



# Periodicity and decidability of translational tilings by rational polygonal sets

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## Abstract

The periodic tiling conjecture asserts that if a region  $\Sigma \subset \mathbb{R}^d$  tiles  $\mathbb{R}^d$  by translations then it admits at least one *fully periodic* tiling. This conjecture is known to hold in  $\mathbb{R}$ , and recently it was disproved in sufficiently high dimensions. In this paper, we study the periodic tiling conjecture for *polygonal sets*: bounded open sets in  $\mathbb{R}^2$  whose boundary is a finite union of line segments. We prove the periodic tiling conjecture for any polygonal tile whose vertices are rational. As a corollary of our argument, we also obtain the decidability of tilings by rational polygonal sets. Moreover, we prove that *any* translational tiling by a rational polygonal tile is weakly-periodic, i.e., can be partitioned into finitely many singly-periodic pieces.

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## 1. Introduction

Let  $\Sigma \subset \mathbb{R}^d$  be a bounded, measurable set. We say that  $\Sigma$  *tiles*  $\mathbb{R}^d$  *by translations* if there exists a countable set  $T \subset \mathbb{R}^d$  such that the translations of  $\Sigma$  along the points

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of  $T$ ,  $\Sigma + t$ ,  $t \in T$ , cover almost every point in  $\mathbb{R}^d$  exactly once. In this case, we write  $\Sigma \oplus T = \mathbb{R}^d$  and refer to  $\Sigma$  as a *translational tile* and to  $T$  as a *tiling of  $\mathbb{R}^d$  by  $\Sigma$* . Let  $\text{Tile}(\Sigma; \mathbb{R}^d)$  denote the set of all the tilings of  $\mathbb{R}^d$  by  $\Sigma$ , i.e.,

$$\text{Tile}(\Sigma; \mathbb{R}^d) = \{T \subset \mathbb{R}^d : \Sigma \oplus T = \mathbb{R}^d\}.$$

Since in this work we consider tilings by *only* translations, in what follows we sometime abbreviate and write “tiling” to refer to tiling by translations and “tile” to refer to translational tile.

The study of translational tiling consists of studying the structure of sets in  $\text{Tile}(\Sigma; \mathbb{R}^d)$ . A major conjecture in this area is *the periodic tiling conjecture* [8,11,15], which asserts that if  $\text{Tile}(\Sigma; \mathbb{R}^d)$  is non-empty, then it must contain at least one *periodic* set, i.e., a set  $T \in \text{Tile}(\Sigma; \mathbb{R}^d)$ , which is invariant under translations by lattice  $\Lambda \subset \mathbb{R}^d$  (here, a lattice is a full rank discrete subgroup).

The periodic tiling conjecture (PTC) is known to hold in  $\mathbb{R}$  [11], for convex domains in all dimensions [12,16] and for topological disks in  $\mathbb{R}^2$  [2,9,10]. On the other hand, recently, the periodic tiling conjecture was disproved in sufficiently high dimensions [7], even under the assumption that the tile is connected [4].

One important motivation to study the structure of tilings and the periodic tiling conjecture in particular is the connection to the *decidability* of tilings. Indeed, in [17] it was shown that if the periodic tiling conjecture were true, then the question of whether a set is a tile or not would be decidable. Recently, in [5], it was shown that translational tilings are *undecidable* if the dimension is unbounded. However, the decidability of translational tilings (by a single tile) in  $\mathbb{R}^d$ , for a fixed  $d \geq 2$ , is still open. In this paper we study the periodicity and decidability of planar translational tilings by *polygonal sets*.

### 1.1. Polygonal tiles

A *polygonal set* is a bounded open set  $\Omega \subset \mathbb{R}^2$ , whose boundary is a finite union of segments. Note that a polygonal set can certainly be disconnected and its connected components are not necessarily simply-connected. We denote the *vertices* of  $\Omega$  by  $V(\Omega)$ , and the *edges* of  $\Omega$  by  $E(\Omega)$ . See Fig. 1.1 for illustration.

We say that a polygonal set  $\Omega$  is *rational* if the set  $V(\Omega) - v$  is contained in  $\mathbb{Q}^2$ , for any  $v \in V(\Omega)$ .

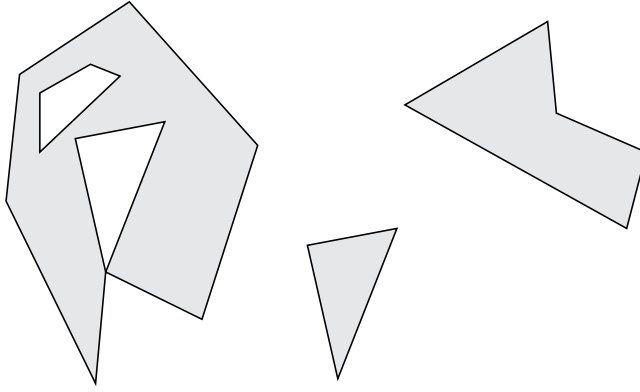
Our first result is that the periodic tiling conjecture holds for rational polygonal tiles.

**Theorem 1.1** (PTC Holds for Rational Polygonal Tiles). *A rational polygonal set  $\Omega \subset \mathbb{R}^2$  tiles  $\mathbb{R}^2$  by translations if and only if  $\text{Tile}(\Omega; \mathbb{R}^2)$  contains a periodic set.*

Using the proof of Theorem 1.1 we also obtain the decidability of tilings in  $\mathbb{R}^2$  by rational polygonal tiles.

**Corollary 1.1.** *There is an algorithm that computes, upon any given rational polygonal set  $\Omega$ , whether  $\text{Tile}(\Omega; \mathbb{R}^2)$  is empty or not.*

In addition, we establish a structural result that applies to *any* set in  $\text{Tile}(\Omega; \mathbb{R}^2)$ . To state our result, we first introduce some refinements of the notion of periodicity.



**Fig. 1.1.** Example of a polygonal set  $\Omega$ . The set  $\Omega$  is shown in gray; it has three connected components. The set  $E(\Omega)$  consists of the 22 segments in black whose union is the boundary of  $\Omega$ ,  $\partial\Omega$ , and  $V(\Omega)$  is the 21 endpoints of those segments.

**Definition 1.2** (*Single-Periodicity, Weak-Periodicity, Double-Periodicity*). Let  $S \subset \mathbb{R}^2$  be a countable set.

- (i) We say that  $S$  is *singly-periodic* (or  $\langle h \rangle$ -periodic) if it is invariant under translations by some non-zero vector  $h \in \mathbb{R}^2 \setminus \{0\}$ .
- (ii) We say that  $S$  is *weakly-periodic* if it can be partitioned into finitely many singly-periodic sets.
- (iii) We say that  $S$  is *double-periodic* (or  $\langle h_1, h_2 \rangle$ -periodic) if it is periodic, i.e., invariant under translations by two linearly independent vectors  $h_1, h_2 \in \mathbb{R}^2$ .

We show that any tiling by a rational polygonal tile must be weakly-periodic.

**Theorem 1.2** (*Structure of Tilings by Rational Polygonal Sets*). Let  $\Omega \subset \mathbb{R}^2$  be a rational polygonal set. Then any  $T \in \text{Tile}(\Omega; \mathbb{R}^2)$  is weakly-periodic.

We will address the periodicity and decidability of polygonal sets with irrational vertices in a subsequent work.

## 1.2. Outline of our arguments

Let  $\Omega \subset \mathbb{R}^2$  be a rational polygonal set. By translation and dilation invariance of tilings, we can assume that  $V(\Omega) \subset \mathbb{Z}^2$ .

