Hook-length Formulas for Skew Shapes via Contour Integrals and Vertex Models

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Abstract

The number of standard Young tableaux of a skew shape λ/μ can be computed as a sum over excited diagrams inside λ. Excited diagrams are in bijection with certain lozenge tilings, with flagged semistandard tableaux and also nonintersecting lattice paths inside λ. We give two new proofs of a multivariate generalization of this formula, which allow us to extend the setup beyond standard Young tableaux and the underlying Schur symmetric polynomials. The first proof uses multiple contour integrals. The second one interprets excited diagrams as configurations of a six-vertex model at a free fermion point, and derives the formula for the number of standard Young tableaux of a skew shape from the Yang-Baxter equation.

1 Introduction

The *hook-length formula* of Frame-Robinson-Thrall [FRT54] for the number of standard Young tableaux often goes with the adjective "celebrated": it is a remarkably rare phenomenon for a class of partially ordered sets to have a product formula for the number of their linear extensions. For a partition λ, the number f^{λ} of standard Young tableaux (SYT) of shape λ is given by

$$f^{\lambda} = |\lambda|! \prod_{(i,j)\in\lambda} \frac{1}{h(i,j)}, \tag{HLF}$$

where $h(i,j) = \lambda_i + \lambda'_j - i - j + 1$ is the hook length of the box (i,j) in the Young diagram of λ. This formula has seen many different proofs, from combinatorial to probabilistic and algebraic, each bringing out different ideas and properties.

The immediate generalization of standard Young tableaux, the skew standard Young tableaux, do not have such nice product formulas. The number $f^{\lambda/\mu}$ of skew standard Young tableaux of shape λ/μ is usually represented via determinants or sums of weighted Littlewood-Richardson coefficients. Ten years ago Naruse [Nar14], following work in [IN09], announced a remarkable formula, which directly generalizes (HLF):

$$f^{\lambda/\mu} = |\lambda/\mu|! \sum_{D\in\mathcal{E}(\lambda/\mu)} \prod_{(i,j)\in\lambda\setminus D} \frac{1}{h(i,j)}, \tag{NHLF}$$

where $\mathcal{E}(\lambda/\mu)$ is the set of so called *excited diagrams* of μ inside λ, and $h(i,j)$ is the hook length of the box (i,j) within λ. The origins of this formula lay within equivariant Schubert calculus.

Formula (NHLF) attracted a lot of attention with its elegance and prompted a flurry of activity bringing in various proofs (including [MPP18b, MPP17, Kon20a, GKMK23]), generalizations (among them [NO19, MZ20, MPP23, ST21, Par22, KS19, MPP22]), wide-ranging applications (see e.g. [HKYY19, JM23, MPP18a, FSTV23, CPP21, Pak21]) and other variations on the theme (e.g. [Kon20b, MPP23]). Its multivariate version appeared in the proofs and applications of [MPP17, MPP19] in the context of lozenge tilings, but in its most explicit and general form it was stated and proved via elaborate but elementary combinatorial manipulations in [GKMK23]:

$$\sum_{T\in \mathrm{SYT}(\lambda/\mu)} \prod_{k=1}^{|\lambda/\mu|} \frac{1}{z(T^{-1}[\geq k])} = \sum_{D\in\mathcal{E}(\lambda/\mu)} \prod_{u\in\lambda\setminus D} \frac{1}{z(H(u))}, \tag{1.1}$$

where $z(D) = \sum_{u\in D} z_{c(u)}$ for every excited diagram D, $H(u)$ is the hook of u within λ, and $T^{-1}[\geq k] = \{u \in \lambda/\mu, T(u) \geq k\}$ is the set of boxes in T occupied by entries $\geq k$. In the case of $\mu = \varnothing$ this formula is due to Pak-Postnikov (see [GKMK23] for a detailed account). Setting all $z_i = 1$ recovers (NHLF).

In the present work, we give two completely self-contained and short proofs of formula (1.1), which are different in nature from the approaches so far and are not combinatorial. Our central identity in Theorem 1.1 below is equivalent[1] to (1.1), and we refer to it as the *skew multivariate hook-length formula (skew-MHLF)*. Given a Young diagram λ with $n = \ell(\lambda)$ nonzero rows and formal variables $t_1, t_2, \ldots$, we set $x_i := t_{\lambda_i+n-i+1}$ for $i = 1, \ldots, n$, and set the remaining t's equal to the variables y: $\{y_1, y_2, \ldots\} := \{t_1, t_2, \ldots\} \setminus \{x_1, \ldots, x_n\}$. For example, for $\lambda = (2, 1)$, we have $n = 2$, $x_1 = t_4$, $x_2 = t_2$, and $y_1 = t_1$, $y_2 = t_3$, $y_3 = t_5$, and so on.

Theorem 1.1. *Let $\mu \subseteq \lambda$ be two Young diagrams, and $t_1, t_2, \ldots$ be formal variables. Then*

$$\sum_{T\in \mathrm{SYT}(\lambda/\mu)} \prod_{k=1}^{|\lambda/\mu|} \frac{1}{t(T^{-1}[< k])} = \sum_{D\in\mathcal{E}(\lambda/\mu)} \prod_{(i,j)\in\lambda\setminus D} \frac{1}{t_{\lambda_i+n-i+1} - t_{j+n-\lambda'_j}}, \tag{MHLF}$$

where for a skew Young diagram $T^{-1}[\geq k] = \lambda/\nu$ occupied by entries $\geq k$ in a SYT *T, we set $T^{-1}[< k] = \nu$ (by agreement, $T^{-1}[< 1] = \mu$), and denote $t(\nu) := \sum_i t_{\lambda_i+n-i+1} - t_{\nu_i+n-i+1}$. Here λ' is the transpose of λ .*

In Sections 2 and 3, we provide the necessary background and a general formalism for obtaining sums over skew standard Young tableaux from Pieri-type rules.

Our first proof of Theorem 1.1 given in Section 4 evaluates a contour integral of a multivariate rational function in two different ways. The first evaluation gives a recursion (Pieri-type formula) which builds up standard Young tableaux one box at a time, and produces the left-hand side of (MHLF). The second evaluation of that integral gives a determinant of weighted lattice path counts, which via the Gessel-Viennot formula is equivalent to a weighted enumeration of non-intersecting lattice paths inside λ, themselves equivalent to the excited diagrams in the right-hand side of (MHLF). We also derive in Proposition 4.12 an analogous multivariate version of the Okounkov-Olshanski formula studied in [MZ20].

[1]By setting $z_{-n+i} = t_{i+1} - t_i$, where $n = \ell(\lambda)$, see Proposition 4.11.

The second proof of Theorem 1.1 given in Section 5 interprets the identity through integrable vertex models. More precisely, we interpret the sum over excited diagrams in the right-hand side of (MHLF) as a partition function in the six-vertex model at a free fermion point. The vertex model lives inside the Young diagram λ, and the boundary conditions depend on μ. Using the R-matrix and Yang-Baxter equation, we show that this partition function obeys a recursive formula, building up the SYTs in the left-hand side of (MHLF).

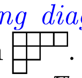
These proofs clear some of the hanging mysteries around the skew hook-length formula (NHLF). Both methods allow to generalize this formula to a sum over semistandard Young tableaux (SSYTs) instead of SYTs, as well as to other tableaux. See Appendix A for one possible generalization. Connecting vertex models to excited diagrams suggests a broad class of boundary conditions for the six-vertex model. It would be interesting to explore the corresponding partition functions beyond the free fermion point. Note also that both proofs suggest explicit ways of generalizing formula (MHLF) to the level of Hall-Littlewood and Macdonald polynomials. As an illustration, in Appendix B we produce an identity at the Macdonald level.

Acknowledgments

We are grateful to Darij Grinberg, Alejandro Morales, Slava Naprienko, Igor Pak, and Alexander Varchenko for helpful discussions. A part of this research was performed in Spring 2024 while the authors were visiting the program "Geometry, Statistical Mechanics, and Integrability" at the Institute for Pure and Applied Mathematics (IPAM), which is supported by the NSF grant DMS-1925919. GP was partially supported by the NSF grant CCF-2302174. LP was partially supported by the NSF grant DMS-2153869 and the Simons Collaboration Grant for Mathematicians 709055.

2 Background and definitions

2.1 Partitions and Young tableaux

A *partition* λ of an integer N is a sequence $\lambda = (\lambda_1, \ldots, \lambda_k)$ of integers $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k \geq 0$, summing up to N, i.e., $|\lambda| \coloneqq \lambda_1 + \cdots + \lambda_k = N$. We denote by $\ell(\lambda) = \max\{i : \lambda_i > 0\}$ the *length* of the partition, and by λ' its conjugate transpose, i.e. $\lambda'_i = \max\{j : \lambda_j \geq i\}$. We represent partitions graphically as *Young diagrams*, with top row having λ_1, and so on. For example $\lambda = (4, 2, 1)$ has Young diagram . We use the same notation λ for the partition and its Young diagram (a set of boxes in $\mathbb{Z}_{\geq 1} \times \mathbb{Z}_{\geq 1}$), and it would be clear from the context which one is meant.

A *skew shape* (*diagram*) λ/μ is the set of boxes in the Young diagram of λ but not in the diagram of μ when both are drawn with top left corner coinciding. We view skew shapes λ/μ as sets of boxes in $\mathbb{Z}_{\geq 1} \times \mathbb{Z}_{\geq 1}$. When using the notation λ/μ, we always assume that $\mu \subseteq \lambda$, that is, $\mu_i \leq \lambda_i$ for all i. When $\mu = \varnothing$, we have $\lambda/\mu = \lambda$. Denote by $|\lambda/\mu|$ the number of boxes in λ/μ (called *size*).

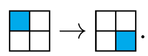
A *standard Young tableaux* (SYT) of shape λ/μ is a bijection $T : \lambda/\mu \to \{1, \ldots, |\lambda/\mu|\}$, such that $T(a, b) \leq T(c, d)$ whenever $a \leq c, b \leq d$. For example, all SYTs of shape $(3, 2)/(1)$ are

$$\begin{ytableau} \none & 1 & 2 \\ 3 & 4 \end{ytableau}, \quad \begin{ytableau} \none & 1 & 3 \\ 2 & 4 \end{ytableau}, \quad \begin{ytableau} \none & 1 & 4 \\ 2 & 3 \end{ytableau}, \quad \begin{ytableau} \none & 2 & 3 \\ 1 & 4 \end{ytableau}, \quad \begin{ytableau} \none & 2 & 4 \\ 1 & 3 \end{ytableau}.$$

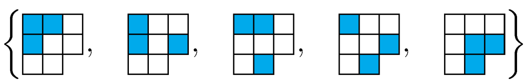
A *semistandard Young tableaux* (SSYT) of shape λ/μ and type α is a map $T : \lambda/\mu \to \{1, \ldots, \ell(\alpha)\}$, such that $T(a, b) < T(c, d)$ for $a < c$, $b \leq d$, $T(a, b) \leq T(a, d)$ for $b \leq d$ and $|T^{-1}(i)| = \alpha_i$. The last condition means that the number of boxes filled with i equals α_i. Here α is a composition (i.e., a partition without the ordering condition). A *flagged* SSYT [Wac85] of shape λ and flag $\mathsf{f} = (f_1, \ldots, f_{\ell(\lambda)})$ is an SSYT T, such that in addition, $T(i, j) \leq f_i$ for every $i = 1, \ldots, \ell(\lambda)$. We denote the sets of SYTs and SSYTs of shape λ/μ by $\mathrm{SYT}(\lambda/\mu)$ and $\mathrm{SSYT}(\lambda/\mu)$ respectively, and the set of flagged SSYTs of shape μ with flag f by $\mathrm{SSYT}(\mu; \mathsf{f})$. By convention, all tableaux and skew tableaux are filled with numbers starting from 1.

2.2 Excited diagrams, lozenge tilings and non-intersecting lattice paths

Excited diagrams have appeared many times in the literature, including e.g. [IN09, KMY09, Kre05, Wac85]. One definition uses the following recursive procedure. Let D be a set of boxes in λ. An *excited move* on a box in D shifts that box by one along its diagonal as long as none of its immediate neighbors below, to the right or down the diagonal are in D:



Then the set $\mathcal{E}(\lambda/\mu)$ is the set of all diagrams in λ which can be obtained from $D = \mu$ after performing a set of the above moves. For example, all excited diagrams in $\mathcal{E}((3, 3, 2)/(2, 1))$ are

$\mathcal{E}((3, 3, 2)/(2, 1)) =$ 

It was observed that excited diagrams in $\mathcal{E}(\lambda/\mu)$ bijectively correspond to flagged SSYTs of shape μ with the flag condition $f_i = \max\{j : \lambda_j - j \geq \mu_i - i\}$. The correspondence is depicted on the left side of Figure 2, and is given as follows. For $D \in \mathcal{E}(\lambda/\mu)$, create an SSYT T given by $T(i, j) = r$, where r is the row index of the location of the initial box (i, j) from μ in the excited diagram D. This is pictured in the second subfigure of Figure 1.

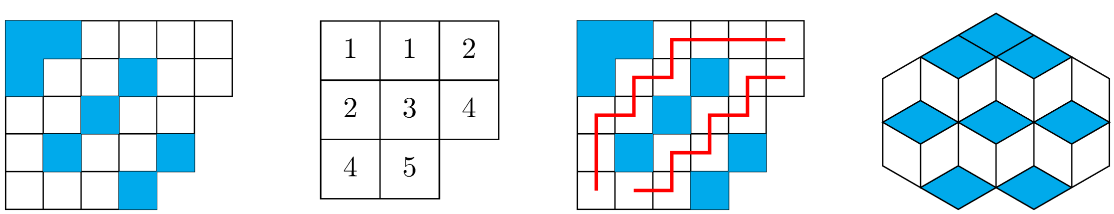


Figure 1: The many faces of excited diagrams. From left to right: An excited diagram in $\mathcal{E}(\lambda/\mu)$ for $\lambda = (6, 6, 5, 5, 4)$ and $\mu = (3, 3, 2)$; the corresponding flagged SSYT (with $f = (3, 4, 5)$); the nonintersecting lattice paths; and the lozenge tiling.

In [MPP17] and separately in [Kre05], it was observed that excited diagrams are also in bijection with non-intersecting lattice paths within λ which start at the lower border and exit at the right border of λ. They are formed exactly by the squares in $\lambda \setminus D$, as illustrated in the third subfigure of Figure 1.

It was then observed in [MPP19] that excited diagrams are in a bijective correspondence with restricted lozenge tilings of a region with lower boundary given by μ, which can be viewed in 3D as a stack of boxes in the corner of a room with base μ and height d, which depends on λ. To see this, let T be the flagged SSYT corresponding to D, we then stack $d - T(i,j) + i$ many boxes on the square (i,j) of μ. The partition λ determines how low each column can be, see the last subfigure in Figure 1.

2.3 Symmetric functions

While the idea of the present paper is not to rely on any symmetric functions formalism and identities, many of them appear in our applications. For the background definitions we refer to [Mac95, Sta01]. The elementary and (complete) homogeneous symmetric polynomials are

$$e_k(x_1,\ldots,x_n) \coloneqq \sum_{1\le i_1<\cdots<i_k\le n} x_{i_1}x_{i_2}\cdots x_{i_k}, \qquad h_k(x_1,\ldots,x_n) \coloneqq \sum_{1\le i_1\le\cdots\le i_k\le n} x_{i_1}x_{i_2}\cdots x_{i_k}.$$

Their generating functions are, respectively,

$$\sum_{r=0}^{n} z^r e_r(x_1,\ldots,x_n) = \prod_{i=1}^{n}(1+zx_i), \qquad \sum_{r=0}^{\infty} z^r h_r(x_1,\ldots,x_n) = \prod_{i=1}^{n}\frac{1}{1-zx_i}.$$

The *factorial Schur polynomials* are defined as follows:

$$s_\mu(x_1,\ldots,x_n \mid a) \coloneqq \frac{1}{\Delta(x)} \det[(x_i-a_1)\cdots(x_i-a_{\mu_j+n-j})]_{i,j=1}^{n}, \tag{2.1}$$

where $a_1, a_2, \ldots$ is an arbitrary sequence of shifts, and

$$\Delta(x) \coloneqq \prod_{1\le i<j\le n}(x_i - x_j) \tag{2.2}$$

is the Vandermonde determinant. When all $a_i = 0$, we obtain the classical Schur polynomials, $s_\mu(x) = s_\mu(x \mid \mathbf{0})$. Factorial Schur polynomials admit many nice properties common with the Schur polynomials [BL89], [Mac92, 6th variation], [MS99, Mol09]. In particular, there is the following combinatorial formula:

$$s_\mu(x_1,\ldots,x_n \mid a) = \sum_{T\in \mathrm{SSYT}(\mu)} \prod_{u\in\mu} (x_{T(u)} - a_{T(u)+c(u)}), \tag{2.3}$$

where for a box $u = (i,j)$ in the Young diagram of μ, the content is $c(u) \coloneqq j - i$, and $T(u)$ is the value of T in that box. The entries in T must be $\le n$.[2] We also employ interpolation Macdonald polynomials in Appendix B which admit a combinatorial formula similar to (2.3), but not a determinantal formula like (2.1).

