17

Birational Rigidity and Alpha Invariants of Fano Varieties

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Abstract

We prove that for every $\epsilon > 0$, there is a birationally super-rigid Fano variety X such that $\frac{1}{2} \leqslant \alpha(X) \leqslant \frac{1}{2}+\epsilon$. Also we show that for every $\epsilon > 0$, there is a Fano variety X and a finite subgroup $G \subset \mathrm{Aut}(X)$ such that X is G-birationally super-rigid, and $\alpha_G(X) < \epsilon$.

17.1 Introduction

Throughout this paper, we assume that all varieties are projective, normal, and defined over $\mathbb{C}$.

Let X be a Fano variety with terminal singularities. If $\mathrm{rk}\,\mathrm{Cl}(X) = 1$, then X is a Mori fiber space. In this case, we say that X is *birationally rigid* if X is not birational to other Mori fiber spaces [8]. Similarly, we say that X is *birationally super-rigid* if it is birationally rigid and $\mathrm{Bir}(X) = \mathrm{Aut}(X)$. Examples of birationally super-rigid smooth Fano varieties include

- smooth hypersurfaces in $\mathbb{P}^{n+1}$ of degree $n+1 \geqslant 4$ [6, 28, 29, 31, 37, 42, 44, 51];
- smooth weighted hypersurfaces in $\mathbb{P}(1^{n+1}, n)$ of degree $2n \geqslant 6$ [43].

Note that these examples of smooth Fano varieties are known to be K-stable [3, 7, 14, 16, 27, 30]. One can prove this by using Tian's criterion. Namely, recall from [41] and [49] that X is K-stable if

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$$\alpha(X) > \frac{\dim(X)}{\dim(X)+1},$$

where $\alpha(X)$ is the α-invariant of X that can be defined as follows:

$$\alpha(X) = \sup\left\{\lambda \in \mathbb{Q} \;\middle|\; \begin{array}{l}\text{the log pair } (X,\lambda D) \text{ is log canonical}\\ \text{for any effective } \mathbb{Q}\text{-divisor } D \sim_{\mathbb{Q}} -K_X\end{array}\right\}.$$

If X is smooth, then X is also K-stable in the case when $\alpha(X) = \frac{\dim(X)}{\dim(X)+1}$ and $\dim(X) \geqslant 2$ [30]. On the other hand, if X is a smooth hypersurface in $\mathbb{P}^{n+1}$ of degree $n+1$, then [7] and [14] give

$$\alpha(X) \geqslant \frac{n}{n+1} = \frac{\dim(X)}{\dim(X)+1}.$$

Similarly, if X is a smooth hypersurface in $\mathbb{P}(1^{n+1}, n)$ of degree $2n \geqslant 2$, then [16] gives

$$\alpha(X) \geqslant \frac{2n-1}{2n} > \frac{n}{n+1} = \frac{\dim(X)}{\dim(X)+1}.$$

This shows that all smooth hypersurfaces in $\mathbb{P}^{n+1}$ of degree $n+1 \geqslant 3$ and all smooth weighted hypersurfaces in $\mathbb{P}(1^{n+1}, n)$ of degree $2n \geqslant 4$ are K-stable. This gives an evidence for the following conjecture.

Conjecture 17.1.1 ([35]) *Let X be a Fano variety with terminal singularities such that* $\operatorname{rk}\operatorname{Cl}(X) = 1$. *Suppose that X is birationally rigid. Then X is K-stable.*

This conjecture has been already verified for many Fano varieties [10, 11, 12, 15, 24, 35, 47, 51], but it is still open in full generality (cf. [40]). On the other hand, we have the following result.

Theorem 17.1.2 ([48]) *Let X be a Fano variety with terminal singularities such that* $\operatorname{rk}\operatorname{Cl}(X) = 1$. *Suppose that X is birationally super-rigid and $\alpha(X) \geqslant \frac{1}{2}$. Then X is K-stable.*

This naturally leads to the following question

Question 17.1.3 ([48]) *Is it true that $\alpha(X) \geqslant \frac{1}{2}$ for any birationally super-rigid Fano variety X?*

In this chapter, we show that the bound $\frac{1}{2}$ is optimal by proving the following theorem.

Theorem 17.1.4 *For every $\epsilon > 0$, there exists a singular Fano variety X with terminal singularities such that* $\operatorname{rk}\operatorname{Cl}(X) = 1$*, the variety X is birationally super-rigid, and*

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$$\frac{1}{2} \leqslant \alpha(X) \leqslant \frac{1}{2} + \epsilon.$$

We also answer a natural equivariant version of Question 17.1.3, which can be stated as follows. Suppose that $\operatorname{rk} \operatorname{Cl}^G(X) = 1$ for a finite subgroup $G \subset \operatorname{Aut}(X)$, so that X is a G-Mori fiber space. Then X is *G-birationally rigid* if it is not G-birational to other G-Mori fiber spaces [21, section 3.1.1]. Similarly, the Fano variety X is said to be G-birationally super-rigid if X is G-birationally rigid, and X does not have non-biregular G-birational self-maps. Finally, we let

$$\alpha_G(X) = \sup\left\{ \lambda \in \mathbb{Q} \;\middle|\; \begin{array}{l} \text{the pair } (X, \lambda D) \text{ is log canonical for every} \\ \text{effective } G\text{-invariant } \mathbb{Q}\text{-divisor } D \sim_{\mathbb{Q}} -K_X \end{array} \right\}.$$

If $\alpha_G(X) > \frac{\dim(X)}{\dim(X)+1}$, then X is K-polystable by [52, Corollary 1.3].

Question 17.1.5 *Is it true that $\alpha_G(X) \geqslant \frac{1}{2}$ for any G-birationally super-rigid Fano variety X?*

The answer to this question is positive in dimension 2.

Exercise 17.1.6 ([9, 18, 46]) *If* $\dim(X) = 2$ *and X is G-birationally super-rigid, then $\alpha_G(X) \geqslant \frac{2}{3}$.*

In dimension 3, we still do not know whether our Question 17.1.5 has a positive answer or not, but many examples suggest that the answer is probably positive.

Example 17.1.7 ([19, 20, 22]) Suppose that $X = \mathbb{P}^3$, and let G be any finite subgroup in $\operatorname{Aut}(X)$. Then X is G-birationally super-rigid if and only if the following four conditions are satisfied:

(i) X does not have G-orbits of length $\leqslant 4$;
(ii) X does not contain G-invariant lines;
(iii) X does not contain G-invariant pairs of skew lines;
(iv) G is not isomorphic to $\mathfrak{A}_5$, $\mathfrak{S}_5$, $\mathrm{PSL}_2(\mathbf{F}_7)$, $\mathfrak{A}_6$, $\boldsymbol{\mu}_2^4 \rtimes \boldsymbol{\mu}_5$, and $\boldsymbol{\mu}_2^4 \rtimes \mathrm{D}_{10}$.

Using this criterion and *[18],* we see that $\alpha_G(X) \geqslant \frac{1}{2}$ if X is G-birationally super-rigid.

In this chapter, we prove that the answer to Question 17.1.5 is very negative in higher dimensions.

Theorem 17.1.8 *For every $\epsilon > 0$, there is a smooth Fano variety X and a finite subgroup $G \subset \operatorname{Aut}(X)$ such that $\operatorname{rk} \operatorname{Pic}^G(X) = 1$, the variety X is G-birationally super-rigid, and $\alpha_G(X) < \epsilon$.*

Let us describe the structure of this chapter. In Section 17.2, we prove Theorem 17.1.4. In Section 17.3, we study equivariant birational geometry of a smooth quadric threefold $Q \subset \mathbb{P}^4$ for the natural action of the symmetric group $\mathfrak{S}_5$, which should be interesting for mathematicians working on finite subgroups of the space Cremona group (cf. [50, section 9]). This example inspired Theorem 17.1.8. In Section 17.4, we present few results used in the proof of Theorem 17.1.8, which is done in Section 17.5.

17.2 The Proof of Theorem 17.1.4

We fix a positive integer $a \geqslant 2$. Then we let X be a quasi-smooth well-formed singular weighted hypersurfaces of degree $2a+1$ in $\mathbb{P}(1^{a+2}, a)$ that is given by the following equation:

$$y^2 x_1 + f_{2a+1}(x_1, \dots, x_{a+2}) = 0,$$

where each x_i is a coordinate of weight 1, y is a coordinate of weight a, and f_{2a+1} is a general homogeneous polynomial of degree $2a+1$. Then

- X is a Fano variety of dimension $N = a+1$;
- the class group of the variety X is of rank 1;
- the singularities of X consist of one singular point $O_y = (0\colon \dots \colon 0\colon 1)$, which is a terminal quotient singularity of type $\frac{1}{a}(1, \dots, 1)$.

Further, it follows from [36] that

$$\alpha(X) \leqslant \frac{a+1}{2a+1} = \frac{1}{2} + \frac{1}{4a+2}.$$

In this section, we prove the following result, which implies Theorem 17.1.4.

Theorem 17.2.1 *The Fano variety X is birationally super-rigid.*

This theorem also answers positively [36, Question 7.2.3].

Remark 17.2.2 *If $a = 2$, then X is known to be birationally super-rigid* [15, 24].

Let $\pi\colon X \dashrightarrow \mathbb{P}^N$ be the projection from the point O_y. Then π contracts the following divisor:

$$D = \{x_1 = 0, f_{2a+1}(x_1, \dots, x_{a+2}) = 0\} \subset \mathbb{P}(1^{a+2}, a).$$

Furthermore, one has the following diagram:

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$$\begin{array}{ccc} \widetilde{X} & \xrightarrow{\nu} & U \\ {\scriptstyle f}\downarrow & \overset{g}{\searrow} & \downarrow{\scriptstyle \theta} \\ X & \overset{\pi}{\dashrightarrow} & \mathbb{P}^N \end{array}$$

where f is the weighted blow-up of the point O_y with weights $(1, \ldots, 1)$, the map g is a morphism, the variety U is a hypersurface in $\mathbb{P}(1^{a+2}, a+1)$ of degree $2a+2$ that is given by

$$z^2 + x_1 f_{2a+1}(x_1, \ldots, x_{a+2}) = 0,$$

the morphism ν is a birational morphism that contracts the strict transform of the divisor D, and the morphism θ is a double cover that is branched over the hypersurface $x_1 f_{2a+1}(x_1, \ldots, x_{a+2}) = 0$. Here, we consider $x_1, \ldots, x_{a+2}$ as coordinates on $\mathbb{P}^N$ and as coordinates of weight 1 on the weighted projective space $\mathbb{P}(1^{a+2}, a+1)$, where z is a coordinate of weight $a+1$.

Now, let us prove Theorem 17.2.1. Assume the contrary, that is, there exists a birational map

$$\Phi\colon X \dashrightarrow W$$

to a Mori fiber space W that is not isomorphism. Let $\mathcal{M}$ be a birational transform of a very ample complete linear system on W via Φ. Let $\lambda \in \mathbb{Q}_{>0}$ be the positive rational number such that

$$K_X + \lambda \mathcal{M} \sim_{\mathbb{Q}} 0.$$

Then, by the Noether–Fano inequality [25], the singularities of the pair $(X, \lambda\mathcal{M})$ are not canonical. Let Z be a center of non-canonical singularities of the log pair $(X, \lambda\mathcal{M})$.

