Model Theory

$M \models T$

A New Kim's Lemma

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Kim's Lemma is a key ingredient in the theory of forking independence in simple theories. It asserts that if a formula divides, then it divides along every Morley sequence in type of the parameters. Variants of Kim's Lemma have formed the core of the theories of independence in two orthogonal generalizations of simplicity — namely, the classes of NTP_2 and $NSOP_1$ theories. We introduce a new variant of Kim's Lemma that simultaneously generalizes the NTP_2 and $NSOP_1$ variants. We explore examples and nonexamples in which this lemma holds, discuss implications with syntactic properties of theories, and ask several questions.

1. Introduction

The simple theories are a class of first-order theories which admit a structure theory built upon a good notion of independence. Nonforking independence was introduced by Shelah [1990] in the context of classification theory for stable theories, but was later shown to be meaningful in a broad class of unstable theories. Shelah's characterization [1980] of simple theories in terms of their saturation spectra, together with Hrushovski's work [2002] on bounded PAC structures and structures of finite S_1-rank, and the work of Cherlin and Hrushovski [2003] on quasi-finite theories, all made use of a circle of ideas concerning independence and amalgamation. These ideas were subsequently distilled and consolidated into the core results of simplicity theory by Kim [1998] and Kim and Pillay [1997], organized around

the good behavior of nonforking independence in this setting. A key ingredient in this theory is a result known as *Kim's Lemma*, which establishes that, in a simple theory, a formula $\varphi(x; b)$ divides over a set A if and only if $\varphi(x; b)$ divides along every Morley sequence over A in tp(b/A). Kim's lemma says that dividing is always witnessed by "generic" indiscernible sequences and changes the existential quantifier in the definition of dividing ("*there is* an A-indiscernible sequence such that...") into a universal one ("*for every* Morley sequence over A..."). Kim [2001] later showed that Kim's lemma characterizes the simple theories.

More recent developments have highlighted the centrality of Kim's lemma to the theory of nonforking independence and its generalizations. In particular, the theories of independence in NTP$_2$ and NSOP$_1$ theories are based on two orthogonal generalizations of Kim's lemma.[1] For NTP$_2$ theories, the equivalence between dividing and dividing along all generic sequences is preserved, but this equivalence requires a stronger notion of genericity. More specifically, Chernikov and Kaplan [2012] showed that, in an NTP$_2$ theory, a formula $\varphi(x; b)$ divides over a model M if and only if $\varphi(x; b)$ divides along Morley sequences for every strictly M-invariant type extending tp(b/M). This variant of Kim's lemma was shown to characterize NTP$_2$ theories in [Chernikov 2014].

On the other hand, in NSOP$_1$ theories, the equivalence between dividing and dividing along generic sequences no longer holds in general. Nonetheless, at the generic scale, there is an analogue of Kim's lemma: a formula $\varphi(x; b)$ divides along *some* generic sequence in tp(b/M) over a model M if and only if it divides along *every* such sequence. More precisely, Kaplan and the second-named author introduced *Kim-dividing*, which is defined so that a formula $\varphi(x; b)$ Kim-divides over a model M if $\varphi(x; b)$ divides along some Morley sequence for a global M-invariant type extending tp(b/M). It was shown in [Kaplan and Ramsey 2020] that, in an NSOP$_1$ theory, $\varphi(x; b)$ Kim-divides over M if and only if it divides along Morley sequences for every global M-invariant type extending tp(b/M) and that, moreover, this variant of Kim's lemma characterizes NSOP$_1$ theories.

We introduce a "New Kim's Lemma" that simultaneously generalizes the Kim's Lemmas for NTP$_2$ and NSOP$_1$ theories. The starting point is an observation about the broom lemma of Chernikov and Kaplan [2012]. This lemma is the key step in showing that, in NTP$_2$ theories, types over models always have global strict invariant extensions, which generate the generic sequences needed to get a Kim's lemma for NTP$_2$ theories. However, an inspection of the proof shows that this fact really bundles together two separate statements. The first is that in NTP$_2$

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[1]As a consequence of Mutchnik's work [2022b], we now know that the properties NSOP$_1$, NSOP$_2$, and NTP$_1$ are equivalent at the level of theories. In this paper, we primarily refer to NSOP$_1$ theories (rather than to NSOP$_2$ or NTP$_1$ theories), since the notion of Kim-independence was originally developed in [Kaplan and Ramsey 2020] under this hypothesis.

theories, Kim-dividing and forking independence coincide over models. The second is that, in any theory whatsoever, types over models extend to global *Kim-strict* invariant types, where Kim-strictness relaxes the nonforking independence condition required for strictness to one that only requires non-Kim-forking; see Theorem 2.26 below.

The statement of New Kim's Lemma, then, suggests itself (see Definition 3.7 below): a formula $\varphi(x; b)$ Kim-divides over a model M if and only if it divides along Morley sequences for every Kim-strictly M-invariant type extending $\operatorname{tp}(b/M)$. This variant of Kim's Lemma coincides with the Chernikov–Kaplan Kim's Lemma in NTP_2 theories (since there, Kim-forking agrees with forking over models, and hence Kim-strict invariant types are strict invariant), and coincides with the Kaplan–Ramsey Kim's Lemma in NSOP_1 theories (since there, Kim-independence is symmetric, so invariant types are automatically Kim-strict).

In Section 3, we survey the Kim's lemmas of the past and introduce our New Kim's Lemma. We also observe that New Kim's Lemma implies that Kim-forking equals Kim-dividing at the level of formulas. In Section 4, we show that our variant of Kim's Lemma holds in some examples of interest, including parametrized dense linear orders and the two-sorted theory of an infinite dimensional vector space over a real closed field with a bilinear form which is alternating and nondegenerate or symmetric and positive-definite. Our choice of examples is motivated by the idea that structures obtained by "generically putting together" NTP_2 and NSOP_1 behavior should satisfy New Kim's Lemma. We show, however, that New Kim's Lemma does *not* hold in the generic triangle-free graph, suggesting that it could serve as a meaningful dividing line among theories.

In Section 5, we try to relate New Kim's Lemma to syntactic properties of formulas. Our approach here reverses the usual order of explanation in neostability theory, which typically begins with a syntactic property (e.g., the tree property, SOP_1, TP_2) and then tries to establish a structure theory for theories without this property. In contrast, we are starting with a structural feature and trying to find a way of characterizing it syntactically. We introduce a new combinatorial configuration, which we provisionally call the bizarre tree property (BTP). The class of NBTP theories (those without BTP) contains both NTP_2 and NSOP_1, and all NBTP theories satisfy New Kim's Lemma. However, we do not obtain an exact characterization.

The antichain tree property (ATP), which was introduced in [Ahn and Kim 2024] and developed in [Ahn et al. 2023], is another combinatorial configuration generalizing TP_2 and SOP_1. We observe that NBTP implies NATP. But it is not clear whether there is an implication in either direction between NATP and New Kim's Lemma, or whether NBTP and NATP are equivalent. Figure 1 summarizes the state of affairs.
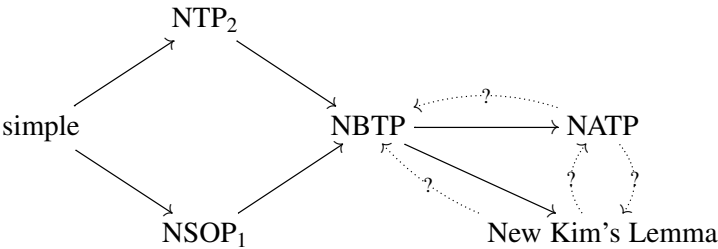
Figure 1. The current state of known implications.

While this paper was in preparation, two closely related preprints appeared:

- Kim and Lee [2023] establish a different variant of Kim's Lemma for NATP theories. Similarly to our work here, they do not prove that this Kim's Lemma characterizes NATP. In the context of NATP, they also study dividing along coheir sequences which are Kim-strict in the sense of this paper.

- Hanson [2023] studies a number of variants of Kim's Lemma which are related to ours. In particular, he succeeds in characterizing the class of NCTP theories by means of a variant of Kim's Lemma. Here CTP is the *comb tree property* (which was introduced by Mutchnik [2022b] under the name ω-DCTP$_2$). The class of NCTP theories contains the NBTP theories and its contained in the NATP theories.

At the moment, the NATP theories are the class beyond NSOP$_1$ and NTP$_2$ with the most developed syntactic theory; it would be very satisfying if these three approaches coincide. We conclude in Section 6 with several questions on where the theory might go from here.

2. Preliminaries

Throughout, T is a complete L-theory and $\mathbb{M} \models T$ is a monster model. As usual, all tuples come from $\mathbb{M}$, all sets are small subsets of $\mathbb{M}$, and all models are small elementary submodels of $\mathbb{M}$.

When α is an ordinal, we view the set $\alpha^{<\omega}$ of all finite sequences from α as a tree, with the tree partial order denoted by $\trianglelefteq$. The root of the tree is the empty sequence $\langle \, \rangle$. For $\rho \in \alpha^\omega$ and $i < \omega$, $\rho \mid i \in \alpha^{<\omega}$ is the restriction of ρ to i. We write $\eta^\frown \nu$ for concatenation of sequences. We write $\eta \perp \nu$ when η and ν are incomparable in the tree order. An *antichain* is a set of pairwise incomparable elements.

2A. *Tree properties.* We will begin by recalling the definitions of a number of tree properties and the known implications between them. The following three tree properties were introduced by Shelah [1990] under different names as part

of his analysis of forking in stable theories.[2] He introduced the "tree property" terminology in [Shelah 1980] and Kim [2001] subsequently dubbed the latter two as TP_1 and TP_2.

Definition 2.1. Let $\varphi(x; y)$ be a formula:

(1) We say $\varphi(x; y)$ has the *tree property* (TP) if there is $k < \omega$ and a tree of tuples $(a_\eta)_{\eta \in \omega^{<\omega}}$ satisfying the following conditions:
   (a) For all $\rho \in \omega^\omega$, $\{\varphi(x; a_{\rho|i}) : i < \omega\}$ is consistent.
   (b) For all $\eta \in \omega^{<\omega}$, $\{\varphi(x; a_{\eta^\frown\langle j\rangle}) : j < \omega\}$ is k-inconsistent.

(2) We say $\varphi(x; y)$ has the *tree property of the first kind* (TP_1) if there is a tree of tuples $(a_\eta)_{\eta \in \omega^{<\omega}}$ satisfying the following conditions:
   (a) For all $\rho \in \omega^\omega$, $\{\varphi(x; a_{\rho|i}) : i < \omega\}$ is consistent.
   (b) For all $\eta, \nu \in \omega^{<\omega}$, if $\eta \perp \nu$, then $\{\varphi(x; a_\eta), \varphi(x; a_\nu)\}$ is inconsistent.

(3) We say $\varphi(x; y)$ has the *tree property of the second kind* (TP_2) if there is $k < \omega$ and an array $(a_{i,j})_{i,j<\omega}$ satisfying the following conditions:
   (a) For all $f : \omega \to \omega$, $\{\varphi(x; a_{i,f(i)}) : i < \omega\}$ is consistent.
   (b) For all $i < \omega$, $\{\varphi(x; a_{i,j}) : j < \omega\}$ is k-inconsistent.

(4) We say T is NTP (NTP$_1$, NTP$_2$) if no formula has TP (TP$_1$, TP$_2$, respectively) modulo T. An NTP theory is also called a *simple theory*.

The next property was introduced by Džamonja and Shelah [2004].

Definition 2.2. [Džamonja and Shelah 2004, Definition 2.2] We say $\varphi(x; y)$ has the 1-*strong order property* (SOP$_1$) if there is a tree of tuples $(a_\eta)_{\eta \in 2^{<\omega}}$ satisfying the following conditions:

- For all $\rho \in 2^\omega$, the set of formulas $\{\varphi(x; a_{\rho|i}) : i < \omega\}$ is consistent.
- For all $\nu, \eta \in 2^{<\omega}$, if $\nu^\frown\langle 0\rangle \trianglelefteq \eta$ then $\{\varphi(x; a_\eta), \varphi(x; a_{\nu^\frown\langle 1\rangle})\}$ is inconsistent.

T is NSOP$_1$ if no formula has SOP$_1$ modulo T.

Our last property was introduced much more recently by Ahn and Kim [2024].

Definition 2.3. [Ahn and Kim 2024, Definition 4.1] We say $\varphi(x; y)$ has the *antichain tree property* (ATP) if there is a tree of tuples $(a_\eta)_{\eta \in 2^{<\omega}}$ satisfying the following conditions:

(1) If $X \subseteq 2^{<\omega}$ is an antichain, then $\{\varphi(x; a_\eta) : \eta \in X\}$ is consistent.

(2) If $\eta \trianglelefteq \nu \in 2^{<\omega}$, then $\{\varphi(x; a_\eta), \varphi(x; a_\nu)\}$ is inconsistent.

T is NATP if no formula has ATP modulo T.

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[2]TP, TP$_1$, and TP$_2$ were first introduced under the rather cumbersome labels $\kappa_{\mathrm{cdt}}(T) = \infty$, $\kappa_{\mathrm{sct}}(T) = \infty$, and $\kappa_{\mathrm{inp}}(T) = \infty$, respectively.
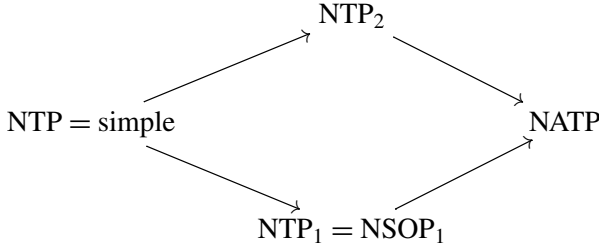
Figure 2. A summary of known implications.

Fact 2.4. Here is a summary of the known implications, which are depicted in Figure 2:

(1) The simple theories are the intersection of the NTP_1 and NTP_2 theories, i.e., T is simple if no formula has TP_1 or TP_2 modulo T [Shelah 1990, Theorem III.7.11].

(2) A theory T is $NSOP_1$ if and only if it is NTP_1 [Mutchnik 2022b, Theorem 1.6].[3]

(3) The NATP theories (properly) contain both the NTP_1 and NTP_2 theories [Ahn and Kim 2024, Propositions 4.4 and 4.6].

2B. *Forking and dividing.* In this section, we introduce a number of refinements of Shelah's notions of forking and dividing, based on the idea that, when a formula divides, it can be useful to study which indiscernible sequences it divides along.

Definition 2.5. Suppose $\varphi(x; b)$ is a formula, C is a set, and $I = (b_i)_{i < \omega}$ is a C-indiscernible sequence in $\operatorname{tp}(b/C)$ (meaning that b_i realizes $\operatorname{tp}(b/C)$ for all $i < \omega$). We say that φ *divides along* I (over C) if $\{\varphi(x; b_i) : i < \omega\}$ is inconsistent.

Definition 2.6. Suppose $\varphi(x; b)$ is a formula and C is a set:

(1) We say $\varphi(x; b)$ *divides* over C if it divides along some C-indiscernible sequence in $\operatorname{tp}(b/C)$.

(2) We say $\varphi(x; b)$ *forks* over C if there are formulas $(\psi_i(x; c_i))_{i < n}$ with $n < \omega$ such that $\varphi(x; b) \models \bigvee_{i < n} \psi_i(x; c_i)$ and each $\psi_i(x; c_i)$ divides over C.

