education sciences

MDPI

Article

Teacher Growth Through Professional Development Centered on the Teaching for Robust Understanding Framework

Eileen Murray [1], Helene S. Leonard [2,*] and Victoria D. Bonaccorso [2]

[1] Urso Educational Consulting, Bloomfield, NJ 07003, USA; emurray@ursoeducationalconsulting.com
[2] Mathematics Department, Montclair State University, Montclair, NJ 07043, USA; victoria.bonaccorso@gmail.com
* Correspondence: leonardh2@montclair.edu

Abstract: Research on teacher learning has continually defined tenants of effective professional development. However, sweeping overhauls and mandates have not yielded the intended results. Instead, transforming approaches to teaching can and should be achieved through innovative and incremental changes that build upon teachers' existing expertise. Informed by Communities of Practice theory, we employed case study methodology to describe, explain, and assess the experiences of one middle school mathematics teacher's longitudinal participation in an innovative and incremental continuous PD model focused on the Teaching for Robust Understanding (TRU) framework. Based on classroom observations and interview data, our findings show evidence of TRU-aligned changes in teaching practice as a result of years of participation in the PD model. These findings strengthen the call for innovative and incremental PD programs.

Keywords: mathematics professional development; teaching for robust understanding; teacher learning; incremental professional development

Academic Editors: Samuel Otten and Luca Tateo

Received: 31 August 2024
Revised: 1 November 2024
Accepted: 23 December 2024
Published: 27 December 2024

Citation: Murray, E. C., Leonard, H. S., & Bonaccorso, V. D. (2025). Teacher Growth Through Professional Development Centered on the Teaching for Robust Understanding Framework. *Education Sciences*, *15*(1), 18. https://doi.org/10.3390/educsci15010018

1. Introduction

> *Honestly, I feel like my teaching has improved. I am thinking differently when I plan, moving differently when I teach, requiring more of my students, and truly laying the foundation for myself and my students' expectations of learning and understanding mathematics in the classroom.—PLC Participant*

Imagine if every mathematics teacher had a professional learning experience that impacted their approach to teaching in this way. What might this mean for our classrooms, our students, and the future of mathematics education? The possibilities are as inspiring as they are profound. Transforming approaches to teaching does not mean ambitious overhauls of a teacher's practice, but can and should be achieved through innovative and incremental changes that build upon teachers' existing expertise.

This paper presents a powerful, innovative, incremental professional development model designed to provide opportunities for continuous growth and reflection by encouraging participants to gradually integrate approaches into their practice. In addition, this paper will share current findings of the model's effectiveness and the story of one teacher's changes in practice as they engaged with this PD model.

2. Background

2.1. Effective Professional Development

The goal of effective professional development (PD) is teacher learning, leading to enhanced student learning (Desimone, 2011, 2023). Over the past several decades, researchers have studied various aspects of PD to determine its effectiveness by examining changes in teacher engagement during PD (e.g., Dyer et al., 2023; Walkoe, 2014), classroom practices (e.g., Alles et al., 2018; Santagata & Bray, 2016; Thurm & Barzel, 2020), and student achievement (e.g., Jacob et al., 2017; Kennedy, 1998; Prast et al., 2018). Although there is no consensus on the elements required for a PD to be effective (e.g., Bayar, 2014; Darling-Hammond et al., 2017; Desimone, 2011, 2023; Garet et al., 2001; Timperley et al., 2007), many researchers discuss common elements that make for powerful professional learning. We have synthesized this work and focus on five characteristics of effective PD that have been shown to positively impact teacher practice and student performance: active learning, coherence, sustained duration, collective participation, and content-focused.

Effective PD should engage teachers in active learning through collective participation over a sustained duration (Darling-Hammond et al., 2017; Garet et al., 2001; Timperley et al., 2007). Active learning directly involves teachers in activities related to their classrooms and students (Darling-Hammond et al., 2017), providing them with opportunities "to observe, receive feedback, analyze student work, or make presentations, rather than passively listening to lectures" (Desimone & Garet, 2015, p. 253). Such activities could involve collaboratively creating, modeling, or reflecting on lessons or analyzing video clips from actual mathematics classrooms, which can empower teachers to acquire new instructional strategies (R. Smith et al., 2020).

In order to engage in active learning, teachers should collectively participate over a sustained period of time. Collective participation in PD refers to teachers working together, often sharing a common goal or interest. Effective PD provides collective experiences for groups of teachers with similar needs and challenges (Desimone & Garet, 2015), such as those from the same grade, subject, or school. This collective participation is not just a process, but a means to build interactive learning communities (Desimone, 2011). These communities enable collaboration, integration, and targeting specific student needs (R. Smith et al., 2020), making teachers feel connected and engaged. One study of PD with mathematics teachers illustrated the power of collective participation through the concept of 'noticing' in a video club (Van Es & Sherin, 2008). The participants—mathematics teachers from the same district, teaching fourth and fifth grades—shared and analyzed video clips of their teaching. This process taught teachers to notice and interpret students' mathematical thinking, and through collective participation, teachers increasingly engaged in noticing practices around students' mathematical thinking (Van Es & Sherin, 2008).

PD that occurs as one-day or short-term sessions often does not produce the desired outcomes (e.g., Darling-Hammond et al., 2017). Instead, PD should be conducted over multiple sessions in an extended period to allow teachers to implement and reflect on new strategies between sessions, which allows for deeper reflection and ample time to change practice. Such sustained duration can be multiple PD sessions focused on the same learning goals, implemented regularly over weeks, months, or even years. Koellner and Jacobs (2015) studied a year-long PD program focused on middle school teachers' mathematical knowledge, instructional practices, and student achievement. Their results showed significant gains in the teachers' mathematical knowledge for teaching (Hill et al., 2008), the richness of their lessons, and their ability to work effectively with students. Students of participating teachers also scored higher on statewide assessments than students of nonparticipating teachers.