Now, let E be the f-exceptional divisor, and let $\widetilde{\mathcal{M}}$ be the strict transform of the mobile linear system $\mathcal{M}$ on the variety $\widetilde{X}$. Then $E \cong \mathbb{P}^a$, and

$$K_{\widetilde{X}} \sim_{\mathbb{Q}} f^*\big(K_X\big) + \tfrac{1}{a}E,$$
$$\lambda\widetilde{\mathcal{M}} \sim_{\mathbb{Q}} f^*\big(\lambda\mathcal{M}\big) - \mu E,$$

for some $\mu \in \mathbb{Q}_{\geqslant 0}$. Therefore, we have

$$K_{\widetilde{X}} + \lambda\widetilde{\mathcal{M}} \sim_{\mathbb{Q}} f^*(K_X + \lambda\mathcal{M}) + \Big(\frac{1}{a} - \mu\Big)E.$$

Thus, if $\mu > \frac{1}{a}$, then O_y is a center of non-canonical singularities of the log pair $(X, \lambda\mathcal{M})$.

Lemma 17.2.3 (cf. [32] for $a = 2$) *Suppose that $O_y \in Z$. Then $\mu > \frac{1}{a}$.*

Proof Suppose that $\mu \leqslant \frac{1}{a}$. Let us seek for a contradiction.

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Case: 1 $Z \neq O_y$. *Let* $\widetilde{Z}$ *be the strict transform of* Z *via* f. *Then* $\mathrm{mult}_{\widetilde{Z}}(\widetilde{\mathcal{M}}) > \frac{1}{\lambda}$ *and hence,*

$$\mathrm{mult}_P\big(\widetilde{\mathcal{M}}\big|_E\big) > \frac{1}{\lambda} \tag{17.1}$$

for any point $P \in \widetilde{Z} \cap E$. *Note that*

$$\lambda\widetilde{\mathcal{M}}\big|_E \sim_{\mathbb{Q}} -\mu E\big|_E \sim_{\mathbb{Q}} a\mu H,$$

where H *is a hyperplane in* $E \cong \mathbb{P}^a$. *Since* $a\mu \leqslant 1$, *this contradicts to* (17.1).

Case: 2 $Z = O_y$. *We write*

$$K_{\widetilde{X}} + \lambda\widetilde{\mathcal{M}} + \Big(\mu - \frac{1}{a}\Big)E \sim_{\mathbb{Q}} f^*\big(K_X + \lambda\mathcal{M}\big).$$

Hence, the singularities of the log pair $(\widetilde{X}, \lambda\widetilde{\mathcal{M}} + (\mu - \frac{1}{a})E)$ *are not canonical at some point* $P \in E$. *Then the singularities of the log pair* $(\widetilde{X}, \lambda\widetilde{\mathcal{M}})$ *are also not canonical at* P, *so that*

$$\mathrm{mult}_P\big(\widetilde{\mathcal{M}}\big) > \frac{1}{\lambda}.$$

Now, we argue as in the previous case to obtain a contradiction. □

One the other hand, we have the following.

Lemma 17.2.4 *One has* $\mu \leqslant \frac{1}{a}$.

Proof One has $g^*(\mathcal{O}_{\mathbb{P}^N}(1)) \sim_{\mathbb{Q}} f^*(-K_X) - \frac{1}{a}E$. Then

$$\Big(f^*(-K_X) - \frac{1}{a}E\Big) \cdot C = 0,$$

for any curve C contracted by g. Thus, if $\mu > \frac{1}{a}$, then

$$\widetilde{M} \cdot C = \frac{1}{\lambda}\big(f^*(-K_X) - \mu E\big) < 0$$

for a general divisor $\widetilde{M} \in \widetilde{\mathcal{M}}$. This is a contradiction, because the linear system $\widetilde{\mathcal{M}}$ is mobile, and the curves contracted by g span a divisor in $\widetilde{X}$ – the proper transform of the divisor D. □

Corollary 17.2.5 *One has* $O_y \notin Z$.

Thus, we see that Z is contained in the smooth locus of the variety X.

Lemma 17.2.6 *One has* $\dim(Z) = a - 1$.

Proof Suppose that $\dim(Z) < a - 1$. Let M_1 and M_2 be sufficiently general divisors in $\mathcal{M}$, and let P be a sufficiently general point in Z. Then

$$(M_1 \cdot M_2)_P > \frac{4}{\lambda^2}$$

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by [26, Corollary 3.4] or [44]. Let $\mathcal{L}$ be the linear subsystem in $|-K_X|$ consisting of all divisors that pass through the point P, and let $H_1,\dots,H_{N-2}$ be sufficiently general divisors in the system $\mathcal{L}$. If $P\notin D$, then the base locus of $\mathcal{L}$ does not contain curves, which gives

$$\frac{2a+1}{a\lambda^2}=M_1\cdot M_2\cdot H_1\cdot\ldots\cdot H_{N-2}\geqslant (M_1\cdot M_2)_P>\frac{4}{\lambda^2},$$

which is a contradiction. Thus, we see that $P\in D$.

Let $L\subset D$ be the curve containing P that is contracted by π. Then L is the only curve contained in the base locus of the linear system $\mathcal{L}$. After a linear change of coordinates, we can assume that

$$P=(0:0:1:0:\ldots:0:1)$$

and $H_i=X\cap\{x_{i+3}=0\}$ for $i=1,\dots,N-2$. Consider the surface S defined as

$$S=\bigcap_{i=1}^{N-2}H_i.$$

We can identify S with a surface in $\mathbb{P}(1,1,1,a)$ given by

$$y^2x_1+f_{2a+1}(x_1,x_2,x_3,0,\ldots,0)=0.$$

Then $L=S\cap\{x_1=x_2=|,0\}$. Let $\mathcal{M}_S=\mathcal{M}|_S$. Then $\lambda\mathcal{M}_S=mL+\lambda\Delta$ for some non-negative rational number $m\in\mathbb{Q}_{\geqslant 0}$ and some mobile linear system Δ on the surface S. Moreover, applying the inversion of adjunction [38, Theorem 5.50], we see that $(S,\lambda\mathcal{M}_S)$ is not log canonical at P.

Let H be a general curve in $|\mathcal{O}_S(1)|$, and let H_L be a general curve in $|\mathcal{O}_S(1)|$ that contains L. Then $H\cdot L=\frac{1}{a}$ and

$$S\cap H_L=L+R,$$

where R is a curve in S such that $L\not\subset\operatorname{Supp}(R)$. One can check that $L\cdot R=2$ and $H\cdot R=2$. Thus, using $(L+R)\cdot L=H\cdot L=\frac{1}{a}$, we get

$$L^2=-2+\frac{1}{a},$$

which can also be shown using the subadjunction formula on S.

Now, using Corti's inequality [26, Theorem 3.1], we get

$$\begin{aligned}&4(1-m)<\lambda^2\big(\Delta_1\cdot\Delta_2\big)_P\leqslant\lambda^2\Delta_1\cdot\Delta_2=\\&=\big(H-mL\big)^2=H^2-2mH\cdot L+m^2L^2=\frac{2a+1}{a}-\frac{2m}{a}+m^2\left(-2+\frac{1}{a}\right),\end{aligned}$$

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which gives

$$0 > \frac{(2a-1)(m-1)^2}{a}.$$

This is a contradiction, since $a \geqslant 2$. □

Therefore, we see that $\dim(Z) = \dim(X) - 2$. Then

$$\mathrm{mult}_Z\big(\mathcal{M}\big) > \frac{1}{\lambda}.$$

Let M_1 and M_2 be general divisors in $\mathcal{M}$. Then

$$\begin{aligned}\frac{3}{\lambda^2} > \frac{2a+1}{\lambda^2 a} &= \big(-K_X\big)^{N-2} \cdot M_1 \cdot M_2 \\ &\geqslant \mathrm{mult}_Z^2(\mathcal{M})\big(-K_X\big)^{N-2} \cdot Z > \frac{1}{\lambda^2}\big(-K_X\big)^{N-2} \cdot Z,\end{aligned}$$

so that $(-K_X)^{N-2} \cdot Z \in \{1,2\}$.

Now, let $H_1, \ldots, H_{N-2}$ be general divisors in $|-K_X|$. After a linear change of coordinates, one can assume that $H_i = X \cap \{x_{i+4} = 0\}$ for $i = 1, \ldots, N-3$. Let V be the threefold defined as

$$V = \bigcap_{i=1}^{N-3} H_i.$$

Then we can identify V with the hypersurface in $\mathbb{P}(1^4, a)$ given by

$$y^2 x_1 + f_{2a+1}(x_1, \ldots, x_4, 0, \ldots, 0) = 0.$$

Let $C = V \cap Z$, $\mathcal{M}_V = \mathcal{M}|_V$, and let H be a general surface in $|\mathcal{O}_V(1)|$. Then

- C is an irreducible curve such that $C \cdot H \in \{1,2\}$;
- C is contained in the smooth locus of the hypersurface V;
- C is a center of non-canonical singularities of the log pair $(V, \lambda\mathcal{M}_V)$.

We set $\mu = \lambda \mathrm{mult}_C(\mathcal{M}_V)$. Then $\mu > 1$.

Lemma 17.2.7 *One has $C \cdot H \neq 1$.*

Proof Suppose that $C \cdot H = 1$. We can choose coordinates on $\mathbb{P}(1^4, a)$ such that

$$C = \big\{x_2 = 0, x_3 = 0, y + F(x_1, \ldots, x_4) = 0\big\} \subset \mathbb{P}(1^4, a),$$

where $F(x_1, \ldots x_4)$ is a homogeneous polynomial of degree a. Note that $C \cong \mathbb{P}^1$.

Now, we let $\beta \colon \widetilde{V} \to V$ be the blow-up of the curve C, and let E be the β-exceptional divisor. We claim that $E^3 = a - 1$. Indeed, let $\widetilde{S}_1$, $\widetilde{S}_2$, $\widetilde{S}_3$ and be

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the strict transforms on $\widetilde{V}$ of the surfaces that are cut out on V by the equations $y+F(x_1,\ldots,x_4)=0$, $x_2=0$, and $x_3=0$, respectively. Then

$$0=\widetilde{S}_1\cdot\widetilde{S}_2\cdot\widetilde{S}_3=(a\beta^*(H)-E)\cdot(\beta^*(H)-E)^2$$
$$=aH^3+(a+2)\beta^*(H)\cdot E^2-E^3=a-1-E^3,$$

which gives $E^3=a-1$ as claimed.

Let $\mathcal{M}_{\widetilde{V}}$ be the strict transform of the linear system $\mathcal{M}_V$ on the threefold $\widetilde{V}$. Then

$$\lambda\mathcal{M}_{\widetilde{V}}\sim_{\mathbb{Q}}\beta^*(H)-\mu E.$$

One the other hand, since $\mathcal{M}_{\widetilde{V}}$ is mobile and $a\beta^*(H)-E$ is nef, we get

$$0\leqslant\big(a\beta^*(H)-E\big)\cdot\big(\beta^*(H)-\mu E\big)^2=aH^3+(2\mu+a\mu^2)\beta^*(H)\cdot E^2-\mu^2E^3$$
$$=(\mu-1)^2-2a(\mu^2-1)<0,$$

which is a contradiction. □

Thus, we see that $C\cdot H=2$. Then we can change coordinates on $\mathbb{P}(1^4,a)$ such that

(A) either

$$C=\big\{x_4=0,x_1x_2+x_3^2=0,y+F_a(x_1,\ldots,x_4)=0\big\}\subset\mathbb{P}(1^4,a)$$

for some homogeneous polynomial $F_a(x_1,\ldots,x_4)$ of degree a,

(B) or

$$C=\big\{x_2=0,x_3=0,y^2+F_{2a}(x_1,\ldots,x_4)=0\big\}\subset\mathbb{P}(1^4,a)$$

for some homogeneous polynomial $F_{2a}(x_1,\ldots,x_4)$ of degree $2a$.

In case (A), we have $C\cong\mathbb{P}^1$. In case (B), the curve C may have singularities.