[2] It is possible to insert infinitely many variables into $s_\mu(\cdots \mid a)$ within the formalism of symmetric functions and drop the condition $T(u) \le n$, but we do not need this here.

3 How to get sums over skew standard Young tableaux

3.1 General formalism

Here we present a general formalism for obtaining summation formulas over skew standard Young tableaux (SYTs) which come from certain vanishing properties and Pieri rules. The main statement of this subsection, Proposition 3.2, appeared in the particular case of the factorial Schur functions in [MS99, Proposition 3.2], with essentially the same proof.

Assume that $\mathsf{Z}_\mu(\lambda)$ is a function of two Young diagrams. It may be complex-valued, and can in addition depend on some parameters.

Remark 3.1. In applications, we obtain $\mathsf{Z}_\mu(\lambda)$ as a specialization of a symmetric polynomial $F_\mu(x_1,\ldots,x_n)$ (like the factorial Schur polynomial $s_\mu(x_1,\ldots,x_n \mid a)$ (2.1)) into the variables $x_i = x_i(\lambda)$, $1 \le i \le n$, which depend on λ. See Section 3.2 for a list of examples. However, a connection to symmetric polynomials is not necessary for the results of the present Section 3.1.

We assume that the quantities $\mathsf{Z}_\mu(\lambda)$ satisfy the following conditions:

- (*Vanishing property*) $\mathsf{Z}_\mu(\lambda) = 0$ if $\mu \not\subseteq \lambda$, and $\mathsf{Z}_\mu(\mu) \ne 0$ for all μ.
- (*Pieri rule*) There exist quantities $\mathsf{p}_\mu(\lambda)$ and constants $\mathsf{C}_{\nu/\mu}$ such that for all μ and λ we have

$$\mathsf{p}_\mu(\lambda)\,\mathsf{Z}_\mu(\lambda) = \sum_{\nu=\mu+\square} \mathsf{C}_{\nu/\mu}\,\mathsf{Z}_\nu(\lambda). \tag{3.1}$$

  Here the sum is over all ν with $|\nu| = |\mu| + 1$ which are obtained from μ by adding a box, and such that $\nu \subseteq \lambda$. We also assume that $\mathsf{p}_\mu(\lambda) \ne 0$ for all $\mu \subseteq \lambda$ with $\mu \ne \lambda$. Note that the vanishing property and the Pieri rule imply that $\mathsf{p}_\lambda(\lambda)$ must be zero.

Proposition 3.2. *Under the vanishing property and the Pieri rule (3.1), we have for any pair of Young diagrams $\mu \subseteq \lambda$:*

$$\mathsf{Z}_\lambda(\lambda) \sum_{T\in \mathrm{SYT}(\lambda/\mu)} \prod_{k=1}^{|\lambda/\mu|} \frac{\mathsf{C}_{T[=k]}}{\mathsf{p}_{T[<k]}(\lambda)} = \mathsf{Z}_\mu(\lambda). \tag{3.2}$$

Here, for a skew Young diagram $T^{-1}[\ge k] = \lambda/\nu$ occupied by entries $\ge k$ in a standard tableau $T \in \mathrm{SYT}(\lambda/\mu)$, we set $T^{-1}[< k] = \nu$ (by agreement, $T^{-1}[< 1] = \mu$).

Remark 3.3 (More general Pieri rule)**.** The Pieri rule may be extended to add more than one box at a time. If $S^+(\mu)$ denotes the set of allowed Young diagrams $\nu \supset \mu$, then (3.1) can be generalized to

$$\mathsf{p}_\mu(\lambda)\,\mathsf{Z}_\mu(\lambda) = \sum_{\nu\in S^+(\mu)} \mathsf{C}_{\nu/\mu}\,\mathsf{Z}_\nu(\lambda). \tag{3.3}$$

If we apply (3.3) as in Proposition 3.2, we obtain a sum over plane partitions T whose equal entries occupy shapes contained in $S^+(\alpha)/\alpha$. This can produce sums over semistandard Young tableaux (SSYTs), strict increasing tableaux (SIT) as in [MPP22], or other types of tableaux of skew shape λ/μ. We present an example in Appendix A.

Proof of Proposition 3.2. We have from (3.1):

$$\mathsf{Z}_\mu(\lambda) = \sum_{\nu=\mu+\square} \frac{\mathsf{C}_{\nu/\mu}}{\mathsf{p}_\mu(\lambda)} \mathsf{Z}_\nu(\lambda).$$

Continuing this process for each $\mathsf{Z}_\nu(\lambda)$, we add more boxes to the Young diagrams until we reach the Young diagram λ. Then we cannot add any more boxes due to the vanishing property of the interpolation symmetric functions. As a result, we obtain the desired sum over the skew standard Young tableaux of shape λ/μ. This completes the proof. $\square$

Under very general assumptions, Proposition 3.2 represents $\mathsf{Z}_\mu(\lambda)$ as a sum over SYTs as in the left-hand side of the multivariate hook-formula (MHLF). The right-hand side $\mathsf{Z}_\mu(\lambda)$ in formulas like (MHLF) usually has a combinatorial interpretation. Finding such an interpretation is a problem on its own.

3.2 Interpolation symmetric polynomials

Many examples of families $\{\mathsf{Z}_\mu(\lambda)\}$ satisfying vanishing and Pieri rule are provided by *interpolation polynomials* $F_\mu(x_1, \ldots, x_n)$ appearing in the theory of symmetric functions. Specializing the variables, we obtain

$$\mathsf{Z}_\mu(\lambda) = F_\mu(x_1(\lambda), \ldots, x_n(\lambda)). \tag{3.4}$$

Interpolation properties of $F_\mu(x_1, \ldots, x_n)$ lead to the vanishing, and the Pieri rule is inherited from symmetric polynomials. Examples based on interpolation symmetric polynomials include:

- *Factorial Schur polynomials* $s_\mu(x_1, \ldots, x_n \mid a)$ (2.1) is the main example we consider in the present paper. Note that in both our approaches (via integrals and vertex models), we reprove the required properties of factorial Schur polynomials from scratch, without using the theory of symmetric functions. From this point of view, the essence of the skew hook-length formula (MHLF) is the identification of the specialization $\mathsf{Z}_\mu(\lambda)$ (3.4), where $x_i(\lambda) = a_{\lambda_i+n-i+1}$, $1 \le i \le n$ (the a's are the shifts in the factorial Schur polynomials), with a sum over excited diagrams. The two proofs of this identification we present here did not explicitly appear in the literature.

- *Interpolation Macdonald polynomials* $I_\mu(x_1, \ldots, x_n; q, t)$ and the corresponding symmetric functions [KS97, Kno97, Sah96, Oko98a, Oko98b], see also [Ols19]. For interpolation Macdonald polynomials, the quantities

$$\begin{bmatrix} \lambda \\ \mu \end{bmatrix}_{q,t} = \frac{I_\mu(x_1^{(q,t)}(\lambda), \ldots, x_n^{(q,t)}(\lambda); q, t)}{I_\mu(x_1^{(q,t)}(\mu), \ldots, x_n^{(q,t)}(\mu); q, t)} \tag{3.5}$$

  are multivariate (q,t)-analogues of the binomial coefficients [Oko97]. Note that the normalization in (3.5) differs from the one in our sum over SYTs (3.2). We discuss the Macdonald example in further detail in Appendix B. In particular, the specialization which ensures the interpolation is defined in (B.1).

- *Balanced elliptic interpolation functions* [Rai06], see also [CG06].

- *Factorial Hall-Littlewood polynomials* considered in [NN23].
- *Factorial Grothendieck polynomials* [McN06], see also [MPP22].
- *Inhomogeneous spin q-Whittaker polynomials* [Kor24].

4 Proof by contour integrals

4.1 A family of integrals

Let $f_j(u \mid a)$ be a family of polynomials in one variable u depending on parameters $a = (a_1, a_2, \dots)$. Define the following n-fold contour integral indexed by a partition μ with $n \geq \ell(\mu)$:

$$F_\mu(x \mid a) = F_\mu(x_1, \dots, x_n \mid a) \coloneqq (-1)^{\binom{n}{2}} \frac{1}{(2\pi\sqrt{-1})^n} \oint_\gamma \cdots \oint_\gamma \prod_{i=1}^{n} \frac{f_{\mu_i+n-i}(u_i \mid a)}{\prod_{j=1}^{n}(u_i - x_j)} \Delta(u)\, du_1 \cdots du_n, \tag{4.1}$$

where γ is a positively oriented contour which contains all the poles $x_1, \dots, x_n$, and $\Delta(u)$ is the Vandermonde determinant (2.2).

In this section, we evaluate the integral (4.1) in two ways, via the residues at the x_j's, and via the residues at $u_i = \infty$, $1 \leq i \leq n$. Choosing appropriate polynomials f_j produces a proof of the skew hook-length formula (MHLF) (Theorem 1.1). Namely, throughout the rest of this section, we set

$$f_j(u) \coloneqq f_j(u \mid a) = \prod_{i=1}^{j} (u - a_i). \tag{4.2}$$

Taking other polynomials f_j in (4.1) produces generalizations of the skew hook-length formula (MHLF). We discuss one such generalization in Appendix A.

4.2 Determinant in disguise?

The integral F_μ defined via (4.1)–(4.2) can be identified with the factorial Schur polynomial s_μ (2.1). This fact is not needed for our proof of (MHLF), but we include it for completeness.

Theorem 4.1. *With $f_j(u)$ defined in (4.2) we have that $F_\mu(x_1, \dots, x_n \mid a) = s_\mu(x_1, \dots, x_n \mid a)$, where s_μ is the factorial Schur polynomial given by the determinantal formula (2.1).*

Proof. We evaluate the integral (4.1) by the residue formula at poles $u_i = x_{\sigma(i)}$ for all possible assignments of the poles to the variables, which are encoded by $\sigma \in \{1, \dots, n\}^n$. Note that if $\sigma(k) = \sigma(l)$ for some $k \neq l$, then

$$Res_{u = x_\sigma} = \frac{f_{\mu_i+n-i}(x_{\sigma(i)})}{\prod_{j \neq \sigma(i)}(x_{\sigma(i)} - x_j)} \prod_{i<j} (x_{\sigma(i)} - x_{\sigma(j)}) = 0,$$

as the Vandermonde factor vanishes. Thus, nonzero residues appear only when σ is a permutation.

We then observe that

$$
\begin{aligned}
Res_{u=x_\sigma} &= \prod_i \frac{f_{\mu_i+n-i}(x_{\sigma(i)})}{\prod_{j\neq\sigma(i)}(x_{\sigma(i)}-x_j)} \prod_{i<j}(x_{\sigma(i)}-x_{\sigma(j)}) \\
&= \prod_i f_{\mu_i+n-i}(x_{\sigma(i)}) \frac{\operatorname{sgn}(\sigma)\prod_{k<l}(x_k-x_l)}{\prod_k \prod_{l\neq k}(x_k-x_l)} \\
&= \prod_i f_{\mu_i+n-i}(x_{\sigma(i)}) \frac{\operatorname{sgn}(\sigma)}{\prod_{k>l}(x_k-x_l)},
\end{aligned}
$$

and summing over all permutations σ gives $\frac{1}{\Delta(x)}\det[f_{\mu_i+n-i}(x_j)]_{i,j=1}^n$. Identifying that determinant with (2.1) and $s_\mu(x \mid a)$ completes the proof. □

The rest of this section does not rely on Theorem 4.1.

4.3 Pieri rule

Here we show that the integrals F_μ (4.1)–(4.2) satisfy a Pieri rule. Let $\epsilon_i = (0^{i-1}, 1, 0^{n-i})$ be the i-th elementary vector.

Proposition 4.2. *We have*

$$
\sum_{i=1}^n F_{\mu+\epsilon_i}(x \mid a) = \left(\sum_{i=1}^n x_i - \sum_{i=1}^n a_{\mu_i+n-i+1}\right) F_\mu(x \mid a), \tag{4.3}
$$

where $F_{\mu+\epsilon_i}(x \mid a) = 0$ if $\mu+\epsilon_i$ is not a partition (i.e., does not weakly decrease).

The integral F_μ (4.1)–(4.2) is defined for any sequence μ which not necessarily a partition. Moreover, note that if $\mu_i + 1 = \mu_{i+1}$ for some i, then $\mu_i + n - i = \mu_{i+1} + n - (i+1)$, and so the product of the f_{μ_i+n-i}'s contains two identical terms. Therefore, the integrand becomes antisymmetric in u_i, u_{i+1} thanks to the factor $u_i - u_{i+1}$ coming from the Vandermonde. Since all integration contours are the same, the integral vanishes when $\mu_i + 1 = \mu_{i+1}$. Thus, in (4.3), only the terms for which $\mu + \epsilon_i$ is a partition survive.

Proof of Proposition 4.2. Identity (4.3) follows from the computation:[3]

$$
\begin{aligned}
\sum_{i=1}^n F_{\mu+\epsilon_i}(x \mid a) &= \sum_{i=1}^n (-1)^{\binom{n}{2}} \frac{1}{(2\pi\sqrt{-1})^n} \oint_\gamma \cdots \oint_\gamma \prod_{j=1}^n \frac{f_{\mu_j+\mathbf{1}_{i=j}+n-j}(u_j \mid a)}{\prod_{k=1}^n (u_j - x_k)} \Delta(u)\, du_1 \cdots du_n \\
&= (-1)^{\binom{n}{2}} \frac{1}{(2\pi\sqrt{-1})^n} \oint_\gamma \cdots \oint_\gamma \prod_{j=1}^n \frac{f_{\mu_j+n-j}(u_j \mid a)}{\prod_{k=1}^n (u_j - x_k)} \Big(\sum_{i=1}^n u_j - a_{\mu_i+n-i+1}\Big) \Delta(u)\, du_1 \cdots du_n \\
&= \Big(\sum_{i=1}^n x_i - \sum_{i=1}^n a_{\mu_i+n-i+1}\Big) F_\mu(x \mid a).
\end{aligned}
$$

[3] Notation $\mathbf{1}_A$ means the indicator function of the condition A.

In the first line, we used the fact that

$$f_{\mu_j+\mathbf{1}_{i=j}+n-j}(u_j \mid a) = f_{\mu_j+n-j}(u_j \mid a) \times \begin{cases} (u_i - a_{\mu_i+n-i+1}), & \text{if } j = i, \\ 1, & \text{if } j \neq i, \end{cases}$$

which implies that

$$\sum_{i=1}^{n} \prod_{j=1}^{n} f_{\mu_j+\mathbf{1}_{i=j}+n-j}(u_j \mid a) = \Big(\sum_{i=1}^{n} u_i - a_{\mu_i+n-i+1}\Big) \prod_{j=1}^{n} f_{\mu_j+n-j}(u_j \mid a). \tag{4.4}$$

In the second line, we used the fact that nonzero residues appear only at permutations: $u_i = x_{\sigma(i)}$, $1 \le i \le n$. The latter implies that $\sum_{i=1}^{n} u_i = \sum_{i=1}^{n} x_i$. This completes the proof. $\square$

The observation (4.4) is the crux of the proof of Proposition 4.2. We use a similar idea to get a generalization of the Pieri rule (and the skew hook-length formula) in Appendix A.

