CHARMED ROOTS AND THE KROWERAS COMPLEMENT

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ABSTRACT. Although both noncrossing partitions and nonnesting partitions are uniformly enumerated for Weyl groups, the exact relationship between these two sets of combinatorial objects remains frustratingly mysterious. In this paper, we give a precise combinatorial answer in the case of the symmetric group: for any standard Coxeter element, we construct an equivariant bijection between noncrossing partitions under the *Kreweras complement* and nonnesting partitions under a Coxeter-theoretically natural cyclic action we call the *Kroweras complement*. Our equivariant bijection is the unique bijection that is both equivariant and support-preserving, and is built using local rules depending on a new definition of *charmed roots*. Charmed roots are determined by the choice of Coxeter element—in the special case of the linear Coxeter element $(1, 2, \ldots, n)$, we recover one of the standard bijections between noncrossing and nonnesting partitions.

1. INTRODUCTION

1.1. **Noncrossing and nonnesting partitions.** Let $W \subseteq \mathrm{GL}(V)$ be a finite complex reflection group acting in its reflection representation on a complex vector space V of dimension r with reflections T [Hum90, LT09]. It is well-known that the ring of W-invariants $\mathbb{C}[V]^W$ is a polynomial ring generated by invariants of degrees $d_1 \leq d_2 \leq \cdots \leq d_r$. The *Coxeter number* of a well-generated W (that is, W is generated by r reflections) is $h = d_r$ and the *W-Catalan number* is

$$\tag{1} \mathrm{Cat}(W) := \prod_{i=1}^{r} \frac{h + d_i}{d_i}.$$

1.1.1. *Noncrossing partitions.* The *absolute order* on W is the poset defined as the oriented Cayley graph of W generated by T, where the identity of W is the minimal element. For W well-generated, a *Coxeter element* c is defined to be a regular element of order $h = d_r$; any Coxeter element has rank r in the absolute order. The *c-noncrossing partition lattice* $\mathrm{NC}(W, c)$ is the interval $[e, c]_T$ in the absolute order (it is indeed a lattice) [BW02, BW08, Bes03]. For any Coxeter elements c, c', we have $\mathrm{NC}(W, c) \simeq \mathrm{NC}(W, c')$ as posets because there exists an automorphism of W (sometimes outer) fixing the set of reflections and sending c to c' [RRS17]. The *support* of a noncrossing partition $\pi \in \mathrm{NC}(W, c)$ is the set $\mathrm{Supp}(\pi)$ of simple reflections required to write a reduced word in simple reflections for π.

Until very recently, the number of noncrossing partitions had only been computed case-by-case; a uniform proof was found in the case of real W in [GLTW22].

Theorem 1.1 ([Bes15, Section 13])**.** *For W a well-generated finite complex reflection group with Coxeter element c, $|\mathrm{NC}(W, c)| = \mathrm{Cat}(W)$.*

Section 3 further discusses c-noncrossing partitions.

1.1.2. *Nonnesting partitions.* Let now W be a Weyl group (a crystallographic real reflection group), with positive roots Φ^+. The positive root poset is the partial order on Φ^+ defined by $\alpha \leq \beta$ iff $\beta - \alpha$ is a nonnegative sum of positive roots. The *nonnesting partitions* $\mathrm{NN}(W)$ are the order ideals in the positive root poset [Rei97, Remark 2]. The *support* of a nonnesting partition $p \in \mathrm{NN}(W)$ is the set $\mathrm{Supp}(p)$ of simple roots that lie in p (as an order ideal of Φ^+).

The number of nonnesting partitions has been computed uniformly by combining a bijection of Cellini and Papi [CP02, Theorem 1] with an argument of Haiman [Hai94, Theorem 7.4.2] and an observation of Thiel [Thi16, Lemma 8.2].

Theorem 1.2 ([CP02, Hai94, Thi16])**.** For W a Weyl group, $|\mathrm{NN}(W)| = \mathrm{Cat}(W)$.

Section 4 further discusses nonnesting partitions.

1.2. **History and Incongruity.** Despite the fact that they are both counted by $\mathrm{Cat}(W)$, there are several Incongruities between $\mathrm{NC}(W, c)$ and $\mathrm{NN}(W)$:

(1) $\mathrm{NC}(W, c)$ is defined for well-generated complex reflection groups, while $\mathrm{NN}(W)$ is only defined for Weyl groups;
(2) the definition of $\mathrm{NC}(W, c)$ requires the choice of a Coxeter element, while $\mathrm{NN}(W)$ has no such dependence;
(3) When c is a *standard Coxeter element*[1] in a real reflection group—that is, when c is a product of the simple reflections in some order—and s is initial in c, the map $w \mapsto s^{-1}ws$ gives a bijection $\mathrm{NC}(W, c) \simeq \mathrm{NC}(W, s^{-1}cs)$ and leads to the *Kreweras complement* and the *Cambrian recurrence* for real W, while there is no obvious similar action of the initial s on $\mathrm{NN}(W)$.

Two additional Incongruities have only been very recently resolved in [GLTW22]:

(4) $\mathrm{NC}(W, c)$ had only *Fuss–Catalan* and *Fuss–Dogolon* generalizations, while $\mathrm{NN}(W)$ was easily generalized to rational parameters p coprime to the Coxeter number h;
(5) the enumeration of $\mathrm{NC}(W, c)$ was only case-by-case, while the enumeration of $\mathrm{NN}(W)$ was uniform.

The exact relationship between noncrossing and nonnesting partitions, therefore, remains frustratingly mysterious, and is perhaps the biggest open question in Coxeter–Catalan combinatorics:

Problem 1.3. Fix W a finite Weyl group and c a Coxeter element. Find a "natural" bijection between $\mathrm{NC}(W, c)$ and $\mathrm{NN}(W)$.

Historically—if one interprets "natural" as meaning "uniform"—finding such a bijection was viewed as a promising way to give a uniform proof that $|\mathrm{NC}(W, c)| = \mathrm{Cat}(W)$. But even with the recent uniform enumeration of $\mathrm{NC}(W, c)$, Problem 1.3 remains interesting in its own right.

To our taste, there are two approaches to Problem 1.3: the first approach is based on the case-by-case combinatorial models available in the classical types A, B, D [GF09, SR10, CM11, Kim11, Stu08, Ath98]; the second approach was pioneered in [AST13] based on observations in [Pan09, BR11], and uses a coincidence of cyclic actions (partially resolving Incongruity (3) above) to induce a bijection. Our main theorem will refine both of these approaches in the special case of the symmetric group S_n, recovering one of the standard bijections between noncrossing and nonnesting partitions in the case of the linear Coxeter element $(1, 2, \ldots, n)$.

1.3. **Cyclic actions on noncrossing and nonnesting partitions.**

1.3.1. *Armstrong-Stump-Thomas.* For W a well-generated complex reflection group, we define the *c-Kreweras complement* on the noncrossing partition lattice as the anti-automorphism [Arm09, Section 4.2], [Kre72]

$$(2) \qquad \mathrm{Krew}_c : \mathrm{NC}(W, c) \to \mathrm{NC}(W, c)$$

$$\pi \mapsto \pi^{-1}c.$$

Since $\mathrm{Krew}_c^2(\pi) = c^{-1}\pi c$ and c has order h, Krew_c has order h if $-1 \in W$, and $2h$ otherwise.

For W a Weyl group, *rowmotion* on nonnesting partitions is the map

$$(3) \qquad \mathrm{Row} : \mathrm{NN}(W) \to \mathrm{NN}(W)$$

$$p \mapsto \min_{\Phi^+}\{\alpha \mid \alpha \not\leq \beta \text{ for any } \beta \in p\}.$$

Panyushev conjectured that the order of Row on $\mathrm{NN}(W)$ was h if $-1 \in W$ and $2h$ otherwise [Pan09], and Bessis and Reiner refined Panyushev's conjecture by observing that Row had the same orbit structure on $\mathrm{NN}(W)$ as Krew on $\mathrm{NC}(W, c)$ [BR11]. This conjecture was proven by Armstrong,

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[1]Outside of this introduction, we will always assume that c is a standard Coxeter element.

Stump, and Thomas, using a uniformly-stated—but only case-by-case verified—inductive bijection between nonnesting and noncrossing partitions [AST13].

Thus, partitioning the sets $\mathrm{NC}(W, c)$ and $\mathrm{NN}(W)$ into orbits under Krew_c and Row refines the problem of finding a "natural" bijection. But Incongruity (2) remains—while the definitions of $\mathrm{NC}(W, c)$ and Krew_c depend on the choice of a Coxeter element, the set $\mathrm{NN}(W)$ and its action Row do not depend on any such choice.

1.3.2. *Flips and the Kreweras Complement on Noncrossing Partitions.* Let W be a finite Coxeter group. For c a standard Coxeter element and c a particular choice of reduced word for c, the *c-sorting word* $\mathsf{w}(\mathsf{c})$ for $w \in W$ is the leftmost reduced word in simple reflections for w in c^∞. An element is *c-sortable* if its c-sorting word uses a weakly decreasing subset of simple reflections in each successive copy of c. The *weak order* is the poset given by the Cayley graph of W generated by S, with the longest element $w_\circ$. The restriction of the weak order to the c-sortable elements is Nathan Reading's c-Cambrian lattice, with edges labeled by roots. The inversion sequence $\mathrm{inv}(\mathsf{w_\circ}(\mathsf{c})) = [\alpha^{(1)}, \alpha^{(2)}, \ldots, \alpha^{(N)}]$ gives total ordering on the set of roots of W (see Section 2.4 for more details on this ordering).

In [TW19], a subset of the authors showed that it is possible to compute the c-Kreweras complement in "slow motion" as a sequence of local *flips* on the edges of the c-Cambrian lattice in $\mathrm{inv}(\mathsf{w_\circ}(\mathsf{c}))$ order.

Theorem 1.4 ([TW19, Theorem 1.2]). *For W a real reflection group, $\mathrm{Krew}_c(\pi)$ can be computed as a sequence of flips on $\mathrm{Camb}(W, c)$.*

1.3.3. *Toggles and the Kroweras Complement on Nonnesting Partitions.* As with the c-Kreweras complement, rowmotion can also be written as a sequence of local moves [CFDF95, SW12]. A *toggle* $\mathrm{tog}_\alpha(p)$ of a nonnesting partition p at a positive root α either adds α to p (when $\alpha \notin p$) or removes α from p (when $\alpha \in p$), provided that the result is again a nonnesting partition. For nonnesting partitions, Row can be computed by toggling each root of the root poset in order of height (or by *row*).

We are now in a position to address Incongruity (2): in contrast to the c-Kreweras complement, Row has no dependence on the Coxeter element c. We resolve Incongruity (2) by mirroring the sequence of local moves used in the "slow motion" c-Kreweras complement of Theorem 1.4—but replacing flips by toggles [Wil13, Wil14]:

$$(4) \qquad \mathrm{Krow}_c : \mathrm{NN}(W) \to \mathrm{NN}(W)$$

$$p \mapsto \left(\mathrm{tog}_{\alpha^{(N)}} \circ \cdots \circ \mathrm{tog}_{\alpha^{(2)}} \circ \mathrm{tog}_{\alpha^{(1)}} \right)(p).$$

As further explained in Section 4.2, we call this map the *c-Kroweras complement*.

1.4. **Main Theorem.** In Section 5, we construct a novel bijection between c-noncrossing partitions and nonnesting partitions for the symmetric group S_n using the combinatorics of certain kissing families of lattice paths whose local behavior is dependent on a novel definition of *c-charmed roots*. When c is a standard Coxeter element of S_n, c consists of a single cycle $(w_1, w_2, \ldots, w_m, w_{m+1} \ldots, w_n)$ with an initial increasing subsequence $1 = w_1 < w_2 < \cdots < w_m = n$, followed by a decreasing sequence $n = w_m > w_{m+1} > \cdots > w_n$ of the remaining unused entries. We say that the reflection (w_i, w_j) is *c-charmed* if $1 < i < m$ and $m < j \leq n$ (see Section 2.3 for a more complete discussion), and *c-ordinary* otherwise.

Our bijection is the unique equivariant support-preserving bijection between $\mathrm{NC}(W, c)$ under the c-Kreweras complement and $\mathrm{NN}(W)$ under the c-Kroweras complement, and is based on *intimate families* of lattice paths (see Section 5.2).

Theorem 1.5. *Let S_n be the symmetric group, and fix a standard Coxeter element $c \in S_n$. Then there is a unique bijection $\mathrm{Charm}_c \colon \mathrm{NC}(S_n, c) \to \mathrm{NN}(S_n)$ satisfying*

- $\mathrm{Charm}_c \circ \mathrm{Krew}_c = \mathrm{Krow}_c \circ \mathrm{Charm}_c$ *and*
- $\mathrm{Supp} = \mathrm{Supp} \circ \mathrm{Charm}_c$.

As we describe in Corollary 6.3, it is easy to directly read the blocks of the noncrossing partition from the corresponding intimate family. We prove Theorem 1.5 in Section 6 using *Cambrian induction*—that is, we will show

- the *base case*: the theorem holds for the linear Coxeter element $c_1 = s_1 s_2 \cdots s_{n-1}$. In this case, Krow_{c_1} is order ideal promotion, as defined by Striker and Williams [SW12], and Charm_{c_1} recovers a standard bijection; and
- the *inductive step*: suppose Theorem 1.5 holds for c. We argue that if s is initial in c and $c' = scs$, then Theorem 1.5 also holds for c'—we understand how $\mathrm{Charm}_{c'}$ differs from Charm_c using the relation between the set of c-charmed and c'-charmed roots in Proposition 2.7.

Since all standard Coxeter elements are conjugate by a sequence of conjugations by initial simple reflections, the theorem, therefore, holds for all standard Coxeter elements. Our proof strategy is summarized in Figure 1.

A priori, there is no reason for an arbitrary product of toggles to have well-behaved order. As a consequence of Theorem 1.5, we have the following surprising result.

Corollary 1.6. For any standard Coxeter element $c \in S_n$, the order of Krow_c on $\mathrm{NN}(W)$ is $2h$.