In both cases, let $\beta\colon\widetilde{V}\to V$ be the blow-up of the curve C, and let E be the β-exceptional divisor. Then, arguing as in the proof of Lemma 17.2.7, we get

$$E^3=\begin{cases}2a-4 \text{ in case (A)},\\ -2 \text{ in case (B)}.\end{cases}$$

Let $\mathcal{M}_{\widetilde{V}}$ be the strict transform of the linear system $\mathcal{M}_V$ on the threefold $\widetilde{V}$. Then

$$\lambda\mathcal{M}_{\widetilde{V}}\sim_{\mathbb{Q}}\beta^*(H)-\mu E.$$

Moreover, in case (A), the divisor $a\beta^*(H)-E$ is nef, so that

$$\big(a\beta^*(H)-E\big)\cdot\big(\beta^*(H)-\mu E\big)^2\geqslant 0$$

because $\mathcal{M}_{\widetilde{V}}$ is mobile. But

$$\begin{aligned}\big(a\beta^*(H)-E\big)\cdot\big(\beta^*(H)-\mu E\big)^2 &= aH^3+(2\mu+a\mu^2)\beta^*(H)\cdot E^2-\mu^2E^3\\ &= (2\mu-1)^2-2a(2\mu^2-1)<0,\end{aligned}$$

because $\mu > 1$. Likewise, in case (B), the divisor $2a\beta^*(H)-E$ is nef, which gives

$$\begin{aligned}0\leqslant\big(2a\beta^*(H)-E\big)\cdot\big(\beta^*(H)-\mu E\big)^2 &= 2aH^3+(2\mu+2a\mu^2)\beta^*(H)\cdot E^2-\mu^2E^3 =\\ &= -2(\mu-1)(1-\mu+2a\,(1+\mu))<0.\end{aligned}$$

Thus, we get a contradiction in both cases (A) and (B). This completes the proof of Theorem 17.2.1.

17.3 $\mathfrak{S}_5$-Invariant Quadric Threefold

Let Q be a smooth quadric hypersurface in $\mathbb{P}^4$. We can choose coordinates x_0,x_1,x_2,x_3,x_4 on the projective space $\mathbb{P}^4$ such that Q is given by the following equation:

$$\sum_{i=0}^{4}x_i^2=0.$$

In particular, we see that Q is faithfully acted on by the symmetric group $\mathfrak{S}_5$, which permutes the coordinates x_0,x_1,x_2,x_3,x_4. Then $\alpha_{\mathfrak{S}_5}(Q)\leqslant\frac{1}{3}$, because $\mathfrak{S}_5$ leaves invariant the hyperplane sections of the quadric Q that is cut out by $x_0+x_1+x_2+x_3+x_4=0$. In fact, arguing as in [18], one can show that $\alpha_{\mathfrak{S}_5}(Q)=\frac{1}{3}$.

Keeping in mind the results obtained in [20], one can expect that Q is $\mathfrak{S}_5$-birationally rigid. However, this is not the case – the quadric hypersurface Q contains two $\mathfrak{S}_5$-orbits of length 5, and each of them leads to a G-birational transformation of the quadric into other $\mathfrak{S}_5$-Mori fiber space. Namely, let Σ_5 be a $\mathfrak{S}_5$-orbit of length 5 in X, and let $\pi: X\to Q$ be the blow-up of this $\mathfrak{S}_5$-orbit. Then there exists the following $\mathfrak{S}_5$-equivariant commutative diagram:

$$\begin{array}{ccccc} & U & \overset{\zeta}{\dashrightarrow} & W & \\ {\scriptstyle\pi}\swarrow & & & & \searrow{\scriptstyle\phi} \\ Q & & \overset{\chi}{\dashrightarrow} & & Y\end{array}$$

where ζ is a small birational map that flops the proper transforms of 10 conics that contain three points in Σ_5, ϕ is a birational morphism that contracts the proper transforms of 5 hyperplane sections of the quadric Q that pass through 4 points in Σ_5, and Y is a cubic threefold in $\mathbb{P}^4$ such that it has 5 isolated ordinary double points and $\mathrm{rk}\,\mathrm{Cl}(Y) = 1$. Since Y is a $\mathfrak{S}_5$-Mori fiber space, we see that Q is not $\mathfrak{S}_5$-birationally rigid. Note that the cubic threefold Y is given in $\mathbb{P}^4$ by

$$x_0x_1x_2 + x_0x_1x_3 + x_0x_1x_4 + x_0x_2x_3 + x_0x_2x_4 + x_0x_3x_4 + x_1x_2x_3 + x_1x_2x_4 + x_1x_3x_4 + x_2x_3x_4 = 0.$$

This is not difficult to prove; see [4] and [5].

The goal of this section is to prove the following result.

Theorem 17.3.1 *The only $\mathfrak{S}_5$-Mori fiber spaces that are $\mathfrak{S}_5$-birational to Q are Q and Y.*

Let us prove Theorem 17.3.1. Let $\iota \in \mathrm{Aut}(Q)$ be the Galois involution of the double cover $Q \to \mathbb{P}^3$ given by the projection from the point $(1:1:1:1:1)$. Then ι commutes with the $\mathfrak{S}_5$-action on Q. It is well known [13, 17] that Theorem 17.3.1 follows from the following technical result.

Theorem 17.3.2 *Let $\mathcal{M}_Q$ be any non-empty mobile $\mathfrak{S}_5$-invariant linear system on the quadric Q, and let $\mathcal{M}_Y$ and $\mathcal{M}'_Y$ be its proper transform on the cubic threefolds Y via χ and $\chi \circ \iota$, respectively. Choose positive rational numbers λ, μ, μ' such that*

$$\lambda\mathcal{M}_Q \sim_{\mathbb{Q}} -K_Q,$$
$$\mu\mathcal{M}_Y \sim_{\mathbb{Q}} -K_Y,$$
$$\mu'\mathcal{M}'_Y \sim_{\mathbb{Q}} -K_{Y'}.$$

Then one of the log pair $(Q, \lambda\mathcal{M}_Q)$, $(Y, \mu\mathcal{M}_Y)$, or $(Y', \mu'\mathcal{M}'_Y)$ has canonical singularities.

To prove Theorem 17.3.2, let us use all notations and assumptions of this theorem. We must prove that at least one of the log pair $(Q, \lambda\mathcal{M}_Q)$, $(Y, \mu\mathcal{M}_Y)$, or $(Y, \mu'\mathcal{M}'_Y)$ has canonical singularities. Set $\Sigma'_5 = \iota(\Sigma_5)$. Then Σ'_5 is the second $\mathfrak{S}_5$-orbit in the quadric Q.

Remark 17.3.3 *Let G be a stabilizer in $\mathfrak{S}_5$ of a point in $P \in \Sigma_5 \cup \Sigma'_5$. Then $G \cong \mathfrak{S}_4$ and its induced linear action on the Zariski tangent space $T_P(Q)$ is an irreducible representation.*

Now using this remark, [1, Lemma 2.4] and [26, Theorem 3.10], we can easily derive the required assertion from the following two propositions, arguing as in the proof of [13, Theorem 1.2].

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Proposition 17.3.4 *The log pair* $(Q, \lambda\mathcal{M}_Q)$ *is canonical away from* $\Sigma_5 \cup \Sigma_5'$.

Proposition 17.3.5 *The log pairs* $(Y, \mu\mathcal{M}_Y)$ *and* $(Y, \mu'\mathcal{M}_Y')$ *are canonical away from* $\mathrm{Sing}(Y)$.

In the remaining part of this section, we will prove Propositions 17.3.4 and 17.3.5. For both proofs, we need the following technical observation, which improves [20, Lemma 2.2].

Remark 17.3.6 *Let* X *be a variety with terminal singularities, let* D *be an effective* $\mathbb{Q}$*-Cartier divisor on the variety* X*, let* $\varphi\colon \widetilde{X} \to X$ *be birational morphism such that* $\widetilde{X}$ *is normal, let* $\widetilde{D}$ *be the proper transform on* $\widetilde{X}$ *of the divisor* D*, and let* $E_1, \ldots, E_n$ *be* φ*-exceptional divisors. Then*

$$K_{\widetilde{X}} + \widetilde{D} + \sum_{i=1}^{n} a(E_i; X, D)E_i \sim_{\mathbb{Q}} \varphi^*\big(K_X + D\big),$$

where each $a(E_i; X, D)$ *is a rational number known as the discrepancy of the pair* (X, D) *along* E_i*. Let* E *be one of the* φ*-exceptional divisors. Then*

$$a\big(E; X, D\big) = a(E; X) - \mathrm{ord}_E(D),$$

where $a(E; X)$ *is the discrepancy of* X *along* E*. Let* $a = a(E; X)$*. If* $a(E; X, D) < 0$*, then*

$$a\Big(E; X, \Big(1 + \frac{1}{a}\Big)D\Big) = a(E; X) - \Big(1 + \frac{1}{a}\Big)\mathrm{ord}_E(D) < \mathrm{ord}_E(D)$$
$$- \Big(1 + \frac{1}{a}\Big)\mathrm{ord}_E(D) = -\frac{\mathrm{ord}_E(D)}{a} < -1,$$

so that the log pair $(X, (1 + \frac{1}{a})D)$ *is not log canonical along* $\varphi(E)$*. In particular, if* $a(E; X, D) < 0$ *and* $\varphi(E)$ *is a smooth point of the variety* X*, then the log pair*

$$\left(X, \frac{\dim(X)}{\dim(X) - 1}D\right)$$

is not log canonical at the point $\varphi(E)$*.*

To prove Proposition 17.3.4, we have to present few standard basic facts about the $\mathfrak{S}_5$-equivariant geometry of the quadric Q. Observe that Q contains two $\mathfrak{S}_5$-orbits Σ_{10} and Σ_{10}' of length 10.

Lemma 17.3.7 *If* Σ *is a* $\mathfrak{S}_5$*-orbit in* Q *with* $|\Sigma| < 20$*, then* Σ *is one of the orbits* Σ_5*,* Σ_5'*,* Σ_{10}*,* Σ_{10}'*.*

Proof Left to the reader. □

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Let $H = \{x_0 + x_1 + x_2 + x_3 + x_4 = 0\} \subset \mathbb{P}^4$ and $S_2 = H \cap Q$. Then S_2 is smooth and $\mathfrak{S}_5$-invariant. Moreover, the surface S_2 does not contain Σ_5, Σ'_5, Σ_{10}, Σ'_{10}. Let $\mathcal{B}_6$ be the curve in Q given by

$$\begin{cases} x_0 + x_1 + x_2 + x_3 + x_4 = 0, \\ x_0^2 + x_1^2 + x_2^2 + x_3^2 + x_4^2 = 0, \\ x_0^3 + x_1^3 + x_2^3 + x_3^3 + x_4^3 = 0. \end{cases}$$

Then $\mathcal{B}_6$ is the unique smooth curve of genus 4 that admits an effective action of the group $\mathfrak{S}_5$, which is known as the Bring's curve (see [21, Remark 5.4.2]). Note that $\mathcal{B}_6 \subset Q \cap H$.