4.4 Lattice paths and SSYTs

Let us now evaluate the same integral $F_\mu(x \mid a)$ (4.1)–(4.2) using the residues at $u_i = \infty$. We then interpret the result in terms of weighted non-intersecting lattice paths. We consider a slightly more general setup. Let $b = (b_1, b_2, \ldots)$ be a family of parameters, the polynomials $f_j(x \mid b)$ be defined by (4.2), as before, and $\mathsf{m} = (m_1, m_2, \ldots)$ be a sequence of nonnegative integers. We define

$$F_{\mu,\mathsf{m}}(x \mid b) := (-1)^{\binom{n}{2}} \frac{1}{(2\pi\sqrt{-1})^n} \oint_\gamma \cdots \oint_\gamma \prod_{i=1}^{n} \frac{f_{\mu_i+m_i-i}(u_i \mid b)}{\prod_{j=1}^{m_i}(u_i - x_j)} \Delta(u)\, du_1 \cdots du_n, \tag{4.5}$$

where the contour γ encompasses all the poles $x_1, \ldots, x_n$. We recover the original definition (4.1) by setting $b_i = a_i$ and $m_j = n$ for all i, j.

Proposition 4.3. *We have*

$$F_{\mu,\mathsf{m}}(x \mid b) = \det[P_{i,j}^{\mu,\mathsf{m}}(x \mid b)]_{i,j=1}^{n}, \tag{4.6}$$

where

$$P_{i,j}^{\mu,\mathsf{m}}(x \mid b) := \sum_{r=0}^{\mu_i+j-i} (-1)^r e_r(b_1, \ldots, b_{\mu_i+m_i-i})\, h_{\mu_i+j-i-r}(x_1, \ldots, x_{m_i}). \tag{4.7}$$

Proof. The integral (4.5) becomes, after changing the variables $u_i = 1/v_i$:

$$\frac{1}{(2\pi\sqrt{-1})^n} \oint_{\gamma'} \cdots \oint_{\gamma'} (-1)^{\binom{n}{2}} \prod_{i=1}^{n} \frac{f_{\mu_i+m_i-i}(1/v_i \mid b)}{(-v_i^2) \prod_{j=1}^{m_i}(1/v_i - x_j)} \Delta(1/v)\, dv_1 \cdots dv_n,$$

where the integration contour γ' goes around 0 in the negative direction, and leaves the poles $x_1^{-1}, \ldots, x_n^{-1}$ outside. The integrand becomes

$$\begin{aligned} &(-1)^{\binom{n}{2}+n} \Delta(1/v) \prod_{i=1}^{n} \frac{(1 - v_i b_1) \ldots (1 - v_i b_{\mu_i+m_i-i}) v_i^{-\mu_i-m_i+i}}{v_i^{2-m_i}(1 - v_i x_1) \ldots (1 - v_i x_{m_i})} \\ &\qquad = (-1)^n \Delta(v) \prod_{i=1}^{n} \frac{1}{v_i^{\mu_i+n-i+1}} \cdot \frac{(1 - v_i b_1) \ldots (1 - v_i b_{\mu_i+m_i-i})}{(1 - v_i x_1) \ldots (1 - v_i x_{m_i})}. \end{aligned} \tag{4.8}$$

Expanding the Vandermonde determinant as $\Delta(v) = \sum_{\sigma\in S_n} \mathrm{sgn}(\sigma) v_1^{n-\sigma_1} \dots v_n^{n-\sigma_n}$, and further using the generating functions for the elementary and complete symmetric functions, we can continue (4.8) as

$$= (-1)^n \sum_{\sigma\in S_n} \mathrm{sgn}(\sigma) \prod_{i=1}^{n} \frac{1}{v_i^{\mu_i+\sigma_i-i+1}} \sum_{r,k=0}^{\infty} u_i^{r+k}(-1)^r e_r(b_1,\dots,b_{\mu_i+m_i-i}) h_k(x_1,\dots,x_{m_i}).$$

Taking the residue at all $v_i = 0$ (note that $(-1)^n$ is absorbed by changing the orientation of the contours), we arrive at the condition $k = \mu_i + \sigma_i - i - r$. Replacing σ_i by j leads to a determinant of the $P_{i,j}^{\mu,\mathsf{m}}$'s, which coincides with the desired expression. □

Remark 4.4. If $b = (0, 0, \dots)$ and $\mathsf{m} = (n, n, \dots)$, we have $F_{\mu,\mathsf{m}}(x \mid b) = s_\mu(x_1, \dots, x_n)$. Since in this case $P_{i,j}^{\mu,\mathsf{m}} = h_{\mu_i+j-i}(x_1, \dots, x_n)$, we recover the classical Jacobi-Trudy identity [Mac95, (II.3.4)].

Let us interpret (4.7) as a partition function of weighted lattice paths.

Lemma 4.5. *Let $x_1, x_2, \dots$ be indeterminates and $\dots, b_{-1}, b_0, b_1, \dots$ be parameters. Consider directed lattice paths L starting at $(-s+1, 1)$ and ending at $(k+1, n)$, which make up and right steps (several such paths are in Figure 2, right). To each horizontal (right) step $(r,t) \to (r+1,t)$ we assign the weight $w(r,t) := x_t - b_{r+t}$. Define the weight of a path by $w(L) := \prod_{u\in L} w(u)$, where the product is over all horizontal steps of L. Then*

$$\sum_{L:(-s+1,1)\to(k+1,n)} w(L) = \sum_{r=0}^{k+i} (-1)^r e_r(b_{-i+2}, \dots, b_0, b_1, \dots, b_{k+n}) h_{k+i-r}(x_1, \dots, x_n). \tag{4.9}$$

Proof. The right-hand side of (4.9) can be rewritten as

$$\sum_r (-1)^r \sum_{i_1<\dots<i_r, j_1\le j_2\le\cdots} b_{i_1} \cdots b_{i_r} x_{j_1} x_{j_2} \cdots$$

The indices of the b's define r diagonal strips $D_\ell = \{(u,v) : i_k - 1 \le u+v \le i_\ell\}$. Set $D = \cup_\ell D_\ell$. We create a path L by greedily picking the horizontal steps $j_1, j_2, \dots$ from the vertical line at $(-s+1, 1)$ to the right as follows. If $(-s+1, j_1) \to (-s+2, j_1) \notin D$, then we add this step to L with weight x_{j_1} and continue. If it is in D, then we find the largest index $\ell < i$, such that $(-s+1, j_1) \to (-\ell+1, j_1) \in D$, but $(-\ell+1, j_1) \to (-\ell+2, j_1) \notin D$, then we add $(-s+1, j_1) \to (-\ell+2, j_1)$ to L with weight $b_{j_1-i+1} \cdots b_{j_1-\ell+1} x_{j_1}$ and note that we must have $i_1 = j_1 - i + 1$. We then continue with j_2 starting from $(-\ell+2, j_1)$ and build up L via its horizontal steps. We see that the weight we picked up this way is obtained by selecting the corresponding terms from each of the brackets $(x_j - b_{i+j})$ along the path, picking up a weight b if the horizontal step is in D and a weight x otherwise.

This is a bijection with the monomials in the left-hand side of (4.9). Indeed, to see the inverse map, taking a path L we select a term from each bracket. If we select the term b_ℓ at the t'th horizontal step (so tth bracket), we set $D_p = \{(u,v) : \ell - 1 \le u+v \le \ell\}$, where p is the number of b terms selected so far. We also set $i_p := \ell$. The brackets where x's were selected then produce the j indices. □

Applying Lemma 4.5 with $k = \mu_i - i$, $s = j$ and $n = m_j$, we obtain the following interpretation:

Corollary 4.6. *With the notation from Lemma 4.5 and (4.7), setting $b_i = 0$ for $i \le 0$, we have*

$$P^{\mu,\mathsf{m}}_{i,j}(x \mid b) = \sum_{L\colon (-j+1,1)\to(\mu_i-i+1,m_i)} w(L). \tag{4.10}$$

Theorem 4.7. *We have the following combinatorial formula for the functions $F_{\mu,\mathsf{m}}$ (4.5):*

$$F_{\mu,\mathsf{m}}(x \mid b) = \sum_{T\in \mathrm{SSYT}(\mu;\mathsf{m})} \prod_{v\in\mu} (x_{T(v)} - b_{T(v)+c(v)}), \tag{4.11}$$

where $\mathrm{SSYT}(\mu;\mathsf{m})$ *is the set of all flagged semistandard Young tableaux (see Section 2.1) of shape μ and flag $\mathsf{m} = (m_1, m_2, \ldots)$.*

Proof. The statement follows by combining Proposition 4.3 and Corollary 4.6. Starting from (4.6), let us rewrite this determinantal formula as a sum over lattice paths. This is possible thanks to the Lindström-Gessel-Viennot lemma [Lin73, GV85]. The lattice paths corresponding to the determinant have weights $P^{\mu,\mathsf{m}}_{i,j}(x \mid b)$, start at $(-i+1, 1)$, and end at $(\mu_j - j+1, m_j)$. These lattice paths are in a well-known bijection with SSYTs: the path starting at $(-i+1, 1)$ is the ith row of the SSYT T, and the entries are the levels of the horizontal steps. The non-intersecting condition ensures that the columns are strictly increasing. If (i, j) is a cell in an SSYT T, then $T(i, j)$ is equal to the height of the path starting at $(-i + 1, 1)$ at its j'th step. Thus, $T(i, j) = t$ if the step is $(-i + j, t) - (i + j + 1, t)$, and the weight of this step is $w(-i + j, t) = x_t - b_{-i+j+t}$. See Figure 2 for an illustration. Since $c(i, j) = j - i$, this weight matches the tableau weight in the right-hand side of (4.11).

Note that nonintersecting conditions force the lattice path starting at $(-i + 1, 1)$ to initially take $i - 1$ vertical steps and continue through $(-i + 1, i)$, which ensures that all indices of b appearing in (4.11) are at least 1. $\square$

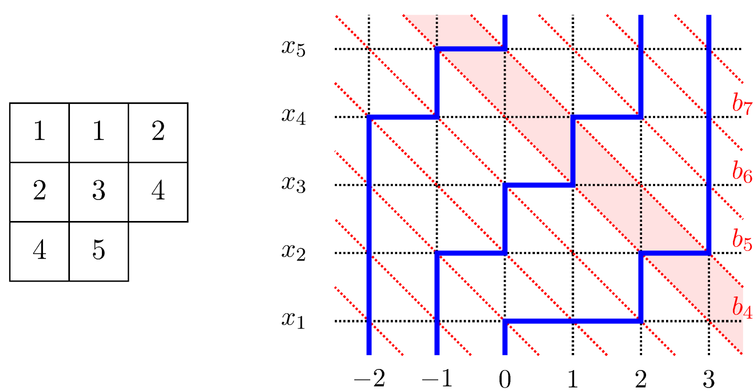


Figure 2: A semistandard tableaux of total weight $wt(T) = (x_1 - b_1)(x_1 - b_2)(x_2 - b_4)(x_2 - b_1)(x_3 - b_3)(x_4 - b_5)(x_4 - b_2)(x_5 - b_4)$ and its corresponding non-intersecting lattice path configuration.

As a hint to our final step, and for completeness, let us obtain as a corollary the combinatorial formula for factorial Schur polynomials (2.3):

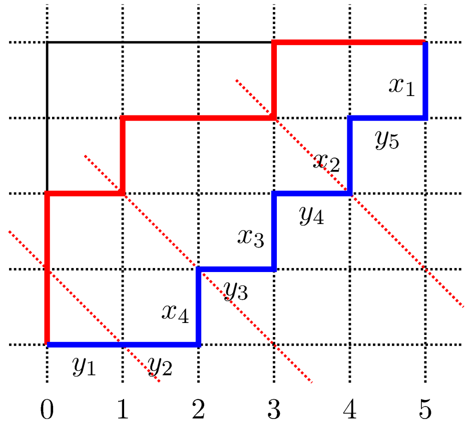


Figure 3: The parameter sequence a^λ. Here $\lambda = (5, 4, 3, 2)$ is the blue path, with labels on the steps giving $a^\lambda = (y_1, y_2, x_4, y_3, x_3, y_4, x_2, y_5, x_1)$. The shape $\mu = (3, 1)$ is drawn in red, and the diagonal lines that start from the end of the rows of μ are shown meeting λ in rows $m_1 = 2$, $m_2 = 3$, $m_3 = 4$.

Corollary 4.8. *For* $\mathsf{m} = (n, n, n, \dots)$ *and* μ *a partition, formula* (4.11) *reduces to the one for the factorial Schur polynomials* (2.3)*:*

$$s_\mu(x \mid a) = F_{\mu,\mathsf{m}}(x \mid a) = \sum_{T \in \mathrm{SSYT}(\mu)} \prod_{v \in \mu} (x_{T(v)} - a_{T(v)+c(v)}).$$

Proof. Theorem 4.7 applied to parameters a with $m_i = n$ gives the desired right side, as the flag condition becomes trivial and we are summing over all SSYT of shape μ. To see the left side, we invoke Theorem 4.1 with $m_i = n$, and observe that integral matches (4.5). $\square$

4.5 Partition functions of excited diagrams

Here we specialize the parameters a in the integral $F_\mu(x \mid a)$ (4.1) into a sequence containing x's and y's, where the order of the variables is determined by another Young diagram λ. Namely, let the boundary of λ be a lattice path L from $(0, 1)$ to (λ_1, n), encoded as a sequence of U(p) and H(orizontal) steps, so $L_{\lambda_i+n-i+1} = U$ are the vertical (up) steps for $i = 1, \dots, n$. We write a variable y_j for a horizontal step at column j and a variable x_i for the vertical step at height i. See Figure 3 for an illustration. In detail, reading along L, we record a sequence of x's and y's as the entries for a^λ:

$$a^\lambda_{\lambda_i+n-i+1} \coloneqq x_i, \quad \text{and} \quad a^\lambda_r = y_{j_r} \text{ for } \lambda_{i+1}+n-i+1 \le r \le \lambda_i+n-i, \text{ setting } j_r \coloneqq r-i. \tag{4.12}$$

We now consider the combinatorial interpretation of $F_\mu(x \mid a^\lambda)$ as excited diagrams.

Proposition 4.9. *Let* $\mathcal{E}(\lambda/\mu)$ *be the set of excited diagrams of* μ *inside* λ*. Then*

$$F_\mu(x \mid a^\lambda) = \sum_{D \in \mathcal{E}(\lambda/\mu)} \prod_{(i,j) \in \lambda \setminus D} (x_i - y_j).$$

Proof. Substituting a^λ into the initial integral formula (4.1), we obtain many cancellations. Denote $m_i := \min\{\ell : \lambda_\ell < \mu_i + \ell - i\} - 1$, so m_i is the row index where the diagonal from (i, μ_i) meets the outer boundary of the Young diagram λ (see Figure 3). We then observe that

$$(a_1^\lambda, a_2^\lambda, \ldots, a_{\mu_i+n-i}^\lambda) = (y_1, \ldots, x_n, \ldots, x_{m_i+1}, \ldots, y_{\mu_i+m_i-i}),$$

that is, the last x variable appearing is x_{m_i+1}. We can cancel some of the terms in the integrand of formula (4.1) of F_μ as

$$\frac{(u_i - y_1)\cdots(u_i - x_n)\cdots(u_i - x_{m_i+1})\cdots(u_i - y_{\mu_i+m_i-i})}{(u_i - x_n)\cdots(u_i - x_1)} = \frac{(u_i - y_1)\cdots(u_i - y_{\mu_i+m_i-i})}{(u_i - x_{m_i})\cdots(u_i - x_1)}.$$

Then the integral formula becomes

$$F_\mu(x \mid a^\lambda) = \frac{(-1)^{\binom{n}{2}}}{(2\pi\sqrt{-1})^n} \oint_\gamma \cdots \oint_\gamma \prod_i \frac{(u_i - y_1)\cdots(u_i - y_{\mu_i+m_i-i})}{(u_i - x_{m_i})\cdots(u_i - x_1)} \Delta(u)\, du_1 \ldots du_n = F_{\mu,\mathsf{m}}(x \mid y),$$

where $F_{\mu,\mathsf{m}}(x \mid y)$ is the generalized integral (4.5), with parameters b replaced by y.