Remark 1.7. The statement of the main theorem in Armstrong-Stump-Thomas [AST13] can be obtained from the statement of our Theorem 1.5 by replacing Krow_c by Row, and replacing the symmetric group by any finite Weyl group. In particular, in the result from [AST13], only one toggle sequence is ever employed in the poset of positive roots. Thus, our main theorem is much richer than the [AST13] result (when restricted to the symmetric group) because we consider a different toggle sequence for each standard Coxeter element. Indeed, the fact that the Coxeter element only plays a role on the noncrossing side of the bijection means that in [AST13], it is sufficient to consider one preferred Coxeter element (a *bipartite* Coxeter element), and find one corresponding bijection—whereas here we construct truly different bijections for each Coxeter element.

1.5. **Future work.** Using initial and final simple reflections, the notion of c-charmed root can be uniformly extended to all finite Weyl groups. We expect that our c-charmed bijections can also be extended; the first step is to find analogs of Proposition 4.7—that is, to determine the correct toggle group elements that conjugate Krow_c to $\mathrm{Krow}_{c'}$ for $c' = scs$. As a simple first case, we show in Section 7.1 that we can use a simple folding argument to extend our work to type C_n.

Although there are now explicit constructions of noncrossing and nonnesting partitions for Weyl groups and any rational parameter p coprime to h [GLTW22, Hai94], only in type A for the linear Coxeter element do we have explicit diagrammatic models coming from rational Dyck paths [ARW13, BR16, Bod18, Bod19]. It would be natural to try to extend our charmed bijections to these rational models.

2. BACKGROUND

Write $[n] \coloneqq \{1, 2, \ldots, n\}$; we work modulo n so that for $0 < i < n$, we may write $-i$ for $n - i$ and $n + 1$ for 1. Since this paper only deals with the permutation group S_n, we will specialize our explanations unless the general case causes no additional complications. For a more general introduction on Coxeter systems, we refer the reader to [Hum90].

2.1. **Coxeter groups.** Fix (W, S) a finite Coxeter system, where S is the set of *simple reflections*. For $W = S_n$, we reserve the symbols s_i for the simple transpositions $(i, i+1)$.

The *length* of an element $w \in W$ is
$$\ell_S(w) \coloneqq \min\{\ell \in \mathbb{N} \mid w = t_1 \cdots t_\ell \text{ with } t_i \in S\}.$$

For $W = S_n$ and $w \in W$, $\ell_S(w)$ is equal to the number of *inversions* of w:
$$\ell_S(w) = \{(i, j) \mid 1 \le i < j \le n \text{ and } w(i) > w(j)\}.$$

For $w = t_1 \cdots t_\ell$ with $t_i \in S$ and $\ell = \ell_S(w)$, the expression $\mathsf{w} = [t_1, \ldots, t_k]$ is a *reduced S-word* for w. The *(right) weak order* on W is defined by
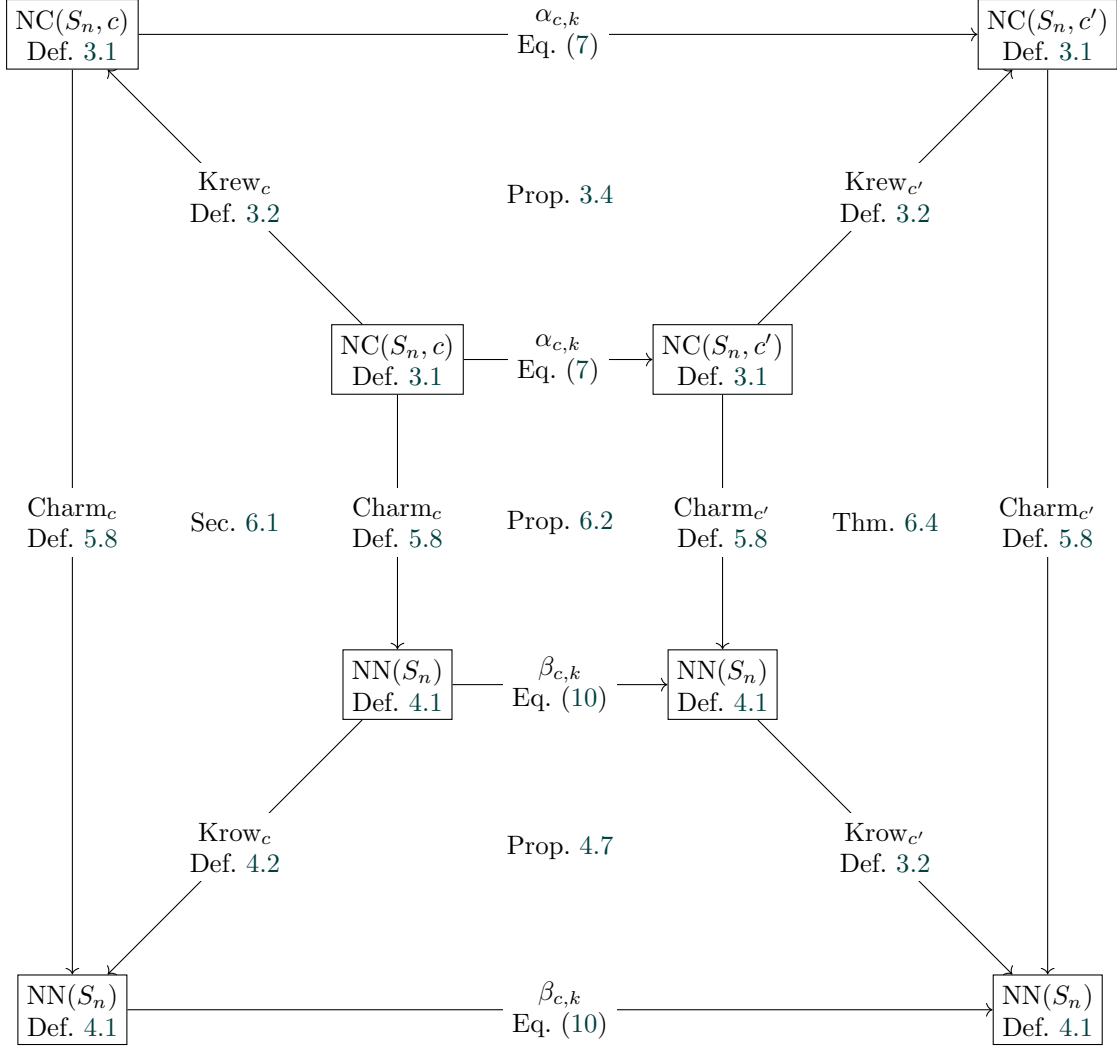$$v \le_S w \iff \ell_S(w) = \ell_S(v) + \ell_S(v^{-1}w).$$

FIGURE 1. The commutative cube underlying the structure of the paper. Each vertex of the cube is the set of noncrossing partitions with respect to some Coxeter element, or the set of nonnesting partitions. The edges of the cube are bijections between these sets. Each face of the cube is commutative—Theorem 6.4 uses Cambrian induction to infer commutativity of the rightmost face from commutativity of all other faces. Figure 12 replaces each vertex of this commutative cube by an element of the corresponding set used in the running example of the paper.

There is a unique *longest element* $w_\circ$ of maximum length, which is the reverse permutation in S_n; this longest element is greater than all other elements in weak order.

The *reflections* in W are the elements of the set $T := \{wsw^{-1} \mid s \in S, \ w \in W\}$. We use the reflections of W to define the *absolute length* of an element $w \in W$ to be

$$\ell_T(w) := \min\{\ell \in \mathbb{N} \mid w = r_1 \cdots r_\ell \colon r_i \in T\}.$$

It is immediate that $\ell_T(w) \leq \ell_S(w)$. The *absolute order* on W is the partial order on W induced by absolute length (in the same way as length induces weak order):

$$v \leq_T w \iff \ell_T(w) = \ell_T(v) + \ell_T(v^{-1}w).$$

For S_n, the set of reflections is the set of all transpositions (i, j). The absolute length of a permutation $w \in S_n$ is n minus the number of cycles of w.

The *support* of an element $w \in W$ is the set of simple reflections below w in absolute order:

$$\mathrm{Supp}(w) = \{s \in S \mid s \leq_T w\}.$$

2.2. **Roots.** Let $\{e_i\}_{i=1}^n$ be the standard basis of $\mathbb{R}^n$. The *positive roots* of type A_{n-1} are the following normal vectors to the reflecting hyperplanes in the braid arrangement

$$(5) \qquad\qquad \alpha_{i,j} := e_j - e_i \text{ with } 1 \leq i < j \leq n.$$

To reduce subscripts, we will often denote $\alpha_{i,j}$ by its corresponding reflection $(i,j) \in S_n$; we will always use the convention that $i < j$ with this notation. The *root poset* of type A_{n-1} is the poset $\Phi^+ := \Phi^+(A_{n-1})$ defined on the positive roots with covering relations $(i+1,j) \lessdot (i,j) \lessdot (i,j+1)$. The minimal elements of Φ^+ are the *simple roots* α_i for $1 \leq i < n$, which we also denote by $(i, i+1)$. The *support* of a root (i,j) is the set of simple roots $\text{Supp}(i,j)$ less than (i,j) in the root poset. The root poset of type A_8 is illustrated in Figure 2.
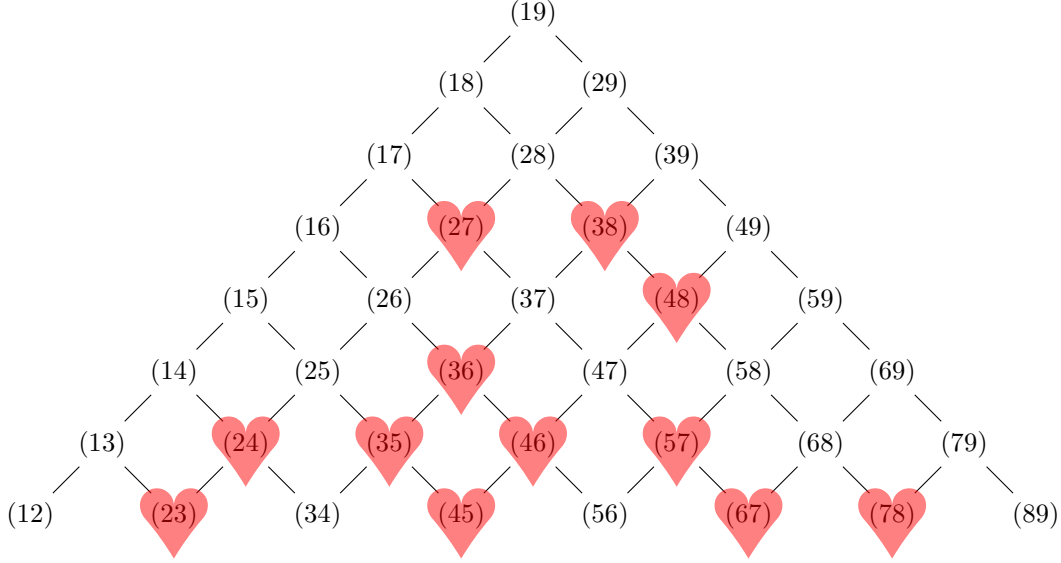

FIGURE 2. The Hasse diagram of the positive root poset Φ^+ of type A_8. The label (ij) represents the positive root $e_j - e_i$. For $c = s_2 s_1 s_3 s_6 s_5 s_4 s_8 s_7$, the c-charmed roots from Definition 2.4 are marked using hearts.

2.3. **Coxeter elements and charmed roots.**

Definition 2.1. A *standard Coxeter element* c is a product of the simple reflections in any order.

To ease notation, we reserve the symbols r_i to refer to a fixed ordering $\mathsf{c} := [r_1, r_2, \ldots, r_{n-1}]$ of S and define $c := r_1 r_2 \cdots r_{n-1}$. We also reserve the symbol $s_k = r_1$ for the first simple reflection in the chosen reduced word, and write $c' = r_2 \cdots r_{n-1} r_1$. We will drop the word "standard" when talking about standard Coxeter elements, as we only use standard Coxeter elements in this paper.

We say that a simple reflection s is *initial* in c if $\ell_S(sc) \leq \ell_S(c)$, and that a simple reflection s is *final* in c if $\ell_S(cs) \leq \ell_S(c)$. If $s = s_k = (k, k+1)$ is initial in c, then $c' = scs$ is also a Coxeter element of S_n, and we will denote this by the writing $c \xrightarrow{k} c'$. It is a basic fact for finite Coxeter groups—relying only on the fact that the underlying Dynkin diagram is a tree—that all standard Coxeter elements are conjugate [Hum90, Section 3.16].

Lemma 2.2 ([Rea07, Lemma 1.7]). Let c and c' be two Coxeter elements of a finite Coxeter group W. Then c and c' are conjugate by a sequence of conjugations by initial simple reflections:

$$c = c^{(0)} \xrightarrow{k_1} c^{(1)} \xrightarrow{k_2} c^{(2)} \xrightarrow{k_3} \cdots \xrightarrow{k_m} c^{(m)} = c'.$$

In S_n and for $1 \leq k < n$, we write c_k for the Coxeter element with s_k as its unique initial simple reflection—that is, $c_k = s_k s_{k+1} \cdots s_{n-1} s_{k-1} s_{k-2} \cdots s_1$. As a special case, the *linear Coxeter element* $c_1 = s_1 s_2 \cdots s_{n-1}$ has cycle notation $(1, 2, \ldots, n)$.

More generally, it is easy to show that the cycle notation of any Coxeter element in the symmetric group has a particularly simple form: $c \in S_n$ consists of a single cycle with an initial increasing

subsequence starting at 1 and ending at n, followed by a decreasing sequence of the remaining unused entries.

Proposition 2.3. An element $w \in S_n$ is a Coxeter element if and only if it has cycle notation

$$w = (w_1, w_2, \ldots, w_m, w_{m+1} \ldots, w_n),$$

where $1 = w_1 < w_2 < \cdots < w_m = n$ and $n = w_i > w_{m+1} > \cdots > w_n > w_1 = 1$.

Let c be a Coxeter element with cycle notation $(w_1, w_2, \ldots, w_m, w_{m+1} \ldots, w_n)$, where $1 = w_1 < w_2 < \cdots < w_m = n$ and $n = w_m > w_{m+1} > \cdots > w_n > w_1 = 1$. Write

$$L_c := \{w_2, \ldots, w_{m-1}\} \text{ and } R_c := \{w_{m+1}, \ldots, w_n\}.$$

Definition 2.4. For $1 < i < j < n$, we say that a root (i, j) is *c-charmed* if $i \in L_c$ and $j \in R_c$ or if $i \in R_c$ and $j \in L_c$ and *c-ordinary* otherwise. We write $\heartsuit_c$ for the set of c-charmed roots.