(3) The notation $a \mathop{\smile}\limits_{C}^{d} b$ means that $\operatorname{tp}(a/Cb)$ contains no formula that divides over C and, similarly, $a \mathop{\smile}\limits_{A}^{f} b$ means that $\operatorname{tp}(a/Cb)$ contains no formula that forks over A.

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[3]The theorem as stated in [Mutchnik 2022b] says that every $NSOP_2$ theory is $NSOP_1$. Prior to the appearance of this result, it was well-known that $NSOP_1$ implies $NSOP_2$ and $NSOP_2$ is equivalent to NTP_1, see, e.g., [Kim and Kim 2011].

We will be primarily concerned with extremely "generic" sequences, i.e., Morley sequences for global invariant types.

Definition 2.7. A *global partial type* $\pi(x)$ is a consistent set of formulas over $\mathbb{M}$. A *global type* is a global partial type which is complete. For a set C, we say the global partial type $\pi(x)$ is *C-invariant* if, for all formulas $\varphi(x; y)$, if $b \equiv_C b'$, then $\varphi(x; b) \in \pi$ if and only if $\varphi(x; b') \in \pi$.

An important class of examples of global C-invariant types are the types that are finitely satisfiable in C. In any theory T, if $M \models T$, every type over M has a global extension which is finitely satisfiable in M (and therefore M-invariant). See Remark 2.9 below.

Over a general set C, there may be no global C-invariant types whatsoever. For this reason, when we want to work with invariant types (such as in the definition of Kim-dividing below), we usually work over a model.

Definition 2.8. Suppose $M \models T$:

(1) We write $a \underset{M}{\overset{i}{\downarrow}} b$ if tp(a/Mb) extends to a global M-invariant type.
(2) We write $a \underset{M}{\overset{u}{\downarrow}} b$ if tp(a/Mb) extends to a global type finitely satisfiable in M.

Remark 2.9. The u superscript comes from "ultrafilter", since global M-finitely satisfiable types all arise from the following construction: if $p(x) \in S_x(M)$, then $\{\varphi(M) : \varphi(x) \in p\} \subseteq \mathcal{P}(M^x)$ generates a filter on M^x. If $\mathcal{D}$ is an ultrafilter on M^x extending this filter, then

$$\mathrm{Av}(\mathcal{D}, \mathbb{M}) = \{\varphi(x) \in L(\mathbb{M}) : \varphi(M) \in \mathcal{D}\}$$

is a global type extending p which is finitely satisfiable in M. We write $\mathrm{Av}(\mathcal{D}, B)$ for $\mathrm{Av}(\mathcal{D}, \mathbb{M})$ restricted to parameters coming from B.

Definition 2.10. If q is a global C-invariant type, then a *Morley sequence* over C for q is a sequence $(a_i)_{i<\omega}$ such that $a_i \models q|_{Ca_{<i}}$ for all $i < \omega$.

Fact 2.11. By invariance, every Morley sequence over C for q is C-indiscernible. Furthermore, for a fixed global C-invariant type q extending tp(b/C), if $\varphi(x; b)$ divides along some Morley sequence over C for q, then it divides along every Morley sequence over C for q.

Definition 2.12. Suppose $\varphi(x; b)$ is a formula and M is a model:

(1) We say $\varphi(x; b)$ *Kim-divides* over M if it divides along a Morley sequence over M for some global M-invariant type extending tp(b/M).
(2) We say $\varphi(x; b)$ *Kim-forks* over M if there are formulas $(\psi_i(x; c_i))_{i<n}$ with $n < \omega$ such that $\varphi(x; b) \models \bigvee_{i<n} \psi_i(x; c_i)$ and each $\psi_i(x; c_i)$ Kim-divides over M.

(3) The notation $a \underset{M}{\overset{Kd}{\downarrow}} b$ means that $\text{tp}(a/Mb)$ contains no formula that Kim-divides over M and, similarly, $a \underset{M}{\overset{K}{\downarrow}} b$ means that $\text{tp}(a/Mb)$ contains no formula that Kim-forks over M.

Kim-independence was introduced by Kaplan and Ramsey [2020], in the context of NSOP_1 theories. They showed that if T is NSOP_1, then Kim-forking is equivalent to Kim-dividing, and $\underset{}{\overset{K}{\downarrow}}$ satisfies many of the good properties of $\underset{}{\overset{f}{\downarrow}}$ in simple theories. The definition of Kim-dividing was inspired by an earlier suggestion of Kim [2009] for studying independence in NTP_1 theories.

Remark 2.13. In a general theory, Kim-dividing as we have defined it is not always preserved under adding dummy parameters. That is, suppose $\varphi(x; y)$ is a formula, and write $\widehat{\varphi}(x; y, z)$ for the same formula consider in a larger variable context by appending unused variables z. It is possible that there are tuples b and c such that $\varphi(x; b)$ Kim-divides over M but $\widehat{\varphi}(x; b, c)$ does not Kim-divide over M. The reason is that $\underset{}{\overset{i}{\downarrow}}$ does not satisfy left-extension in general. More explicitly, if $q(y)$ is a global M-invariant type extending $\text{tp}(b/M)$ (and witnessing the Kim-dividing of $\varphi(x; b)$), there may be no global M-invariant type $r(y, z)$ extending both $q(y)$ and $\text{tp}(bc/M)$. Hanson [2023, Appendix C] has produced an explicit example of this behavior.

As a result, we have to be careful about parameters when working with Kim-dividing in arbitrary theories. For example, if $\varphi(x; b)$ Kim-forks, we cannot assume in general that the witnessing Kim-dividing formulas $(\psi_i(x; c_i))_{i<n}$ have the same tuple of parameters. This will cause us some trouble in Section 2C below.

All this suggests to us that our definition of Kim-dividing may not be the "right" one outside of the context of NSOP_1 theories. If T is NSOP_1, then a formula Kim-divides over a model M if and only if it Kim-divides along a coheir sequence over M (a Morley sequence for a global type finitely satisfiable in M). And if Kim-dividing were defined as dividing along a coheir sequence, then the issue with dummy parameters would not arise, since $\underset{}{\overset{u}{\downarrow}}$ always satisfies left-extension. However, focusing only on coheir sequences seems potentially too restrictive, and the definition of Kim-dividing in terms of invariant Morley sequences is well-established, so we retain it for this paper.

The diagram below depicts the implications between the notions of independence defined in this section:

$$a \underset{M}{\overset{u}{\downarrow}} b \longrightarrow a \underset{M}{\overset{i}{\downarrow}} b \longrightarrow a \underset{M}{\overset{f}{\downarrow}} b \longrightarrow a \underset{M}{\overset{K}{\downarrow}} b$$

$$\downarrow \qquad\qquad\qquad \downarrow$$

$$a \underset{M}{\overset{d}{\downarrow}} b \longrightarrow a \underset{M}{\overset{Kd}{\downarrow}} b$$

Fact 2.14 [Chernikov and Kaplan 2012; Adler 2014]. In NTP$_2$ theories, a formula $\varphi(x; b)$ divides over a model M if and only if it Kim-divides over M. Further, forking and dividing coincide over models. So when T is NTP$_2$, $\mathop{\smile}\limits_{M}^{f} = \mathop{\smile}\limits_{M}^{d} = \mathop{\smile}\limits_{M}^{K} = \mathop{\smile}\limits_{M}^{Kd}$.

It is a fact that simple theories are characterized by symmetry of $\mathop{\smile}^{f}$ [Kim 2001, Theorem 2.4]. So in a simple theory, if p is a global M-invariant type and a realizes $p|_{MB}$, then $B \mathop{\smile}\limits_{M}^{f} a$ (since $a \mathop{\smile}\limits_{M}^{i} B$ implies $a \mathop{\smile}\limits_{M}^{f} B$ and $\mathop{\smile}^{f}$ is symmetric). Outside of the simple context, it can be useful to consider invariant types which always satisfy this instance of symmetry. These "strict" invariant types play an important role in Chernikov and Kaplan's analysis [2012] of forking in NTP$_2$ theories.

Similarly, NSOP$_1$ theories are characterized by symmetry of $\mathop{\smile}^{K}$, so it makes sense in our context to consider "Kim-strict" invariant types, which are the analogue of strict invariant types for Kim-forking.

Definition 2.15. Suppose $p \in S(\mathbb{M})$ is a global M-invariant type:

(1) We say p is a *strict invariant type* over M when, for any set B, if $a \models p|_{MB}$, then $B \mathop{\smile}\limits_{M}^{f} a$.

(2) We say p is a *Kim-strict invariant type* over M when, for any set B, if $a \models p|_{MB}$, then $B \mathop{\smile}\limits_{M}^{K} a$.

(3) A formula $\varphi(x; b)$ *strictly divides* over M if it divides along a Morley sequence for some global strictly M-invariant type extending $\mathrm{tp}(b/M)$.

(4) A formula $\varphi(x; b)$ *Kim-strictly divides* over M if it divides along a Morley sequence for some global Kim-strictly M-invariant type extending $\mathrm{tp}(b/M)$.

Finally, for each of the variants of dividing defined above, we can also consider changing the quantifier from dividing along *some* to dividing along *every* indiscernible sequence of the appropriate kind.

Definition 2.16. We say a formula $\varphi(x; b)$ *universally Kim-divides* over M if it divides along Morley sequences for *every* global M-invariant type extending $\mathrm{tp}(b/M)$.[4] Similarly, we say $\varphi(x; b)$ *universally strictly divides* over M if it divides along Morley sequences for *every* global strict M-invariant type extending $\mathrm{tp}(b/M)$, and we say $\varphi(x; b)$ *universally Kim-strictly divides* over M if it divides along Morley sequences for *every* global Kim-strict M-invariant type extending $\mathrm{tp}(b/M)$.

Remark 2.17. For completeness, we could say a formula $\varphi(x; b)$ *universally divides* over C if it divides along *every* C-indiscernible sequence in $\mathrm{tp}(b/C)$. Note,

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[4]Universal Kim-dividing is called "strong Kim-dividing" in [Kaplan et al. 2019] and "Conant-dividing" in [Mutchnik 2022a].

however, that since the constant sequence with $b_i = b$ for all i is C-indiscernible, a universally dividing formula is inconsistent.

2C. *The broom lemma.* It is clear that universal Kim-dividing implies Kim-dividing, since every type over a model M extends to a global M-invariant type (see Remark 2.9). However, it is not so clear that universal (Kim-)strict dividing implies (Kim-)strict dividing.

Chernikov and Kaplan [2012] proved that in an NTP$_2$ theory, every type over a model M extends to a global strictly M-invariant type, using a device they called the broom lemma. It turns out that their argument applies to all theories, if we replace strict invariance with Kim-strict invariance.

A key step in the Chernikov–Kaplan argument is that forking implies quasi-dividing in the sense of the following definition.

Definition 2.18. A formula $\varphi(x; b)$ *quasi-divides* over M if the conjunction of finitely many conjugates of $\varphi(x; b)$ over M is inconsistent. That is, if there exist $(b_i)_{i < k}$ with $k < \omega$ and $b_i \equiv_M b$ for all $i < k$ such that $\bigwedge_{i < k} \varphi(x; b_i)$ is inconsistent.

Remark 2.19. We could say that $\varphi(x; b)$ *quasi-forks* over M if there are formulas $(\psi_i(x; c_i))_{i < n}$ with $n < \omega$ such that $\varphi(x; b) \models \bigvee_{i < n} \psi_i(x; c_i)$ and each $\psi_i(x; c_i)$ quasi-divides over M. It is worth noting that $a \underset{M}{\overset{i}{\smile}} b$ if and only if tp(a/Mb) contains no formula which quasi-forks. But we will not make use of this fact.

The original broom lemma argument from [Chernikov and Kaplan 2012] does not appear to generalize directly to our context. But Adler [2014] used a variant of the broom lemma, which he called the vacuum cleaner lemma, to give a simplified proof of some of the Chernikov–Kaplan results on NTP$_2$ theories. Adler's proof [2014, Lemma 3] goes through verbatim to prove the following result, in the context of an arbitrary theory T.

Lemma 2.20 (vacuum cleaner for Kim-dividing). *Let $\pi(x)$ be an M-invariant partial type and suppose*

$$\pi(x) \models \psi(x; b) \vee \bigvee_{i < n} \varphi_i(x; c),$$

where $b \underset{M}{\overset{i}{\smile}} c$ and each $\varphi_i(x; c)$ Kim-divides over M. Then $\pi(x) \models \psi(x; b)$.[5]

Corollary 2.21. *Suppose $\theta(x; b) \models \bigvee_{i < n} \varphi_i(x; c)$, where each $\varphi_i(x; c)$ Kim-divides over M. Then $\theta(x; b)$ quasi-divides over M.*

Proof. Let $\pi(x) = \{\theta(x; b') : b' \equiv_M b\}$ and let ψ be $\bot$. By Lemma 2.20, $\pi(x)$ is inconsistent, so, by compactness, $\theta(x; b)$ quasi-divides. $\qquad\square$

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[5] A similar modified broom lemma played a key role in Mutchnik's proof [2022b] of the equivalence of NSOP$_1$ and NSOP$_2$.

[Corollary 2.21](#) seems to say that Kim-forking formulas quasi-divide. But, as noted in [Remark 2.13](#) above, we cannot assume in general that in the finite disjunction $\bigvee_{i<n} \varphi_i(x; c_i)$ witnessing Kim-forking, all of the Kim-dividing formulas have the same tuple of parameters c. Unfortunately, this assumption seems crucial in Adler's proof of the vacuum cleaner lemma. As in [Remark 2.13](#), this would not be an issue if we defined Kim-dividing in terms of dividing along coheir sequences.

Nevertheless, it is true in general that Kim-forking formulas quasi-divide. We present an alternative proof, based on an idea due to Hanson.

Lemma 2.22. *Let $\varphi(x; b)$ be a formula. Suppose that the conjunction of finitely many conjugates of $\varphi(x; b)$ over M entails a formula which quasi-divides over M. Then $\varphi(x; b)$ quasi-divides over M.*

Proof. By hypothesis, there exist $(b_i)_{i<k}$ with $b_i \equiv_M b$ for all $i < k$ such that $\bigwedge_{i<k} \varphi(x; b_i) \models \psi(x; c)$, and $\psi(x; c)$ quasi-divides over M. Then there exist $(c_j)_{j<n}$ with $c_j \equiv_M c$ for all $j < n$ such that $\bigwedge_{j<n} \psi(x; c_j)$ is inconsistent.

For each $j < n$, pick $(b_{i,j})_{i<k}$ such that $b_{0,j} \cdots b_{(k-1),j} c_j \equiv_M b_0 \cdots b_{k-1} c$. Then

$$\bigwedge_{j<n} \bigwedge_{i<k} \varphi(x; b_{i,j}) \models \bigwedge_{j<n} \psi(x; c_j).$$

For all $i < k$ and $j < n$, $b_{i,j} \equiv_M b_i \equiv_M b$, so this is a finite conjunction of conjugates of $\varphi(x; b)$ over M which is inconsistent. $\square$

Lemma 2.23. *Suppose $\varphi(x; b)$ Kim-divides over M. Then for any $(b_i)_{i<\ell}$ such that $b_i \equiv_M b$ for all $i < \ell$, $\bigvee_{i<\ell} \varphi(x; b_i)$ quasi-divides over M.*

Proof. Write $\Phi(x; \bar{b})$ for the formula $\bigvee_{i<\ell} \varphi(x; b_i)$. Our goal is to show that $\Phi(x; \bar{b})$ quasi-divides. Let $q(y)$ be a global M-invariant type extending $\mathrm{tp}(b/M)$ and witnessing that $\varphi(x; b)$ Kim-divides over M. Let k be such that, if $(b'_i)_{i<\omega}$ is a Morley sequence for q over M, $\{\varphi(x; b'_i) : i < k\}$ is inconsistent.