Lemma 17.3.8 *Let C be a $\mathfrak{S}_5$-invariant curve in Q such that* $\deg(C) \leqslant 6$. *Then $C = \mathcal{B}_6$.*

Proof We may assume that C is $\mathfrak{S}_5$-irreducible, that is, the symmetric group $\mathfrak{S}_5$ acts transitively on the set of its irreducible components. Then S_2 contains C, since otherwise $|S_2 \cap C| \leqslant S_2 \cdot C = 12$, which contradicts Lemma 17.3.7. Thus, if $C \neq \mathcal{B}_6$, then

$$|C \cap \mathcal{B}_6| \leqslant C \cdot \mathcal{B}_6 = 18,$$

which is impossible by Lemma 17.3.7, since S_2 does not contain Σ_5, Σ'_5, Σ_{10}, and Σ'_{10}. □

Corollary 17.3.9 *The log pair $(Q, \lambda\mathcal{M}_Q)$ has log canonical singularities.*

Proof Suppose that the log pair $(Q, \lambda\mathcal{M}_Q)$ is not log canonical. Let us seek for a contradiction. If the log pair $(Q, \lambda\mathcal{M}_Q)$ is log canonical outside of finitely many points, then it is log canonical outside of a single point by the Kollár–Shokurov connectedness, which must be $\mathfrak{S}_5$-invariant point. The latter contradicts Lemma 17.3.7. Thus, we see that there is a $\mathfrak{S}_5$-irreducible curve C such that the log pair $(Q, \lambda\mathcal{M}_Q)$ is not log canonical at general points of its irreducible components. Then

$$\left(M_1 \cdot M_2\right)_C > \frac{4}{\lambda^2}$$

by [26, Theorem 3.1], where M_1 and M_2 are general surfaces in $\mathcal{M}_Q$. Using this, we get $\deg(C) < \frac{9}{2}$, which is impossible by Lemma 17.3.8. □

Observe that $S_2 \cong \mathbb{P}^1 \times \mathbb{P}^2$, and the induced $\mathfrak{S}_5$-action on S_2 is faithful.

Lemma 17.3.10 (cf. [23, Theorem 7.5]) *One has $\alpha_{\mathfrak{S}_5}(S_2) = \frac{3}{2}$.*

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Proof Observe that $\mathrm{Pic}^{\mathfrak{S}_5}(S_2) = \mathbb{Z}[H|_{S_2}]$ and $\mathcal{B}_6 \in |3H|_{S_2}|$. But $|H|_{S_2}|$ and $|2H|_{S_2}|$ do not contain any $\mathfrak{S}_5$-invariant curves. Hence, we have $\alpha_{\mathfrak{S}_5}(S_2) = \frac{3}{2}$ by [9, Lemma 5.1] and Lemma 17.3.7. □

Now we are ready to prove Proposition 17.3.4.

Proof of Proposition 17.3.4 Suppose $(Q, \lambda\mathcal{M}_Q)$ is not canonical. Denote by Σ its non-canonical locus. To complete the proof, we have to show that $\Sigma \subseteq \Sigma_5 \cup \Sigma_5'$.

First, let us show that the set Σ consists of finitely many points. Indeed, suppose that Σ contains a $\mathfrak{S}_5$-irreducible curve C. Then

$$\mathrm{mult}_C\big(\mathcal{M}_Q\big) > \frac{1}{\lambda}, \tag{17.2}$$

which easily implies that $\deg(C) < 18$. Arguing as in the proof of Lemma 17.3.8, we see that $C \subseteq S_2$. Then (17.2) gives $\deg(C) < 6$, which is impossible by Lemma 17.3.8. Hence, we see that Σ is finite.

If $\Sigma \cap S_2 \neq \varnothing$, then the log pair $(S_2, \lambda\mathcal{M}_Q|_{S_2})$ is not log canonical by the inversion of adjunction, which is impossible by Lemma 17.3.10. Thus, we have $\Sigma \cap S_2 = \varnothing$.

Applying Remark 17.3.6, we see that $(Q, \frac{3\lambda}{2}\mathcal{M}_Q)$ is not log canonical at every point of the set Σ. Take $\varepsilon \in \mathbb{Q}_{>0}$ such that $\Sigma \subset \mathrm{Nklt}(Q, \frac{3\lambda-\varepsilon}{2}\mathcal{M}_Q)$. Set $\Omega = \mathrm{Nklt}(Q, \frac{3\lambda-\varepsilon}{2}\mathcal{M}_Q)$. Then Ω is $\mathfrak{S}_5$-invariant. Moreover, arguing as in the proof of Corollary 17.3.9, we see that the locus Ω does not contain curves, so that Ω is a finite set. Now, applying Nadel vanishing theorem, we get $h^1(Q, \mathcal{J} \otimes \mathcal{O}_Q(2H|_Q)) = 0$, where $\mathcal{J}$ is the multiplier ideal sheaf of the log pair $(Q, \frac{3-\varepsilon}{2}\lambda\mathcal{M}_Q)$. This gives

$$|\Sigma| \leqslant |\Omega| \leqslant h^0\Big(Q, \mathcal{O}_Q\big(2H|_Q\big)\Big) = 14,$$

because $\mathrm{Supp}(\mathcal{J}) = \Omega$. Now, using Lemma 17.3.7, we see that one of the following possibilities holds:

- $\Sigma \subseteq \Sigma_5 \cup \Sigma_5'$;
- $\Omega = \Sigma = \Sigma_{10}$;
- $\Omega = \Sigma = \Sigma_{10}'$.

If $\Sigma \subseteq \Sigma_5 \cup \Sigma_5'$, we are done. Hence, without loss of generality, we may assume that $\Omega = \Sigma = \Sigma_{10}$. Let us show that this assumption leads to a contradiction.

Let $\mathcal{D}$ be the linear subsystem in $|2H|$ that consists of all surfaces in $|2H|$ that pass through Σ_{10}. By counting parameters, we get $\dim(\mathcal{D}) \geqslant 4$. Arguing as in the proof of Lemma 17.3.8, we see that the base locus of the linear system $\mathcal{D}$ contains no curves. Using [26, Corollary 3.4] or [44], we get

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$$\frac{36}{\lambda^2} = D \cdot M_1 \cdot M_2 \geqslant \sum_{P \in \Sigma_{10}} \big(M_1 \cdot M_2\big)_P > \sum_{P \in \Sigma_{10}} \frac{4}{\lambda^2} = \frac{40}{\lambda^2},$$

which is absurd. This completes the proof of Proposition 17.3.4. □

Now, let us present a few facts about the threefold Y. Its singular locus consists of five nodes:

$$P_1 = (1:0:0:0:0),$$
$$P_2 = (0:1:0:0:0),$$
$$P_3 = (0:0:1:0:0),$$
$$P_4 = (0:0:0:1:0),$$
$$P_5 = (0:0:0:0:1).$$

Note that $(3:3:3:3:-2) \in Y \setminus \mathrm{Sing}(Y)$. Let Θ_5 be the $\mathfrak{S}_5$-orbit of this point. Then $|\Theta_5| = 5$. For every $1 \leqslant i < j \leqslant 5$, we let ℓ_{ij} be the line in $\mathbb{P}^4$ that passes through the nodes P_i and P_j. Let $\mathcal{L}_{10}$ be the union of these lines. Then $\mathcal{L}_{10} \subset Y$, and $\mathcal{L}_{10} \cap H$ is a $\mathfrak{S}_5$-orbit Θ_{10} of length 10. The cubic Y contains two more $\mathfrak{S}_5$-orbits of length 10, which we denote by Θ'_{10} and Θ''_{10}.

Lemma 17.3.11 *The orbits* $\mathrm{Sing}(Y)$*,* Θ_5*,* Θ_{10}*,* Θ'_{10}*,* Θ''_{10} *are all* $\mathfrak{S}_5$*-orbit in* Y *of length* < 20*.*

Proof Left to the reader. □

Let $S_3 = Y \cap H$. Then S_3 is a smooth cubic surface known as the Clebsch diagonal cubic surface. It follows from [21, Lemma 6.3.12] that $\Theta_{10} \subset S_3$, but S_3 does not contain $\mathrm{Sing}(Y)$, Θ_5, Θ'_{10}, Θ''_{10}. Observe also that S_3 contains the curve $\mathcal{B}_6$.

Lemma 17.3.12 *Let* C *be a* $\mathfrak{S}_5$*-invariant curve in* Y *such that* $\deg(C) \leqslant 10$*. Then* $C = \mathcal{B}_6$ *or* $\mathcal{L}_{10}$*.*

Proof If $C \subset S_3$, the assertion follows from [21, Theorem 6.3.18]. Hence, we assume that $C \not\subset S_3$. Then, arguing as in the proof of Lemma 17.3.8, we conclude that and $C \cdot H = \Theta_{10}$.

We suppose that the curve C is irreducible. Then C has to be singular at every point $P \in \Theta_{10}$, because the stabilizer in $\mathfrak{S}_5$ of the point P acts faithfully on the Zariski tangent space $T_P(C)$. Thus, if C is irreducible, then $10 = C \cdot H \geqslant 2|\Theta_{10}|$, which is absurd.

We see that C is reducible and $\deg(C) = 10$. Let C_1 be an irreducible components of the curve C, and let G be the stabilizer in $\mathfrak{S}_5$ of the curve C_1. Then one of the following four cases holds:

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(1) $G \cong \mathfrak{A}_5$ and C is a union of 2 irreducible curves of degree 5;
(2) $G \cong \mathfrak{S}_4$ and C is a union of 5 irreducible conics;
(3) $G \cong \mathfrak{A}_4$ the C is a union of 10 lines;
(4) $G \cong \mathfrak{S}_3 \times \boldsymbol{\mu}_2$ and C is a union of 10 lines.

In case (1), Θ_{10} splits as two G-orbits of length 5, which is not the case by [21, Lemma 6.3.12]. In cases (2) and (3), the only two-dimensional G-invariant linear subspace of $\mathbb{P}^4$ is contained in the $\mathfrak{S}_5$-invariant hyperplane H, so that C_1 is contained in S_3, which contradicts our assumption. In case (4), one can easily see that $C = \mathcal{L}_{10}$. □

Now, we are ready to prove Proposition 17.3.5.

Proof of Proposition 17.3.5 It is enough to prove that $(Y, \mu\mathcal{M}_Y)$ is canonical away from $\mathrm{Sing}(Y)$. Suppose that this log pair is not canonical. Let Σ be its non-canonical locus.

First, we claim that Σ is a finite set. Indeed, suppose that Σ contains a $\mathfrak{S}_5$-irreducible curve. Then $\mathrm{mult}_C(\mathcal{M}) > \frac{1}{\mu}$. If $C \subset S_3$, this implies that $\deg(C) < 6$, which is impossible by Lemma 17.3.12. Thus, we see that $C \not\subset S_3$. Then $\deg(C) < 12$, so that $H \cdot C < 12$. Using Lemmas 17.3.11 and 17.3.12, we conclude that $C = \mathcal{L}_{10}$. Let H' be the hyperplane in $\mathbb{P}^4$ that contains the nodes P_1, P_2, P_3, and P_4, and let M be a general surface in $\mathcal{M}_Y$. Then

$$H' \cdot M = m\big(\ell_{12} + \ell_{13} + \ell_{14} + \ell_{23} + \ell_{24} + \ell_{34}\big) + \Delta,$$

where a is an integer such that $a \geqslant \mathrm{mult}_C(\mathcal{M})$, and Δ is an effective one-cycle whose support does not contain the lines $\ell_{12}, \ell_{13}, \ell_{14}, \ell_{23}, \ell_{24}$, and ℓ_{34}. Therefore, we have

$$\frac{6}{\mu} = \frac{2}{\mu}H^3 = H \cdot H' \cdot M = 6m + H \cdot \Delta \geqslant 6m \geqslant 6\mathrm{mult}_C(\mathcal{M}) > \frac{6}{\mu},$$

which is absurd. Thus, we see that Σ is a finite set.

Let Σ_1 be the subset in Σ that consists of all smooth points of Y. We have to show that $\Sigma_1 = \varnothing$. If $\Sigma_1 \cap S_3 \neq \varnothing$, then the log pair $(S_3, \mu\mathcal{M}_Y|_{S_3})$ is not log canonical, which implies that $\alpha_{\mathfrak{S}_5}(S_3) < 2$. The latter contradicts [9, Example 1.11]. Thus, we have $\Sigma_1 \cap S_3 = \varnothing$.