- (1) In Section 2, by sliding the *sliding components* of a tiling by  $\Omega$ , we show that there exists a tiling  $T$  of  $\mathbb{R}^2$  by  $\Omega$  that is contained in  $\mathbb{Z}^2$ .
- (2) In Section 3, we *discretize*  $\Omega$ : We show that there are a natural number  $N$  and a finite set  $F \subset N^{-1}\mathbb{Z}^2$  such that every  $0 \in T \subset \mathbb{R}^2$  is a tiling of  $N^{-1}\mathbb{Z}^2$  by  $F$  if and only if it is an integer tiling of  $\mathbb{R}^2$  by  $\Omega$ .

- (3) Then, using [1,6], where the periodic tiling conjecture was proved to hold in  $\mathbb{Z}^2$ , we show the existence of a periodic  $T \subset \mathbb{Z}^2$  which is a tiling of  $N^{-1}\mathbb{Z}^2$  by  $F$ . [Theorem 1.1](#) and [Corollary 1.1](#) then follow from combining this with (1) and (2).
- (4) While the proof of [Theorem 1.1](#) uses off-the-shelf results from [1,6], the proof of [Theorem 1.2](#) is more delicate and requires novel refinement of the technology introduced in [6]. Indeed, by combining (2) and [6, Theorem 1.4], we merely obtain that any *integer* tiling in  $\text{Tile}(\Omega; \mathbb{R}^2)$  is weakly periodic. The main obstacle to transferring [Theorem 1.2](#) from its integer counterpart is the possible *sliding* components obstructing periodicity. In Section 4, we address this obstacle by analyzing the connection between the sliding components of *any* set in  $\text{Tile}(\Omega; \mathbb{R}^2)$  and the singly-periodic components of sets in  $\text{Tile}(F; N^{-1}\mathbb{Z}^2)$ , and conclude [Theorem 1.2](#).

### 1.3. Notation

Let  $G = (G, +)$  be an Abelian group. For  $A \subset G$ ,  $g \in G$  we use the notation  $A + g$  for the set  $\{a + g : a \in A\}$ ; the set  $A - g$  is defined similarly. For  $A, B \subset G$ , we write

$$A + B := \{a + b : a \in A; b \in B\};$$

the set  $A - B$  is defined similarly.

For  $\Sigma \subset \mathbb{R}^d$ , we denote the closure of  $\Sigma$  by  $\overline{\Sigma}$  and the boundary of  $\Sigma$  by  $\partial\Sigma$ .

## 2. From a tiling to an integer tiling

Let  $\Omega \subset \mathbb{R}^2$  be a rational polygonal set. As tilings and periodicity are both invariant under affine transformations, by translating  $\Omega$  to make the origin one of its vertices and dilating by the least common multiple of the denominators of both coordinates of all the vertices we can assume without loss of generality that the vertices of  $\Omega$  are in  $\mathbb{Z}^2$ . In other words, we may assume that  $\Omega$  is an *integer* set. If  $\Omega$  tiles we say that it is an *integer tile*.

**Definition 2.1** (*Integer Tiling*). A tiling  $T \in \text{Tile}(\Omega; \mathbb{R}^2)$  is called an *integer tiling* if  $V(\Omega) + T$  is entirely contained in  $\mathbb{Z}^2$ .

The goal of this section is showing that any tiling of  $\mathbb{R}^2$  by  $\Omega$  gives rise to an integer tiling  $T'$  of  $\mathbb{R}^2$  by  $\Omega$ , as stated in the following lemma.

**Lemma 2.2** (*Existence of Integer Tiling*). Let  $\Omega \subset \mathbb{R}^2$  be an integer polygonal tile. Then there exists  $T \in \text{Tile}(\Omega; \mathbb{R}^2)$  such that  $T \subset \mathbb{Z}^2$ .

We will use the vertex-adjacent equivalence relationship:

**Definition 2.3** (*Integer Tile, Vertex-Sharing Equivalence Class*). Let  $\Omega$  be an integer tile. For  $t, t'$  in  $\mathbb{R}^2$  we say that  $t \approx_\Omega t'$  if  $(V(\Omega) + t) \cap (V(\Omega) + t')$  is not empty. If  $T$  is a tiling of  $\mathbb{R}^2$  by  $\Omega$ , the relationship  $\approx_\Omega$  induces (by transitive closure) an equivalence relationship in  $T$ . We denote this equivalence relationship by  $t \sim_\Omega t'$ .

### 2.1. Reduction

The following lemma shows that to prove [Lemma 2.2](#) it suffices to show that every polygonal tile can be repaired to a tile with a unique vertex sharing component.

**Lemma 2.4** (*Vertex-Sharing Tiling by an Integer Tile Is an Integer Tiling*). *Let  $\Omega$  be an integer tile, so that  $T$  is a tiling of  $\mathbb{R}^2$  by  $\Omega$  containing 0. If  $t \sim_{\Omega} t'$  for any elements  $t, t'$  of  $T$ , then  $T \subset \mathbb{Z}^2$ .*

**Proof.** If  $\Omega$  is an integer tile  $t \approx_{\Omega} t'$  implies that  $t - t' \in \mathbb{Z}^2$ . Let  $t \in T$ . By assumption, since  $0 \in T$ , we have  $0 \sim_{\Omega} t$ . This means that there is a path  $0 = t_1, t_2, \dots, t_n = t$  with  $t_j \in T$  and  $t_i \approx_{\Omega} t_{i+1}$  for  $i = 1, \dots, n - 1$ . Therefore,

$$t - 0 = \sum_{j=1}^{n-1} t_{j+1} - t_j$$

is in  $\mathbb{Z}^2$ , as claimed.  $\square$

### 2.2. Sliding

[Lemma 2.4](#) reduces the proof of [Lemma 2.2](#) to showing, for any integer tile  $\Omega \subset \mathbb{R}^2$ , the existence of a tiling  $T \in \text{Tile}(\Omega; \mathbb{R}^2)$  that is  $\sim_{\Omega}$ -connected, i.e., has a single  $\sim_{\Omega}$ -equivalence class. This will be achieved in this subsection, by developing a *sliding* machinery, which allows one to shift the  $\sim_{\Omega}$ -components of a given tiling  $T \in \text{Tile}(\Omega; \mathbb{R}^2)$ , while preserving the tiling, to eventually merge all of the components into a tiling  $T' \in \text{Tile}(\Omega; \mathbb{R}^2)$  with a *single*  $\sim_{\Omega}$ -equivalence class (see [Fig. 2.2](#) for an illustration).

This is done by first analyzing the structure of the  $\sim_{\Omega}$ -tiling-components, and showing that they are invariant under *sliding* in some direction  $\vec{v} \in \mathbb{R}^2$ , i.e.,  $\Omega + T_i + \delta \vec{v} = \Omega + T_i$ , for every  $\delta \in \mathbb{R}$  and  $T_i \in T / \sim_{\Omega}$  (see [Fig. 2.1](#) for an illustration).

