Positivity conditions on the annulus via the double-layer potential kernel

by

Michael T. Jury (Gainesville, FL)
and Georgios Tsikalas (St. Louis, MO)

Abstract. We introduce and study a scale of operator classes on the annulus that is motivated by the $\mathcal{C}_\rho$ classes of ρ-contractions of Nagy and Foiaş. In particular, our classes are defined in terms of the contractivity of the double-layer potential integral operator over the annulus. We prove that if, in addition, complete contractivity is assumed, then one obtains a complete characterization involving certain variants of the $\mathcal{C}_\rho$ classes. Recent work of Crouzeix–Greenbaum and Schwenninger–de Vries allows us to also obtain relevant K-spectral estimates, generalizing and improving existing results from the literature on the annulus. Finally, we exhibit a special case where these estimates can be significantly strengthened.

1. Introduction. Let $\rho > 0$. Denote by $\mathcal{C}_\rho$ the class of all bounded Hilbert space operators $T \in \mathcal{B}(H)$ that have a unitary ρ-dilation, i.e. there exists a Hilbert space $K \supset H$ and a unitary $U \in \mathcal{B}(K)$ such that

$$T^n = \rho P_H U^n|_H, \quad n = 1, 2, \ldots,$$

where P_H is the orthogonal projection of K onto H. The elements of $\mathcal{C}_\rho$ are referred to as ρ-*contractions*. The class $\mathcal{C}_\rho$ was introduced by Sz.-Nagy and Foiaş [32] (see also [33, Chapter 1]) and has subsequently been investigated by many authors (see e.g. [8] and the references therein). It is known that $\mathcal{C}_1$ is precisely the set of contractions on H [31], while $\mathcal{C}_2$ is the set of operators whose numerical range $W(T) = \{\langle Tx, x \rangle \mid \|x\| = 1\}$ is contained in $\overline{\mathbb{D}} = \{|z| \leq 1\}$ [4, 3].

Now, for any open, bounded domain $\Omega \subset \mathbb{C}$, let $\mathcal{A}(\Omega)$ denote the uniform algebra of continuous functions f on $\overline{\Omega}$ that are holomorphic on Ω. Also, let $T \in \mathcal{B}(H)$ be such that $\sigma(T) \subset \mathbb{D}$. An alternative characterization of $\mathcal{C}_\rho$ (see

[233]

[9, p. 315]) then states that T is a ρ-contraction if and only if the operator
$$S_{\mathbb{D},\rho} : \mathcal{A}(\mathbb{D}) \to \mathcal{B}(H),$$

(1.1)
$$f \mapsto \frac{1}{\rho}\left[f(T) + (C\overline{f})(T)^* + \frac{\rho - 2}{2\pi i} \int_{\partial\mathbb{D}} f(\zeta)\, d\zeta \right]$$

$$= \frac{1}{\rho}[f(T) + (\rho - 1)f(0)],$$

is contractive, where

$$(C\overline{f})(z) = \frac{1}{2\pi i} \int_{\partial\mathbb{D}} \frac{\overline{f(\zeta)}}{\zeta - z}\, d\zeta \quad (z \in \mathbb{D})$$

denotes the Cauchy transform of $\overline{f}$. The mapping $S_{\mathbb{D},\rho}$ has proved itself a valuable tool for better understanding $\mathcal{C}_\rho$. As a recent example, Clouâtre, Ostermann and Ransford [10], building on ideas from [7], used the contractivity of $S_{\mathbb{D},\rho}$ as a stepping stone for an alternative, simple proof of the fact that, for any $\rho \geq 1$ and $T \in \mathcal{C}_\rho$,

$$\|p(T)\| \leq \rho \sup_{z \in \overline{\mathbb{D}}} |p(z)|, \quad \forall p \in C[z].$$

In other words, $\overline{\mathbb{D}}$ is a ρ-spectral set for T whenever T is a ρ-contraction (with $\rho \geq 1$), a result originally proved by Okubo and Ando via dilation theory [24].

More generally, the unit disk $\mathbb{D}$ can be replaced with any (smoothly bounded) open convex set $\Omega \subset \mathbb{C}$. In this setting, the contractivity of the *double-layer potential* integral operator

$$S_{\Omega,2} : f \mapsto \tfrac{1}{2}[f(T) + (C\overline{f})(T)^*]$$

(which is equivalent to the inclusion $\overline{W(T)} \subset \Omega$) has served as a basis for establishing K-spectral estimates for $W(T)$; an approach originally due to Delyon and Delyon [18], it was further refined by Crouzeix in [13] and culminated in the Crouzeix–Palencia paper [17] (see also [28, 6] and the recent preprint [29] for further developments), where it was shown that $\overline{W(T)}$ is always a $(1 + \sqrt{2})$-spectral set for any $T \in \mathcal{B}(H)$.

More recently, the study of K-spectral estimates through the contractivity of $S_{\Omega,2}$ has expanded beyond convex domains (see [7, 15]). One important example in that direction, and what actually served as the main motivation for this paper, revolves around the annulus $A_R = \{1/R < |z| < R\}$ and the associated operator class $\mathbb{QA}_R$, termed the *quantum annulus*, that consists of all invertible Hilbert space operators T satisfying $\|T\|, \|T^{-1}\| \leq R$ (for recent results regarding $\mathbb{QA}_R$, see also [2, 22, 34, 35]). In particular, in [15, Section 5] it was shown that the mapping

$$S_{R,0} : \mathcal{A}(A_R) \to \mathcal{B}(H), \quad f \mapsto \tfrac{1}{2}[f(T) + (C\overline{f})(T)^*],$$

where $(C\overline{f})(z) = \frac{1}{2\pi i}\int_{A_R}\frac{\overline{f(\zeta)}}{\zeta - z}\,d\zeta$, is always contractive if $T \in \mathbb{Q}\mathbb{A}_R$. This observation, combined with the contractivity of $f \mapsto C\overline{f}$ over A_R and an abstract functional analysis lemma (see [15, Theorem 2] or [28, Lemma 1.1]), suffices to establish that, given any $T \in \mathbb{Q}\mathbb{A}_R$, we have

$$\|f(T)\| \le (1 + \sqrt{2}) \sup_{z \in \overline{A_R}} |f(z)|, \quad \forall f \in \mathcal{A}(A_R),$$

i.e. $\overline{A_R}$ is a $(1 + \sqrt{2})$-spectral set for T whenever $T \in \mathbb{Q}\mathbb{A}_R$ (it is known that the optimal value of this spectral constant cannot be less than 2; see [34]).

Now, observe that in the previous discussion, the class $\mathbb{Q}\mathbb{A}_R$ enters the picture only through the contractivity of $S_{R,0}$. Thus, if one wants to gain a better understanding of K-spectral estimates over A_R, the following question emerges naturally: for which operators T is $S_{R,0}$ contractive? This line of inquiry, together with the form of the mapping (1.1), is what motivated our definition of the operator class $\mathbb{D}\mathbb{L}\mathbb{A}_R(c)$ (where $c > -2$); an operator $T \in \mathcal{B}(H)$ with $\sigma(T) \subset A_R$ belongs to $\mathbb{D}\mathbb{L}\mathbb{A}_R(c)$ if and only if the mapping

$$S_{R,c} : \mathcal{A}(A_R) \to \mathcal{B}(H),$$

$$f = \sum_{n=-\infty}^{\infty} a_n z^n \mapsto \frac{1}{2+c}[f(T) + (C\overline{f})(T)^* + c a_0],$$

is contractive (see Section 3 for the general definition and Theorem 3.12 for the equivalence between the two when $\sigma(T) \subset A_R$). The goal of the present work is to study the operator class $\mathbb{D}\mathbb{L}\mathbb{A}_R(c)$ and its completely contractive analogue $\mathbb{C}\mathbb{D}\mathbb{L}\mathbb{A}_R(c)$ (see Section 4). Note that the study of completely bounded maps and dilations in the setting of the double-layer potential kernel and Crouzeix's conjecture has already been successfully initiated in the papers [27] and [10, Section 6]. Also, we point out that while the inclusion $\mathbb{C}\mathbb{D}\mathbb{L}\mathbb{A}_R(c) \subseteq \mathbb{D}\mathbb{L}\mathbb{A}_R(c)$ follows immediately from the definitions, it is not known to us whether it is strict or not (see Question 4.7).

To state our first main result, a characterization of $\mathbb{C}\mathbb{D}\mathbb{L}\mathbb{A}_R(c)$, we require the following generalization of the $\mathcal{C}_\rho$ classes, introduced by Langer [33, p. 53] (see also [30]).

DEFINITION 1.1. Assume $A \in \mathcal{B}(H)$ is a bounded, positive operator that is also bounded below. The class $\mathcal{C}_A$ contains all operators $T \in \mathcal{B}(H)$ with the property that there exists a Hilbert space $K \supset H$ and a unitary $U \in \mathcal{B}(K)$ such that

$$A^{-1/2}T^n A^{-1/2} = P_H U^n|_H, \quad n = 1, 2, \ldots.$$

Our characterization then proceeds as follows. Note that the inequality $T \ge 0$ indicates that the Hilbert space operator T is positive, while $T > 0$ means that it is strictly positive.

THEOREM 1.2. *Let $T \in \mathcal{B}(H)$ and $c > -2$, with $\sigma(T) \subset A_R$. Then $T \in \mathbb{CDLA}_R(c)$ if and only if $T/R \in \mathcal{C}_{2+c-A}$ and $T^{-1}/R \in \mathcal{C}_A$ for some $A \in \mathcal{B}(H)$ such that $A > 0$ and $2 + c - A > 0$.*

Dropping the assumption $\sigma(T) \subset A_R$ leads us to Theorem 4.4.

As remarked previously, the only way the class $\mathbb{QA}_R$ enters the proof of the $K = 1 + \sqrt{2}$ spectral estimate in [15] is through the inclusion $\mathbb{QA}_R \subset \mathbb{CDLA}_R(0)$. This suggests that the spectral constant for $\mathbb{QA}_R$ may coincide with the one for $\mathbb{CDLA}_R(0)$. Utilizing the solution of an extremal problem over A_R due to McCullough and Shen [23], we are able to prove a partial result in the setting of 2×2 matrices that supports this idea.

THEOREM 1.3. *Let $c \geq 0$ and assume $T \in \mathbb{DLA}_R(c)$ is a 2×2 matrix with a single eigenvalue. Then $\overline{A_R}$ is a $K(R)$-spectral set for T, where*

$$K(R) = 2 + c\frac{R^2 - 1}{R^2 + 1} \leq 2 + c.$$

Note that one can also take advantage of the machinery established in [7] and [15] to prove general K-spectral estimates for $\mathbb{DLA}_R(c)$ and $\mathbb{CDLA}_R(c)$; see Theorem 5.3 (and also Remark 5.5). In fact, our approach yields sharper estimates for certain operator classes; see Remark 5.4.

Our paper is organized as follows: Section 2 contains a few preliminary lemmata on the $\mathcal{C}_A$ classes. In Section 3, we explore the basic properties of $\mathbb{DLA}_R(c)$, mostly related to certain inclusion and monotonicity results. In Section 4, we characterize $\mathbb{CDLA}_R(c)$ through Theorems 1.2 and 4.4. Finally, Section 5 contains the proofs of Theorems 1.3 and 5.3.

2. Preliminaries

2.1. The $\mathcal{C}_A$ classes. This section contains alternate characterizations of $\mathcal{C}_A$ that are usually easier to work with. We have included proofs for the convenience of the reader.

LEMMA 2.1. *Assume $A \in \mathcal{B}(H)$ is a bounded, strictly positive operator and let $T \in \mathcal{B}(H)$. Then $T \in \mathcal{C}_A$ if and only if $\sigma(T) \subset \overline{\mathbb{D}}$ and*

$$(2.1) \qquad 2\Re(1 - zT)^{-1} + A - 2 \geq 0, \quad \forall z \in \mathbb{D}.$$

Proof. Assume first that $T \in \mathcal{C}_A$. Since $\mathcal{C}_A \subset \mathcal{C}_{\|A\|}$, we have $\sigma(T) \subset \overline{\mathbb{D}}$ (see e.g. [33, p. 43]). Also, in view of [33, p. 53], $T \in \mathcal{C}_A$ is equivalent to

$$\langle Ah, h \rangle - 2\Re\langle z(A - I)Th, h \rangle + |z|^2 \langle (A - 2I)Th, Th \rangle \geq 0$$

for all $h \in H, |z| \leq 1$. This inequality can be rewritten as

$$\langle A(I - zT)h, (I - zT)h \rangle - 2\langle (I - zT)h, (I - zT)h \rangle + 2\Re\langle (I - zT)h, h \rangle \geq 0$$

for all $h \in H, |z| \leq 1$. Setting $h = (I - zT)^{-1}h$, we obtain

$$\langle (A - 2 + (I - zT)^{-1} + (1 - \overline{z}T^*)^{-1})h, h \rangle \geq 0$$

for all $h \in H, |z| \leq 1$, which is the desired conclusion.

For the converse, simply roll back the previous steps. ∎

Note that one could define $\mathcal{C}_A$ more generally, for A bounded and self-adjoint, through (2.1). Using this definition, it is easy to see that $\mathcal{C}_A$ is non-empty if and only if $A \geq 0$ (in which case it contains the zero operator).