Example 2.5. Consider the Coxeter element $c = s_2 s_1 s_3 s_6 s_5 s_4 s_8 s_7$ in S_9; s_2, s_6, and s_8 are initial in c, while s_1, s_4, and s_7 are final. The element c has cycle notation $(1, 3, 4, 7, 9, 8, 6, 5, 2)$, so that

$$L_c = \{3, 4, 7\} \text{ and } R_c = \{8, 6, 5, 2\}.$$

As a special case, a simple root $(i, i+1)$ for $1 < i < n-1$ is c-charmed if and only if it is initial or final in c; more precisely, $(i, i+1)$ is initial when $i+1 \in L_c$ and $i \in R_c$, and $(i, i+1)$ is final when $i \in L_c$ and $i+1 \in R_c$. In figures, we depict c-charmed roots with a $\heartsuit$ and ordinary roots by a circle.

We visualize the cycle notation of c by drawing it as points labeled $w_1, w_2, \ldots, w_n$ counterclockwise around a circle. We visualize a root (i, j) by connecting the vertices labeled i and j by a line segment. For any $a, b, c, d \in [n]$, we say that (a, b) *crosses* (c, d) if and only if (a, b) and (c, d) are crossing in their interior (note that (a, b) does not cross itself). For $1 < i < j < n$, it is easy to check that a root (i, j) is c-charmed if and only if (i, j) crosses $(i-1, j+1)$.

Example 2.6. Figure 3 illustrates this visualization of the cycle notation $(1, 3, 4, 7, 9, 8, 6, 5, 2)$ of $c = s_2 s_1 s_3 s_6 s_5 s_4 s_8 s_7$.
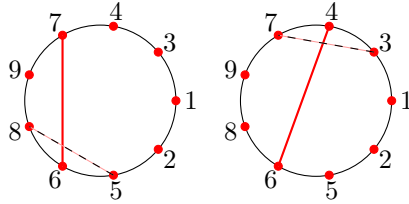

FIGURE 3. The visualization of the cycle notation $(1, 3, 4, 7, 9, 8, 6, 5, 2)$ of $c = s_2 s_1 s_3 s_6 s_5 s_4 s_8 s_7$. *Left:* the initial c-charmed simple root $(6, 7)$ (in red) intersecting the root $(5, 8) = (6-1, 8+1)$ (dashed). *Right:* the c-charmed root $(4, 6)$ (in red) intersecting the root $(3, 7) = (4-1, 6+1)$ (dashed).

For $s = s_k = (k, k+1)$, conjugating c by s interchanges the positions of k and $k+1$ in the cycle notation of c. When s is initial in c, then $c' = scs$ is again a Coxeter element and we recall that we write this using the notation $c \xrightarrow{k} c'$. We obtain the following relationship between c-charmed roots and c'-charmed roots.

Proposition 2.7. Let $c \xrightarrow{k} c'$. Then for $1 < i < j < n$,

$$(i, j) \text{ is } c\text{-charmed if and only if it is } \begin{cases} c'\text{-charmed} & \text{and } |\{i, j\} \cap \{k, k+1\}| = 0 \mod 2, \\ c'\text{-ordinary} & \text{and } |\{i, j\} \cap \{k, k+1\}| = 1 \mod 2. \end{cases}$$

2.4. **Coxeter-sorting words and Auslander–Reiten quivers.** Let $c \in S_n$ be a Coxeter element, and fix $\mathsf{c} = [r_1, r_2 \ldots, r_{n-1}]$ a reduced word for c. Define the *c-sorting word* of the long element $w_\circ$ to be the leftmost reduced word $\mathsf{w}_\circ(\mathsf{c})$ for $w_\circ$ in c^∞. Up to commutations, $\mathsf{w}_\circ(\mathsf{c})$ does not depend on the choice of reduced word for c, and so we shall denote it by $\mathsf{w}_\circ(c)$. Write the c-sorting word for the long element $\mathsf{w}_\circ(c) = [t_1, t_2, \ldots, t_N]$, with each $t_i \in S$ a simple reflection and $N = |T| = \binom{n}{2}$ the number of reflections in S_n. For $a = 1, 2, \ldots, N$, define the inversion

$$(6) \qquad\qquad \alpha^{(a)} := t_1 t_2 \cdots t_{a-1}(\alpha_{t_a}),$$

where α_{t_a} is the simple root corresponding to the simple reflection t_a. Since $w_\circ$ has every positive root as an inversion, each positive root appears exactly once in the inversion sequence $\mathrm{inv}(\mathsf{w}_\circ(c)) := [\alpha^{(1)}, \alpha^{(2)}, \ldots, \alpha^{(N)}]$.

Example 2.8. For $\mathsf{c} = [s_2, s_1, s_3, s_6, s_5, s_4, s_8, s_7]$, we have

$$\mathsf{w}_\circ(c) = s_2 s_1 s_3 s_6 s_5 s_4 s_8 s_7 s_2 s_1 s_3 s_6 s_5 s_4 s_8 s_7 s_2 s_1 s_3 s_6 s_5 s_4 s_8 s_7 s_2 s_1 s_3 s_6 s_5 s_4 s_8 s_7 s_2 s_6 s_5 s_8 \text{ and}$$

$$\mathrm{inv}(\mathsf{w}_\circ(c)) = [(23), (13), (24), (67), (57), (27), (89), (69), (14), (34), (17), (59),$$
$$(29), (19), (68), (58), (37), (47), (39), (28), (18), (38), (56), (26),$$
$$(49), (79), (48), (16), (36), (46), (25), (15), (78), (35), (45), (12)].$$

It is helpful to arrange the letters in the Coxeter-sorting word $\mathsf{w}_\circ(c) = [t_1, t_2, \ldots, t_N]$ in a quiver $\mathrm{AR}(c)$—if we replace each letter t_a by the reflection corresponding to the positive root $\alpha^{(a)}$ in the inversion sequence, this is known as the *Auslander–Reiten quiver* $\mathrm{AR}(c)$ [Sch14, Sections 3.1.1 and 3.1.2.2]. A combinatorial description of $\mathrm{AR}(c)$ is given as follows:

- the vertices are the roots (i, j) for $1 \le i < j \le n$;
- if $(i, i+1)$ is initial in c, then $(i, i+1)$ appears as the leftmost vertex in the ith row from the top;
- if $(i, i+1)$ is final in c, then $(i, i+1)$ appears as the rightmost vertex in the ith row from the bottom.
- for $i < j$ we draw a south-east arrow $(i, j) \longrightarrow (i, c(j))$ if $i < c(j)$; and
- for $i < j$ we draw a north-east arrow $(i, j) \longrightarrow (c(i), j)$ if $c(i) < j$.

It will be useful to consider the example Auslander–Reiten quiver shown in Figure 4 while digesting this description.

Representation-theoretically, the vertices of $\mathrm{AR}(c)$ are the indecomposable representations of Q_c, but these are naturally labelled by positive roots using Gabriel's theorem. We now provide some more details on the structure of $\mathrm{AR}(c)$.

It follows immediately from the above description that the labels (i, j) appearing on each SW-NE diagonal have j fixed, while those appearing on each NW-SE diagonal have i fixed; we therefore index the diagonals by the corresponding label. One SW-NE diagonal appears for each value of j except 1; one NW-SE diagonal appears for each value of i except n. It is also immediate that the NW-SE diagonals appear in left-to-right order given by c, and similarly for the SW-NE diagonals. For j in $L \cap [2, n-1]$, the righthand end of the SW-NE diagonal labelled by j is $(c^{-1}(j), j)$. The lefthand end of the SW-NE diagonal labelled by j begins one step to the right from $(c^{-1}(j), j)$, at $(j, c(j))$. For i in $R \cap [2, n-1]$, the righthand end of the NW-SE diagonal labelled by i ends at $(i, c^{-1}(i))$ and the SW-NE-diagonal labelled by i starts one step to the right, at $(c(i), i)$.

Let $s = s_k$ be initial in c. It is convenient to write $c = (a_1, \ldots a_n)$ where $a_1 = k$. We have that $\{a_1, \ldots, a_k\} = \{1, \ldots, k\}$, $a_{k+1} = k + 1$, and $\{a_{k+1}, \ldots, a_n\} = \{k + 1, \ldots, n\}$. The $(k+1)$ SW-NE diagonals starting with the diagonal containing containing $(k, k+1)$ are numbered $k = a_1, \ldots, a_k, a_{k+1} = k + 1$ in that order. The $(n - k + 2)$ NW-SE diagonals starting with the diagonal containing $(k, k+1)$ are numbered $k + 1 = a_{k+1}, \ldots, a_n, a_1 = k$ in that order. In addition to Figure 4, it will also be useful to consult Figure 5, which provides a schematic description of a generic AR quiver with s_k initial.

The diagonals labelled by k and $k + 1$ form a rectangle with three missing corners. We have already observed that the top and bottom corners of the rectangle are missing; these are the positions where the two diagonals labelled k, or the two diagonals labelled $k + 1$, should cross. We see that the righthand corner is also missing; it is at the intersection of the SW-NE diagonal labelled k and the NW-SE diagonal labelled $k + 1$, so that its label would be $(k + 1, k)$, which is

not a valid label; however, the labels to its NW and SW are the (valid) labels $(c^{-1}(k+1),k)$ and $(k+1,c^{-1}(k))$, so all the rest of the roots along the edges of the rectangle are indeed present.

We will need the following lemma, which characterizes the labels whichappear outside this rectangle.

Lemma 2.9. Let $s = s_k$ be initial in c. In $\mathrm{AR}(c)$, all the roots above the SW-NE diagonal labelled $k+1$ or below the SW-NE diagonal labelled by k have both entries less than k. Similarly, all the roots above the NW-SE diagonal labelled $k+1$ or below the NW-SE diagonal labelled k have both entries greater than $k+1$.

Proof. Recall that each set of diagonals appears in the order given by c. This implies that the NW-SE diagonals with labels greater than k are between the one labelled $k+1$ and the one labelled k. The SW-NE diagonals less than $k+1$ are between the one labelled $k+1$ and the one labelled k. It follows that the roots in the region above the SW-NE diagonal labelled by $k+1$ lie on diagonals which are both labelled by numbers less than k. A similar argument applies to the three other regions. $\square$

Let $c' = s_k c s_k$, so $c \xrightarrow{k} c'$. In this case, we can relate the Auslander–Reiten quivers $\mathrm{AR}(c)$ and $\mathrm{AR}(c')$.

Proposition 2.10. Let $c \xrightarrow{k} c'$. Then $\mathrm{AR}(c')$ is obtained from $\mathrm{AR}(c)$ by replacing every label k by $k+1$ and vice versa, while also removing the root labelled $(k, k+1)$ and replacing it where the SW-NE diagonal labelled $k+1$ and the NW-SE diagonal labelled k now cross.

Proof. Another way to describe the transformation of the roots (other than $(k, k+1)$) is that we replace the label (i,j) by $(s(i), s(j))$. Since $c' = scs$, if $c(j) = \tilde{j}$, then $c'(s(j)) = s(\tilde{j})$. All the edges of the quiver except those involving $(k, k+1)$ will remain, because the action of s will not change whether or not the inequalities from the description of the Auslander–Reiten quiver hold. In $\mathrm{AR}(c')$, the label $(k, k+1)$ must lie at the intersection of the SW-NE diagonal labeled by $k+1$ and the NW-SE diagonal labelled by k; as we have already observed, this position was empty in the Auslander–Reiten quiver of c. $\square$
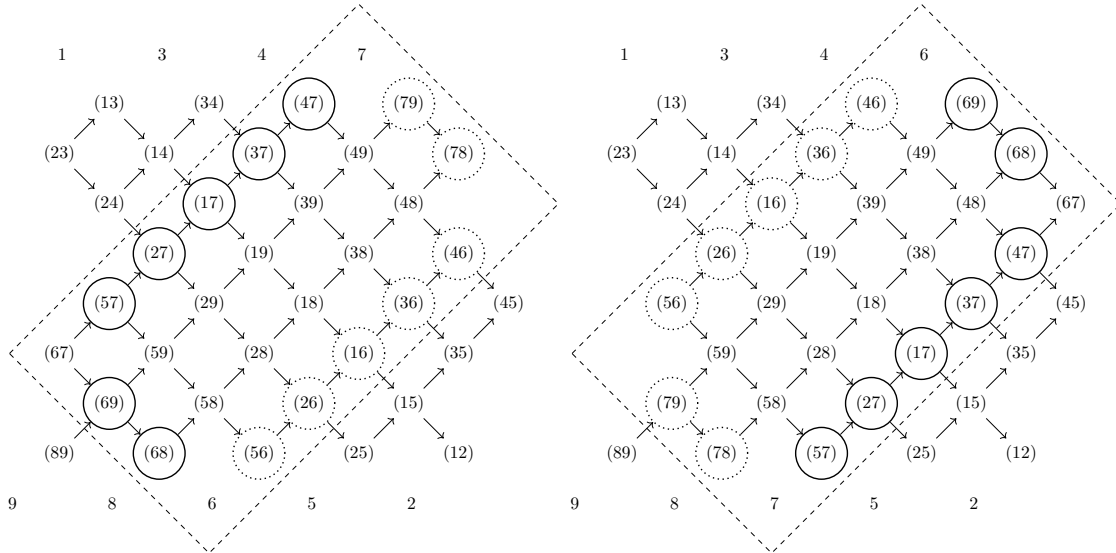