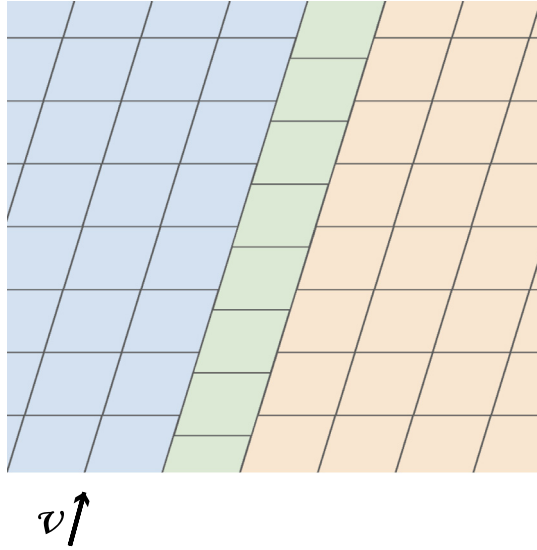
**Lemma 2.5** (*Tiling-Components Slide*). *Let  $T$  be a tiling of  $\mathbb{R}^2$  by a polygonal tile  $\Omega$ , let  $T_i, i \in I$ , be the equivalence classes of*

$$T = \bigsqcup_{i \in I} T_i \tag{2.1}$$

*by  $\sim_{\Omega}$ . Then there exists a unit vector  $\vec{v} \in \mathbb{R}^2$  such that each set  $W_i := \Omega + T_i$  is  $\vec{v}\mathbb{R}$ -invariant.*

**Remark 2.6.** See also [3, Theorem 1.7] for a related decomposition into “rational components” of tilings of a torus.

**Proof.** For  $D \subset \mathbb{R}^d$  we say that  $x \in \partial D$  is a *proper vertex* of  $D$  if for no neighborhood  $B_{\epsilon}(x)$ , the intersection of  $B_{\epsilon}(x) \cap \partial D$  is a segment.



**Fig. 2.1.** In blue, green and orange, three *sliding components*,  $W_1, W_2, W_3$ , of a tiling, with a sliding direction  $v$ .

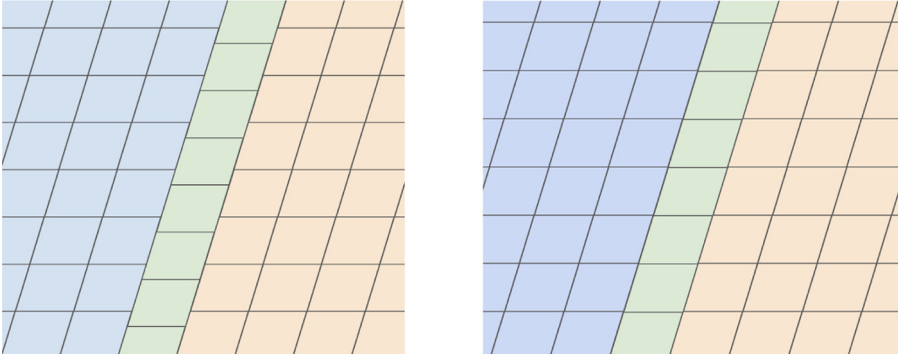
We first show that if  $x_0 \in \partial(\Omega + S)$  is a proper vertex of  $\Omega + S$ , then  $S$  is not  $\sim_\Omega$ -closed, i.e., there exist  $t \in T \setminus S$  and  $s \in S$  such that  $s \approx_\Omega t$ .

Indeed, for all  $\epsilon$  there exists a  $t_\epsilon$  in  $T \setminus S$  so that  $B_\epsilon(x_0) \cap (\Omega + t_\epsilon) \neq \emptyset$ . Since the set  $T$  is uniformly discrete and  $\Omega$  is bounded, there must exist  $t \in T \setminus S$  so that  $(\Omega + t) \cap B_\epsilon(x_0) \neq \emptyset$  for all  $\epsilon > 0$  (or, equivalently,  $x_0 \in \overline{\Omega} + t$ ). Let  $\{t_1, \dots, t_n\}$  be the set of all elements in  $T \setminus S$  with the property that  $x_0 \in \overline{\Omega} + t_i$ , and  $\{s_1, \dots, s_m\}$  be the set of the elements in  $S$  with the property that  $x_0 \in \overline{\Omega} + s_i$ . (Note that none of these sets is empty.) Then we must be in one of the following three scenarios:

1.  $m = 1$ , and  $x_0 \notin V(\Omega + s_1)$ ;
2.  $n = 1$ , and  $x_0 \notin V(\Omega + t_1)$ ;
3. There are  $1 \leq j \leq n$  and  $1 \leq k \leq m$  such that  $x_0 \in V(\Omega + t_j) \cap V(\Omega + s_k)$ .

In the first case, we have that  $x_0 \in \overline{\Omega} + s$  for  $s \in S$  only if  $s = s_1$ . Therefore  $x_0$  is not a proper vertex of  $\Omega + S$  as  $x_0 \notin V(\Omega) + s_1$ . In the second case, by the same argument we show that  $x_0$  is not a proper vertex of  $\Omega + (T \setminus S)$ . This leads to a contradiction, since  $\partial(\Omega + (T \setminus S)) = \partial(\Omega + S)$ . In the third case, our conclusion holds; indeed, in this case,  $t_j \approx_\Omega s_k$ .

Now, if an open set  $D$  has no proper vertices, then  $\partial D$  is a closed subset of  $\mathbb{R}^2$  which is locally a line, and therefore a union of parallel lines in direction  $\vec{v} \in \mathbb{R}^2$ . Since  $\partial D$  is invariant under translations by  $\vec{v}$ , also  $D$  is invariant under translations by  $\vec{v}$ . Indeed, since  $\partial D$  does not contain  $x$  for any point  $x \in D$ , we must have that  $\partial D$  does not intersect  $x + \vec{v}\mathbb{R}$ . In particular,  $x + \vec{v}\mathbb{R} \subset D$ .



**Fig. 2.2.** The  $\sim_\Omega$ -classes from Fig. 2.1 before sliding (left) and after sliding (right).

We conclude that  $\mathbb{R}^2 = \bigsqcup_{i \in I} \Omega + T_i$ , where  $\Omega + T_i$  is  $\vec{v}_i$  invariant. However, as  $\Omega + T_i$  is not empty, and  $\Omega + T_i \cap \Omega + T_j = \emptyset$  if  $i \neq j$ , all of the vectors  $\vec{v}_i$  must be parallel. This concludes the proof.  $\square$

Using Lemma 2.5, one can now slide the  $\sim_\Omega$ -components of a given tiling  $T \in \text{Tile}(\Omega; \mathbb{R}^2)$  to merge them into a single  $\sim_\Omega$ -component of another tiling  $T' \in \text{Tile}(\Omega; \mathbb{R}^2)$ .

**Lemma 2.7** (*Merge Components by Sliding*). *Suppose that  $T$  is a tiling of  $\mathbb{R}^2$  by a polygonal tile  $\Omega$ , and let  $T_1, T_2, \dots$  and  $\vec{v} \in \mathbb{R}^2 \setminus \{0\}$  be as in Lemma 2.5. Then there exist real coefficients  $s_1, s_2, \dots$  such that*

$$T' := \bigsqcup T_i + s_i \vec{v} \in \text{Tile}(\Omega; \mathbb{R}^2)$$

and  $T'$  has a single  $\sim_\Omega$ -equivalence class.