Now, using Remark 17.3.6, we see that $(Y, \frac{3\mu}{2}\mathcal{M}_Y)$ is not log canonical at every point of the set Σ_1. Moreover, arguing exactly as in the proof of Corollary 17.3.9 and using Lemma 17.3.12, we see that each point of the subset Σ_1 is an isolated center of non-log canonical singularities of the pair $(Y, \frac{3\mu}{2}\mathcal{M}_Y)$. Now, using Nadel vanishing theorem as we did in the proof of Proposition 17.3.4, we see that $|\Sigma_1| \leqslant 5$. Therefore, we have $\Sigma_1 = \Theta_5$ by Lemma 17.3.11.

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Let M_1 and M_2 be general surfaces in $\mathcal{M}_Y$. Then it follows from [26, Corollary 3.4] or [44] that

$$\big(M_1 \cdot M_2\big)_Q > \frac{4}{\mu^2},$$

for every point $Q \in \Theta_5$. Let Q_1, Q_2, and Q_3 be three points in Θ_5, let Π be the plane in $\mathbb{P}^4$ that contains these three points, and let $\mathcal{C} = Y|_\Pi$. Then $\mathcal{C}$ is a smooth irreducible cubic curve. Write

$$M_1 \cdot M_2 = \epsilon\mathcal{C} + \Omega,$$

where ϵ is a non-negative rational number, and Ω is an effective one-cycle whose support does not contain $\mathcal{C}$. Let H' be a general hyperplane section of the cubic hypersurface Y that contains $\mathcal{C}$. Then H' does not contain any irreducible component of the one-cycle Ω. Thus, we have

$$\frac{12}{\mu^2} - 3\epsilon = H' \cdot \Omega \geqslant \operatorname{mult}_{Q_1}\big(\Omega\big) + \operatorname{mult}_{Q_2}\big(\Omega\big) + \operatorname{mult}_{Q_3}\big(\Omega\big) > 3\Big(\frac{4}{\mu^2} - \epsilon\Big)$$
$$= \frac{12}{\mu^2} - 3\epsilon,$$

which is absurd. This completes the proof of Proposition 17.3.5. □

This completes the proof of Theorem 17.3.1, which also implies that Q is $\mathfrak{S}_5$-solid [2, 13, 17].

17.4 Preliminary Results

In this section, we prove a few results that will be used towards the proof of Theorem 17.1.8.

Let X be a variety with at most Kawamata log terminal singularities that is faithfully acted on by a finite group G. The following result is a consequence of the technique developed in [45, section 3].

Lemma 17.4.1 *Suppose X is smooth. Let Z be a G-irreducible subvariety of X of codimension m, let H be an ample divisor on X, and let $D_1, D_2, \ldots, D_m$ be effective divisors on X such that*

$$D_1 \sim_{\mathbb{Q}} D_2 \sim_{\mathbb{Q}} \cdots \sim_{\mathbb{Q}} D_m \sim_{\mathbb{Q}} H,$$

and Z is a G-irreducible component of the intersection $\cap_{i=1}^m \operatorname{Supp}(D_i)$. Let $Y \subset X$ be an effective cycle of codimension $c \leqslant m$. Then

$$\frac{\operatorname{mult}_Z(Y)}{\deg(Y)} \leqslant \left(\deg(Z) \cdot \min_S \prod_{i\in S} \operatorname{mult}_Z(D_i)\right)^{-1},$$

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where the minimum is taken over all subsets $S \subseteq \{1, \ldots, m\}$ of cardinality $m - c$.

Proof We may assume that Y is irreducible and $Z \subseteq Y$. We construct a sequence of irreducible subvarieties $Y_c, \ldots, Y_m$ and a permutation $D'_1, \ldots, D'_m$ of $D_1, \ldots, D_m$ such that

- $Y_c = Y$;
- $\mathrm{codim}_X(Y_i) = i$;
- $Y_i \not\subset \mathrm{Supp}(D'_{i+1})$;
- Y_{i+1} is a component of $Y_i \cdot D'_{i+1}$ that contains Z;
- for all $c \leqslant i \leqslant m - 1$ one has

$$\frac{\mathrm{mult}_Z(Y_{i+1})}{\deg(Y_{i+1})} \geqslant \mathrm{mult}_Z(D'_{i+1}) \cdot \frac{\mathrm{mult}_Z(Y_i)}{\deg(Y_i)}.$$

Once this is done, the lemma follows immediately from the trivial equality $Y_m = Z$.

Suppose that $Y_c, \ldots, Y_i$ and $D'_{c+1}, \ldots, D'_i$ have been constructed for some $i < m$. Then

$$Y_i \subseteq \bigcap_{j=c+1}^{i} \mathrm{Supp}(D_j).$$

Since $\cap_{i=1}^m \mathrm{Supp}(D_i)$ has codimension m in a neighborhood of Z by assumption and

$$\mathrm{codim}_X(Y_i) = i < m,$$

then there exists some D_j, which is necessarily different from $D'_{c+1}, \ldots, D'_i$, which gives $Y_i \not\subseteq D_j$. We may then take $D'_{i+1} = D_j$ and Y_{i+1} an irreducible component of $(Y_i \cdot D_j)$ such that

$$\frac{\mathrm{mult}_Z(Y_{i+1})}{\deg(Y_{i+1})} \geqslant \frac{\mathrm{mult}_Z(Y_i \cdot D_j)}{\deg(Y_i \cdot D_j)} \geqslant \mathrm{mult}_Z D_j \cdot \frac{\mathrm{mult}_Z(Y_i)}{\deg(Y_i)}.$$

By induction, this finishes the construction. □

Now, let D be either an effective $\mathbb{Q}$-divisor on X (a boundary) or a movable (mobile) boundary:

$$D = \sum_{i=1}^{r} a_i \mathcal{M}_i,$$

where each $a_i \in \mathbb{Q}_{\geqslant 0}$, and each $\mathcal{M}_i$ is a linear system on X that does not have fixed components. Suppose, in addition, that D is G-invariant.

Lemma 17.4.2 *Suppose that (X, D) is not log canonical, and D is ample. Then there exists positive rational number $\epsilon < 1$ such that the following assertions hold:*

- *If D is a $\mathbb{Q}$-divisor, there exists a G-invariant effective $\mathbb{Q}$-divisor $D' \sim_{\mathbb{Q}} (1-\epsilon)D$ such that the log pair (X, D') has log canonical singularities, and $\mathrm{Nklt}(X, D')$ is a non-empty disjoint union of minimal log canonical centers of the log pair (X, D').*
- *If D is a mobile boundary, there exists a G-invariant mobile boundary $D' \sim_{\mathbb{Q}} (1-\epsilon)D$ such that the log pair (X, D') has log canonical singularities, and $\mathrm{Nklt}(X, D')$ is a non-empty disjoint union of minimal log canonical centers of the log pair (X, D').*

Furthermore, irreducible components of $\mathrm{Nklt}(X, D')$ *are normal, and G transitively permutes them.*

Proof This is an equivariant version of *the tie breaking*. See [21, Lemma 2.4.10] or [33] and [34]. □

Lemma 17.4.3 *Let H be a very ample divisor in* $\mathrm{Pic}(X)$*, and let L be a divisor in* $\mathrm{Pic}(X)$ *such that the divisor $L - (K_X + D + \dim(X)H)$ is ample. Then $|L|$ contains a non-empty G-invariant linear subsystem $\mathcal{L}$ such that* $\mathrm{Nklt}(X, D) = \mathrm{Bs}(\mathcal{L})$.

Proof Let $\mathcal{J} = \mathcal{J}(X, D)$ be the multiplier ideal. Then the support of $\mathcal{O}_X/\mathcal{J}$ is exactly $\mathrm{Nklt}(X, D)$. By [39, Proposition 9.4.26], $\mathcal{J} \otimes \mathcal{O}_X(L)$ is generated by global sections. The G-invariant linear system $\mathcal{L} = |\mathcal{J} \otimes \mathcal{O}_X(L)|$ then satisfies the statement of the lemma. □

Now, we fix $d, n \in \mathbb{Z}_{>0}$. Let $\mathbb{W}$ be the subgroup in $\mathrm{GL}_{n+1}(\mathbb{C})$ consisting of all permutation matrices, let $\mathbb{T}$ be the subgroup in $\mathrm{GL}_{n+1}(\mathbb{C})$ consisting of diagonal matrices whose (non-zero) entries are the dth roots of unity, and let $\mathbb{G}$ be the subgroup in $\mathrm{GL}_{n+1}(\mathbb{C})$ generated by $\mathbb{T}$ and $\mathbb{W}$. Then $\mathbb{W} \cong \mathfrak{S}_{n+1}$, $\mathbb{T} \cong \boldsymbol{\mu}_d^{n+1}$, and

$$\mathbb{G} \cong \mathbb{T} \rtimes \mathbb{W} \cong \boldsymbol{\mu}_d^{n+1} \rtimes \mathfrak{S}_{n+1}.$$

Let W, T, and G be the images in $\mathrm{PGL}_{n+1}(\mathbb{C})$ via the quotient map of the groups $\mathbb{W}$, $\mathbb{T}$, and $\mathbb{G}$, respectively. Then $W \cong \mathfrak{S}_{n+1}$, $T \cong \boldsymbol{\mu}_d^n$, and $G \cong T \rtimes W$. Note that G leaves invariant the Fermat hypersurface

$$X_d := \left\{ \sum_{i=0}^{d} x_i^d = 0 \right\} \subset \mathbb{P}^n,$$

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where $x_0, \dots, x_n$ are homogeneous coordinates on $\mathbb{P}^n$. If $n \geqslant 2$ and $d \geqslant 3$, then $G = \mathrm{Aut}(\mathbb{P}^n, X_d)$.

The examples for Theorem 17.1.8 are complete intersections in $\mathbb{P}^n$ of some Fermat hypersurfaces. The main result of this section is the following proposition. We will use it in the next section.

Proposition 17.4.4 *Let $\mathcal{M}$ be a W-invariant linear subsystem in $|\mathcal{O}_{\mathbb{P}^n}(m)|$, let Z be an irreducible component of the intersection* $\mathrm{Bs}(\mathcal{M})$*, and let $\mathscr{Z}$ be the W-irreducible subvariety in $\mathbb{P}^n$, whose irreducible component is Z. Then at least one of the following two cases holds:*

(1) a general point in Z has at most d different coordinates, and $\dim(Z) \leqslant m-1$*;*
(2) the subvariety Z is an irreducible component of a set-theoretic intersection of W-invariant hypersurfaces of degree at most m, and $\dim(Z) \geqslant n-m$.

Moreover, in case (1)*, if $m \leqslant n$ and $n \geqslant 4$, then either $\mathscr{Z}$ has at least $n+1$ irreducible components, or $\mathscr{Z} = Z = (1:1:\dots:1)$.*

In particular, the base locus of a W-invariant linear subsystem in $|\mathcal{O}_{\mathbb{P}^n}(m)|$ either has dimension at most $m-1$ or has codimension at most m. This can be illustrated by the following example.

Example 17.4.5 In the assumptions and notations of Proposition 17.4.4, suppose $m=1$ and $Y=\mathbb{P}^n$. Then either $\mathcal{M} = |\mathcal{O}_{\mathbb{P}^n}(1)|$, so it is base point free, or one of the following two cases holds:

(1) $\mathcal{M}$ is the linear system of hyperplanes containing the W-invariant point $(1:1:\dots:1)$;
(2) $\mathcal{M}$ is the W-invariant hyperplane $X_1 = \{x_0 + \cdots + x_n = 0\} \subset \mathbb{P}^n$.

In case (1), we have $\mathscr{Z} = Z = (1:1:\dots:1)$. In case (2), we have $\mathscr{Z} = Z = X_1$.

To prove Proposition 17.4.4, we need to prove a few auxiliary results.

