbb{S}^2) \cap C_\beta^{2,1}(\overline{\Omega} \times (0, T_1), \mathbb{R}^2 \times \mathbb{S}^2),$$

but

$$(u, d) \notin C^\infty(\Omega \times (0, T_1], \mathbb{R}^2 \times \mathbb{S}^2) \cap C_\beta^{2,1}(\overline{\Omega} \times (0, T_1], \mathbb{R}^2 \times \mathbb{S}^2).$$

Now, we would like to extend this weak solution in time. To do so, we shall investigate and define new “initial” data at $t = T_1$.

We claim that $(u, d) \in C^0([0, T_1], L^2(\Omega))$. Indeed, we test (1.1)₃ with $\phi \in H_0^2(\Omega, \mathbb{R}^3)$ and obtain

$$\begin{aligned} |\langle d_t, \phi \rangle| &\lesssim \|\nabla d\|_{L^2(\Omega)} \|\nabla \phi\|_{L^2(\Omega)} + (\|u\|_{L^2(\Omega)} \|\nabla d\|_{L^2(\Omega)} + \|\nabla d\|_{L^2(\Omega)}^2) \|\phi\|_{C^0(\Omega)} \\ &\lesssim (\|\nabla d\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)} \|\nabla d\|_{L^2(\Omega)} + \|\nabla d\|_{L^2(\Omega)}^2) \|\phi\|_{H^2(\Omega)}, \end{aligned}$$

where we have used $\|\phi\|_{C^0(\Omega)} \lesssim \|\phi\|_{H^2(\Omega)}$ by the fact that $H_0^2(\Omega) \subseteq C^0(\Omega)$. Therefore, $d_t \in L^2([0, T_1], H^{-2}(\Omega))$, which together with the fact that $d \in L^2([0, T_1], H^1(\Omega))$, implies that $d \in C^0([0, T_1], L^2(\Omega))$. Similarly, we test (1.1)₁ with $\phi \in H_0^3(\Omega, \mathbb{R}^2)$ and $\nabla \cdot \phi = 0$ to obtain

$$\begin{aligned} |\langle u_t, \phi \rangle| &\lesssim \|\nabla u\|_{L^2(\Omega)} \|\nabla \phi\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)} \|\nabla u\|_{L^2(\Omega)} \|\phi\|_{C^0(\Omega)} + \|\nabla u\|_{L^2(\Omega)}^2 \|\nabla \phi\|_{C^0(\Omega)} \\ &\lesssim (\|\nabla u\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)} \|\nabla u\|_{L^2(\Omega)} + \|\nabla u\|_{L^2(\Omega)}^2) \|\phi\|_{H^3(\Omega)}, \end{aligned}$$

where we have used $\|\phi\|_{C^1(\Omega)} \lesssim \|\phi\|_{H^3(\Omega)}$ by the fact that $H_0^3(\Omega) \subseteq C^1(\Omega)$. Since $\nabla \cdot u_t = 0$, it follows that $u_t \in L^2([0, T_1], H^{-3}(\Omega))$. This, together with the fact that $u \in L^2([0, T_1], H^1(\Omega))$, implies that $u \in C^0([0, T_1], L^2(\Omega))$, and thus the claim is valid.

Now $(u, d) \in C^0([0, T_1], L^2(\Omega))$ means that we can define

$$(u(T_1), d(T_1)) = \lim_{t \nearrow T_1} (u(t), d(t)) \text{ in } L^2(\Omega).$$

Then, the energy estimate in Lemma 5.1 yields $\nabla d \in L^\infty([0, T_1], L^2(\Omega))$. Thus, we have the weak convergence $\nabla d(t) \rightharpoonup \nabla d(T_1)$ in $L^2(\Omega)$. In particular, $u(T_1) \in \mathbf{L}$ and $d(T_1) \in H^1(\Omega)$. Moreover, $u(T_1)$ and $d(T_1)$ satisfy the partially free boundary condition (1.2) since $u(t)$ and $d(t)$ satisfy it.

Now we can use $(u(T_1), d(T_1))$ as initial data in the above procedure to obtain a weak extension of (u, d) beyond T_1 that solves (1.1)–(1.4). We may be confronted with another singular time, and we will continue to process a weak extension in this scheme. We want to show that there can be, at most finitely many, such singular times, and we can construct an external weak solution beyond afterward.

Notice that in the study of heat flow of harmonic maps, each singularity carries a loss of energy. Here, we will prove a similar result: at each singular time, the energy is lost by at least ϵ_1^2 . By definition, T_1 is the first singular time of (u, d) , and then Lemma 6.1 states that we must have

$$\limsup_{t \nearrow T_1} \max_{x \in \bar{\Omega}} \int_{\Omega \cap B_{2R_0}(x)} (|u|^2 + |\nabla d|^2)(\cdot, t) > \epsilon_1^2.$$

This means that there exists $t_i \nearrow T_1$ and $x_0 \in \bar{\Omega}$ such that for any $R > 0$,

$$\limsup_{t_i \nearrow T_1} \int_{\Omega \cap B_{2R_0}(x_0)} (|u|^2 + |\nabla d|^2)(\cdot, t_i) > \epsilon_1^2.$$

And we observe that

$$\begin{aligned} \int_{\Omega} (|u|^2 + |\nabla d|^2)(\cdot, T_1) &= \lim_{R \rightarrow 0} \int_{\Omega \setminus B_R(x_0)} (|u|^2 + |\nabla d|^2)(\cdot, T_1) \\ &\leq \lim_{R \rightarrow 0} \liminf_{t_i \nearrow T_1} \int_{\Omega \setminus B_R(x_0)} (|u|^2 + |\nabla d|^2)(\cdot, t_i) \\ &\leq \lim_{R \rightarrow 0} \liminf_{t_i \nearrow T_1} \int_{\Omega} (|u|^2 + |\nabla d|^2)(\cdot, t_i) - \lim_{R \rightarrow 0} \limsup_{t_i \nearrow T_1} \int_{\Omega \cap B_R(x_0)} (|u|^2 + |\nabla d|^2)(\cdot, t_i) \\ &\leq \liminf_{t_i \nearrow T_1} \int_{\Omega} (|u|^2 + |\nabla d|^2)(\cdot, t_i) - \epsilon_1^2 \leq E_0 - \epsilon_1^2. \end{aligned}$$

This shows that each singular time takes away energy at least ϵ_1^2 , so the number of singular time is bounded by E_0/ϵ_1^2 . Let $0 < T_L < \infty$ be the last singular time, then we must have

$$E(T_L) = \int_{\Omega} (|u|^2 + |\nabla d|^2)(\cdot, T_L) < \epsilon_1^2.$$

Consequently, if we take $(u(T_L), d(T_L))$ as the initial condition and construct a weak solution to (1.1)–(1.3), this weak extension will be an eternal weak solution.

Step 3: Uniqueness of global weak solution. Uniqueness of regular solution to (1.1) with Dirichlet boundary condition was shown in [37], and their techniques involve proving

$$\sqrt{t}(\|u(t)\|_{L^\infty(\Omega)} + \|\nabla d(t)\|_{L^\infty(\Omega)}) \rightarrow 0$$

as $t \rightarrow 0$, the L^p - L^q regularity of the heat and Stokes operators. We shall follow the same argument as in [37]. Suppose that there are two weak solutions (u_i, d_i) , $i = 1, 2$ to (1.1)–(1.3) with (1.4) such that $u_i \in L^\infty([0, T], L^2(\Omega, \mathbb{R}^2)) \cap L^2([0, T], H^1(\Omega, \mathbb{R}^2))$ and $d_i \in L^\infty([0, T], H^1(\Omega, \mathbb{S}^2)) \cap L^2([0, T], H^2(\Omega, \mathbb{S}^2))$.