**Proof.** Applying an affine transformation, we may assume without loss of generality that  $v = (0, 1)$ . Denote  $W_i := \Omega + T_i$ . Let  $L = \mathbb{R} \times \{0\}$ , and  $L_i := \{(x, 0) : \exists y, (x, y) \in W_i\}$ . Then, since  $\Omega$  is open and  $T = \bigsqcup T_i$  is a tiling by  $\Omega$ , the sets  $L_i$  are open, non-overlapping and non-empty. Since the set  $T$  is a tiling by  $\Omega$ , the distance between any two elements in  $T$  is at least

$$\delta := \min\{|x| : x \in \mathbb{R}^2, \Omega \cap (\Omega + x) \text{ has zero measure}\} > 0.$$

Thus, for any  $R$  there are finitely many  $L_i$  at distance at most  $R$  from 0, and so, by reordering if needed, we may assume that the sets  $L_i$  are ordered so that  $d(L_i, 0) \leq d(L_{i+1}, 0)$ . This implies that for  $i > 1$ , one has  $\overline{L_i} \cap \bigcup_{j < i} \overline{L_j} \neq \emptyset$ .

We will define the numbers  $s_i$  by induction. Set  $s_1 = 0$ . For  $i > 1$  suppose that  $s_1, \dots, s_{i-1}$  were already chosen to satisfy the claim. We will now define  $s_i$  from the values of  $s_1, \dots, s_{i-1}$ . From the ordering of  $L_1, L_2, \dots$  it follows that there must exist  $j < i$  and  $x \in L$  such that  $x \in \partial L_i \cap \partial L_j$ , or equivalently,

$$x + v\mathbb{R} \subset \partial(\overline{\Omega} + T_i) \cap \partial(\overline{\Omega} + T_j).$$

Thus, there are  $t_i \in T_i, t_j \in T_j$  such that both  $V(\Omega + t_i) \cap (x + \vec{v}\mathbb{R})$  and  $V(\Omega + t_j) \cap (x + \vec{v}\mathbb{R})$  are non-empty. We can therefore choose  $v_i \in V(\Omega + t_i) \cap (x + \vec{v}\mathbb{R})$  and  $v_j \in V(\Omega + t_j) \cap (x + \vec{v}\mathbb{R})$ . We then set  $s_i \in \mathbb{R}$  to satisfy

$$(s_j - s_i)\vec{v} = v_i - v_j,$$

which ensures that an element of  $T_i + s_i\vec{v}$  is  $\sim_\Omega$ -connected to an element of  $T_j + s_j\vec{v}$ , while preserving the  $\sim_\Omega$ -connectedness of  $T_i$  and  $T_j$ .  $\square$

Combining [Lemmas 2.4](#) and [2.7](#) we obtain [Lemma 2.2](#), as needed.

### 3. Encoding an integer tile as a discrete tile

In this section, we prove [Theorems 1.1](#) and [Corollary 1.1](#) by a reduction from the continuous setup to the discrete setup.

#### 3.1. Discrete translational tilings

Let  $(G, +)$  be a finitely generated Abelian group. A finite set  $F \subset G$  tiles  $G$  by translations if there exists a set  $T \subset G$  such that  $F \oplus T = G$ , namely, the translations of  $F$  along  $T$ :  $F + t, t \in T$ , cover every point in  $G$  exactly once. As in the continuous setup, we denote

$$\text{Tile}(F; G) := \{T \subset G : F \oplus T = G\}.$$

The discrete version of the periodic tiling conjecture is known to hold in  $\mathbb{Z}$  [[14](#)] and  $\mathbb{Z}^2$  [[1,6](#)], but was recently disproved in  $\mathbb{Z}^2 \times G_0$  for some high-rank finite groups  $G_0$ , and thus (using [[13](#)]) in  $\mathbb{Z}^d$  for sufficiently large  $d$  [[7](#)].

#### 3.2. Discretization

In this subsection, we encode any rational polygonal set  $\Omega$  as a finite set in a two-dimensional lattice, in a way that preserves the tiling properties of  $\Omega$ . This will enable us to utilize the proof of the two-dimensional discrete periodic tiling conjecture [[1,6](#)] to obtain [Theorem 1.1](#) and deduce [Corollary 1.1](#).

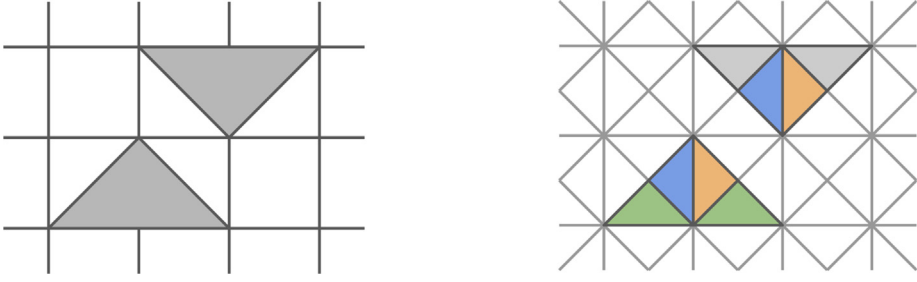
Let  $\Omega$  be an integer set. Consider all integer translates of the finitely many segments in  $E(\Omega)$ . These segments partition  $[0, 1]^2$  into  $M$  open polygonal sets  $P_0, \dots, P_{M-1}$  (up to null sets), see [Fig. 3.1](#).

Observe that for each  $v \in \mathbb{Z}^2$  and each  $P_i$ , either  $(v + P_i)$  is in  $\Omega$  or is disjoint from  $\Omega$ . Let  $N = N(\Omega) = 3k + 4$ , where  $k \in \mathbb{N}$  is such that  $M \leq k^2$ . To each  $P_i$  we associate a subset  $S_i$  of  $\{0, \dots, N - 1\}^2$  as follows:

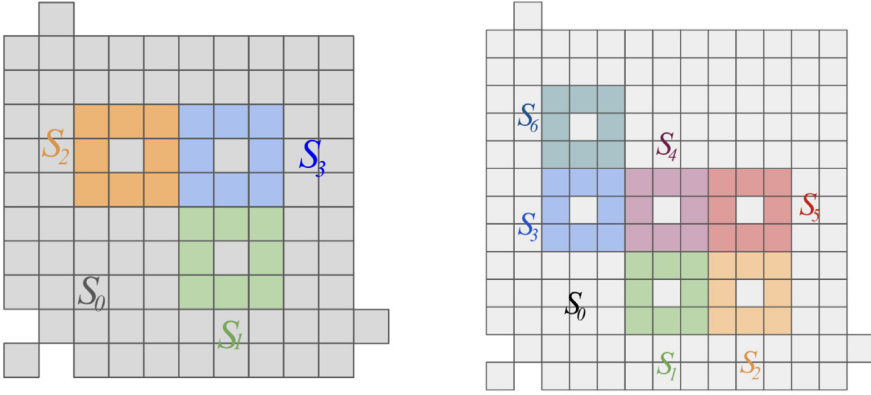
1. For  $1 \leq i \leq M - 1$ , let  $(a_i, b_i) \in \{1, \dots, k\}^2$  be the unique pair such that  $(a_i - 1) + k(b_i - 1) = i$ . We define the set  $S_i$  to consist of the 8 points

$$(\tilde{a}, \tilde{b}) \in \{0, \dots, N - 1\}^2 \setminus \{(3a_i, 3b_i)\}$$

which satisfy  $|\tilde{a} - 3a_i| \leq 1$  and  $|\tilde{b} - 3b_i| \leq 1$ .



**Fig. 3.1.** (Left) An integer set  $\Omega$  and (Right) its associated partition into translations of subsets of  $[0, 1]^2$ , with  $P_0$  in gray;  $P_1$  in green;  $P_2$  in orange; and  $P_3$  in blue.



(A) When  $M = 4$ ,  $k = 2$  and  $N = 10$  (e.g., for  $\Omega$  as in Figure 3.1).