Write $\ell^{\leq m}_*$ for the set of functions $\eta \colon n \to \ell$ with $0 < n \leq m$, that is, $\ell^{\leq m}_* = \ell^{\leq m} \setminus \{\langle\,\rangle\}$. We will prove by induction that for all $m \leq k$, we can find $(b_\eta)_{\eta \in \ell^{\leq m}_*}$ such that:

(1) For each $\rho \in \ell^m$, $(b_\rho, b_{\rho|_{m-1}}, \dots, b_{\rho|_1})$ begins a Morley sequence in q over M.

(2) For each $\eta \in \ell^{<m}$, writing $\bar{b}'_\eta$ for the tuple $(b_{\eta^\frown \langle i \rangle})_{i<\ell}$, we have $\bar{b}'_\eta \equiv_M \bar{b}$.

In the base case, when $m = 0$, $\ell^{\leq m}_*$ is empty, and the conditions are satisfied vacuously.

For the inductive step, suppose we are given $F_0 = (b_\eta)_{\eta \in \ell^{\leq m}_*}$ satisfying the conditions, with $m < k$. Let b''_0 realize $q|_{MF_0}$. By condition (1), we now have that for each $\rho \in \ell^m$, $(b_\rho, b_{\rho|_{m-1}}, \dots, b_{\rho|_1}, b''_0)$ begins a Morley sequence in q over M.

Since $b''_0 \equiv_M b_0$, we can pick $(b''_i)_{0<i<\ell}$ so that $(b''_i)_{i<\ell} \equiv_M \bar{b}$. Now, for each $0 < i < \ell$, pick F_i so that $F_i b''_i \equiv_M F_0 b''_0$. Reindex so that we have a forest indexed

by $\ell_*^{\leq(m+1)}$, with $(b_i'')_{i<\ell}$ as the "bottom layer" $\bar{b}'_{\langle\rangle}$. This completes the inductive construction.

Now we have $(b_\eta)_{\eta\in\ell_*^{\leq k}}$ satisfying (1) and (2). Observe that

$$\bigwedge_{\eta\in\ell^{<k}}\bigvee_{i<\ell}\varphi(x;b_{\eta^\frown\langle i\rangle})\models\bigvee_{\rho\in\ell^k}\bigwedge_{1\leq i\leq k}\varphi(x;b_{\rho|_i}).$$

By (1), for each $\rho\in\ell^k$, $\bigwedge_{1\leq i\leq k}\varphi(x;b_{\rho|_i})$ is inconsistent. Thus the left-hand side, which is $\bigwedge_{\eta\in\ell^{<k}}\Phi(x;\bar{b}'_\eta)$, is inconsistent. By (2), this shows that $\Phi(x;\bar{b})$ quasi-divides over M. $\qquad\square$

Lemma 2.24. *Suppose $(\varphi_i(x;b_i))_{i<n}$ are formulas, each of which Kim-divides over M. For each $i < n$, let $\theta_i(x;c_i)$ be a disjunction of finitely many conjugates of $\varphi_i(x;b_i)$. Then $\bigvee_{i<n}\theta_i(x;c_i)$ quasi-divides over M.*

Proof. By induction on n. When $n = 0$, the disjunction is $\bot$, which quasi-divides over M. For the inductive step, we consider $\bigvee_{i<n+1}\theta_i(x;c_i)$. Now $\theta_n(x;c_n)$ is a disjunction of finitely many conjugates of $\varphi_n(x;b_n)$. By Lemma 2.23, $\theta_n(x;c_n)$ quasi-divides over M, so there are $(c_{nj})_{j<k}$ with $c_{nj}\equiv_M c_n$ for all $j < k$ such that $\bigwedge_{j<k}\theta_n(x;c_{nj})$ is inconsistent.

For each $j < k$, pick $(c_{ij})_{i<n}$ such that $c_{0j}\cdots c_{nj}\equiv_M c_0\cdots c_n$. Consider the conjunction

$$\bigwedge_{j<k}\bigvee_{i<n+1}\theta_i(x;c_{ij}).$$

Whenever this formula is true, there must be some $j < k$ such that some disjunct $\theta_i(x;c_{ij})$ with $i\neq n$ is true, since $\bigwedge_{j<k}\theta_n(x;c_{nj})$ is inconsistent. Thus

$$\bigwedge_{j<k}\bigvee_{i<n+1}\theta_i(x;c_{ij})\models\bigvee_{i<n}\bigvee_{j<k}\theta_i(x;c_{ij}).$$

Since each formula $\bigvee_{j<k}\theta_i(x;c_{ij})$ is a disjunction of finitely many conjugates of $\varphi_i(x;b_i)$, by induction $\bigvee_{i<n}\bigvee_{j<k}\theta_i(x;c_{ij})$ quasi-divides over M. By Lemma 2.22, $\bigvee_{i<n+1}\theta_i(x;c_{ij})$ quasi-divides over M. $\qquad\square$

Corollary 2.25. *Every formula which Kim-forks over M quasi-divides over M.*

Proof. Suppose $\varphi(x;b)$ Kim-forks over M. Then $\varphi(x;b)\models\bigvee_{i<n}\psi_i(x;c_i)$ such that each $\psi_i(x;c_i)$ Kim-divides over M. By Lemma 2.24 (taking each θ_i to be $\psi_i(x;c_i)$), $\bigvee_{i<n}\psi(x;c_i)$ quasi-divides over M, and hence so does $\varphi(x;b)$ by Lemma 2.22. $\qquad\square$

Theorem 2.26. *Every type over $M\models T$ has a Kim-strict M-invariant global extension.*

Proof. Given $p(x) = \text{tp}(a/M)$, consider the following collection of formulas:

$$p(x) \cup \{\psi(x; c) \leftrightarrow \psi(x; c') : c \equiv_M c'\} \cup \{\neg\varphi(x; b) : \varphi(a; y) \text{ Kim-forks over } M\}.$$

We must show that this is a consistent partial type. Suppose not; then, by compactness,

$$p(x) \cup \{\psi(x; c) \leftrightarrow \psi(x; c') : c \equiv_M c'\} \models \varphi(x; b),$$

for some formula $\varphi(x; y)$ such that $\varphi(a; y)$ Kim-forks over M.

By Corollary 2.25, there are $(a_i)_{i<m}$ such that $a_i \equiv_M a$ for all $i < m$ and $\{\varphi(a_i, y) : i < m\}$ is inconsistent. Let $r(x_0, \ldots, x_{m-1})$ be a global M-invariant type extending $\text{tp}(a_0, \ldots, a_{m-1}/M)$, and for $j < m$, let $r(x_j)$ be the restriction of r to formulas with free variables from x_j. Then each $r(x_j)$ is a global M-invariant type extending $p(x_j)$, so $r(x_j) \models \varphi(x_j, b)$. Thus,

$$r(x_0, \ldots, x_{m-1}) \models \bigwedge_{j<m} \varphi(x_j; b),$$

and therefore $\exists y \bigwedge_{j<m} \varphi(x_j, y) \in r$. This contradicts the fact that r extends $\text{tp}(a_0, \ldots, a_{m-1}/M)$. $\square$

Corollary 2.27. *If $\varphi(x; b)$ universally Kim-strictly divides over M, then it Kim-strictly divides over M.*
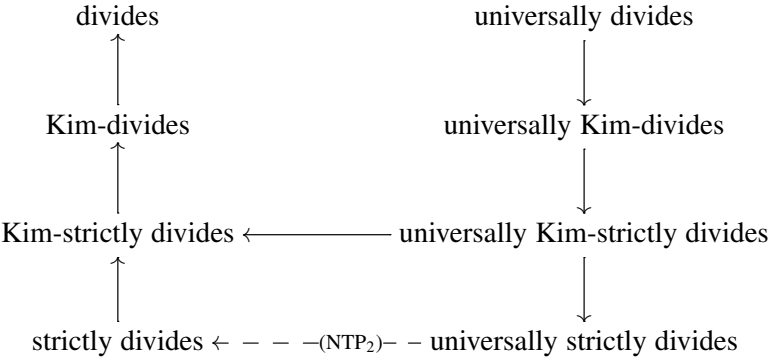
Proof. By Theorem 2.26, $\text{tp}(b/M)$ has a Kim-strict M-invariant global extension $q(y)$. Since $\varphi(x; b)$ universally Kim-strictly divides over M, it divides along Morley sequences for q. Thus, it Kim-strictly divides over M. $\square$

Note that the only properties of Kim-forking used in the proof of Theorem 2.26 are (a) that the Kim-forking formulas form an ideal (i.e., they are closed under finite disjunctions), and (b) that every Kim-forking formula quasi-divides. In unpublished work, Hanson has shown that there is a largest M-invariant ideal which contains only quasi-dividing formulas, called the "fracturing" ideal. The proof of Theorem 2.26 works just as well to show that $\text{tp}(a/M)$ extends to a global M-invariant extension containing no formula $\varphi(x; b)$ such that $\varphi(a; y)$ fractures.

Remark 2.28. Chernikov and Kaplan [2012, Subsection 5.1] present an example, due to Martin Ziegler, of a theory T in which there is a model $M \models T$ and a type over M with no global extension that is strict invariant over M. This shows that, in general, Theorem 2.26 cannot be improved to establish the existence of global strict invariant types over models in arbitrary theories.

We conclude this section with a diagram showing the implications between the various notions of dividing (over models) introduced in Section 2B. All implications

hold in an arbitrary theory, except for the implication from universally strictly divides to strictly divides, which requires NTP_2:

$$
\begin{array}{ccc}
\text{divides} & & \text{universally divides} \\
\uparrow & & \downarrow \\
\text{Kim-divides} & & \text{universally Kim-divides} \\
\uparrow & & \downarrow \\
\text{Kim-strictly divides} & \longleftarrow & \text{universally Kim-strictly divides} \\
\uparrow & & \downarrow \\
\text{strictly divides} & \xleftarrow{\;-\;-\;-(NTP_2)-\;-\;} & \text{universally strictly divides}
\end{array}
$$

3. A diversity of Kim's lemmas

In this section, we survey the characterizations of simplicity, $NSOP_1$, and NTP_2 by Kim's Lemmas, and we introduce our new Kim's Lemma. We begin with the original Kim's Lemma in the context of simple theories.

Theorem 3.1 [Kim 1998, Proposition 2.1; 2001, Theorem 2.4]. *The following are equivalent*:

(1) *T is simple.*

(2) *For all sets C, if a formula $\varphi(x; b)$ divides over C, then it divides along every $\bigcup^f$-Morley sequence over C.*

In this paper, we are primarily interested in Morley sequences for global invariant types over models (rather than $\bigcup^f$-Morley sequences over arbitrary sets), so we are led to consider the following variant of (2):

(3) For all models M, if a formula $\varphi(x; b)$ divides over M, then it universally Kim-divides over M.

Note that (3) is a weakening of (2), since it restricts to the special case of models, and since every Morley sequence for a global M-invariant type is a $\bigcup^f$-Morley sequence over M. But (3) is still strong enough to characterize simplicity.

The equivalence of (1) and (3) has not (to our knowledge) appeared explicitly in the literature, but it does follow directly from facts in the literature. We have already observed that (1) implies (2) and (2) implies (3). Conversely, (3) implies, in particular, that Kim-dividing implies universal Kim-dividing, so T is $NSOP_1$ (by the Kim's Lemma for $NSOP_1$ theories, Theorem 3.3 below). Thus T is an $NSOP_1$ theory in which dividing and Kim-dividing coincide over models, so T is simple by [Kaplan and Ramsey 2020, Proposition 8.4].

For the reader's convenience, and to give an indication of the typical flavor of arguments relating variants of Kim's Lemma to combinatorial configurations like the tree property, we will also give a self-contained proof of the equivalence of (1) and (3).

Proof. (1)$\Longrightarrow$(3). Suppose (3) fails, so there is a model $M \models T$, a formula $\varphi(x; b)$ that divides over M, and a global M-invariant type $q \supseteq \mathrm{tp}(b/M)$ such that $\varphi(x; b)$ does not divide along Morley sequences over M for q. Let $(b_i)_{i<\omega}$ be an M-indiscernible sequence in $\mathrm{tp}(b/M)$ such that $\{\varphi(x; b_i) : i < \omega\}$ is inconsistent (and hence k-inconsistent for some k). By induction, we will build for each $n < \omega$, a tree $(c_\eta)_{\eta \in \omega^{\leq n}}$ satisfying the following:

- For all $\eta \in \omega^{<n}$, $(c_{\eta^\frown\langle i \rangle})_{i<\omega} \equiv_M (b_i)_{i<\omega}$.

- For all $\nu \in \omega^n$, $(c_\nu, c_{\nu\restriction(n-1)}, \ldots, c_{\nu\restriction 0})$ begins a Morley sequence in q over M.

For $n = 0$, we define $c_{\langle\,\rangle} = b$. The conditions are trivially satisfied.

For the inductive step, we are given a tree $(c_{\eta,0})_{\eta \in \omega^{\leq n}}$. Since $c_{\langle\,\rangle,0} \models q|_M$, we have $c_{\langle\,\rangle,0} \equiv_M b$, and there is a sequence $(c_{\langle\,\rangle,i})_{i<\omega}$ beginning with $c_{\langle\,\rangle,0}$ such that $(c_{\langle\,\rangle,i})_{i<\omega} \equiv_M (b_i)_{i<\omega}$. For each i, $c_{\langle\,\rangle,i} \equiv_M c_{\langle\,\rangle,0}$, so we can choose a tree $(c_{\eta,i})_{\eta \in \omega^{\leq n}}$ with root $c_{\langle\,\rangle,i}$ such that $(c_{\eta,0})_{\eta \in \omega^{\leq n}} \equiv_M (c_{\eta,i})_{\eta \in \omega^{\leq n}}$. Let $c_{\langle\,\rangle}$ be a realization of $q|_{M\{c_{\eta,i}:\eta \in \omega^{\leq n}, i<\omega\}}$. Then we reindex to define a tree $(c_\eta)_{\eta \in \omega^{\leq n+1}}$ by setting $c_{\langle i \rangle^\frown \eta} = c_{\eta,i}$ for all $i < \omega$ and $\eta \in \omega^{\leq n}$.

Note that, for each n, the tree $(c_\eta)_{\eta \in \omega^{\leq n}}$ that we constructed has the following properties. First, for each $\eta \in \omega^{<n}$, $\{\varphi(x; c_{\eta^\frown\langle i \rangle}) : i < \omega\}$ is k-inconsistent, by the first bullet point above. Secondly, for all $\nu \in \omega^n$, $\{\varphi(x; c_{\nu\restriction\ell}) : \ell \leq n\}$ is consistent, by the second bullet point and our assumption on q. By compactness, $\varphi(x; y)$ has TP, and T is not simple.

(3)$\Longrightarrow$(1) Suppose T has TP witnessed by $\varphi(x; y)$, $k < \omega$, and $(a_\eta)_{\eta \in \omega^{<\omega}}$. Fix a Skolemization T^{Sk} of T. The same data shows that $\varphi(x; y)$ has TP modulo T^{Sk}.