First we claim that $A_i(t) = \sup_{0 < s < t} \sqrt{s}(\|u_i(s)\|_{L^\infty(\Omega)} + \|\nabla d_i(s)\|_{L^\infty(\Omega)}) < +\infty$. For $(x_0, t_0) \in \Omega \times (0, T)$, we take $0 < \tau \leq \sqrt{t_0}$. Since (u, d) solves (1.1) on $\Omega \times [0, T]$, by a scaling argument, taking $(v, g)(y, s) = (\tau u, d)(x + \tau y, \tau^2 + \tau^2 s)$ for $(y, s) \in P' = [-1, 0] \times \frac{1}{\tau} \Omega$, one has (v, g) solves (1.1) on P' . Thus, one can apply Theorem 1.2 to conclude that $(v, g) \in C^\infty(P') \cap C_\alpha^{2,1}(\overline{P'})$. This means $\|v\|_{L^\infty(P')} + \|\nabla g\|_{L^\infty(P')} \leq C(u, d) < +\infty$, where $C(u, d)$ is independent of the scaling size τ . Back to the original scales, we obtain

$$\sup_{0 < \tau \leq \sqrt{t_0}} \tau(\|u(\tau^2)\|_{L^\infty(\Omega)} + \|\nabla d(\tau^2)\|_{L^\infty(\Omega)}) < +\infty.$$

Next, we claim that $A_i(t) = \sup_{0 < s < t} \sqrt{s}(\|u_i(s)\|_{L^\infty(\Omega)} + \|\nabla d_i(s)\|_{L^\infty(\Omega)}) \rightarrow 0$ as $t \rightarrow 0$. From the global energy equality in Lemma 5.1, we see that the energy $E_i(t) = \int_\Omega |u_i(t)|^2 + |\nabla d_i(t)|^2$ is monotone decreasing with respect to $t \geq 0$. As a consequence,

$$\lim_{t \rightarrow 0} E_i(t) \leq E(0) = \int_\Omega |u_0|^2 + |\nabla d_0|^2.$$

On the other hand, since $(u_i(t), \nabla d_i(t))$ converges weakly to $(u_0, \nabla d_0)$ in $L^2(\Omega)$ as $t \rightarrow 0$, lower semicontinuity also gives $\lim_{t \rightarrow 0} E_i(t) \geq E_0$, and hence,

$$\lim_{t \rightarrow 0} E_i(t) = E_0.$$

So, we have $E_i(t) \in C([0, T])$. It then follows that

$$\lim_{t \rightarrow 0} \sup_{x \in \Omega} \int_{B_t(x) \times [0, t^2]} \sum_{i=1}^2 (|u_i|^2 + |\nabla d_i|^2) = 0,$$

and by the smoothness of u_i and ∇d_i , we have

$$\lim_{t \rightarrow 0} \sup_{0 < s < t} \sqrt{s}(\|u_i(s)\|_{L^\infty(\Omega)} + \|\nabla d_i(s)\|_{L^\infty(\Omega)}) = 0.$$

Finally, we claim that $\lim_{t \rightarrow 0} A_i(t) = 0$ for $i = 1, 2$ implies that $(u_1, d_1) \equiv (u_2, d_2)$ on some $\Omega \times [0, t_0]$. Let $\tilde{u} = u_1 - u_2$ and $\tilde{d} = d_1 - d_2$, then we take the difference of the corresponding

equations and obtain

$$\begin{aligned}
 \partial_t \tilde{u} - \mathbb{A} \tilde{u} &= -\mathbb{P} \nabla \cdot (\tilde{u} \otimes u_1 + u_2 \otimes \tilde{u} + \nabla \tilde{d} \otimes \nabla d_1 + \nabla d_2 \otimes \nabla \tilde{d}) \\
 \nabla \cdot \tilde{u} &= 0 \\
 \partial_t \tilde{d} - \Delta \tilde{d} &= [(\nabla d_1 + \nabla d_2) \cdot \nabla \tilde{d} d_1 + |\nabla d_2|^2 \tilde{d}] - [\tilde{u} \cdot \nabla d_1 + u_2 \cdot \nabla \tilde{d}] \\
 (\tilde{u}, \tilde{d})|_{t=0} &= 0 \\
 (\tilde{u}, \tilde{d}) &\text{ satisfies (1.2).}
 \end{aligned} \tag{6.13}$$

Now for $\delta \in (0, 1)$, we define

$$D_\delta(t) = t^{\frac{1-\delta}{2}} (\|u_1(t)\|_{L^{2/\delta}(\Omega)} + \|u_2(t)\|_{L^{2/\delta}(\Omega)} + \|\nabla d_1(t)\|_{L^{2/\delta}(\Omega)} + \|\nabla d_2(t)\|_{L^{2/\delta}(\Omega)}).$$

By the interpolation inequality, we obtain

$$D_\delta(t) \leq (E_1(t) + E_2(t))^\delta (A_1(t) + A_2(t))^{1-\delta}.$$

Recall that by Duhamel's formula, we have

$$\begin{aligned}
 \tilde{u}(t) &= - \int_0^t e^{-(t-s)\mathbb{A}} \mathbb{P} \nabla \cdot (\tilde{u} \otimes u_1 + u_2 \otimes \tilde{u} + \nabla \tilde{d} \otimes \nabla d_1 + \nabla d_2 \otimes \nabla \tilde{d})(s), \\
 \tilde{d}(t) &= \int_0^t e^{-(t-s)\Delta} [(\nabla d_1 + \nabla d_2) \cdot \nabla \tilde{d} d_1 + |\nabla d_2|^2 \tilde{d} - \tilde{u} \cdot \nabla d_1 - u_2 \cdot \nabla \tilde{d}](s).
 \end{aligned}$$

Thus, we apply Lemma 2.5 with $q = 2/\delta$ and $p = 1$, together with Hölder inequality, and obtain

$$\begin{aligned}
 \|\tilde{d}(t)\|_{L^{2/\delta}(\Omega)} &\lesssim \int_0^t (t-s)^{-\frac{2-\delta}{2}} \left[\sum_{i=1}^2 \|\nabla d_i(s)\|_{L^2(\Omega)} + \|u_i(s)\|_{L^2(\Omega)} \right]^2 ds \\
 &\lesssim \left(\int_0^t (t-s)^{-\frac{2-\delta}{2}} ds \right) \sup_{0 \leq s \leq t} (E_1 + E_2)(s) \lesssim t^{\delta/2} \sup_{0 \leq s \leq t} (E_1 + E_2)(s).
 \end{aligned}$$

Moreover, we can apply Lemma 2.5 with $q = 2/\delta$ and $p = 2/(\delta + 1)$, together with the Hölder inequality, to obtain

$$\begin{aligned}
 &\|\tilde{d}(t)\|_{L^{2/\delta}(\Omega)} \\
 &\lesssim \int_0^t (t-s)^{-1/2} \left[\sum_{i=1}^2 \|\nabla d_i(s)\|_{L^{2/\delta}(\Omega)} + \|u_i(s)\|_{L^{2/\delta}(\Omega)} \right] (\|\nabla \tilde{d}(s)\|_{L^2(\Omega)} + \|\tilde{u}(s)\|_{L^2(\Omega)}) \\
 &+ \int_0^t (t-s)^{-1/2} \|\nabla d_2\|_{L^\infty(\Omega)} \|\nabla d_2\|_{L^2(\Omega)} \|\tilde{d}(s)\|_{L^{2/\delta}(\Omega)} \\
 &\lesssim \left(\int_0^t (t-s)^{-1/2} s^{(\delta-1)/2} ds \right) \left(\sup_{0 < s \leq t} D_\delta(s) \right) \left(\sup_{0 \leq s \leq t} \|\nabla \tilde{d}(s)\|_{L^2(\Omega)} + \|\tilde{u}(s)\|_{L^2(\Omega)} \right) \\
 &+ \left(\int_0^t (t-s)^{-1/2} s^{(\delta-1)/2} ds \right) \left(\sup_{0 < s \leq t} A_2(s) \right) \left(\sup_{0 \leq s \leq t} (E_1 + E_2)(s) \right) \left(\sup_{0 < s \leq t} s^{-\delta/2} \|\tilde{d}(s)\|_{L^{2/\delta}(\Omega)} \right)
 \end{aligned}$$