FIGURE 4. The Coxeter element $c = s_2 s_1 s_3 s_6 s_5 s_4 s_8 s_7$ has cycle notation $(1, 3, 4, 7, 9, 8, 6, 5, 2)$. The simple reflection $s = s_6$ is initial in c, and we set $c' = scs$, with cycle notation $(1, 3, 4, 6, 9, 8, 7, 5, 2)$. The quivers $\mathrm{AR}(c)$ (*left*) and $\mathrm{AR}(c')$ (*right*) are illustrated above. The roots in T_6 defined in Equation (9) are marked by solid circles in $\mathrm{AR}(c)$ and $\mathrm{AR}(c')$, while the roots in T'_6 are marked by dotted circles.
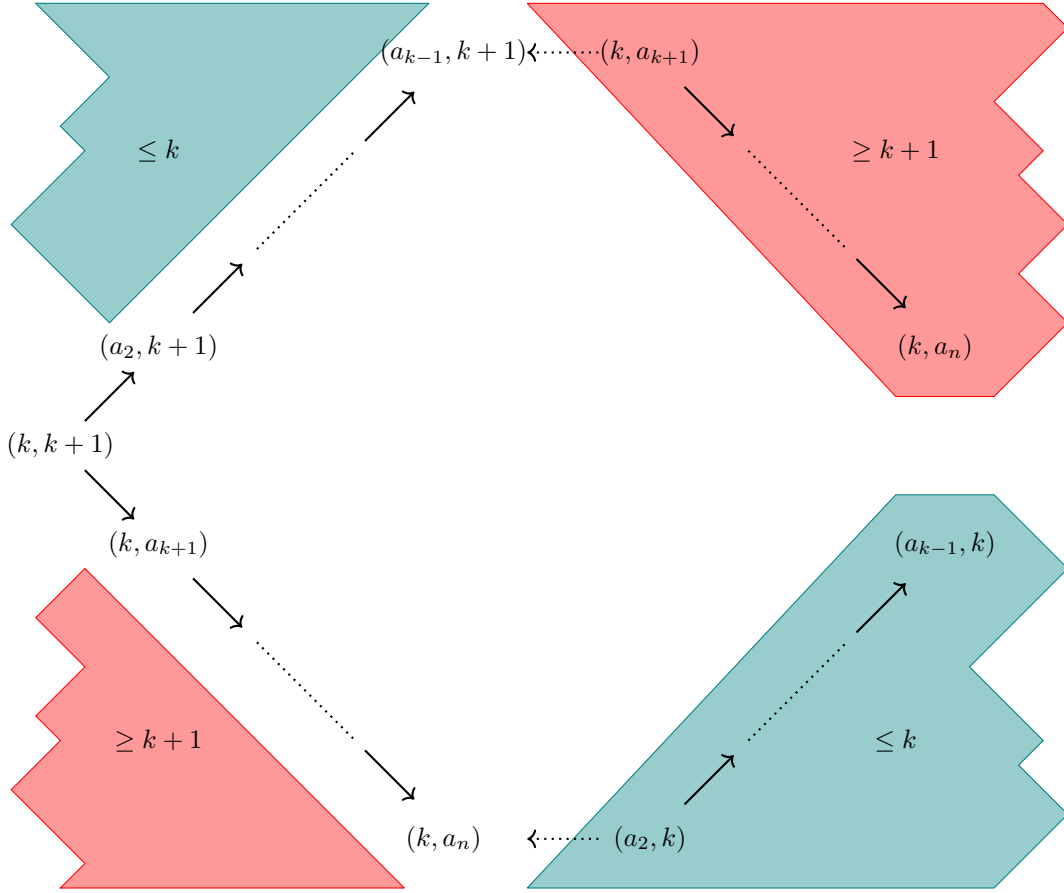
FIGURE 5. This provides a generic description of $\mathrm{AR}(c)$ where s_k is initial in c. Here c has cycle notation $(a_1, \ldots, a_k, a_{k+1}, \ldots, a_n)$ with $a_1 = k$, $a_{k+1} = k + 1$, $\{a_1, \ldots, a_k\} = \{1, \ldots, k\}$, and $\{a_{k+1}, \ldots, a_n\} = \{k+1, \ldots, n\}$. The regions shaded in green consist of roots with both labels $\leq k$, while the regions shaded in red consist of roots with both labels $\geq k + 1$, as guaranteed by Lemma 2.9.

3. NONCROSSING PARTITIONS AND THE KREWERAS COMPLEMENT

In this section, we define *noncrossing partitions* and the *Kreweras complement*.

3.1. Noncrossing partitions.

Definition 3.1. Let (W, S) be a Coxeter system and c be a Coxeter element. The *c-noncrossing partition lattice* is the absolute order interval

$$\mathrm{NC}(W, c) \coloneqq [1, c]_T = \{\pi \in W \mid 1 \leq_T \pi \leq_T c\}.$$

The *support* of a noncrossing partition $\pi \in \mathrm{NC}(W, c)$ is the set $\mathrm{Supp}(\pi)$ of simple reflections required to write a reduced word in simple reflections for π.

We now recall the usual graphical realization of noncrossing partitions in S_n. When c is a Coxeter element of S_n with cycle notation $(a_1, a_2, \ldots, a_n)$, we depict a c-noncrossing partition $\pi \in \mathrm{NC}(S_n, c)$ using the visualization of the cycle notation $(a_1, a_2, \ldots, a_n)$ by drawing the convex hull of the sets of vertices which belong to the same cycle of π. Examples are illustrated on the left and right of Figure 6.

As the name suggests, the noncrossing partitions $\mathrm{NC}(S_n, c)$ are exactly the set partitions of $[n]$ with the property that the convex hulls of their blocks are disjoint using the cyclic ordering $(a_1, a_2, \ldots, a_n)$; the absolute order interval is obtained by partially ordering these set partitions by reverse refinement.
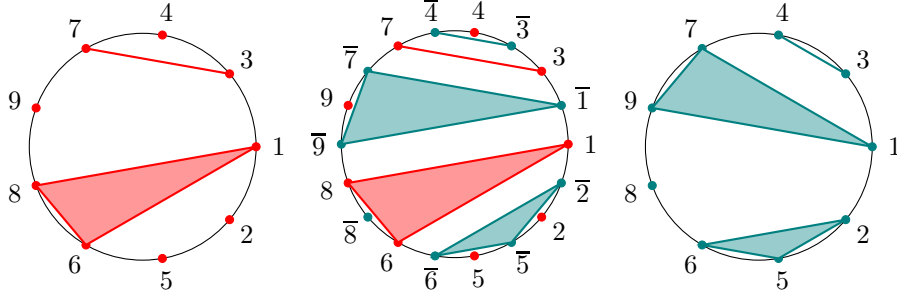
FIGURE 6. Fix the Coxeter element $c = (134798652)$. *Left:* the c-noncrossing partition $\pi = \{\{1,6,8\},\{3,7\}\}$. *Right:* the c-noncrossing partition $\mathrm{Krew}_c(\pi) = \{\{1,7,9\},\{2,5,6\},\{3,4\}\}$. *Middle:* the combinatorial construction of the Kreweras complement.

3.2. The Kreweras complement.

Definition 3.2. As in Equation (2), we define the c-Kreweras complement on $\mathrm{NC}(S_n, c)$ by

$$\mathrm{Krew}_c(\pi) = \pi^{-1}c.$$

We compute the c-Kreweras complement using the graphical realization of noncrossing partitions as follows: between each point a_i and a_{i+1}, add a new "barred" point labeled $\overline{a}_i$. Draw the graphical depiction of $\pi \in \mathrm{NC}(S_n, c)$ using only unbarred points, and define $\overline{\pi}'$ to be the coarsest noncrossing partition using only barred points such that $\overline{\pi}' \cup \pi$ is noncrossing. Finally, define $\mathrm{Krew}_c(\pi) = \pi'$ to be the combinatorial noncrossing partition obtained from $\overline{\pi}'$ by dropping the bars and rotating clockwise by π/n radians. This procedure is illustrated in Figure 6.

Proposition 3.3. The square of the c-Kreweras complement Krew_c acts as clockwise rotation by $2\pi/n$ radians. Furthermore, Krew_c is a poset anti-automorphism of $\mathrm{NC}(S_n, c)$.

Proof. Obviously, Krew_c^2 is the desired rotation, since Krew_c is an involution up to relabeling $a_i \to a_{i-1}$. Since rot is bijective, Krew_c is bijective. Krew_c reverses the ordering of $\mathrm{NC}(S_n, c)$ because if a given noncrossing partition π_1 is coarser than some other partition π_2, then the coarsest element $\overline{\pi}_1'$ such that $\pi_1 \cup \overline{\pi}_1'$ is still noncrossing must be finer than the coarsest element $\overline{\pi}_2'$ such that $\pi_2 \cup \overline{\pi}_2'$ is still noncrossing. $\square$

There is an easy relation between the Kreweras complement on $\mathrm{NC}(S_n, c)$ and the Kreweras complement on $\mathrm{NC}(S_n, c')$.

Proposition 3.4. Let $c \xrightarrow{k} c'$ and define

$$\alpha_{c,k} \colon \mathrm{NC}(S_n, c) \to \mathrm{NC}(S_n, c')$$

(7)
$$\alpha_{c,k}(\pi) := s_k \pi s_k.$$

Then $\mathrm{Krew}_{c'} \circ \alpha_{c,k} = \alpha_{c,k} \circ \mathrm{Krew}_c$.

Proof. Write $s = s_k$. Since $\mathrm{Krew}_c(\pi) = \pi^{-1}c$, we have

$$\alpha_{c,k}(\mathrm{Krew}_c(\pi)) = s\,\mathrm{Krew}_c(\pi)s = s\pi^{-1}cs = (s\pi^{-1}s)(scs)$$
$$= \mathrm{Krew}_{c'}(s\pi s) = \mathrm{Krew}_{c'}(\alpha_{c,k}(\pi)). \qquad \square$$

4. Nonnesting partitions and the Kroweras complement

In this section, we define *nonnesting partitions* and the *Kroweras complement*.

4.1. Nonnesting partitions.
Let $(P, \leq)$ be a poset. An *order ideal* of P is a subset $I \subseteq P$ such that if $x \in I$ and $y \leq x$, then $y \in I$. Let $J(P)$ denote the set of order ideals of P.

Definition 4.1. For W a Weyl group with positive root poset Φ^+, the *nonnesting partitions* are the order ideals in the set

$$\mathrm{NN}(S_n) := J(\Phi^+).$$

The *support* of a nonnesting partition $p \in \mathrm{NN}(W)$ is the set $\mathrm{Supp}(p)$ of simple roots that lie in p.

Definition 4.1 illustrates Incongruity (2) from Section 1.2: unlike $\mathrm{NC}(W, c)$, the definition of nonnesting partitions does *not* require the choice of a Coxeter element (or, equivalently, an ordering of the simple roots).

4.2. **The Kroweras complement.** There is a natural map *rowmotion* defined on $J(P)$ that takes an order ideal I to the order ideal $\mathrm{Row}(I)$ generated by the minimal elements of P not in I [SW12] (see Equation (3)). In [Pan09], Panyushev conjectured that rowmotion had order h or $2h$ on the set of nonnesting partitions. This conjecture was refined by Bessis and Reiner to the statement that there should be an equivariant bijection between noncrossing partitions under Krew_c and nonnesting partitions under Row [BR11]. Such a bijection was recursively constructed by Armstrong, Stump, and Thomas for bipartite Coxeter element [AST13].

Let c be a Coxeter element and let $\mathrm{inv}(\mathsf{w}_\circ(c)) = [\alpha^{(1)}, \alpha^{(2)}, \dots, \alpha^{(N)}]$ be the inversion sequence associated to the c-sorting word for the long element $w_\circ$ in Equation (6). In [TW19], a subset of the authors showed that it is possible to compute the Kreweras complement as a sequence of flips in $\mathrm{inv}(\mathsf{w}_\circ(c))$ order on the c-Cambrian lattice under the tag line "rowmotion in slow motion."

For a poset $(P, \leq)$, define the *toggle* at $x \in P$ to be the bijection $\mathrm{tog}_x \colon J(P) \to J(P)$ defined by

$$(8) \qquad \mathrm{tog}_x(I) := \begin{cases} I \cup \{x\} & \text{if } x \notin I \text{ and } I \cup \{x\} \in J(P) \\ I \setminus \{x\} & \text{if } x \in I \text{ and } I \setminus \{x\} \in J(P) \\ I & \text{otherwise} \end{cases} .$$

Any toggle tog_x is an involution, and tog_x commutes with tog_y if x does not cover y and y does not cover x. Write

$$\mathrm{tog}_{[x_1, x_2, \dots, x_m]} = \mathrm{tog}_{x_m} \circ \mathrm{tog}_{x_{m-1}} \circ \cdots \circ \mathrm{tog}_{x_1}.$$

It turns out that Row can also be written as a product of toggles [CFDF95, SW12]; for nonnesting partitions, Row can be computed by toggling each root of the root poset in order of height (or *row*). It is Coxeter-theoretically natural to consider what happens if one instead applies the "slow motion" Kreweras complement for nonnesting partitions—using toggles instead of flips.

Definition 4.2. As in Equation (2), we define the *c-Kroweras complement* on $\mathrm{NN}(S_n)$ by

$$\mathrm{Krow}_c(p) := \mathrm{tog}_{\mathrm{inv}(\mathsf{w}_\circ(c))}(p).$$

Example 4.3. Using the data from Example 2.8, we have that

$$\mathrm{Krow}_c = \mathrm{tog}_{[(23),(13),(24),\dots,(15),(78),(35),(45),(12)]}$$
$$= \mathrm{tog}_{(12)} \circ \mathrm{tog}_{(45)} \circ \mathrm{tog}_{(35)} \circ \mathrm{tog}_{(78)} \circ \mathrm{tog}_{(15)} \circ \cdots \circ t_{(24)} \circ \mathrm{tog}_{(13)} \circ \mathrm{tog}_{(23)}.$$

Remark 4.4. The map Krow_c does not depend on the choice of reduced word c for c: all reduced words for c are related by a sequence of commutations of non-adjacent transpositions, and each commutation only affects the sequence $\alpha^{(k)}$ by interchanging pairs of roots that are not adjacent in the root poset. By the remark after Equation (8), this, therefore, does not change the composition of toggles. We may therefore view Krow_c as the composition of toggles according to any linear extension of the poset obtained from the transitive closure of the Auslander–Reiten quiver $\mathrm{AR}(c)$.

Remark 4.5. For the linear Coxeter element $c_1 = (1, 2, \dots, n)$, Krow_{c_1} toggles roots in lexicographic order $[(1, 2), (1, 3), \dots, (n - 1, n)]$, which is equivalent under commuting toggles to the sequence that toggles the elements of the root poset from left to right. In [SW12], this map was termed *promotion* because—viewing a nonnesting partition as a standard Young tableau of shape $2 \times n$—Krow_{c_1} recovers Schützenberger promotion on standard Young tableaux. The main result of [SW12] showed that promotion on nonnesting partitions was conjugate by toggles to rowmotion. Our K-*row*-eras complement is a reference to this second map.