We can now apply Theorem 4.7 to interpret $F_{\mu,\mathsf{m}}$ as a sum over flagged SSYT. Our final step is to identify these flagged SSYT with excited diagrams with the corresponding weight. The map from an excited diagram $D \in \mathcal{E}(\lambda/\mu)$ to a flagged SSYT T of shape μ and flag $f_i = \max\{j : \lambda_j - j \geq \mu_i - i\}$ was discussed in Section 2.2. We observe that $f_i = m_i$, so we have the same set of SSYTs. Finally, if for a box $v = (i, j) \in \mu$ we have $T(i, j) = t$, then $T(i, j) + c(i, j) = t + j - i$ is the column index of the corresponding excited box and $x_{T(v)} - y_{T(v)+c(v)} = x_t - y_{t+j-i}$ and $(t, t + j - i) \in \lambda \setminus D$ is the corresponding box. This completes the proof. ☐

The interpolation property of the factorial Schur polynomials $s_\mu(x \mid a) = F_\mu(x \mid a)$ (see Theorem 4.1) can be derived directly from Proposition 4.9. This property is originally due to [Oko98a] (see also [Oko96, OO97]), and can be alternatively shown using the double alternant formula (2.1).

Corollary 4.10. *Let λ be a partition and a^λ be defined in (4.12). Then*

$$F_\mu(x \mid a^\lambda) = 0 \text{ if } \mu \not\subseteq \lambda, \qquad \text{and} \qquad F_\lambda(x \mid a^\lambda) = \prod_{(i,j)\in\lambda} (x_i - y_j).$$

Proof. For $\mu \not\subseteq \lambda$, there are no allowable flagged SSYT/excited diagrams, and thus $F_\mu(x \mid a^\lambda) = 0$. For $\lambda = \mu$, the only possible flagged tableau is the one with $T(i, j) = i$, whose weight for the box (i, j) is $x_i - y_j$. This completes the proof. ☐

4.6 Proof of the generalized hook-length formula and Theorem 1.1

The generalized (multivariate) skew hook-length formulas (1.1), (MHLF) follow from evaluating $F_\mu(x \mid a^\lambda)$ in two different ways. One is recursively by the Pieri rule adding boxes to μ until it reaches λ, and the other is the combinatorial interpretation for s_μ given in Proposition 4.9. This approach closely follows the general formalism of Section 3.1. First, let us establish the equivalence of the two formulas:

Proposition 4.11. *Formulas* (1.1) *and* (MHLF) *are equivalent.*

Proof. Throughout the proof, we use the notation from the Introduction (Section 1), more precisely, the definitions after formula (1.1) and in Theorem 1.1.

We start from the right-hand sides. Given shapes $\mu \subset \lambda$, set $n = \ell(\lambda)$ and $N = |\lambda/\mu|$. Given $z_{-n+1}, \ldots, z_{\lambda_1-1}$ as in (1.1), set $t_0 = 0$ and $t_i = z_{-n} + \cdots + z_{-n+i-1}$ for $i \geq 1$, so that $z_{-n+i} = t_{i+1} - t_i$. Next, rename $t_{\lambda_i+n+1-i} = x_i$, and denote the rest of $t_1, t_2 \ldots$ by $y_1, y_2, \ldots$. That is, $t = a^\lambda$ (4.12). Observe that with this notation, if $u = (i,j)$ is a box in λ, then we have telescoping along the hook:

$$\begin{aligned} z(H(u)) &= z_{j-\lambda'_j} + z_{j-\lambda'_j+1} + \cdots + z_{\lambda_i-i} \\ &= t_{\lambda_i-i+1} - t_{\lambda_i+n-i} + t_{\lambda_i+n-i} - t_{\lambda_i+n-i-1} - \cdots - t_{j+n-\lambda'_j} \\ &= x_i - y_j. \end{aligned}$$

In the last equality, we noted that $t_{j+n-\lambda'_j} = y_j$, as this is the jth horizontal step of L, the outer boundary of the Young diagram λ.

For the left-hand sides, pick $T \in \mathrm{SYT}(\lambda/\mu)$, and let $\nu = T^{-1}[< k]$ be an intermediate shape occupied by the entries $< k$ in T. We have

$$\begin{aligned} t(\nu) &= \sum_{i=1}^{n} (x_i - t_{\nu_i+n-i+1}) = \sum_{i=1}^{n} (x_{n+1-i} - t_{\nu_i+n-i+1}) \\ &= \sum_{i=1}^{n} (t_{\lambda_i+n-i+1} - t_{\nu_i+n-i+1}) = \sum_{i=1}^{n} (z_{\nu_i-i+1} + \cdots + z_{\lambda_i-i}) = z(\lambda/\nu), \end{aligned}$$

which completes the proof. $\square$

Applying Proposition 3.2 with the Pieri rule and the vanishing property for F_μ given in Proposition 4.2 and Corollary 4.10, respectively, we obtain (with $t = a^\lambda$):

$$\sum_{\mu \subset \mu^1 \subset \mu^2 \cdots \subset \mu^N = \lambda} \frac{1}{t(\mu) t(\mu^1) \cdots t(\mu^{N-1})} F_\lambda(x \mid t) = F_\mu(x \mid a^\lambda), \tag{4.13}$$

where $\mu^i/\mu^{i-1} = (1)$ for all i, and so each such sequence corresponds to a standard Young tableau of shape λ/μ. Dividing both sides by $F_\lambda(x \mid a^\lambda)$ (given by Corollary 4.10), we identify the ratio $F_\mu(x \mid a^\lambda)/F_\lambda(x \mid a^\lambda)$ as a sum over excited diagrams $\mathcal{E}(\lambda/\mu)$, thanks to Proposition 4.9. This completes the proof of Theorem 1.1.

4.7 The generalized Okounkov-Olshanski formula

In [MZ20], Morales and Zhu obtained a variant (via reverse excited diagrams in a shifted shape, or certain SSYTs) of (NHLF) which they coined as the Okounkov-Olshanski formula (OOF). The derivations above can be used to give a multivariate version of formula (4.13), too. This derivation was suggested by Alejandro Morales.

We start with equation (4.13) and the same notation. Let us apply the combinatorial formula (Corollary 4.8) to $F_\mu(x \mid a) = s_\mu(x \mid a) = s_\mu(x_n, \ldots, x_1 \mid a)$, where the variables can be reversed because these polynomials are symmetric in the x's. Then, let us substitute $a = a^\lambda$. We obtain:

$$\sum_{\mu \subset \mu^1 \subset \mu^2 \cdots \subset \mu^N = \lambda} \frac{1}{t(\mu)t(\mu^1)\cdots t(\mu^{N-1})} F_\lambda(x \mid a^\lambda) = \sum_{T \in \mathrm{SSYT}(\mu)} \prod_{u \in \mu} \big(x_{n+1-T(u)} - a^\lambda_{T(u)+c(u)}\big).$$

The terms in the product on the RHS can be written in terms of the parameters t_j:

$$\sum_{\mu \subset \mu^1 \subset \mu^2 \cdots \subset \mu^N = \lambda} \frac{1}{t(\mu)t(\mu^1)\cdots t(\mu^{N-1})} F_\lambda(x \mid a^\lambda) = \sum_{T \in \mathrm{SSYT}(\mu)} \prod_{u \in \mu} \big(t_{\lambda_{n+1-T(u)}+T(u)} - t_{T(u)+c(u)}\big).$$

Substituting the value for $F_\lambda(x \mid a^\lambda)$ from Corollary 4.10 we arrive at the following formula.

Proposition 4.12. *Let $\mu \subset \lambda$ be two Young diagrams and $t_1, t_2, \ldots$ formal variables. Then*

$$\begin{aligned}\sum_{T \in \mathrm{SYT}(\lambda/\mu)} \prod_{k=1}^{|\lambda/\mu|} \frac{1}{t(T^{-1}[<k])} = \prod_{(i,j) \in \lambda} & \frac{1}{t_{\lambda_i+n-i+1} - t_{j+n-\lambda'_j}} \\ & \times \Big(\sum_{T \in \mathrm{SSYT}(\mu)} \prod_{u \in \mu} \big(t_{\lambda_{n+1-T(u)}+T(u)} - t_{T(u)+c(u)}\big)\Big),\end{aligned} \tag{MOOF}$$

where the notation follows Theorem 1.1.

The tableaux appearing in the Okounkov-Olshanski formula are not all $\mathrm{SSYT}(\mu)$, as the terms involved vanish for some of them. Morales and Zhu have found several different characterizations of these tableaux and it remains to be understood whether any of these interpretation have nice meanings for the indices $\lambda_{n+1-T(u)} + T(u)$ and $T(u) + c(u)$. When we substitute $t_i = i$, then $t_{\lambda_{n+1-T(u)}+T(u)} - t_{T(u)+c(u)} = \lambda_{n+1-T(u)} - c(u)$ are the arm lengths of certain cells in the reversed shifted excited diagrams of [MZ20]. The multivariate formula above reduces to the original Okounkov-Olshanski formula for $f^{\lambda/\mu}$ in [Oko98b].

5 Proof via free fermion five-vertex model

In this section, we present the proof of Theorem 1.1 using the free fermion five-vertex model. This proof is completely independent of the one by contour integrals (Section 4), but we comment on the identification of certain quantities arising in both approaches.

5.1 Vertex weights and the Yang–Baxter equation

We begin by recalling the five-vertex weights which are related to the factorial Schur polynomials [Las07, McN09, BMN14]. See also [ABPW23, Section 4.1] and [Nap24] for generalizations connecting free fermion six-vertex model to most known Schur-type functions.

Consider the following vertex weights w_x:

$$\begin{aligned} &w_x(\;) = w_x(0,0;0,0) = x, \quad && w_x(\;) = w_x(1,1;1,1) = 0, \quad && w_x(\;) = w_x(1,0;1,0) = 1, \\ &w_x(\;) = w_x(0,1;0,1) = 1, \quad && w_x(\;) = w_x(0,1;1,0) = 1, \quad && w_x(\;) = w_x(1,0;0,1) = 1. \end{aligned} \tag{5.1}$$

We also need the following dual weights $\check{w}_y$:

$$\begin{gathered}\check{w}_y(0,0;0,0) = 1, \quad \check{w}_y(1,1;1,1) = 0, \quad \check{w}_y(1,0;1,0) = \tfrac{1}{y}, \\ \check{w}_y(0,1;0,1) = \tfrac{1}{y}, \quad \check{w}_y(0,1;1,0) = \tfrac{1}{y}, \quad \check{w}_y(1,0;0,1) = \tfrac{1}{y}. \end{gathered} \tag{5.2}$$

We set $w_x(i_1,j_1;i_2,j_2) = \check{w}_y(i_1,j_1;i_2,j_2) = 0$ for all choices of $i_1,j_1,i_2,j_2 \in \{0,1\}$ not listed in (5.1) and (5.2). Clearly, $\check{w}_y(i_1,j_1;i_2,j_2) = y^{-1}w_y(i_1,j_1;i_2,j_2)$, but these weights play two very different roles, so we will keep this separate notation.

The weights (5.1)–(5.2) satisfy the Yang–Baxter equation with the following weights $r = r_z$:

$$\begin{gathered} r_z(0,0;0,0) = 1, \quad r_z(1,1;1,1) = 1, \quad r_z(1,0;1,0) = z, \\ r_z(0,1;0,1) = 0, \quad r_z(0,1;1,0) = 1, \quad r_z(1,0;0,1) = 1. \end{gathered} \tag{5.3}$$

Like for (5.1)–(5.2), the weights r_z (5.3) are nonzero on *five* out of six configurations which conserve the total number of incoming and outgoing paths at a vertex (that is, $i_1 + j_1 = i_2 + j_2$). However, under r_z, the paths are allowed to meet at a vertex.

Remark 5.1. Each of the weights $w_x, \check{w}_y$, and r_z satisfies the free fermion condition

$$w(0,0;0,0)\,w(1,1;1,1) + w(1,0;1,0)\,w(0,1;0,1) = w(0,1;1,0)\,w(1,0;0,1),$$

which allows to write many partition functions (i.e., sums of products of vertex weights over all configurations of paths in a region with fixed boundary conditions) as determinants. See [Nap24] for the most general case of free fermion six-vertex model. Note that partition functions for the general six-vertex model also take determinantal form for special boundary conditions. The most well-known example of this phenomenon is the Izergin–Korepin determinant [Kor82, Ize87].

The *spectral parameters* x, y, z in (5.1)–(5.3) may be thought of as generic complex numbers, and the Yang–Baxter equation holds under the condition that $z = y - x$:

Proposition 5.2 (Yang–Baxter equation)**.** *For any $i_1, i_2, i_3, j_1, j_2, j_3 \in \{0,1\}$ and all x, y, t with $x \neq t$, we have*

$$\begin{aligned} &\sum\nolimits_{k_1,k_2,k_3} w_{x-y}(i_3,k_1;k_3,j_1)\,\check{w}_{x-t}(i_2,i_1;k_2,k_1)\,r_{y-t}(k_3,k_2;j_3,j_2) \\ &\qquad = \sum\nolimits_{k_1',k_2',k_3'} w_{x-y}(k_3',i_1;j_3,k_1')\,\check{w}_{x-t}(k_2',k_1';j_2,j_1)\,r_{y-t}(i_3,i_2;k_3',k_2'). \end{aligned} \tag{5.4}$$

where all sums are over $k_1,k_2,k_3 \in \{0,1\}$ or $k_1',k_2',k_3' \in \{0,1\}$. See Figure 4 for illustration.

Proof. For each $i_1,i_2,i_3,j_1,j_2,j_3 \in \{0,1\}$, equation (5.4) is an identity of rational functions in x, y, t which is directly checked. ∎

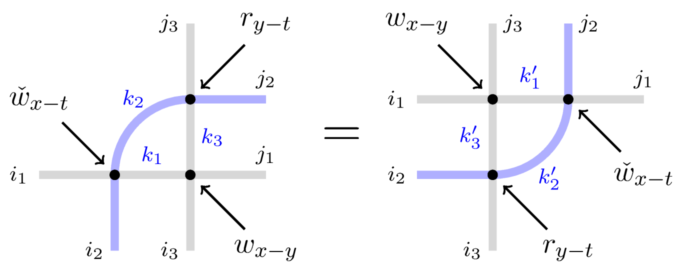


Figure 4: Graphical representation of the Yang–Baxter equation (5.4) which states that for any fixed boundary conditions $i_1, i_2, i_3, j_1, j_2, j_3 \in \{0, 1\}$, the partition functions on the left and on the right are equal. The Yang–Baxter equation is nontrivial only if $i_1 + i_2 + i_3 = j_1 + j_2 + j_3$.

5.2 Excited diagrams as configurations of the five-vertex model

Fix two Young diagrams μ and λ such that $\mu \subset \lambda$. Recall the set of excited diagrams $\mathcal{E}(\lambda/\mu)$ described in Section 2.2. Let $x_1, x_2, \ldots, y_1, y_2, \ldots$ be generic complex numbers such that $x_i \neq y_j$ for all i, j. Define the following sum over excited diagrams:

$$\mathsf{Z}_\mu(\lambda) \coloneqq \sum_{D \in \mathcal{E}(\lambda/\mu)} \prod_{(i,j) \in D} (x_i - y_j). \tag{5.5}$$

When $\mu \not\subseteq \lambda$, we set $\mathsf{Z}_\mu(\lambda) = 0$.

Clearly,

$$\mathsf{Z}_\mu(\lambda) = \prod_{(i,j) \in \lambda} (x_i - y_j) \sum_{D \in \mathcal{E}(\lambda/\mu)} \prod_{(i,j) \in \lambda \setminus D} \frac{1}{x_i - y_j}, \tag{5.6}$$

and the sum in the right-hand side is the same as the right-hand side of the multivariate hook-length formula (MHLF). Our goal in the present Section 5 is to find a representation of $\mathsf{Z}_\mu(\lambda)$ as a sum over skew standard Young tableaux. For this, we will verify the vanishing property and a Pieri rule for $\mathsf{Z}_\mu(\lambda)$, following the general strategy outlined in Section 3.1.

Remark 5.3 (Connection to factorial Schur polynomials)**.** The argument in this section is independent from the rest of the paper, and does not rely on properties of factorial Schur polynomials. More precisely, $\mathsf{Z}_\mu(\lambda)$ is a *specialized* factorial Schur polynomial $s_\mu(x \mid a^\lambda) = F_\mu(x \mid a^\lambda)$. Here we use only these specialized quantities, and not the general parameters a.

Remark 5.4. The vanishing property $\mathsf{Z}_\mu(\lambda) = 0$ for $\mu \not\subseteq \lambda$ is a part of the definition (5.5), and we also immediately have

$$\mathsf{Z}_\lambda(\lambda) = \prod_{(i,j) \in \lambda} (x_i - y_j), \tag{5.7}$$

which is nonzero by our assumptions.