Lemma 17.4.6 *Fix $s \in \{1,\dots,n\}$, and take positive integers $a_1,\dots,a_s$ such that $n = a_1 + \cdots + a_s$. Let N be the number of unordered partitions of the set $\{1,\dots,n\}$ into subsets of $a_1,\dots,a_s$ elements, respectively. Then $N \geqslant n$ unless $s=1$, $s=n$, or $n=4$, $s=2$, $a_1=a_2=2$.*

Proof We may assume $a_1 \leqslant a_2 \leqslant \cdots \leqslant a_k$. If $a_1 = \cdots = a_i < a_{i+1}$ for some $i \in \{1,\cdots,k-1\}$, then

$$N \geqslant \binom{n}{ia_1} \geqslant n.$$

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Hence, we may assume that $a_1 = \cdots = a_s = r$ for some $r \in \{2, \ldots, n-1\}$. Then $s = \frac{n}{r} \geqslant 2$ and

$$N = \frac{n!}{(r!)^s \cdot s!} \geqslant \frac{n(n-1)\cdot \ldots \cdot (n-r+1)}{r! \cdot s}.$$

Thus, since $n \geqslant 2s$, we get

$$N \geqslant (n-1) \cdot \frac{n}{2s} \cdot \prod_{j=0}^{r-3} \frac{n-2-j}{r-j} \geqslant n-1,$$

with equality only if $n = 2s$ and $n-2 = r$, that is, when $r = s = 2$. Since N is a positive integer, the assertion follows. □

For the second result, we need the following two conventions. A *color set* is a finite multiset, where elements (i.e. colors) may appear with multiplicities. If $K = (V, E)$ is a graph and $\mathscr{C}$ is a color set, then a *coloring* of the graph K by $\mathscr{C}$ is a map $\phi\colon V \to \mathscr{C}$ such that

- every color is used at most once, that is, we have $|\phi^{-1}(c)| \leqslant 1$ for every $c \in \mathscr{C}$;
- every pair of adjacent vertices has different color, that is, we have $\phi(u) \neq \phi(v)$ as integers whenever $(u, v) \in E$.

Lemma 17.4.7 *Let $K = (V, E)$ be a graph such that K contains at least $s \geqslant 1$ connected components, and let $\mathscr{C}$ be a color set of size at least $|V|$ such that $\mathscr{C}$ has at least $|V| - s + 1$ different colors. Then there exists a coloring of the graph K by $\mathscr{C}$.*

Proof We use induction on $|V| - s \geqslant 0$. The result is clear when $|V| - s = 0$, since in this case there are no edges in K. Suppose now that the result has been proved for smaller values of $|V| - s$. We can assume that every connected component of K contains at least two vertices, since we can assign any color to isolated points. In particular, the number $|V| - s$ drops if we remove connected components from V. It is also clear that we may assume $s \geqslant 2$ and at least one of the colors has multiplicity $\geqslant 2$ (otherwise there are already $|V|$ different colors).

Now, we let $K_1 = (V_1, E_1)$ be a connected component of the graph K, and we set $r = |V_1| \geqslant 2$. Let $K' = (V', E')$ be the subgraph of the graph K that is obtained by removing the component K_1. We may choose a subset $\mathscr{C}_1 \subseteq \mathscr{C}$ that consists of r distinct colors (each with multiplicity 1) such that the complement $\mathscr{C} \backslash \mathscr{C}_1$ has at least $|V| - s + 2 - r$ different color (here we use the assumption that at least one color in $\mathscr{C}$ has multiplicity $\geqslant 2$). Note that we can color the graph K_1 by $\mathscr{C}_1$. By induction hypothesis, we can also color K' by $\mathscr{C} \backslash \mathscr{C}_1$,

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since $|V| - s + 2 - r = |V'| - (s-1) + 1$. This gives us a coloring of K by $\mathscr{C}$. □

Let us identify $H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(m))$ with the subspace in $\mathbb{C}[x_0, \ldots, x_n]$ consisting of all homogeneous polynomials of degree m. For $f \in H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(m))$ and a (possibly reducible) subvariety $Y \subset \mathbb{P}^n$, we define $f|_Y$ to be the image of the polynomial f in $H^0(Y, \mathcal{O}_{\mathbb{P}^n}(m)|_Y)$ via the restriction morphism. For any $f \in H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(m))$, we denote by $\mathcal{M}_f$ the linear subsystem in $|\mathcal{O}_{\mathbb{P}^n}(m)|$ that is given by the subspace in $H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(m))$ spanned by $\tau^*(f)$ for all $\tau \in \mathbb{W}$. Finally, we fix $V = \{0, 1, \ldots, n\}$. For every graph $K = (V, E)$, let $c(K)$ be the number of its connected components, and let

$$f_K = \pm \prod_{(i,j)\in E} (x_i - x_j) \in H^0\big(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(|E|)\big).$$

Lemma 17.4.8 *Let Y be an intersection in $\mathbb{P}^n$ of some W-invariant hypersurfaces, and fix $\ell \in \mathbb{Z}_{>0}$. Take some $g \in H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(\ell))$ such that $g|_Y$ is not $\mathbb{W}$-invariant, and let $K = (V, E)$ be a graph. Then there exists a graph $K' = (V, E')$ containing K as a subgraph and $g' \in H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(\ell - 1))$ such that $c(K') \geqslant c(K) - 1$, $g'|_Y \neq 0$ and*

$$\mathrm{Bs}\big(\mathcal{M}_h\big) \subseteq \mathrm{Bs}\big(\mathcal{M}_{h'}\big)$$

for $h = f_K g$ and $h' = f_{K'} g'$.

Proof Since $g|_Y$ is not $\mathbb{W}$-invariant, there is a transposition $\tau = (ij) \in \mathbb{W}$ such that $\tau^*(g)|_Y \neq g|_Y$. Then $\tau^*(g) - g = (x_i - x_j)g'$ for some $g' \in H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(\ell - 1))$ such that $g'|_Y \neq 0$, since otherwise we would have $\tau^*(g)|_Y = g|_Y$.

Let $\tau(K)$ be the graph obtained from K by switching the labeling of the vertices i and j without changing any edges, and let K' be the graph obtained by adding the edge (ij) to $K \cup \tau(K)$ (take the union of edges). Then $c(K') \geqslant c(K) - 1$.

Let $h = f_K g$ and $h' = f_{K'} g'$. Then $\tau^*(h) = f_{\tau(K)}\tau^*(g)$, and $\mathrm{Bs}(\mathcal{M}_h) \subseteq \mathrm{Bs}(\mathcal{M}_{h'})$, because h' has the same factors (ignoring multiplicities) as

$$f_{K\cup\tau(K)} \cdot (x_i - x_j)g' = f_{K\cup\tau(K)}\big(\tau^*(g) - g\big) = f_{K-\tau(K)}\tau^*(h) - f_{\tau(K)-K}h,$$

where $K - \tau(K)$ is the graph obtained by removing from K the edges of $\tau(K)$. □

Corollary 17.4.9 *Let Y be an intersection in $\mathbb{P}^n$ of W-invariant hypersurfaces, and let f be a polynomial in $H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(m))$ such that $f|_Y \neq 0$. Then there are*

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$r \in \{0, 1, \ldots, m\}$, a graph $K = (V, E)$, and a $\mathbb{W}$-invariant polynomial $g_0 \in H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(m - r))$ such that $g_0|_Y \neq 0$, $c(K) \geqslant n + 1 - r$, and

$$\mathrm{Bs}\big(\mathcal{M}_f\big) \cap Y \subseteq \mathrm{Bs}\big(\mathcal{M}_g\big) \tag{17.3}$$

for $g = f_K g_0$.

Proof Let us apply Lemma 17.4.8 repeatedly starting with the graph $(V, \varnothing)$ and $g = f$. This process must stop after at most m steps. Therefore, we obtain a graph $K(V, E)$ and a polynomial $g = f_K h$ such that $\deg(h) = m - r$ for $r \leqslant m$, $c(K) \geqslant n + 1 - r$, the restriction $h|_Y$ is $\mathbb{W}$-invariant, and

$$\mathrm{Bs}\big(\mathcal{M}_f\big) \subseteq \mathrm{Bs}\big(\mathcal{M}_g\big).$$

Then we can replace h by a W-invariant polynomial g_0 of the same degree such that $g_0|_Y = h|_Y$. □

Proof of Proposition 17.4.4 The assertions on $\dim(Z)$ and the assertion on the number of irreducible components of the subvariety $\mathscr{X}$ follow from Lemma 17.4.6. Thus, we have to prove that

(1) either a general point in Z has at most m different coordinates, or
(2) the subvariety Z is an irreducible component of a set-theoretic intersection of W-invariant hypersurfaces of degree at most m.

Let $D_1, \ldots, D_k$ be W-invariant hypersurfaces of degree at most m that contain Z, and let Y be the set theoretic intersection $D_1 \cap \cdots \cap D_k$ (if there exist no such hypersurfaces, we set $Y = X$). We may assume $Z \subsetneqq Y$ (otherwise (2) clearly holds). Hence, there is $f \in \mathcal{M}$ such that $f|_Y \neq 0$. Note that this gives $Z \subseteq \mathrm{Bs}(\mathcal{M}_f) \cap Y$.

By Corollary 17.4.9, we find a graph $K = (V, E)$ with $c(K) \geqslant n + 1 - r$ and a $\mathbb{W}$-invariant polynomial $g_0 \in H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(m - r))$ such that $g_0|_Y \neq 0$ and (17.3) holds, where $r \in \{0, 1, \ldots, m\}$. By the construction of Y, we see that $g_0|_Z \neq 0$; thus, (17.3) gives

$$Z \subseteq \mathrm{Bs}\big(\mathcal{M}_{f_K}\big).$$

Pick a general point $z \in Z$ with coordinates $[z_0 : \ldots : z_n]$ and consider the color set $\mathcal{C} = \{z_0, \ldots, z_n\}$. If (1) does not hold, then there are at least $m \geqslant r$ different colors in $\mathcal{C}$. By Lemma 17.4.7, we may color the graph K by $\mathcal{C}$. After unwinding the definitions, this implies that there is $\sigma \in \mathbb{W}$ such that $\sigma^*(f_K)$ does not vanish on Z. But this is a contradiction as $\sigma^*(f_K) \in \mathcal{M}_{f_K}$. So, we conclude that (1) holds in this case and this completes the proof of the proposition. □

Let us apply Proposition 17.4.4. Recall that X_d is the Fermat hypersurface in $\mathbb{P}^n$ of degree d.

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Proposition 17.4.10 *If $d \leqslant n \leqslant 3d-1$, then $\alpha_G(X) \geqslant 1$. If $n \geqslant 3d$, then $\alpha_G(X) = \frac{2d}{n+1-d}$.*

Proof We suppose that $n \geqslant d$. Let H be a hyperplane section of X_d, let $r = \min\{2d, n+1-d\}$, and let D be a G-invariant effective divisor on X_d such that $D \sim_{\mathbb{Q}} rH$. We have $\alpha_G(X) \leqslant \frac{2d}{n+1-d}$, where the right-hand side is computed by the G-invariant Fermat hypersurface X_{2d}. Hence, both statements of the proposition would follow once we prove that the log pair (X, D) is log canonical. Suppose that (X, D) is not log canonical. Let us seek for a contradiction.

Let $\lambda = \mathrm{lct}(X, D)$ and $Z = \mathrm{Nklt}(X, \lambda D)$. Then $(X, \lambda D)$ is log canonical, $\lambda < 1$, and $Z \neq \varnothing$. Applying Lemma 17.4.2, we may assume that Z is a disjoint union of irreducible normal subvarieties. But, on the other hand, since $-(K_X+\lambda D)$ is ample, applying Kollár–Shokurov's connectedness, we conclude that Z is an irreducible subvariety. By Lemma 17.4.3, there exists a G-invariant linear subsystem $\mathcal{L} \subset |(3d-2)H|$ such that $Z = \mathrm{Bs}(\mathcal{L})$.