Example 4.6. Let $c = s_1 s_3 s_2 \in S_4$. The c-sorting word for $w_\circ$ is $s_1 s_3 s_2 s_1 s_3 s_2$, and the corresponding ordering of positive roots is

$$\alpha^{(1)} = (1, 2), \quad \alpha^{(2)} = (3, 4), \quad \alpha^{(3)} = (1, 4), \quad \alpha^{(4)} = (2, 4), \quad \alpha^{(5)} = (1, 3), \quad \alpha^{(6)} = (2, 3).$$
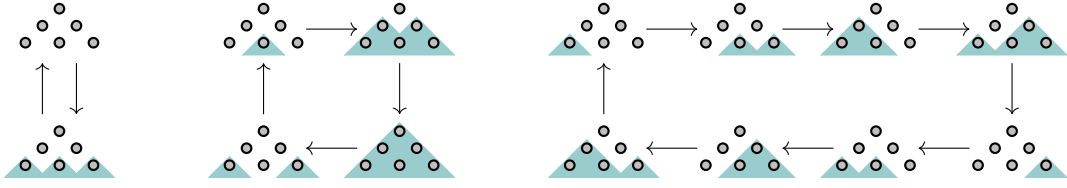
Figure 7 illustrates the orbits of Krow_c on $\mathrm{NN}(4)$.

FIGURE 7. Orbits of $\mathrm{Krow}_{s_1 s_3 s_2} = \mathrm{tog}_{(2,3)} \mathrm{tog}_{(1,3)} \mathrm{tog}_{(2,4)} \mathrm{tog}_{(1,4)} \mathrm{tog}_{(3,4)} \mathrm{tog}_{(1,2)}$ on the nonnesting partitions $\mathrm{NN}(4)$.

In contrast to the noncrossing side, we must work harder on the nonnesting side to relate the toggle sequence Krow_c with the toggle sequence $\mathrm{Krow}_{c'}$. Our goal is to find a conjugating sequence β of toggles with $\beta \circ \mathrm{Krow}_c \circ \beta^{-1} = \mathrm{Krow}_{c'}$.

As $s = s_k$ is initial in c, we fix the cycle notation for c as $(a_1, \ldots, a_k, a_{k+1}, \ldots, a_n)$ with $a_1 = k$, $a_{k+1} = k+1$, $\{a_1, \ldots, a_k\} = \{1, \ldots, k\}$, and $\{a_{k+1}, \ldots, a_n\} = \{k+1, \ldots, n\}$.

Define the sequences of roots

$$\mathsf{T}_{(k,\bullet)} := [(k, a_n), \ldots, (k, a_{k+3}), (k, a_{k+2})],$$
$$\mathsf{T}_{(\bullet,k+1)} := [(a_k, k+1), \ldots, (a_3, k+1), (a_2, k+1)],$$
$$(9) \qquad \mathsf{T}_k := \mathsf{T}_{(k,\bullet)} \sqcup \mathsf{T}_{(\bullet,k+1)}, \text{ and}$$
$$\mathsf{T}_k' := s\mathsf{T}_k = [(k+1, a_n), \ldots, (k+1, a_{k+3}), (k+1, a_{k+2}), (a_k, k), \ldots, (a_3, k), (a_2, k)],$$

where we use $\sqcup$ to denote concatenation. For example, the roots in T_6 and T_6' are circled in Figure 4.

Proposition 4.7. Let $c \xrightarrow{k} c'$ and define

$$\beta_{c,k} : \mathrm{NN}(S_n) \to \mathrm{NN}(S_n)$$
$$(10) \qquad \beta_{c,k} := \mathrm{tog}_{(k,k+1)} \circ \mathrm{tog}_{\mathsf{T}_k} \circ \mathrm{tog}_{(k,k+1)}.$$

Then $\mathrm{Krow}_{c'} \circ \beta_{c,k} = \beta_{c,k} \circ \mathrm{Krow}_c$.

Proof. We show that $\mathrm{Krow}_c \circ \beta_{c,k}^{-1} = \beta_{c,k}^{-1} \circ \mathrm{Krow}_{c'}$. By the description of $\mathrm{AR}(c)$ in Section 2.4 and Remark 4.4, $\beta_{c,k}$ is the restriction of Krow_c to the roots that lie in T_k. Let $\mathrm{Krow}_{c,k}$ be the toggle sequence obtained from Krow_c by removing the toggles of the form tog_t for $t \in \mathsf{T}_k$. Then $\mathrm{Krow}_c \circ (\beta_{c,k})^{-1} = \mathrm{Krow}_{c,k}$, since (as shown in Figure 5 and proved in Lemma 2.9) the roots which precede those of T_k (other than $(k, k+1)$) are of the form (i, j) with $i, j \leq k$ or $i, j \geq k+1$, and consequently none of these roots cover or are covered by roots in T_k, so their toggles commute with those from T_k.

Write $s = s_k$. By Proposition 2.10, $\mathrm{Krow}_{c'}$ is obtained from Krow_c by moving $\mathrm{tog}_{(k,k+1)}$ from the beginning of the toggle sequence to the end, and replacing $\mathrm{tog}_{(i,j)}$ by $\mathrm{tog}_{(s(i),s(j))}$. Almost all roots are unchanged by this replacement, which has the simple effect of swapping the roots in T_k with the roots in T_k'.

So if $\mathrm{Krow}_{c',k}$ is the toggle sequence obtained from $\mathrm{Krow}_{c'}$ by removing the toggles of the form tog_t for $t \in \mathsf{T}_k$, then the same reasoning as above shows that $(\beta_{c,k})^{-1} \circ \mathrm{Krow}_{c'} = \mathrm{Krow}_{c',k}$.

It remains to show that $\mathrm{Krow}_{c,k} = \mathrm{Krow}_{c',k}$. Every tog_t for $t \in \mathsf{T}_k'$ commutes with any tog_r with r in the rectangle defined by the diagonals labeled by k and $k+1$— except those tog_r with $r \in \mathsf{T}_k$. But these toggles no longer appear in either $\mathrm{Krow}_{c,k}$ or $\mathrm{Krow}_{c',k}$, so that we can commute the toggles from T_k' in $\mathrm{Krow}_{c,k}$ to their positions in $\mathrm{Krow}_{c',k}$. Finally, since $\mathrm{tog}_{(k,k+2)}$ and $\mathrm{tog}_{(k-1,k+1)}$ have been removed from $\mathrm{Krow}_{c,k}$ and $\mathrm{Krow}_{c',k}$, $\mathrm{tog}_{(k,k+1)}$ commutes with all remaining toggles. $\square$

5. CHARMED BIJECTIONS

In this section, we define a general family of *charmed bijections* between *balanced pairs* of subsets and nonnesting partitions. Our charmed bijections depend on a choice of decoration of the roots in $\Phi^+ = \Phi^+(A_{n-1})$, and use certain *intimate* families of lattice paths as intermediate objects. Specializing these decorations to those arising from the choice of Coxeter element c, we obtain the *c-charmed bijections* between $\mathrm{NC}(S_n, c)$ and $\mathrm{NN}(S_n)$.

5.1. **Balanced pairs and noncrossing partitions.**

Definition 5.1. Say that a pair of sets (O, I) with $O, I \subseteq [n]$ is *balanced* if $|O| = |I|$ and $|O \cap [k]| \geq |I \cap [k]|$ for all $1 \leq k \leq n$. Write $\mathrm{Bal}(n)$ for all balanced pairs of subsets of $[n]$.

We first show that balanced pairs are naturally in bijection with c-noncrossing partitions. Let $\pi \in \mathrm{NC}(S_n, c)$. We define $O(\pi)$ to be the set of integers i for which there exists an $j > i$ in the same block as i, and we define the set $I(\pi)$ to be the set of integers j such that there exists an $i < j$ in the same block as j. It is immediate from the definition that $(O, I) \in \mathrm{Bal}(n)$.

Proposition 5.2. The map[2] $\pi \mapsto (O(\pi), I(\pi))$ is a bijection between $\mathrm{NC}(S_n, c)$ and $\mathrm{Bal}(n)$.

Proof. We construct its inverse. For a given pair $(O, I) \in \mathrm{Bal}(n)$, we can construct a $\pi \in \mathrm{NC}(S_n, c)$ with $O(\pi) = O$ and $I(\pi) = I$ as follows. The singletons of π are the integers that are neither in O nor I. We place each integer in $O \setminus I$ in its own block and call these blocks *open*. Then we add iteratively the remaining integers of $O \cup I$ to the open blocks, starting with the smallest integer, such that the intermediate partition is always noncrossing. This is achieved by adding an integer x to the first open block we visit when walking from x towards n via 1 in the cycle notation of c. If an integer in $I \setminus O$ is added to a block we call this block *closed* and thereafter do not add any integers to it. By construction we have $O(\pi) = O$ and $I(\pi) = I$. $\square$

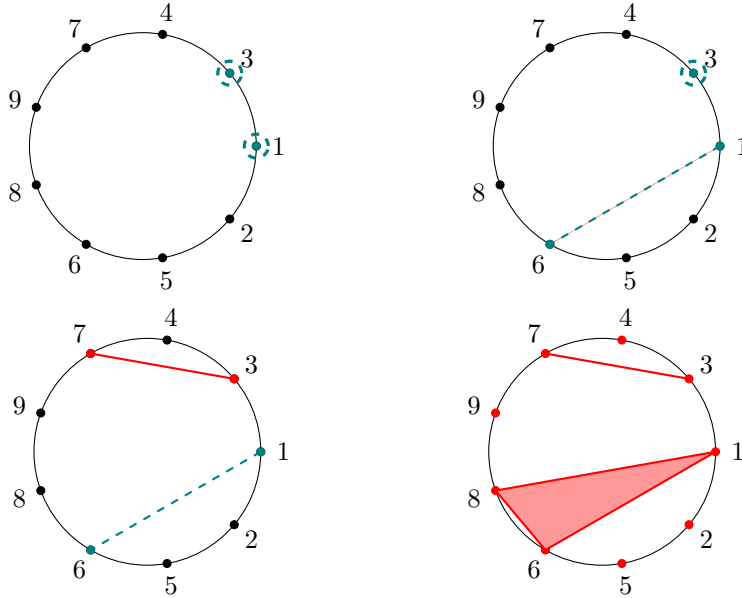The bijection of Proposition 5.2 is illustrated in Figure 8.

FIGURE 8.   The construction of a noncrossing partition in $\mathrm{NC}(S_9, c)$ for $c = s_2 s_1 s_3 s_6 s_5 s_4 s_8 s_7 = (1\,3\,4\,7\,9\,8\,6\,5\,2)$ starting with the outgoing set $O = \{1, 3, 6\}$ and incoming set $I = \{6, 7, 8\}$. We depict open blocks in teal and dashed, and closed blocks in red. For better visibility, we circled open and closed singletons using the same color code.

5.2. **Intimate families.** Fix n a positive integer and consider the type A_{n-1} positive root poset $\Phi^+ := \Phi^+(A_{n-1})$ on the positive roots $\{(i, j) \mid 1 \leq i < j \leq n\}$. We draw the root (i, j) in the plane with coordinates $(i + (j - i)/2, j - i)$ and—since the label (i, j) is implied by the position—we may omit the labels on the roots. For $1 \leq i \leq n$, we draw n additional points labeled by i at coordinates $(i - 1, 0)$ and call these extra points *integral vertices*. For $\heartsuit \subseteq \Phi^+$, we call a root (i, j) *charmed* if $(i, j) \in \heartsuit$ and *ordinary* otherwise. We depict charmed roots using hearts $\heartsuit$ and ordinary roots using circles—an example of c-charmed roots is illustrated in Figure 2.

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[2]To be precise, both O and I depend on c, however since c will always be clear from the context we omit it in the notation.
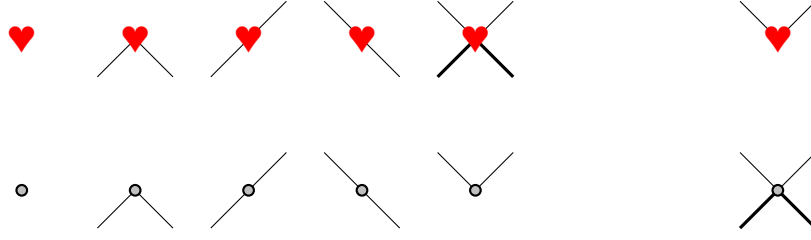
FIGURE 9. *Left:* the five allowed local configurations for charmed families. *Right:* the two local configurations are forbidden for charmed families.

A *path* is a lattice path with step set $\{(.5, 1), (.5, -1)\}$ that starts and ends at an integral vertex and stays strictly above the x-axis. We call $(.5, 1)$-steps *up*, and $(.5, -1)$-steps *down*; a *peak* (resp. *valley*) of a path is a root contained in an up step to its left (resp. right) and a down step to its right (resp. left). Two paths are *kissing* if they do not cross or share edges—they may *meet* at a vertex, where they are said to *kiss*. (Note, though, that kisses are not required for two paths to be kissing.) A family of paths is *kissing* if they are pairwise kissing. A path *feints* at a root (i, j) if (i, j) is a valley, but the path does not kiss any path at (i, j). These definitions are illustrated on the right of Figure 9: the top configuration is a feint at a charmed root, while the bottom one is a kiss at an ordinary root.

For $\heartsuit \subseteq \Phi^+$, a family $\mathcal{L}$ of kissing paths is called $\heartsuit$-*charmed* if:

- paths only kiss at charmed roots and
- paths only feint at ordinary roots.

In other words, a family of paths is charmed if it avoids the two local configurations shown on the right of Figure 9.

Definition 5.3. A family $\mathcal{L}$ of $\heartsuit$-charmed kissing paths is called $\heartsuit$-*intimate* if:

- every ordinary root either lies above all paths in $\mathcal{L}$ or is contained in some path in $\mathcal{L}$ and
- no path contains a root above a charmed peak of a path in $\mathcal{L}$, unless that charmed peak is the location of a kiss.

5.3. **Balanced pairs and intimate families.** We now relate balanced pairs and intimate families of paths. For a family $\mathcal{L}$ of paths, we call an integral vertex on the x-axis *outgoing* if it is incident to an up step and *incoming* if it is incident to a down step (an integral vertex can be both outgoing and incoming). Denote by $\text{Out}(\mathcal{L})$ (resp. $\text{In}(\mathcal{L})$) the set of labels of outgoing (resp. incoming) vertices of $\mathcal{L}$. It is clear that $(\text{Out}(\mathcal{L}), \text{In}(\mathcal{L}))$ is balanced.