In the present Section 5.2, we identify the sum $\mathsf{Z}_\mu(\lambda)$ in (5.5) as a partition function of the five-vertex model with the weights (5.1). This is done in several steps.

Domain. Let $\Omega = \{(i, j) : i, j \geq 1\}$ be the set of all boxes in the bottom right quadrant. Here i and j are the row and column coordinates, with i increasing down, and j increasing to the right. We will represent boxes by vertices in the five-vertex model, and in this way the domain Ω becomes the quadrant $\mathbb{Z}^2_{\geq 1}$ in the square grid. Let

$$\Omega_\lambda \coloneqq \{(i, j) : 1 \leq j \leq \lambda_i \text{ for all } i \geq 1\} \subseteq \Omega$$

be the set of all boxes in the Young diagram λ, identified with a subset of the square grid. See Figure 5, right, for an illustration of Ω_λ for $\lambda = (6, 6, 5, 5, 4)$.

Weights. Assign spectral parameters $x_1, x_2, \ldots$ to the rows i, and spectral parameters $y_1, y_2, \ldots$ to the columns j. Let the weight at each vertex $(i, j) \in \Omega_\lambda$ be $w_{x_i - y_j}$.

Consider a configuration of paths of the five-vertex model in Ω_λ, such that the paths travel in the up-right direction and are allowed to enter and exit Ω_λ only through the southeast broken line border of the Young diagram λ (i.e., not through the west and north straight boundaries of the quadrant Ω in which λ is placed). Two paths are nonintersecting; that is, they are not allowed to pass through the same vertex because $w_{x_i - y_j}(1, 1; 1, 1) = 0$. Each configuration of paths is identified with an excited diagram D whose boxes are precisely the empty vertices $(0, 0; 0, 0)$. Recall that $w_{x_i - y_j}(0, 0; 0, 0) = x_i - y_j$, and the weights of all other vertices are 1. Thus, the weight of a five-vertex path configuration is equal to $\prod_{(i,j) \in D}(x_i - y_j)$. See Figure 5 for an illustration.

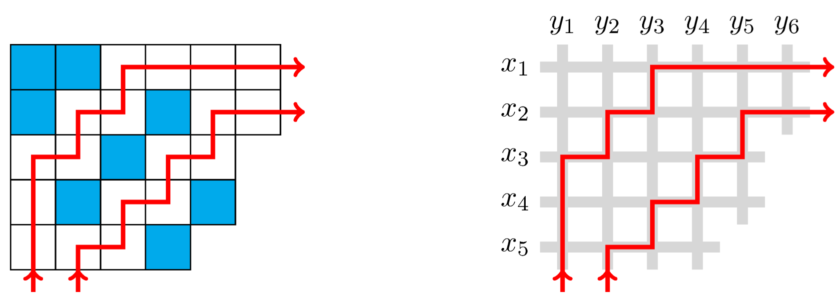


Figure 5: A configuration of the five-vertex model in the domain Ω_λ for $\lambda = (6, 6, 5, 5, 4)$ superimposed on an excited diagram (left), and the same path configuration in the domain Ω_λ (right). In the right picture, we also indicated the spectral parameters x_i, y_j along the lines. The weight of this configuration is $(x_1 - y_1)(x_1 - y_2)(x_2 - y_1)(x_2 - y_4)(x_3 - y_3)(x_4 - y_2)(x_4 - y_5)(x_5 - y_4)$. The left picture is essentially the same as the third picture in Figure 1.

Boundary conditions. An elementary diagonal move (Section 2.2) of a box in an excited diagram is the same as the flip $(\text{right}, \text{up}) \to (\text{up}, \text{right})$ of a path in the vertex model path configuration. Therefore, the set of all excited diagrams $D \in \mathcal{E}(\lambda/\mu)$ (for some $\mu \subseteq \lambda$) is in bijection with the set of all path configurations in Ω_λ with a fixed boundary condition. Here by a boundary condition we mean a binary string along the southeast border of λ, where 1 encodes a entering/exiting path, and 0 means no path. For example, the boundary condition for the five-vertex model in Figure 5 is 11000000011.

For an arbitrary Young diagram μ, let us define a rim-hook decomposition of $\Omega \setminus \Omega_\mu$ into parallel translations of the first (infinite) outer rim-hook

$$R_\mu^{(1)} := \bigcup_{i=1}^{\infty} \{(i,j) \colon \mu_i + 1 \le j \le \mu_{i-1} + 1\},$$

where, by agreement, $\mu_0 = +\infty$. Define by $R_\mu^{(k)}$, $k \ge 2$, the parallel translation of $R_\mu^{(1)}$ by the vector $(i,j) = (k-1, k-1)$ (that is, by $k-1$ in the southeast direction). We refer to the $R_\mu^{(k)}$'s as the *μ-rim-hooks*. We have

$$\Omega \setminus \Omega_\mu = \bigcup_{k=1}^{\infty} R_\mu^{(k)}.$$

See Figure 6 for an illustration.

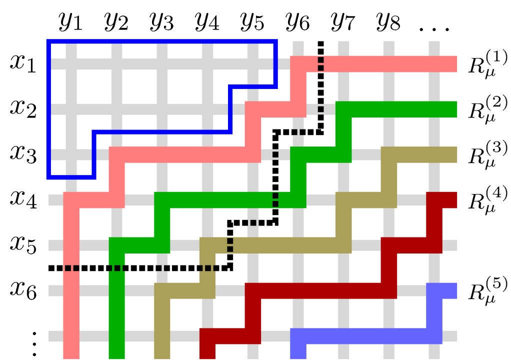


Figure 6: Rim-hook decomposition of the part of the square lattice $\Omega \setminus \Omega_\mu$ into the union of $R_\mu^{(k)}$, $k \ge 1$. Here Here $\mu = (5,4,1)$. The dotted line indicates the southeast border of another Young diagram, $\lambda = (6,6,5,5,4)$.

Definition 5.5. For two Young diagrams $\mu \subseteq \lambda$, the *μ-boundary condition* on Ω_λ is a binary string $B(\lambda/\mu)$ of length $\lambda_1 + \ell(\lambda)$ which records intersections of of the southeast border of λ with the μ-rim-hooks $R_\mu^{(k)}$, $k \ge 1$. Namely, when a length 1 segment of the boundary of λ intersects any μ-rim-hook $R_\mu^{(k)}$, we put 1 in the position of the string $B(\lambda/\mu)$ corresponding to this boundary segment. When a λ-boundary segment does not intersect a μ-rim-hook, we put 0 in $B(\lambda/\mu)$.

For example, in Figure 6 the μ-boundary condition is $B(66554/541) = 11011010001$. Note that the diagram $\mu = (5,4,1)$ in Figure 6 differs from the inner diagram in Figure 5, which results in a different binary string $B(66554/332) = 11000000011$.

Clearly, the boundary of λ intersects each rim-hook $R_\mu^{(k)}$ an even number of times. However, not every binary string of length $\lambda_1 + \ell(\lambda)$ with an even number of 1's is a valid μ-boundary condition (see Proposition 5.13 below for a precise description).

Proposition 5.6. *For any $\mu \subseteq \lambda$, the sum over excited diagrams $\mathsf{Z}_\mu(\lambda)$ (5.5) is equal to the partition function of the five-vertex model in Ω_λ with the weight $w_{x_i - y_j}$ (5.1) at each vertex $(i,j) \in \Omega_\lambda$, boundary conditions $B(\lambda/\mu)$ along the southeast border of Ω_λ, and empty boundary conditions along its west and north boundaries.*

Proof. This statement follows from the discussion above in the present Section 5.2. Indeed, observe that the configuration of rim-hooks $R_\mu^{(k)}$ inside Ω_λ is the same as a distinguished five-vertex model paths configuration. This distinguished configurations is minimal in the sense that all empty vertices are pushed in the northwest direction. The minimal configuration is identified with an initial excited diagram $D = \mu \in \mathcal{E}(\lambda/\mu)$. A move of a box in an excited diagram (Section 2.2) corresponds to a flip of a path in the five-vertex model. All five-vertex model path configurations are obtained from the minimal one by a sequence of flips. Thus, the five-vertex model partition function is equal to the sum over all $D \in \mathcal{E}(\lambda/\mu)$. This completes the proof. $\square$

Remark 5.7. The five-vertex model configurations in Proposition 5.6 are the same as the non-intersecting lattice path configurations discussed in Section 4.4 (and which we enumerated by a determinantal formula). Note that in the present Section 5, a key role in the analysis of the five-vertex model is played by the boundary conditions along the southeast border of Ω_λ. In particular, the dependence of these boundary conditions on μ is crucial for the Pieri rule.

5.3 Yang–Baxter moves sweeping a Young diagram

In this subsection, we apply the Yang–Baxter equation (Proposition 5.2) to express $\mathsf{Z}_\mu(\lambda)$ as a partition function in a larger domain with an additional strand of vertices along its southeast border. Throughout this subsection, t is an auxiliary spectral parameter assumed to be a generic complex number. First, let us add a strand to the northwest boundary of Ω_λ.

Definition 5.8 (Domain $\Omega_\lambda^{\ulcorner}$)**.** Fix $\mu \subseteq \lambda$, and consider a larger domain $\Omega_\lambda^{\ulcorner}$ obtained by adding a single *new strand* of $\lambda_1 + \ell(\lambda)$ vertices along the northwest boundary of Ω_λ. Let the additional vertices $(i, 0)$, $1 \le i \le \ell(\lambda)$, have weights $\check{w}_{x_i - t}$, and $(0, j)$, $1 \le j \le \lambda_1$, have weights $r_{y_j - t}$. The northwest boundary of $\Omega_\lambda^{\ulcorner}$ and the boundary conditions on the new strand are empty, while the southeast border carries the binary string $B(\lambda/\mu)$. Inside Ω_λ, the weights are $w_{x_i - y_j}$, as before. See Figure 7, left, for an illustration.

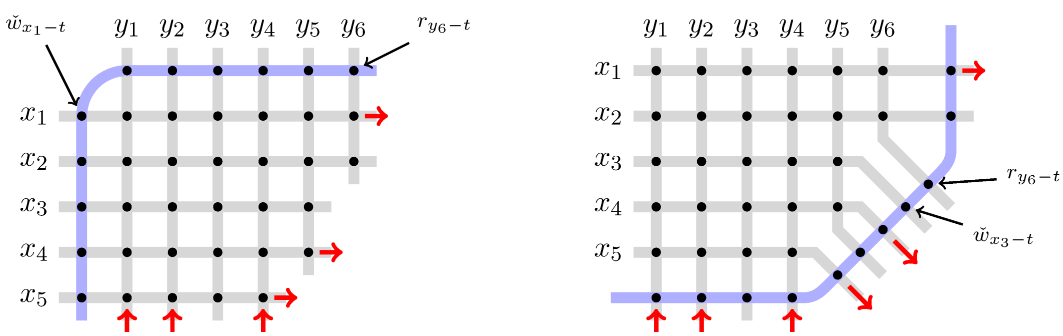


Figure 7: The domains $\Omega_\lambda^{\ulcorner}$ (left) and $\Omega_\lambda^{\lrcorner}$ (right), see Definitions 5.8 and 5.10. The partition functions in these domains depend on x_i, y_j, and t. They are equal to each other by the Yang–Baxter equation. Here $\lambda = (6, 6, 5, 5, 4)$, $\mu = (5, 4, 1)$, and the boundary binary string is $B(\lambda/\mu) = 11011010001$. For each 1 in the binary string, we draw an incoming or an outgoing arrow for, respectively, a vertical or a horizontal edge. In $\Omega_\lambda^{\lrcorner}$, we modified the way to draw the southeast border (while preserving the same intersections) for better visibility.

Lemma 5.9. *The partition function of the vertex model in $\Omega_\lambda^{\ulcorner}$ is equal to $\mathsf{Z}_\mu(\lambda)$.*

Proof. Due to the arrow preservation at each vertex $(i,0)$, $1 \le i \le \ell(\lambda)$, the empty boundary conditions along the west boundary of $\Omega_\lambda^{\ulcorner}$ lead to the empty boundary conditions entering Ω_λ. Note that $\breve{w}_{x_i-t}(0,0;0,0) = 1$, so the extra vertices at $(i,0)$ contribute a factor of 1 to the partition function. Similarly, the arrow preservation and the fact that $r_{y_j-t}(0,0;0,0) = 1$ imply that the north boundary of Ω_λ gets an empty boundary condition, and the extra vertices at $(0,j)$ also contribute a factor of 1. Thus, the partition function in $\Omega_\lambda^{\ulcorner}$ reduces to the one in Ω_λ, which is equal to $\mathsf{Z}_\mu(\lambda)$. ☐

The lattice configuration in the extended domain $\Omega_\lambda^{\ulcorner}$ now allows to apply the Yang–Baxter equation (Proposition 5.2). That is, we start in $\Omega_\lambda^{\ulcorner}$ at the triangle formed by the vertices $(1,1), (0,1)$, and $(1,0)$, and apply the Yang–Baxter equation to move the new strand one step in the southeast direction. Continuing in this way, the strand sweeps the Young diagram λ, and in the end it is located below the southeast border of Ω_λ. This results in a new domain for the vertex model:

Definition 5.10 (Domain $\Omega_\lambda^{\lrcorner}$)**.** Let $\Omega_\lambda^{\lrcorner}$ be obtained from the domain Ω_λ by adding one more vertex to each horizontal and vertical edge along the southeast border of λ. Let these new vertices be connected by a single *new strand*. When the new strand intersects a horizontal edge carrying a spectral parameter x_i or a vertical edge carrying a spectral parameter y_j, we assign the weight $\breve{w}_{x_i-t}$ or r_{y_j-t}, respectively, to the new vertex on this edge. The southeast border of the new domain $\Omega_\lambda^{\lrcorner}$ carries the binary string $B(\lambda/\mu)$, while the northwest boundary and the boundary conditions on the new strand are empty. The weights inside Ω_λ are $w_{x_i-y_j}$, as before. See Figure 7, right, for an illustration.

Combining Lemma 5.9 with the Yang–Baxter equation, we immediately obtain:

Proposition 5.11. *The partition function of the vertex model in $\Omega_\lambda^{\lrcorner}$ is equal to $\mathsf{Z}_\mu(\lambda)$.*

Remark 5.12. In $\Omega_\lambda^{\ulcorner}$, the new strand may be thought of as the boundary of an empty Young diagram κ. Each application of the Yang–Baxter equation when passing from $\Omega_\lambda^{\ulcorner}$ to $\Omega_\lambda^{\lrcorner}$ may be thought of as adding a box to κ. When κ becomes λ, the new strand is located below the southeast border of Ω_λ. Since the Yang–Baxter equation is a local transformation, the order of adding boxes to κ in this growing process is irrelevant.

5.4 Boundary binary strings via Maya diagrams

For a Young diagram λ, denote

$$I(\lambda) := \{-\ell(\lambda), -\ell(\lambda)+1, \ldots, \lambda_1 - 2, \lambda_1 - 1\} \subset \mathbb{Z}, \qquad |I(\lambda)| = \lambda_1 + \ell(\lambda). \tag{5.8}$$

Encode the southeast border of λ via its (zero-charge) *Maya diagram*

$$X(\lambda) := \big\{\lambda_i - i \colon 1 \le i \le \ell(\lambda)\big\} \subset I(\lambda). \tag{5.9}$$

The vertical and horizontal edges along the southeast border of λ correspond, respectively, to the elements of $X^c(\lambda) := I(\lambda) \setminus X(\lambda)$ and $X(\lambda)$. It is well-known that

$$X^c(\lambda) = \big\{-\lambda'_j + j - 1 \colon 1 \le j \le \lambda_1\big\}, \tag{5.10}$$

where λ' is the transposed Young diagram of λ.

We have $I_\varnothing = X(\varnothing) = \varnothing$. For our running example $\lambda = (6,6,5,5,4)$, we have

$$I(\lambda) = \{-5,-4,\ldots,4,5\}, \qquad X(\lambda) = \{5,4,2,1,-1\}, \qquad X^c(\lambda) = \{-5,-4,-3,-2,0,3\}.$$

Maya diagrams help demystify the boundary binary string $B(\lambda/\mu)$ from Definition 5.5:

Proposition 5.13. *For any $\mu \subseteq \lambda$, we have*

$$B(\lambda/\mu) = X(\lambda)\,\Delta\,X(\mu) \subseteq I(\lambda), \tag{5.11}$$

where Δ denotes the symmetric difference of sets, and we interpret the binary string as a subset of $I(\lambda)$.