(B) When  $M = 7$ ,  $k = 3$  and  $N = 13$ .

**Fig. 3.2.** The associated sets  $S_0, \dots, S_{M-1}$  in two different cases. (Here, a point  $(a, b) \in \{0, \dots, N\}^2$  is represented by a square  $(a, b) + [0, 1]^2$ ).

2. For  $i = 0$ , we first define  $\tilde{S}_0 := \{0, \dots, N-1\}^2 \setminus \bigcup_{j=1}^{M-1} S_j$ . The set  $S_0$  is defined as  $\tilde{S}_0 \cup \{(N, 1), (1, N)\} \setminus \{(0, 1), (1, 0)\}$ .

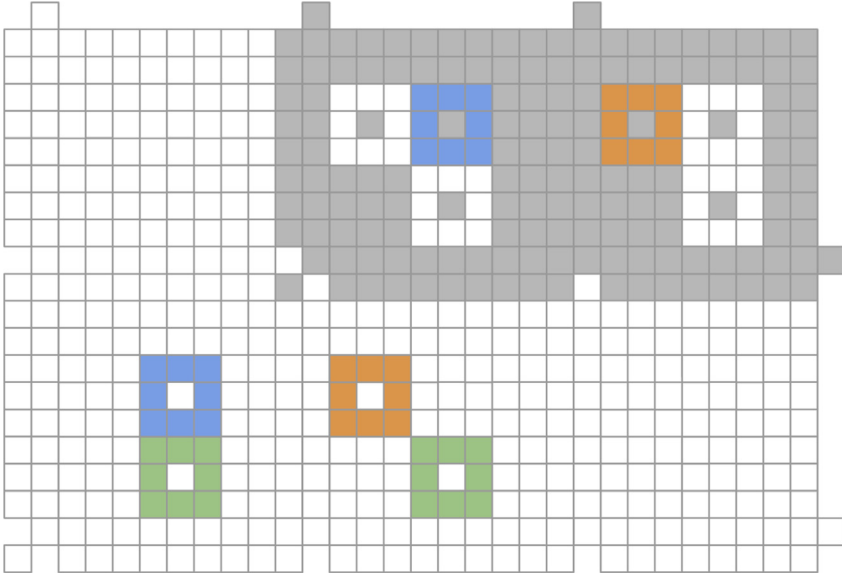
See Fig. 3.2.

Now we associate to  $\Omega$  the (discrete) set:

$$F = F(\Omega) := \bigcup_{v \in \mathbb{Z}^2} \bigcup_{\substack{i=0, \dots, M-1 \\ (v + P_i) \subset \Omega}} \left( v + \frac{1}{N} S_i \right) \subset \frac{1}{N} \mathbb{Z}^2. \quad (3.1)$$

(see Fig. 3.3.) Thus, for every  $v \in \mathbb{Z}^2$ ,  $S_i \in \{S_0, \dots, S_{M-1}\}$ , either  $(v + \frac{1}{N} S_i) \subset F$  or  $(v + \frac{1}{N} S_i) \cap F$  is empty.

We show that this  $F$  “encodes” the tiling properties of  $\Omega$ :



**Fig. 3.3.** The tile  $F(\Omega) \subset N^{-1}\mathbb{Z}^2$  for  $\Omega$  as in Fig. 3.1. A point  $(a, b) \in N^{-1}\mathbb{Z}^2$  is represented here by a square  $(a, b) + [0, \frac{1}{N}]^2$ . The colored squares correspond to the elements in  $F \subset N^{-1}\mathbb{Z}^2$ , while the white squares correspond to the elements in  $N^{-1}\mathbb{Z}^2$  that are not in  $F$ .

**Lemma 3.1.** *Let  $\Omega$  be an integer set, and let  $N = N(\Omega)$  and  $F = F(\Omega)$  be as constructed above. Then we have:*

- (i) *The set  $\text{Tile}(\Omega; \mathbb{R}^2)$  is empty if and only if the set  $\text{Tile}(F; \frac{1}{N}\mathbb{Z}^2)$  is empty.*
- (ii) *For every set  $0 \in T \subset \mathbb{R}^2$ ,  $T \in \text{Tile}(F; \frac{1}{N}\mathbb{Z}^2)$  if and only if  $T \in \text{Tile}(\Omega; \mathbb{R}^2)$  with  $T \subset \mathbb{Z}^2$ .*
- (iii) *If  $T_0, T_1$  are subsets of  $\mathbb{Z}^2$ , and  $\vec{v} \in \mathbb{Z}^2$ , then  $\Omega \oplus T_0 + \vec{v} = \Omega \oplus T_1$  if and only if  $F \oplus T_0 + \vec{v} = F \oplus T_1$ .*

**Proof.** Observe that (i) follows from (ii) and Lemma 2.2. Indeed, if  $\text{Tile}(\Omega; \mathbb{R}^2)$  is non-empty then by Lemma 2.2 there exists  $0 \in T \in \text{Tile}(\Omega; \mathbb{R}^2)$  such that  $T \subset \mathbb{Z}^2$ , thus, (ii) implies that  $T \in \text{Tile}(F; \frac{1}{N}\mathbb{Z}^2)$ . Conversely, if  $\text{Tile}(F; \frac{1}{N}\mathbb{Z}^2)$  is non-empty then it contains a set  $T$  such that  $0 \in T$ ; thus, again, by (ii),  $T \in \text{Tile}(\Omega; \mathbb{R}^2)$ .

We prove (ii). Suppose first that  $0 \in T \in \text{Tile}(\Omega; \mathbb{R}^2)$  and  $T \subset \mathbb{Z}^2$ . Then, for any  $P_i \in \{P_0, \dots, P_{M-1}\}$  and  $x \in \mathbb{Z}^2$  there exists a unique translate  $t_{i,x} \in T$  so that  $(x + P_i) \subset (\Omega + t_{i,x})$ . Note that by the definition of  $S_i$  we have  $(x + P_i) \subset (\Omega + t_{i,x})$  if and only if  $(x + \frac{1}{N}S_i) \subset (F + t_{i,x})$ . This implies that

$$T \oplus F = \bigsqcup_{\substack{x \in \mathbb{Z}^2 \\ i \in \{0, \dots, M-1\}}} (x + \frac{1}{N}S_i) = \frac{1}{N}\mathbb{Z}^2$$

i.e.,  $T \in \text{Tile}(F; (\frac{1}{N}\mathbb{Z})^2)$ .

Conversely, suppose that  $0 \in T \in \text{Tile}(F; \frac{1}{N}\mathbb{Z}^2)$ , then, by construction of  $S_0$  and since  $F$  is a tile of  $\frac{1}{N}\mathbb{Z}^2$ , we must have that the set

$$F_0 := \bigsqcup_{v \in \mathbb{Z}^2} \bigsqcup_{(x+P_0) \subset \Omega} (x + \frac{1}{N}S_0) \subset F$$

is non-empty. Thus, since  $T \in \text{Tile}(F; \frac{1}{N}\mathbb{Z}^2)$ , by construction of  $F$ , we have that if  $t, t'$  are two distinct elements of  $T$  such that  $F_0 + [0, \frac{1}{N}]^2 + t$  and  $F_0 + [0, \frac{1}{N}]^2 + t'$  share an edge then  $t' - t \in \mathbb{Z}^2$ . This means that every set in  $\text{Tile}(F; \frac{1}{N}\mathbb{Z}^2)$  must be contained in some coset of  $\mathbb{Z}^2$  in  $\frac{1}{N}\mathbb{Z}^2$ . In particular, as by assumption  $0 \in T$ , we must have  $T \subset \mathbb{Z}^2$ .