Effective PD should also focus on content and coherence. Coherence in PD includes aligning PD with other aspects of a teacher's profession, such as "the school curriculum and goals, teacher knowledge and beliefs, the needs of students, and school, district, and state reforms and policies" (Desimone & Garet, 2015, p. 253). Coherent PD builds on teachers' prior learning, focuses on relevant aspects of their curriculum, and encourages sustained, collaborative communication with colleagues in similar contexts. PD alignment with broader policy trends and research has been identified as a key factor in PD's impact on student achievement (Timperley et al., 2007) and teacher practice (Santagata & Bray, 2016). For instance, Santagata and Bray (Santagata & Bray, 2016) observed positive changes in participants' teaching practices due to a coherent PD model that used classroom video recordings focused on students' misconceptions about fractions. Teachers were initially unaware of some student misconceptions, but recognizing these led to changes in their instructional strategies. Teachers responded more effectively to students' errors, helping students engage with the underlying mathematical ideas and increasing peer collaboration to explore mathematical concepts. The coherence of the PD with teachers' classroom experiences and curriculum was found to positively impact teaching practice.

In addition, Timperley et al. (2007) determined that "the content of the learning was far and away the most influential factor in determining whether professional learning would result in improved mathematical outcomes for students" (p. 91). Timperley et al. (2007) also found that engaging in various learning activities is crucial for PD participants to learn content effectively. Effective PD should therefore center on the subject matter and how students learn and engage with specific content. Situating content within teachers' classroom contexts enables them to deepen their understanding of the subject matter (Desimone, 2011; Desimone & Garet, 2015). In content-focused PD, teachers may explore new curricula, analyze student work, or study specific pedagogical approaches (Darling-Hammond et al., 2017). Borko et al. (2008) studied a PD model centered around analyzing student video cases, where each session focused on a different mathematical task. In each session, teachers completed the same tasks their students would, then analyzed videos of students working on these tasks. The researchers found that focusing on specific mathematical content (e.g., proportional reasoning, ratios, rates of change) helped teachers better understand students' solution strategies, even when these solutions diverged from the expected approaches. The participants also engaged in increasingly reflective and productive conversations, focusing more on mathematical content as the PD progressed. Teachers reported that the content topics were meaningful, motivating them to improve their practice and better serve their students.

The elements of effective PD described here are not mutually exclusive; they work in tandem. The various studies on the effective characteristics of PD frequently highlight a particular element but also display other effective elements. Therefore, rather than pinpointing which element is responsible for the effectiveness of a PD program, these five elements should be viewed holistically as a guide for designing effective PD.

2.2. Incremental, Innovative Professional Development

Incremental, innovative professional development is ongoing, continuous teacher learning that involves gradual, step-by-step improvement in teaching practices and skills. This approach segments the teacher development process into small, manageable phases or increments. Each phase builds upon the previous one, ensuring each step is functional and contributes to overall progress. Incremental, innovative PD should embody the characteristics of effective PD, as described above, and target specific skills or knowledge areas within a cohesive framework for teaching and learning. By focusing sessions in this way, educators can implement new strategies gradually, reflect on their effectiveness, and make adjustments before moving on to the next increment. This process mirrors the effective practice of collaborative reflective teaching cycles (e.g., Murray, 2011, 2015; M. S. Smith, 2001) where teachers plan, implement, and reflect on lessons that help them implement high-quality instruction aligned with characteristics of powerful learning environments. Incremental, innovative PD contrasts traditional PD models that involve longer, less frequent sessions covering broader topics aimed at ambitious instructional changes, which have not been shown to significantly change teacher knowledge or student achievement (e.g., Garet et al., 2010).

Incremental, innovative PD is needed to provide continuous opportunities for learning and application, which can lead to sustained improvements in teaching practices and student outcomes. Star (2016) emphasizes the need for teachers to enhance rigor and coherence in response to new standards. While this requires significant changes in knowledge and practice, Star argues broad, sweeping reforms are unrealistic. Teachers face real challenges to change, such as the pressure for teacher-led instruction and the constraints of conventional curricula. Therefore, incremental changes (e.g., improving questioning strategies) can be gateway practices to more extensive reforms. Small, manageable changes to existing practices are easier for teachers to implement and build upon and are thus more likely to be adopted, leading to more significant changes in the future. For example, a focus on multiple mathematical strategies allows teachers to shift from answer correctness to exploring methods and their applications. This small, manageable change helps teachers and students engage in deeper mathematical discussions and develop a richer understanding of problem-solving. Overall, Star argues incremental changes may lead to substantial, lasting improvements in practice. Otten et al. (2022) also argue for incremental, innovative PD to redefine success, rethink starting points, and provide new mechanisms for sustained change at scale. Success in incremental PD may be seen through the high levels of teacher buy-in coupled with modest, impactful improvements in teaching practices. According to these researchers, practices at the root of incremental, innovative PD are more likely to be adopted because they are aligned with current practices and routines while acknowledging real-world constraints of teaching. Additionally, the success of incremental, innovative PD comes from teachers' ability to implement meaningful improvements in their teaching practices because, unlike ambitious PD, such practices can be small adjustments.

Rooted in the understanding of the real-world constraints teachers face (e.g., limited planning time, expectations from students, parents, and administrators), incremental PD is more realistic and effective. Ambitious PD is often at odds with these constraints, and may not be built on teachers' commonplace, daily, and comfortable pedagogical practices. Therefore, rather than overhauling teaching methods, incremental PD allows teachers to make small improvements to existing practices (e.g., comparing worked examples or adding a novel problem to a problem set; (Otten et al., 2022)).