Remark 5.14. In (5.11) and throughout the rest of Section 5, we slightly abuse the notation by appending $\mu \subseteq \lambda$ by zeros, if necessary, such that the set $X(\mu) = \{\mu_i - i\colon i = 1,2,\ldots\}$ is treated a subset of $I(\lambda)$. Note that $I(\mu)$ may be strictly inside $I(\lambda)$, but we never deal with the set $I(\mu)$ of the inner Young diagram μ.

The number of elements in $X(\lambda)$ is equal to $\ell(\lambda)$. One can check that for any $\mu \subseteq \lambda$, the number of elements of $X(\mu)$ (viewed as a subset of $I(\lambda)$) is also equal to $\ell(\lambda)$.

Continuing with our example $\lambda = (6,6,5,5,4)$, $\mu = (5,4,1)$, we have

$$X(\mu) = \{4,2,-2,-4,-5\} \subseteq I(\lambda), \qquad X(\lambda)\,\Delta\,X(\mu) = \{-5,-4,-2,-1,1,5\},$$

which agrees with $B(\lambda/\mu) = 11011010001$.

Proof of Proposition 5.13. Throughout the proof, we treat all equalities between subsets of $\mathbb{Z}$ as valid only when intersecting with $I(\lambda) = \{-\ell(\lambda),\ldots,\lambda_1 - 1,\lambda_1\}$ (but do not explicitly include this intersection in the notation).

We argue by induction, by adding one box to μ. The base case is $\mu = \varnothing$. The binary string $B(\lambda/\varnothing)$ arises from the usual hook decomposition of λ (cf. Figure 6). One readily sees that

$$B(\lambda/\varnothing) = \{\lambda_i - i\colon \lambda_i - i \ge 0\} \cup \left\{i - 1 - \lambda'_i\colon \lambda'_i - i + 1 \ge 1\right\} = X(\lambda)\,\Delta\,\mathbb{Z}_{\le 0},$$

as desired.

Now let $\mu,\nu \subseteq \lambda$ are such that $\nu = \mu + \square$. In terms of Maya diagrams, this means that for some k,

$$k \in X(\mu), \quad k+1 \notin X(\mu), \qquad X(\nu) = (X(\mu) \cup \{k+1\}) \setminus \{k\}\,. \tag{5.12}$$

There are four cases depending on whether k and $k+1$ belong to $X(\lambda)$. They are illustrated by local pictures in Figure 8 (an example of a global rim-hook configuration is in Figure 6). The four cases correspond to four possible directions of the southeast border of λ through k and $k+1$. Indeed, in (a) we have $k, k+1 \notin X(\lambda)$, and the boundary goes horizontally. The other cases are (b) $k \notin X(\lambda)$, $k+1 \in X(\lambda)$; (c) $k \in X(\lambda)$, $k+1 \notin X(\lambda)$; and (d) $k, k+1 \in X(\lambda)$.

From the induction assumption, it follows that the configuration of μ-rim-hooks around the part of the southeast border of λ through k and $k+1$ is the same in all four cases. Adding a box to μ changes the μ-rim-hook configuration to ν-rim-hooks in the same way in all cases, which results in the corresponding change of the boundary binary string $B(\lambda/\mu) \to B(\lambda/\nu)$. This completes the proof. $\square$

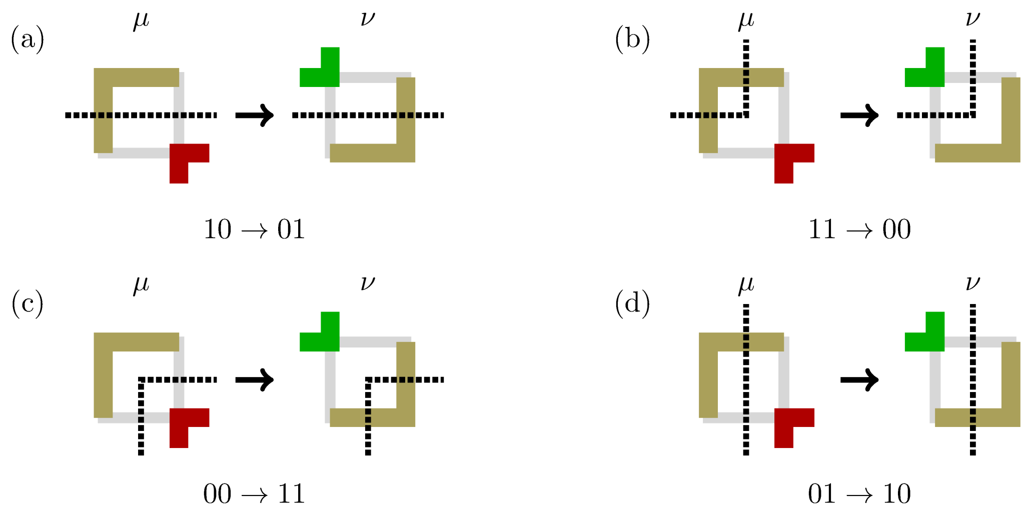


Figure 8: Four cases of adding a box $\nu = \mu + \square$ in the proof of Proposition 5.13. The dashed line is the southeast boundary of λ, and thick lines are μ- or ν-rim-hooks. Below each case, we indicate the local change in the boundary binary string, $B(\lambda/\mu) \to B(\lambda/\nu)$.

5.5 Vertical strip expansion of the five-vertex partition function

Definition 5.15. Fix a Young diagram λ. Let us define a *transfer matrix* $\mathcal{R}_\lambda^t$ which depends on the spectral parameter t (and also on x_i, y_j, but we suppress this in the notation), and has rows and columns indexed by Young diagrams $\mu, \nu \subseteq \lambda$. The value $\mathcal{R}_\lambda^t(\mu,\nu)$ is a partition function of a single-row vertex model whose vertices are indexed by $I(\lambda)$ (5.8). The vertex weight at each $k \in I(\lambda)$ has the form

$$\begin{cases} r_{x_i - t}, & k = \lambda_i - i \in X(\lambda); \\ r_{y_j - t}, & k = -\lambda'_j + j - 1 \in X^c(\lambda). \end{cases} \tag{5.13}$$

The boundary conditions on the left and right of the row are empty, and boundary conditions on the top and bottom are given by $X(\mu)$ and $X(\nu)$ (viewed as subsets of $I(\lambda)$), respectively.

Remark 5.16. The choice of a spectral parameter $x_i - t$ or $y_j - t$ at a point in $k \in I(\lambda)$ can be uniformly written as

$$\mathrm{parameter}(k) \coloneqq x_{|X(\lambda)\cap\mathbb{Z}_{\ge k}|}\mathbf{1}_{k\in X(\lambda)} + y_{|X^c(\lambda)\cap\mathbb{Z}_{\le k}|}\mathbf{1}_{k\in X^c(\lambda)} - t.$$

Clearly, for each μ, ν, there is at most one path configuration with these boundary conditions. If there are no path configurations, we set $\mathcal{R}_\lambda^t(\mu,\nu) = 0$, and otherwise we let $\mathcal{R}_\lambda^t(\mu,\nu)$ to be the product of the weights of all vertices along $I(\lambda)$. See Figure 9 for an illustration.

Recall that a vertical strip is a skew Young diagram which has at most one box in each row. We have the following expansion of the five-vertex model partition functions $\mathsf{Z}_\mu(\lambda)$:

Proposition 5.17. *For any $\mu \subseteq \lambda$, we have*

$$\mathsf{Z}_\mu(\lambda) = \frac{1}{(x_1 - t)\ldots(x_{\ell(\lambda)} - t)} \sum_{\substack{\nu\subseteq\lambda \\ \nu=\mu+\text{vertical strip}}} \mathcal{R}_\lambda^t(\mu,\nu)\,\mathsf{Z}_\nu(\lambda), \tag{5.14}$$

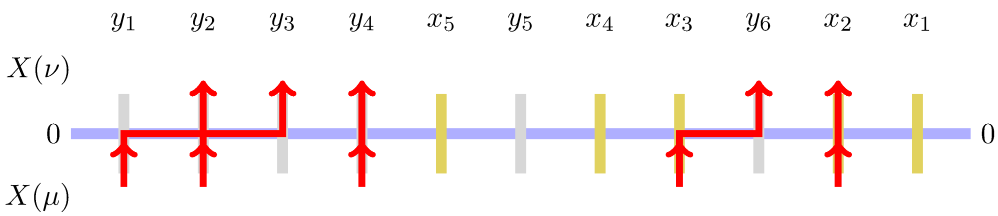


Figure 9: The one-row partition function for $\mathcal{R}^t_\lambda(\mu,\nu)$ with $\lambda = (6,6,5,5,4)$, $\mu = (5,4,1)$, and $\nu = (5,5,1,1,1)$. The different colors of the vertical edges correspond to the different spectral parameters $x_i - t$ or $y_j - t$ in the vertex weights, see (5.13). The sequence of spectral parameters depends only on λ.

where $\mathcal{R}^t_\lambda$ is the transfer matrix from Definition 5.15. The vertical strip in (5.14) can be empty.

Proof of Proposition 5.17. We start from Proposition 5.11 which states that $\mathsf{Z}_\mu(\lambda)$ is the partition function of the vertex model in the domain $\Omega^{\blacksquare}_\lambda$ (see Definition 5.10), with the boundary conditions $B(\lambda/\mu)$ along the extra new strand of vertices carrying the weights r_{y_j-t} and $\check{w}_{x_i-t}$ (see Figure 7, right). Peeling off this extra strand and summing over the binary strings between the strand and the southeast border of λ, we immediately get the following expansion:

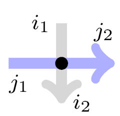
$$\mathsf{Z}_\mu(\lambda) = \sum_{\nu\subseteq\lambda} \mathcal{T}^t_\lambda(\mu,\nu)\,\mathsf{Z}_\nu(\lambda). \tag{5.15}$$

Indeed, $\mathsf{Z}_\nu(\lambda)$ is the partition function of the five-vertex model in Ω_λ with some boundary conditions. The coefficients $\mathcal{T}^t_\lambda(\mu,\nu)$ are determined from one-row partition functions with the following data:

- The vertices on the row are indexed by $I(\lambda)$.
- At each $\lambda_i - i \in X(\lambda)$, we put the *reversed* weight $\check{w}_{x_i-t}$. Namely, paths at this vertex are oriented *down and right*.
- At each $-\lambda'_j + j - 1 \in X^c(\lambda)$, we put the *usual* weight r_{y_j-t}, with the *up and right* path orientation.
- The boundary conditions on the left and right of the row are empty.
- The boundary conditions on the top and bottom of the row are given by the binary strings $B(\lambda/\mu)$ and $B(\lambda/\nu)$, respectively.

See Figure 10 for an illustration.

In the partition function $\mathcal{T}^t_\lambda(\mu,\nu)$, we now *reverse* the orientation of all vertical edges carrying the weights $\check{w}_{x_i-t}$. We obtain new weights which have the form

$\check{w}_{x_i-t}\Big($ 

$\Big) = \dfrac{r_{x_i-t}(1-i_2, j_1; 1-i_1, j_2)}{x_i - t}, \qquad i_1, j_1, i_2, j_2 \in \{0,1\}, \quad i = 1,2,\ldots,\ell(\lambda).$

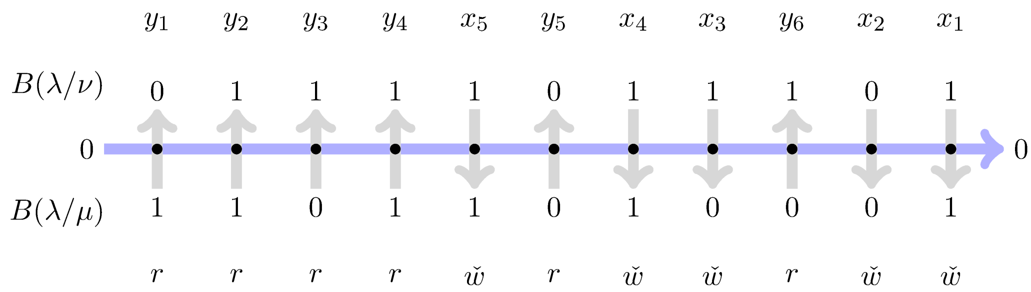


Figure 10: The one-row partition function for the coefficients $\mathcal{T}^t_\lambda(\mu,\nu)$ in (5.15) with the same λ,μ,ν as in Figure 9. The up and down arrows indicate the orientation of the vertical paths at the vertices. Note that the horizontal paths are always oriented to the right. Zeroes and ones indicate the boundary conditions $B(\lambda/\mu)$ and $B(\lambda/\nu)$.

By Proposition 5.13, this reversal modifies the bottom and top boundary conditions to

$$B(\lambda/\mu)\,\Delta\,X(\lambda) = X(\mu), \qquad B(\lambda/\nu)\,\Delta\,X(\lambda) = X(\nu).$$

Thus, we conclude that

$$\mathcal{T}^t_\lambda(\mu,\nu) = \frac{\mathcal{R}^t_\lambda(\mu,\nu)}{(x_1-t)\dots(x_{\ell(\lambda)}-t)}.$$

It remains to show that the sum over $\nu \subseteq \lambda$ in (5.15) is restricted to ν obtained from μ by adding a vertical strip. This follows from the fact that $r_z(0,1;0,1)=0$, which implies that horizontal paths in Figure 9 cannot travel by more than one horizontal step. This restriction implies that $\mathcal{R}^t_\lambda(\mu,\nu)$ vanishes unless $\nu = \mu + \text{vertical strip}$, and so we are done. $\square$

5.6 Pieri rule and proof Theorem 1.1

We are now ready to establish the Pieri rule for the five-vertex partition functions $\mathsf{Z}_\mu(\lambda)$ (5.5). Together with vanishing (Remark 5.4), the general approach of Section 3.1 then guarantees that $\mathsf{Z}_\mu(\lambda)$ is expressed as a sum over skew standard Young tableaux of shape λ/μ. This would complete the proof of the multivariate hook-length formula (MHLF).

Definition 5.18. Let $\mu \subseteq \lambda$ be two Young diagrams. Define

$$\mathsf{p}_\mu(\lambda) := \sum_{k\in X^c(\mu)\cap X(\lambda)} x_{|X(\lambda)\cap\mathbb{Z}_{\geq k}|} - \sum_{k\in X(\mu)\cap X^c(\lambda)} y_{|X^c(\lambda)\cap\mathbb{Z}_{\leq k}|}. \tag{5.16}$$

For example, for $\lambda=(6,6,5,5,4)$ and $\mu=(5,4,1)$, we have

$$\mathsf{p}_\mu(\lambda) = (x_1+x_4+x_5) - (y_1+y_2+y_4).$$

Proposition 5.19 (Pieri rule for five-vertex partition functions)**.** *Let $\mu\subseteq\lambda$ be two Young diagrams. Then we have*

$$\mathsf{p}_\mu(\lambda)\,\mathsf{Z}_\mu(\lambda) = \sum_{\substack{\nu\subseteq\lambda\\ \nu=\mu+\square}} \mathsf{Z}_\nu(\lambda).$$

Proof. We employ Proposition 5.17 and consider the behavior of identity (5.14) as $t \to \infty$. Since $\mathsf{Z}_\mu(\lambda)$ and $\mathsf{Z}_\nu(\lambda)$ do not depend on t, it suffices to look at the transfer matrix $\mathcal{R}^t_\lambda(\mu, \nu)$ defined as the one-row partition function (see Figure 9).

Recall (Remark 5.14) that the number of paths in the one-row vertex model for $\mathcal{R}^t_\lambda(\mu, \nu)$ is equal to $\ell(\lambda)$. First, observe that

$$\mathcal{R}^t_\lambda(\mu, \mu) = \prod_{k \in X(\mu)} \begin{cases} x_i - t, & k = \lambda_i - i \in X(\lambda); \\ y_j - t, & k = -\lambda'_j + j - 1 \in X^c(\lambda), \end{cases} \tag{5.17}$$

which behaves as $t \to +\infty$ as follows:

$$(-t)^{\ell(\lambda)} + (-t)^{\ell(\lambda)-1}\Bigg(\sum_{k \in X(\mu)\cap X(\lambda)} x_{|X(\lambda)\cap \mathbb{Z}_{\geq k}|} + \sum_{k \in X(\mu)\cap X^c(\lambda)} y_{|X^c(\lambda)\cap \mathbb{Z}_{\leq k}|}\Bigg) + O(t^{\ell(\lambda)-2}). \tag{5.18}$$

Indeed, the factors in (5.17) are in one-to-one correspondence with the summands by $(-t)^{\ell(\lambda)-1}$ in (5.18), cf. Remark 5.16.