LEMMA 2.2. *Assume $A \in \mathcal{B}(H)$ is a bounded, strictly positive operator and let $T \in \mathcal{B}(H)$ be such that $\sigma(T) \subset \mathbb{D}$. Then $T \in \mathcal{C}_A$ if and only if*

$$(2.2) \qquad (1 - e^{i\theta}T)^{-1} + (1 - e^{-i\theta}T^*)^{-1} + A - 2 \geq 0, \quad \forall \theta \in [0, 2\pi).$$

Proof. Fix an arbitrary $h \in H$ and define

$$\Phi_h : \mathbb{D} \to \mathbb{C}, \qquad z \mapsto \langle (2\Re(1 - zT)^{-1} + A - 2)h, h \rangle.$$

Since $\sigma(T) \subset \mathbb{D}$, Φ_h is a harmonic function on $\mathbb{D}$ that extends continuously to $\overline{\mathbb{D}}$. By the minimum principle for harmonic functions, we then deduce that $\Phi_h(z) \geq 0$, for all $z \in \mathbb{D}$, if and only if

$$\Phi_h(e^{i\theta}) = \langle (2\Re(1 - e^{i\theta}T)^{-1} + A - 2)h, h \rangle \geq 0, \quad \forall \theta \in [0, 2\pi).$$

Since h was arbitrary, we are done. ∎

2.2. Completely bounded maps. Let $A \subset \mathcal{B}(H)$ denote an *operator algebra*, i.e. a unital subalgebra of the C^*-algebra of bounded linear operators on some Hilbert space H. Given a natural number $n \geq 1$, we denote by $M_n(A)$ the algebra of $n \times n$ matrices with entries from A, which we view as a subalgebra of bounded linear operators acting on $H^{(n)} = H \oplus \cdots \oplus H$ (n summands). In particular, $M_n(A)$ is endowed with a norm under this identification. Given a map $\Phi : A \to \mathcal{B}(H)$, for each $n \geq 1$, we may define the coordinatewise map $\Phi^{(n)} : M_n(A) \to \mathcal{B}(K^{(n)})$ as

$$\Phi^{(n)}([a_{ij}]) = [\Phi(a_{ij})], \qquad [a_{ij}] \in M_n(A).$$

If Φ is linear (or anti-linear), we say that Φ is *completely bounded* if the quantity

$$\|\Phi\|_{\mathrm{cb}} = \sup_n \|\Phi^{(n)}\|$$

is finite. We say that Φ is *completely contractive* if $\|\Phi\|_{\mathrm{cb}} \leq 1$. Furthermore, Φ will be called *positive* if it maps positive elements of A to positive operators in $\mathcal{B}(K)$. The map Φ will be said to be *completely positive* if $\Phi^{(n)}$ is positive for every $n \geq 1$. For more details on these concepts, see [26].

3. The $\mathbb{DLA}_R(c)$ class

3.1. The $\mathcal{C}_{s,t}(R)$ classes. Before we dive into the study of $\mathbb{DLA}_R(c)$, we will establish a few basic facts regarding $\mathcal{C}_{s,t}(R)$. These are operator classes which form (as will be seen in the next subsection) strict subsets of certain $\mathbb{DLA}_R(c)$ classes.

DEFINITION 3.1. Let $R > 1$ and $s, t > 0$. Define
$$\mathcal{C}_{s,t}(R) = \{T \in \mathcal{B}(H) \mid T/R \in \mathcal{C}_s \text{ and } T^{-1}/R \in \mathcal{C}_t\}.$$

First, we record $\mathcal{C}_{s,t}(R)$-membership criteria concerning normal matrices. These are easy consequences of known results about $\mathcal{C}_\rho$.

LEMMA 3.2. *Let $s > 0$ and assume $T \in \mathcal{B}(H)$ is in $\mathcal{C}_s$.*

(i) *If $s \geq 1$, then $\sigma(T) \subset \{|z| \leq 1\}$.*
(ii) *If $s \leq 1$, then $\sigma(T) \subset \{|z| \leq s/(2-s)\}$.*

Proof. This is [5, Lemma 5]. ∎

PROPOSITION 3.3. *Fix $R > 1$ and assume $s, t > 0$. Also, let $N \in \mathcal{B}(H)$ be normal.*

(i) *If $s, t \geq 1$, then $\mathcal{C}_{s,t}(R)$ is always non-empty. Moreover, $N \in \mathcal{C}_{s,t}(R)$ if and only if $1/R^2 \leq N^*N \leq R^2$.*
(ii) *If $s \geq 1$ and $t \leq 1$, then $\mathcal{C}_{s,t}(R)$ is non-empty if and only if*
$$\frac{2}{R^2+1} \leq t.$$
    Moreover, $N \in \mathcal{C}_{s,t}(R)$ if and only if $(2-t)^2/(R^2t^2) \leq N^*N \leq R^2$.
(iii) *If $s \leq 1$ and $t \geq 1$, then $\mathcal{C}_{s,t}(R)$ is non-empty if and only if*
$$\frac{2}{R^2+1} \leq s.$$
    Moreover, $N \in \mathcal{C}_{s,t}(R)$ if and only if $1/R^2 \leq N^*N \leq R^2s^2/(2-s)^2$.
(iv) *If $s \leq 1$ and $t \leq 1$, then $\mathcal{C}_{s,t}(R)$ is non-empty if and only if*
$$\frac{1}{R}\frac{2-t}{t} \leq R\frac{s}{2-s}.$$
    Moreover, $N \in \mathcal{C}_{s,t}(R)$ if and only if
$$\frac{(2-t)^2}{R^2t^2} \leq N^*N \leq R^2\frac{s^2}{(2-s)^2}.$$

Proof. We only consider the case $s \geq 1$, $t \leq 1$ (the remaining three cases can be proved in essentially identical ways). The assertion about N is an immediate consequence of [5, Theorem 6]. Note also that, if $\frac{2}{R^2+1} \leq t$, there exists a scalar $a \in \mathbb{C}$ satisfying $\frac{2-t}{Rt} \leq |a| \leq R$. This scalar lies in $\mathcal{C}_{s,t}(R)$, hence this class is non-empty. Conversely, assume that $\frac{2}{R^2+1} > t$ and there exists $T \in \mathcal{C}_{s,t}(R)$. Since $T/R \in \mathcal{C}_s$, Lemma 3.2 tells us that

$\sigma(T/R) \subset \{|z| \leq 1\}$, hence $\sigma(T) \subset \{|z| \leq R\}$. Also, the fact that T^{-1}/R is in $\mathcal{C}_t$ implies (in view of the same lemma) that $\sigma((2-t)t^{-1}T^{-1}/R) \subset \{|z| \leq 1\}$. Thus, we can conclude

$$\sigma(T) \subset \{R^{-1}(2-t)t^{-1} \leq |z| \leq R\} = \emptyset,$$

as $\frac{2}{R^2+1} > t$, a contradiction. Therefore, $\mathcal{C}_{s,t}(R)$ must be empty. ∎

We now show that the $\mathcal{C}_{s,t}(R)$ classes are, in a certain sense, "rigid" with respect to the parameters s, t. To do this, we require $\mathcal{C}_{s,t}(R)$-membership conditions for 2×2 matrices with a single eigenvalue.

LEMMA 3.4. *Fix* $R > 1$. *Also, let* $s, t > 0$ *and* $T = \begin{pmatrix} a & b \\ 0 & a \end{pmatrix}$, *where* $a, b \in \mathbb{C}$.

(i) *Assume* $s, t \geq 1$. *Then* $T \in \mathcal{C}_{s,t}(R)$ *if and only if*

$$R|b| \leq (R - |a|)((2-s)|a| + sR)$$

and

$$R|b| \leq (R|a| - 1)(2 - t + tR|a|).$$

(ii) *Assume* $s \geq 1$ *and* $t \leq 1$. *Then* $T \in \mathcal{C}_{s,t}(R)$ *if and only if*

$$R|b| \leq (R - |a|)((2-s)|a| + sR)$$

and

$$R|b| \leq (R|a| + 1)(tR|a| - 2 + t).$$

(iii) *Assume* $s \leq 1$ *and* $t \geq 1$. *Then* $T \in \mathcal{C}_{s,t}(R)$ *if and only if*

$$R|b| \leq (R + |a|)(sR - (2-s)|a|)$$

and

$$R|b| \leq (R|a| - 1)(2 - t + tR|a|).$$

(iv) *Assume* $s \leq 1$ *and* $t \leq 1$. *Then* $T \in \mathcal{C}_{s,t}(R)$ *if and only if*

$$R|b| \leq (R + |a|)(sR - (2-s)|a|)$$

and

$$R|b| \leq (R|a| + 1)(tR|a| - 2 + t).$$

Proof. The lemma is a consequence of the following observation: if $s \geq 1$, then $\begin{pmatrix} a/R & b/R \\ 0 & a/R \end{pmatrix} \in \mathcal{C}_s$ if and only if $|a| \leq R$ and

$$(3.1) \qquad R|b| - sR^2 + (2-s)|a|^2 \leq 2(1-s)R|a|,$$

while if $0 < s < 1$ we have $\begin{pmatrix} a/R & b/R \\ 0 & a/R \end{pmatrix} \in \mathcal{C}_s$ if and only if $|a| \leq R\frac{s}{2-s}$ and

$$(3.2) \qquad R|b| - sR^2 + (2-s)|a|^2 \leq 2(s-1)R|a|.$$

So, assume first that $s \geq 1$. In view of [25, Theorem 3.1], we know that $\begin{pmatrix} a/R & b/R \\ 0 & a/R \end{pmatrix} \in \mathcal{C}_s$ if and only if $|a| \leq R$ and

$$(3.3) \qquad |b|^2/R^2 \leq |s + (s-2)|a|^2/R^2 - 2(s-1)|a|/R|^2.$$

Observe that

$$s + (s-2)|a|^2/R^2 - 2(s-1)|a|/R = (s-1)(|a|-R)^2/R^2 + 1 - |a|^2/R^2,$$

which is non-negative because $|a| \leq R$. We thus conclude, after taking square roots, that (3.3) is equivalent to (3.1), as desired.

Assume now that $0 < s < 1$. Using [25, Theorem 3.1] again, we deduce that $\left(\begin{smallmatrix} a/R & b/R \\ 0 & a/R \end{smallmatrix} \right) \in \mathcal{C}_s$ if and only if $|a| \leq R\frac{s}{2-s}$ and

$$(3.4) \qquad |b|^2/R^2 \leq |s + (s-2)|a|^2/R^2 - 2(1-s)|a|/R|^2.$$

Observe that

$$
\begin{aligned}
s + (s-2)|a|^2/R^2 &- 2(1-s)|a|/R \\
&= (s-1)(|a|+R)^2/R^2 + 1 - |a|^2/R^2 \\
&= (|a|/R + 1)((s-1)(|a|/R + 1) + 1 - |a|/R) \\
&= (|a|/R + 1)((s-2)|a|/R + s),
\end{aligned}
$$

which is non-negative because $|a| \leq R\frac{s}{2-s}$. Hence, (3.4) is equivalent to (3.2) and the proof is complete. ∎

THEOREM 3.5. *Assume $s, t > 0$ and $R > 1$ are such that $\mathcal{C}_{s,t}(R)$ is non-empty. Then there exists $T \in \mathcal{C}_{s,t}(R)$ such that for every $\epsilon > 0$ and every $s', t' > 0$ we have*

$$T \notin \mathcal{C}_{s',t-\epsilon}(R) \quad and \quad T \notin \mathcal{C}_{s-\epsilon,t'}(R).$$

Proof. We only deal with the case $s \geq 1, t \leq 1$, as the computations required for the remaining three cases are very similar in nature.

So, let $s \geq 1, t \leq 1$ and assume also that $(2-t)/(Rt) \leq R$, hence $\mathcal{C}_{s,t}(R)$ is non-empty. Note that if $(2-t)/(Rt) = R$, then any class either of the form $\mathcal{C}_{s',t-\epsilon}(R)$ or of the form $\mathcal{C}_{s-\epsilon,t'}(R)$ is empty. Obviously, the conclusion of the theorem holds in this case. Thus, we may actually assume that $(2-t)/(Rt) < R$. We are looking for an operator that satisfies the conditions in the statement of the theorem and has the form $T = \left(\begin{smallmatrix} a & b \\ 0 & a \end{smallmatrix} \right)$, where $a, b \in \mathbb{C}$. In view of Lemma 3.4, we know that such a T belongs to $\mathcal{C}_{s,t}(R)$ if and only if $(2-t)/(Rt) \leq |a| \leq R$ and

$$(3.5) \qquad R|b| - sR^2 + (2-s)|a|^2 \leq 2(1-s)R|a|$$

and

$$(3.6) \qquad R|b| - t|a|^2 R^2 + (2-t) \leq 2(t-1)R|a|.$$

Our goal is to find a pair (a_0, b_0) such that $(2-t)/(Rt) < |a_0| < R$ and both (3.5) and (3.6) are equalities. Indeed, assuming such a_0 and b_0 exist, let T_0 denote the corresponding operator and consider any $\epsilon > 0$. We then have $T_0 \in \mathcal{C}_{s,t}(R)$ and also

$$R|b_0| - (s - \epsilon)R^2 + (2 - (s - \epsilon))|a_0|^2 + 2((s - \epsilon) - 1)R|a_0|$$
$$= R|b_0| - sR^2 + (2 - s)|a_0|^2 + 2(s - 1)R|a_0| + \epsilon(R^2 + |a_0|^2 - 2R|a_0|)$$
$$= \epsilon(R - |a_0|)^2 > 0,$$

which shows that $T_0 \notin \mathcal{C}_{s-\epsilon,t'}(R)$, for any $t' > 0$. An analogous argument involving the second inequality also shows that $T_0 \notin \mathcal{C}_{s',t-\epsilon}(R)$ for any $s' > 0$, as desired.

Now, assume that we have equality in both (3.5) and (3.6) (without any extra restrictions). We can then extract a quadratic equation involving $|a|$ only:

$$(3.7) \qquad (s - tR^2 - 2)|a|^2 + 2(2 - s - t)R|a| + 2 - t + sR^2 = 0.$$

Notice that if we are able to find a solution $|a_0|$ of the above equation that also satisfies $(2 - t)/(Rt) < |a_0| < R$, our proof will be complete. Indeed, $|b_0|$ is then uniquely determined as

$$|b_0| = sR + \frac{(s - 2)|a_0|^2}{R} + 2(1 - s)|a_0| = t|a_0|^2 R + \frac{t - 2}{R} + 2(t - 1)|a_0|$$

and the associated T_0 has the desired properties.