By compactness, we can obtain a tree $(a_\eta)_{\eta \in \kappa^{<\omega}}$, where $\kappa > 2^{|T|}$, and which satisfies the obvious extensions of the defining conditions of the tree property.

We build an array $(b_{i,j})_{i,j<\omega}$ and $\rho \in \kappa^\omega$ with the following properties (in T^{Sk}):

- $b_{i,0} = a_{\rho\restriction(i+1)}$ for all $i < \omega$ (and therefore $\{\varphi(x; b_{i,0}) : i < \omega\}$ is consistent).

- For all $i < \omega$, $\{\varphi(x; b_{i,j}) : j < \omega\}$ is k-inconsistent.

- For all $i < \omega$, $(b_{i,j})_{j<\omega}$ is indiscernible over $(b_{\ell,0})_{\ell<i}$.

We proceed by recursion on i. Given $\rho\restriction n$ and $(b_{i,j})_{i<n, j<\omega}$, let $\eta = \rho\restriction n$, and consider the sequence $(a_{\eta^\frown\langle\alpha\rangle})_{\alpha<\kappa}$. By the conditions on κ, we can find a subsequence $I = (a_{\eta^\frown\langle\alpha_j\rangle})_{j<\omega}$ such that each $a_{\eta^\frown\langle\alpha_j\rangle}$ satisfies the same complete type $p(y)$ over $(b_{i,0})_{i<n}$. Let $(b_{n,j})_{j<\omega}$ be a sequence which is indiscernible and locally based on I over $(b_{i,0})_{i<n}$ (i.e., realizes the Ehrenfeucht–Mostowski type

of I over $(b_{i,0})_{i<n}$). It follows that each $b_{n,j}$ satisfies $p(y)$, so we can assume that $b_{n,0} = a_{\eta^\frown\langle\alpha_0\rangle}$ and let $\rho(n) = \alpha_0$. It also follows that $\{\varphi(x; b_{n,j}) : j < \omega\}$ is k-inconsistent. This completes the construction.

For each $i < \omega$, let $\bar{b}_i = (b_{i,j})_{j<\omega}$, and let $J = (\bar{b}_i)_{i<\omega}$. Let $J' = (\bar{b}'_i)_{i<\omega+\omega}$ be a sequence which is indiscernible and locally based on J (over $\varnothing$). Writing each $\bar{b}'_i$ as $(b'_{i,j})_{j<\omega}$, we retain consistency of $\{\varphi(x; b'_{i,0}) : i < \omega+\omega\}$, k-inconsistency of $\{\varphi(x; b'_{i,j}) : j < \omega\}$ for all $i < \omega+\omega$, and indiscernibility of $(b'_{i,j})_{j<\omega}$ over $(b'_{\ell,0})_{\ell<i}$ for all $i < \omega+\omega$.

Let M be the Skolem hull of $(b'_{i,0})_{i<\omega}$. By indiscernibility, $\mathrm{tp}(b'_{\omega,0}/M(b'_{i,0})_{i>\omega})$ is finitely satisfiable in M and therefore extends to a global M-finitely satisfiable (and therefore M-invariant) type q. Moreover, by indiscernibility, $b'_{\omega+i,0} \models q|_{M(b'_{n,0})_{n>\omega+i}}$ for all i, which shows that for all n, $(b'_{\omega+n,0}, \ldots, b'_{\omega,0})$ begins a Morley sequence for q over M. By construction, $\{\varphi(x; b'_{\omega+i,0}) : i < \omega\}$ is consistent, so $\varphi(x; b'_{\omega,0})$ does not divide along Morley sequences for q over M. However, $\varphi(x; b'_{\omega,0})$ does divide along the M-indiscernible sequence $\bar{b}'_\omega$.

Taking the reduct back to T, the restriction $q|_L$ of q to L-formulas is still finitely satisfiable in M, Morley sequences in q are also Morley sequences in $q|_L$, and the M-indiscernible sequence $\bar{b}'_\omega$ remains M-indiscernible in the reduct. Thus $\varphi(x; b'_{\omega,0})$ divides but does not universally Kim-divide with respect to T, and (3) fails.   □

Example 3.2. Let T_E be the theory of an equivalence relation E with infinitely many classes, each of which is infinite. T_E is a simple theory (in fact, it is ω-stable). Let $M \models T_E$, and let b be an element of $\mathbb{M}$ in an equivalence class which is not represented in M. There are three types of M-indiscernible sequence $(b_i)_{i<\omega}$ in $\mathrm{tp}(b/M)$: (a) constant sequences, in which $b_i = b_j$ for all $i, j < \omega$, (b) sequences contained in one equivalence class, in which $b_i \neq b_j$ but $b_i E b_j$ for all $i \neq j$, and (c) sequences that move across equivalence classes, in which $\neg b_i E b_j$ for all $i \neq j$.

The formula xEb divides along sequences of type (c), but not along sequences of type (a) or (b). Is there a general explanation for this behavior? Kim's Lemma gives the answer: the dividing formula xEb universally Kim-divides, and every Morley sequence for a global M-invariant type extending $\mathrm{tp}(b/M)$ has type (c).

Indeed, if $q(y)$ is a global M-invariant type extending $\mathrm{tp}(b/M)$, we will show that q cannot contain the formula yEc for any $c \in \mathbb{M}$. If cEm for some $m \in M$, then since $\neg yEm \in \mathrm{tp}(b/M)$, $\neg yEc \in q$. And if the equivalence class of c is not represented in M, then letting c' be another element inequivalent to c whose equivalence class is not represented in M, q cannot contain both yEc and yEc', but $\mathrm{tp}(c/M) = \mathrm{tp}(c'/M)$, so by invariance q does not contain yEc. It follows that a Morley sequence for q has type (c).

Next, we turn to the Kim's Lemma characterization of NSOP_1 theories.

Theorem 3.3 [Kaplan and Ramsey 2020, Theorem 3.16]. *The following are equivalent*:

(1) *T is* NSOP$_1$.

(2) *For all models M, if a formula $\varphi(x; b)$ Kim-divides over M, then it universally Kim-divides over M.*

Example 3.4. T_{feq}^*, the generic theory of parametrized equivalence relations, is NSOP$_1$ and has TP$_2$. It is the complete theory of the Fraïssé limit of the Fraïssé class $\mathcal{K}_{\text{feq}}$. The language has two sorts, O and P, and one ternary relation $y E_x z$, where the subscript x has type P and y and z have type O. A finite structure A is in $\mathcal{K}_{\text{feq}}$ if for all $a \in P(A)$, E_a defines an equivalence relation on $O(A)$.

Let $M \models T_{\text{feq}}^*$, let $c \in P(\mathbb{M}) \setminus P(M)$, and let $b \in O(\mathbb{M})$ such that the E_c-class of b is not represented in $O(M)$. The formula $x E_c b$ divides over M, along any M-indiscernible sequence $(b_i, c_i)_{i < \omega}$ such that $c_i = c$ for all i and $\neg b_i E_c b_j$ for all $i \neq j$. But if $p(y, z)$ is a global M-invariant type extending $\text{tp}(bc/M)$ and $I = (b_i, c_i)_{i < \omega}$ is a Morley sequence for p, then $c_i \neq c_j$ for all $i \neq j$, and $x E_c b$ does not divide along I. Indeed, by compactness and the genericity properties of the Fraïssé limit, if $(c_i)_{i < \omega}$ is any sequence of pairwise distinct elements of $P(\mathbb{M})$, and C_i is an E_{c_i} class for each $i < \omega$, then we can find $a \in O(\mathbb{M})$ such that $a \in C_i$ for all $i \in \omega$. It follows that $x E_c b$ does not Kim-divide, and hence does not universally Kim-divide, so the Kim's Lemma for simple theories fails in T_{feq}^*.

Now let $m \in P(M)$, and let $b' \in O(\mathbb{M})$ such that the E_m-class of b' is not represented in $O(M)$. Then the formula $x E_m b'$ Kim-divides over M, and, as predicted by the Kim's Lemma for NSOP$_1$ theories, it universally Kim-divides over M. Indeed, if $p(y, z)$ is any global M-invariant type extending $\text{tp}(b'm/M)$, and $I = (b_i, m_i)_{i < \omega}$ is a Morley sequence for p, then $m_i = m$ for all $i \in \omega$ and $(b_i)_{i < \omega}$ is an indiscernible sequence of type (c) for E_m, according to the terminology in Example 3.2. Thus $x E_m b'$ divides along I.

Finally, we turn to the Kim's Lemma characterization of NTP$_2$ theories.

Theorem 3.5 [Chernikov and Kaplan 2012, Lemma 3.14; Chernikov 2014, Theorem 4.9]. *The following are equivalent*:

(1) *T is* NTP$_2$.

(2) *For all models M, if a formula $\varphi(x; b)$ divides over M, then it universally Kim-strictly divides over M.*

Note that the notion of Kim-strict dividing does not appear in [Chernikov and Kaplan 2012] or [Chernikov 2014]. Instead, Chernikov and Kaplan prove that (1) is equivalent to (3):

(3) For all models M, if a formula $\varphi(x; b)$ divides over M, then it universally strictly divides over M.

But since Kim-strict invariant types coincide with strict invariant types in NTP_2 theories (by Fact 2.14), and universal Kim-strict dividing implies universal strict dividing in arbitrary theories, it follows immediately that (1), (2), and (3) are all equivalent. We have chosen to focus on Kim-strict dividing because it behaves better outside of the NTP_2 context (by Theorem 2.26 and Remark 2.28).

Example 3.6. DLO, the theory of dense linear orders without endpoints, is NTP_2 (in fact, it is NIP) and has SOP_1 (in fact, it has SOP). Let $M \models \text{DLO}$, and $b < c$ be two elements in $\mathbb{M} \setminus M$ living in the same cut in M (so there is no $m \in M$ with $b < m < c$). Now $q(y, z) = \text{tp}(bc/M)$ has three global M-invariant extensions. By quantifier elimination, each is determined by the order relations between y and z and the elements $d \in \mathbb{M}$ living in the same cut in M as b and c:

(1) Let q_1 be the global type containing $d < y < z$ for all such d. A Morley sequence $(b_i, c_i)_{i < \omega}$ for q_1 has

$$b_0 < c_0 < b_1 < c_1 < b_2 < c_2 < \cdots .$$

(2) Let q_2 be the global type containing $y < z < d$ for all such d. A Morley sequence $(b_i, c_i)_{i < \omega}$ for q_2 has

$$\cdots < b_2 < c_2 < b_1 < c_1 < b_0 < c_0.$$

(3) Let q_3 be the global type containing $y < d < z$ for all such d. A Morley sequence $(b_i, c_i)_{i < \omega}$ for q_3 has

$$\cdots < b_2 < b_1 < b_0 < c_0 < c_1 < c_2 < \cdots .$$

The formula $b < x < c$ divides along Morley sequences for q_1 and q_2, but not along Morley sequences for q_3. This shows that the Kim's Lemma for NSOP_1 theories fails in DLO: Kim-dividing does not imply universal Kim-dividing. But the Kim's Lemma for NTP_2 theories explains which Morley sequences we should expect a dividing formula to divide along. Indeed, the dividing formula $b < x < c$ universally Kim-strictly divides, and we will show that q_1 and q_2 are Kim-strict, while q_3 is not.

Suppose $A \subseteq \mathbb{M}$, and suppose $b'c' \models q_i|_{MA}$ for some $i \in \{1, 2, 3\}$. If $i = 1$ or 2, then there is no $a \in A$ such that $b < a < c$, and it follows that $A \underset{M}{\overset{K}{\smile}} b'c'$. So q_1 and q_2 are Kim-strict.

On the other hand, if $i = 3$, and if A contains an element a living in the same cut in M and b and c, then $b' < a < c'$. Thus $\text{tp}(A/Mb'c')$ contains the Kim-dividing formula $b' < x < c'$, and $A \underset{M}{\overset{K}{\not\smile}} b'c'$. So q_3 is not Kim-strict.

We can now fill in the diagram from the end of Section 2 with the implications coming from the variants of Kim's Lemma which hold in various contexts, as well

as our New Kim's Lemma:
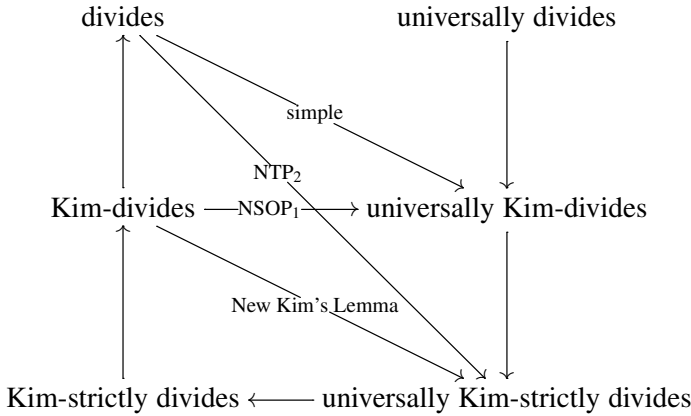
Definition 3.7. *T* satisfies *New Kim's Lemma* if for all models M, if a formula $\varphi(x; b)$ Kim-divides over M, then it universally Kim-strictly divides over M.

We will give some examples and nonexamples of New Kim's Lemma in the next section. For now, let us observe a simple consequence. Variants of Kim's Lemma allow us to prove that the relevant notions of forking and dividing coincide, and the usual proof works here as well.

Proposition 3.8. *Suppose T satisfies New Kim's Lemma and $M \models T$. Then a formula $\varphi(x; b)$ Kim-forks over M if and only if it Kim-divides over M.*

Proof. Kim-dividing implies Kim-forking by definition. So suppose $\varphi(x; b)$ Kim-forks over M. Then $\varphi(x; b) \models \bigvee_{j<n} \psi_j(x; c_j)$, where each $\psi_j(x; c_j)$ Kim-divides over M.

By Theorem 2.26, let $q(y, z_1, \ldots, z_n)$ be a global Kim-strict invariant type extending $\operatorname{tp}(bc_0 \cdots c_{n-1}/M)$, and let $I = (b^i, c_0^i, \ldots, c_{n-1}^i)_{i<\omega}$ be a Morley sequence for q over M.

For all $j < n$, $I_j = (c_j^i)_{i<\omega}$ is also a Morley sequence over M for a global Kim-strict invariant type, namely the restriction of q to formulas in the single variable z_j. By New Kim's Lemma, $\psi_j(x; c_j)$ divides along I_j.

Suppose for contradiction that $\varphi(x; b)$ does not divide along $I_* = (b^i)_{i<\omega}$. Then there exists a satisfying $\{\varphi(x; b^i) : i < \omega\}$. For each $i < \omega$, since $b^i c_0^i \cdots c_{n-1}^i \equiv_M bc_0 \cdots c_{n-1}$, there exists $j < n$ such that $\models \psi_j(a; c_j^i)$. By the pigeonhole principle, there is some $j < n$ such that for infinitely many $i < \omega$, a satisfies $\psi_j(x; c_j^i)$. This contradicts the fact that $\psi_j(x; c_j)$ divides along I_j. Thus $\varphi(x; b)$ divides along I_*. Since I_* is a Morley sequence over M for the restriction of q to formulas in the single variable y, $\varphi(x; b)$ Kim-divides over M. $\qquad\square$

4. Examples

4A. *Parametrized linear orders.* In this section, we introduce the theory DLO_p of parametrized dense linear orders without endpoints, and we show that it satisfies New Kim's Lemma. The choice of this example is motivated by the examples in Section 3: DLO_p is to DLO (Example 3.6) as T_{feq}^* (Example 3.4) is to T_E (Example 3.2).