These small, manageable changes are a key mechanism for sustained change within a teacher's daily routine. They can bring modest benefits without requiring major pedagogical shifts and are thus scalable across educational contexts (Otten et al., 2022). Incremental, innovative PD honors teachers' existing expertise by avoiding deficit thinking. Teachers are positioned as experts in their field, and incremental PD acknowledges the value of their current practices and routines.

2.3. AIM-TRU Professional Development Model

This article describes our incremental, innovative PD model, Analyzing Instruction in Mathematics Using the Teaching for Robust Understanding Framework (AIM-TRU). AIM-TRU is a PD model anchored in communities of practice and aligned with the Teaching for Robust Understanding (TRU) (Schoenfeld, 2015) framework.

Communities of practice (CoPs) are "groups of people who share a concern or a passion for something they do and learn how to do it better as they interact regularly" (Wenger-Trayner & Wenger-Trayner, 2015, p. 2). CoPs mutually engage in an activity, are connected by a joint enterprise, and engage with a shared repertoire of resources (Wenger, 1998a, 1998b). Participants in CoPs may be newcomers and more knowledgeable others, and may move from peripheral to full participation in the community (Kelly, 2006; Lave & Wenger, 1991). In the context of Blinded teacher communities, a participant new to the PD may begin by observing the norms of interaction and discourse within the various PD activities and interact congenially at first. Over time, this peripheral participation can shift towards full participation as they learn the community norms and can congenially contribute to discussions.

Learning within CoPs is described by moving from peripheral to full participation (Kelly, 2006; Lave & Wenger, 1991). Evidence of this learning is seen as changes in participation and reification (Wenger, 1998a), where participation is the experiential process of taking part in the community, and reification shapes that experience by transforming it into a more concrete form. Blinded teacher communities allow members to address challenges that arise in their instructional practice and provide space for them to reflect and create professional narratives through dialogue with others. This dialogue represents the patterns of participation, and changes within these patterns are reified by specific actions, including where participants choose to focus to enhance their understanding of powerful lessons and best teaching practices.

To support and focus participant focus, we anchor our PD model with the TRU framework developed by researchers at the University of Berkeley in collaboration with school districts and other partners. TRU provides an exhaustive, research-based response to the question: What are the attributes of equitable and robust learning environments—environments in which all students are supported in becoming knowledgeable, flexible, and resourceful disciplinary thinkers (Schoenfeld et al., 2023, p. 3)? By rooting our PD in this framework of teaching and learning, the model's effectiveness improves because teachers have a common language for thinking about and discussing the work of teaching. Because teaching is a deeply personal act that connects to an individual's culture, dispositions, beliefs, and histories (e.g., Star, 2016; Voss et al., 2013), a common language allows teachers to collaborate across differences and provides a viewing window into classrooms for ongoing reflective processes.

TRU acknowledges the reality of teachers' constraints and begins with familiar elements of instruction. By rooting our model in TRU, teachers can build upon their practices gradually, leading to significant and lasting changes in instruction. This also helps educators improve their ability to plan and reflect as they think critically about instructional change through the TRU dimensions. Additionally, planning does not need to be overly ambitious. Our model and guiding questions for teachers are flexible, offering opportunities for gradual yet meaningful improvements in instruction.

In addition to the teaching framework, AIM-TRU engages mathematics teachers in investigating and enacting formative assessment lessons (FALs) to deepen instructional knowledge and support incremental shifts in practice aligned with the TRU framework. FALs are high-quality, research-based lessons developed by the TRU team and over 1000 partnering classrooms, which support teachers in creating TRU-aligned classroom experiences (see map.mathshell.org). Grounded in tenets of effective professional development (e.g., Desimone & Garet, 2015; Garet et al., 2001; Timperley et al., 2007) described above, AIM-TRU provides opportunities for teacher communities to collectively generate professional knowledge for teaching and learning mathematics by analyzing video of FALs enacted in real-world classrooms and generating possible teacher moves.

Teachers participating in AIM-TRU use the five inter-related dimensions of classroom teaching that are necessary and sufficient to produce equitable environments and support rich learning opportunities for all students (Wenger, 1998a): the Content (CT); Cognitive Demand (CD); Equitable Access (EA); Agency, Ownership, and Identity (AOI); and Formative Assessment (FA) (see Table 1).

Table 1. The Teaching for Robust Understanding Framework.

Dimension	Description
The Content	Effective mathematics classrooms rely on rich mathematical content that emphasizes important concepts presented coherently. This content should reflect the logical structure of mathematical ideas, promoting meaningful connections (e.g., Council of Chief State School Officers & National Governors Association, 2010).
Cognitive Demand	The complexity of mathematical tasks students engage with influences their depth of understanding. High cognitive demand tasks encourage productive struggle and conceptual development, leading to enhanced learning opportunities and higher achievement (e.g., Jackson et al., 2015).
Equitable Access	Ensuring all students have access to rich mathematical content and instruction is crucial for student success. Differential outcomes should be reframed as differential opportunities to learn, with teachers employing strategies like multiple entry points to promote equitable engagement (e.g., Hodge & Cobb, 2019).
Agency, Ownership, & Identity	Students' mathematical identities are closely tied to their participation in the classroom. Teachers can shape positive identities by fostering agency in students as mathematical meaning-makers, encouraging evaluation of mathematical arguments, and establishing norms centered around mathematical reasoning (e.g., Boaler & Greeno, 2000).
Formative Assessment	Effective use of formative assessment in the classroom supports positive learning outcomes. Formative assessment informs instruction and can involve tasks or real-time interactions. This type of assessment promotes a growth mindset, metacognitive behaviors, and teacher adjustments to enhance teaching and learning (e.g., Black & Wiliam, 1988; Granberg et al., 2021).