Assume first that $s = tR^2 + 2$. Equation (3.7) then turns into the linear equation

$$2(2 - s - t)R|a| + 2 - t + sR^2 = 0,$$

where $s = tR^2 + 2$. Let a_0 be any number satisfying the previous equation, i.e.

$$|a_0| = \frac{2(R^2 + 1) + t(R^4 - 1)}{2R(R^2 + 1)t}.$$

One can then verify that $|a_0| < R$ is equivalent to the inequality $\frac{2}{R^2+1} < t$, which is equivalent to $(2 - t)/(Rt) < R$, hence it must be true. An analogous argument applies to the inequality $|a_0| > (2 - t)/(Rt)$.

Assume now that $s \neq tR^2 + 2$. After some calculations, we obtain (for (3.7)) the (non-negative) discriminant

$$\Delta = 4[(R^2 + 1)^2 st - 2(R^2 + 1)(s + t) + 4(R^2 + 1)].$$

Choose a_0 to be any number satisfying

$$|a_0| = \frac{2R(s + t - 2) - \sqrt{\Delta}}{2(s - tR^2 - 2)}.$$

Suppose first that $s > tR^2 + 2$. The inequality $|a_0| < R$ is equivalent to

$$tR^2 + 2 = \frac{R^2(R^2 + 1)^2 t^2 + 2(R^2 + 1)t - 4(R^2 + 1)}{(R^2 + 1)((R^2 + 1)t - 2)} < s,$$

hence it must be true. The proof when $s < tR^2 + 2$ is entirely analogous.

We now prove that $|a_0| > (2-t)/(tR)$. The computations here are somewhat more unpleasant. First, we assume that $s > tR^2 + 2$, in which case $|a_0| > (2-t)/(tR)$ is equivalent to

$$\frac{[(R^2+1)t-2]^2}{R^2}s^2$$
$$+ \left(-\frac{16}{R^2} + \frac{8}{R^2}(R^2+2)t + 2(R^2+1)[1-2/R^2]t^2 - (R^2+1)^2t^3\right)s$$
$$+ \frac{16}{R^2} - \frac{16}{R^2}t + 4(1/R^2 - R^2 - 1)t^2 + 2(R^2+1)t^3 > 0.$$

But this can be rewritten as

$$\frac{1}{R^2}((R^2+1)t-2)(s-tR^2-2)([(R^2+1)t-2]s + 2(2-t)) \geq 0,$$

which holds for $s > tR^2 + 2$ and $(R^2+1) > (R^2+1)t > 2$, as desired. The proof when $s < tR^2 + 2$ is entirely analogous. ∎

3.2. Basic properties of $\mathbb{DLA}_R(c)$. We now define a new operator class attached to the annulus A_R.

DEFINITION 3.6. Let $c \in \mathbb{R}$ and $R > 1$. Then $\mathbb{DLA}_R(c)$ denotes the class of all operators $T \in \mathcal{B}(H)$ such that

(i) $\sigma(T) \subset \overline{A_R}$,
(ii) $2\Re[(1-zT/R)^{-1} + (1-wT^{-1}/R)^{-1}] - 2 + c \geq 0$ for all $z, w \in \mathbb{D}$.

First, we prove a few elementary properties of $\mathbb{DLA}_R(c)$, including membership criteria for normal operators.

LEMMA 3.7. *Let $c > -2$, $R > 1$ and assume $A \in \mathcal{B}(H)$ is a self-adjoint operator such that $A > 0$ and $2 + c - A > 0$. If $T \in \mathcal{B}(H)$ is such that $T/R \in \mathcal{C}_{2+c-A}$ and $T^{-1}/R \in \mathcal{C}_A$, then $T \in \mathbb{DLA}_R(c)$.*

Proof. Assume $T \in \mathcal{B}(H)$ satisfies our hypotheses. Lemma 2.1 allows us to deduce that $\sigma(T/R)$, $\sigma(T^{-1}/R) \subset \overline{\mathbb{D}}$, hence $\sigma(T) \subset \overline{A_R}$. Also, (2.1) implies that

$$2\Re(1-zT/R)^{-1} + 2 + c - A - 2 \geq 0$$

and

$$2\Re(1-wT^{-1}/R)^{-1} + A - 2 \geq 0,$$

for all $z, w \in \mathbb{D}$. Adding these two inequalities concludes the proof. ∎

PROPOSITION 3.8. *Let $c \in \mathbb{R}$, $R > 1$ and assume $N \in \mathcal{B}(H)$ is normal.*

(i) *If $c \geq 0$, then $\mathbb{DLA}_R(c)$ is non-empty for every $R > 1$.*

(ii) *If $-2 < c < 0$, then $\mathbb{DLA}_R(c)$ is non-empty if and only if there exists $s \in (0, 2 + c)$ such that*

$$\max\left\{1, \frac{2-s}{s}\right\} \leq R^2 \min\left\{1, \frac{2+c-s}{s-c}\right\}.$$

In fact, $\mathbb{DLA}_R(c)$ is non-empty if and only if it contains a scalar $a \in \mathbb{C}$.

(iii) *If $c \leq -2$, then $\mathbb{DLA}_R(c)$ is empty for every $R > 1$.*

(iv) *$\mathbb{DLA}_R(c) \subset \mathcal{C}_{2-c,2-c}(R)$ for every $c > -2$.*

(v) *Assume $c > -2$. Then $N \in \mathbb{DLA}_R(c)$ if and only if*

$$\sigma(N) \subset \bigcup_{0 < s < 2+c} \mathcal{C}_{2+c-s,s}(R).$$

(vi) *If, in addition, we assume $c \geq 0$, then $N \in \mathbb{DLA}_R(c)$ is equivalent to $R^{-2} \leq N^*N \leq R^2$.*

Proof. First, we prove that if $c > -2$, then $\mathbb{DLA}_R(c)$ is non-empty if and only if there exists $s \in (0, 2 + c)$ such that $\mathcal{C}_{2+c-s,s}(R)$ is non-empty. Once this has been shown, the last assertion in (ii) will follow immediately from Lemma 3.7 and the fact that $\mathcal{C}_{2+c-s,s}(R)$ is non-empty if and only if it contains a scalar (see Proposition 3.3).

First, observe that Lemma 3.7 implies $\mathcal{C}_{2+c-s,s}(R) \subset \mathbb{DLA}_R(c)$ for every $s \in (0, 2 + c)$, hence one direction is obvious. For the converse, assume $\mathbb{DLA}_R(c)$ is non-empty. Thus, there exists $T \in \mathcal{B}(H)$ such that

(3.8) $$\left\langle \left(2\Re[(1 - zT/R)^{-1} + (1 - wT^{-1}/R)^{-1}] - 2 + c\right)h, h \right\rangle \geq 0$$

for all $z, w \in \mathbb{D}$ and $x \in H$. Now, let $\lambda \in \mathbb{C}^*$ be in the approximate point spectrum of T (it is always non-empty, as it contains the topological boundary of $\sigma(T)$; see [20, Problem 63]). Thus, there exists a sequence $\{h_n\} \subset H$ such that $\|h_n\| = 1$ $(n = 1, 2, \dots)$ and $(\lambda - T)h_n \to 0$ as $n \to \infty$. Hence we easily obtain

$$\lim_n (\lambda^k - T^k)h_n = 0,$$

for all $k \in \mathbb{Z}$. Thus, $\langle T^k h_n, h_n \rangle \to \lambda^k$ for all $k \in \mathbb{Z}$, and so

(3.9) $$\left\langle \sum_{k=0}^m (zT/R)^k h_n, h_n \right\rangle \to \sum_{k=0}^m (z\lambda/R)^k$$

and

(3.10) $$\left\langle \sum_{k=0}^m (wT^{-1}/R)^k h_n, h_n \right\rangle \to \sum_{k=0}^m (w/(R\lambda))^k$$

as $n \to \infty$, for all $m \geq 0$ and $z, w \in \mathbb{D}$.

Now, fix $z, w \in \mathbb{D}$. We have

$$\sum_{k=0}^{m} (zT/R)^k \to (1 - zT/R)^{-1} \quad \text{and} \quad \sum_{k=0}^{m} (wT^{-1}/R)^k \to (1 - wT^{-1}/R)^{-1}$$

in the operator norm as $m \to \infty$, since $\sigma(zT/R), \sigma(wT^{-1}/R) \subset \mathbb{D}$. Hence, in view of (3.9) and (3.10), we conclude that

$$(3.11) \qquad \lim_{n \to \infty} \langle (1 - zT/R)^{-1} h_n, h_n \rangle = \lim_{m,n \to \infty} \left\langle \sum_{k=0}^{m} (zT/R)^k h_n, h_n \right\rangle$$
$$= (1 - z\lambda/R)^{-1}$$

and

$$(3.12) \qquad \lim_{n \to \infty} \langle (1 - wT^{-1}/R)^{-1} h_n, h_n \rangle = \lim_{m,n \to \infty} \left\langle \sum_{k=0}^{m} (wT^{-1}/R)^k h_n, h_n \right\rangle$$
$$= (1 - w/(R\lambda))^{-1}.$$

We now set $h = h_n$ in (3.8) and let $n \to \infty$. In view of the real-part versions of (3.11) and (3.12), we deduce that

$$2\Re[(1 - z\lambda/R)^{-1} + (1 - w/(\lambda R))^{-1}] - 2 + c \geq 0$$

for all $z, w \in \mathbb{D}$. This can also be written as

$$\inf_{z \in \mathbb{D}}[2\Re(1 - z\lambda/R)^{-1} + c] \geq \sup_{w \in \mathbb{D}}[-2\Re(1 - w/(\lambda R))^{-1} + 2]$$

and thus we can deduce the existence of $s \in \mathbb{R}$ such that

$$2\Re(1 - z\lambda/R)^{-1} + (1 - w/(\lambda R))^{-1} + c \geq s \geq -2\Re(1 - w/(\lambda R))^{-1} + 2$$

for all $z, w \in \mathbb{D}$. In view of Lemma 2.1, we find that $\lambda/R \in \mathcal{C}_{2+c-s}$ and $1/(\lambda R) \in \mathcal{C}_s$, which concludes the proof of our assertion. If $c \geq 0$, then we can always choose $s = 1$, as $\mathcal{C}_{2+c-1,1}(R)$ will be non-empty by Lemma 3.3(i). This gives (i) and if $-2 < c < 0$, we obtain (ii) by the same lemma.

For (iii), assume $c \leq -2$ and let $T \in \mathbb{DLA}_R(c)$. Set $w = 0$ in the definition of $\mathbb{DLA}_R(c)$ to deduce that $T/R \in \mathcal{C}_{c+2}$, which implies that $c = -2$ (since $\mathcal{C}_\rho$ is empty for $\rho < 0$). But then $T/R \in \mathcal{C}_0$ and so $T \equiv 0$, which contradicts the inclusion $\sigma(T) \subset \overline{A_R}$. Thus, $\mathbb{DLA}_R(c)$ must be empty.

For (iv), assume $c > -2$ and let $T \in \mathbb{DLA}_R(c)$. Setting $w = 0$ in the definition of $\mathbb{DLA}_R(c)$ gives $T/R \in \mathcal{C}_{2-c}$, while setting $z = 0$ (and letting $w \in \mathbb{D}$) gives $T^{-1}/R \in \mathcal{C}_{2-c}$. Thus, $\mathbb{DLA}_R(c) \subset \mathcal{C}_{2-c,2-c}(R)$.

For (v), let $c > -2$. Since N is normal, it has a spectral decomposition (see [11, Chapter IX])

$$N = \int_{\sigma(N)} \lambda \, dE(\lambda).$$

Using this, we find that $N \in \mathbb{DLA}_R(c)$ if and only if its spectrum is contained in $\overline{A_R}$ and

$$2\Re[(1 - zN/R)^{-1} + (1 - wN^{-1}/R)^{-1}] - 2 + c$$
$$= \int_{\sigma(N)} (2\Re[(1 - z\lambda/R)^{-1} + (1 - w/(\lambda R))^{-1}] - 2 + c)\, dE(\lambda) \geq 0$$

for all $z, w \in \mathbb{D}$, which can be equivalently restated as

$$2\Re[(1 - z\lambda/R)^{-1} + (1 - w/(\lambda R))^{-1}] - 2 + c \geq 0$$

for all $z, w \in \mathbb{D}$ and all $\lambda \in \sigma(N)$. Mimicking our argument from the proof of (i), we deduce, for every $\lambda \in \sigma(N)$, the existence of $s \in (0, 2+c)$ (depending on λ) such that $\lambda \in \mathcal{C}_{2+c-s,s}(R)$. Hence

$$\sigma(N) \subset \bigcup_{0 < s < 2+c} \mathcal{C}_{2+c-s,s}(R),$$

which concludes the proof.

For (vi), assume $c \geq 0$. Since N is normal, $R^{-2} \leq N^*N \leq R^2$ is equivalent to $\sigma(N) \subset A_R$, which is necessary for membership in $\mathbb{DLA}_R(c)$. Conversely, if $R^{-2} \leq N^*N \leq R^2$, then

$$\sigma(N) \subset \mathcal{C}_{1,1}(R) \subset \bigcup_{0 < s < 2+c} \mathcal{C}_{2+c-s,s}(R),$$

where the last inclusion holds because $c \geq 0$. Thus, $N \in \mathbb{DLA}_R(c)$. ∎

Under the extra assumption $\sigma(T) \subset A_R$, we can restrict the parameters z, w in the definition of $\mathbb{DLA}_R(c)$ to the boundary of the disk (this is the $\mathbb{DLA}_R(c)$-version of Lemma 2.2).