We now show that  $T \in \text{Tile}(\Omega; \mathbb{R}^2)$ . Since  $0 \in T \in \text{Tile}(F; \frac{1}{N}\mathbb{Z}^2)$ , for every  $S_i \in \{S_0, \dots, S_{M-1}\}$  and  $x \in \mathbb{Z}^2$ , there exists a unique  $t_{i,x} \in T$  such that  $(x + \frac{1}{N}S_i) \subset (F + t_{i,x})$ . Thus, by construction of  $S_i$ , we have that  $(x + P_i) \subset (\Omega + t_{i,x})$ ; hence

$$T \oplus \Omega = \bigsqcup_{\substack{x \in \mathbb{Z}^2 \\ i \in \{0, \dots, M-1\}}} (x + P_i) = \mathbb{R}^2$$

i.e.,  $T \in \text{Tile}(\Omega; \mathbb{R}^2)$ .

It remains to prove (iii). For  $i = 0, \dots, M-1$ , let

$$X_{i,0} := \{x \in \mathbb{Z}^2 : x + P_i \subset \Omega \oplus T_0 + \vec{v}\}, \quad X_{i,1} := \{x \in \mathbb{Z}^2 : x + P_i \subset \Omega \oplus T_1\},$$

and

$$Y_{i,0} := \{x \in \mathbb{Z}^2 : x + N^{-1}S_i \subset F \oplus T_0 + \vec{v}\}, \quad Y_{i,1} := \{x + N^{-1}S_i \subset F \oplus T_1\}.$$

As argued above, we have  $X_{i,j} = Y_{i,j}$  for every  $i = 1, \dots, M-1$  and  $j = 0, 1$ . We therefore have that  $X_{i,0} = X_{i,1}$  for every  $i = 0, \dots, M-1$  if and only if  $Y_{i,0} = Y_{i,1}$  for every  $i = 0, \dots, M-1$ . In other words,  $\Omega \oplus T_0 + \vec{v} = \Omega \oplus T_1$  if and only if  $F \oplus T_0 + \vec{v} = F \oplus T_1$ , as claimed.  $\square$

### 3.3.

Combining [Lemmas 3.1](#) with [\[6, Theorem 1.5\]](#), we can now prove [Theorem 1.1](#), as follows.

**Proof of Theorem 1.1.** Let  $\Omega$  be a rational polygonal tile and, by translation invariance, assume without loss of generality that  $0 \in V(\Omega)$ . Then, as  $V(\Omega) \subset \mathbb{Q}^2$  is finite, there exists  $n \in \mathbb{N}$  such that  $V(n\Omega) \subset \mathbb{Z}^2$ . Let  $\tilde{\Omega} := n\Omega$ . Note that  $T \in \text{Tile}(\Omega; \mathbb{R}^2)$  if and only if  $nT \in \text{Tile}(\tilde{\Omega}; \mathbb{R}^2)$  and  $T$  is doubly-periodic if and only if  $nT$  is doubly-periodic. Thus, we can equivalently prove the statement for  $\tilde{\Omega}$ .

Let  $N \in \mathbb{N} = N(\tilde{\Omega})$  and  $F = F(\tilde{\Omega}) \subset N^{-1}\mathbb{Z}^2$  be as in [Lemma 3.1](#). Then, by [Lemma 3.1\(i\)](#) and dilation invariance, as by assumption  $\text{Tile}(\tilde{\Omega}; \mathbb{R}^2)$  is non-empty,  $\text{Tile}(NF; \mathbb{Z}^2)$  is non-empty, i.e.,  $NF \subset \mathbb{Z}^2$  tiles  $\mathbb{Z}^2$ .

By [\[6, Theorem 1.5\]](#), we then have the existence of a doubly-periodic  $\tilde{T} \subset \mathbb{Z}^2$  such that  $\tilde{T} \in \text{Tile}(NF; \mathbb{Z}^2)$ . By Translation invariance, we can assume without loss of generality that  $0 \in \tilde{T}$ .

Letting  $T := N^{-1}\tilde{T} \subset N^{-1}\mathbb{Z}^2$  we have that  $0 \in T \in \text{Tile}(F; N^{-1}\mathbb{Z}^2)$ . By [Lemma 3.1\(ii\)](#), we then have that  $T \in \text{Tile}(\tilde{\Omega}; \mathbb{R}^2)$  and  $T \subset \mathbb{Z}^2$ . As  $T$  is doubly-periodic, this concludes the proof.  $\square$

3.4.

We now deduce [Corollary 1.1](#), as follows.

**Proof of Corollary 1.1.** We describe an algorithm that computes whether a given rational polygonal set tiles  $\mathbb{R}^2$  by translations: Let  $\Omega$  be a rational polygonal set. The algorithm first computes the number  $N$  and the finite set  $F \subset N^{-1}\mathbb{Z}^2$ , as described in the proof of [Lemma 3.1](#). By [Lemma 3.1\(i\)](#),  $\Omega$  tiles  $\mathbb{R}^2$  if and only if  $F$  tiles  $N^{-1}\mathbb{Z}^2$ .

The algorithm then computes all the ways to tile finite regions in  $N^{-1}\mathbb{Z}^2$  by translated copies of  $F$  (“patches”), with growing diameter. If  $F$  does not tile, then, by compactness, the largest possible diameter of any patch formed by translated copies of the  $F$  must be finite. Thus the algorithm will detect that  $F$ , and thus  $\Omega$ , is not a tile after finitely many steps. If  $F$  tiles, then a periodic tiling exists; and so, after finitely many steps the algorithm will detect a tiling of a patch that satisfies periodic boundary conditions (i.e., the patch a fundamental domain of some lattice in  $N^{-1}\mathbb{Z}^2$ ), and will then output that  $F$ , and thus  $\Omega$  is a tile.  $\square$

## 4. Structure of tilings by a rational tile

In this section, using [Lemma 3.1](#) and analyzing the structure of discrete tilings by  $F(\Omega)$ , and its connection to the structure of tilings by the original rational polygonal set, we prove [Theorem 1.2](#).

We begin by recalling the discrete analog of [Theorem 1.2](#), which was established in [[6](#), Theorem 1.4].

**Theorem 4.1** (*Tilings in  $\mathbb{Z}^2$  are Weakly-Periodic*). *Let  $F \subset \mathbb{Z}^2$  be a tile. Then there exist  $c = c(F)$  and  $d = d(F)$  such that there are  $m < c$  pairwise incommensurable vectors  $h_1, \dots, h_m \in \mathbb{Z}^2 \setminus \{0\}$  with  $|h_j| \leq d^m$  such that every tiling  $T$  of  $\mathbb{Z}^2$  by  $F$  admits a decomposition*

$$T = T_1 \sqcup \dots \sqcup T_m \quad (4.1)$$

*such that  $T_j$  is  $\langle h_j \rangle$ -periodic. In fact, there exists a lattice  $\Lambda \subset \mathbb{Z}^2$  such that for every  $x \in \mathbb{Z}^2$  there is  $1 \leq j \leq m$  such that the set  $(\Lambda + x) \cap T$  is  $\langle h_j \rangle$ -periodic.*

**Proof.** See [[6](#), Theorem 1.4].  $\square$

Observe that by combining [Theorem 4.1](#) and [Lemma 3.1\(ii\)](#), we can immediately deduce that any integer tiling in  $\text{Tile}(\Omega; \mathbb{R}^2)$  is weakly periodic. The main barrier to transferring [Theorem 1.2](#) from its integer counterpart is the possible *sliding* components obstructing periodicity. The proof of [Theorem 1.2](#) consists of showing that the decomposition of [Theorem 4.1](#) is compatible with the sliding decomposition in [Lemma 2.5](#), in the sense that each of the singly-periodic pieces in the decomposition (4.1) can be further decomposed into pieces, each contained in a sliding component of the partition (2.1). In order to show that, we introduce the notion *earthquake*<sup>1</sup> decomposition of discrete tilings.