Each meeting of the teacher community uses the following components: (a) unpacking the big mathematical ideas in an FAL, (b) making observations about a video case demonstrating students' mathematical thinking while engaging in an FAL, and (c) sets of reflective discussion questions about the video case and the mathematics, based on one of the TRU dimensions (see Figure 1). During the reflective discussion of the video analysis, teachers often co-construct understandings about TRU-aligned teaching practices through dialogue (Leonard et al., 2022).

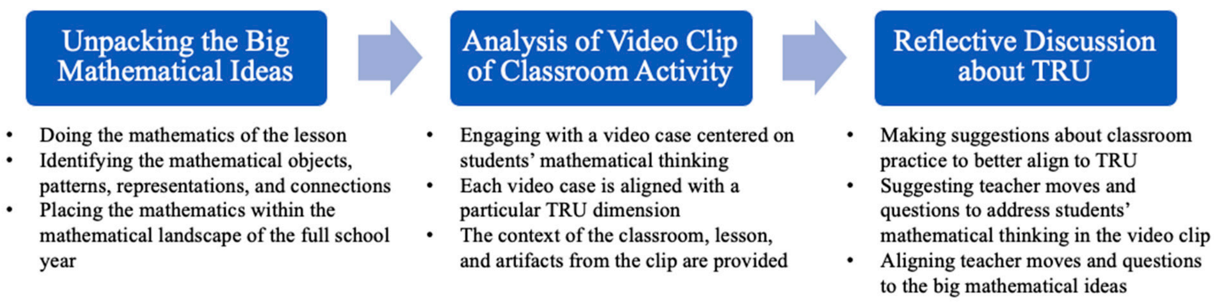

Figure 1. AIM-TRU PD model components.

AIM-TRU sessions layer the TRU framework throughout each component, making engagement accessible for new and returning teacher participants. Embodying the idea of "low floor, high ceiling" mathematics tasks, teachers with no prior TRU or FAL knowledge can actively participate in an FAL's cooperative hands-on mathematical tasks. Through intentional scaffolding with tools for the TRU frameworks, teachers can more easily enter discourse about TRU-aligned teaching as they draw connections to student thinking and their classroom landscape. Focusing on one dimension of TRU, teachers engage in a video case study of students' mathematical thinking and the teacher moves. After watching a video of students working through the task(s), participants discuss possible alterations in classroom practice and/or teacher moves that would lead to stronger connections to the big mathematical ideas and a more TRU-aligned classroom using a tool created specifically for this purpose.

As participating teachers engage in repeated sessions with the same structure, they are encouraged to implement an FAL in their classroom. Teachers may then choose to record the implementation of the lesson and choose a video clip for the teacher community to investigate. In our research, we found teachers who participated over multiple years learned about FALs, implemented lessons, collectively reflected on the lesson, and iterated the process (e.g., DiNapoli et al., 2023). Some teacher participants' roles evolved within the model as they transitioned from participant to facilitator.

3. Methods

This work is part of a larger research study where we investigated how teacher learning within CoPs leads to change in classroom practice. The full study was conducted across three sites in the northeastern and midwestern areas of the United States. The CoP at each site included approximately 30 members across three professional learning communities (PLCs), each with 8–10 secondary teachers (student ages 12–18). We collected data across four years for the study, and this paper presents a subset of data from one of the northeast sites.

For the full data set, we use frame analysis (Goffman, 1974) with transcripts of PLC conversations following the classroom video. In other papers (e.g., DiNapoli et al., 2023; Leonard et al., 2022), we note changes in participation and reification patterns, which indicate meaningful learning over time. Specifically, we found teacher participants moved from congenial conversations, generally agreeable exchanges where teachers were polite and avoided conflict (e.g., Selkrig & Keamy, 2015), towards collegial conversations, where they constructively disagreed by suggesting and arguing for something different. These shifts prompted participating teachers to more frequently suggest TRU-aligned teaching approaches and to propose solutions to problems, along with the rationale behind those moves. This movement towards TRU-aligned teaching approaches and the increased collegiality is evidence of teacher learning in the CoPs and participants' reification of the TRU framework, thus illustrating the effectiveness of our PD model.

For this paper, we apply evaluative case study methodology (Merriam, 1998) to describe, explain, and assess the experiences of one middle school mathematics teacher's longitudinal participation. Our case study focuses on Ms. Chaves (pseudonym), a nine-year veteran teacher in a suburban district in the northeast United States. She primarily teaches 8th-grade mathematics and algebra. Before joining the teacher community, Ms. Chaves participated in various professional developments about classroom practice. We chose Ms. Chaves to be our focus for this study because of her active involvement in the AIM-TRU PD model as both a participant and facilitator throughout the four years of this study.

Evaluative case studies call for triangulation of multiple sources of evidence (e.g., Yin, 2018). Therefore, in addition to analyzing the transcripts of PLC conversations, we recorded and analyzed one classroom observation video from years one and four, and a semi-structured follow-up interview with Ms. Chaves, which took place in January of year 4. For the classroom observations, we used thematic analysis to describe teaching practices and the TRU observation rubric (Figure 2) to determine the TRU-alignment of classroom practices. The observation rubric for whole class activities, developed by the TRU team, evaluates lessons across the framework's five dimensions using a 5-point scale, with scores ranging from 1 to 3, including half-point intervals. The analysis of the semi-structured follow-up interview allowed us to corroborate our findings to develop convergent evidence around changes in Ms. Chaves' instructional practices and the potential impact of the PD model on these changes.