The language L has two sorts, O and P, and one ternary relation $y <_x z$, where the subscript x has type P and y and z have type O. For an L-structure A, we write A_P and A_O for the two sorts. Let $L_<$ be the language $\{<\}$, where $<$ is a binary relation. Given $a \in A_P$, we write A_a for the $L_<$-structure $(A_O, <_a)$.

Let $\mathcal{K}$ be the class of all finite structures A such that for all $a \in A_P$, $<_a$ is a linear order on A_O. This is a special case of the parametrization construction introduced in [Chernikov and Ramsey 2016, Section 6.3], applied to the class of finite linear orders. By [loc. cit., Lemma 6.3], $\mathcal{K}$ is a Fraïssé class with disjoint amalgamation. Let DLO_p be the theory of its Fraïssé limit. By disjoint amalgamation, DLO_p has trivial acl. By [loc. cit., Lemma 6.4], if $M \models \mathrm{DLO}_p$, then for all $m \in M_P$, $M_m \models \mathrm{DLO}$.

If $C \subseteq \mathbb{M}_O$ and $\varphi(x)$ is an $L_<$-formula with parameters in C, then, for each $m \in \mathbb{M}_P$, we write $\varphi_m(x)$ for the L-formula obtained by replacing each instance of $<$ with $<_m$. Likewise, if $q(x)$ is a partial $L_<$-type over C, we write $q_m(x)$ for $\{\varphi_m(x) : \varphi(x) \in q\}$. Note that $q_m(x)$ is a partial L-type over Cm.

Fact 4.1 [Chernikov and Ramsey 2016, Lemma 6.5]. *Suppose $C \subseteq \mathbb{M}_O$, $(b_i)_{i \in I}$ is a family of distinct elements of $\mathbb{M}_P$, and for each $i \in I$, $p^i(x)$ is a consistent nonalgebraic $L_<$-type over C in $\mathbb{M}_{b_i}$. Then $\bigcup_{i \in I} p_{b_i}^i(x)$ is a consistent partial L-type over $C(b_i)_{i \in I}$.*

Recall that a *coheir sequence* over A is a Morley sequence for a global type finitely satisfiable in A. The following lemma is a general fact that is easy and well-known.

Lemma 4.2. *Suppose $M \models T$ and $I = (a_i)_{i < \omega}$ is a coheir sequence over M. Then given any b, there exists $(b_i)_{i < \omega}$ such that $(a_i, b_i)_{i < \omega}$ is a coheir sequence over M and $\mathrm{tp}(a_i b_i / M) = \mathrm{tp}(a_0 b / M)$ for all $i < \omega$.*

Proof. Suppose $(a_i)_{i < \omega}$ is a coheir sequence for the global M-finitely satisfiable type $p(x)$. Let N be an $|M|^+$-saturated model containing M. Let a^* realize $p \mid N$, so $a^* \underset{M}{\overset{u}{\downarrow}} N$. By left extension for $\overset{u}{\downarrow}$, we can find b^* such that $\mathrm{tp}(a^* b^* / M) = \mathrm{tp}(a_0 b / M)$ and such that $a^* b^* \underset{M}{\overset{u}{\downarrow}} N$. By saturation of N, $\mathrm{tp}(a^* b^* / N)$ has a unique global M-invariant extension $q(x, y)$, which is finitely satisfiable in M. Likewise, $p(x) \subseteq q(x, y)$, since the restriction of q to formulas in context x is the unique global M-invariant extension of $\mathrm{tp}(a^* / N) = p \mid N$. Let $(a_i^* b_i^*)_{i < \omega}$ be a Morley sequence for q over M. Since $(a_i^*)_{i < \omega}$ and $(a_i)_{i < \omega}$ are both Morley sequences for

p over M, there is an automorphism σ of $\mathbb{M}$ over M such that $\sigma(a_i^*) = a_i$ for all i. Let $b_i = \sigma(b_i^*)$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Lemma 4.3. *Let $M \models \mathrm{DLO}_p$. Then $A \underset{M}{\overset{Kd}{\bigcup}} B$ if and only if*

(1) *$A \cap B \subseteq M$, and*

(2) *for every $m \in M_P$, and for all $b <_m a <_m b'$ with $a \in A_O \setminus M_O$ and $b, b' \in B_O \setminus M_O$, there exists $m' \in M_O$ such that $b <_m m' <_m b'$ (i.e., b and b' live in different $<_m$-cuts in M_O).*

Condition (2) in the statement of Lemma 4.3 can be more succinctly stated as: for every $m \in M_P$, $A_O \underset{M_O}{\overset{f}{\bigcup}} B_O$ in the $L_<$-structure $\mathbb{M}_m$. Nevertheless, we will prove and use the more concrete characterization.

Proof. Suppose $A \underset{M}{\overset{Kd}{\bigcup}} B$. In any theory, $A \underset{M}{\overset{Kd}{\bigcup}} B$ implies $A \cap B \subseteq M$, so we have (1). For (2), assume for contradiction that $b <_m a <_m b'$, with $m \in M_P$, $a \in A_O \setminus M_O$, and $b, b' \in B_O \setminus M_O$, and b and b' live in the same $<_m$-cut in M_O, i.e., there is no $m' \in M_O$ such that $b <_m m' <_m b'$.

We will find a global type $q(y, y')$ extending $\mathrm{tp}(bb'/M)$ and finitely satisfiable in M, such that the formula $\varphi(x; b, b')$: $b <_m x <_m b'$ divides along Morley sequences for q over M. We may assume that the set $C = \{c \in M_O \mid c <_m b\}$ is nonempty and has no greatest element. The other case, when the set $D = \{d \in M_O \mid b <_m d\}$ is nonempty and has no least element, is symmetrical.

Consider the filter on $M_O^{yy'}$ generated by

$$\{\psi(M) : \psi(y, y') \in \mathrm{tp}(bb'/M)\} \cup \{(e, e') \mid e' \in C\}.$$

By quantifier elimination, a set Y in this filter contains the intersection of:

(1) $\{(e, e') \mid e' \in C\}$,

(2) A set $\{(e, e') \mid c <_m e <_m e' <_m d\}$ for some $c \in C$ and $d \notin C$, or $\{(e, e') \mid c <_m e <_m e'\}$ for some $c \in C$, and

(3) finitely many nonempty sets in $M_O^{yy'}$, each defined in terms of an order $<_{m'}$ for $m' \neq m$ in M_P.

Since C has no greatest element, we can pick some $c' \in C$ with $c <_m c'$. Then replacing (1) and (2) in the intersection with $\{(e, e') \mid c <_m e <_m e' <_m c'\}$, the intersection of these sets is nonempty, by the extension axioms for the Fraïssé limit, and contained in Y. Thus the filter is proper and extends to an ultrafilter $\mathcal{D}$.

Let $q = \mathrm{Av}(\mathcal{D}, \mathbb{M})$. Suppose $I = (b_i, b_i')_{i < \omega}$ is a Morley sequence for q over M. Since each b_i realizes $\mathrm{tp}(b/M)$,

$$\{(e, e') \in M_O^{yy'} \mid e' <_m b_i\} = \{(e, e') \in M_O^{yy'} \mid e' \in C\} \in \mathcal{D}.$$

So $b_{i+1} <_m b_{i+1}' <_m b_i$ for all $i < \omega$. Thus the $<_m$-intervals (b_i, b_i') are pairwise disjoint, and $b <_m x <_m b'$ divides along I.

Let b^* be a tuple enumerating $B \setminus \{b, b'\}$. By Lemma 4.2, there exists $(b_i^*)_{i<\omega}$ such that $J = (b_i, b_i', b_i^*)_{i<\omega}$ is a coheir sequence over M and $\operatorname{tp}(b_i, b_i', b_i^*) = \operatorname{tp}(b, b', b_*) = \operatorname{tp}(B/M)$ for all $i < \omega$. The formula $b <_m x <_m b'$ is contained in $\operatorname{tp}(A/MB)$ and divides along J, which contradicts $A \underset{M}{\overset{Kd}{\downarrow}} B$.

Conversely, suppose conditions (1) and (2) hold. We may assume that A is disjoint from M (and hence also from B, by (1)), since $(A \setminus M) \underset{M}{\overset{Kd}{\downarrow}} B$ implies $A \underset{M}{\overset{Kd}{\downarrow}} B$.

Let $p(x, x', y) = \operatorname{tp}(AB/M)$, where x enumerates A_O, x' enumerates A_P, and y enumerates B. Let $(B_i)_{i<\omega}$ be a Morley sequence for a global M-invariant type extending $\operatorname{tp}(B/M)$. Let $C = M \cup \bigcup_{i<\omega} B_i$. It suffices to show that $q(x, x') = \bigcup_{i<\omega} p(x, x', B_i)$ is a consistent partial type over C.

Let $q_O(x)$ be the subset of q that only mentions the variables x (those of type O). For each $c \in C_P$, let $q_c(x)$ be set of atomic and negated atomic formulas in $q_O(x)$ involving the relation $<_c$. Then there is a partial $L_<$-type $q^c(x)$ over C_O in $\mathbb{M}_c$ such that $(q^c)_c$ is equivalent to q_c. We will show that each $q^c(x)$ is consistent.

If $c \notin M$, then since the B_i are pairwise disjoint over M, there is a unique $i < \omega$ such that $c \in (B_i)_P$. Then $q_c(x)$ is contained in $p(x, x', B_i)$, which is consistent, and hence $q^c(x)$ is consistent as well.

Suppose $c \in M$, and assume for contradiction that $q^c(x)$ is inconsistent. By compactness and density of $\mathbb{M}_c$, there is some variable z from x and some $b_i \in B_i$ and $b_j' \in B_j$ for $i, j < \omega$ such that $b_i \leq b_j'$ in $\mathbb{M}_c$, but $q^c(x)$ entails $b_j' < z < b_i$. Let b and b' be the elements of B corresponding to b_i and b_j', respectively, and let a be the element of A corresponding to the variable z. Then $b' < a < b$, so by (2) there is some $m' \in M_O$ such that $b' < m' < b$ in $\mathbb{M}_c$. But since $B_i \equiv_M B_j \equiv_M B$, $b_j' < m' < b_i$, contradicting $b_i \leq b_j'$.

Since A is disjoint from B, each type $q^c(x)$ is nonalgebraic, so by Fact 4.1, $\bigcup_{c \in C_P} q_c(x)$ is consistent. By quantifier elimination, $q_O(x)$ is consistent.

Let A' realize $q_O(x)$. It remains to show that $q(A', x')$ is consistent. Each variable in x' is of type P. Since each atomic formula contains at most one variable of type P, it suffices to show that for each variable z in x', the set $r(z)$ of all atomic and negated atomic formulas from $q(A', x')$ involving the relation $<_z$ is consistent.

The type $r(z)$ specifies a linear order on $A'M_O$, which extends to a linear order on $A'M_O(B_i)_O$ for all $i < \omega$. Using the amalgamation property for linear orders, we can find a linear order on $A'C_O$ extending each of the given linear orders. By compactness and the extension axioms for the Fraïssé limit, we can find $c \in \mathbb{M}_P$ such that $<_c$ induces this linear order on $A'C_O$. This completes the proof.   □

Theorem 4.4. DLO_p *satisfies the New Kim's Lemma.*

Proof. Suppose $\varphi(x; b)$ Kim-divides over $M \models T$. To show that $\varphi(x; b)$ universally Kim-strictly divides, let $p(y)$ be a global Kim-strict M-invariant type extending $\operatorname{tp}(b/M)$, and let $I = (b_n)_{n<\omega}$ be a Morley sequence for p. Suppose for contradiction

that $\{\varphi(x; b_n) : n < \omega\}$ is consistent, realized by a. By Ramsey's theorem, compactness, and an automorphism, we may assume that $(b_n)_{n<\omega}$ is Ma-indiscernible. Now it suffices to show that $a \underset{M}{\overset{Kd}{\smile}} b_0$, since this will contradict the fact that the Kim-dividing formula $\varphi(x; b_0)$ is in $\mathrm{tp}(a/Mb_0)$.

Let A be the set enumerated by a, and let B_n be the set enumerated by b_n for all n. Since $B_1 \underset{M}{\overset{i}{\smile}} B_0$, $B_1 \cap B_0 \subseteq M$. If $c \in A \cap B_0$, then since $AB_0 \equiv_M AB_1$, also $c \in B_1$, so $c \in M$. Thus $A \cap B_0 \subseteq M$.

Now suppose $m \in M_P$ and $d_0 <_m c <_m d_0'$, with $c \in A_O \setminus M_O$ and $d_0, d_0' \in (B_0)_O \setminus M_O$. Suppose for contradiction that there is no $m' \in M_O$ such that $d_0 <_m m' <_m d_0'$. Let d_1 and d_1' be the elements of $(B_1)_O$ corresponding to d_0 and d_0' in $(B_0)_O$. Since $B_1 \equiv_{MA} B_0$, $d_1 <_m c <_m d_1'$, and there is no $m' \in M_O$ such that $d_1 <_m m' <_m d_1'$.

Since p is Kim-strict, $B_0 \underset{M}{\overset{Kd}{\smile}} B_1$ and $B_1 \underset{M}{\overset{Kd}{\smile}} B_0$. By Lemma 4.3, d_0, d_0', d_1, and d_1' are distinct, neither d_0 nor d_0' are in the $<_m$-interval (d_1, d_1'), and neither d_1 nor d_1' are in the $<_m$-interval (d_0, d_0'). It follows that the $<_m$-intervals (d_0, d_0') and (d_1, d_1') are disjoint. This contradicts the fact that c is in both of them.

So there is $m' \in M_O$ such that $d_0 <_m m' <_m d_0'$. By Lemma 4.3, $A \underset{M}{\overset{Kd}{\smile}} B_0$. $\square$

4B. *Bilinear forms over real closed fields.*

Let T_∞^{RCF} be the two-sorted theory of an infinite-dimensional vector space over a real closed field with a bilinear form, which is assumed to be either alternating and nondegenerate, or symmetric and positive-definite. This is really two theories, one for each type of bilinear form, but our arguments are identical in both cases so we will not notationally distinguish them. The language has a sort V for the vector space, equipped with the language of abelian groups, a sort R for the real closed field of scalars, equipped with the language of ordered rings, a function symbol $\cdot : R \times V \to V$ for scalar multiplication, and a function symbol $[-, -] : V \times V \to R$ for the bilinear form.

By [Granger 1999], T_∞^{RCF} is the model companion of the theory of a vector space over a real closed field with an alternating (or symmetric and positive-definite) bilinear form. By [Dobrowolski 2023], this theory additionally has quantifier-elimination in an expanded language, containing, for each n, a predicate I_n on V^n, such that $I_n(v_1, \ldots, v_n)$ holds if and only if $v_1, \ldots, v_n$ are linearly independent, as well as $(n+1)$-ary "coordinate functions" $F_{n,i} : V^{n+1} \to R$ for each $1 \leq i \leq n$. These functions are which are interpreted so that, if $v_1, \ldots, v_n$ are linearly independent and $w = \sum_{i=1}^n \alpha_i v_i$, then $F_{n,i}(\bar{v}, w) = \alpha_i$, and $F_{n,i}(\bar{v}, w) = 0$ otherwise.

When A is a subset of $\mathbb{M} \models T_\infty^{\mathrm{RCF}}$, we write A_R for the elements of the field sort and A_V for the elements of the vector space sort.