$$\begin{aligned} & \lesssim t^{\delta/2} \left[\sup_{0 < s \leq t} D_\delta(s) + \left(\sup_{0 < s \leq t} A_2(s) \right) \left(\sup_{0 < s \leq t} (E_1 + E_2)(s) \right) \right] \\ & \times \sup_{0 < s \leq t} \left[\|\nabla \tilde{d}(s)\|_{L^2(\Omega)} + \|\tilde{u}(s)\|_{L^2(\Omega)} + s^{-\delta/2} \|\tilde{d}(s)\|_{L^{2/\delta}(\Omega)} \right], \end{aligned} \quad (6.14)$$

which converges to 0 as $t \rightarrow 0$. Also, we apply Lemma 2.5 with $q = 2$ and $p = 2/(\delta + 1)$, together with Hölder inequality, to obtain

$$\begin{aligned} & \|\nabla \tilde{d}(t)\|_{L^{2/\delta}(\Omega)} \\ & \lesssim \int_0^t (t-s)^{-(1+\delta)/2} (\|\nabla d_1(s)\|_{L^{2/\delta}(\Omega)} + \|\nabla d_2(s)\|_{L^{2/\delta}(\Omega)}) \|\nabla \tilde{d}(s)\|_{L^2(\Omega)} ds \\ & + \int_0^t (t-s)^{-(1+\delta)/2} (\|\nabla d_1(s)\|_{L^{2/\delta}(\Omega)} \|\tilde{u}(s)\|_{L^2(\Omega)} + \|u_2(s)\|_{L^{2/\delta}(\Omega)} \|\nabla \tilde{d}(s)\|_{L^2(\Omega)}) ds \\ & + \int_0^t (t-s)^{-(1+\delta)/2} \|\nabla d_2(s)\|_{L^\infty(\Omega)} \|\nabla d_2(s)\|_{L^2(\Omega)} \|\tilde{d}(s)\|_{L^{2/\delta}(\Omega)} ds \\ & \lesssim \left(\int_0^t (t-s)^{-(1+\delta)/2} s^{(\delta-1)/2} ds \right) \left(\sup_{0 < s \leq t} D_\delta(s) \right) \left(\sup_{0 \leq s \leq t} (\|\tilde{u}(s)\|_{L^2(\Omega)} + \|\nabla \tilde{d}(s)\|_{L^2(\Omega)}) \right) \\ & + \left(\int_0^t (t-s)^{-(1+\delta)/2} s^{(\delta-1)/2} ds \right) \left(\sup_{0 < s \leq t} A_2(s) \right) \left(\sup_{0 < s \leq t} (E_1 + E_2)(s) \right) \left(\sup_{0 < s \leq t} s^{-\delta/2} \|\tilde{d}(s)\|_{L^{2/\delta}(\Omega)} \right) \\ & \leq C \left(\sup_{0 < s \leq t} D_\delta(s) \right) \left(\sup_{0 \leq s \leq t} (\|\tilde{u}(s)\|_{L^2(\Omega)} + \|\nabla \tilde{d}(s)\|_{L^2(\Omega)}) \right) \\ & + C \left(\sup_{0 < s \leq t} A_2(s) \right) \left(\sup_{0 \leq s \leq t} (E_1 + E_2)(s) \right) \left(\sup_{0 < s \leq t} s^{-\delta/2} \|\tilde{d}(s)\|_{L^{2/\delta}(\Omega)} \right), \end{aligned} \quad (6.15)$$

where we have used the fact that $\int_0^t (t-s)^{-(\delta+1)/2} s^{(\delta-1)/2} ds = \int_0^1 (1-s)^{-(1+\delta)/2} s^{(\delta-1)/2} < +\infty$, and likewise we conclude that it converges to 0 as $t \rightarrow 0$.

Now, we apply Lemma 2.6 with $2 > q = \frac{2}{1+\delta} \frac{2}{2-\delta} > p = \frac{2}{1+\delta}$, $\gamma = 1 - \frac{p}{q} = \frac{\delta}{2}$, and obtain

$$\begin{aligned} & \|\tilde{u}\|_{L^{\frac{2}{1+\delta} \frac{2}{2-\delta}}(\Omega)} \\ & \lesssim \int_0^t (t-s)^{-(\delta+1)/2} (\|u_1(s)\|_{L^{2/\delta}(\Omega)} + \|u_2(s)\|_{L^{2/\delta}(\Omega)}) \|\tilde{u}(s)\|_{L^2(\Omega)} ds \\ & + \int_0^t (t-s)^{-(\delta+1)/2} (\|\nabla d_1(s)\|_{L^{2/\delta}(\Omega)} + \|\nabla d_2(s)\|_{L^{2/\delta}(\Omega)}) \|\nabla \tilde{d}(s)\|_{L^2(\Omega)} ds \\ & \lesssim \left(\int_0^t (t-s)^{-(\delta+1)/2} s^{(\delta-1)/2} ds \right) \left(\sup_{0 < s \leq t} D_\delta(s) \right) \left(\sup_{0 \leq s \leq t} (\|\tilde{u}(s)\|_{L^2(\Omega)} + \|\nabla \tilde{d}(s)\|_{L^2(\Omega)}) \right) \\ & \leq C \left(\sup_{0 < s \leq t} D_\delta(s) \right) \left(\sup_{0 \leq s \leq t} (\|\tilde{u}(s)\|_{L^q(\Omega)} + \|\nabla \tilde{d}(s)\|_{L^2(\Omega)}) \right). \end{aligned} \quad (6.16)$$

Finally, for $0 < t \leq t_0$, we define

$$\Phi(t) = \sup_{0 < s \leq t} \left(\|\nabla \tilde{d}(s)\|_{L^2(\Omega)} + s^{-\frac{\delta}{2}} \|\tilde{d}(s)\|_{L^{\frac{2}{1+\delta}}(\Omega)}^{\frac{2}{2-\delta}} + \|\tilde{u}(s)\|_{L^{\frac{2}{1+\delta}}(\Omega)}^{\frac{2}{2-\delta}} \right).$$

From estimates (6.14)–(6.16), and the Hölder inequality $\|\cdot\|_{L^2(\Omega)} \lesssim \|\cdot\|_{L^{\frac{2}{1+\delta}}(\Omega)}^{\frac{2}{2-\delta}}$, we have

$$\Phi(t) \leq C \left[\sup_{0 < s \leq t} D_\delta(s) + \left(\sup_{0 < s \leq t} (A_1 + A_2)(s) \right) \left(\sup_{0 \leq s \leq t} (E_1 + E_2)(s) \right) \right] \Phi(t) \leq \frac{1}{2} \Phi(t)$$

as long as $t_0 > 0$ is sufficiently small such that

$$C \left[\sup_{0 < s \leq t} D_\delta(s) + \left(\sup_{0 < s \leq t} (A_1 + A_2)(s) \right) \left(\sup_{0 \leq s \leq t} (E_1 + E_2)(s) \right) \right] \leq C(\epsilon^{1-\delta} + \epsilon) \leq \frac{1}{2}.$$

As a consequence, $\Phi(t) \equiv 0$ on $(0, t_0]$ and hence $(u_1, d_1) \equiv (u_2, d_2)$ on $\Omega \times (0, t_0]$. Furthermore, this implies that our choice for weak extension beyond singular time is also unique.