Rubric Score	The Mathematics	Cognitive Demand	Access to Mathematical Content	Agency, Authority, and Identity	Uses of Assessment
	How accurate, coherent, and well justified is the mathematical content?	*To what extent are students supported in grappling with and making sense of mathematical concepts?*	*To what extent does the teacher support access to the content of the lesson for all students?*	*To what extent are students the source of ideas and discussion of them? How are student contributions framed?*	*To what extent is students' mathematical thinking surfaced; to what extent does instruction build on student ideas when potentially valuable or address misunderstandings when they arise?*
1	Classroom activities are unfocused or skills-oriented, lacking opportunities for engagement with key grade level content (as specified in the Common Core Standards)	Classroom activities are structured so that students mostly apply memorized procedures and/or work routine exercises.	There is differential access to or participation in the mathematical content, and no apparent effort to address this issue.	The teacher initiates conversations. Students' speech turns are short (one sentence or less), and constrained by what the teacher says or does.	Student reasoning is not actively surfaced or pursued. Teacher actions are limited to corrective feedback or encouragement.
2	Activities are at grade level but are primarily skills-oriented, with few opportunities for making connections (e.g., between procedures and concepts) or for mathematical coherence (see glossary).	Classroom activities offer possibilities of conceptual richness or problem solving challenge, but teaching interactions tend to "scaffold away" the challenges, removing opportunities for productive struggle.	There is uneven access or participation, but the teacher makes some efforts to provide mathematical access to a wide range of students.	Students have a chance to explain some of their thinking, but the teacher is the primary driver of conversations and arbiter of correctness. In class discussions, student ideas are not explored or built upon.	The teacher refers to student thinking, perhaps even to common mistakes, but specific students' ideas are not built on (when potentially valuable) or used to address challenges (when problematic).
3	Classroom activities support meaningful connections between procedures, concepts and contexts (where appropriate) and provide opportunities for building a coherent view of mathematics.	The teacher's hints or scaffolds support students in productive struggle in building understandings and engaging in mathematical practices.	The teacher actively supports and to some degree achieves broad and meaningful mathematical participation; **OR** what appear to be established participation structures result in such engagement.	Students explain their ideas and reasoning. The teacher may ascribe ownership for students' ideas in exposition, **AND/OR** students respond to and build on each other's ideas.	The teacher solicits student thinking and subsequent instruction responds to those ideas, by building on productive beginnings or addressing emerging misunderstandings.

Figure 2. TRU scoring rubric for whole class activities (Schoenfeld et al., 2014).

4. Results: Evidence of AIM-TRU PD Model Effectiveness

4.1. Shifts in FAL Implementation

Ms. Chaves engaged as a teacher-participant for the first year of the study and led as a facilitator-participant for the following three years. In this section, we describe various shifts in Ms. Chaves' practice through her transition. In year 1, Ms. Chaves shared that she would use tasks, pedagogical strategies, and routines from the FALs by merging them with her existing classroom practices: "[I was] looking at those [FAL activities] like, oh, this would be good. And I would just pull it and plop it in, trying to figure it out as I went". FALs are comprised of three parts; the pre-assessment designed to reveal each student's understanding and misunderstandings of the target concept; the lesson, including a rich and complex task designed to expose students' different ideas and ways of thinking; and the post-assessment designed to showcase learning from engaging in the lesson. However, the classroom video indicated that she eliminated the pre-assessment and whole-class introduction, which would have increased students' access to mathematics by providing them opportunities to engage with the content prior to small group work.

When implementing a card sort activity, Ms. Chaves lowered the cognitive demand of the activity by removing a pair of matches she deemed too difficult and providing students with information about the number of matches.

In contrast, during the fourth year, Ms. Chaves used the FAL pre-assessment and whole-class introduction and utilized pedagogical strategies (e.g., mini-whiteboards) to formatively assess students' mathematical thinking. By using the prescribed whole-class introduction, she was able to facilitate class discussion based on perceived misunderstandings, prompting students to justify their thinking and reasoning. In addition, she did not remove any part of the main lesson task. This increased fidelity to lesson implementation positioned Ms. Chaves to demonstrate teaching practices more closely aligned with the TRU framework, which was evident in the video of her classroom while implementing FALs.

This shift in FAL implementation is also seen through the allotted time Ms. Chaves provided for students to engage and productively struggle with the full FAL tasks. During the semi-structured interview, she shared her interpretation of this growth through the years:

> I've done one lesson … three or four years, I finally feel like I let it breathe enough. And all of a sudden, these kids figured out things throughout the lesson that they had never in previous years. It was like, oh, my gosh, what just happened? The answer is I gave them more time. I didn't try to rush. (follow-up interview)

Compared with year 1 where we observed Ms. Chaves' tendency to alter the FAL and reduce its cognitive demand, we now see through the year 4 classroom observation and her reflection in the interview how she shifted her practice by providing students with more time, space, and structured opportunities to make connections and persist in their learning, resulting in her perceiving improved mathematical understanding compared to previous years. In her post-year 4 interview, she also mentioned that the PD is "the only one I've participated in that's been long-term and sustained". This self-reporting of the impact of the sustained duration of the PD provides evidence that the opportunity to collaborate with others enhanced her teaching practice since it allowed her to implement lessons multiple times while participating in a CoP.

4.2. General Changes in Classroom Practice

4.2.1. Homework

A classroom video of Ms. Chaves from the first year of her participation shows her facilitating a homework review from the previous day. Students were asked to self-check solutions by comparing their answers to a posted key. Ms. Chaves followed up with the students: "Does anyone have any questions?" Seeing none, she moved on to the next part of the lesson. During an interview, Ms. Chaves reflected on her beliefs about homework at the time:

> It's because I was supposed to, and everyone in our district, starting in sixth grade, does this sort of homework, and this is how much it is. All of the teams give the same. Its due on this day. This is how we grade it. (follow up interview)

This statement indicates that Ms. Chaves believed she was conforming to the institutional expectations for assigning, reviewing, and grading homework. We coded this classroom video excerpt using the TRU observation rubric (see Black & Wiliam, 1988). We assigned low scores in four dimensions: cognitive demand (CD); equitable access (EA); agency, ownership, identity (AOI); and formative assessment (FA). Ms. Chaves' homework assignments did offer students a chance to engage in productive problem-solving struggles (high CD). However, during the classroom observation, this opportunity was not supported or expanded upon through classroom practices (low CD). Her district's standardized homework approach resulted in unequal access for students, as some students lacked the necessary background knowledge to begin the tasks (low EA). Furthermore, students were restricted to working individually on solutions and had no opportunity for peer-to-peer discussion (low AOI). This approach also restricted assessment to providing corrective feedback focused on student work (low FA).