Next, for any ν with $|\nu| > |\mu|$, we have

$$\mathcal{R}^t_\lambda(\mu, \nu) = (-t)^{\ell(\lambda)-|\nu|+|\mu|} + O(t^{\ell(\lambda)-|\nu|+|\mu|-1}), \qquad t \to \infty. \tag{5.19}$$

Indeed, $|\nu| - |\mu|$ is the number of occupied horizontal edges in the vertex model for $\mathcal{R}^t_\lambda(\mu, \nu)$. Placing each extra occupied horizontal edge exchanges one weight $r_{y_j - t}(1, 0; 1, 0) = y_j - t$ or $r_{x_i - t}(1, 0; 1, 0) = x_i - t$ (growing with t) by a product of other r weights. All other r weights are equal to 0 or 1 (see (5.3)). This produces (5.19).

Let us now combine the asymptotics (5.18), (5.19) with the prefactor in (5.14),

$$\frac{1}{(x_1 - t)\dots(x_{\ell(\lambda)} - t)} = (-t)^{-\ell(\lambda)}\left(1 + t^{-1}\sum_{i=1}^{\ell(\lambda)} x_i\right) + O(t^{-\ell(\lambda)-2}), \qquad t \to \infty.$$

We see that we can cancel out the overall multiplicative factor $(-t)^{\ell(\lambda)}$. After that, the constant terms in both sides are equal to $\mathsf{Z}_\mu(\lambda)$, which cancel out. Equating the terms of order t^{-1}, we obtain the desired Pieri rule. $\square$

Proof of Theorem 1.1. The Pieri rule of Proposition 5.19 together with the vanishing (Remark 5.4) and the general result of Proposition 3.2 imply that

$$\sum_{T \in \mathrm{SYT}(\lambda/\mu)} \prod_{m=1}^{|\lambda/\mu|} \frac{1}{\mathsf{p}_{T^{-1}[<m]}(\lambda)} = \sum_{D \in \mathcal{E}(\lambda/\mu)} \prod_{(i,j)\in\lambda\setminus D} \frac{1}{x_i - y_j}.$$

Note that the Pieri coefficients $\mathsf{C}_{\nu/\mu}$ are equal to 1 in our case. We also employed the definition of $\mathsf{Z}_\mu(\lambda)$ as a sum over excited diagrams (5.6), and cancelled out the factor $\mathsf{Z}_\lambda(\lambda)$ (5.7).

For any m, let us denote $T^{-1}[< m]$ by ν. Starting from (5.16), we can rewrite

$$\begin{aligned}
\mathsf{p}_\nu(\lambda) &= \sum_{k \in X^c(\nu) \cap X(\lambda)} x_{|X(\lambda) \cap \mathbb{Z}_{\geq k}|} - \sum_{k \in X(\nu) \cap X^c(\lambda)} y_{|X^c(\lambda) \cap \mathbb{Z}_{\leq k}|} \\
&= \sum_{i=1}^{\ell(\lambda)} x_i - \left(\sum_{k \in X(\nu) \cap X(\lambda)} x_{|X(\lambda) \cap \mathbb{Z}_{\geq k}|} + \sum_{k \in X(\nu) \cap X^c(\lambda)} y_{|X^c(\lambda) \cap \mathbb{Z}_{\leq k}|} \right) \\
&= \sum_{i=1}^{\ell(\lambda)} x_i - \sum_{j=1}^{\ell(\lambda)} b_{\nu_j - j}.
\end{aligned}$$

Here

$$b_j \coloneqq \begin{cases} x_i, & j = \lambda_i - i; \\ y_k, & j = n_k, \end{cases}$$

with the notation

$$\{n_1 < \ldots < n_{\lambda_1}\} = \{-\ell(\lambda), -\ell(\lambda)+1, \ldots, \lambda_1 - 2, \lambda_1 - 1\} \setminus \{\lambda_1 - 1, \ldots, \lambda_{\ell(\lambda)} - \ell(\lambda)\}.$$

We see that the expressions $\mathsf{p}_\nu(\lambda) = \sum_{i=1}^{\ell(\lambda)} x_i - \sum_{j=1}^{\ell(\lambda)} b_{\nu_j - j}$, where $\mu \subseteq \nu \subseteq \lambda$, coincide with the factors in the denominator in the left-hand side of the multivariate hook-length formula (MHLF). This completes the proof of Theorem 1.1. $\square$

A A semistandard variant

Let us modify the polynomials $f_j(u)$ (4.2) from Section 4, and investigate the resulting contour integrals defined in the same way as in (4.1). Denote the integrals by J to avoid confusion. Let β be a parameter, $a = (a_1, a_2, \ldots)$ be a sequence of parameters as before, and $\mathsf{m} = (m_1, m_2, \ldots)$ be a sequence of nonnegative integers. Set

$$f_r^\beta(u \mid a) = \prod_{i=1}^{r} (u + a_i + \beta u a_i),$$

and

$$J_{\mu,\mathsf{m}}(x \mid a) \coloneqq \frac{(-1)^{\binom{n}{2}}}{(2\pi\sqrt{-1})^n} \oint_\gamma \cdots \oint_\gamma \prod_{i=1}^{n} \frac{f^\beta_{\mu_i + m_i - i}(u_i \mid a)}{\prod_{j=1}^{m_i}(u_i - x_j)} \Delta(u)\, du_1 \cdots du_n. \tag{A.1}$$

The contours γ are the same as in Section 4, they go around all the poles x_i in the positive direction. Set $m_i = n$ for all i, and omit m from the notation.

Arguing as in Section 4.3, we obtain a Pieri-type rule:

$$\begin{aligned}
\sum_{\epsilon \in \{0,1\}^n} \beta^{|\epsilon|} J_{\mu+\epsilon}(x \mid a) &= \frac{1}{(2\pi\sqrt{-1})^n} \oint_\gamma \cdots \oint_\gamma \prod_{i=1}^{n} \frac{f^\beta_{\mu_i+n-i}(u_i \mid a)}{\prod_{j=1}^{n}(u_i - x_j)} \\
&\times \prod_{i=1}^{n} \left(1 + \beta(u_i + a_{\mu_i+1+n-i} + \beta u_i a_{\mu_i+1+n-i})\right) \Delta(u)\, du_1 \cdots du_n
\end{aligned}$$

$$= \frac{1}{(2\pi\sqrt{-1})^n} \oint_\gamma \cdots \oint_\gamma \prod_{i=1}^n \frac{f^\beta_{\mu_i+n-i}(u_i \mid a)}{\prod_{j=1}^n (u_i - x_j)} \prod_{i=1}^n (1+\beta u_i)(1+\beta a_{\mu_i+1+n-i}) \Delta(u)\, du_1 \cdots du_n$$
$$= J_\mu(x \mid a) \prod_{i=1}^n (1+\beta x_i)(1+\beta a_{\mu_i+1+n-i}).$$

Here $|\epsilon| = \sum_{i=1}^n \epsilon_i$. Next, if $\hat\mu = \mu + \epsilon$ is not a partition, i.e. $\mu_i + \epsilon_i < \mu_{i+1} + \epsilon_{i+1}$, we must have $\mu_i = \mu_{i+1}$, $\epsilon_i = 0$ and $\epsilon_{i+1} = 1$, and so $f_{\hat\mu_i+n-i}(u \mid a) = f_{\hat\mu_{i+1}+n-(i+1)}(u \mid a)$. This makes the integral 0 by skew symmetry. Therefore, the Pieri-type rule takes the form:

$$\sum_\nu J_\nu(x \mid a) = J_\mu(x \mid a) \prod_{i=1}^n (1+\beta x_i)(1+\beta a_{\mu_i+1+n-i}), \tag{A.2}$$

where the sum is over all partitions ν obtained from μ by adding a (possibly employ) vertical strip.

Similarly to Theorem 4.1, integral (A.1) can be rewritten as a determinant of the f_i^β's:

$$\begin{aligned} J_\mu(x \mid a) &= \frac{(-1)^{\binom{n}{2}}}{(2\pi\sqrt{-1})^n} \oint_\gamma \cdots \oint_\gamma \prod_{i=1}^n \frac{f^\beta_{\mu_i+n-i}(u_i \mid a)}{\prod_{j=1}^n (u_i - x_j)} \Delta(u)\, du_1 \cdots du_n \\ &= \frac{1}{\Delta(x)} \det\left[\prod_{r=1}^{\mu_j+n-j} (x_i + a_r + \beta x_i a_r) \right]_{i,j=1}^n . \end{aligned} \tag{A.3}$$

In particular, when $\mu = \varnothing$ only the maximal degree terms in this determinant survive, so

$$J_\varnothing(x \mid a) = \prod_{i=1}^n (1+\beta a_i)^{n-i}. \tag{A.4}$$

Remark A.1. The determinantal formula (A.3) is similar to the one for factorial Grothendieck polynomials of [McN06] or [HJK+24]. However, in order to obtain the Grothendieck polynomials one needs to replace the polynomials $f^\beta_{\mu_i+n-i}(u \mid a)$ with $(1+\beta u)^{i-1} \prod_{j=1}^{\mu_i+n-i} (u + a_j + \beta u a_j)$. The approach outlined here would lead to the identities in [MPP22] after some tedious manipulations.

Consider now the vanishing of $J_\mu(x \mid a)$ for certain values of x. Let $x_i^\lambda := -\frac{a_{\lambda_i+n-i+1}}{1+\beta a_{\lambda_i+n-i+1}}$. Then $f^\beta_{\mu_i+n-i}(x_j^\lambda \mid a) = 0$ if $\lambda_j + n - j + 1 + 1 \le \mu_i + n - i$.

Let λ be such that for some i, we have $\lambda_i + n - i < \mu_i + n - i$, i.e., $\lambda_i < \mu_i$. Then we have $f^\beta_{\mu_r+n-r}(x_j^\lambda \mid a) = 0$ for $r \le i$ and $j \ge i$, which implies that $\det[f^\beta_{\mu_i+n-i}(x_j^\lambda \mid a)]_{i,j=1}^n = 0$. On the other hand, if $\lambda = \mu$, then the matrix is lower triangular. This implies

Lemma A.2 (Vanishing property)**.** *Let $x_i^\lambda := -\frac{a_{\lambda_i+n-i+1}}{1+\beta a_{\lambda_i+n-i+1}}$. Then*

$$J_\mu(x^\lambda \mid a) = \begin{cases} 0, & \text{if } \mu \not\subset \lambda; \\ \displaystyle\prod_{i=1}^n \prod_{j=1}^{\mu_i+n-i} \frac{a_j - a_{\lambda_i+n-i+1}}{(1+\beta a_{\lambda_i+n-i+1})}, & \text{if } \lambda = \mu. \end{cases}$$

The Pieri-type rule (A.2) can be rewritten as $(\cdots)J_\mu = \sum_{\nu\supset\mu} J_\nu$, where the sum is over all ν such that ν/μ is a nonempty vertical strip. Iterating this identity as in Section 3.1, we get the following result.

Theorem A.3. *Let $\mu \subset \lambda$ and set $x_i^\lambda = -\frac{a_{\lambda_i+n-i+1}}{1+\beta a_{\lambda_i+n-i+1}}$. For a Young diagram ν, let*

$$Y(\nu) := \prod_i (1+\beta x_i^\lambda)(1+\beta a_{\nu_i+n-i}) - 1.$$

Then we have

$$\frac{J_\mu(x^\lambda \mid a)}{J_\lambda(x^\lambda \mid a)} = \sum_{T\in \mathrm{SSYT}(\lambda'/\mu')} \prod_{k=1}^{|\lambda/\mu|} \frac{1}{Y(T[<k])}, \tag{A.5}$$

where the sum is over all SSYT *of shape λ'/μ', and $T[<k]=\nu$ means that the shape λ/ν is filled with entries $\geq k$. By agreement, $T[<1]=\mu$.*

When $\mu = \varnothing$, the RHS of (A.5) is a sum over $\mathrm{SSYT}(\lambda')$, and the LHS is the product

$$\frac{J_\varnothing(x^\lambda \mid a)}{J_\lambda(x^\lambda \mid a)} = \prod_{i=1}^n \frac{(1+\beta a_{\lambda_i+n-i+1})^{n-i}(1+\beta a_i)^{n-i}}{\prod_{j=1}^{n-i}(a_j - a_{\lambda_i+n-i+1})}.$$

To see excited diagrams in the left-hand side of (A.5), let $z_i = -\frac{a_i}{1+\beta a_i}$. One can check that

$$J_\mu(x \mid a) = \prod_{i=1}^n \prod_{j=1}^{\mu_i+n-i} \frac{1}{1+\beta a_j} F_\mu(x \mid z),$$

where $F_\mu(x \mid z)$ is the factorial Schur function from Section 4. Then $x^\lambda = z^\lambda$, and we can rewrite (A.5) in terms of excited diagrams:

Theorem A.4. *Let $x_1, x_2, \ldots, y_1, y_2, \ldots$ be two sets of indeterminates, and set*

$$a_{\lambda_i+n-i+1} = -\frac{x_i}{1+\beta x_i}, \qquad a_{\ell_j} = -\frac{y_j}{1+\beta y_j},$$

where $\ell = [1,\ldots,n+\lambda_1-1] \setminus \{\lambda_j+n-j+1 \colon 1\leq j\leq n\}$. With this notation, we have

$$\sum_{D\in\mathcal{E}(\lambda/\mu)} \prod_{(i,j)\in\lambda\setminus D} (x_i - y_j) = \prod_{i=1}^n \prod_{j=1}^{\mu_i+n-i} \frac{(a_j - a_{\lambda_i+n-i+1})(1+\beta a_j)}{(1+\beta a_{\lambda_i+n-i+1})} \sum_{T\in\mathrm{SSYT}(\lambda'/\mu')} \prod_{k=1}^{|\lambda/\mu|} \frac{1}{Y(T[<k])}.$$

Remark A.5. Observe that $Y(\nu) = \beta \sum_{i=1}^n (x_i^\lambda + a_{\nu_i+n-i}) + O(\beta^2)$. If we let $\beta \to 0$, and perform cancelations with the factors $a_j - a_r$ which are of the form $\beta(y-x)$, the surviving terms above would be the ones where T has a maximal number of different entries, so it is an SYT. This recovers the original formula of Theorem 1.1. We do not observe any substitutions that directly connect the formula in Theorem A.4 to the expression in [MPP23, Theorem 9.3].

B Skew hook-length formula from Macdonald polynomials

Here we consider the example of interpolation Macdonald polynomials [KS97, Kno97, Sah96, Oko98a, Oko98b], and apply the general formalism of Section 3 to obtain a "skew hook-length type" formula involving summation over skew standard Young tableaux. The discussion in the current Appendix B does not rely on contour integral or vertex model techniques of Sections 4 and 5, respectively.

We denote the interpolation Macdonald polynomials by $I_\mu(x_1, \ldots, x_n; q, t)$. Note that we work only with symmetric polynomials and not symmetric functions, so we drop the index n (which is fixed) from the notation $I_{\mu|n}$ used in [Ols19]. Throughout the current Appendix B, we assume that $n \geq \ell(\mu)$.

The polynomials I_μ are inhomogeneous symmetric polynomials of degree $|\mu|$ whose top degree homogeneous part is the Macdonald symmetric polynomial $P_\mu(x_1, x_2, \ldots, x_n; q, t)$ [Mac95, Ch. VI]. The substitution which ensures vanishing properties is

$$\mathsf{x}^{(q,t)}(\lambda) = \bigl(x_1^{(q,t)}(\lambda), \ldots, x_n^{(q,t)}(\lambda)\bigr) := \bigl(q^{-\lambda_1}, q^{-\lambda_2} t, \ldots, q^{-\lambda_n} t^{n-1}\bigr). \tag{B.1}$$

Note that here we use the normalization from [Ols19], which means that the substitution must be as in (B.1) (there are other equivalent variants in the literature).