Remark 4.5. As a consequence of quantifier elimination and elementary linear algebra, the field sort R is stably embedded. More precisely, suppose C is a substructure of $\mathbb{M}$. If $\varphi(x)$ is a formula with parameters from C such that every variable is in the field sort R, then $\varphi(x)$ is equivalent to a formula $\psi(x)$ in the

language of ordered rings with parameters from C_R. Consequently, for any tuple a from $\mathbb{M}_R$ and any substructure C, $\mathrm{tp}_{\mathrm{RCF}}(a/C_R)$ entails $\mathrm{tp}(a/C)$.

If W is a set of vectors, we write $\langle W \rangle$ for the linear span of W with scalars from the field $\mathbb{M}_R$ (so $\langle W \rangle$ is a large set). By $\dim(W)$, we mean the dimension of $\langle W \rangle$ as a vector space over $\mathbb{M}_R$.

Suppose A, B, and C are substructures of $\mathbb{M}$. We write $A \underset{C}{\overset{\mathrm{RCF}}{\smile}} B$ to mean that A_R and B_R are forking-independent over C_R in the reduct of $\mathbb{M}_R$ to a model of RCF. We write $A \underset{C}{\overset{V}{\smile}} B$ to mean $\langle A_V \rangle \cap \langle B_V \rangle \subseteq \langle C_V \rangle$. Our goal is to show that T_∞^{RCF} satisfies New Kim's Lemma, which will involve characterizing $\overset{Kd}{\smile}$ in this theory in terms of $\overset{\mathrm{RCF}}{\smile}$ and $\overset{V}{\smile}$. The argument is the analogue of [Kaplan and Ramsey 2020, Proposition 9.37] (incorporating the corrections of [Dobrowolski 2023, Proposition 8.12]). A similar characterization of Kim-independence in the theory of a bilinear form on a vector space over an NSOP_1 field occurs in [Bossut 2023].

We begin with another general lemma which, in conjunction with Lemma 4.2, will allow us to upgrade a coheir sequence in $\mathrm{tp}_{\mathrm{RCF}}(B_R/M_R)$ in RCF to a coheir sequence in $\mathrm{tp}(B/M)$ in T_∞^{RCF}.

Lemma 4.6. *Suppose $L \subseteq L'$ are languages, T' is an L'-theory and $T = T' \upharpoonright L$. If $A \subseteq B$ and $I = (c_i)_{i<\omega}$ is a coheir sequence over A in T, then there is $I' \models \mathrm{tp}_L(I/A)$ which is a coheir sequence in T' over B.*

Proof. If $(c_i)_{i<\omega}$ is a coheir sequence over A in T, there is some ultrafilter $\mathcal{D}$ on A^n, where n is the length of c_0, such that I is a Morley sequence over A in the global A-finitely satisfiable type $\mathrm{Av}_L(\mathcal{D}, \mathbb{M})$. To see this, stretch I to $(c_i)_{i<\omega+1}$ and observe that the family of sets $\{\varphi(A; c_{<\omega}) : \varphi(x; c_{<\omega}) \in \mathrm{tp}(c_\omega/Ac_{<\omega})\} \subseteq \mathcal{P}(A^n)$ generates a filter and hence extends to an ultrafilter $\mathcal{D}$. It is easily checked that this $\mathcal{D}$ works. Let $\mathcal{E}$ be the ultrafilter on B^n induced by $\mathcal{D}$, i.e., a subset $X \subseteq B^n$ satisfies $X \in \mathcal{E}$ if and only if $X \cap A^n \in \mathcal{D}$. Then we can take I' to be Morley over B in the global B-finitely satisfiable type $\mathrm{Av}_{L'}(\mathcal{E}, \mathbb{M})$. $\square$

Lemma 4.7. *If $M \models T_\infty^{\mathrm{RCF}}$ and $A \underset{M}{\overset{Kd}{\smile}} B$, then $A \underset{M}{\overset{\mathrm{RCF}}{\smile}} B$.*

Proof. Because RCF is an NTP_2 theory, any dividing formula divides along some coheir sequence by [Chernikov and Kaplan 2012, Lemma 3.12]. So if $A \underset{M}{\overset{\mathrm{RCF}}{\not\smile}} B$, then there is a formula $\varphi(x; b)$ in $\mathrm{tp}_{\mathrm{RCF}}(A_R/M_R B_R)$ and a coheir sequence $I = (B_i)$ over M_R in $\mathrm{tp}_{\mathrm{RCF}}(B_R/M_R)$ such that $\varphi(x; b)$ divides along I. By Lemmas 4.6 and 4.2, there is a coheir sequence $I' = (B_i')_{i<\omega}$ over M in $\mathrm{tp}(B/M)$ such that $((B_i')_R)_{i<\omega} \equiv_{M_R}^{\mathrm{RCF}} I$. Then $\varphi(x; b)$ divides along I', and $A \underset{M}{\overset{Kd}{\not\smile}} B$. $\square$

Lemma 4.8. *Suppose $M \models T_\infty^{\mathrm{RCF}}$:*

(1) *If $A \underset{M}{\overset{u}{\smile}} B$, then $A \underset{M}{\overset{V}{\smile}} B$.*

(2) *If $(B_i)_{i<\omega}$ is a $\underset{M}{\overset{V}{\bigsmile}}$-independent sequence (i.e., $B_i \underset{M}{\overset{V}{\bigsmile}} B_0 \cdots B_{i-1}$ for all $i < \omega$), and there exists A' such that $A'B_i \equiv_M AB$ for all $i < \omega$, then $A \underset{M}{\overset{V}{\bigsmile}} B$.*

(3) *If $A \underset{M}{\overset{Kd}{\bigsmile}} B$, then $A \underset{M}{\overset{V}{\bigsmile}} B$.*

Proof. Suppose that $A \underset{M}{\overset{V}{\not\bigsmile}} B$. Then $\langle A_V \rangle \cap \langle B_V \rangle \not\subseteq \langle M_V \rangle$, so there exists a vector v, a finite linearly independent tuple a from A_V, and a finite linearly independent tuple b from B_V such that $v \in \langle a \rangle \cap \langle b \rangle$ and $v \notin \langle M_V \rangle$. Let $C = \langle b \rangle \cap \langle M_V \rangle$, and note that C is a subspace of the finite-dimensional space $\langle b \rangle$. Let c be a finite basis for C. Note that the formula $\varphi(x; b, c)$:

$$\exists w (I_{|a|}(x) \wedge \neg I_{|a|+1}(x, w) \wedge \neg I_{|b|+1}(b, w) \wedge I_{|c|+1}(c, w)),$$

which asserts that x is linearly independent and $\langle x \rangle \cap \langle b \rangle \not\subseteq \langle c \rangle$, is in $\mathrm{tp}(a/Mb)$. With the above notation set, we now prove (1) and (2).

For (1), assume for contradiction that $A \underset{M}{\overset{u}{\bigsmile}} B$. Since $\mathrm{tp}(a/Mb)$ is finitely satisfiable in M, there is some $a' \in M_V$ satisfying $\varphi(a'; b, c)$. Let w' be the witness to the existential quantifier. Then $w' \in \langle a' \rangle \subseteq \langle M_V \rangle$ and $w' \in \langle b \rangle$, so $w' \in \langle b \rangle \cap \langle M_V \rangle = C$. But $w' \notin \langle c \rangle$, contradiction.

For (2), assume for contradiction that there exists a $\underset{M}{\overset{V}{\bigsmile}}$-independent sequence $(B_i)_{i<\omega}$ and A' such that $A'B_i \equiv_M AB$ for all $i < \omega$. Let $(b_i)_{i<\omega}$ be the restriction of this sequence to the tuples b_i from B_i corresponding to the tuple b in B, and let a' be the tuple from A' corresponding to the tuple a in A. Let $k = \dim(\langle a' \rangle) = |a'|$, and let $v_0, \ldots, v_k$ be such that $v_i \in \langle a' \rangle \cap \langle b_i \rangle \setminus \langle c \rangle$ for all $i < k+1$. Since these $k+1$ vectors are all in $\langle a' \rangle$, they are not linearly independent, and we can write one of them, say v_j, as a linear combination of $v_0, \ldots, v_{j-1}$. Then $v_j \in \langle b_j \rangle \cap \langle b_0, \ldots, b_{j-1} \rangle \setminus \langle c \rangle$. But since $b_j \underset{M}{\overset{V}{\bigsmile}} b_0 \cdots b_{j-1}$, $\langle b_j \rangle \cap \langle b_0, \ldots, b_{j-1} \rangle \subseteq \langle b_j \rangle \cap \langle M_V \rangle = \langle c \rangle$, contradiction.

For (3), let $(B_i)_{i<\omega}$ be a coheir sequence in $\mathrm{tp}(B/M)$. Since $A \underset{M}{\overset{Kd}{\bigsmile}} B$, by compactness there exists A' such that $A'B_i \equiv_M AB$ for all $i < \omega$. By (1), $(B_i)_{i<\omega}$ is a $\underset{M}{\overset{V}{\bigsmile}}$-independent sequence, and by (2), $A \underset{M}{\overset{V}{\bigsmile}} B$. $\qquad\square$

Theorem 4.9. *If $M \models T_\infty^{\mathrm{RCF}}$, $A = \mathrm{acl}(AM)$, $B = \mathrm{acl}(BM)$, then $A \underset{M}{\overset{Kd}{\bigsmile}} B$ if and only if $A \underset{M}{\overset{\mathrm{RCF}}{\bigsmile}} B$ and $A \underset{M}{\overset{V}{\bigsmile}} B$.*

Proof. One direction is Lemmas 4.7 and 4.8(3).

In the other direction, suppose that $A \underset{M}{\overset{\mathrm{RCF}}{\bigsmile}} B$ and $A \underset{M}{\overset{V}{\bigsmile}} B$. Let $(B_i)_{i<\omega}$ be a Morley sequence over M for a global M-invariant type extending $\mathrm{tp}(B/M)$. Since $A \underset{M}{\overset{\mathrm{RCF}}{\bigsmile}} B$, we can find A'_R such that $A'_R(B_i)_R \equiv_{M_R}^{\mathrm{RCF}} A_R B_R$ for all $i < \omega$. By Remark 4.5, $A'_R B_i \equiv_M A_R B$ for all $i < \omega$. Let $\tilde{R}$ be the field $(\mathrm{acl}(A'_R(B_i)_{i<\omega}))_R$.

Let $\bar{m} = (m_i)_{i<\alpha}$ be a tuple from M_V which is a basis of $\langle M_V \rangle$. Choose $\bar{a} = (a_i)_{i<\beta}$ from A_V such that $\bar{a}\bar{m}$ is a basis of $\langle A_V \rangle$ and choose $\bar{b}_i = (b_{i,j})_{j<\gamma}$ from $(B_i)_V$ such that $\bar{m}\bar{b}_i$ is a basis of $\langle (B_i)_V \rangle$. Since $(B_i)_{i<\omega}$ is a $\underset{M}{\overset{i}{\bigsmile}}$-independent sequence, by Lemma 4.8(3) it is also a $\underset{M}{\overset{V}{\bigsmile}}$-independent sequence. Thus $\bar{m}$ and

$(\bar{b}_i)_{i<\omega}$ are linearly independent. Let $\tilde{V} = \langle \bar{m}(\bar{b}_i)_{i<\omega} \rangle_{\tilde{R}}$, the vector space over $\tilde{R}$ spanned by this basis. Note that, unlike $\langle \bar{m}(\bar{b}_i)_{i<\omega} \rangle$, this is a small set, and it contains $(B_i)_{i<\omega}$, since $\tilde{R}$ contains the values of the coordinate functions $F_{n,i}$ on tuples from $(B_i)_{i<\omega}$. Let $\tilde{N}$ be the substructure of $\mathbb{M}$ with $\tilde{N}_R = \tilde{R}$ and $\tilde{N}_V = \tilde{V}$. Note that if we give the symbols θ_n and $F_{n,i}$ their intended interpretations in $\tilde{N}$, they agree with the interpretations of these symbols in $\mathbb{M}$.

Let $\bar{a}' = (a'_i)_{i<\beta}$ be a tuple of new vectors (not in $\mathbb{M}_V$) of the same length as $\bar{a}$. Let W be the $\tilde{R}$-vector space extending $\tilde{V}$ with basis $\bar{a}'$, $\bar{m}$, and $(\bar{b}_i)_{i<\omega}$. We build a structure N extending $\tilde{N}$ with $N_R = \tilde{N}_R = \tilde{R}$ and $N_V = W$. The field structure and vector space structure have been determined, so it remains to define the bilinear form $[-,-]^N$. To do this, it suffices to define the form on every pair of basis vectors for W such that at least one comes from $\bar{a}'$, and extend linearly.

For all $i < \alpha$, $i' < \beta$, $j' < \beta$, $j < \omega$, and $k < \gamma$, set

$$[a'_{i'}, a'_{j'}]^N = [a_{i'}, a_{j'}]^{\mathbb{M}}, \quad [a'_{i'}, m_i]^N = [a_{i'}, m_i]^{\mathbb{M}}, \quad [a'_{i'}, b_{j,k}]^N = [a_i, b_{0,k}]^{\mathbb{M}}.$$

These conditions uniquely determine a bilinear form on all pairs of vectors from W, which is alternating or symmetric and positive-definite, as required by T_∞^{RCF}. We can extend the language to include the θ_n and $F_{n,i}$ in the natural way, and the interpretations of these symbols agree with those on $\tilde{N}$, since $\tilde{N}_R = N_R$.

Now we can embed N into $\mathbb{M}$ over $\tilde{N}$. Let A'_V be the image under this embedding of the subset of N corresponding to A_V, and let $A' = (A'_R, A'_V)$. It follows by construction and quantifier elimination that $A'B_i \equiv_M AB$ for all $i < \omega$. Thus $A \underset{M}{\overset{Kd}{\smile}} B$. □

Theorem 4.10. *The theory T_∞^{RCF} satisfies New Kim's Lemma.*

Proof. Let $M \models T_\infty^{\mathrm{RCF}}$ and suppose $\varphi(x; b)$ Kim-divides over M. Let $I = (b_i)_{i<\omega}$ be a Morley sequence over M for a global Kim-strict M-invariant type $q(y) \supseteq \mathrm{tp}(b/M)$. We would like to show that $\varphi(x; b_i)$ divides along I. Assume, towards contradiction, that there exists a realizing $\{\varphi(x; b_i) : i < \omega\}$. By Ramsey's theorem, compactness, and an automorphism, we may assume that $(b_i)_{i<\omega}$ is indiscernible over $A = \mathrm{acl}(Ma)$. For each $i < \omega$, let $B_i = \mathrm{acl}(Mb_i)$, with each B_i enumerated in such a way that $(B_i)_{i<\omega}$ remains indiscernible over A.

Since $(b_i)_{i<\omega}$ is a $\underset{M}{\overset{i}{\smile}}$-independent sequence, it is a $\underset{M}{\overset{K}{\smile}}$-independent sequence, and thus $(B_i)_{i<\omega}$ is a $\underset{M}{\overset{K}{\smile}}$-independent sequence. By Lemma 4.8(3), $(B_i)_{i<\omega}$ is a $\underset{M}{\overset{V}{\smile}}$-independent sequence, and since $AB_i \equiv_M AB_0$ for all $i < \omega$, $A \underset{M}{\overset{V}{\smile}} B_0$ by Lemma 4.8(2).