<sup>1</sup> The terminology of *earthquakes* also appears in [[9](#)], but there it is used in the context of continuous tilings and the definition (although closely related) is slightly different.

#### 4.1. Earthquake decomposition

**Definition 4.1** (*Earthquake Decomposition of a Discrete Tiling*). Let  $F \subset N^{-1}\mathbb{Z}^2$  be finite. Suppose  $T \in \text{Tile}(F; N^{-1}\mathbb{Z}^2)$ , and  $\vec{v} \in N^{-1}\mathbb{Z}^2$  then we define the  $\vec{v}$ -discrete earthquake decomposition of  $T$  by  $F$  as a decomposition

$$T = \bigsqcup_{P \in \text{Plates}(T, \vec{v})} P,$$

where:

- (i) For each set  $P \in \text{Plates}(T, \vec{v})$  the set  $F \oplus P$  is  $\mathbb{Z}\vec{v}$ -periodic.
- (ii) If  $P \in \text{Plates}(T, \vec{v})$  and  $P' \subset P$  satisfies (i) then  $P' \in \{P, \emptyset\}$ .

The set<sup>2</sup>  $\text{Plates}(T, \vec{v})$  is the set of *tectonic plates* arising from earthquakes in the direction  $\vec{v}$ .

The following Lemma shows that the discrete decomposition into singly-periodic sets is compatible with a discrete earthquake decomposition.

**Lemma 4.2.** *Let  $F \subset N^{-1}\mathbb{Z}^2$  be a tile of  $N^{-1}\mathbb{Z}^2$ ,  $T \in \text{Tile}(F; N^{-1}\mathbb{Z}^2)$  and  $\text{Plates}(T, \vec{v})$  be the  $\vec{v}$ -earthquake decomposition of  $T$ . Then there are pairwise incommensurable vectors  $h_1, \dots, h_m \in \mathbb{Z}^2 \setminus \{0\}$  such that for each  $P \in \text{Plates}(T, \vec{v})$  there is a decomposition  $P = \bigsqcup_{j=1}^m P_j$  such that  $P_j$  is  $\langle h_j \rangle$ -periodic for every  $1 \leq j \leq m$ .*

**Proof.** Let  $\tilde{F} := NF$ . Then, as  $T \in \text{Tile}(F; N^{-1}\mathbb{Z}^2)$  if and only if  $NT \in \text{Tile}(\tilde{F}; \mathbb{Z}^2)$ , it suffices to prove the statement for any  $T \in \text{Tile}(\tilde{F}; \mathbb{Z}^2)$ . By translating, we may assume without loss of generality that  $\tilde{F}$  contains the origin. Fix some  $P \in \text{Plates}(T, \vec{v})$ , and define  $T^k \in \text{Tile}(\tilde{F}, \mathbb{Z}^2)$  as

$$T^k := P \sqcup \bigsqcup_{P' \in \text{Plates}(T, \vec{v}) \setminus \{P\}} P' + k\vec{v}.$$

Note that  $T^k$  is obtained from  $T$  by an *earthquake* that shifts all the plates but  $P$  by  $k\vec{v}$ , and that  $P \in \text{Plates}(T^k, \vec{v})$  for all  $k$  by the construction of  $T^k$ .

By Theorem 4.1, there is  $M \in \mathbb{N}$  and pairwise incommensurable vectors  $\tilde{h}_1, \dots, \tilde{h}_m \in M\mathbb{Z}^2$  with the property that for every coset  $\Gamma = (x + M\mathbb{Z}^2) \in \mathbb{Z}^2/M\mathbb{Z}^2$  there is  $1 \leq j = j(k, \Gamma) \leq m$  such that  $\Gamma \cap T^k$  is  $\langle \tilde{h}_j \rangle$ -periodic. By adding a vector (if needed) we can assume without loss of generality that  $\tilde{h}_1$  is parallel to  $\vec{v}$ , and by enlarging  $\tilde{h}_1$  (if needed) we can further assume that  $\tilde{h}_1 = \ell M\vec{v}$  for some  $\ell \in \mathbb{N}$ .

Let  $\Gamma = (x + M\mathbb{Z}^2) \in \mathbb{Z}^2/\Lambda$ . For each  $k \in \mathbb{Z}$ , let  $h(k, \Gamma) := \tilde{h}_{j(k, \Gamma)}$ . By the pigeonhole principle, there exist  $0 \leq k < k' \leq m$  such that  $h(Mk, \Gamma) = h(Mk', \Gamma)$ . We let  $0 \leq k_\Gamma \leq m - 1$  be the smallest such that there exists  $k_\Gamma < k' \leq m$  with  $h(Mk_\Gamma, \Gamma) = h(Mk', \Gamma)$  and set  $k_\Gamma < k'_\Gamma \leq m$  to be the smallest among all possible  $k'$  with this property. Let  $h(\Gamma) := h(Mk_\Gamma, \Gamma)$ .

<sup>2</sup> Note that a decomposition satisfying (ii) is the maximal one amongst all decompositions satisfying (i) and therefore always exists and is unique.

For each  $j = 2, \dots, m$ , let  $\mathcal{P}_j \subset P$  be the set of points  $x$  in  $P$  such that  $h(x + M\mathbb{Z}^2) = \tilde{h}_j$ , and let  $P_j \subset \mathcal{P}_j$  be the set of points  $x$  in  $\mathcal{P}_j$  such that the coset  $x + \tilde{h}_j\mathbb{Z}$  is entirely contained in  $P$ . Clearly, for each  $2 \leq j \leq m$ , the set  $P_j$  is  $\langle \tilde{h}_j \rangle$ -periodic.

We will show that there exists  $n \in \mathbb{N}$ , independent of the choice of  $P \in \text{Plates}(T, \vec{v})$ , such that each of the sets  $\tilde{P}_j := \mathcal{P}_j \setminus P_j$ ,  $j = 2, \dots, m$ , is  $\langle n\tilde{h}_1 \rangle$ -periodic. Then, the result will follow by setting  $h_1 := n\tilde{h}_1$ ,  $h_j := \tilde{h}_j$  for  $j = 2, \dots, m$ , and

$$P_1 := P \setminus \left( \sqcup_{j=2}^m P_j \right) = \{x \in P : h(x + M\mathbb{Z}^2) = \tilde{h}_1\} \cup \left( \sqcup_{j=2}^m \tilde{P}_j \right),$$

which is  $\langle h_1 \rangle$ -periodic. It thus suffices to show for each  $j = 2, \dots, m$  that the set  $\tilde{P}_j$  is  $\langle n\ell M\vec{v} \rangle$ -periodic with  $n = \prod_{\Gamma \in \mathbb{Z}^2/M\mathbb{Z}^2} (k_\Gamma - k'_\Gamma)$ .