Step 4: Blow-up analysis. We have established (1), (2), (4), and the first half of (3) in Theorem 1.3. It remains to carry out the blow-up analysis at each singular time. There exists $0 < t_0 < T_1, t_m \nearrow T_1, r_m \searrow 0$ such that

$$\epsilon_1^2 = \sup_{x \in \bar{\Omega}, t_0 \leq t \leq t_m} \int_{\Omega \cap B_{r_m}(x)} (|u|^2 + |\nabla d|^2), \quad (6.17)$$

and we use Lemma 6.1 in the opposite way to obtain $\{x_m\}_{m=1}^\infty \subseteq \Omega$ such that

$$\int_{\Omega \cap B_{2r_m}(x_m)} (|u|^2 + |\nabla d|^2)(\cdot, t_m - \theta_0 r_m^2) \geq \frac{1}{2} \max_{x \in \bar{\Omega}} \int_{\Omega \cap B_{2r_m}(x)} (|u|^2 + |\nabla d|^2)(\cdot, t_m - \theta_0 r_m^2) \geq \frac{1}{2} \epsilon_1^2. \quad (6.18)$$

Energy estimate in Lemma 5.1, (6.17), and the Ladyzhenskaya's inequality states that

$$\int_{\Omega \times [t_0, t_1]} (|u|^4 + |\nabla d|^4) \leq C(\epsilon_1, E_0). \quad (6.19)$$

Denote $\Omega_m = r_m^{-1}(\Omega \setminus \{x_m\})$. Define the blow-up sequence $(u_m, d_m) : \Omega_m \times [\frac{t_0 - t_m}{r_m^2}, 0]$ by

$$u_m(x, t) = r_m u(x_m + r_m x, t_m + r_m^2 t), \quad d_m(x, t) = d(x_m + r_m x, t_m + r_m^2 t).$$

It follows that (u_m, d_m) solves (1.1) on $\Omega_m \times [\frac{t_0 - t_m}{r_m^2}, 0]$, and (6.17), (6.18), and (6.19) give us the following:

$$\begin{aligned} \int_{\Omega_m \cap B_2(0)} (|u_m|^2 + |\nabla d_m|^2)(-\theta_0) &\geq \frac{1}{2} \epsilon_1^2, \\ \int_{\Omega_m \cap B_1(x)} (|u_m|^2 + |\nabla d_m|^2)(t) &\leq \epsilon_1^2, \quad \forall x \in \Omega_m, \frac{t_0 - t_m}{r_m^2} \leq t \leq 0, \\ \int_{\Omega_m \times [-\frac{t_0 - t_m}{r_m^2}, 0]} |u_m|^4 + |\nabla d_m|^4 &\leq C(\epsilon_1, E_0). \end{aligned}$$

By possibly passing to subsequences, we may assume without loss of generality that $x_m \rightarrow x_0 \in \overline{\Omega}$ for some $x_0 \in \overline{\Omega}$.

Case 1: $x_0 \in \Omega$. Then, we can assume $r_0 < \text{dist}(x_0, \partial\Omega)$ and $\Omega_m \rightarrow \mathbb{R}^2$. Also, we have $\frac{t_0 - t_m}{r_0^2} \rightarrow -\infty$. Consequently, regularity result in Theorem 1.2 states that there exists a smooth solution $(u'_\infty, d'_\infty) : \mathbb{R}^2 \times (-\infty, 0] \rightarrow \mathbb{R}^2 \times \mathbb{S}^2$ such that it solves (1.1) and

$$(u_m, d_m) \rightarrow (u'_\infty, d'_\infty) \text{ in } C_{\text{loc}}^2(\mathbb{R}^2 \times [-\infty, 0]).$$

Because of the regularity of the two-dimensional Navier–Stokes equation and the phenomenon of separation of sphere in harmonic map heat flow, we would like to show that the singularity is attributed to ∇d . First, we want to show that $u'_\infty \equiv 0$. Indeed, take any parabolic cylinder $P_R \subseteq \mathbb{R}^2 \times [-\infty, 0]$, since $u \in L^4(\Omega \times [0, T_1])$, we have

$$\int_{P_R} |u'_\infty|^4 = \lim_{m \rightarrow \infty} \int_{P_R} |u_m|^4 = \lim_{m \rightarrow \infty} \int_{B_{Rr_m}(x_m)} \int_{[t_m - R^2 r_m^2, t_m]} |u|^4 = 0.$$

Next, we claim that d'_∞ is a nontrivial and smooth harmonic map with finite energy. In fact, since $\Delta d + |\nabla d|^2 d \in L^2(\Omega \times [0, T_1])$, we have, for any compact $K \subseteq \mathbb{R}^2$,

$$\begin{aligned} \int_{-2\theta_0}^0 \int_K |\Delta d'_\infty + |\nabla d'_\infty|^2 d'_\infty|^2 &\leq \liminf_m \int_{-2\theta_0}^0 \int_{\Omega_m} |\Delta d_m + |\nabla d_m|^2 d_m|^2 \\ &= \lim_{m \rightarrow \infty} \int_{t_m - 2\theta_0 r_m^2}^{t_m} \int_\Omega |\Delta d + |\nabla d|^2 d|^2 = 0. \end{aligned}$$

This means $\partial_t d'_\infty + u \cdot \nabla d'_\infty = 0$ on $\mathbb{R}^2 \times [-2\theta_0, 0]$. Hence, $\partial_t d_m = u_m = 0$ and $d_\infty \in C^2(\mathbb{R}^2, \mathbb{S}^2)$ is a harmonic map. Also, notice that

$$\int_{B_2} |\nabla d'_\infty|^2 = \lim_{m \rightarrow \infty} \int_{B_2} (|u_m|^2 + |\nabla d_m|^2)(\cdot, -\theta_0) \geq \frac{\epsilon_1^2}{4},$$

and hence, d'_∞ is a nontrivial map. By the lower semicontinuity, for any $B_R \subseteq \mathbb{R}^2$,

$$\int_{B_R} |\nabla d'_\infty|^2 \leq \liminf_{m \rightarrow \infty} \int_{B_R} |\nabla d_m|^2(\cdot, -\theta_0) = \liminf_{m \rightarrow \infty} \int_{B_{r_m R}(x_m)} |\nabla d|^2(t_m - \theta_0 r_m^2) \leq E_0.$$

This implies that d'_∞ has finite energy. Studies of harmonic maps (see, e.g., [54] and [46]) show that d'_∞ can be lifted to be a nonconstant harmonic map from $\mathbb{S}^2$ to $\mathbb{S}^2$. In particular, the degree of d'_∞ is nonzero and

$$\int_{\mathbb{R}^2} |\nabla d'_\infty|^2 \geq 8\pi |\deg(d'_\infty)| \geq 8\pi.$$

Case 2: $x_0 \in \partial\Omega$. If further $\lim_{m \rightarrow \infty} \frac{|x_m - x_0|}{r_m} = \infty$, then $\Omega_m \rightarrow \mathbb{R}^2$. Then, the same reasoning in Case 1 shows that $(u_m, d_m) \rightarrow (0, d'_\infty)$ in $C_{\text{loc}}^2(\mathbb{R}^2)$, and $d'_\infty \in C^\infty(\mathbb{R}^2, \mathbb{S}^2)$ is a nontrivial harmonic map with finite energy. The other situation $\lim_{m \rightarrow \infty} \frac{|x_m - x_0|}{r_m} < \infty$ implies that Ω_m converges to a half-plane and it will give singularity at boundary: assume without loss of generality that $\frac{x_m - x_0}{r_m} \rightarrow (0, 0) \in \mathbb{R}^2$ and $\Omega_m \rightarrow \mathbb{R}_+^2 = \{(x_1, x_2) : x_2 > 0\}$. Because $d_m(x) = d(x_m + r_m x)$ for $x \in \partial\Omega_m$, we

can show similarly that $(u_m, d_m) \rightarrow (0, d'_\infty)$ in $C^2_{\text{loc}}(\mathbb{R}^2_+)$, where $d'_\infty : \mathbb{R}^2_+ \rightarrow \mathbb{S}^2$ is a nontrivial harmonic map with finite energy and d'_∞ satisfies (1.2)₂ at $\partial\mathbb{R}^2_+ = \{(x_1, x_2) : x_2 = 0\}$. The reflection given in Appendix A allows us to use reflection to extend d'_∞ to be a nontrivial harmonic map on the whole space $\mathbb{R}^2$. Moreover, the reflection symmetry directly states that we have half the energy

$$\int_{\mathbb{R}^2_a} |\nabla d'_\infty|^2 = \frac{1}{2} \int_{\mathbb{R}^2} |\nabla d'_\infty|^2 \geq 4\pi |\deg(d'_\infty)| \geq 4\pi. \quad \square$$