Let V be the vector subspace in $H^0(X, \mathcal{O}_X((3d-2)H))$ that corresponds to the linear system $\mathcal{L}$. Then V is a $\mathbb{G}$-subrepresentation in $H^0(X, \mathcal{O}_X((3d-2)H))$. As $\mathbb{T}$-representation, we have

$$V = \bigoplus_{\chi} V_\chi,$$

where the summand runs over all characters χ of the group $\mathbb{T}$. For each χ, we have $V_\chi = \mathbf{x}_\chi \cdot W_\chi$, where $\mathbf{x}_\chi$ is a monomial of degree at most $d-1$ in each homogeneous coordinate $x_0, x_1, \ldots, x_n$, while $\mathbb{T}$ acts trivially on W_χ. Each W_χ is the image in $H^0(X, \mathcal{O}_X(mdH))$ of a subspace of

$$\mathrm{Sym}^m(U) \subseteq H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(md)),$$

where $U = \mathrm{span}(x_0^d, \ldots, x_n^d) \subseteq H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d))$ and $m \leqslant 2$, because $md = \deg(W_\chi) \leqslant 3d-2$.

Since the action of the group $\mathbb{G}$ on the vector space $H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$ is irreducible, we see that the subvariety Z is not contained in a hyperplane, so Z is not contained in $\{\mathbf{x}_\chi = 0\}$ for any χ. Then Z is a set-theoretic intersection of zeroes of all polynomials in all W_χ. Since Z is invariant under the G-action, we see that $\sigma^*(f)$ vanishes on Z for any $f \in W_\chi$ and any $\sigma \in \mathbb{W} \cong \mathfrak{S}_{n+1}$.

Now, let us consider a morphism $\upsilon \colon \mathbb{P}^n \to \mathbb{P}^n$ defined as

$$\upsilon(x_0 : \ldots : x_n) = \left(x_0^d : \ldots : x_n^d\right).$$

Then the induced action of G on $\mathrm{Im}(f) = \mathbb{P}^n$ is isomorphic (as an action) to the permutational action of the group $W \cong \mathfrak{S}_{n+1}$ on $\mathbb{P}^n$. Further, we observe that

$$(1 : \ldots : 1) \notin \upsilon(X_d).$$

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Moreover, since Z is connected and W-invariant, so is $\upsilon(Z)$. Thus, from the previous discussion, we conclude that Z is the base locus of some W-invariant linear system of degree at most $2d$, generated by all the polynomials in W_χ. Then $\upsilon(Z)$ is the base locus of some W-invariant linear system of degree at most 2. Applying Proposition 17.4.4 to $\upsilon(Z)$, we see that Z is an irreducible component of a set-theoretic intersection of G-invariant hypersurfaces of degree at most $2d$. Then

$$Z = X_d \cap X_{2d}.$$

On the other hand, since the log pair (X_d, D) is not log canonical along Z, we have $\mathrm{mult}_Z(D) > 1$. This contradicts to $Z \sim_{\mathbb{Q}} 2dH$ and $r \leqslant 2d$. □

Similarly, we prove the following result.

Proposition 17.4.11 *Let $X = X_d \cap X_{2d} \cap \ldots \cap X_{rd}$ for $r \geqslant 1$, let H be a hyperplane section of X, and let D be a G-invariant effective $\mathbb{Q}$-divisor on X such that $D \sim_{\mathbb{Q}} qH$ for a positive rational number $q < (r+1)d$. Suppose, in addition, that* $\dim(X) \geqslant 1$*, $n \geqslant 4$, and $dH - (K_X + D)$ is nef. Then the log pair (X, D) is log canonical.*

Proof Replacing q by $\lceil q \rceil$, and D by $\frac{\lceil q \rceil}{q} D$, we may assume that the number q is actually an integer. Suppose that (X, D) is not log canonical. Let us seek for a contradiction.

Let $\lambda = \mathrm{lct}(X, D)$ and $Z = \mathrm{Nklt}(X, \lambda D)$. Then $(X, \lambda D)$ is log canonical, $\lambda < 1$ and $Z \neq \varnothing$. Applying Lemma 17.4.2, we may assume that Z is a disjoint union of irreducible normal subvarieties, and Z is G-irreducible. Moreover, using Lemma 17.4.3, we see that $Z = \mathrm{Bs}(\mathcal{L})$ for some G-invariant linear subsystem $\mathcal{L} \subset |aH|$, where

$$a = \frac{r(r+1)}{2} d + q - (r-1)$$

that satisfies $K_X + D + (n-r)H \sim_{\mathbb{Q}} aH$.

Now, let V be the vector subspace in $H^0(X, \mathcal{O}_X(aH))$ that corresponds to the linear system $\mathcal{L}$. Then V is a $\mathbb{G}$-subrepresentation in $H^0(X, \mathcal{O}_X(aH))$. As before, we have

$$V = \bigoplus_{\chi} V_\chi,$$

where the summand runs over all characters χ of the group $\mathbb{T}$. For each χ, we have $V_\chi = \mathbf{x}_\chi \cdot W_\chi$, where $\mathbf{x}_\chi$ is a monomial of degree at most $d-1$ in each homogeneous coordinate $x_0, x_1, \ldots, x_n$, and each W_χ is the image in $H^0(X, \mathcal{O}_X(\ell dH))$ of a subspace of

$$\mathrm{Sym}^\ell(U) \subseteq H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(\ell d)),$$

where $U = \mathrm{span}(x_0^d, \ldots, x_n^d) \subseteq H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d))$ and $\ell \leqslant \lfloor \frac{a}{d} \rfloor$.

Now, let $Z_1, \ldots, Z_s$ be the T-irreducible components of the locus Z. Then we claim that $s \leqslant n$. Indeed, using Nadel vanishing theorem and the nefness of the divisor $dH - (K_X + D)$, we get

$$H^1\Big(X, \mathcal{J}\big(X, \lambda D\big) \otimes \mathcal{O}_X\big(dH\big)\Big) = 0,$$

where $\mathcal{J}(X, \lambda D)$ is the multiplier ideal sheaf of the log pair $(X, \lambda D)$. Now, let Υ be the subscheme defined by the multiplier ideal sheaf $\mathcal{J}(X, \lambda D)$ of the log pair $(X, \lambda D)$, and let Υ_i be its irreducible component supported on Z_i for $i \in \{1, \ldots, s\}$. Then the natural restriction

$$H^0\Big(X, \mathcal{O}_X\big(dH\big)\Big) \to H^0\Big(\Upsilon, \mathcal{O}_\Upsilon\big(dH|_\Upsilon\big)\Big)$$

is surjective. Taking the $\mathbb{T}$-invariant parts, we see that

$$s \leqslant \sum_{i=1}^{s} \dim\left(H^0\Big(\Upsilon_i, \mathcal{O}_{\Upsilon_i}\big(dH|_{\Upsilon_i}\big)\Big)^{\mathbb{T}}\right) \leqslant \dim\left(H^0\Big(\Upsilon, \mathcal{O}_\Upsilon\big(dH|_\Upsilon\big)\Big)^{\mathbb{T}}\right) \leqslant$$
$$\leqslant \dim\left(H^0\Big(X, \mathcal{O}_X\big(dH\big)\Big)^{\mathbb{T}}\right) = \dim(U) - 1 = n.$$

Here, the first inequality holds because $H^0(\Upsilon_i, \mathcal{O}_{\Upsilon_i}(dH|_{\Upsilon_i}))^{\mathbb{T}}$ contains $U|_{\Upsilon_i} \neq 0$.

We claim that no T-irreducible components of Z are contained in coordinate hyperplanes. Indeed, otherwise, such a component would be contained in the (unique) minimal T-invariant linear subspace in $\mathbb{P}^n$, which would imply that Z has at least $n + 1$ T-irreducible components.

Let $m = \lfloor \frac{a}{d} \rfloor$. Arguing as in the proof of Proposition 17.4.10, we see that Z is the base locus of the W-invariant subsystem of $|\mathcal{O}_{\mathbb{P}^n}(md)|$ generated by $\mathrm{Bs}(W_\chi)$ and hypersurfaces containing X. Now, using Proposition 17.4.4 and the same morphism $\upsilon\colon \mathbb{P}^n \to \mathbb{P}^n$ as in Proposition 17.4.10, we conclude (as in the proof of Proposition 17.4.10) that Z is a G-irreducible component of the set-theoretic intersection of some G-invariant hypersurfaces of degree at most md, because the other possibility in Proposition 17.4.4 is excluded, since $\upsilon(Z)$ has at most n irreducible components and

$$\big\{x_0^d = x_1^d = \cdots = x_n^d\big\} \not\subset X.$$

In particular, we see that the pair (X, D) is not log canonical along $Y = X_d \cap X_{2d} \cap \ldots \cap X_{md}$, and hence $\mathrm{mult}_Y(D) > 1$. But as $n > m$ under our assumption (we leave to the reader to verify this), we see that Y is irreducible. Now, applying Lemma 17.4.1 to $D_k = \frac{1}{kd}(X_{kd} \cdot X)$ for $k = r + 1, \ldots, m$, we get

$$\operatorname{mult}_Y(D) \leqslant \frac{\deg(D)}{\deg(Y)} \prod_{i=k+2}^{m} kd \leqslant \frac{\deg(H)}{\deg(Y)} \prod_{k=r+1}^{m} kd = 1,$$

which is a contradiction. □

17.5 The Proof of Theorem 17.1.8

Let us use all assumptions and notations of Section 17.4. Let X be the complete intersection in the projective space $\mathbb{P}^n$ of the Fermat hypersurfaces $X_{2d}, X_{3d}, \ldots, X_{rd}$ for some integer $r \geqslant 2$, and let H be a hyperplane section of the variety X. Suppose that

$$-K_X \sim qH$$

for some $q \leqslant \frac{(r+1)d}{2}$. Then $\alpha_G(X) \leqslant \frac{d}{q}$, since $-K_X \sim_{\mathbb{Q}} \frac{q}{d} X_d|_X$. So, we can make $\alpha_G(X)$ arbitrarily small by choosing $q = \lfloor \frac{1}{2}(r+1)d \rfloor$ and letting $r \gg 0$. Therefore, to prove Theorem 17.1.8, it remains to show that X is G-birationally super-rigid.

In order to prove this, we use a similar strategy as in Proposition 17.4.11. Let $\mathcal{M}$ be a G-invariant mobile linear system, and let λ be a positive rational number such that

$$K_X + \lambda\mathcal{M} \sim_{\mathbb{Q}} 0.$$

As in the proof of Theorem 17.1.4, we need to show that $(X, \lambda\mathcal{M})$ has canonical singularities. Suppose the singularities of the pair $(X, \lambda\mathcal{M})$ are non-canonical. Let us seek for a contradiction.

Let B be a center of non-canonical singularities of the log pair $(X, \lambda\mathcal{M})$. Let us create some non-log canonical behavior using the center B. In Lemma 17.5.1, we first treat the case when B is contained in some special divisor $Y \subset X$, so that $(Y, \lambda\mathcal{M}|_Y)$ is not log canonical by the inversion of adjunction. As in the proof of Proposition 17.4.11, we will use Nadel vanishing to get an estimate of the possible number of irreducible components of the non-log canonical locus, and then use Proposition 17.4.4 to derive a contradiction.