We began to see a shift in Ms. Chaves' homework practice through her facilitation of the PLC in year 2. During one meeting, the community was discussing what teacher moves they could make to ensure students are able contributors and mathematical thinkers. As the teachers grappled with this problem of practice, Ms. Chaves added, "having those students kind of defend how they know, right? Instead of just stating it, in the other words, saying yes". This statement shows a change in how Ms. Chaves viewed the purpose of students' production of mathematical solutions, which contributed to the incremental shift in her homework practice.

In the year 4 classroom video, we see how Ms. Chaves does not assign homework but rather ends the previous lesson with the pre-assessment of the FAL. Not only does this move illustrate the shift in her FAL implementation since the pre-assessment is the first part of each lesson, but this also illustrates a second incremental shift around homework practices. In her interview, she shared that she thought her previous homework practices were inequitable because "whatever these kids are going home to may or may not be conducive to them doing [home]work" and to "then penalize a child [for not doing homework] . . . seems like a one-two punch". She explained that over the past four years, she gradually refined her approach to homework by reducing the amount assigned and focusing more on helping students connect mathematical concepts. She believed these adjustments made the practice more equitable and better supported students' retention of mathematical understanding. Additionally, if students encounter difficulties with homework, she now uses class time for collaborative problem-solving instead of providing an answer key. This change encourages student discussions about concepts and connections and promotes self-evaluation of their mathematical thinking. As she explained in her interview, "I'm letting them come to those conclusions by themselves now". Her current approach to homework better aligns with the TRU framework: she addresses EA by acknowledging that home conditions might not always support homework completion, enhances CD by allowing students to struggle productively, fosters AOI by encouraging collaborative problem-solving, and increases FA by eliciting and discussing student thinking.

4.2.2. Questioning

There was also evidence of changes in the way Ms. Chaves supported students to come to mathematical conclusions through purposeful questioning within the FAL lesson plans. As presented in the previous section, in year 1, Ms. Chaves would modify the intended FAL by removing particular aspects of the lesson plan. During the year 1 classroom observation, we observed how Ms. Chaves modified the FAL's card sort activity by removing a pair of matching cards and providing students with information about the number of matches. To introduce the task, the FAL includes a collaborative warmup activity with three algebraic expressions and three written sentences. Students are asked to match the algebraic expression with the written sentence. The warmup is designed to have students tackle the dilemma that not every card will have a match. Ms. Chaves did not have students engage in this part of the FAL. Instead, Ms. Chaves displayed the slide in Figure 3.

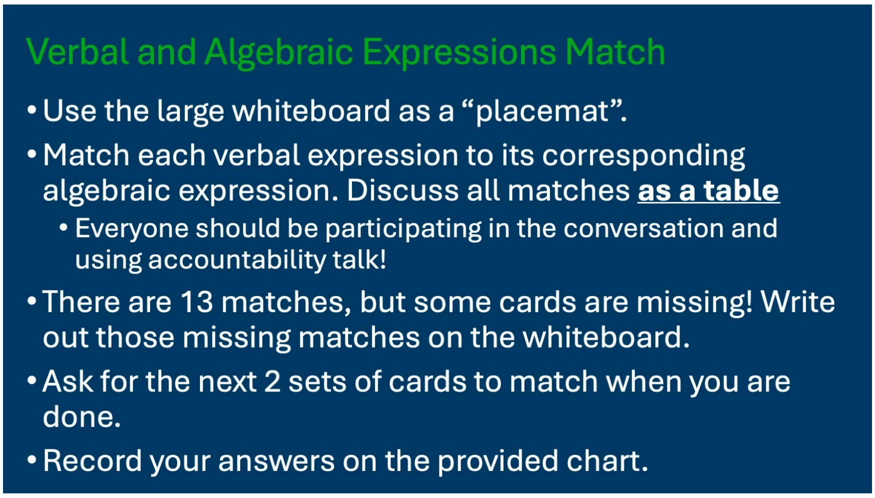

Figure 3. Ms. Chaves' teacher generated instructions.

During her interview, Ms. Chaves reflected on when she first implemented FALs by stating "I figured I could live in a place where I could piece this in with other things and like, put it all together". By doing this, Ms. Chaves eliminated the possibility that students could group multiple expression cards together and she removed an opportunity for students to productively struggle and the meaningful discussion that could take place with planned teacher questioning and prompts.

During year 4, when Ms. Chaves self-reported following the FAL lesson plans, more closely, we verified this through the classroom observation. While launching an FAL on classifying rational and irrational numbers, Ms. Chaves asked students to determine if 0.123 was a terminating, non-terminating repeating, or non-terminating non-repeating number. The FAL lesson plan prompts the teacher to write 0.123 on a large sticky note and model a classification activity using this number. In this classification activity, the teacher is prompted to ask particular questions, and the lesson plan provides potential student responses (see Figure 4).

In Ms. Chaves's class, after a student shared their choice she asked the student, "Why did you choose that one?" The student gave their justification, and Ms. Chaves prompted another student, "Is that what you are thinking, would you like to add on?" She followed up with the entire class by asking if students had "any argument" for the other two choices. Ms. Chaves proceeded to ask if the number was rational or irrational. She provided time for students to think individually and then solicited student responses. Students responded with various ideas, including "both", and when the students did not agree on a choice, she did not disclose the correct answer. Instead, she said, "We will be figuring this out in our task". The video illustrates how Ms. Chaves followed the teacher guide, and when using the TRU observation rubric, both FA and CD were rated at the highest level of TRU alignment because she used students' emerging understandings to build on student thinking and engaged and supported students in productive struggle by not scaffolding away challenges, respectively.