Let us recall the vanishing property and a tableau formula for I_μ [Oko98a, Oko98b].

Proposition B.1. **1.** *We have $I_\mu(\mathsf{x}^{(q,t)}(\lambda); q, t) = 0$ unless $\mu \subseteq \lambda$.*

2. *The interpolation Macdonald polynomials I_μ admit the following tableau formula:*

$$I_\mu(x_1, \ldots, x_n; q, t) = \sum_{R \in RTab(\mu, n)} \psi_R(q;t) \prod_{(i,j)\in\mu} \bigl(x_{R(i,j)} - q^{1-j} t^{R(i,j)+i-2}\bigr), \tag{B.2}$$

where the sum is over all reverse semistandard tableaux of shape μ with values in $\{1, \ldots, n\}$ (that is, the values in the tableau must weakly decay along the rows and strictly decay down the columns). The coefficients $\psi_R(q;t)$ (where we view R as a sequence of horizontal strips) are rational functions in q, t given in [Mac95, Ch. VI, (6.24)(ii) and (7.11')].

3. *We have*

$$I_\lambda(\mathsf{x}^{(q,t)}(\lambda); q, t) = \prod_{(i,j)\in\lambda} \bigl(q^{-\lambda_i} t^{i-1} - q^{1-j} t^{\lambda'_j - 1}\bigr), \tag{B.3}$$

where λ' is the transposed Young diagram of λ.

Formula (B.3) follows from (B.2) since for $x = \mathsf{x}^{(q,t)}(\lambda)$, there is a unique reverse tableau $R(i,j) = \lambda'_j - i + 1$ contributing a nonzero term to the sum, and for it we have $\psi_R(q;t) = 1$.

A (one-box) Pieri formula for the (non-specialized) interpolation polynomials I_μ has the form

$$I_\mu(x_1, \ldots, x_n; q, t) \cdot \sum_{i=1}^{n} \bigl(x_i - q^{-\mu_i} t^{i-1}\bigr) = \sum_{\nu = \mu + \square} \varphi_{\nu/\mu}(q;t)\, I_\nu(x_1, \ldots, x_n; q, t), \tag{B.4}$$

where $\varphi_{\nu/\mu}(q;t)$ are the rational functions in q, t given in [Mac95, Ch. VI, (6.24)(i)]. Identity (B.4) follows by comparing the degrees and top homogeneous components in both sides, and using the uniqueness of interpolation.

Remark B.2. The Pieri rule can be generalized to a skew Cauchy type identity (also sometimes called Pieri rule) involving summation over horizontal strips [Ols19, Lemmas 5.5 and 5.9]:

$$I_\mu(x_1,\dots,x_n;q,t)\cdot\prod_{i=1}^{n}\frac{(x_iyt;q)_\infty}{(x_iy;q)_\infty}$$
$$=\sum_{\nu=\mu+\text{horizontal strip}} I_\nu(x_1,\dots,x_n;q,t)\cdot\varphi_{\nu/\mu}(q;t)\,y^{|\nu|-\mu}\prod_{i=1}^{n}\frac{(yq^{-\mu_i}t^i;q)_\infty}{(yq^{-\nu_i}t^{i-1};q)_\infty}.$$

Applying Proposition 3.2 together with the properties of the interpolation Macdonald polynomials in Proposition B.1, we immediately obtain the following skew hook-length type formula:

Proposition B.3 (Skew hook-length type formula with Macdonald parameters)**.** *For any $\mu\subseteq\lambda$, we have*

$$\sum_{T\in\mathrm{SYT}(\lambda/\mu)}\varphi_T(q;t)\prod_{k=1}^{|\lambda/\mu|}\left(\sum_{i=1}^{\ell(\lambda)}t^{i-1}\left(q^{-\lambda_i}-q^{-T^{-1}[<k]_i}\right)\right)^{-1}$$
$$=\prod_{(i,j)\in\lambda}\left(q^{-\lambda_i}t^{i-1}-q^{1-j}t^{\lambda'_j-1}\right)^{-1}\sum_{R\in RTab(\mu,\ell(\lambda))}\psi_R(q;t)\prod_{(i,j)\in\mu}t^{R_{i,j}-1}\left(q^{-\lambda_{R(i,j)}}-q^{1-j}t^{i-1}\right),$$

where the left-hand sum is over skew standard Young tableaux T of shape λ/μ, and the right-hand side sum is over reverse semistandard tableaux R of shape μ with entries in $\{1,\dots,\ell(\lambda)\}$.

References

[ABPW23] A. Aggarwal, A. Borodin, L. Petrov, and M. Wheeler, *Free Fermion Six Vertex Model: Symmetric Functions and Random Domino Tilings*, Selecta Math. **29** (2023), article 36. arXiv:2109.06718 [math.PR]. ↑16

[BL89] L.C. Biedenharn and J.D. Louck, *A new class of symmetric polynomials defined in terms of tableaux*, Adv. Appl. Math. **10** (1989), no. 4, 396–438. ↑5

[BMN14] D. Bump, P. McNamara, and M. Nakasuji, *Factorial Schur functions and the Yang-Baxter equation*, Rikkyo-daigaku-sugaku-zasshi **63** (2014), no. 1-2, 23–45. arXiv:1108.3087 [math.CO]. ↑16

[CG06] H. Coskun and R.A. Gustafson, *Well-poised Macdonald functions W_λ and Jackson coefficients ω_λ on BC_n*, Jack, Hall-Littlewood and Macdonald polynomials, 2006, pp. 127–155. arXiv:math.CO/0412153. ↑7

[CPP21] S.H. Chan, I. Pak, and G. Panova, *Sorting probability for large Young diagrams*, Discrete Anal. **2021** (2021), no. 24, 57 pp. arXiv:2005.08390 [math.CO]. ↑2

[FRT54] J.S. Frame, G. de B. Robinson, and R.M. Thrall, *The hook graphs of the symmetric group*, Canad. J. Math. **6** (1954), 316–324. ↑1

[FSTV23] G. Felder, A. Smirnov, V. Tarasov, and A. Varchenko, *Hypergeometric integrals, hook formulas and Whittaker vectors*, arXiv preprint (2023). arXiv:2308.05766 [math.RT]. ↑2

[GKMK23] D. Grinberg, N. Korniichuk, K. Molokanov, and S. Khomych, *The Pak–Postnikov and Naruse skew hook length formulas: a new proof*, arXiv preprint (2023). arXiv:2310.18275 [math.CO]. ↑2

[GV85] I. Gessel and G. Viennot, *Binomial determinants, paths, and hook length formulae*, Adv. Math. **58** (1985), no. 3, 300–321. ↑12

[HJK+24] B.-H. Hwang, J. Jang, J.S. Kim, M. Song, and U.-K. Song, *Refined canonical stable Grothendieck polynomials and their duals, Part I*, Adv. Math. **446** (2024), 109670. arXiv:2104.04251 [math.CO]. ↑29

[HKYY19] B. H. Hwang, J. S. Kim, M. Yoo, and S. M. Yun, *Reverse plane partitions of skew staircase shapes and q-Euler numbers*, J. Combin. Theory Ser. A **168** (2019), 120–163. arXiv:1711.02337 [math.CO]. ↑2

[IN09] T. Ikeda and H. Naruse, *Excited Young diagrams and equivariant Schubert calculus*, Trans. AMS **361** (2009), no. 10, 5193–5221. arXiv:math/0703637 [math.AG]. ↑1, 4

[Ize87] A. Izergin, *Partition function of the six-vertex model in a finite volume*, Sov. Phys. Dokl. **32** (1987), 878–879. ↑17

[JM23] P. Jiradilok and T. McConville, *Roots of descent polynomials and an algebraic inequality on hook lengths*, Electron. J. Combin. **30** (2023), no. 4, P4.41. arXiv:1910.14631 [math.CO]. ↑2

[KMY09] A. Knutson, E. Miller, and A. Yong, *Gröbner geometry of vertex decompositions and of flagged tableaux*, J. Reine Angew. Math. **630** (2009), 1–31. ↑4

[Kno97] F. Knop, *Symmetric and non-symmetric quantum Capelli polynomials*, Comment. Math. Helv. **72** (1997), no. 1, 84–100. arXiv:q-alg/9603028. ↑7, 31

[Kon20a] M. Konvalinka, *A bijective proof of the hook-length formula for skew shapes*, European J. Combin. **88** (2020), 103104. arXiv:1703.08414 [math.CO]. ↑2

[Kon20b] M. Konvalinka, *Hook, line and sinker: A bijective proof of the skew shifted hook-length formula*, Eur. J. Combin. **86** (2020), 103079. arXiv:1809.02389 [math.CO]. ↑2

[Kor24] S. Korotkikh, *Representation theoretic interpretation and interpolation properties of inhomogeneous spin q-Whittaker polynomials*, Selecta Math. **30** (2024), no. 40. arXiv:2204.06166 [math.CO]. ↑8

[Kor82] V. Korepin, *Calculation of norms of Bethe wave functions*, Comm. Math. Phys. **86** (1982), 391–418. ↑17

[Kre05] V. Kreiman, *Schubert classes in the equivariant K-theory and equivariant cohomology of the Grassmannian*, arXiv preprint (2005). arXiv:math/0512204 [math.AG]. ↑4

[KS19] A.N. Kirillov and T. Scrimshaw, *Hook-content formula using excited Young diagrams*, arXiv preprint (2019). arXiv:1904.00371 [math.CO]. ↑2

[KS97] F. Knop and S. Sahi, *A recursion and a combinatorial formula for Jack polynomials*, Inventiones mathematicae **128** (1997), 9–22. arXiv:q-alg/9610016. ↑7, 31

[Las07] A. Lascoux, *The 6 vertex model and Schubert polynomials*, SIGMA **3** (2007), 029. arXiv:math/0610719 [math.CO]. ↑16

[Lin73] B. Lindström, *On the vector representations of induced matroids*, Bulletin of the London Mathematical Society **5** (1973), no. 1, 85–90. ↑12

[Mac92] I.G. Macdonald, *Schur functions: Theme and variations*, Sém. Lothar. Combin **28** (1992), 5–39. ↑5

[Mac95] I.G. Macdonald, *Symmetric functions and Hall polynomials*, 2nd ed., Oxford University Press, 1995. ↑5, 11, 31

[McN06] P. McNamara, *Factorial Grothendieck Polynomials*, Electron. J. Comb. **13** (2006), no. R71. arXiv:math/0508192 [math.CO]. ↑8, 29

[McN09] P. McNamara, *Factorial Schur functions via the six vertex model*, arXiv preprint (2009). arXiv:0910.5288 [math.CO]. ↑16

[Mol09] A. Molev, *Comultiplication rules for the double Schur functions and Cauchy identities*, Electron. J. Comb. **R13** (2009). arXiv:0807.2127 [math.CO]. ↑5

[MPP17] A.H. Morales, I. Pak, and G. Panova, *Hook formulas for skew shapes II. Combinatorial proofs and enumerative applications*, SIAM J. Discr. Math. **31** (2017), no. 3, 1953–1989. arXiv:1610.04744 [math.CO]. ↑2, 4

[MPP18a] A.H. Morales, I. Pak, and G. Panova, *Asymptotics of the number of standard Young tableaux of skew shape*, European J. Combin. **70** (2018), 26–49. ↑2

[MPP18b] A.H. Morales, I. Pak, and G. Panova, *Hook formulas for skew shapes I. q-analogues and bijections*, J. Combin. Theory Ser. A **154** (2018), 350–405. arXiv:1512.08348 [math.CO]. ↑2

[MPP19] A.H. Morales, I. Pak, and G. Panova, *Hook formulas for skew shapes III. Multivariate and product formulas*, Algebraic Combinatorics **2** (2019), no. 5, 815–861. arXiv:1707.00931 [math.CO]. ↑2, 4

[MPP22] A.H. Morales, I. Pak, and G. Panova, *Hook Formulas for Skew Shapes IV. Increasing Tableaux and Factorial Grothendieck Polynomials*, J. Math. Sci. **261** (2022), 630–657. arXiv:2108.10140 [math.CO]. Published in Zapiski Nauchnykh Seminarov POMI, Vol. 507, 2021, pp. 59–98. ↑2, 6, 8, 29

[MPP23] A.H. Morales, G. Panova, and G. Park, *Minimal skew semistandard tableaux and the Hillman–Grassl correspondence*, arXiv preprint (2023). arXiv:2311.09209 [math.CO]. ↑2, 30

[MS99] A. Molev and B. Sagan, *A Littlewood-Richardson rule for factorial Schur functions*, Trans. Amer. Math. Soc. **351** (1999), 4429–4443. arXiv:q-alg/9707028. ↑5, 6

[MZ20] A.H. Morales and D.G. Zhu, *On the Okounkov-Olshanski formula for standard tableaux of skew shapes*, arXiv preprint (2020). arXiv:2007.05006 [math.CO]. ↑2, 15, 16

[Nap24] S. Naprienko, *Free Fermionic Schur Functions*, Adv. Math. **436** (2024), 109413. arXiv:2301.12110. ↑16, 17

[Nar14] H. Naruse, *Schubert calculus and hook formula*, 2014. Talk slides at 73rd Sém. Lothar. Combin., Strobl, Austria, available at https://tinyurl.com/z6paqzu. ↑1

[NN23] M. Nakagawa and H. Naruse, *The universal factorial Hall–Littlewood P- and Q-functions*, Fund. Math. **263** (2023), 133–166. arXiv:1705.04791 [math.AT]. ↑8

[NO19] H. Naruse and S. Okada, *Skew hook formula for d-complete posets via equivariant K-theory*, Algebraic Combin. **2** (2019), no. 4, 541–571. ↑2

[Oko96] A. Okounkov, *Quantum immanants and higher Capelli identities*, Transform. Groups **1** (1996), no. 1-2, 99–126. arXiv:q-alg/9602028. ↑14

[Oko97] A. Okounkov, *Binomial formula for Macdonald polynomials and applications*, Math. Res. Lett. **4** (1997), no. 4, 533–553. arXiv:q-alg/9608021. ↑7

[Oko98a] A. Okounkov, *On Newton interpolation of symmetric functions: A characterization of interpolation Macdonald polynomials*, Adv. Appl. Math. **20** (1998), 395–428. ↑7, 14, 31

[Oko98b] A. Okounkov, *(Shifted) Macdonald polynomials: q-Integral representation and combinatorial formula*, Compos. Math. **112** (1998), 147–182. arXiv:q-alg/9605013. ↑7, 16, 31

[Ols19] G. Olshanski, *Interpolation Macdonald polynomials and Cauchy-type identities*, Jour. Comb. Th. A **162** (2019), 65–117. arXiv:1712.08018 [math.CO]. ↑7, 31, 32

[OO97] A. Okounkov and G. Olshanski, *Shifted Schur functions*, Algebra i Analiz **9** (1997), no. 2, 73–146. translation in St. Petersburg Math. J. **9** (1998), no. 2, 239—300, arXiv:q-alg/9605042. ↑14

[Pak21] Igor Pak, *Skew shape asymptotics, a case-based introduction*, Seminaire Lotharingien de Combinatoire **84** (2021), no. B84a. ↑2

[Par22] G. Park, *Naruse hook formula for linear extensions of mobile posets*, Electron. J. Combin. **29** (2022), no. 3, P3.57. arXiv:2104.11166 [math.CO]. ↑2

[Rai06] E.M. Rains, *BC_n-symmetric abelian functions*, Duke Math. J. **135** (2006), no. 1, 99–180. arXiv:math/0402113 [math.CO]. ↑7

[Sah96] S. Sahi, *Interpolation, integrality, and a generalization of Macdonald's polynomials*, Intern. Math. Res. Notices **1996** (1996), no. 10, 457–471. ↑7, 31

[ST21] T. Suzuki and Y. Toyosawa, *On hook formulas for cylindric skew diagrams*, arXiv preprint (2021). arXiv:2106.09254 [math.CO]. ↑2

[Sta01] R. Stanley, *Enumerative Combinatorics. Vol. 2*, Cambridge University Press, Cambridge, 2001. With a foreword by Gian-Carlo Rota and appendix 1 by Sergey Fomin. ↑5

[Wac85] M.L. Wachs, *Flagged Schur functions, Schubert polynomials, and symmetrizing operators*, J. Combin. Theory Ser. A **40** (1985), no. 2, 276–289. ↑4

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