LEMMA 3.9. *Assume $c > -2$ and $T \in \mathcal{B}(H)$ is such that $\sigma(T) \subset A_R$. Then $T \in \mathbb{DLA}_R(c)$ if and only if*

$$2\Re[(1 - e^{i\theta}T/R)^{-1} + (1 - e^{i\psi}T^{-1}/R)^{-1}] - 2 + c \geq 0, \quad \forall \theta, \psi \in [0, 2\pi).$$

Proof. First, assume $T \in \mathbb{DLA}_R(c)$. Fix an arbitrary $h \in H$ and $w \in \mathbb{D}$ and define

$$\Phi_{h,w} : \mathbb{D} \to \mathbb{C},$$
$$z \mapsto \langle (2\Re[(1 - zT/R)^{-1} + (1 - wT^{-1}/R)^{-1}] - 2 + c)h, h \rangle.$$

Since $\sigma(T) \subset A_R$, $\Phi_{h,w}$ is a harmonic function on $\mathbb{D}$ that extends continuously to $\overline{\mathbb{D}}$. By the minimum principle for harmonic functions, we then obtain

$$\Phi_{h,w}(z) = \langle (2\Re[(1 - zT/R)^{-1} + (1 - wT^{-1}/R)^{-1}] - 2 + c)h, h \rangle \geq 0$$

for all $z \in \mathbb{D}$, if and only if

$$\Phi_{h,w}(e^{i\theta}) = \langle (2\Re[(1 - e^{i\theta}T/R)^{-1} + (1 - wT^{-1}/R)^{-1}] - 2 + c)h, h \rangle \geq 0$$

for all $\theta \in [0, 2\pi)$. Since h and w were arbitrary, we conclude that
$$2\Re[(1 - e^{i\theta}T/R)^{-1} + (1 - wT^{-1}/R)^{-1}] - 2 + c \geq 0$$
for all $\theta \in [0, 2\pi)$ and all $w \in \mathbb{D}$. We can now apply the minimum principle to the function
$$w \mapsto \langle (2\Re[(1 - e^{i\theta}T/R)^{-1} + (1 - wT^{-1}/R)^{-1}] - 2 + c)h, h \rangle$$
to conclude the proof.

For the converse, simply roll back the steps in the previous proof. ∎

Now, recall that the $\mathcal{C}_\rho$ classes are strictly monotone with respect to ρ, i.e. $\rho < \rho'$ implies $\mathcal{C}_\rho \subsetneq \mathcal{C}_{\rho'}$. We are going to prove an analogous monotonicity result for $\mathbb{DLA}_R(c)$.

THEOREM 3.10. *If* $-2 < c < c'$ *and* $\mathbb{DLA}_R(c')$ *is non-empty, then*
$$\mathbb{DLA}_R(c) \subsetneq \mathbb{DLA}_R(c').$$

Proof. If $c < c'$, it is obvious by the definition of $\mathbb{DLA}_R(c)$ that $\mathbb{DLA}_R(c) \subseteq \mathbb{DLA}_R(c')$. To show that the inclusion is actually strict, we are going to divide the proof into two cases.

First, assume there exist $s, t > 0$ such that $s \geq 1$, $s + t = 2 + c'$ and $\mathcal{C}_{s,t}(R)$ is non-empty. Note that (in view of Lemma 3.3 and monotonicity of the $\mathcal{C}_\rho$ classes), if such s, t exist and $t \geq 1$, we can replace them by new parameters s', t' such that $s' \geq 1$, $t' < 1$, $s' + t' = 2 + c'$, $\frac{2}{R^2+1} < t$ and $\mathcal{C}_{s',t'}(R)$ is non-empty. So, we may also assume that $t < 1$ and $\frac{2}{R^2+1} < t$.

Now, we will take advantage of the matrices we calculated in the proof of Theorem 3.5. Recall first that, in view of Lemma 3.4, a matrix
$$T = \begin{pmatrix} a & b \\ 0 & a \end{pmatrix}$$
with $a, b > 0$ lies in $\mathcal{C}_{s,t}(R)$ if and only if $(2 - t)/(Rt) \leq a \leq R$ and
$$(3.13) \qquad 2\Re\left[\frac{1}{1 - ae^{i\theta}/R}\right] + (s - 2) \geq \frac{b}{R|1 - ae^{i\theta}/R|^2}$$
for $\theta = 0$ and
$$(3.14) \qquad 2\Re\left[\frac{1}{1 - e^{i\psi}/(aR)}\right] + (t - 2) \geq \frac{b}{a^2}\frac{1}{R|1 - e^{i\psi}/(aR)|^2}$$
for $\psi = \pi$. Since $\frac{2}{R^2+1} < t$, the proof of Theorem 3.5 tells us that we can find a, b such that $(2 - t)/(Rt) < a < R$ and we have equality in (3.13) for $\theta = 0$ and in (3.14) for $\psi = \pi$. Since, for these values of a, b, we have $T \in \mathcal{C}_{s,t}(R)$ and $s + t = 2 + c'$, Lemma 3.7 tells us that $T \in \mathbb{DLA}_R(c')$.

Now, we claim that T cannot lie in $\mathbb{DLA}_R(c)$. Indeed, assume $T \in \mathbb{DLA}_R(c)$. Taking determinants in the inequality
$$2\Re[(1 - zT/R)^{-1} + (1 - wT^{-1}/R)^{-1}] - 2 + c \geq 0$$

implies that

$$2\Re\left[\frac{1}{1-ae^{i\theta}/R}+\frac{1}{1-e^{i\psi}/(aR)}\right]-2+c$$

$$\geq\left|\frac{be^{i\theta}}{R(1-ae^{i\theta}/R)^2}-\frac{b}{a^2}\frac{e^{i\psi}}{R(1-e^{i\psi}/(aR))^2}\right|$$

for all θ and ψ. In particular, we can choose $\theta_0 = 0$ and $\psi_0 = \pi$, which implies (since we have equality in both (3.13) and (3.14))

$$2\Re\left[\frac{1}{1-ae^{i\theta_0}/R}+\frac{1}{1-e^{i\psi_0}/(aR)}\right]-2+c$$

$$\geq\left|\frac{be^{i\theta_0}}{R(1-ae^{i\theta_0}/R)^2}-\frac{b}{a^2}\frac{e^{i\psi_0}}{R(1-e^{i\psi_0}/(aR))^2}\right|$$

$$=\left|\frac{be^{i\theta_0}}{R(1-ae^{i\theta_0}/R)^2}\right|+\left|\frac{b}{a^2}\frac{e^{i\psi_0}}{R(1-e^{i\psi_0}/(aR))^2}\right|$$

$$=2\Re\left[\frac{1}{1-ae^{i\theta_0}/R}\right]+(s-2)+2\Re\left[\frac{1}{1-e^{i\psi_0}/(aR)}\right]+(t-2)$$

$$=2\Re\left[\frac{1}{1-ae^{i\theta_0}/R}\right]+2\Re\left[\frac{1}{1-e^{i\psi_0}/(aR)}\right]+c'-2,$$

a contradiction, since $c < c'$. Thus, $T \notin \mathbb{DLA}_R(c')$.

Now, assume that there do not exist $s, t > 0$ such that $s \geq 1, s+t = 2+c'$ and $\mathcal{C}_{s,t}(R)$ is non-empty. Since $T \in \mathcal{C}_{s,t}(R)$ if and only if $T^{-1} \in \mathcal{C}_{t,s}(R)$, we also deduce that there do not exist $s, t > 0$ such that $t \geq 1, s + t = 2 + c'$ and $\mathcal{C}_{s,t}(R)$ is non-empty. Thus, $-2 < c' < 0$. Now, $\mathbb{DLA}_R(c')$ is non-empty, so Proposition 3.8(i) tells us that there exists $s \in (0, 2+c')$ such that $\mathcal{C}_{s,2+c'-s}(R)$ is non-empty. In view of our previous remarks, $s, 2 + c' - s < 1$. Lemma 3.3 then tells us that

$$\frac{1}{R}\frac{s-c'}{2+c'-s}\leq R\frac{s}{2-s}.$$

Note that the right-hand side tends to 0 as $s \to 0$, while the left-hand side remains bounded below by a strictly positive number. Thus, shrinking s, we may replace it by a positive number δ such that $\frac{1}{R}\frac{\delta-c'}{2+c'-\delta} = R\frac{\delta}{2-\delta}$. In view of Lemma 3.3 again, we obtain that $\mathcal{C}_{\delta,2+c'-\delta}(R)$ is non-empty. Setting $a = R\frac{\delta}{2-\delta}$, we obtain, from the same lemma, that $a \in \mathcal{C}_{\delta,2+c'-\delta}(R) \subset \mathbb{DLA}_R(c')$.

Now, we shall show that $a \notin \mathbb{DLA}_R(c)$. Assume instead $a \in \mathbb{DLA}_R(c)$. Proposition 3.8 then implies that we can find $s_0 \in (0, 2 + c)$ such that $a \in \mathcal{C}_{s_0,2+c-s_0}(R)$. But $c < c'$, hence either $s_0 < \delta$ or $2 + c - s_0 < 2 + c' - \delta$. Assume that $s_0 < \delta$, then

$$R\frac{s_0}{2-s_0} < R\frac{\delta}{2-\delta} = a,$$

which contradicts $a \in \mathcal{C}_{s_0, 2+c-s_0}(R)$. Similarly, if $2 + c - s_0 < 2 + c' - \delta$ we can write

$$\frac{1}{R} \frac{2 - (2 + c - s_0)}{2 + c - s_0} > \frac{1}{R} \frac{2 - (2 + c' - \delta)}{2 + c' - \delta} = \frac{1}{R} \frac{\delta - c'}{2 + c' - \delta} = a,$$

which again contradicts $a \in \mathcal{C}_{s_0, 2+c-s_0}(R)$. Thus, $a \notin \mathbb{DLA}_R(c)$ and we are done. ∎

It is well-known (see e.g. [36]) that given any $T \in \mathcal{C}_\rho$ ($\rho > 0$) and any $f \in \mathcal{A}(\mathbb{D})$ such that $\|f\|_\infty \leq 1$ and $f(0) = 0$, we have $f(T) \in \mathcal{C}_\rho$. We end this subsection with a proposition that is motivated by this result.

PROPOSITION 3.11. *Assume $c > -2$ and $T \in \mathbb{DLA}_R(c)$. Then for any $f, g \in \mathcal{A}(\mathbb{D})$ that are bounded by 1 and satisfy $f(0) = g(0) = 0$, we have*

$$2\Re\big[(1 - zf(T/R))^{-1} + (1 - wg(T^{-1}/R))^{-1}\big] - 2 + c \geq 0$$

for all $z, w \in \mathbb{D}$.

Proof. By assumption, we know that

$$(3.15) \qquad \Re\left[\sum_{n=1}^{\infty}\left(\frac{Tz}{R}\right)^n + 1 + \sum_{n=1}^{\infty}\left(\frac{T^{-1}w}{R}\right)^n\right] \geq \frac{-c}{2}$$

for all $z, w \in \mathbb{D}$. Fix $w \in \mathbb{D}$, let $x \in H$ and choose a decreasing null sequence $\{\epsilon_k\}$. Set

$$S_k = S_k(w) = 1 + c/2 + \epsilon_k + \Re\sum_{n=1}^{\infty}\left(\frac{T^{-1}w}{R}\right)^n \in \mathcal{B}(H).$$

Inequality (3.15) now tells us that the holomorphic function

$$F_k : \mathbb{D} \to \mathbb{C}, \qquad z \mapsto \sum_{n=1}^{\infty} \frac{z^n}{R^n}\langle T^n x, x\rangle + \langle S_k x, x\rangle,$$

has positive real part and satisfies $F_k(0) = \langle S_k x, x\rangle > 0$ (put $z = 0$ in (3.15)). Thus, Herglotz's theorem implies the existence of a positive measure $\mu_{x,w,k}$ on the unit circle such that

$$F_k(0) + \sum_{n=1}^{\infty} \frac{z^n}{R^n}\langle T^n x, x\rangle = \int \frac{1 + ze^{-i\theta}}{1 - ze^{-i\theta}}\, d\mu_{x,w,k}(\theta)$$

for all $z \in \mathbb{D}$. Expanding the integrand and equating coefficients, we obtain

$$\frac{1}{R^n}\langle T^n x, x\rangle = 2\int e^{-in\theta}\, d\mu_{x,w,k}(\theta)$$

for every $n \geq 1$. Thus, if p is any polynomial such that $p(0) = 0$, we can write

$$\langle p(T/R)x, x\rangle = 2\int p(e^{-i\theta})\, d\mu_{x,w,k}(\theta).$$

Replacing p by p^n, we obtain

$$\langle p^n(T/R)x, x\rangle = 2\int p^n(e^{-i\theta})\,d\mu_{x,w,k}(\theta)$$

for every $n \geq 1$. Thus, if we also assume that $|p| \leq 1$, we deduce that $p(T/R)$ has its spectrum inside $\mathbb{D}$ (since the same must be true for T/R) and we can write

$$\left\langle \left(S_k + \sum_{n=1}^{\infty} z^n p^n(T/R)\right)x, x\right\rangle = F_k(0) + 2\sum_{n=1}^{\infty} z^n \int p(e^{-i\theta})^n\,d\mu_{x,w,k}(\theta)$$

$$= \int \frac{1 + zp(e^{-i\theta})}{1 - zp(e^{-i\theta})}\,d\mu_{x,w,k}(\theta).$$

Now, if $f \in \mathcal{A}(\mathbb{D})$ is bounded by 1 and satisfies $f(0) = 0$, a standard approximation argument shows that

$$\left\langle \left(S_k + \sum_{n=1}^{\infty} z^n f^n(T/R)\right)x, x\right\rangle = \int \frac{1 + zf(e^{i\theta})}{1 - zf(e^{i\theta})}\,d\mu_{x,w,k}(\theta).$$

The integrand has positive real part for all z and θ, hence

$$\Re\left[\sum_{n=1}^{\infty}(zf(T/R))^n + 1 + \epsilon_k + \sum_{n=1}^{\infty}(wT^{-1}/R)^n\right] \geq \frac{-c}{2}$$

for all k and for all $z, w \in \mathbb{D}$. Letting $k \to \infty$, we obtain

$$\Re\left[\sum_{n=1}^{\infty}(zf(T/R))^n + 1 + \sum_{n=1}^{\infty}(wT^{-1}/R)^n\right] \geq \frac{-c}{2}$$

for all $z, w \in \mathbb{D}$. Using this last inequality, we may repeat the previous argument with the roles of z and w swapped, thus obtaining the desired result. ∎

3.3. The double-layer potential kernel. In this subsection, we exhibit the connection between $\mathbb{DLA}_R(c)$ and the double-layer potential kernel over the annulus, as described in the introduction.