We now claim that $((B_i)_R)_{i<\omega}$ is a (Kim-)strict Morley sequence over M_R in RCF. Let N be an $|M|^+$-saturated model containing M and $(B_i)_{i<\omega}$. Let b_ω realize $q|_N$, and let $B_\omega = \mathrm{acl}(Mb_\omega)$. Since $(b_i)_{i\leq\omega}$ is a Morley sequence over M, and

hence M-indiscernible, and $(B_i)_{i<\omega}$ is M-indiscernible, we can enumerate B_ω in such a way that $(B_i)_{i\leq\omega}$ remains M-indiscernible.

Since q is Kim-strict, $b_\omega \underset{M}{\overset{K}{\bigcup}} N$ and $N \underset{M}{\overset{K}{\bigcup}} b_\omega$, so $B_\omega \underset{M}{\overset{K}{\bigcup}} N$ and $N \underset{M}{\overset{K}{\bigcup}} B_\omega$, and hence $B_\omega \underset{M}{\overset{\text{RCF}}{\bigcup}} N$ and $N \underset{M}{\overset{\text{RCF}}{\bigcup}} B_\omega$, by Theorem 4.9. Since RCF is an NIP theory, $\underset{M}{\overset{i}{\bigcup}} = \underset{M}{\overset{f}{\bigcup}}$ in RCF; see [Simon 2015, Corollary 5.22]. Thus $\mathrm{tp}_{\text{RCF}}((B_\omega)_R/N_R)$ extends to a global M_R-invariant type q_* which is strict over M_R in RCF. Indeed, suppose for contradiction that $C_R \subseteq \mathbb{M}_R$, $B'_R \models q_*|_{N_R C_R}$, and $C_R \underset{M_R}{\overset{f}{\not\bigcup}} B'_R$ in RCF. Then $c \underset{M_R}{\overset{f}{\not\bigcup}} B'_R$ for some finite tuple c from C_R, whose type over M_R is realized by $c' \in N_R$. Then $c' \underset{M_R}{\overset{f}{\not\bigcup}} B'_R$ in RCF by invariance of q_*, contradicting $N \underset{M}{\overset{\text{RCF}}{\bigcup}} B_\omega$.

By M-indiscernibility of $(B_i)_{i\leq\omega}$, $(B_i)_R \models q_*|_{M_R(B_{<i})_R}$ for all i, so $((B_i)_R)_{i<\omega}$ is a strict Morley sequence over M_R in RCF. Since $((B_i)_R)_{i<\omega}$ is A_R-indiscernible, it follows that $A \underset{M}{\overset{\text{RCF}}{\bigcup}} B_0$ by the NTP$_2$ Kim's Lemma (Theorem 3.5).

Since $A \underset{M}{\overset{\text{RCF}}{\bigcup}} B_0$ and $A \underset{M}{\overset{V}{\bigcup}} B_0$, by Theorem 4.9, $A \underset{M}{\overset{Kd}{\bigcup}} B_0$. This contradicts the fact that $\mathrm{tp}(A/MB_0)$ contains the formula $\varphi(x; b_0)$, which Kim-divides over M, since $b_0 \equiv_M b$. □

4C. *Nonexample: the Henson graph.*

The Henson graph, or generic triangle-free graph, is the Fraïssé limit of the class of finite triangle-free graphs. Its complete theory $T_\triangle$ is SOP$_3$ and NSOP$_4$. Conant [2017] analyzed forking and dividing in $T_\triangle$ in detail. We will use the following characterization of $\underset{}{\overset{f}{\bigcup}}$.

Fact 4.11 [Conant 2017, Theorem 5.3]. Suppose that A and B are sets in $\mathbb{M} \models T_\triangle$ and $M \models T_\triangle$. Then $A \underset{M}{\overset{f}{\bigcup}} B$ if and only if $A \cap B \subseteq M$ and for all $a \in A$ and $b \neq c \in B \setminus M$, if aRb and aRc, then there exists $m \in M$ such that mRb and mRc.

We will show that a very weak variant of Kim's Lemma fails in $T_\triangle$: strict dividing does not imply universal strict dividing. Since strict dividing implies Kim-dividing and universal Kim-strict dividing implies universal strict dividing, it follows that $T_\triangle$ fails to satisfy New Kim's Lemma.

Theorem 4.12. *Modulo $T_\triangle$, there is a formula which strictly divides but does not universally strictly divide. Thus $T_\triangle$ does not satisfy New Kim's Lemma.*

Proof. Let $M \models T$. Let b and c be elements of $\mathbb{M} \setminus M$ with $\neg bRc$, such that b has a single neighbor in M, call it m, and c has no neighbors in M.[6] Consider the formula $\varphi(x; b, c)$: $xRb \wedge xRc$. It suffices to find two strict global M-invariant types $p(y, z)$ and $q(y, z)$ extending $\mathrm{tp}(b, c/M)$ such that $\varphi(x; b, c)$ divides along Morley sequences for p but does not divide along Morley sequences for q.

Let $p(y, z)$ extend $\mathrm{tp}(bc/M)$ by including, for each $d \in \mathbb{M} \setminus M$, $y \neq d$, $z \neq d$ and $\neg yRd$. Additionally, include zRd if $d \models \mathrm{tp}(b/M)$ and $\neg zRd$ otherwise. We claim this defines a consistent partial type. Any inconsistency would come from a

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[6]Really, all we will use is that the set of neighbors of b in M is nonempty and disjoint from the set of neighbors of c in M.

triangle involving the variables and elements of $\mathbb{M}$. Such a triangle cannot contain y, since $\neg y R z$ and y has an edge to exactly one element of $\mathbb{M}$, namely m. Since z only has edges to realizations of $\mathrm{tp}(b/M)$, any triangle containing z contains two realizations of $\mathrm{tp}(b/M)$. But no two realizations of $\mathrm{tp}(b/M)$ are adjacent, since they are both adjacent to m.

By quantifier elimination, this partial type determines a complete M-invariant type over $\mathbb{M}$. Letting $I = (b_i, c_i)_{i<\omega}$ be a Morley sequence for p over M, $\varphi(x; b, c)$ divides along I, since $\{\varphi(x; b_1, c_1), \varphi(x; b_2, c_2)\}$ entails $\{x R b_1, x R c_2\}$, and $b_1 R c_2$.

Now let $q(y, z)$ extend $\mathrm{tp}(bc/M)$ by including, for each $d \in \mathbb{M} \setminus M$, $y \neq d$, $z \neq d$, $\neg y R d$, and $\neg z R d$. This defines a consistent partial type, since the only edge from a variable to an element of $\mathbb{M}$ is the single edge from y to m. Again, by quantifier elimination, this determines a complete M-invariant type over $\mathbb{M}$. And if $J = (b_i, c_i)_{i<\omega}$ is a Morley sequence for q over M, then $\varphi(x; b, c)$ does not divide along J. Indeed, since there are no edges among the vertices $\{b_i, c_i : i < \omega\}$, $\{\varphi(x; b_i, c_i) : i < \omega\}$ does not induce any triangles.

It remains to show that both p and q are strict. Let $A \subseteq \mathbb{M}$, $b_0, c_0 \models p|_{MA}$, and $b_1, c_1 \models q|_{MA}$. We would like to show that for $i \in \{0, 1\}$, $A \underset{M}{\overset{f}{\downarrow}} b_i c_i$. In each case, $A \cap \{b_i, c_i\} = \varnothing \subseteq M$, and there is no $a \in A$ such that $a R b_i$ and $a R c_i$ (since b_i is not adjacent to any element of $A \setminus M$, and c_i is not adjacent to any element of M). By Fact 4.11, $A \underset{M}{\overset{f}{\downarrow}} b_i c_i$. □

5. Syntax

In this section, we isolate a tree property, provisionally called BTP, which generalizes TP_2 and SOP_1, and we show that NBTP theories satisfy New Kim's Lemma. We also show that NBTP theories are NATP. We have not succeeded in proving that New Kim's Lemma characterizes NBTP theories.

For ordinals $\alpha, \beta \leq \omega$, write $\alpha_*^{<\beta}$ for the forest obtained by removing the root from $\alpha^{<\beta}$:

- A *left-leaning path* in $\alpha_*^{<\beta}$ is a sequence (λ_n) such that if $\lambda_n = \eta^\frown\langle i\rangle$, then $\eta^\frown\langle j\rangle \vartriangleleft \lambda_{n+1}$ for some $j \leq i$.

- A *right-veering path* in $\alpha_*^{<\beta}$ is a sequence (ρ_n) such that if $\rho_n = \eta^\frown\langle i\rangle$, then $\eta^\frown\langle j\rangle \trianglelefteq \rho_{n+1}$ for some $j > i$.

Note that to get to the next element in a left-leaning path, one *optionally* moves leftward to a sibling and then moves *strictly* upward to a descendent, while in a right-veering path, one moves *strictly* rightward to a sibling, and then *optionally* moves upward to a descendent.

Definition 5.1. A formula $\varphi(x; y)$ has k-BTP (k-*bizarre tree property*) with $k < \omega$ if there exists a forest of tuples $(a_\eta)_{\eta \in \omega_*^{<\omega}}$ satisfying the following conditions:

- For every left-leaning path $(\lambda_n)_{n<\omega}$, $\{\varphi(x; a_{\lambda_n}) : n < \omega\}$ is consistent.
- For every right-veering path $(\rho_n)_{n\in\omega}$, $\{\varphi(x; a_{\rho_n}) : n < \omega\}$ is k-inconsistent.

A theory T has BTP if there is some formula $\varphi(x; y)$ and some $k < \omega$ such that φ has k-BTP. Otherwise, T is NBTP.

Theorem 5.2. *Suppose T is NBTP. Then T satisfies New Kim's Lemma.*

Proof. We prove the contrapositive. If New Kim's Lemma fails, then we have a formula $\varphi(x; b)$, a model $M \models T$, and global M-invariant types $p(y)$ and $q(y)$ extending $\mathrm{tp}(b/M)$ such that $p(y)$ is Kim-strict and $\varphi(x; b)$ divides along Morley sequences for q but not along Morley sequences for p. Fix $k < \omega$ such that if $(b_i)_{i<\omega}$ is a Morley sequence for q, then $\{\varphi(x; b_i) : i < \omega\}$ is k-inconsistent.

For arbitrary m and n in ω, we will build a finite forest $(a_\eta)_{\eta \in m_*^{<n}}$ such that:

- For every left-leaning path $(\lambda_i)_{1\leq i\leq \ell}$ in $m_*^{<n}$, $(a_{\lambda_\ell}, \ldots, a_{\lambda_1})$ starts a Morley sequence for p over M, and hence $\{\varphi(x; a_{\lambda_i}) : 1 \leq i \leq \ell\}$ is consistent.
- For every right-veering path $(\rho_i)_{1\leq i\leq \ell}$ in $m_*^{<n}$, $(a_{\rho_\ell}, \ldots, a_{\rho_1})$ starts a Morley sequence for q over M, and hence $\{\varphi(x; a_{\lambda_i}) : 1 \leq i \leq \ell\}$ is k-inconsistent.

By compactness, this will suffice to show that $\varphi(x; y)$ has k-BTP.

Fix $m < \omega$ with $m > 0$, and proceed by induction on n. The base cases $n = 0$ and $n = 1$ are trivial, since $m_*^{<n}$ is empty.

Suppose we are given $F_0 = (a_\eta)_{\eta \in m_*^{<n}}$ satisfying the induction hypothesis. Let b_0 realize $p|_{MF_0}$. Since p is Kim-strict, $F_0 \underset{M}{\overset{K}{\smile}} b_0$. Let $r(z, y) = \mathrm{tp}(F_0 b_0/M)$.

By induction on $1 \leq \ell \leq m$, we now find $(b_i, F_i)_{i<\ell}$ such that:

(1) $F_i \equiv_M F_0$ for all $i < \ell$.

(2) b_i realizes $p|_{MF_j}$ if $i \leq j$.

(3) $(b_i, b_{i+1}, \ldots, b_{\ell-1})$ starts a Morley sequence in q over MF_j if $i > j$.

In the base case $\ell = 1$, b_0 and F_0 satisfy the conditions.

Given $(b_i, F_i)_{i<\ell}$ satisfying (1)–(3) for $\ell < m$, let b_ℓ realize $q|_{M(b_i, F_i)_{i<\ell}}$. Then (3) is satisfied for $\ell+1$. Since $r(z, b_0) = \mathrm{tp}(F_0/Mb_0)$ does not Kim-divide over M and $(b_i)_{i<\ell+1}$ starts a Morley sequence for a global M-invariant type, $\bigcup_{i<\ell+1} r(z, b_i)$ is consistent. Let F_ℓ realize this type. Then (1) is satisfied for $\ell+1$. Now since $r(F_0, z) = \mathrm{tp}(b_0/MF_0) = p|_{MF_0}$, p is M-invariant, and $F_\ell \equiv_M F_0$, we have, for all $i < \ell+1$, $\mathrm{tp}(b_i/MF_\ell) = r(F_\ell, z) = p|_{MF_\ell}$, and thus (2) is satisfied for $\ell+1$.

Having constructed $(b_i, F_i)_{i<m}$, we reindex to define the forest $(a'_\eta)_{\eta \in m_*^{<n+1}}$. By (1), we can write $F_i = (a_\eta^i)_{\eta \in m_*^{<n}}$, and each F_i satisfies the induction hypothesis. Set $a'_{\langle i \rangle} = b_{m-i-1}$, and $a'_{\langle i \rangle \frown \eta} = a_\eta^{m-i-1}$. Note that the reindexing by $(m - i - 1)$ means that our sequence $(b_i, F_i)_{i<m}$ proceeds leftward in the new forest.

A left-leaning path in the new forest begins with at most one element b_i at the bottom level and is followed by some left-leaning path in F_j with $i \leq j$. By (2)

and induction, the reverse sequence starts a Morley sequence for p over M. A right-veering path in the new forest may begin with elements $b_{i_1}, \ldots, b_{i_\ell}$ at the bottom level, with $i_1 > \cdots > i_\ell$, and is followed by a right-veering path in some F_j with $i_\ell > j$. By (3) and induction, the reverse sequence starts a Morley sequence for q over M. □

We now situate NBTP relative to the other tree properties.

Proposition 5.3. *If T is* NTP$_2$, *then T is* NBTP.

Proof. Assume $\varphi(x; y)$ has k-BTP, witnessed by $(a_\eta)_{\eta \in \omega_*^{<\omega}}$. Consider the array $(b_{i,j})_{i,j<\omega}$ with $b_{i,j} = a_{(0^i)^\frown \langle j \rangle}$, where 0^i denotes the string of length i consisting of all 0's.

For all $f : \omega \to \omega$, the sequence $(\lambda_i)_{i<\omega}$ with $\lambda_i = (0^i)^\frown \langle f(i) \rangle$ is a left-leaning path. So $\{\varphi(x; b_{i,f(i)}) : i < \omega\} = \{\varphi(x; a_{\lambda_i}) : i < \omega\}$ is consistent.

For all $i < \omega$, the sequence $(\rho_j)_{j<\omega}$ with $\rho_j = (0^i)^\frown \langle j \rangle$ is a right-veering path. So $\{\varphi(x; b_{i,j}) : j < \omega\} = \{\varphi(x; a_{\rho_j}) : j < \omega\}$ is k-inconsistent.

Thus $\varphi(x; y)$ has TP$_2$. □

When $k > 2$, a witness to k-BTP does not directly contain a witness to SOP$_1$, but rather a variant of SOP$_1$ with k-inconsistency instead of 2-inconsistency. So for the implication from NSOP$_1$ to NBTP, we will use the following alternative characterization of SOP$_1$ from [Kaplan and Ramsey 2020].