Display the following slide:

Classifying Rational and Irrational Numbers

	Rational Numbers	Irrational Numbers
Terminating decimal		
Non-terminating repeating decimal		
Non-terminating non-repeating decimal		

Projector Resources Rational and Irrational Numbers 1 P-1

Explain to students that this lesson they will make a poster classifying rational and irrational numbers.

Check students' understanding of the terminology used for decimal numbers:

Write 0.123 on a large sticky note.

Model the classification activity using the number 0.123.

- *Teacher: Remind me what the little bar over the digits means.*
  - *Possible student responses: It is a repeating decimal that begins 0.123123123... ;the digits continue in a repeating pattern; it does not terminate.*
- *Teacher: In which row of the table does 0.123 go? Why?*
  - *Possible student responses: Row 2, because the decimal does not terminate but does repeat.*
- *Teacher: Ok. So this number is a non-terminating repeating decimal because the bar shows it has endless repeats of the same three digits. [Write this on the card.]*
- *Teacher: Show me on your whiteboard: is 0.123 rational or irrational?*
  - *Possible student response: Rational.*
- *Students may offer different opinions on the rationality of 0.123. If there is dispute, accept students' answers for either classification at this stage in the lesson. Make it clear to students that the issue is unresolved and will be discussed again later in the lesson.*

Figure 4. Classifying rational and irrational numbers Teacher guide.

4.2.3. Accountability Talk

Another classroom practice we noted during year 1 was Ms. Chaves's use of accountability talk. Accountability talk is a structured classroom discussion where students are responsible for their contributions. This practice is beneficial for mathematics instruction because students engage deeply with the materials as they explain their reasoning, justify their answers, and critically evaluate the ideas of their peers (e.g., Rüede et al., 2023). Ms. Chaves used this practice to prompt discussions in small groups, as she reminded the students that they had structures of accountability talk. She indicated during her interview as she reflected on year 1 that she supported student accountability talk by hanging a poster of prompts on the wall for reference. However, there was room for incremental improvements to help this practice become more TRU-aligned. In particular, the poster and her reminders supported student engagement and discourse, but the placement of the posters may have limited students' access.

As similar issue surfaced during the sixth PLC session in year 2 during Ms. Chaves' facilitation. In this session, the community was exploring an FAL focused on interpreting algebraic expressions. After watching the classroom video, the teachers noticed how students may not have had the mathematical vocabulary needed to engage productively in the task, which impacted the nature and quality of their discourse. One teacher suggested giving students a "cheat sheet" or anchor chart to refer back to when the vocabulary we needed. To summarize the conversation, Ms. Chaves stated, "What I'm hearing is those scaffoldings are going to get them to the point where they can make the connection, right, because that was the goal. The goal is to be connecting these different representations". This facilitation move illustrates the incremental shifts occurring in Ms. Chaves' considerations of how to support student discourse and problem-solving by providing them with physical resources they could refer to at the moment.

The classroom video of Ms. Chaves' classroom from year 4 revealed how her practice of accountability talk evolved. During the particular lesson, Ms. Chaves references accountability talk stems from students' desks. In the follow-up interview, she described how these stems included prompts for responding to peers, seeking clarifications, and sharing new ideas. By incorporating these prompts, students were encouraged to participate in discussions, challenge one another, and justify their own mathematical reasoning. Unlike in year 1, where the accountable talk stems were displayed on the wall, they were now placed on student desks, which allowed for greater student engagement with these practices. The desk resources increased EA in Ms. Chaves's classroom. Specifically, the physical proximity of the students made the stems easier to access and use during instruction, thus supporting the immediate application of accountability talk. This accessibility can also reduce cognitive load and allow students to focus on discourse and problem-solving rather than the effort required to locate or reference the material (e.g., Goffman, 1974). Furthermore, by providing each student a desk copy of the stems, Ms. Chaves ensured that all students, regardless of their physical location in the classroom or visual challenges, had equitable access to the support they needed. The new desk resource also enhanced AOI by assisting students in sharing their ideas and expanding on each other's understanding by allowing them to reference the accountability talk stems whenever necessary. This supports student autonomy, enabling students to practice accountability talk independently, and self-correct when necessary. Research highlights the importance of students having immediate access to reference materials to reinforce learning and foster independence in problem-solving (e.g., Roth et al., 1999).

In addition to the physical proximity, in her interview, Ms. Chaves noted that she now spends more time at the beginning of each school year preparing students to engage in collaborative tasks and how to communicate in her classroom.

I start out my year now with a good three weeks of: How do we work together in math? Even when I am allotted four days on my calendar. Additionally, I spend half a week and then two more full weeks, where we are learning structures. We are learning how to communicate; we are learning how to talk to each other. Additionally, they are like math-adjacent sorts of tasks. However, I do not think we are starting unit one until the beginning of week four. Additionally, being able to front-load how to work together has helped give me the confidence to let the kids keep working on something for a longer period of time because they are better at working together.