Recall that, given any bounded open $\Omega \subset \mathbb{C}$ and any $T \in \mathcal{B}(H)$ such that $\sigma(T) \subset \Omega$, we may consider the Cauchy transforms of f and $\overline{f}$,

$$f(T) = (Cf)(T) = \frac{1}{2\pi i}\int_{\partial\Omega} f(\sigma)(\sigma - T)^{-1}\,d\sigma,$$

$$(C\overline{f})(T) = \frac{1}{2\pi i}\int_{\partial\Omega} \overline{f(\sigma)}(\sigma - T)^{-1}\,d\sigma.$$

We also define the transform of f by the double-layer potential kernel

$$S(f, T) = \int_{\partial\Omega} \mu(\sigma(s), T)f(\sigma(s))\,ds,$$

where s denotes the arc length of $\sigma = \sigma(s)$ on the (counter-clockwise) oriented boundary $\partial\Omega$ and $\mu(\sigma(s), T)$ is the self-adjoint operator defined (for $\sigma(s) \notin \sigma(T)$) as

$$\mu(\sigma(s), T) = \frac{1}{2\pi i}(\sigma'(s)(\sigma(s) - T)^{-1} - \overline{\sigma'(s)}(\overline{\sigma(s)} - T^*)^{-1}).$$

Note that $S(f, T) = f(T) + (C\overline{f})(T)^*$ and thus $\int_{\partial\Omega} \mu(\sigma, T)\, ds = 2I$.

The definition of $\mathbb{DLA}_R(c)$ can now be recast (for $\sigma(T) \subset A_R$) as follows.

THEOREM 3.12. *Assume $R > 1$, $c > -2$ and $T \in \mathcal{B}(H)$ satisfies $\sigma(T) \subset A_R$. Then $T \in \mathbb{DLA}_R(c)$ if and only if the mapping*

$$S_{R,c} : \mathcal{A}(A_R) \to \mathcal{B}(H),$$
$$f = \sum_{n \in \mathbb{Z}} a_n z^n \mapsto \frac{1}{2 + c}\Big[\int_{\partial A_R} \mu(\sigma, T) f(\sigma)\, ds + c a_0\Big],$$

is contractive.

Proof. Write $\partial A_R = \Gamma_1 \cup \Gamma_{-1}$, where Γ_1 (the outer circle) is counter-clockwise oriented, while Γ_{-1} (the inner circle) is clockwise oriented. Also, in view of [26, Corollary 2.9, Proposition 2.12], the fact that $S_{R,c}$ is unital allows us to deduce that $S_{R,c}$ is contractive if and only if the operator

$$(3.16) \quad S'_{R,c} : \mathcal{A}(A_R) + \mathcal{A}(A_R)^* \to \mathcal{B}(H),$$
$$f + \overline{g} = \sum_{n \in \mathbb{Z}} a_n z^n + \sum_{n \in \mathbb{Z}} \overline{b_n} \overline{z}^n \mapsto S_{R,c}(f) + S_{R,c}(g)^*$$
$$= \frac{1}{2 + c}\Big[\int_{\partial A_R} \mu(\sigma, T)(f + \overline{g})(\sigma)\, ds + c(a_0 + \overline{b_0})\Big],$$

is positive. Let $C(\partial A_R)$ denote the algebra of continuous functions on ∂A_R and recall that the closure of $\mathcal{A}(A_R) + \mathcal{A}(A_R)^*$ in $C(\partial A_R)$ is the codimension 1 subspace (see [26, p. 80])

$$\mathcal{M}_R = \Big\{f \in C(\partial A_R) : \frac{1}{2\pi R}\int_{\Gamma_1} f(\sigma)\, ds = \frac{R}{2\pi}\int_{\Gamma_{-1}} f(\sigma)\, ds\Big\}.$$

Thus, our goal is to show that $T \in \mathbb{DLA}_R(c)$ if and only if $\tilde{S}_{R,c}$ is positive over $\mathcal{M}_R$, where

$$\tilde{S}_{R,c} : \mathcal{M}_R \to \mathcal{B}(H),$$
$$f \mapsto \frac{1}{2 + c}\Big[\int_{\partial A_R} \mu(\sigma, T) f(\sigma)\, ds + \frac{c}{2\pi i}\int_{\Gamma_1} \frac{f}{\sigma}\sigma'\, ds\Big]$$

(note that $S'_{R,c} \equiv \tilde{S}_{R,c}$ over $\mathcal{A}(A_R) + \mathcal{A}(A_R)^*$). We require the following lemma.

LEMMA 3.13. *In the setting of Theorem 3.12, $T \in \mathbb{DLA}_R(c)$ if and only if*

$$R\mu(\sigma_1, T) + R^{-1}\mu(\sigma_2, T) + \frac{c}{2\pi} \geq 0$$

for all $\sigma_1 \in \Gamma_1$ and $\sigma_2 \in \Gamma_{-1}$.

Proof. For $\sigma_1 \in \Gamma_1$, the arc-length parametrization gives $\sigma_1 = Re^{i\theta}$, $\sigma_1' = ie^{i\theta}$, while for $\sigma_2 \in \Gamma_{-1}$ we obtain $\sigma_2 = R^{-1}e^{-i\psi}, \sigma_2' = -ie^{-i\psi}$, where $\theta, \psi \in [0, 2\pi)$. Thus, we may write

$$2\pi[R\mu(\sigma_1, T) + R^{-1}\mu(\sigma_2, T)] + c$$
$$= 2\Re[Re^{i\theta}(Re^{i\theta} - T)^{-1}] + 2\Re[-R^{-1}e^{-i\psi}(R^{-1}e^{-i\psi} - T)^{-1}] + c$$
$$= 2\Re[(1 - e^{-i\theta}T/R)^{-1} + (1 - e^{-i\psi}T^{-1}/R)^{-1}] + c - 2,$$

which is positive for all $\theta, \psi \in [0, 2\pi)$ if and only if $T \in \mathbb{DLA}_R(c)$. ∎

Now, assume that $\tilde{S}_{R,c}$ is positive and that $T \notin \mathbb{DLA}_R(c)$. In view of Lemma 3.13, there exist $\eta_1 \in \Gamma_1$, $\eta_2 \in \Gamma_{-1}$ and a unit vector $v \in H$ such that

(3.17) $$\left\langle \left(R\mu(\eta_1, T) + R^{-1}\mu(\eta_2, T) + \frac{c}{2\pi}\right)v, v\right\rangle = 2k < 0.$$

Since $\sigma(T) \subset A_R$, we know that both of the maps $\sigma_1 \mapsto \langle\mu(\sigma_1, T)v, v\rangle$ and $\sigma_2 \mapsto \langle\mu(\sigma_2, T)v, v\rangle$ are continuous. Thus, in view of (3.17), we can find small arcs $I_1 \subset \Gamma_1$ and $I_{-1} \subset \Gamma_{-1}$ of equal length and centered at η_1 and η_2 respectively such that

$$\left\langle \left(R\mu(\sigma_1, T) + R^{-1}\mu(\sigma_2, T) + \frac{c}{2\pi}\right)v, v\right\rangle \leq k < 0$$

for all $\sigma_1 \in I_1$ and all $\sigma_2 \in I_{-1}$. From this, we easily deduce the existence of $t \in \mathbb{R}$ such that

(3.18) $$\left\langle \left(R\mu(\sigma_1, T) + \frac{c}{2\pi}\right)v, v\right\rangle \leq k/2 + t$$

and

(3.19) $$\langle R^{-1}\mu(\sigma_2, T)v, v\rangle \leq k/2 - t$$

for all $\sigma_1 \in I_1$ and all $\sigma_2 \in I_{-1}$. Now, take $g : \partial A_R \to \mathbb{C}$ to be a continuous function such that $0 \leq g \leq 1$, $g(z) = 0$ for z outside $I_1 \cup I_{-1}$ and $d = \frac{1}{2\pi R}\int_{\Gamma_1} g(\sigma)\,ds = \frac{R}{2\pi}\int_{\Gamma_{-1}} g(\sigma)\,ds > 0$. Then $g \in \mathcal{M}_R$ and so $\tilde{S}_{R,c}(g)$ must be a positive operator. However, observe that by (3.18) and (3.19),

$$(2+c)\langle \tilde{S}_{R,c}(g)v,v\rangle = \int_{\partial A_R} \langle \mu(\sigma,T)v,v\rangle g(\sigma)\,ds + \frac{c}{2\pi R}\int_{\Gamma_1} g(\sigma_1)\,ds$$

$$= \int_{\Gamma_1} \langle R\mu(\sigma_1,T)v,v\rangle \frac{g(\sigma_1)}{R}\,ds$$

$$+ \int_{\Gamma_{-1}} \langle R^{-1}\mu(\sigma_2,T)v,v\rangle Rg(\sigma_2)\,ds + \frac{c}{2\pi R}\int_{\Gamma_1} g(\sigma_1)\,ds$$

$$\leq (k/2+t)\frac{1}{R}\int_{\Gamma_1} g(\sigma_1)\,ds + (k/2-t)R\int_{\Gamma_{-1}} g(\sigma_2)\,ds$$

$$= 2\pi dk < 0,$$

a contradiction. Thus, $T \in \mathbb{DLA}_R(c)$.

Conversely, assume $T \in \mathbb{DLA}_R(c)$. Fix $v \in H$. Lemma 3.13 tells us that we can find $t \in \mathbb{R}$ such that

$$(3.20) \qquad \left\langle \left(R\mu(\sigma_1,T) + \frac{c}{2\pi} \right)v,v \right\rangle \geq t$$

and

$$(3.21) \qquad \langle R^{-1}\mu(\sigma_2,T)v,v\rangle \geq -t,$$

for all $\sigma_1 \in \Gamma_1$ and all $\sigma_2 \in \Gamma_{-1}$. Now, let $f \in \mathcal{M}_R$ be positive. Since $\frac{1}{2\pi R}\int_{\Gamma_1} f(\sigma)\,ds = \frac{R}{2\pi}\int_{\Gamma_{-1}} f(\sigma)\,ds$, we may write (in view of (3.20) and (3.21))

$$(2+c)\langle \tilde{S}_{R,c}(f)v,v\rangle = \int_{\Gamma_1} \left\langle \left(R\mu(\sigma_1,T) + \frac{c}{2\pi} \right)v,v \right\rangle \frac{f(\sigma_1)}{R}\,ds$$

$$+ \int_{\Gamma_{-1}} \langle R^{-1}\mu(\sigma_2,T)v,v\rangle Rf(\sigma_2)\,ds$$

$$\geq \frac{t}{R}\int_{\Gamma_1} f(\sigma_1)\,ds - tR\int_{\Gamma_{-1}} f(\sigma_2)\,ds = 0.$$

Since $v \in H$ was arbitrary, $\tilde{S}_{R,c}(f)$ has to be a positive operator and we are done. ∎

4. The $\mathbb{CDLA}_R(c)$ class. In this section, we introduce and characterize $\mathbb{CDLA}_R(c)$, the "complete version" of $\mathbb{DLA}_R(c)$. Our main result is Theorem 1.2. One can also work with operators satisfying $\sigma(T) \subset \overline{A_R}$ to obtain the more general Theorem 4.4.

DEFINITION 4.1. Assume $R > 1$, $c > -2$ and $T \in \mathcal{B}(H)$ satisfies $\sigma(T) \subset A_R$. Then T is in $\mathbb{CDLA}_R(c)$ if the mapping

$$S_{R,c} : \mathcal{A}(A_R) \to \mathcal{B}(H), \quad f = \sum_{n\in\mathbb{Z}} a_n z^n \mapsto \frac{1}{2+c}\Big[\int_{\partial A_R} \mu(\sigma,T)f(\sigma)\,ds + ca_0 \Big],$$

is completely contractive.

We first establish one direction of Theorem 1.2.

LEMMA 4.2. *Assume $R > 1$, $c > -2$ and $T \in \mathcal{B}(H)$ satisfies $\sigma(T) \subset A_R$. If there exists $A \in \mathcal{B}(H)$ such that $A > 0$, $2 + c - A > 0$ and $T/R \in \mathcal{C}_{2+c-A}$ and $T^{-1}/R \in \mathcal{C}_A$, then $T \in \mathbb{CDLA}_R(c)$.*

Proof. We again write $\partial A_R = \Gamma_1 \cup \Gamma_{-1}$, where Γ_1 (the outer circle) is counter-clockwise oriented, while Γ_{-1} (the inner circle) is clockwise oriented.

Now, define the self-adjoint operator

$$\nu_A(\sigma, T) := \mu(\sigma, T) + \frac{\sigma'}{\sigma} \frac{(c - A)}{2\pi i}, \quad \forall \sigma \in \partial \Gamma_1,$$

and

$$\nu_A(\sigma, T) := \mu(\sigma, T) - \frac{\sigma'}{\sigma} \frac{A}{2\pi i}, \quad \forall \sigma \in \partial \Gamma_{-1}.$$

Note that if $\sigma \in \Gamma_1$, we have $\sigma = R e^{i\theta}$ and $s = R\theta$, thus

$$\nu_A(\sigma, T) = \mu(R e^{i\theta}, T) + \frac{c - A}{2\pi R} \geq 0$$

for all θ, as $T/R \in \mathcal{C}_{2+c-A}$. Also, if $\sigma \in \Gamma_{-1}$, we can write $\sigma = R^{-1} e^{-i\phi}$ and $s = R^{-1}\phi$, hence

$$
\begin{aligned}
\nu_A(\sigma, T) &= \mu(R^{-1} e^{-i\phi}, T) + \frac{R}{2\pi} A \\
&= \frac{1}{2\pi} \left(-e^{-i\phi}(R^{-1} e^{-i\phi} - T)^{-1} + e^{i\phi}(R^{-1} e^{i\phi} - T^*)^{-1} \right) + \frac{R}{2\pi} A \\
&= \frac{R}{2\pi} \left(2\Re(R^{-1} T^{-1} e^{-i\phi}(I - T^{-1} e^{-i\phi} R^{-1})^{-1}) + A \right) \\
&= \frac{R}{2\pi} \left(2\Re(I - T^{-1} e^{-i\phi} R^{-1})^{-1} + A - 2 \right) \geq 0
\end{aligned}
$$

for all ϕ, as $T^{-1}/R \in \mathcal{C}_A$.