7 | ETERNAL BEHAVIOR: PROOF OF THEOREMS 1.5 AND 1.6

Proof of Theorem 1.5. To prove (1), Lemma 5.1 states that there exists $t_k \rightarrow \infty$ such that for $(u_k, d_k) = (u(\cdot, t_k), d(\cdot, t_k))$,

$$\begin{aligned} \int_{\Omega} |u_k|^2 + |\nabla d_k|^2 &\leq E_0, \\ \lim_{k \rightarrow \infty} \int_{\Omega} |\mathbb{D}u_k|^2 + |\Delta d_k + |\nabla d_k|^2 d_k|^2 &= 0. \end{aligned}$$

As a consequence, we have $\int_{\Omega} |\mathbb{D}u_k|^2 \rightarrow 0$ as $k \rightarrow \infty$, and thus, by weak limit, there exists $t_k \rightarrow \infty$ such that $u_k \rightharpoonup u_\infty$ in $H^1(\Omega)$ such that $\mathbb{D}u_\infty \equiv 0$. In the situation of nonaxisymmetric domain Ω , we use the Korn's inequality in Lemma 2.12 to deduce that $\nabla u_\infty \equiv 0$, which together with Poincaré inequality for H^1 -vector field with tangential boundary condition, that is, $v \cdot \hat{n} = 0$ on $\partial\Omega$, implies that $u_\infty \equiv 0$ in $H^1(\Omega)$. Meanwhile, $\{d_k\}_{k=1}^\infty \subseteq H^1(\Omega, \mathbb{S}^2)$ is a bounded sequence of approximated harmonic maps from Ω to $\mathbb{S}^2$. Also, $\{d_k\}_{k=1}^\infty$ satisfies partially boundary condition (1.2)₂, and the tension field $\Delta d_k + |\nabla d_k|^2 d_k$ converges to 0 in $L^2(\Omega)$. By the energy identity result by Qing [45] and Lin–Wang [36], we can conclude that there exists a harmonic map $d_\infty \in C^{2,\beta}(\Omega, \mathbb{S}^2)$ with d_∞ satisfying (1.2)₂, and there exist finitely many interior points $\{x_i\}_{i=1}^l \subseteq \Omega$ and boundary points $\{y_i\}_{i=1}^{l'} \subseteq \partial\Omega$ such that

$$|\nabla d_k|^2 dx \rightharpoonup |\nabla d_\infty|^2 dx + \sum_{i=1}^l 8\pi m_i \delta_{x_i} + \sum_{i=1}^{l'} 4\pi m'_i \delta_{y_i}$$

for some subsequence $\{m_i\}_{i=1}^l, \{m'_i\}_{i=1}^{l'} \subseteq \mathbb{N}$. Note that this $d_\infty : \Omega \rightarrow \mathbb{S}^2$ has a different meaning from $d_\infty : \mathbb{R}^2(\text{or } \mathbb{R}^2_a) \rightarrow \mathbb{S}^2$ that is used in the blow-up analysis for Theorem 1.3.

In the end, it remains to prove (2). We first observe that the energy is insufficient to evolve finite time singularities. Suppose for the purpose of contradiction that it blows up near the first singular time T_1 . Then, (3) in Theorem 1.3 implies that there exists a nontrivial harmonic map $w \in C^\infty(\mathbb{R}^2, \mathbb{S}^2)$ and

$$8\pi \leq \int_{\mathbb{R}^2} |\nabla w|^2 \leq 2 \lim_{t \nearrow T_1} \int_{\Omega} (|u|^2 + |\nabla d|^2)(\cdot, t) \leq 2 \int_{\Omega} |u_0|^2 + |\nabla d_0|^2 \leq 8\pi, \quad (7.1)$$

where the factor 2 includes the possibility that we have singularity on the boundary. Thus, there is no loss of energy, but then Lemma 5.1 implies that

$$\int_0^{T_1} \int_{\Omega} |\mathbb{D}u|^2 + |\Delta d + |\nabla d|^2 d|^2 = 0, \quad (7.2)$$

and hence, $\mathbb{D}u = d_t \equiv 0$ in Ω_{T_1} . Therefore, $d(\cdot, t) = d_0 \in C^{2,\beta}(\Omega, \mathbb{S}^2)$, $0 \leq t \leq T_1$, is a harmonic map, which contradicts the assumption that T_1 is a singular time.

Moreover, we would like to show that there is no blow-up of $\phi(t) = \max_{x \in \bar{\Omega}, \tau \leq t} (|u| + |\nabla d|)(x, \tau)$ at infinity. Suppose for the purpose of contradiction that there exists $t_k \rightarrow \infty$ and $x_k \in \bar{\Omega}$, such that

$$\lambda_k = \phi(t_k) = (|u| + |\nabla d|)(x_k, t_k) \rightarrow \infty.$$

Define $\Omega_k = \lambda_k(\Omega \setminus \{x_k\})$ and $(u_k, d_k) : \Omega_k \times [-t_k \lambda_k^2, 0] \rightarrow \mathbb{R}^2 \times \mathbb{S}^2$ by

$$u_k(x, t) = \frac{1}{\lambda_k} u\left(x_k + \frac{x}{\lambda_k}, t_k + \frac{t}{\lambda_k^2}\right), \quad d_k(x, t) = d\left(x_k + \frac{x}{\lambda_k}, t_k + \frac{t}{\lambda_k^2}\right).$$

It follows that (u_k, d_k) solves (1.1) on $\Omega_k \times [-t_k \lambda_k^2, 0]$ and

$$1 = (|u_k| + |\nabla d_k|)(0, 0) \geq (|u_k| + |\nabla d_k|)(x, t), \quad \forall (x, t) \in \Omega_k \times [-t_k \lambda_k^2, 0].$$

With the same procedure as in Theorem 1.3(3), we conclude that there are two cases: either (i) $\Omega_k \rightarrow \mathbb{R}^2$ and $(u_k, d_k) \rightarrow (0, w')$ in $C_{\text{loc}}^2(\mathbb{R}^2)$ where $w' \in C^\infty(\mathbb{R}^2, \mathbb{S}^2)$ is a nontrivial harmonic map with finite energy, or (ii) $\Omega_k \rightarrow \mathbb{R}_a^2$ for some half-plane $\mathbb{R}_a := \{a_1 x_1 + a_2 x_2 > a_0\}$ and $(u_k, d_k) \rightarrow (0, w')$ in $C_{\text{loc}}^2(\mathbb{R}_a^2)$ where $w' : \mathbb{R}_a^2 \rightarrow \mathbb{S}^2$ is a nontrivial harmonic map with finite energy and satisfies (1.2)₂ at $\partial \mathbb{R}_a^2$. Again, reflection symmetry in Appendix A allows us to extend w' to be a nontrivial harmonic map on the whole space $\mathbb{R}^2$.

For cases (i) and (ii), we perform the same scheme as in (7.1) and (7.2) (with T_1 replaced by ∞), to conclude that

$$\int_0^\infty \int_\Omega |\mathbb{D}u|^2 + |\Delta d + |\nabla d|^2 d|^2 = 0,$$

and the same reasoning yields $\mathbb{D}u = d_t \equiv 0$ on $\Omega \times [0, \infty)$. So, $d(t) = d_0 \in C^{2,\beta}(\Omega, \mathbb{S}^2)$, $0 \leq t < \infty$, is a harmonic map. This implies that $\phi(t)$ is constant and contradicts the initial assumption that $\phi(t_k) \rightarrow \infty$.