Lemma 5.4. Let $\heartsuit \subseteq \Phi^+$ and let $(O, I) \in \text{Bal}(n)$. Then there is a unique $\heartsuit$-inimate family $\mathcal{L}_{(O,I)}$ with $\text{Out}(\mathcal{L}_{(O,I)}) = O$ and $\text{In}(\mathcal{L}_{(O,I)}) = I$.

Proof. We first construct a well-formed word of parentheses from the subsets O and I. For i from 1 to n, write:

- a $(_i$ parenthesis if $i \in O \setminus I$,
- a $)_i$ parenthesis if $i \in I \setminus O$, and
- $)_i(_i$ parentheses if $i \in O \cap I$.

We now construct an $\heartsuit$-intimate family $\mathcal{L}$ recursively, starting with the empty family of paths. At each step, we pick neighbouring parentheses of the form $(_i)_j$, delete them, and add the path P that starts at i and ends at j that takes a down step whenever possible without violating the condition that the family is charmed, and an up step otherwise. Then $\mathcal{L} \cup \{P\}$ is intimate:

- If there were an ordinary root below $\mathcal{L} \cup \{P\}$ that wasn't part of a path, then that root would lie between p and $\mathcal{L}$, since $\mathcal{L}$ was intimate. But then P took an up step instead of a possible down step, contradicting the definition of P.
- If a previously constructed path P' in $\mathcal{L}$ started at an integral vertex after i, ended before j, and had a charmed peak which is not the location of a kiss, then our new path P will kiss P' at that charmed peak.

The order of choosing two neighbouring parentheses is irrelevant. The family produced is unique, since if at any point a path uses a step different from those prescribed by the algorithm above, then the resulting family of paths will be non-intimate. This non-intimacy will persist, regardless of how the family is extended. □

An example of the algorithm used in the proof of Lemma 5.4 is given in Figure 10.
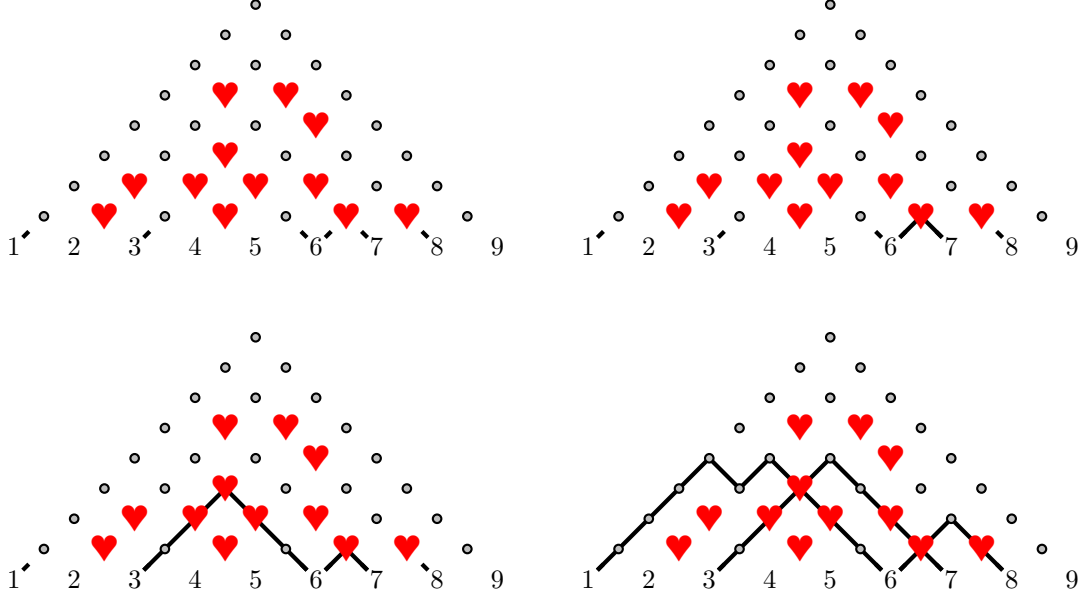

FIGURE 10.   The construction of an intimate family of paths with outgoing set $\{1, 3, 6\}$ and incoming set $\{6, 7, 8\}$. The corresponding word of parentheses is $(_1(_3)_6(_6)_7)_8$.

5.4. **Intimate families and nonnesting partitions.** Let $\mathcal{L}$ be an ♥-intimate family of paths. We define the order ideal $J(\mathcal{L})$ of $\mathcal{L}$ to be the set of all roots (i, j) which lie on or below a path in $\mathcal{L}$. It is clear that $J(\mathcal{L})$ is an order ideal and hence is in $\mathrm{NN}(S_n)$.

Lemma 5.5. Let $♥ \subseteq \Phi^+$ and $J \in \mathrm{NN}(S_n)$. Then there exists a unique ♥-intimate family $\mathcal{L}_J$ with $J(\mathcal{L}_J) = J$.

Proof. We construct an ♥-intimate family $\mathcal{L}$ recursively, starting with the empty family of paths. At each step, we add a maximal path p to $\mathcal{L}$ such that all roots contained in p lie in J. We then replace J by the order ideal generated by all ordinary roots in J not contained in a path of $\mathcal{L}$, all charmed feints of paths of $\mathcal{L}$, and roots in J not lying below a path of $\mathcal{L}$. The recursion stops when J is empty. It is clear that the resulting family $\mathcal{L}$ is the unique ♥-intimate family of paths with order ideal J. □

An example of the algorithm used in the proof of Lemma 5.5 is given in Figure 11.

5.5. **Charmed bijections between balanced pairs and nonnesting partitions.** As a direct consequence of Lemmas 5.4 and 5.5, we obtain the following family of bijections between balanced pairs and nonnesting partitions.

Proposition 5.6. Fix a collection of charmed roots $♥ \subseteq \Phi^+$. Then the map $J_♥ \colon \mathrm{Bal}(n) \to \mathrm{NN}(S_n)$ defined by $J_♥(O, I) = J(\mathcal{L}_{(O,I)})$ is a bijection.

Remark 5.7. Charmed roots along the upper boundary of Φ^+ do not affect the bijection of Proposition 5.6. On the other hand, each of the $2^{\binom{n-2}{2}}$ charming choices for the the roots (i, j) with $1 < i < j < n$ gives rise to a distinct bijection between $\mathrm{NN}(S_n)$ and $\mathrm{Bal}(n)$, as we now prove.

Let $♥ \subseteq \Phi^+$ and $p \in \mathrm{NN}(S_n)$. We need the following two observations.

(1) The preimage $J_♥^{-1}(p)$ depends only on p and $♥ \cap p$, i.e., $J_♥^{-1}(p) = J_{♥ \cap p}^{-1}(p)$.
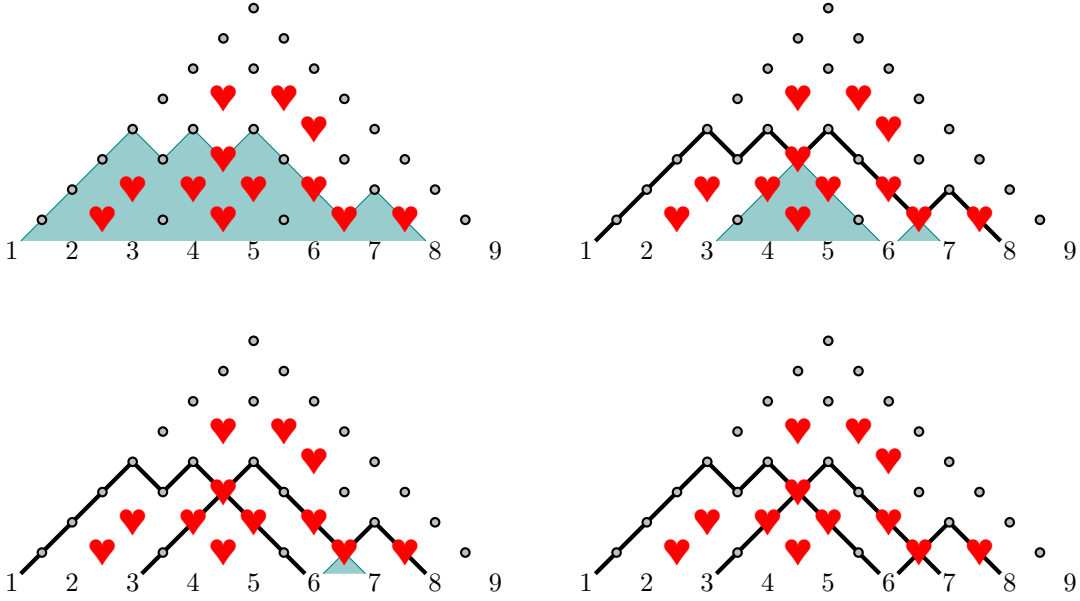
FIGURE 11. The construction of the intimate family of lattice paths with order ideal J, where J contains the roots in the grey-shaded region of the top left picture. At each step, the order ideal under consideration consists of the roots contained in the teal-shaded region.

(2) Let (O, I) be the preimage $J_{\heartsuit}^{-1}(p)$ and P be the first path in the recursive construction of the $\heartsuit$-intimate family of paths as in the proof of Lemma 5.5 and p' be the corresponding order ideal. Denote by $(O', I') = J_{\heartsuit}^{-1}(p') = J_{\heartsuit \cap p'}^{-1}(p')$. Then O is the union of O' and the starting point of P, and I is the union of I' and the end point of P.

Let $\heartsuit_1$ and $\heartsuit_2$ be different subsets of $\{(i, j) \mid 1 < i < j < n\}$ and assume that the bijections $J_{\heartsuit_1}$ and $J_{\heartsuit_2}$ are equal. Denote by (i_0, j_0) a root with $j_0 - i_0$ minimal such that it is in exactly one of the two sets $\heartsuit_1$ or $\heartsuit_2$; without loss of generality, we assume $(i_0, j_0) \in \heartsuit_1 \setminus \heartsuit_2$. Let p be the order ideal generated by the roots $(i_0 - 1, j_0)$ and $(i_0, j_0 + 1)$ and let $(O, I) = J_{\heartsuit_1}^{-1}(p) = J_{\heartsuit_2}^{-1}(p)$. Denote by P_1 and p_1' (resp. P_2 and p_2') the path and order ideal appearing in the recursive construction of $J_{\heartsuit_1}^{-1}(p)$ (resp. $J_{\heartsuit_2}^{-1}(p)$) as described above. The paths P_1 and P_2 coincide since both are maximal. This implies by the second observation $J_{\heartsuit_1}^{-1}(p_1') = J_{\heartsuit_2}^{-1}(p_2')$. Since $(i_0, j_0) \notin \heartsuit_2$, the order ideal p_2' is by construction contained in the set of roots $\{(i, j) \in \Phi^+; (i, j) < (i_0, j_0)\}$. By minimality of (i_0, j_0) we therefore have $p_2' \cap \heartsuit_1 = p_2' \cap \heartsuit_2$. The first above observation implies

$$J_{\heartsuit_2}^{-1}(p_2') = J_{\heartsuit_2 \cap p_2'}^{-1}(p_2') = J_{\heartsuit_1 \cap p_2'}^{-1}(p_2') = J_{\heartsuit_1}^{-1}(p_2'),$$

and hence $J_{\heartsuit_1}^{-1}(p_1') = J_{\heartsuit_1}^{-1}(p_2')$. However since $J_{\heartsuit_1}$ is a bijection and $p_1' \neq p_2'$ this is a contradiction.

5.6. **Charmed bijections between noncrossing and nonnesting partitions.** Let $c \in S_n$ be a Coxeter element. Using Proposition 5.2 and Proposition 5.6, we now construct a bijection between c-noncrossing partitions and nonnesting partitions *that depends on the choice of Coxeter element c.* This addresses Incongruity (2) from Section 1.2.

Definition 5.8. The *c-charmed bijection* between c-noncrossing partitions and nonnesting partitions is given by

$$\mathrm{Charm}_c \colon \mathrm{NC}(S_n, c) \to \mathrm{NN}(S_n)$$
$$\pi \mapsto J_{\heartsuit_c}(O(\pi), I(\pi)),$$

where the set $\heartsuit_c$ is defined in Definition 2.4 and the map $J_{\heartsuit_c}$ is defined in Proposition 5.6.

In Section 6, we will show that Charm_c is the desired bijection from the introduction. We give also an alternative description for the inverse in Corollary 6.3 which allows one to directly read the blocks of the noncrossing partition from the corresponding intimate family.