Next, consider the coordinatewise map $S_{R,c}^{(m)} : M_m(\mathcal{A}(A_R)) \to \mathcal{B}(H^{(m)})$, for $m \geq 1$. Here, $M_m(\mathcal{A}(A_R))$ denotes the algebra of all matrix-valued functions $F : A_R \to \mathbb{C}^{m \times m}$ that are (coordinatewise) analytic and admit a continuous extension to $\overline{A_R}$. For any such $F = \sum_{n \in \mathbb{Z}} A_n \otimes z^n$, we can write

$$
\begin{aligned}
(2 + c) S_{R,c}^{(m)}(F) &= \int_{\partial A_R} F(\sigma) \otimes \mu(\sigma, T) \, ds + c A_0 \otimes I \\
&= \int_{\partial A_R} F(\sigma) \otimes \mu(\sigma, T) \, ds + \frac{c}{2\pi i} \int_{\Gamma_1} \frac{\sigma'}{\sigma} F(\sigma) \otimes I \, ds
\end{aligned}
$$

$$= \int\limits_{\partial A_R} F(\sigma) \otimes \mu(\sigma, T)\, ds + \frac{c}{2\pi i} \int\limits_{\Gamma_1} \frac{\sigma'}{\sigma} F(\sigma) \otimes I\, ds$$

$$- \frac{A}{2\pi i} \int\limits_{\partial A_R} \frac{\sigma'}{\sigma} F(\sigma) \otimes I\, ds$$

$$= \int\limits_{\partial A_R} F(\sigma) \otimes \nu_A(\sigma, T)\, ds.$$

But now, since $\nu_A(\sigma, T) \geq 0$ for every σ in ∂A_R and also $\int_{A_R} \nu_A(\sigma, T)\, ds = (2 + c)I$, one can show (see e.g. [16, proof of Lemma 2.2]) that $S_{R,c}^{(m)}$ is contractive for every $m \geq 1$. This concludes the proof. ∎

We now prove a lemma which states that the $\mathcal{C}_A$ classes do not contain any invertible operators if A is not invertible.

LEMMA 4.3. *Let* $A \in \mathcal{B}(H)$ *be a positive operator that is not invertible. Assume also that* $T \in \mathcal{B}(H)$ *satisfies* $\sigma(T) \subset \mathbb{D}$ *and*

$$2\Re(1 - zT)^{-1} + A - 2 \geq 0, \quad \forall z \in \mathbb{D}.$$

Then T *is not invertible.*

Proof. Arguing as in the proof of Theorem 3.12, one can show that T satisfies

$$2\Re(1 - zT)^{-1} + A - 2 \geq 0, \quad \forall z \in \mathbb{D}$$

if and only if the mapping

$$S_A : \mathcal{A}(\mathbb{D}) \to \mathcal{B}(H), \quad f \mapsto \int\limits_{\partial \mathbb{D}} \mu(\sigma, T) f(\sigma)\, ds + f(0)(A - 2),$$

is contractive. But then we know (see e.g. [26, Chapters 2, 3]) that S_A is contractive if and only if it has a completely positive extension to all of $C(\partial \mathbb{D})$. Let $\tilde{S}_A$ denote such an extension. The non-unital version of Stinespring's Theorem [26, Theorem 4.1] then implies the existence of a Hilbert space $K \supset H$, a unital $*$-homomorphism $\pi : C(\partial \mathbb{D}) \to \mathcal{B}(K)$ and a bounded operator $V : H \to K$ such that

$$(4.1) \qquad \tilde{S}_A(f) = V^* \pi(f) V, \quad \forall f \in C(\partial \mathbb{D}).$$

Set $U = \pi(z)$. It is easy to see that U is then a unitary operator. Setting $f \equiv 1$ in (4.1) gives us $A = V^*V$, thus the polar decomposition of V is given by $YA^{1/2}$, where Y is some partial isometry. But observe also that putting $f = z$ in (4.1) gives

$$T = A^{1/2} Y^* U Y A^{1/2}.$$

Since $A^{1/2}$ is not invertible, we conclude that T cannot be invertible. ∎

Before we finish the proof of Theorem 1.2, a few dilation-theoretic observations are in order. Assume $T \in \mathcal{B}(H)$ with $\sigma(T) \subset A_R$. It is then

well-known that $T \in \mathbb{CDLA}_R(c)$ if and only if the mapping (3.16) is completely positive. Further, by Stinespring's Theorem, this is equivalent to the existence of a Hilbert space $K \supset H$ and a unital $*$-homomorphism $\pi : C(\partial A_R) \to \mathcal{B}(K)$ such that, for $f = \sum a_n z^n \in \mathcal{A}(A_R)$,

$$(2+c)S_{R,c}(f) = f(T) + (C\overline{f})(T)^* + ca_0 = (2+c)P_H\pi(f)|_H.$$

Set $\pi(z) = N$. Then N is a normal operator satisfying $\sigma(N) \subset \partial A_R$ and our previous equality becomes, for $f(z) = z^n$,

$$T^n + (C\overline{z^n})(T)^* = (2+c)P_H N^n|_H, \qquad \forall n \neq 0.$$

After some computations, one verifies that

$$(C\overline{z^n})(\zeta) = \frac{1}{2\pi i} \int\limits_{\partial A_R} \frac{\overline{z^n}}{z-\zeta}\, dz = R^{-2|n|}\zeta^{-n},$$

hence

$$(4.2) \qquad T^n + R^{-2|n|}T^{-n*} = (2+c)P_H N^n|_H, \qquad \forall n \neq 0.$$

On the other hand, Theorem 1.2 tells us that $T \in \mathbb{CDLA}_R(c)$ if and only if there exists $A \in \mathcal{B}(H)$ such that $A > 0$, $2 + c - A > 0$ and $T/R \in \mathcal{C}_{2+c-A}$ and $T^{-1}/R \in \mathcal{C}_A$. In view of Definition 1.1, this is equivalent to the existence of a Hilbert space $K' \supset H$ and unitaries $U_1, U_{-1} \in \mathcal{B}(K')$ such that

$$(4.3) \qquad R^{-n}T^n = (2+c-A)^{1/2}P_H U_1^n (2+c-A)^{1/2}|_H, \qquad \forall n \geq 1,$$

and

$$(4.4) \qquad R^{-n}T^{-n} = A^{1/2}P_H U_{-1}^n A^{1/2}|_H, \qquad \forall n \geq 1.$$

Thus, the content of Theorem 1.2 is that (4.2) holds if and only if there exists $A \in \mathcal{B}(H)$ with $0 < A < 2+c$ such that (4.3) and (4.4) hold. It would be of interest to find a direct proof of this assertion, using only the dilations U_1, U_{-1} and N.

Proof of Theorem 1.2. It remains to establish the converse of Lemma 4.2.

Accordingly, assume $T \in \mathbb{CDLA}_R(c)$, with $\sigma(T) \subset A_R$. By Arveson's Theorem, $S_{R,c}$ extends to a completely positive map $\Psi_{R,c} : C(\partial A_R) \to \mathcal{B}(H)$. Next, consider the mapping

$$\psi_{R,c} : C(\partial A_R) \to \mathcal{B}(H),$$

$$f \mapsto \frac{1}{2+c}\left[\int\limits_{\partial A_R} \mu(\sigma,T)f(\sigma)\, ds + \frac{c}{2\pi i}\int\limits_{\Gamma_1} \frac{f}{\sigma}\sigma'\, ds\right],$$

which is an alternate (not necessarily positive) extension of $S_{R,c}$. Since $\|\mu(\sigma,T)\|$ is uniformly bounded with respect to σ (because of the assumption $\sigma(T) \subset A_R$), we can estimate

$$\left\|\int\limits_{\partial A_R} \mu(\sigma,T)f(\sigma)\, ds\right\| \leq \|f\|_\infty \int\limits_{\partial A_R} \|\mu(\sigma,T)\|\, ds \leq M\|f\|_\infty, \qquad \forall f \in C(\partial A_R).$$

Thus, $\psi_{R,c}$ is bounded. Observe also that both $\psi_{R,c}$ and $\Psi_{R,c}$ are actually extensions of the (completely positive) map $\tilde{S}_{R,c} : \mathcal{M}_R \to \mathbb{C}$ defined in the proof of Theorem 3.12. For $\psi_{R,c}$ this is obvious, while for $\Psi_{R,c}$ it holds because the completely contractive map $S_{R,c}$ has a unique completely positive extension to the closure of $\mathcal{A}(A_R) + \mathcal{A}(A_R)^*$.

Now, define $S : C(\partial A_R) \to \mathcal{B}(H)$ as $S = \psi_{R,c} - \Psi_{R,c}$. Fix an orthonormal basis $\{e_j\}$ of H and put

$$S_{ij} : C(\partial A_R) \to \mathbb{C}, \quad f \mapsto \langle S(f)e_j, e_i \rangle.$$

The map S_{ij} is then a bounded linear functional that vanishes on $\mathcal{M}_R$, which is a codimension 1 subspace of $C(\partial A_R)$. Hence, each S_{ij} lies in the 1-dimensional annihilator $\mathrm{Ann}[\mathcal{M}_R] \subset \{L : C(\partial A_R) \to \mathbb{C} \text{ linear, bounded}\}$ of $\mathcal{M}_R$. But we also know that the (non-zero) map

$$\phi : C(\partial A_R) \to \mathbb{C}, \quad f \mapsto \frac{1}{(2+c)} \frac{1}{2\pi i} \int_{\partial A_R} \frac{f(\zeta)}{\zeta} \, d\zeta,$$

is in $\mathrm{Ann}[\mathcal{M}_R]$. Thus, for all i, j, there exists $k_{ij} \in \mathbb{C}$ such that $S_{ij} = k_{ij}\phi$. Define the (a priori unbounded) operator A acting on H by $\langle Ae_j, e_i \rangle = k_{ij}$, for all i, j, and let A_J denote its compression to $H_J := \mathrm{span}\{e_j : j \in J\}$, where J is any finite subset of $\mathbb{N}$. Hence,

$$(4.5) \qquad \frac{1}{2+c}\left[\frac{1}{2\pi i}\int_{\partial A_R} \frac{f(\zeta)}{\zeta}\, d\zeta\right] A_J = P_{H_J} S(f)|_{H_J}$$

for every $f \in C(\partial A_R)$ and every J. Set $f = f_0$ in this last equality, where $f_0 \equiv 1$ on Γ_1 and $f_0 \equiv 0$ on Γ_{-1}. This gives

$$A_J = (2+c)P_{H_J} S(f_0)|_{H_J}$$

for all J. Since $S(f_0) = \psi(f_0) - \Psi(f_0)$ is bounded and self-adjoint, it follows that A is a bounded, self-adjoint operator. Also, in view of (4.5), we deduce that

$$\Psi_{R,c}(f) = \psi_{R,c}(f) - S(f) = \int_{\partial A_R} \nu_{A,c}(\sigma, T)f(\sigma)\, ds$$

for every continuous f, where

$$\nu_{A,c}(\sigma, T) = \begin{cases} \mu(\sigma, T) + \dfrac{1}{2\pi i}\dfrac{\sigma'}{\sigma}(c - A) & \text{for } \sigma \in \Gamma_1, \\[2mm] \mu(\sigma, T) - \dfrac{1}{2\pi i}\dfrac{\sigma'}{\sigma}(A) & \text{for } \sigma \in \Gamma_{-1}. \end{cases}$$

Since $\Psi_{R,c}$ is (completely) positive on $C(\partial A_R)$, we know that $\nu_{A,c}(\sigma, T) \geq 0$ for every $\sigma \in \partial A_R$, hence (as in the proof of Lemma 4.2) $T/R \in \mathcal{C}_{2+c-A}$ and $T^{-1}/R \in \mathcal{C}_A$. The fact that $\mathcal{C}_{2+c-A}$ and $\mathcal{C}_A$ are non-empty immediately implies $0 \leq A \leq 2 + c$ (see the remark after Lemma 2.1). But we also know

that T is invertible, so Lemma 4.3 tells us that both A and $2 + c - A$ have to be invertible as well. This concludes the proof. ∎

We now drop the assumption $\sigma(T) \subset A_R$.

THEOREM 4.4. *Assume* $R > 1$, $c > -2$ *and* $T \in \mathcal{B}(H)$ *satisfies* $\sigma(T) \subset \overline{A_R}$. *Then there exists* $A \in \mathcal{B}(H)$ *such that* $A > 0$, $2 + c - A > 0$ *and* $T/R \in \mathcal{C}_{2+c-A}$ *and* $T^{-1}/R \in \mathcal{C}_A$ *if and only if* $T \in \mathbb{CDLA}_{R'}(c)$ *for every* $R' > R$.

Proof. First, assume T is in $\mathbb{CDLA}_{R'}(c)$ for every $R' > R$ and fix a decreasing sequence $\epsilon_k \to 0$. Put $R_k = R + \epsilon_k$. Since $\sigma(T) \subset A_{R_k}$, Theorem 1.2 tells us that there exists a sequence $\{A_k\} \subset \mathcal{B}(H)$ such that $0 < A_k < 2 + c$ and $T/R_k \in \mathcal{C}_{2+c-A_k}$ and $T^{-1}/R_k \in \mathcal{C}_{A_k}$, for all $k \geq 1$. Now, from $T/R_k \in \mathcal{C}_{2+c-A_k}$ we get

$$(4.6) \qquad \langle (2\Re(1 - zT/R_k)^{-1} + c - A_k)v, v \rangle \geq 0$$

for all $z \in \mathbb{D}, v \in H$ and $k \geq 1$. Since $\{A_k\}$ is uniformly bounded, we may replace it, without loss of generality, by a WOT-convergent subsequence. Note also that $\sigma(zT/R_k) \subset \mathbb{D}$ for all z, k. Letting $k \to \infty$ in (4.6) (while keeping z and v fixed) implies

$$\langle (2\Re(1 - zT/R)^{-1} + c - A)v, v \rangle \geq 0,$$

where A is the WOT limit of $\{A_k\}$ (notice that A has to be self-adjoint, being the WOT limit of self-adjoint operators). Since this last inequality holds for any $z \in \mathbb{D}$ and $v \in H$, we conclude that $T/R \in \mathcal{C}_{2+c-A}$, while an entirely analogous argument shows that $T^{-1}/R \in \mathcal{C}_A$. Finally, the fact that both $\mathcal{C}_{2+c-A}$ and $\mathcal{C}_A$ contain an invertible operator implies, as seen previously, that $0 < A < 2 + c$.