Fact 5.4 [Kaplan and Ramsey 2020, Proposition 2.4]. *T has* SOP$_1$ *if and only if there exists $k < \omega$ and an array $(c_{i,j})_{i<\omega, j<2}$ such that:*

- $c_{n,0} \equiv_{(c_{i,j})_{i<n, j<2}} c_{n,1}$ *for all $n < \omega$.*
- $\{\varphi(x; c_{i,0}) : i < \omega\}$ *is consistent.*
- $\{\varphi(x; c_{i,1}) : i < \omega\}$ *is k-inconsistent.*

Proposition 5.5. *If T is* NSOP$_1$, *then T is* NBTP.

Proof. Assume $\varphi(x; y)$ has k-BTP, witnessed by $(a_\eta)_{\eta \in \omega_*^{<\omega}}$. Consider the binary subtree $(b_\eta)_{\eta \in 2^{<\omega}}$ with $b_\eta = a_{\langle 0 \rangle^\frown \eta}$. This tree does not witness SOP$_1$, but it does have the following properties, which will be sufficient to obtain SOP$_1$:

- For any $\rho \in 2^\omega$, $\{\varphi(x; b_{\rho \upharpoonright n}) : n < \omega\}$ is consistent (since the corresponding sequence in our original forest is a left-leaning path).
- For any $\mu_1, \ldots, \mu_k \in 2^{<\omega}$ such that $\mu_i^\frown \langle 1 \rangle \trianglelefteq \mu_{i+1}$ for all $1 \le i < k$, $\{\varphi(x; b_{\mu_i^\frown \langle 0 \rangle}) : 1 \le i \le k\}$ is inconsistent (since the corresponding sequence in our original forest is a right-veering path of length k).

By compactness, we can obtain a tree $(b_\eta)_{\eta \in 2^{<\kappa}}$, where $\kappa > |S_y(T)|$, which satisfies the obvious extensions of the two properties above.

Following the proof of [Chernikov and Ramsey 2016, Proposition 5.2], we define $(\eta_i, v_i)_{i<\kappa}$ in $2^{<\kappa}$ by recursion. Given $(\eta_i, v_i)_{i<n}$ (and setting $\eta_{-1} = \langle \rangle$ when $n = 0$), let $\mu_\alpha = \eta_{n-1}{}^\frown (1^\alpha){}^\frown \langle 0 \rangle$ for all $\alpha < \kappa$. Since $\kappa > |S_y(T)|$, there are $\alpha < \beta < \kappa$ such that b_{μ_α} and b_{μ_β} have the same type over $(b_{\eta_i}, b_{v_i})_{i<n}$. Let $v_n = \mu_\alpha$ and $\eta_n = \mu_\beta$. Directly from the construction, we have the following properties:

(1) $b_{\eta_n} \equiv_{(b_{\eta_i}, b_{v_i})_{i<n}} b_{v_n}$ for all n.

(2) If $i < j$, then $\eta_i \trianglelefteq \eta_j, v_j$.

(3) For all i, $(\eta_i \wedge v_i){}^\frown \langle 1 \rangle \trianglelefteq \eta_i$ and $(\eta_i \wedge v_i){}^\frown \langle 0 \rangle = v_i$.

Now, in the statement of Fact 5.4, set $c_{i,0} = b_{\eta_i}$ and $c_{i,1} = b_{v_i}$ for all $i < \omega$. We have $c_{n,0} \equiv_{(c_{i,j})_{i<n, j<2}} c_{n,1}$ by (1). Since $(\eta_i)_{i<\omega}$ is a chain in $2^{<\kappa}$ by (2), $\{\varphi(x; c_{i,0}) : i < \omega\} = \{\varphi(x; b_{\eta_i}) : i < \omega\}$ is consistent. And setting $\mu_i = (\eta_i \wedge v_i)$ for all i, note that by (2) and (3), $v_i = \mu_i{}^\frown \langle 0 \rangle$, and $\mu_i{}^\frown \langle 1 \rangle \trianglelefteq \eta_i \trianglelefteq (\eta_j \wedge v_j) = \mu_j$ when $i < j$. So $\{\varphi(x; c_{i,1}) : i < \omega\} = \{\varphi(x; b_{\mu_i{}^\frown \langle 0 \rangle}) : i < \omega\}$ is k-inconsistent. Thus T has SOP$_1$. $\qquad\qquad\square$

Proposition 5.6. *If T is NBTP, then T is NATP.*

Proof. Assume $\varphi(x; y)$ has ATP, witnessed by $(a_\eta)_{\eta \in 2^{<\omega}}$.

Define a map $e \colon \omega^{<\omega} \to 2^{<\omega}$ by recursion on the length of the input sequence:

$$e(\langle \rangle) = \langle \rangle,$$
$$e(\eta{}^\frown \langle i \rangle) = e(\eta){}^\frown \langle 0 \rangle{}^\frown (1^{2i}).$$

Note that if $\eta \trianglelefteq v$, then $e(\eta) \trianglelefteq e(v)$.

Now define $f \colon \omega^{<\omega} \to 2^{<\omega}$ by $f(\eta) = e(\eta){}^\frown \langle 1 \rangle$, and consider the tree $(b_\eta)_{\eta \in \omega_*^{<\omega}}$ with $b_\eta = a_{f(\eta)}$.

If $(\lambda_n)_{n<\omega}$ is a left-leaning path, we claim that $\{f(\lambda_n) : n < \omega\}$ is an antichain in $2^{<\omega}$, and hence $\{\varphi(x; b_{\lambda_n}) : n < \omega\} = \{\varphi(x; a_{f(\lambda_n)}) : n < \omega\}$ is consistent.

So fix $n < m$ in ω. Writing $\lambda_n = \eta{}^\frown \langle i \rangle$, we have $\eta{}^\frown \langle j \rangle \triangleleft \lambda_{n+1}$ for some $j \leq i$. Now if $\eta{}^\frown \langle j \rangle \triangleleft v$, then also $\eta{}^\frown \langle j \rangle \triangleleft v'$ whenever v' is a descendent of v or a descendent of a leftward sibling of v. Since $(n+1) \leq m$, it follows that $\eta{}^\frown \langle j \rangle \triangleleft \lambda_m$. Let $j' < \omega$ be such that $\eta{}^\frown \langle j \rangle{}^\frown \langle j' \rangle \trianglelefteq \lambda_m$.

Now $f(\lambda_n) = e(\eta{}^\frown \langle i \rangle){}^\frown \langle 1 \rangle = e(\eta){}^\frown \langle 0 \rangle{}^\frown (1^{2i+1})$. On the other hand, $f(\lambda_m) = e(\lambda_m){}^\frown \langle 1 \rangle$ has as an initial segment $e(\eta{}^\frown \langle j \rangle{}^\frown \langle j' \rangle) = e(\eta){}^\frown \langle 0 \rangle{}^\frown (1^{2j}){}^\frown \langle 0 \rangle{}^\frown (1^{2j'})$. Since $2i + 1 \neq 2j$, $f(\lambda_n) \perp f(\lambda_m)$, as desired.

If $(\rho_n)_{n<\omega}$ is a right-veering path, we claim that $f(\rho_n) \trianglelefteq f(\rho_{n+1})$ for all $n < \omega$. From this, it follows that the values $\{f(\rho_n) : n < \omega\}$ are pairwise comparable, and hence $\{\varphi(x; b_{\rho_n}) : n < \omega\} = \{\varphi(x; a_{f(\rho_n)}) : n < \omega\}$ is 2-inconsistent.

So fix $n < \omega$. Writing $\rho_n = \eta^\frown \langle i \rangle$, we have $\eta^\frown \langle j \rangle \trianglelefteq \rho_{n+1}$ for some $j > i$. Now $f(\rho_n) = e(\eta^\frown \langle i \rangle)^\frown \langle 1 \rangle = e(\eta)^\frown \langle 0 \rangle^\frown \langle 1^{2i+1} \rangle$. On the other hand, $f(\rho_{n+1}) = e(\rho_{n+1})^\frown \langle 1 \rangle$ has as an initial segment $e(\eta^\frown \langle j \rangle) = e(\eta)^\frown \langle 0 \rangle^\frown \langle 1^{2j} \rangle$. Since $2i + 1 < 2j$, $f(\rho_n) \trianglelefteq f(\rho_{n+1})$, as desired.

Thus $\varphi(x; y)$ has 2-BTP. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

6. Questions

We have left open several natural directions for future work. In our view, the main problem is to find a syntactic characterization of the theories satisfying New Kim's Lemma. We have shown that NBTP implies New Kim's Lemma, but it is open whether this implication reverses. No implication in either direction is known between New Kim's Lemma and NATP. In light of Hanson's preprint [Hanson 2023], we are also interested in the relationship between New Kim's Lemma and the property NCTP explored there.

Question 6.1. Is New Kim's Lemma equivalent to one or more of the syntactic properties NATP, NBTP, or NCTP?

However, it is conceivable that there simply is no syntactic property that characterizes New Kim's Lemma. One way of making this precise is to recall the following very general definition, due to Shelah.

Definition 6.2 [Shelah 2000, Definition 5.17]. • For $n < \omega$, an *n-code* (for a partial type) is a pair $A = (A_+, A_-)$ of disjoint subsets of $[n] = \{0, \ldots, n-1\}$. Given a formula $\varphi(x; y)$ and tuples $a_0, \ldots, a_{n-1} \in \mathbb{M}^y$, the partial type coded by $A = (A_+, A_-)$ is

$$q_A(x) = \{\varphi(x; a_i) : i \in A_+\} \cup \{\neg\varphi(x; a_i) : i \in A_-\}.$$

- For $n < \omega$, an *n-pattern* (of consistency and inconsistency) is a pair (C, I) of disjoint sets of *n*-codes. A *finite pattern* is an *n*-pattern for some $n < \omega$. We say that a formula $\varphi(x; y)$ *exhibits* the *n*-pattern (C, I) if there are tuples $a_0, \ldots, a_{n-1} \in \mathbb{M}^y$ such that for every code $A \in C$, $q_A(x)$ is consistent, and for every code $A \in I$, $q_A(x)$ is inconsistent.

- A property of formulas P is *definable by patterns* if there is a set $\mathcal{F}$ of finite patterns such that $\varphi(x; y)$ has property P if and only if $\varphi(x; y)$ exhibits every pattern in $\mathcal{F}$.

- A property Q of theories is *definable by patterns* if there is a property P of formulas which is defined by patterns, and T has property Q if and only if there is some formula $\varphi(x; y)$ which has property P.[7]

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[7] Shelah calls a property of theories which is definable by patterns "weakly simply high straight". This is a special case of a related notion that Shelah calls "straightly defined".

Each of the properties TP, TP$_1$, TP$_2$, SOP$_1$, ATP, and BTP considered in this paper are definable by patterns: let $\mathcal{F}$ consist of one pattern for each finite subset of the infinite pattern of consistency and inconsistency defining the property, and apply compactness.

Question 6.3. Is the class of theories in which New Kim's Lemma fails definable by patterns?

It would be nice to have a larger stock of examples of theories satisfying New Kim's Lemma. To this end, we would like it to be easier to check that New Kim's Lemma holds, and to have more constructions for producing theories satisfying New Kim's Lemma.

Question 6.4. Does it suffice to show that New Kim's Lemma holds for formulas in a single free variable to establish that it holds for all formulas?

The analogous fact is known for each of the properties NTP, NTP$_1$, NTP$_2$, NSOP$_1$, and NATP: to prove that a theory has one of these properties, it suffices to check that no formula $\varphi(x; y)$ has the corresponding property, where x is a single variable. These arguments typically push against the syntactic definition of the property, so it is hard to envision what a solution to this question might look like without first resolving Question 6.3. In light of this, it makes sense to ask Question 6.4 with New Kim's Lemma replaced by NBTP.

The theory DLO$_p$ examined in Section 4A is a special case of a general construction, developed in [Chernikov and Ramsey 2016], for "parametrizing" arbitrary Fraïssé limits with disjoint amalgamation. As shown in [loc. cit., Corollary 6.3], the parametrization of a Fraïssé limit with a simple theory is always NSOP$_1$. It seems likely that the arguments in Section 4A generalize to provide a positive answer to the following question.

Question 6.5. Suppose $\mathcal{K}$ is a Fraïssé class with disjoint amalgamation, and let $\mathcal{K}_{\mathrm{pfc}}$ be the parametrized version of $\mathcal{K}$, as defined in [loc. cit., Section 6.3]. Let T and T_{pfc} be the theories of the Fraïssé limits of $\mathcal{K}$ and $\mathcal{K}_{\mathrm{pfc}}$, respectively. If T satisfies New Kim's Lemma (or if T is NTP$_2$), does T_{pfc} satisfy New Kim's Lemma?

There is a theme in the literature that "generic constructions" (i.e., those involving taking a model companion) often produce properly NSOP$_1$ theories. For example, interpolative fusion, introduced in [Kruckman et al. 2021], is a general method for "generically putting together" multiple theories over a common reduct. Tran, Walsberg, and Kruckman [Kruckman et al. 2022] showed that the interpolative fusion of stable theories over a stable base theory is always NSOP$_1$ (and, under mild hypotheses, the interpolative fusion of NSOP$_1$ theories over a stable base theory is always NSOP$_1$).

If theories satisfying New Kim's Lemma are to generalize NSOP$_1$ theories in an analogous way to how NTP$_2$ theories generalize simple theories, and how NIP theories generalize stable theories, then the following seems like a reasonable conjecture.

Question 6.6. Does the interpolative fusion of NIP theories over a stable base theory always satisfy New Kim's Lemma?

Questions 6.5 and 6.6 are also meaningful with New Kim's Lemma replaced by NBTP.

Finally, since the Kim's Lemma surveyed in Section 3 form the cornerstones of the theories of independence in simple, NSOP$_1$, and NTP$_2$ theories, one might hope that a satisfying theory of Kim-independence, generalizing the theory of $\underset{}{\overset{f}{\smile}}$ in NTP$_2$ theories and of $\underset{}{\overset{K}{\smile}}$ in NSOP$_1$ theories, could be developed on the basis of New Kim's Lemma. A natural first step would be the chain condition.

Definition 6.7. We say $\underset{}{\overset{K}{\smile}}$ satisfies the *chain condition over models* if whenever $M \models T$, $a \underset{M}{\overset{K}{\smile}} b$, and $I = (b_i)_{i < \omega}$ is a Morley sequence for a global M-invariant type extending $\mathrm{tp}(b/M)$, there exists a' such that $a'b_i \equiv_M ab$ for all $i < \omega$, I is Ma'-indiscernible, and $a' \underset{M}{\overset{K}{\smile}} I$.

Question 6.8. If T satisfies New Kim's Lemma, does $\underset{}{\overset{K}{\smile}}$ satisfy the chain condition over models?

One motivation for this question is that $\underset{}{\overset{f}{\smile}}$ satisfies the chain condition over models in NTP$_2$ theories, see [Ben Yaacov and Chernikov 2014, Theorem 2.9] (and the chain condition is the key step in the proof of the variant of the independence theorem for NTP$_2$ theories in that paper). The proof of the chain condition in [loc. cit.] uses both the Kim's Lemma for NTP$_2$ theories and the syntactic definition of NTP$_2$. So here again, if Question 6.8 has a positive answer, it may be necessary to first resolve Question 6.3.

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