6. Proof of the main theorem

In this section, we prove Theorem 1.5, which states that the bijection Charm_c from Section 5.6 is the unique support-preserving bijection between noncrossing and nonnesting partitions that is equivariant with respect to Krew_c and Krow_c.
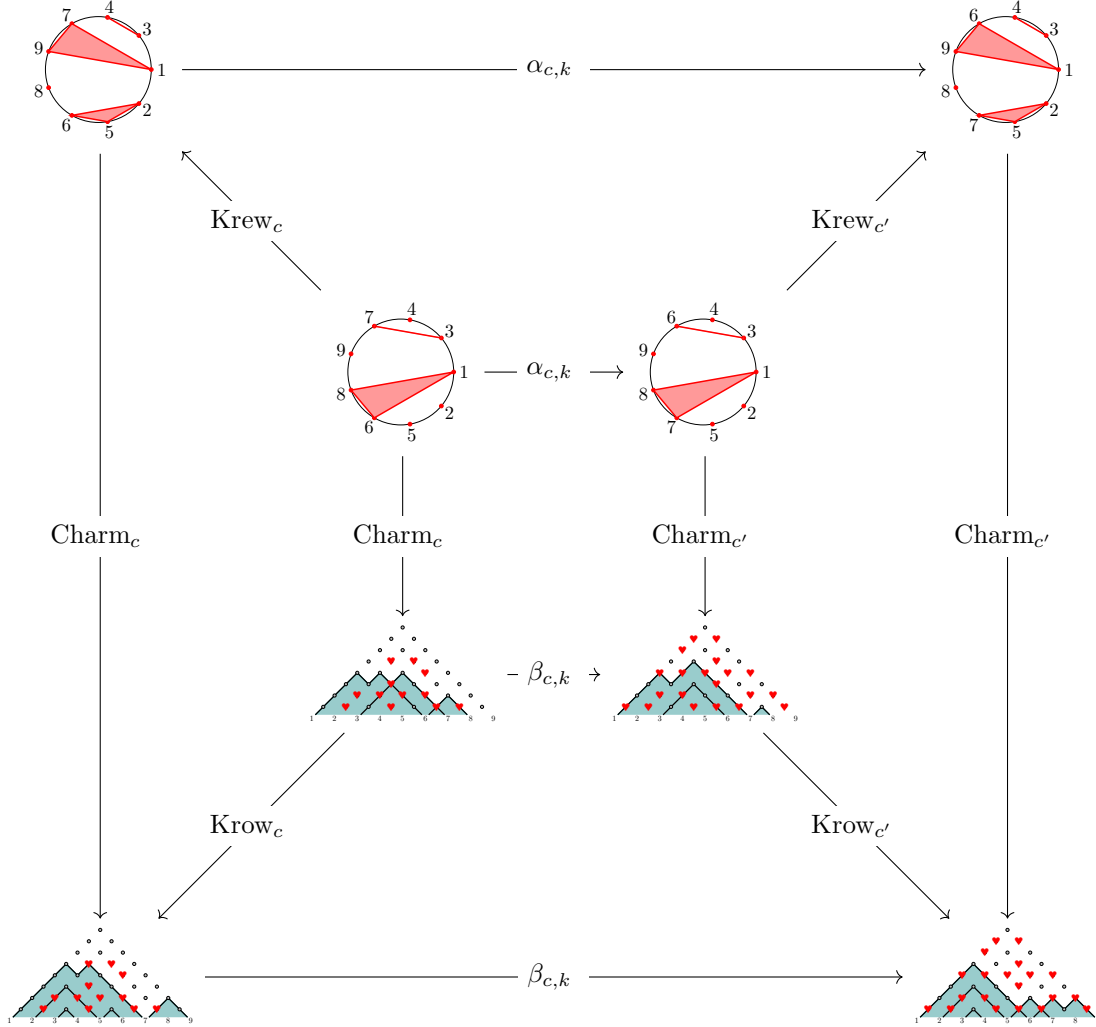

FIGURE 12. Figure 1, where the vertices previously labeled by the sets of non-crossing and nonnesting partitions are now replaced by our running example for the choice of Coxeter element $c = s_2 s_1 s_3 s_6 s_5 s_4 s_8 s_7 \in S_9$ with cycle notation $(1\,3\,4\,7\,9\,8\,6\,5\,2)$.

6.1. Cambrian induction.
Fix the symmetric group S_n, let c be a Coxeter element, and let $s = s_k = (k, k+1)$ be initial in c. Then $c' = scs$ is also a Coxeter element of S_n, and recall from Section 2.3 that we write $c \xrightarrow{k} c'$.

We will prove Theorem 1.5 using *Cambrian induction*—that is, we will show that the theorem holds for a particular Coxeter element c_1 (the *base case*), and then we will show that if $c \to c'$ and the theorem holds for c, then the theorem also holds for c' (the *inductive step*). By Lemma 2.2, all Coxeter elements in S_n are conjugate by a sequence of conjugations by initial simple reflections, so the theorem holds for all Coxeter elements.

Remark 6.1. Note that this proof technique differs from the usual Cambrian induction used in [Rea07, AST13]: the base case of our proof is the statement for the linear Coxeter element

in S_n, and not the statement for the Coxeter group S_2—that is, we do not need to descend to parabolic subgroups.

6.2. Base case. For the base case, we use the linear Coxeter element $c_1 = s_1 s_2 \cdots s_{n-1}$, with one-line notation $(1\,2\,\ldots\,n)$. We show that Charm_c is an equivariant bijection between $\mathrm{NC}(S_n, c_1)$ under Krew_{c_1} and $\mathrm{NN}(S_n)$ under Krow_{c_1}. In this case, Krow_{c_1} is order ideal promotion, as defined by Striker and Williams [SW12] (see Remark 4.5), which toggles roots in lexicographic order. There are well-known equivariant bijections (see [Rho10, proof of Theorem 8.2] and [SW12, Theorems 3.9 and 3.10])

$$\begin{pmatrix} \mathrm{NC}(S_n, c_1), \\ \text{Kreweras complement} \end{pmatrix} \xrightarrow{\text{Match}} \begin{pmatrix} \text{noncrossing matchings of } [2n], \\ \text{clockwise rotation} \end{pmatrix} \xrightarrow{\text{Dyck}} \begin{pmatrix} \mathrm{NN}(S_n), \\ \text{promotion} \end{pmatrix}.$$

The bijection Match from a noncrossing partition π on $\{1, 2, \ldots, n\}$ to a noncrossing matching on $\{1, \overline{1}, 2, \overline{2}, \ldots, n, \overline{n}\}$ goes by a "thickening" operation, as illustrated at the top left of Figure 13; thickening produces even-length cycles, and to get a matching from an even-length cycle, we take the edges starting from a non-barred letter and going clockwise (which end on a barred letter by construction). To produce a nonnesting partition from a noncrossing matching, the bijection Dyck reads each vertex $1, \overline{1}, 2, \overline{2}, \ldots, n, \overline{n}$ in counter-clockwise order—each letter that opens a new block in the noncrossing matching becomes a NE step in a Dyck path, while each letter closing a block becomes a SE step. For c_1, there are no charmed roots, so the paths in the corresponding intimate family are shifts of the initial Dyck path, removing the lower two strips of steps each time. It is clear from this description that the resulting balanced pair (O, I) recovers the balanced pair $(O(\pi), I(\pi))$ for the original noncrossing partition π, so that $\mathrm{Dyck} \circ \mathrm{Match} = \phi_{c_1}$. This is illustrated on the bottom of Figure 13.
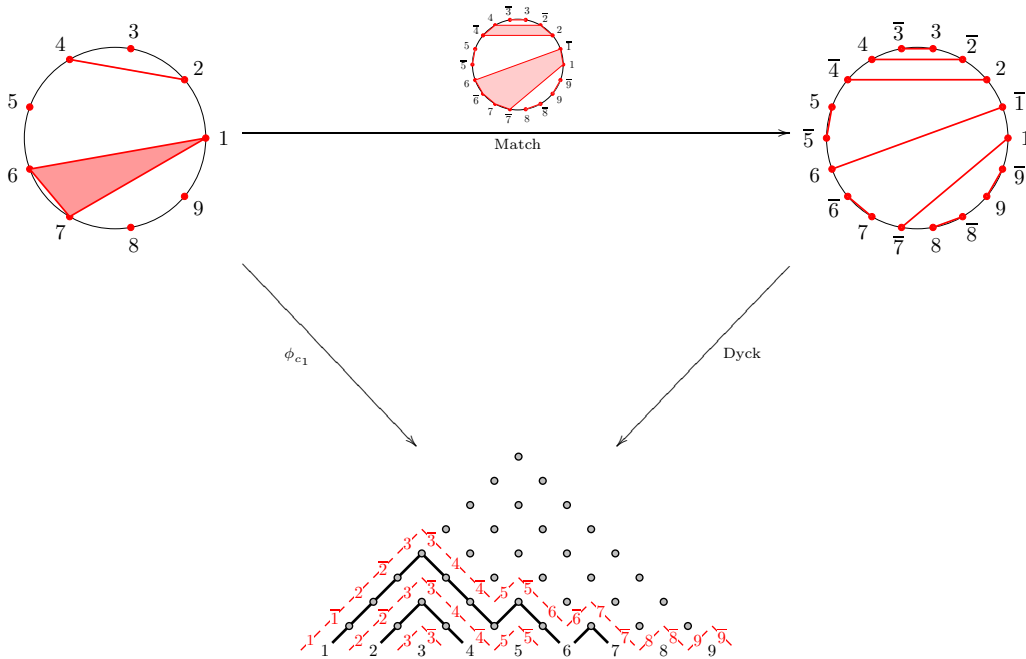

FIGURE 13. The equivariant bijection Match from noncrossing partitions for $c_1 = (1, 2, \ldots, n)$ under the Kreweras complement to noncrossing matchings under clockwise rotation and the equivariant bijection Dyck from noncrossing matchings under clockwise rotation to nonnesting partitions under order ideal promotion. As there are no charmed roots for c_1, this recovers our c_1-charmed bijection ϕ_{c_1}.

6.3. Inductive step. For the inductive step, we show that if $c \xrightarrow{k} c'$ and Charm_c is an equivariant bijection between $\mathrm{NC}(S_n, c)$ under Krew_c and $\mathrm{NN}(S_n)$ under Krow_c, then the statement also holds for c'.

Proposition 6.2. Let $c \xrightarrow{k} c'$. Then $\beta_{c,k} \circ \mathrm{Charm}_c = \mathrm{Charm}_{c'} \circ \alpha_{c,k}$.

Proof. Fix $\pi \in \mathrm{NC}(S_n, c)$ and write $\pi' = \alpha_{c,k}(\pi) \in \mathrm{NC}(S_n, c')$. We will write $(O(\pi), I(\pi))$ and $(O(\pi'), I(\pi'))$ for the corresponding balanced pairs from Proposition 5.2. By definition, we see that $(O(\pi'), I(\pi'))$ is obtained from $(O(\pi), I(\pi))$ by simply interchanging k and $k+1$. Denote by $\mathcal{L}$ (resp. $\mathcal{L}'$) the intimate family of lattice paths obtained when calculating $\mathrm{Charm}_c(\pi)$ (resp. $(\mathrm{Charm}_{c'} \circ \alpha_{c,k})(\pi)$). By Lemma 5.4, $\mathcal{L}'$ is the unique $\heartsuit_{c'}$-intimate family with $\mathrm{Out}(\mathcal{L}') = O(\pi')$ and $\mathrm{In}(\mathcal{L}') = I(\pi')$. Denote by $\mathcal{L}^t$ the unique $\heartsuit_{c'}$-intimate family with $J(\mathcal{L}^t) = \beta_{c,k} \circ \mathrm{Charm}_c(\pi)$. Since $\mathcal{L}'$ is by Lemma 5.5 further the unique $\heartsuit_{c'}$-intimate family with $J(\mathcal{L}') = \mathrm{Charm}_{c'} \circ \alpha_{c,k}(\pi)$, it suffices to show that $\mathrm{Out}(\mathcal{L}^t) = O(\pi') = \mathrm{Out}(\mathcal{L}')$ and $\mathrm{In}(\mathcal{L}^t) = I(\pi') = \mathrm{In}(\mathcal{L}')$.

Let $\mathsf{T}_k, \mathsf{T}_k'$ be the sequences of roots defined in Equation (9). We prove the above assertion by showing that

- $\mathcal{L}$ and $\mathcal{L}^t$ coincide on all roots outside of $\mathsf{T}_k \cup \mathsf{T}_k'$ and that
- $\mathrm{Out}(\mathcal{L}^t) \cap \{k, k+1\}$ (resp. $\mathrm{In}(\mathcal{L}^t) \cap \{k, k+1\}$) is obtained from $\mathrm{Out}(\mathcal{L}) \cap \{k, k+1\}$ (resp. $\mathrm{In}(\mathcal{L}) \cap \{k, k+1\}$) by interchanging k and $k+1$.

Since $\heartsuit_c$ and $\heartsuit_{c'}$ coincide on all roots outside of $\mathsf{T}_k \cup \mathsf{T}_k'$ by Proposition 2.7, it suffices by the construction of $\mathcal{L}$ and $\mathcal{L}^t$ as described in Lemma 5.5 to show that $\mathcal{L}$ and $\mathcal{L}^t$ "enter" and "leave" the region of roots in $\mathsf{T}_k \cup \mathsf{T}_k'$ at the same roots; by symmetry, it suffices to show this only for the region of roots in $(\mathsf{T}_k \cup \mathsf{T}_k') \cap \{(i, j) \mid i < j \leq k+1\}$. We show this by induction on the number of paths in $\mathcal{L}$ which include roots in $(\mathsf{T}_k \cup \mathsf{T}_k') \cap \{(i, j) \mid i < j \leq k+1\}$.

Since $(k, k+1)$ is initial in c, we have that $k+1 \in L_c$ and $k \in R_c$, so that the roots in $\mathsf{T}_{(k, \bullet)}$ that are charmed form an initial subsequence of $\mathsf{T}_{(k, \bullet)}$. Similarly, the roots in $\mathsf{T}_{(\bullet, k+1)}$ that are charmed form an initial subsequence of $\mathsf{T}_{(\bullet, k+1)}$. Note that the toggles tog_t for $t \in \mathsf{T}_{(k, \bullet)}$ commute with the toggles tog_r for $r \in \mathsf{T}_{(\bullet, k+1)}$. Furthermore for $i < k$, a root (i, k) is c-charmed (or c'-charmed resp.) if and only if $(i, k+1)$ is not c-charmed (or c'-charmed resp.). Using this, we can verify our assertion in the base case: Figure 14 shows how the restriction of $\mathcal{L}$ to the region $(\mathsf{T}_k \cup \mathsf{T}_k') \cap \{(i, j) \mid i < j \leq k+1\}$ is mapped to $\mathcal{L}^t$. Note, that it suffices to regard the cases shown in Figure 14 because we assume that $\mathcal{L}$ is an intimate family of paths; for example the left-most path in Figure 14 would not be intimate if there would be an ordinary root below the dashed line.
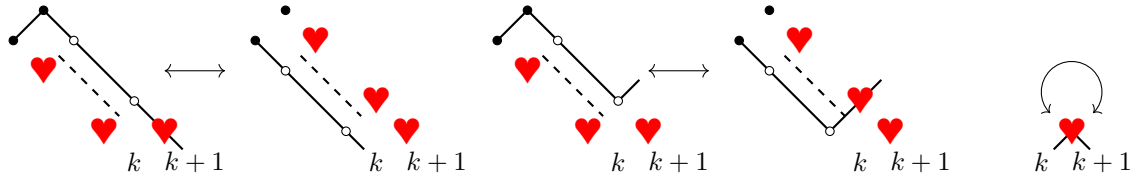

FIGURE 14.   We use the following conventions to depict the different cases concisely. The roots along the dashed lines are all charmed on one side and all ordinary on the other side. The root (i, k) represented by a black dot can be either charmed or ordinary if it is reached by a north-east step. If the root (i, k) is reached by a south-east step then it is charmed if and only if the root $(i-1, k)$ is ordinary.

For the induction step, we regard first the top path in $\mathsf{T}_k \cup \mathsf{T}_k' \cap \{(i, j) \mid i < j \leq k+1\}$, see Figure 15 for all possible cases. We observe that in all cases, the second path from the top behaves similarly to the top path, i.e., it would be mapped to the same path if we omit the top path. This shows the claim by the induction hypothesis. $\square$

We can give an alternative description for reading the blocks of the noncrossing partition from the corresponding intimate family by following the above proof.