For the converse, observe that (in view of Lemma 2.1) having $T/R \in \mathcal{C}_{2+c-A}$ and $T^{-1}/R \in \mathcal{C}_A$ implies that $T/R' \in \mathcal{C}_{2+c-A}$ and $T^{-1}/R' \in \mathcal{C}_A$ for every $R' > R$. Since $\sigma(T) \subset A_{R'}$, Theorem 1.2 allows us to deduce that $T \in \mathbb{CDLA}_{R'}(c)$ for every $R' > R$. ∎

Now, we record the following analogue of Proposition 3.8 for $\mathbb{CDLA}_R(c)$. The proof is essentially an application of Proposition 3.8 combined with Lemma 4.2, so we omit the details.

PROPOSITION 4.5. *Let* $c \in \mathbb{R}$ *and* $R > 1$. *Then* $\mathbb{CDLA}_R(c)$ *is non-empty if and only if* $\mathbb{DLA}_R(c)$ *is non-empty. Further, given normal* $N \in \mathcal{B}(H)$, *we have* $N \in \mathbb{CDLA}_R(c)$ *if and only if* $N \in \mathbb{DLA}_R(c)$.

It is also worth noting that, like $\mathbb{DLA}_R(c)$, $\mathbb{CDLA}_R(c)$ is (eventually) strictly monotone with respect to c. This has essentially already been given to us by the proof of Theorem 3.10 together with Lemma 4.2.

THEOREM 4.6. *If* $-2 < c < c'$ *and* $\mathbb{CDLA}_R(c')$ *is non-empty, then*

$$\mathbb{CDLA}_R(c) \subsetneq \mathbb{CDLA}_R(c').$$

Proof. If $c < c'$, it is obvious by the definition of $\mathbb{CDLA}_R(c)$ that $\mathbb{CDLA}_R(c)$ $\subseteq \mathbb{CDLA}_R(c')$. To show that the inclusion is actually strict, we divide the proof into two cases, similar to Theorem 3.10.

First, assume that there exist $s, t > 0$ such that $s \geq 1$, $s + t = 2 + c'$ and $\mathcal{C}_{s,t}(R)$ is non-empty. As in the proof of Theorem 3.10, we may assume that $t < 1$ and $\frac{2}{R^2+1} < t$. In this setting, we were able to construct $T \in \mathcal{C}_{s,t}(R) \subset$ $\mathbb{CDLA}_R(c')$ such that $T \notin \mathbb{DLA}_R(c)$, hence also $T \notin \mathbb{CDLA}_R(c)$. We thus obtain strict inclusion.

On the other hand, assume that there do not exist $s, t > 0$ such that $s \geq 1, s + t = 2 + c'$ and $\mathcal{C}_{s,t}(R)$ is non-empty. In this setting, we found $\delta > 0$ and $a \in \mathbb{C}$ such that $a \in \mathcal{C}_{\delta, 2+c'-\delta}(R) \subset \mathbb{CDLA}_R(c')$, but $a \notin \mathbb{CDLA}_R(c)$, as desired. ∎

We end with a question. While the inclusion $\mathbb{CDLA}_R(c) \subseteq \mathbb{DLA}_R(c)$ is obvious, we have not been able to determine whether it is actually strict or not.

QUESTION 4.7. Let $R > 1, c > -2$ and assume $\mathbb{CDLA}_R(c)$ is non-empty. Is it true that

$$\mathbb{CDLA}_R(c) \subsetneq \mathbb{DLA}_R(c)?$$

A negative answer to the above question would imply that, given $\sigma(T) \subset A_R$, having

$$2\Re[(1 - zT/R)^{-1} + (1 - wT^{-1}/R)^{-1}] - 2 + c \geq 0, \quad \forall z, w \in \mathbb{D}$$

is equivalent to the existence of $0 < A < 2 + c$ such that

$$2\Re(1 - zT/R)^{-1} + c - A \geq 0, \quad \forall z \in \mathbb{D}$$

and

$$2\Re(1 - wT^{-1}/R)^{-1} + A - 2 \geq 0, \quad \forall w \in \mathbb{D}.$$

While this seems unlikely to hold, the computational difficulty in verifying membership conditions of the form $T/R \in \mathcal{C}_{2+c-A}(R)$ and $T^{-1}/R \in \mathcal{C}_A(R)$, for arbitrary $0 < A < 2 + c$ and T non-normal, does not make it easy to come up with a counterexample.

5. K-spectral estimates

5.1. General estimates. Recall that, given a compact set $X \subset \mathbb{C}$ and $T \in \mathcal{B}(H)$ such that $\sigma(T) \subset X$, X is said to be a K-*spectral set* for T if

$$\|f(T)\| \leq K \sup_{z \in X} |f(z)|$$

for every rational function f with poles outside of X. The set X will be called a *complete K-spectral set* if the above inequality holds for all matrices with rational coefficients.

In this short subsection, we show how the methods established in [7] and [15] can be used to derive K-spectral estimates for $\mathbb{DLA}_R(c)$ and $\mathbb{CDLA}_R(c)$. We shall need a few preliminary lemmata, the scalar-valued versions of which are all contained in [15].

LEMMA 5.1. *The map*

$$\alpha : \mathcal{A}(A_R) \to \mathcal{A}(A_R), \qquad f \mapsto C\overline{f},$$

is completely contractive for every $R > 1$.

Proof. By [15, Lemma 8], we know that $\|\alpha(f)\| \leq \|f\|$ whenever f is a scalar-valued rational function in A_R that is bounded by 1. A standard approximation argument shows that α must be contractive. The proof of [16, Lemma 2.1] then implies that α is completely contractive. ∎

LEMMA 5.2. *Let $R > 1$ and $T \in \mathcal{B}(H)$ be such that $\sigma(T) \subset A_R$. Assume that there exists a bounded linear functional $\gamma : \mathcal{A}(A_R) \to \mathbb{C}$ and a constant $p > 0$ such that the mapping*

$$f \mapsto \frac{1}{2p}(f(T) + \alpha(f)(T)^* + \gamma(f))$$

is completely contractive on $\mathcal{A}(A_R)$. Then $\overline{A_R}$ is a complete K-spectral set for T with constant $K = p + \sqrt{1 + p^2 + \|\gamma\|_{\mathrm{cb}}}$.

Proof. In the setting of [15, Theorem 2], replace Ω by A_R, c_1 by 1 (this is possible because of Lemma 5.1), c_2 by p and $\hat{\gamma}$ by $\|\gamma\|_{\mathrm{cb}}$. While the proof of Theorem 2 in [15] was given in the scalar-valued setting, it can be repeated, mutatis mutandis, in the matrix-valued setting to give the exact same K-spectral estimate (see also [10, Remark (i) after the proof of Theorem 1.1]), where

$$K = c_2 + \sqrt{c_2^2 + c_1 + \hat{\gamma}} = p + \sqrt{1 + p^2 + \|\gamma\|_{\mathrm{cb}}}. \quad ∎$$

We are now in a position to show:

THEOREM 5.3. *Let $c > -2$ and assume $T \in \mathbb{DLA}_R(c)$ (resp. $T \in \mathbb{CDLA}_R(c)$). Then $\overline{A_R}$ is a K-spectral (resp. complete K-spectral) set for T, where*

$$K = 1 + \frac{c}{2} + \sqrt{\left(1 + \frac{c}{2}\right)^2 + 1 + |c|}.$$

Proof. Let $T \in \mathbb{CDLA}_R(c)$ (the case $T \in \mathbb{DLA}_R(c)$ is essentially contained in [15, Theorem 2]). By assumption, the mapping

$$f \mapsto \frac{1}{2 + c}(f(T) + \alpha(f)(T)^* + ca_0)$$

is completely contractive on $\mathcal{A}(A_R)$. If $\gamma : \mathcal{A}(A_R) \to \mathbb{C}$ is given by $\gamma(\sum a_n z^n) = ca_0$, it can be easily verified that $\|\gamma\|_{\mathrm{cb}} = |c|$. Thus, one can apply Lemma 5.2 with $p = 1 + c/2$ and $\gamma(f) = ca_0$ to deduce the desired result. ∎

REMARK 5.4. Let $\mathbb{N}\mathbb{A}_R$ denote the *numerical annulus*, i.e. the class of all $T \in \mathcal{B}(H)$ such that $w(T) \le R$ and $w(T^{-1}) \le R$. K-spectral estimates for $\mathbb{N}\mathbb{A}_R$ have been studied in [14] and, more recently, in [15]. Since

$$\mathbb{N}\mathbb{A}_R \equiv \mathcal{C}_{2,2}(R) \subset \mathbb{D}\mathbb{L}\mathbb{A}_R(2),$$

Theorem 5.3 tells us that $\overline{A_R}$ is a $(2 + \sqrt{7})$-spectral set for T whenever $T \in \mathbb{N}\mathbb{A}_R$. This improves on the estimates from [15, Section 6].

REMARK 5.5. The main result of [29] can be used to obtain sharper spectral estimates in certain cases. Indeed, let $T \in \mathbb{D}\mathbb{L}\mathbb{A}_R(c)$ be a matrix with $\sigma(T) \subset A_R$. In the setting of [29, Theorem 5], choose $A = \mathcal{A}(A_R)$ and set $\gamma(f) = f(T)$ and $\Phi(f) = C\overline{f}$. Since $c \ge 0$, we have $\mathbb{Q}\mathbb{A}_R \subset \mathbb{D}\mathbb{L}\mathbb{A}_R(c)$, which implies that $\|\gamma\| \ge 2$ (see [34]). Now, assume, in addition, the existence of an *extremal pair* $(f_0, x_0) \in \mathcal{A}(A_R) \times H$ for γ (see [29, p. 2]). We have $\|f_0\| = \|x_0\| = 1$ and $\|\gamma\| = \|\gamma(f_0)x_0\|$. Also, there exists an *extremal measure* associated with (f_0, x_0) (see [29, Proposition 3 and the discussion afterwards]). This observation, combined with the contractivity of Φ (Lemma 5.1), allows us to deduce that $|\langle \gamma(\Phi(f_0)f_0)x_0, x_0 \rangle| \le 1$; see e.g. [29, proof of Theorem 11]). Thus, if we define ω in the dual of $\mathcal{A}(A_R)$ as $\omega(\sum_n a_n z^n) = -ca_0$, [29, Theorem 5] implies that

$$\|\gamma\| \le \frac{1}{2}\|\gamma_\Phi - \omega\| + \sqrt{\left(\frac{1}{2}\|\gamma_\Phi - \omega\|\right)^2 + |\langle \gamma(\Phi(f_0)f_0)x_0, x_0 \rangle|}$$

$$\le \frac{2+c}{2}\|S_{R,c}\| + \sqrt{\left(\frac{2+c}{2}\|S_{R,c}\|\right)^2 + 1}$$

$$\le 1 + \frac{c}{2} + \sqrt{\left(1 + \frac{c}{2}\right)^2 + 1},$$

which gives us the sharper constant $K' = 1 + c/2 + \sqrt{(1 + c/2)^2 + 1}$. Note that if T has distinct eigenvalues, then the existence of an extremal function f_0 (that extends continuously to the boundary) can be obtained as in [12, proof of Theorem 2.1] (see [21, Section 3] and [19, Exercise 5, p. 162] for the structure of solutions to extremal Pick problems over the annulus). While the authors deem it very likely that the extremal function continues to enjoy boundary continuity even in mixed Carathéodory–Pick interpolation problems (corresponding to the general case where one might have repeated eigenvalues), they are not aware of any reference that describes the properties of extremal functions for such problems.

5.2. 2×2 matrices. We now show Theorem 1.3 from the Introduction, which offers improved K-spectral estimates for 2×2 matrices with a single eigenvalue. The key ingredient of the proof will be a function-theoretic result

from [23]. To more easily connect with the setting of that paper, we will work with the annulus,

$$\mathscr{A}_q := \{q < |z| < 1\},$$

which is conformally equivalent to $A_{1/\sqrt{q}}$. The definition of $\mathbb{DLA}_R(c)$ can then be updated as follows:

DEFINITION 5.6. Let $c \in \mathbb{R}$ and $0 < q < 1$. Then $\mathscr{DLA}_q(c)$ denotes the class of all operators $T \in \mathcal{B}(H)$ such that

(i) $\sigma(T) \subset \overline{\mathscr{A}_q}$,
(ii) $2\Re[(1 - zT)^{-1} + (1 - wqT^{-1})^{-1}] - 2 + c \geq 0$ for all $z, w \in \mathbb{D}$.