Lemma 17.5.1 *Let $r, d \geqslant 2$, $n \geqslant 4$ be integers, let H be a divisor in $|\mathcal{O}_{\mathbb{P}^n}(1)|_X|$, and set $Y = X \cap X_d$. Assume that $D \sim_{\mathbb{Q}} lH$ is a G-invariant effective divisor on X for some $l \leqslant (r+1)d$ such that the divisor $H - (K_X + D)$ is nef. Assume also that* $\dim X \geqslant 2$. *Then*

(1) if D does not contain Y in its support, then (X, D) is log canonical;

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(2) the non-log canonical locus of (X, D) *is contained in* Y.

Proof As in the proof of Proposition 17.4.11, we may assume that $l \in \mathbb{N}$. Suppose that (1) is proved. To prove (2), write

$$D = t \cdot \frac{l}{d} Y + (1-t) D_0$$

for some $0 \leqslant t \leqslant 1$ and $D_0 \sim_{\mathbb{Q}} lH$ such that $Y \not\subseteq \operatorname{Supp}(D_0)$. Then (X, D_0) is log canonical by (1). Hence, every non-log canonical center of (X, D) is a non-log canonical center of the pair $(X, \frac{l}{d}Y)$. In particular, the non-log canonical locus of (X, D) is contained in Y. This proves (2).

Now, let us prove (1). Suppose that $Y \not\subseteq \operatorname{Supp}(D)$, and the log pair (X, D) is not log canonical. Let us seek for a contradiction. Let $\lambda = \operatorname{lct}(X, D)$ and $Z = \operatorname{Nklt}(X, \lambda D)$. By Lemmas 17.4.2 and 17.4.3, we may further assume that Z is G-irreducible, Z is a disjoint union of its irreducible components, and $Z = \operatorname{Bs}(\mathcal{L})$ for a G-invariant linear system $\mathcal{L} \subset |aH|$, where

$$a = \left(\frac{r(r+1)}{2} - 1\right) d + l - r$$

satisfies

$$K_X + D + (n - r + 1)H \sim_{\mathbb{Q}} aH.$$

Let s be the number of irreducible components of Z, and let $Z_1, \ldots, Z_s$ be these components. By Nadel vanishing applied to the multiplier ideal sheaf $\mathcal{J}\big(X, \lambda D\big)$, we have a surjection

$$H^0\big(X, \mathcal{O}_X(H)\big) \to H^0\big(Z, \mathcal{O}_Z(H|_Z)\big) = \bigoplus_{i=1}^{s} H^0\big(Z_i, \mathcal{O}_{Z_i}(H|_{Z_i})\big),$$

so $s \leqslant h^0(X, \mathcal{O}_X(H)) = n + 1$. But $h^0(Z_i, \mathcal{O}_{Z_i}(H|_{Z_i})) \geqslant 1$ with strict inequality for $\dim(Z_i) > 0$. Thus, if $s = n + 1$, then

$$n + 1 = h^0\big(Z, \mathcal{O}_Z(H|_Z)\big) = \sum_{i=1}^{s} h^0\big(Z_i, \mathcal{O}_{Z_i}(H|_{Z_i})\big),$$

which gives $\dim(Z) = 0$, so that we obtain a contradiction $n + 1 \geqslant |Z| \geqslant d(n+1) > n + 1$ as $d \geqslant 2$, since the length of a G-orbit in X is at least $d(n+1)$. This shows that $s \leqslant n$.

Arguing as in the proof of Proposition 17.4.11, we see that Z is a component of the set-theoretic intersection of some G-invariant hypersurfaces of degree at most md, where $m = \lfloor \frac{a}{d} \rfloor < n$. Set

$$R = X_d \cap X_{2d} \cap \ldots \cap X_{md}.$$

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Then the pair (X, D) is not log canonical along R. So, since one has $Y \not\subseteq \mathrm{Supp}(D)$, we have

$$\mathrm{mult}_R(D|_Y) \geqslant \mathrm{mult}_R(D) > 1.$$

On the other hand, as in the proof of Proposition 17.4.11, we obtain $\mathrm{mult}_R(D|_Y) \leqslant 1$ by Lemma 17.4.1. The obtained contradiction completes the proof of the lemma. □

Finally we treat the general case. Here the main observation is that if the center of non-canonical singularities is not contained in the special divisors, then, as a consequence of Proposition 17.4.4, it has small dimension, and in this case we can prove the G-birational super-rigidity by a similar application of the method of [51].

Theorem 17.5.2 *Let $d \gg 0$, $r \geqslant 2$ be integers. Assume that $-K_X \sim qH$ where H is the hyperplane class and $1 \leqslant q \leqslant \frac{(r+1)d}{2}$. Then X is G-birationally super-rigid.*

Proof Assume the contrary. Then, using the Noether–Fano inequality [8], we obtain a non-canonical log pair $(X, \lambda\mathcal{M})$ such that $\mathcal{M}$ is a mobile linear system, and $\lambda \in \mathbb{Q}_{>0}$ such that $K_X + \lambda\mathcal{M} \sim_{\mathbb{Q}} 0$. Let B be a center of non-canonical singularities of the pair $(X, \lambda\mathcal{M})$. Let us seek for a contradiction.

Observe that the center B is not contained in the Fermat hypersurface X_d. Indeed, otherwise, by the inverse of adjunction, the log pair $(Y, \mathcal{M}|_Y)$ is not log canonical, where $Y = X \cap X_d$. But

$$dH - (K_Y + \lambda\mathcal{M}|_Y) = -(K_X + \lambda\mathcal{M})|_Y \sim_{\mathbb{Q}} 0,$$

which is impossible by Proposition 17.4.11.

We claim that B is not contained in any T-invariant hyperplane. Indeed, suppose $B \subseteq \{x_i = 0\}$. Let $X' = X \cap \{x_i = 0\}$, and let G' be the stabilizer subgroup in G of the hyperplane $\{x_i = 0\}$. Then $G' \cong \boldsymbol{\mu}_d^n \rtimes SS_n$, and X' is G'-invariant. Let $\mathcal{M}' = \mathcal{M}|_{X'}$. Then $(X', \lambda\mathcal{M}')$ is not log canonical along B by the inverse of adjunction. But $K_{X'} + \lambda\mathcal{M}' \sim_{\mathbb{Q}} H$, which gives $B \subset X_d$ by Lemma 17.5.1. However, we already proved that $B \not\subset X_d$.

Now by Remark 17.3.6, the log pair $(X, 2\lambda\mathcal{M})$ is not log canonical along B. Let μ be the smallest positive rational number such that $B \subset \mathrm{Nklt}(X, \mu\mathcal{M})$, and let Z be an irreducible component of the locus $\mathrm{Nklt}(X, \mu\mathcal{M})$ containing B. Then $\mu < 2\lambda$, and it follows from [26, Theorem 3.1] that

$$\mathrm{mult}_B(M_1 \cdot M_2) > 4/\mu^2,$$

for general divisors M_1 and M_2 in the linear system $\mathcal{M}$. Moreover, it follows from Lemma 17.4.3 that the subvariety Z is a component of $\mathrm{Bs}(\mathcal{L})$ for some G-invariant linear system $\mathcal{L} \subseteq |\mathcal{O}_X(a)|$, where

$$a = \left(\frac{r(r+1)}{2} - 1\right) d + 2q - r$$

satisfies

$$K_X + 2\lambda\mathcal{M} + (n - r + 1)H \sim_{\mathbb{Q}} aH.$$

Furthermore, we know that Z is not contained in any T-invariant hyperplane, because B is not contained in any T-invariant hyperplane.

Set $m = \lfloor \frac{a}{d} \rfloor$. Then $m < n$. Now, arguing as in the proof of Proposition 17.4.10 or Proposition 17.4.11, and using Proposition 17.4.4, we see that either the subvariety Z is a component of a set-theoretic intersection of G-invariant hypersurfaces of degree at most md, or $\dim(B) \leqslant \dim(Z) \leqslant m-1$.

Suppose that the subvariety Z is a component of a set-theoretic intersection of G-invariant hypersurfaces of degree at most md. Let Γ' be an irreducible component of $M_1 \cdot M_2$ such that

$$\frac{\mathrm{mult}_Z(\Gamma')}{\deg(\Gamma')} \geqslant \frac{\mathrm{mult}_Z(M_1 \cdot M_2)}{\deg(M_1 \cdot M_2)} > \frac{1}{\mu^2 \deg(M_1 \cdot M_2)}.$$

Since $Z \not\subset X_d$, we observe that $\Gamma = \Gamma' \cdot X_d$ is a codimension-2 cycle on $Y = X \cap X_d$ such that

$$\frac{\mathrm{mult}_{Z'}(\Gamma)}{\deg(\Gamma)} \geqslant \frac{\mathrm{mult}_Z(\Gamma')}{\deg(\Gamma')} > \frac{1}{\mu^2 \deg(M_1 \cdot M_2)},$$

where $Z' = X_d \cap \ldots \cap X_{md} \subseteq Z$. On the other hand, let H_1 and H_2 be general divisors in $|H|$. Then, as in the proof of Proposition 17.4.11, we get

$$\begin{aligned}\frac{\mathrm{mult}_{Z'}(\Gamma)}{\deg(\Gamma)} &\leqslant \frac{1}{\deg Z'} \prod_{k=r+3}^{m} kd \\ &= \frac{1}{d^2(r+1)(r+2)\deg(H_1 \cdot H_2)} < \frac{1}{\mu^2 \deg(M_1 \cdot M_2)},\end{aligned}$$

by Lemma 17.4.1 applied to the divisors $Y \cdot X_{kd}$, for $r+1 \leqslant k \leqslant m$ on the complete intersection Y, where the last inequality follows from $2q \leqslant (r+1)d$. This gives us a contradiction.

Thus, we have $\dim B \leqslant \dim Z \leqslant m-1$. Let P be a sufficiently general point in the center B, and let Y be a general codimension m linear section of the complete intersection X containing P. Set $\mathcal{M}_Y = \mathcal{M}|_Y$. Then the pair $(Y, \lambda\mathcal{M}_Y)$ is not log canonical at P by the inverse of adjunction, but the pair $(Y, 2\lambda\mathcal{M}_Y)$ is log canonical in a punctured neighborhood of the point P. Note that

$$K_Y + 2\lambda\mathcal{M}_Y \sim_{\mathbb{Q}} (m+q)H.$$

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Using [51, Corollary 1.8] and the lower bound in [37, Paragraph 56] for the number of lattice points, we obtain the following inequality:

$$h^0\big(Y, \mathcal{O}_Y((m+q)H|_Y)\big) > \frac{1}{n-m-r}\binom{2(n-m-r)}{n-m-r} \geqslant \frac{1}{(n-m-r)^2}4^{n-m-r}. \tag{17.4}$$

On the other hand, since Y is a complete intersection in $\mathbb{P}^{n-m}$, we also have

$$h^0\big(Y, \mathcal{O}_Y((m+q)H|_Y)\big) \leqslant h^0(\mathbb{P}^{n-m}, \mathcal{O}_{\mathbb{P}^{n-m}}(m+q)) = \binom{n+q}{m+q} < 2^{n+q}. \tag{17.5}$$

Recall that $m = \lfloor \frac{a}{d} \rfloor \leqslant \frac{r(r+1)}{2} + 2r$ is bounded by a constant (that does not depend on d), and, by assumption, we have

$$n = \left(\frac{r(r+1)}{2} - 1\right) d + q - 1 \geqslant \frac{1}{r+1}\left(\frac{r(r+1)}{2} - 1\right) q + q - 1 \geqslant \frac{5}{3}q - 1.$$

Therefore, using (17.5), we obtain

$$h^0\big(Y, \mathcal{O}_Y((m+q)H)\big) < 2^{1.6n+1} < \frac{1}{(n-m-r)^2}4^{n-m-r}$$

when $n \gg 0$, which is equivalent to $d \gg 0$. This contradicts to (17.4). The proof is complete. □

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