Corollary 6.3. Let $\mathcal{L}$ be an $\heartsuit_c$-intimate family and $\pi \in \mathrm{NC}(S_n, c)$ the corresponding noncrossing partition with $(\mathrm{Out}(\mathcal{L}), \mathrm{In}(\mathcal{L})) = (O(\pi), I(\pi))$. The blocks of π consist of the integers which are connected by paths in $\mathcal{L}$ after reinterpreting each kiss between a pair of paths in $\mathcal{L}$ at a charmed root as a crossing.

Proof. We follow analogous steps as in the above proof. For the linear Coxeter element $c_1 = s_1 s_2 \cdots s_{n-1}$, the paths in an $\heartsuit_{c_1}$-intimate family are nonintersecting and by comparing with
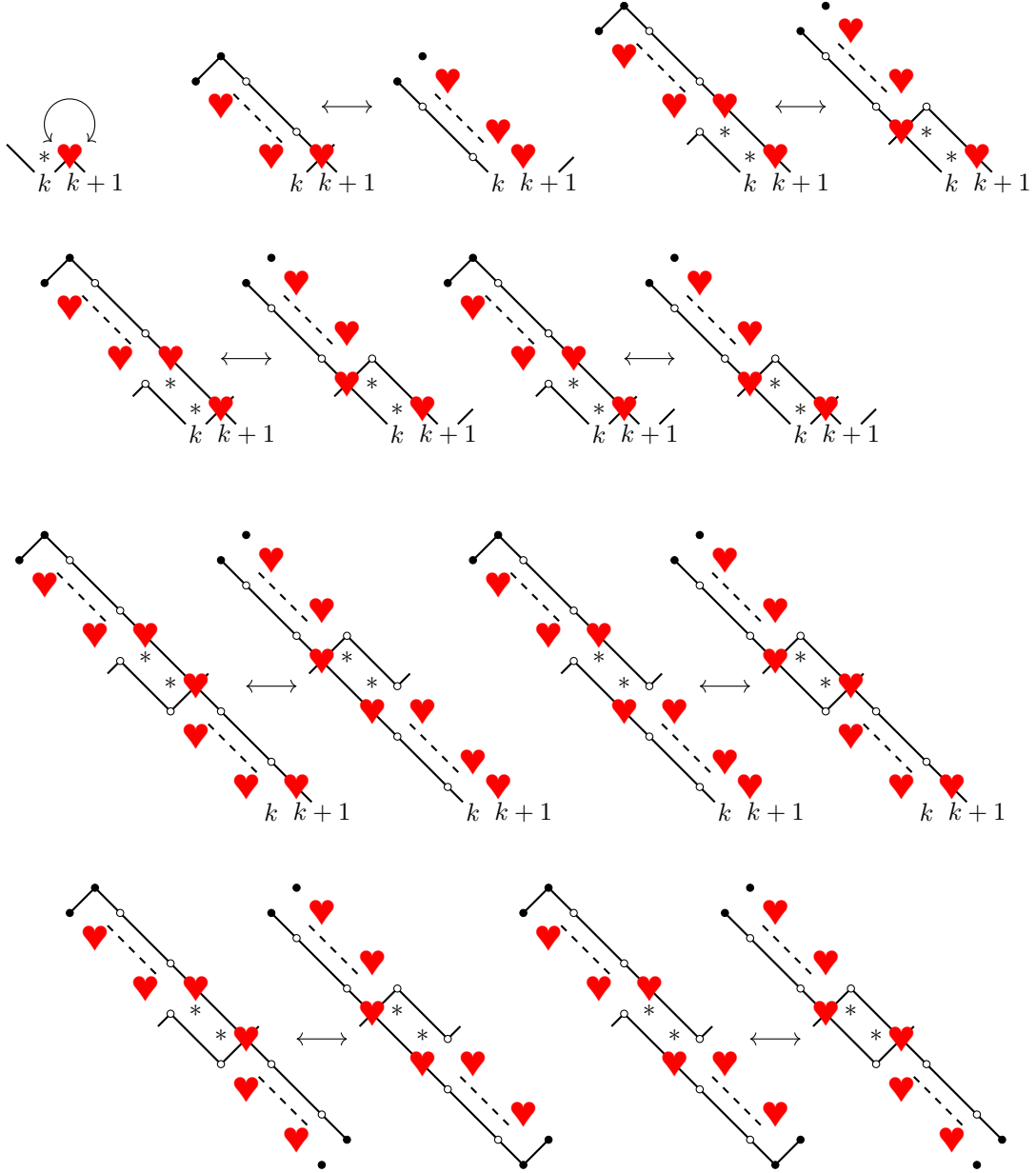
FIGURE 15. All possible configurations of the topmost path(s) in $\mathcal{L}$ together with their counterparts in $\mathcal{L}^t$. We continue the same conventions as in Figure 14. Further, the roots in the regions indicated by $*$s are charmed or ordinary arbitrarily. The root $(i, k+1)$ represented by a black dot at the bottom of the figure can be charmed or ordinary if the path leaves it by a north-east step. If the path leaves $(i, k+1)$ by a south-east step, then $(i, k+1)$ is charmed if and only if $(i+1, k+1)$ is ordinary.

Proposition 5.2 we see that the assertion is true. Next we show the general case by Cambrian induction. For $c \xrightarrow{k} c'$ let $\mathcal{L}'$ be the $\heartsuit_{c'}$-intimate family as defined in the above proof. Then it suffices to show that two integers i, j are connected in $\mathcal{L}$ after reinterpreting kissing paths as crossing paths if and only if $s_k(i), s_k(j)$ are connected in $\mathcal{L}'$ after reinterpreting kissing paths as crossing paths. By the proof of Proposition 6.2 it suffices to check the cases shown in Figure 14 and Figure 15 which imply the assertion. $\qquad\square$
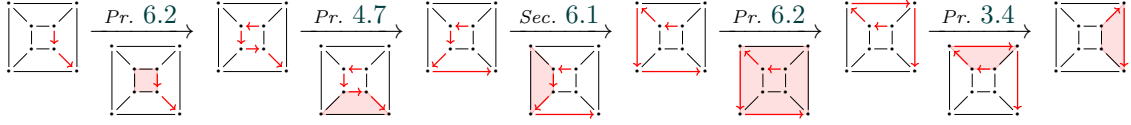
Theorem 6.4. For all Coxeter elements c, the bijection Charm_c is the unique support-preserving bijection between $\mathrm{NC}(S_n, c)$ and $\mathrm{NN}(S_n)$ satisfying $\mathrm{Krow}_c \circ \mathrm{Charm}_c = \mathrm{Charm}_c \circ \mathrm{Krew}_c$.

Proof. We deduce the equivariance of $\mathrm{Charm}_{c'}$ given equivariance of Charm_c using the commutativity of the other faces of the diagram in Figure 1. We compute:

$$
\begin{aligned}
\mathrm{Krow}_{c'} \circ \mathrm{Charm}_{c'} &= \mathrm{Krow}_{c'} \circ \beta_{c,k} \circ \mathrm{Charm}_c \circ \alpha_{c,k}^{-1} & \text{by Proposition 6.2} \\
&= \beta_{c,k} \circ \mathrm{Krow}_c \circ \mathrm{Charm}_c \circ \alpha_{c,k}^{-1} & \text{by Proposition 4.7} \\
&= \beta_{c,k} \circ \mathrm{Charm}_c \circ \mathrm{Krew}_c \circ \alpha_{c,k}^{-1} & \text{by Cambrian induction} \\
&= \mathrm{Charm}_{c'} \circ \alpha_{c,k} \circ \mathrm{Krew}_c \circ \alpha_{c,k}^{-1} & \text{by Proposition 6.2} \\
&= \mathrm{Charm}_{c'} \circ \mathrm{Krew}_{c'} & \text{by Proposition 3.4.}
\end{aligned}
$$

Therefore, if Theorem 1.5 holds for c, then Theorem 1.5 holds for c'. Since we know that the theorem holds for c_1, by Lemma 2.2 we conclude Theorem 1.5 for all (standard) Coxeter elements.

To understand it pictorially, we have

where, with respect to the statements of Proposition 6.2 and Proposition 3.4, we can reverse some arrows because the involved functions are all bijections.

To establish the uniqueness of these bijections, we show that $w \in \mathrm{NC}(S_n, c)$ is uniquely identified by

$$
\mathrm{OrbitSupp}(w) := (\mathrm{Supp}(w), \mathrm{Supp}(\mathrm{Krew}_c(w)), \mathrm{Supp}(\mathrm{Krew}_c^2(w)), \ldots, \mathrm{Supp}(\mathrm{Krew}_c^{2n-1}(w))).
$$

Suppose this claim is established. Making a similar definition for $x \in \mathrm{NN}(S_n)$,

$$
\mathrm{OrbitSupp}(x) := (\mathrm{Supp}(x), \mathrm{Supp}(\rho_c(x)), \mathrm{Supp}(\rho_c^2(x)), \ldots, \mathrm{Supp}(\rho_c^{2n-1}(x))),
$$

the rest of the main theorem tells us that $\phi_c(w)$ satisfies $\mathrm{OrbitSupp}(w) = \mathrm{OrbitSupp}(\phi_c(w))$. Since, by the claim, distinct noncrossing partitions w yield distinct values for $\mathrm{OrbitSupp}(w)$, it must also be that the values of $\mathrm{OrbitSupp}(x)$ are distinct for distinct $x \in \mathrm{NN}(S_n)$. Then ϕ_c is the unique map sending w to the $x \in \mathrm{NN}(S_n)$ such that $\mathrm{OrbitSupp}(w) = \mathrm{OrbitSupp}(x)$.

We now establish the claim that $w \in \mathrm{NC}(S_n, c)$ is uniquely identified by $\mathrm{OrbitSupp}(w)$ (i.e., that the map $\mathrm{OrbitSupp}$ is injective on $\mathrm{NC}(S_n, c)$). Let $w \in \mathrm{NC}(S_n, c)$. Let $1 \leq i < j \leq n$, and suppose that i and j are in different blocks of w. There exists a coarsening w' of w which has only two parts in which i and j are still in different blocks. (Such a coarsening can be obtained by successively choosing a block that contains neither i nor j, but which is adjacent to a block that does contain one of them, and merging the selected block with the adjacent block.) Suppose that in w' the block containing i has k elements, and the block containing j has $n - k$ elements. There is a rotation of w', say $c^{-t} w c^t$ in which the image of the block containing i consists of the numbers $\{1, \ldots, k\}$, while the image of the block containing j consists of $\{k+1, \ldots, n\}$. (This is true since there is a rotation of w' such that the block containing i consists of any k numbers consecutive around the circle, and $\{1, \ldots, k\}$ are the labels of one such set.) It follows that α_k is not in the support of $c^{-t} w' c^t$, and thus it is not in the support of $c^{-t} w c^t$ either. This fact can be read off from $\mathrm{OrbitSupp}(w)$. This allows us to deduce that if $\widetilde{w}$ is a noncrossing partition such that $\mathrm{OrbitSupp}(\widetilde{w}) = \mathrm{OrbitSupp}(w)$, then i and j must be in different blocks, and thus, by applying the same idea with any pair i, j in different blocks, it follows that $\widetilde{w}$ must refine w. Reversing the argument, w must refine $\widetilde{w}$. Thus $\widetilde{w} = w$. $\qquad\square$

7. Future Work

It is natural to wish to extend our bijections to all Dynkin types. A first step is to extend the notion of c-charmed—to this end, let W be an irreducible finite Weyl group with positive roots Φ^+, and let c be a standard Coxeter element. For s a simple reflection, we say that its corresponding simple root α_s is *c-charmed* if s is either initial or final in c. Then a general positive root $\alpha \in \Phi^+$ is *c-charmed* if and only if $s(\alpha)$ is *scs*-charmed.

7.1. **Type** C. In this short subsection, we prove that our techniques immediately extend to type C_n by folding. We note that although the root posets of types B_n and C_n are isomorphic as posets, the labeling of these posets by roots differs in a fundamental way—the labeling of $\Phi^+(C_n)$ is a folding of $\Phi^+(A_{n-1})$, but this is not the case for B_n.

The *hyperoctahedral group* $W(C_n)$ is the group of signed permutations—that is, bijections w from $[\pm n] := \{-n, \ldots, -1, 1, \ldots, n\}$ such that $w(i) = -w(-i)$ for all $1 \le i \le n$. As a Coxeter group, $W(C_n)$ is generated by the simple reflections $s_i^C := (i, i+1)(-i, -i-1) = (\,(i, i+1)\,)$ for $1 \le i < n$ and $s_n^C = (n, -n)$. A simple folding argument gives an embedding $\iota \colon W(C_n) \hookrightarrow S_{2n}$:

$$\iota(s_i^C) = s_i s_{2n-i} = (i, i+1)(2n-i, 2n+1-i) \text{ for } 1 \le i < n \text{ and}$$
$$\iota(s_n^C) = s_n = (n, n+1).$$

Under this folding, any Coxeter element $c^C \in W(C_n)$ is sent to a Coxeter element $c = \iota(c^C)$ of S_{2n}. It is easy to see that $\mathrm{AR}(c^C)$ is a folding of $\mathrm{AR}(c)$ horizontally, and that $\Phi^+(C_n)$ is a folding $\Phi^+(A_{2n})$ vertically—thus, the c^C-Kroweras complement Krow_{c^C} is obtained by folding Krow_c. Similarly, $\iota \colon \mathrm{NC}(W(C_n), c^C) \hookrightarrow \mathrm{NC}(S_{2n}, c)$ sends a noncrossing partition π^C to a noncrossing partition $\pi = \iota(\pi^C)$ that is centrally symmetric, and so the c^C-Kreweras complement is obtained by folding Krew_c. We conclude the following.

Corollary 7.1. Let $W(C_n)$ be the group of signed permutations, and fix a standard Coxeter element $c \in W(C_n)$. Then there is a unique bijection $\mathrm{Charm}_c \colon \mathrm{NC}(W(C_n), c) \to \mathrm{NN}(C_n)$ satisfying

- $\mathrm{Charm}_c \circ \mathrm{Krew}_c = \mathrm{Krow}_c \circ \mathrm{Charm}_c$ and
- $\mathrm{Supp} = \mathrm{Supp} \circ \mathrm{Charm}_c$.

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