Now, for $w \in \mathbb{D}$ and $a \in \mathscr{A}_q$, define

$$\psi_w(z) = \frac{z - w}{1 - \overline{w}z}$$

and

$$\mathcal{F}_{a,w} = \{f : \overline{\mathscr{A}_q} \to \mathbb{D} \mid f \text{ analytic and } f(a) = w\}.$$

LEMMA 5.7. *Let $w \in \mathbb{D}$ and $a \in \mathscr{A}_q$. Then*

$$\sup \{|f'(a)| \mid f \in \mathcal{F}_{a,w}\} \leq (1 - |w|^2)\left(\frac{1}{1 - |a|^2} + \frac{q}{|a|^2 - q^2}\right).$$

Proof. Put

$$k^q(a, a) = \sum_{n \in \mathbb{Z}} \frac{|a|^{2n}}{1 + q^{2n+1}}.$$

The solution to the above extremal problem for $w = 0$ can be found in [23, p. 1119]. In particular, it is known that

$$\sup \{|f'(a)| \mid f \in \mathcal{F}_{a,0}\} = k^q(a, a).$$

Now, if $h \in \mathcal{F}_{a,w}$, it can be easily verified that $\psi_w \circ h \in \mathcal{F}_{a,0}$, hence

$$\frac{|h'(a)|}{1 - |w|^2} = |(\psi_w \circ h)'(a)| \leq k^q(a, a).$$

Since $h \in \mathcal{F}_{a,w}$ was arbitrary, we can deduce that

$$\sup \{|f'(a)| \mid f \in \mathcal{F}_{a,w}\} \leq (1 - |w|^2)k^q(a, a)$$

$$= (1 - |w|^2)\left(\sum_{n=0}^{\infty} \frac{|a|^{2n}}{1 + q^{2n+1}} + \sum_{n=-1}^{-\infty} \frac{|a|^{2n}}{1 + q^{2n+1}}\right)$$

$$\leq (1 - |w|^2)\left(\sum_{n=0}^{\infty} |a|^{2n} + \frac{1}{q}\sum_{n=1}^{\infty} \frac{q^{2n}}{|a|^{2n}}\right)$$

$$= (1 - |w|^2)\left(\frac{1}{1 - |a|^2} + \frac{q}{|a|^2 - q^2}\right). \quad\blacksquare$$

We also require the following computational lemmata.

LEMMA 5.8. *Let* $a, u \in \mathbb{C}$ *and assume that* $T = \left(\begin{smallmatrix} a & u \\ 0 & a \end{smallmatrix} \right) \in \mathscr{DLA}_q(c)$ *with* $q < |a| < 1$. *Then*

$$\left| \frac{e^{i\theta}}{(1 - ae^{i\theta})^2} - \frac{qe^{i\psi}}{a^2(1 - qe^{i\psi}a^{-1})^2} \right| |u| \leq 2\Re\left(\frac{1}{1 - ae^{i\theta}} + \frac{1}{1 - qe^{i\psi}a^{-1}} \right) + c - 2$$

for all θ *and* ψ.

Proof. The proof of Lemma 3.9 carries over to the $\mathscr{DLA}_q(c)$ setting. Thus, the fact that $T \in \mathscr{DLA}_q(c)$ and $\sigma(T) = \{a\} \subset \mathscr{A}_q$ allows us to deduce

$$2\Re[(1 - e^{i\theta}T)^{-1} + (1 - e^{i\psi}qT^{-1})^{-1}] - 2 + c \geq 0, \quad \forall \theta, \psi \in [0, 2\pi).$$

Taking determinants then leads to the desired inequality. ∎

LEMMA 5.9. *Let* $C > 0$ *and* $w \in \mathbb{D}$. *Then*

$$\left\| \begin{pmatrix} w & C(1 - |w|^2) \\ 0 & w \end{pmatrix} \right\| \leq \max\{1, C\}.$$

Proof. Given any matrix of the form $P = \left(\begin{smallmatrix} e & f \\ 0 & e \end{smallmatrix} \right)$, it is well-known that $\|P\| \leq 1$ if and only if $|f| \leq 1 - |e|^2$. Thus, if $C \leq 1$, we immediately obtain

$$\left\| \begin{pmatrix} w & C(1 - |w|^2) \\ 0 & w \end{pmatrix} \right\| \leq 1.$$

Now, assume $C > 1$. Note that

$$C^2 - \begin{pmatrix} w & C(1 - |w|^2) \\ 0 & w \end{pmatrix} \begin{pmatrix} w & C(1 - |w|^2) \\ 0 & w \end{pmatrix}^*$$

$$= \begin{pmatrix} -|w|^2 + C^2(2 - |w|^2)|w|^2 & -C\overline{w}(1 - |w|^2) \\ -Cw(1 - |w|^2) & C^2 - |w|^2 \end{pmatrix} \geq 0,$$

since the $(1,1)$-entry of this last matrix is clearly positive, while its determinant is equal to

$$|w|^2(C^2 - |w|^2)(2C^2 - C^2|w|^2 - 1) - C^2|w|^2(1 - |w|^2)^2$$
$$= |w|^2(C^2 - 1)(2C^2 - |w|^2(1 + C^2)) > 0.$$

This concludes the proof. ∎

We are now prepared for the main result of this subsection.

Proof of Theorem 1.3. First, we will prove the theorem in the setting of $\mathscr{A}_q$.

Let $T \in \mathscr{DLA}_q(c)$ be a 2×2 matrix with a single eigenvalue. Since unitary equivalence respects K-spectral estimates (see e.g. [1, Example 4, p. 107-5]), we may assume that T is of the form $\left(\begin{smallmatrix} a & u \\ 0 & a \end{smallmatrix} \right)$. We may also take

$a > 0$ (as $\mathscr{A}_q$ is invariant under rotations). Finally, it suffices to work with $q < a < 1$ (the general case follows by a standard approximation argument).

Now, let $f : \overline{\mathscr{A}_q} \to \mathbb{D}$ be analytic. We may write

$$\|f(T)\| = \left\|\begin{pmatrix} f(a) & f'(a)u \\ 0 & f(a) \end{pmatrix}\right\| = \left\|\begin{pmatrix} f(a) & |f'(a)u| \\ 0 & f(a) \end{pmatrix}\right\|$$

$$= \left\|\begin{pmatrix} f(a) & \frac{|f'(a)u|}{1-|f(a)|^2}(1-|f(a)|^2) \\ 0 & f(a) \end{pmatrix}\right\|.$$

If $\frac{|f'(a)u|}{1-|f(a)|^2} \leq 1$, Lemma 5.9 gives us $\|f(T)\| \leq 1$, which is stronger than the desired estimate. On the other hand, if $\frac{|f'(a)u|}{1-|f(a)|^2} > 1$, Lemmata 5.7 and 5.9 imply that

$$\|f(T)\| = \left\|\begin{pmatrix} f(a) & \frac{|f'(a)u|}{1-|f(a)|^2}(1-|f(a)|^2) \\ 0 & f(a) \end{pmatrix}\right\| \leq \frac{|f'(a)|}{1-|f(a)|^2}|u|$$

$$\leq \left(\frac{1}{1-a^2} + \frac{q}{a^2-q^2}\right)|u|.$$

Assume that $a \geq \sqrt{q}$. In Lemma 5.8, take $\theta = 0, \psi = \pi$. The resulting bound on $|u|$ then allows us to write

$$\|f(T)\| \leq \left(\frac{1}{1-a^2} + \frac{q}{a^2-q^2}\right)\left(\frac{1}{(1-a)^2} + \frac{q}{a^2(1+q/a)^2}\right)^{-1}$$

$$\times \left(\frac{2}{1-a} + \frac{2}{1+q/a} + c - 2\right)$$

$$\leq 2 + c\frac{1-q}{1+q},$$

where the last inequality can be seen (after some computations) to be equivalent to $a \geq \sqrt{q}$. This concludes the proof in this case. If $a < \sqrt{q}$, one can choose $\theta = \pi$ and $\psi = 0$ in the above calculation and argue in an analogous manner. Thus, we have shown that $\overline{\mathscr{A}_q}$ is a K_q-spectral set for $T \in \mathscr{DLA}_q(c)$ whenever T is a 2×2 matrix with a single eigenvalue, where $K_q = 2 + c\frac{1-q}{1+q}$.

We now convert this estimate to the A_R-setting. Assume $T \in \mathbb{DLA}_R(c)$ is a 2×2 matrix with a single eigenvalue and set $q = R^{-2}$. Then $\tilde{T} := T/R$ is in $\mathscr{DLA}_q(c)$ and is also, evidently, still a 2×2 matrix with a single eigenvalue. In view of our previous result, $\overline{\mathscr{A}_q}$ will be a K-spectral set for $\tilde{T}$, where

$$K = 2 + c\frac{1-q}{1+q} = 2 + c\frac{R^2-1}{R^2+1}.$$

This is easily seen to imply (see e.g. [1, Fact 2, p. 107-3]) that $\overline{A_R}$ is a K-spectral set for T, which concludes our proof. ∎

References

[1] C. Badea and B. Beckermann, *Spectral sets*, in: Handbook of Linear Algebra, L. Hogben (ed.), 2nd ed., CRC Press, 2013.

[2] G. Bello and D. Yakubovich, *An operator model in the annulus*, J. Operator Theory 90 (2023), 25–40.

[3] C. A. Berger, *Normal dilations*, Ph.D. Thesis, Cornell Univ., 1963.

[4] C. A. Berger and J. G. Stampfli, *Mapping theorems for the numerical range*, Amer. J. Math. 89 (1967), 1047–1055.

[5] C. A. Berger and J. G. Stampfli, *Norm relations and skew dilations*, Acta Sci. Math. (Szeged) 28 (1967), 191–195.

[6] K. Bickel, P. Gorkin, A. Greenbaum, T. Ransford, F. L. Schwenninger, and E. Wegert, *Crouzeix's conjecture and related problems*, Comput. Methods Funct. Theory 20 (2020), 701–728.

[7] T. Caldwell, A. Greenbaum, and K. Li, *Some extensions of the Crouzeix–Palencia result*, SIAM J. Matrix Anal. Appl. 39 (2018), 769–780.

[8] G. Cassier, M. Benharrat, and S. Belmouhoub, *Harnack parts of ρ-contractions*, J. Operator Theory 80 (2018), 453–480.

[9] G. Cassier and T. Fack, *Contractions in von Neumann algebras*, J. Funct. Anal. 135 (1996), 297–338.

[10] R. Clouâtre, M. Ostermann, and T. Ransford, *An abstract approach to the Crouzeix conjecture*, J. Operator Theory 90 (2023), 209–221.

[11] J. B. Conway, *A Course in Functional Analysis*, 2nd ed., Grad. Texts in Math. 96, Springer, New York, 1990.

[12] M. Crouzeix, *Bounds for analytical functions of matrices*, Integral Equations Operator Theory 48 (2004), 461–477.

[13] M. Crouzeix, *Numerical range and functional calculus in Hilbert space*, J. Funct. Anal. 244 (2007), 668–690.

[14] M. Crouzeix, *The annulus as a K-spectral set*, Commun. Pure Appl. Anal. 11 (2012), 2291–2303.

[15] M. Crouzeix and A. Greenbaum, *Spectral sets: numerical range and beyond*, SIAM J. Matrix Anal. Appl. 40 (2019), 1087–1101.

[16] M. Crouzeix and D. Kressner, *A bivariate extension of the Crouzeix–Palencia result with an application to Fréchet derivatives of matrix functions*, arXiv: 2007.09784, 2020.

[17] M. Crouzeix and C. Palencia, *The numerical range is a $(1+\sqrt{2})$-spectral set*, SIAM J. Matrix Anal. Appl. 38 (2017), 649–655.

[18] B. Delyon and F. Delyon, *Generalization of von Neumann's spectral sets and integral representation of operators*, Bull. Soc. Math. France 127 (1999), 25–41.

[19] S. D. Fisher, *Function Theory on Planar Domains*, Pure Appl. Math. (New York), Wiley, New York, 1983.

[20] P. R. Halmos, *A Hilbert Space Problem Book*, 2nd ed., Encyclopedia Math. Appl. 17, Springer, New York, 1982.

[21] S. McCullough, *Isometric representations of some quotients of H^∞ of an annulus*, Integral Equations Operator Theory 39 (2001), 335–362.

[22] S. McCullough and J. E. Pascoe, *Geometric dilations and operator annuli*, J. Funct. Anal. 285 (2023), art. 110035, 20 pp.

[23] S. McCullough and L.-C. Shen, *On the Szegő kernel of an annulus*, Proc. Amer. Math. Soc. 121 (1994), 1111–1121.

[24] K. Okubo and T. Ando, *Constants related to operators of class C_ρ*, Manuscripta Math. 16 (1975), 385–394.

[25] K. Okubo and I. Spitkovsky, *On the characterization of 2×2 ρ-contraction matrices*, Linear Algebra Appl. 325 (2001), 177–189.

[26] V. Paulsen, *Completely Bounded Maps and Operator Algebras*, Cambridge Stud. Adv. Math. 78, Cambridge Univ. Press, Cambridge, 2002.

[27] M. Putinar and S. Sandberg, *A skew normal dilation on the numerical range of an operator*, Math. Ann. 331 (2005), 345–357.

[28] T. Ransford and F. L. Schwenninger, *Remarks on the Crouzeix–Palencia proof that the numerical range is a $(1+\sqrt{2})$-spectral set*, SIAM J. Matrix Anal. Appl. 39 (2018), 342–345.

[29] F. L. Schwenninger and J. de Vries, *On abstract spectral constants*, in: Operator and Matrix Theory, Function Spaces, and Applications, Oper. Theory Adv. Appl. 295, Birkhäuser, 2024, 335–350.

[30] C.-Y. Suen, *W_A contractions*, Positivity 2 (1998), 301–310.

[31] B. Sz.-Nagy, *Sur les contractions de l'espace de Hilbert*, Acta Sci. Math. (Szeged) 15 (1953), 87–92.

[32] B. Sz.-Nagy, and C. Foiaş, *On certain classes of power-bounded operators in Hilbert space*, Acta Sci. Math. (Szeged) 27 (1966), 17–25.

[33] B. Sz.-Nagy, and C. Foias, H. Bercovici, and L. Kérchy, *Harmonic Analysis of Operators on Hilbert Space*, 2nd ed., Universitext, Springer, New York, 2010.

[34] G. Tsikalas, *A note on a spectral constant associated with an annulus*, Oper. Matrices 16 (2022), 95–99.

[35] G. Tsikalas, *A von Neumann type inequality for an annulus*, J. Math. Anal. Appl. 506 (2022), art. 125714, 12 pp.

[36] J. P. Williams, *Schwarz norms for operators*, Pacific J. Math. 24 (1968), 181–188.

Michael T. Jury
Department of Mathematics
University of Florida
Gainesville, FL 32611, USA
E-mail: mjury@ufl.edu

Georgios Tsikalas
Department of Mathematics and Statistics
Washington University in St. Louis
St. Louis, MO 63130, USA
E-mail: gtsikalas@wustl.edu