We have shown that $\phi(t)$ is bounded on $t \in (0, \infty)$, and thus regularity results in Theorem 1.2 give that $\|u(\cdot, t)\|_{C^{2,\beta}(\Omega)}$, $\|d(\cdot, t)\|_{C^{2,\beta}(\Omega)}$ stays bounded on $t \in (0, \infty)$. It follows that there exists a sequence $t_k \rightarrow \infty$ such that

$$\begin{aligned} \int_\Omega (|u|^2 + |\nabla d|^2)(x, t_k) &\leq E_0, \\ \int_\Omega (|\mathbb{D}u|^2 + |\Delta d + |\nabla d|^2 d|)(x, t_k) &\rightarrow 0, \\ \|u(\cdot, t_k)\|_{C^{2,\beta}(\Omega)} + \|d(\cdot, t_k)\|_{C^{2,\beta}(\Omega)} &\leq C, \\ (u(\cdot, t_k), d(\cdot, t_k)) &\rightarrow (u_\infty, d_\infty) \text{ in } C^2(\bar{\Omega}, \mathbb{R}^2 \times \mathbb{S}^2) \end{aligned}$$

for some $(u_\infty, d_\infty) \in C^{2,\beta}(\bar{\Omega}, \mathbb{R}^2) \times C^{2,\beta}(\bar{\Omega}, \mathbb{S}^2)$ satisfying partially free boundary condition (1.2) on $\partial \Omega$. $\square$

Proof of Theorem 1.6. From $\mathbb{D}u_\infty \equiv 0 \in L^2(\Omega)$, one can see immediately that u_∞ is a vortex flow of the form $u_\infty = c(x_2, -x_1)$, and it solves the Navier–Stokes equation (1.1)₁. Moreover, we have $u_\infty \cdot \nabla d_\infty = 0$. If $c = 0$, then we have $u_\infty = 0$ and d_∞ is a harmonic map on Ω . If $c \neq 0$, then (1.2)₁ implies $\partial\Omega$ must be circles, and if not, then $u_\infty \equiv 0$. Then, from $u_\infty \cdot \nabla d_\infty = 0$, we know that each entry of d_∞ is radially symmetric.

We now elaborate in more detail in two cases.

Case 1: If $c \neq 0$ and Ω is a disk, since vortex flow $u_\infty \perp \hat{r}$ almost everywhere, we can show that d is a constant harmonic map by an Ordinary Differential Equation. Indeed, by writing harmonic map equation in polar coordinates, we have

$$\partial_{rr}d^i + \frac{1}{r}\partial_r d^i + |\nabla d|^2 d^i = 0,$$

and thus,

$$\sum_i \partial_{rr}d^i \partial_r d^i + \frac{1}{r}|\partial_r d^i|^2 + |\nabla d|^2 d^i \partial_r d^i = 0,$$

where the last term $\sum_i d^i \partial_r d^i = 0$ because $|d|^2 \equiv 1$. By taking $f(r) = |\nabla d|^2$, we have

$$\partial_r f + \frac{2}{r}f = 0,$$

and thus, $f = \frac{\alpha^2}{r^2}$. We have harmonic map d_∞ being smooth so that $f(0) < \infty$ and thus conclude that $f \equiv 0$.

Case 2: If $c \neq 0$ and Ω is an annulus. Then, as above $f = |\nabla d|^2 = \frac{\alpha^2}{r^2}$, but here α can be nonzero, so we have

$$\partial_{rr}d^i + \frac{1}{r}\partial_r d^i + \frac{\alpha^2}{r^2}d^i = 0.$$

We change the variable $r = e^s$. Then, $\frac{d}{dr} = r^{-1} \frac{d}{ds}$ and $\frac{d^2}{dr^2} = r^{-2}(\frac{d}{ds} - 1)\frac{d}{ds}$, which gives

$$\partial_{ss}d^i + \alpha^2 d^i = 0,$$

and hence,

$$d^i = A_i \sin(\alpha s + \phi_i) = A_i \sin(\alpha \ln(r) + \phi_i)$$

for some constant ϕ_i . And the partially free boundary conditions (1.2)₂ imply (1.8). $\square$

APPENDIX A: BOUNDARY CONDITION AND THE BASIC ENERGY LAW

We would like to show that the free boundary condition (1.2) is compatible with the basic energy law for the system (1.1).

We repeat the process in Lemma 5.1 and obtain the basic energy law:

$$\frac{1}{2} \frac{d}{dt} \left(\int_\Omega |u|^2 + |\nabla d|^2 \right) = - \int_\Omega \frac{1}{2} |\nabla u + (\nabla u)^T|^2 - \int_\Omega |\Delta d + |\nabla d|^2 d|^2, \quad (\text{A.1})$$

which describes the property of energy dissipation for the flow of liquid crystals.

Note that the system (1.1) has stress tensor

$$S = \frac{1}{2}(\nabla u + (\nabla u)^T) - P \mathbb{I}_2 + \nabla d \odot \nabla d - \frac{1}{2}|\nabla d|^2 \mathbb{I}_2,$$

so the physical compatibility condition requires $(S \cdot \hat{n})_\tau = 0$. Considering $\hat{n} \cdot \tau = 0$ and the Navier perfect-slip boundary condition (1.2)₂, we have

$$0 = \langle (\nabla d \odot \nabla d) \hat{n}, \tau \rangle = \langle \nabla_{\hat{n}} d, \nabla_\tau d \rangle,$$

which gives the free boundary condition (1.2)₂ : $\nabla_{\hat{n}} d \perp T_d \Sigma$.

In addition, it is worth mentioning that in the case of half-plane $\Omega = \mathbb{R}_+^2$, the free boundary condition (1.2) gives a reflection across $\partial \mathbb{R}_+^2$. First, free boundary condition (1.2) has a simple form in such case:

$$\begin{cases} \partial_{x_2} u_1 = u_2 = 0 \\ \partial_{x_2} d_1 = \partial_{x_2} d_2 = d_3 = 0 \end{cases} \quad \text{on } \partial \mathbb{R}_+^2.$$

By performing even reflection for u_1, d_1, d_2 and odd reflection for u_2, d_3 , we can use this reflection symmetry to extend our solution (u, d) to the whole domain $\mathbb{R}^2$. Explicitly,

$$\tilde{u}(x_1, x_2, t) = \begin{bmatrix} u_1(x_1, -x_2, t) \\ -u_2(x_1, -x_2, t) \end{bmatrix}, \quad \tilde{d}(x_1, x_2, t) = \begin{bmatrix} d_1(x_1, -x_2, t) \\ d_2(x_1, -x_2, t) \\ -d_3(x_1, -x_2, t) \end{bmatrix}, \quad x_2 < 0.$$

It turns out that the partially free boundary condition (1.2) is automatically satisfied. We can further compute that

$$\begin{aligned} \tilde{u} \cdot \nabla \tilde{u} &= \begin{bmatrix} u_1 \partial_{x_1} u_1 + u_2 \partial_{x_2} u_1 \\ -(u_1 \partial_{x_1} u_2 + u_2 \partial_{x_2} u_2) \end{bmatrix} (x_1, -x_2, t), \\ \tilde{u} \cdot \nabla \tilde{d} &= \begin{bmatrix} u_1 \partial_{x_1} d_1 + u_2 \partial_{x_2} d_1 \\ u_1 \partial_{x_1} d_2 + u_2 \partial_{x_2} d_2 \\ -(u_1 \partial_{x_1} d_3 + u_2 \partial_{x_2} d_3) \end{bmatrix} (x_1, -x_2, t), \end{aligned}$$

and

$$\nabla \cdot (\nabla \tilde{d} \odot \nabla \tilde{d}) = \begin{bmatrix} 2\partial_{x_1} d_k \partial_{x_1 x_1} d_k + \partial_{x_2 x_2} d_k \partial_{x_1} d_k + \partial_{x_1 x_2} d_k \partial_{x_2} d_k \\ -(2\partial_{x_2} d_k \partial_{x_2 x_2} d_k + \partial_{x_1 x_1} d_k \partial_{x_2} d_k + \partial_{x_1 x_2} d_k \partial_{x_1} d_k) \end{bmatrix} (x_1, -x_2, t).$$