Let  $2 \leq j \leq m$ . Suppose that  $x \in \tilde{P}_j$  and let  $\Gamma := x + M\mathbb{Z}^2$ . Then, by the definition of  $\tilde{P}_j$ , there is  $k \in \mathbb{Z}$  such that the point  $y := x + k\tilde{h}_j \in \Gamma$  is in  $T \setminus P$ . Observe that as  $\tilde{h}_j \in M\mathbb{Z}^2$  we have  $\Gamma = x + M\mathbb{Z}^2 = y + M\mathbb{Z}^2$ . Thus, as by the definition of  $(k_\Gamma, k'_\Gamma)$  both  $T^{Mk_\Gamma} \cap \Gamma$  and  $T^{Mk'_\Gamma} \cap \Gamma$  are  $\langle \tilde{h}_j \rangle$ -periodic and  $x \in T^{Mk_\Gamma} \cap T^{Mk'_\Gamma} \cap \Gamma$ , we have that

$$y = x + k\tilde{h}_j \in T^{Mk_\Gamma} \cap T^{Mk'_\Gamma} \cap \Gamma.$$

Therefore, the point  $z := y - M k_\Gamma \vec{v}$  is in  $T \cap \Gamma$ . In turn, we obtain that the point

$$z' := z + M k'_\Gamma \vec{v} = y - M(k_\Gamma - k'_\Gamma)\vec{v}$$

is in  $T^{Mk'_\Gamma} \cap \Gamma$ ; and so, by  $\langle \tilde{h}_j \rangle$ -periodicity of  $T^{Mk'_\Gamma} \cap \Gamma$ , we have that the point

$$x' := z' - k\tilde{h}_j = x - M(k_\Gamma - k'_\Gamma)\vec{v}$$

is also in  $T^{Mk'_\Gamma} \cap \Gamma$ . Observe that since  $x \in \tilde{P}_j \subset P$  and  $P \in \text{Plates}(T, \vec{v})$ , the point  $x'$  is in  $P$ , and so, as  $x' \in \Gamma$ , we have that  $x' \in \mathcal{P}_j$ . Moreover, since  $y \in T \setminus P$  we have  $y \notin F + P$ , and so, by the  $\vec{v}$ -periodicity of  $F + P$ , we obtain that neither  $z$  nor  $z'$  is in  $F + P$  and, in particular,  $z' = x' + k\tilde{h}_j \notin P$ . This shows that  $x' = x - M(k_\Gamma - k'_\Gamma)\vec{v}$  is in  $\tilde{P}_j$ . Similarly, one can show that the point  $x + M(k_\Gamma - k'_\Gamma)\vec{v}$  is in  $\tilde{P}_j$ , by simply repeating the above argument with the roles of  $k_\Gamma$  and  $k'_\Gamma$  being swapped. We therefore conclude that  $\tilde{P}_j$  is  $\langle n\tilde{h}_1 \rangle$ -periodic with  $n = \prod_{\Gamma \in \mathbb{Z}^2/M\mathbb{Z}^2} (k_\Gamma - k'_\Gamma)$ .  $\square$

#### 4.2.

Observe that using [Lemma 4.2](#), to complete the proof of [Theorem 1.2](#), it only remains to show that the discrete earthquake decomposition in [Lemma 4.2](#) is compatible with the continuous earthquake structure in  $\Omega$ .

**Proof of Theorem 1.2.** Let  $\Omega$  be a rational polygonal tile, by translation invariance, we can assume without loss of generality that  $0 \in V(\Omega)$ . Let  $T \in \text{Tile}(\Omega; \mathbb{R}^2)$ . By [Lemma 2.5](#), there exists a direction  $\vec{v} \in \mathbb{R}^2 \setminus \{0\}$  such that  $T$  can be decomposed as  $T = \bigsqcup_{i \in I} Q_i$ , where for each  $i \in I$  the set  $\tilde{\Omega} \oplus Q_i$  is  $\vec{v}\mathbb{R}$ -invariant. By invariance under affine transformations, we may assume that  $\vec{v} = (0, 1)$ .

As  $V(\Omega) \subset \mathbb{Q}^2$  is finite, there exists  $n \in \mathbb{N}$  such that  $V(n\Omega) \subset \mathbb{Z}^2$ . Let  $\tilde{\Omega} := n\Omega$ . Note that  $T \in \text{Tile}(\Omega; \mathbb{R}^2)$  if and only if  $nT \in \text{Tile}(\tilde{\Omega}; \mathbb{R}^2)$  and  $T$  is weakly-periodic if and only if  $nT$  is weakly-periodic. Thus, we can equivalently prove the statement for  $\tilde{\Omega}$ .

Let  $T' \in \text{Tile}(\tilde{\Omega}; \mathbb{R}^2)$  be the tiling that arises from merging the  $\sim_{\tilde{\Omega}}$  components of  $T$  by sliding, as described in Lemma 2.7. Then, by construction, there are coefficients  $s_i \in \mathbb{R}$  such that

$$T' = \bigsqcup_{i \in I} Q_i - s_i(0, 1),$$

and by Lemma 2.4  $T'$  is a subset of  $\mathbb{Z}^2$ , i.e.,  $T$  is an integer tiling by  $\tilde{\Omega}$ .

Let  $F = F(\tilde{\Omega})$  be as in (3.1). Then, by Lemma 3.1(ii)  $T' \in \text{Tile}(F; N^{-1}\mathbb{Z}^2)$ , and by Lemma 3.1(iii), if for a subset  $S$  of  $T'$  the set  $\tilde{\Omega} \oplus S$  is  $(0, 1)\mathbb{R}$ -invariant, then  $F \oplus S$  is  $(0, 1)\mathbb{Z}$ -invariant. Therefore, the decomposition of  $T'$  into  $(0, 1)$ -discrete earthquakes by  $F$  is a refinement of the decomposition of  $T'$  into continuous earthquakes by  $\tilde{\Omega}$ . In other words, for  $i \in I$  there exist sets  $P_{i,j} \subset \mathbb{Z}^2$ ,  $j \in J_i$  such that  $Q_i - s_i(0, 1) = \bigsqcup_{j \in J_i} P_{i,j}$  and the  $(0, 1)$ -earthquake decomposition of  $T'$  by  $F$  is  $T' = \bigsqcup_{i,j} P_{i,j}$ .

Thus, by Lemma 4.2, for each  $i \in I$ ,  $j \in J_i$  there exists  $h_{i,j} \in \mathbb{Z}^2$  such that the set  $P_{i,j}$  is  $\langle h_{i,j} \rangle$ -periodic, and the vectors  $h_{i,j}$  take finitely many possible values in  $\mathbb{Z}^2$ . Therefore, each  $Q_i - s_i(0, 1) = \bigsqcup_{j \in J_i} P_{i,j}$  is weakly-periodic. By translation invariance, this implies that each  $Q_i$  is weakly-periodic with the same periods. Therefore, as  $T = \bigsqcup_{i \in I} Q_i$ , the tiling  $T$  is weakly-periodic. We thus have Theorem 1.2.  $\square$

## CRediT authorship contribution statement

**Jaume de Dios Pont:** Conceptualization, Visualization, Writing – original draft, Writing – review & editing. **Jan Grebík:** Conceptualization, Visualization, Writing – original draft, Writing – review & editing. **Rachel Greenfeld:** Conceptualization, Methodology, Project administration, Visualization, Writing – original draft, Writing – review & editing. **José Madrid:** Conceptualization, Writing – original draft, Writing – review & editing.

## Declaration of competing interest

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## Data availability

No data was used for the research described in the article.

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