Also, observe that the partially free boundary condition (1.2) implies that $\nabla_{\hat{n}} P = 0$ on $\partial \mathbb{R}_+^2$. This follows from

$$\begin{aligned} -\partial_{x_2} P &= \partial_t u_2 + u_1 \partial_{x_1} u_2 + u_2 \partial_{x_2} u_2 + \Delta u_2 + \partial_{x_2} d_k \partial_{x_2 x_2} d_k + \partial_{x_1 x_1} d_k \partial_{x_2} d_k \\ &= 0 + \partial_{x_1 x_1} u_2 - \partial_{x_1} (\partial_{x_2} u_1) + \partial_{x_2} d_3 \Delta d_3 \\ &= 0 + \partial_{x_2} d_3 (\partial_t d_3 + u_1 \partial_{x_1} d_3 + u_2 \partial_{x_2} d_3 - |\nabla d|^2 d_3) = 0. \end{aligned}$$

Hence, we perform even reflection for P and get $\tilde{P}(x_1, x_2, t) = P(x_1, -x_2, t)$ for $x_2 < 0$. Then, we can then show that the structure of the system is preserved via the reflection $(u, d, P) \mapsto (\tilde{u}, \tilde{d}, \tilde{P})$,

that is,

$$\begin{cases} \partial_t \tilde{u} + \tilde{u} \cdot \nabla \tilde{u} - \Delta \tilde{u} + \nabla \tilde{P} = -\nabla \cdot (\nabla \tilde{d} \odot \nabla \tilde{d} - \frac{1}{2} |\nabla \tilde{d}|^2 \mathbb{I}_2) \\ \nabla \cdot \tilde{u} = 0 \\ \partial_t \tilde{d} + \tilde{u} \cdot \nabla \tilde{d} = \Delta \tilde{d} + |\nabla \tilde{d}|^2 \tilde{d}. \end{cases}$$

Such a reflection is not possible if we instead consider Navier no-slip boundary condition $u \equiv 0$ on $\partial\Omega$; thus, free boundary condition (1.2) is both physically meaningful and mathematically useful: it allows us to convert boundary estimates to interior estimates and saves half of our labor. In particular, Lemma 3.3 follows directly from Lemma 3.2.

However, it is difficult to tackle the general situation with curved boundary $\partial\Omega$. We assume that $\partial\Omega$ is smooth, and we can flatten the boundary in a way such that in the new coordinate, the velocity field is still divergence-free. One example in [20] is to take a map $\phi : \mathbb{R}_+^2 \cap B_r(z') \rightarrow \Omega \cap B_r(z)$ given by $\phi(x_1, x_2) = (x_1, x_2 + h(x_1))$, where $h(x_1)$ is locally the graph of the boundary $\partial\Omega$. Then, we define the transformed vector field v on $\mathbb{R}_+^2$ by $v = Tu = u \circ \phi - (u \circ \phi) \cdot (h', 0)e_2$. This transformation has the property that $\nabla \cdot u = 0$ implies $\nabla \cdot v = 0$ and tangential vector along $\partial\Omega$ still maps to tangential vector along $\partial\mathbb{R}_+^2$, though the normal vector is not preserved. Thus, if we want to keep both the divergence-free property and the free boundary condition, we may think of reflection over a curved boundary or generalize the system (1.1) to non-Euclidean metric setup; both will produce extra low order terms in the system.

ACKNOWLEDGMENTS

Y. Sire is partially supported by NSF DMS Grant 2154219, “Regularity vs singularity formation in elliptic and parabolic equations.” Y. Zhou is supported in part by the Fundamental Research Funds for the Central Universities.

JOURNAL INFORMATION

The *Journal of the London Mathematical Society* is wholly owned and managed by the London Mathematical Society, a not-for-profit Charity registered with the UK Charity Commission. All surplus income from its publishing programme is used to support mathematicians and mathematics research in the form of research grants, conference grants, prizes, initiatives for early career researchers and the promotion of mathematics.

REFERENCES

1. P. A. Tapia, C. Amrouche, C. Conca, and A. Ghosh, *Stokes and Navier-Stokes equations with Navier boundary conditions*, J. Differential Equations **285** (2021), 258–320.
2. H. Al Baba, C. Amrouche, and M. Escobedo, *Semi-group theory for the Stokes operator with Navier-type boundary conditions on L^p -spaces*, Arch. Ration. Mech. Anal. **223** (2017), no. 2, 881–940.
3. C. Amrouche, M. Escobedo, and A. Ghosh, *Semigroup theory for the Stokes operator with Navier boundary condition on L^p spaces*, Waves in flows—the 2018 Prague-Sum Workshop lectures, Adv. Math. Fluid Mech., Birkhäuser/Springer, Cham, [2021] ©2021, pp. 1–51.
4. C. Amrouche and A. Rejaiba, *L^p -theory for Stokes and Navier-Stokes equations with Navier boundary condition*, J. Differential Equations **256** (2014), no. 4, 1515–1547.
5. H. Beirão da Veiga, *Vorticity and regularity for flows under the Navier boundary condition*, Commun. Pure Appl. Anal. **5** (2006), no. 4, 907–918.
6. H. Beirão da Veiga and L. C. Berselli, *Navier-Stokes equations: Green’s matrices, vorticity direction, and regularity up to the boundary*, J. Differential Equations **246** (2009), no. 2, 597–628.

7. F. Bethuel, *The approximation problem for Sobolev maps between two manifolds*, Acta Math. **167** (1991), no. 3–4, 153–206.
8. D. Bothe, M. Köhne, and J. Prüss, *On a class of energy preserving boundary conditions for incompressible Newtonian flows*, SIAM J. Math. Anal. **45** (2013), no. 6, 3768–3822.
9. L. Caffarelli, R. Kohn, and L. Nirenberg, *Partial regularity of suitable weak solutions of the Navier-Stokes equations*, Comm. Pure Appl. Math. **35** (1982), no. 6, 771–831.
10. G.-Q. Chen and Z. Qian, *A study of the Navier-Stokes equations with the kinematic and Navier boundary conditions*, Indiana Univ. Math. J. **59** (2010), no. 2, 721–760.
11. Y. Chen and F. H. Lin, *Evolution equations with a free boundary condition*, J. Geom. Anal. **8** (1998), no. 2, 179–197.
12. P. Constantin and C. Foias, *Navier-Stokes equations*, Chicago Lectures in Mathematics, University of Chicago Press, Chicago, IL, 1988.
13. L. Desvillettes and C. Villani, *On a variant of Korn's inequality arising in statistical mechanics*, ESAIM Control Optim. Calc. Var. **8** (2002), 603–619. A tribute to J. L. Lions.
14. J. L. Ericksen, *Hydrostatic theory of liquid crystals*, Arch. Rational Mech. Anal. **9** (1962), 371–378.
15. R. Farwig and V. Rostock, *Resolvent estimates of the Stokes system with Navier boundary conditions in general unbounded domains*, Adv. Differential Equations **21** (2016), no. 5/6, 401–428.
16. R. Farwig and V. Rostock, *Maximal regularity of the Stokes system with Navier boundary condition in general unbounded domains*, J. Math. Soc. Japan **71** (2019), no. 4, 1293–1319.
17. A. Ghosh, *Navier-Stokes equations with Navier boundary condition*, Thesis Submitted to the Université de Pau et des Pays de l'Adour, France, and Universidad del País Vasco, Bilbao, 2018.
18. D. Gilbarg and N. S. Trudinger, *Elliptic partial differential equations of second order*, Classics in Mathematics, Springer, Berlin, 2001. Reprint of the 1998 edition.
19. R. S. Hamilton, *Harmonic maps of manifolds with boundary*, Lecture Notes in Mathematics, vol. 471, Springer, Berlin-New York, 1975.
20. M. Hieber and J. Saal, *The Stokes equation in the L^p -setting: well-posedness and regularity properties*, Handbook of mathematical analysis in mechanics of viscous fluids, Springer, Cham, 2018, pp. 117–206.
21. T. Huang, F. Lin, C. Liu, and C. Wang, *Finite time singularity of the nematic liquid crystal flow in dimension three*, Arch. Ration. Mech. Anal. **221** (2016), no. 3, 1223–1254.
22. T. Huang and C. Wang, *Notes on the regularity of harmonic map systems*, Proc. Amer. Math. Soc. **138** (2010), no. 6, 2015–2023.
23. T. Huang and C. Wang, *Blow up criterion for nematic liquid crystal flows*, Comm. Partial Differential Equations **37** (2012), no. 5, 875–884.
24. A. Hyder, A. Segatti, Y. Sire, and C. Wang, *Partial regularity of the heat flow of half-harmonic maps and applications to harmonic maps with free boundary*, Comm. Partial Differential Equations **47** (2022), no. 9, 1845–1882.
25. R. Kakizawa, *Maximal L_p - L_q regularity of the linearized initial-boundary value problem for motion of compressible viscous fluids*, J. Differential Equations **251** (2011), no. 2, 339–372.
26. J. P. Kelliher, *Navier-Stokes equations with Navier boundary conditions for a bounded domain in the plane*, SIAM J. Math. Anal. **38** (2006), no. 1, 210–232.
27. H. Komatsu, *Fractional powers of operators*, Pacific J. Math. **19** (1966), 285–346.
28. F. Kong, C.-C. Lai, and J. Wei, *Global existence and aggregation of chemotaxis-fluid systems in dimension two*, J. Differential Equations **400** (2024), 1–89.
29. O. A. Ladyženskaja, V. A. Solonnikov, and N. N. Ural'ceva, *Linear and quasilinear equations of parabolic type*, Translations of Mathematical Monographs, vol. 23, American Mathematical Society, Providence, RI, 1968. Translated from the Russian by S. Smith.
30. C.-C. Lai, F. Lin, C. Wang, J. Wei, and Y. Zhou, *Finite time blowup for the nematic liquid crystal flow in dimension two*, Comm. Pure Appl. Math. **75** (2022), no. 1, 128–196.
31. F. M. Leslie, *Some constitutive equations for liquid crystals*, Arch. Rational Mech. Anal. **28** (1968), no. 4, 265–283.
32. S. Li, *Geometric regularity criteria for incompressible Navier-Stokes equations with Navier boundary conditions*, Nonlinear Anal. **188** (2019), 202–235.
33. G. M. Lieberman, *Second order parabolic differential equations*, World Scientific Publishing Co., Inc., River Edge, NJ, 1996.

34. F. Lin, J. Lin, and C. Wang, *Liquid crystal flows in two dimensions*, Arch. Ration. Mech. Anal. **197** (2010), no. 1, 297–336.
35. F. Lin, Y. Sire, J. Wei, and Y. Zhou, *Nematic liquid crystal flow with partially free boundary*, Arch. Ration. Mech. Anal. **247** (2023), no. 2, Paper No. 20, 54.
36. F. Lin and C. Wang, *Harmonic and quasi-harmonic spheres. II*, Comm. Anal. Geom. **10** (2002), no. 2, 341–375.
37. F. Lin and C. Wang, *On the uniqueness of heat flow of harmonic maps and hydrodynamic flow of nematic liquid crystals*, Chinese Ann. Math. Ser. B **31** (2010), no. 6, 921–938.
38. F. Lin and C. Wang, *Recent developments of analysis for hydrodynamic flow of nematic liquid crystals*, Philos. Trans. R. Soc. London Ser. A Math. Phys. Eng. Sci. **372** (2014), no. 2029, 20130361, 18.
39. F. Lin and C. Wang, *Global existence of weak solutions of the nematic liquid crystal flow in dimension three*, Comm. Pure Appl. Math. **69** (2016), no. 8, 1532–1571.
40. F.-H. Lin, *Nonlinear theory of defects in nematic liquid crystals; phase transition and flow phenomena*, Comm. Pure Appl. Math. **42** (1989), no. 6, 789–814.
41. F.-H. Lin and C. Liu, *Nonparabolic dissipative systems modeling the flow of liquid crystals*, Comm. Pure Appl. Math. **48** (1995), no. 5, 501–537.
42. F.-H. Lin and C. Liu, *Partial regularity of the dynamic system modeling the flow of liquid crystals*, Discrete Contin. Dynam. Systems **2** (1996), no. 1, 1–22.
43. L. Ma, *Harmonic map heat flow with free boundary*, Comment. Math. Helv. **66** (1991), no. 2, 279–301.
44. J. Neustupa and P. Penel, *Local in time strong solvability of the non-steady Navier-Stokes equations with Navier's boundary condition and the question of the inviscid limit*, C. R. Math. Acad. Sci. Paris **348** (2010), no. 19–20, 1093–1097.
45. J. Qing, *On singularities of the heat flow for harmonic maps from surfaces into spheres*, Comm. Anal. Geom. **3** (1995), no. 1–2, 297–315.
46. J. Sacks and K. Uhlenbeck, *The existence of minimal immersions of 2-spheres*, Ann. of Math. (2) **113** (1981), no. 1, 1–24.
47. G. A. Seregin, *Some estimates near the boundary for solutions to the non-stationary linearized Navier-Stokes equations*, Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI) **271**, Kraev. Zadachi Mat. Fiz. i Smezh. Vopr. Teor. Funkts. **31** (2000), 204–223, 317.
48. G. A. Seregin, T. N. Shilkin, and V. A. Solonnikov, *Boundary partial regularity for the Navier-Stokes equations*, Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI) **310**, Kraev. Zadachi Mat. Fiz. i Smezh. Vopr. Teor. Funkts. **35** [34] (2004), 158–190, 228.
49. Y. Shibata and S. Shimizu, *L_p - L_q maximal regularity and viscous incompressible flows with free surface*, Proc. Japan Acad. Ser. A Math. Sci. **81** (2005), no. 9, 151–155.
50. Y. Shibata and S. Shimizu, *On the L_p - L_q maximal regularity of the Neumann problem for the Stokes equations in a bounded domain*, J. Reine Angew. Math. **615** (2008), 157–209.
51. Y. Shibata and S. Shimizu, *On the maximal L_p - L_q regularity of the Stokes problem with first order boundary condition; model problems*, J. Math. Soc. Japan **64** (2012), no. 2, 561–626.
52. Y. Sire, J. Wei, and Y. Zheng, *Singularity formation in the harmonic map flow with free boundary*, Amer. J. Math. **145** (2023), no. 4, 1273–1314.
53. V. A. Solonnikov, *Estimates for solutions of nonstationary navier-stokes equations*, J. Soviet Math. **8** (1977), 467–529.
54. M. Struwe, *On the evolution of harmonic mappings of Riemannian surfaces*, Comment. Math. Helv. **60** (1985), no. 4, 558–581.
55. M. Struwe, *Heat-flow methods for harmonic maps of surfaces and applications to free boundary problems*, Partial differential equations (Rio de Janeiro, 1986), Lecture Notes in Math., vol. 1324, Springer, Berlin, 1988, pp. 293–319.
56. M. Struwe, *The evolution of harmonic mappings with free boundaries*, Manuscripta Math. **70** (1991), no. 4, 373–384.
57. L. Tartar, *An introduction to Navier-Stokes equation and oceanography*, Lecture Notes of the Unione Matematica Italiana, vol. 1, Springer, Berlin; UMI, Bologna, 2006.
58. M. Winkler, *Aggregation vs. global diffusive behavior in the higher-dimensional keller-segel model*, J. Differential Equations **248** (2010), no. 12, 2889–2905.