Ms. Chaves' emphasis on student discourse and collaboration evolved throughout her participation in this PD. In her interview, Ms. Chaves shared that she perceived her years of engaging with this PD model helped her incrementally solidify her understanding of the dimensions of the TRU framework. When confronted with outside curricular materials or resources, she critically analyzes them with a TRU lens:

> [How can I] make sure all kids have access to the lesson but also make it cognitively demanding and also give the kids agency? If someone comes up to me

> and says, I want you to teach like this now, I am going to naturally throw that up against TRU in my mind. (follow up interview)

5. Discussion and Implications

Ms. Chaves' experienced an evolution in her teaching practice, which she partially attributed to her sustained involvement with the AIM-TRU teacher community. It is important to note that even before participating in the project, Ms. Chaves was considered a master teacher with strong teaching practices. However, by spending time interrogating the TRU framework, familiarizing herself with FALs, and navigating how to reflect on her teaching these lessons using the framework, she reported a change in her teaching in incremental and meaningful ways. Through her transition, Ms. Chaves implemented new strategies gradually, reflected on their effectiveness using the TRU framework, and adjusted before the next implementation to continue her movement towards TRU-aligned teaching practices. Analysis of Ms. Chaves' changing teaching practices and her reflections suggests that her continued participation in our PD. The PD's incremental and innovative design (Garet et al., 2001; Timperley et al., 2007; Desimone & Garet, 2015) provided her the opportunity to reify ideas about powerful mathematics teaching and the implementation of high-quality materials.

By engaging in prolonged and collaborative analysis of teaching practices focused on equity, Ms. Chaves revised her approach to create a more equitable learning environment and adjusted her homework expectations for students based on her self-reflection of the inequities in her system. She utilized her professional development experiences to increase formative assessment opportunities in her classroom, encouraging students to discuss homework and connect mathematical concepts to address challenges. While Ms. Chaves' structural changes for accountability talk stems may seem minor, the incremental shift from a whole-class anchor chart to individual small-group reference sheets provided additional support for more students to engage in mathematical discourse. The resources for small groups facilitated access to mathematical discussions and supported collaborative thinking.

Effective professional development involves sustained engagement and active learning experiences that connect to classroom practice (e.g., Desimone & Garet, 2015). Ms. Chaves' involvement in Blinded was that it provided space and time for her to read, analyze, and reflect on FAL implementation and the TRU framework within a community of practice. Through PLC discussions, her understanding of how to use FALs and her perception of the impact of modifying these resources evolved as she re-evaluated how to integrate new classroom materials by assessing their alignment with the TRU framework. This evaluation gave her tools to move beyond established institutional norms and adopt more TRU-aligned teaching practices. Her thorough examination of instructional routines and teaching practices using the TRU framework contributed to how she selected and implemented materials and strategies that promote powerful mathematics teaching.

We link the incremental changes in Ms. Chaves' practice over time to her participation in our PD while acknowledging the influence of her unique situation and prior experiences. An evaluative case study with a single teacher has several limitations. Specifically, our findings may not apply broadly to other teachers or contexts. Without multiple cases, it is challenging to determine if Ms. Chaves' unique characteristics, instructional training, prior PD experiences, or teaching style influenced the impact of the AIM-TRU PD. The changes Ms. Chaves experienced may have also been influenced by external factors, such as school environment, student dynamics, or policy changes. Therefore, the context-specific insights gained limit the generalizability of the results to different educational settings, and we acknowledge the risk of overgeneralizing our findings based on this single case.

Nevertheless, we believe her story provides evidence that these shifts are possible for other educators. The structure of the AIM-TRU PD model situates all teachers, new and experienced, as knowledgeable entities in the room. AIM-TRU provides scaffolds for facilitators to help focus all teachers' attention on TRU-aligned teaching practices and permit the participants to engage in deep conversations related to the observed classroom, their own practice, and anticipated moves they might make. Through discussions, teachers begin to see themselves as capable of implementing FALs and even use videos of their classrooms for the CoP to reflect on. Moreover, FALs are accessible to students and teachers in any context. Curriculum guides and school district-adopted materials can vary greatly across school contexts. The nature of FALs allows teachers with any curriculum to supplement their resources with the FALs across multiple units.

The transferability of these particular teaching materials across multiple contexts can be coupled with the flexibility of implementing Blinded. Facilitator guides and resources are available for online and in-person sessions. Each session is outlined to include familiarity with TRU and FALs while using these tools as a common understanding to interrogate mathematics teaching and learning. Teachers in large and small groups can collaborate to build a CoP with Blinded as a guide to their teacher learning. We have found success in teacher learning using AIM-TRU in large, multidistrict groups, small, in-district groups, and fellowship programs (see Bonaccorso et al., 2023; DiNapoli et al., 2023). The full grant project works with participants from rural, urban, and suburban schools implementing FALs and analyzing the outcomes within their individual CoP. The open-source resources for facilitating and implementing Blinded also feature closed-captioning on the videos, editable digital resources to meet individual needs, and printed materials were created with an eye for accessibility in mind.

Our incremental approach to professional development may be more effective for teacher learning and educational improvement than large-scale overhauls. AIM-TRU focuses on small, manageable changes that teachers can easily integrate into their routines. These tweaks are not sweeping overhauls but enhancements to current practices, increasing the likelihood of adoption and leading to lasting instructional improvements. Our approach accounts for teachers' constraints and builds on familiar instructional elements. By aligning with the TRU framework, our model supports gradual but meaningful improvements in planning and reflecting on instructional change.

Finally, the interpretations from this study may be influenced by our perspectives and long-standing relationship with Ms. Chaves. We acknowledge the subjectivity the in-depth, qualitative nature of an evaluative case study introduces and have worked diligently to provide results that may serve as a foundation for future research.

Author Contributions: Conceptualization, methodology, E.C.M., H.S.L., & V.D.B.; Data analysis, H.S.L., & V.D.B.; writing—original draft preparation, review, and editing, E.C.M., H.S.L., & V.D.B. All authors have read and agreed to the published version of the manuscript.

Funding: This research was funded by National Science Foundation, grant #1908319.

Institutional Review Board Statement: The study was conducted in accordance with the Declaration of Helsinki and approved by the Institutional Review Board of Montclair State University (IRB-FY17-18-903 and 1 June 2018).

Informed Consent Statement: Informed consent was obtained from all subjects involved in the study.

Data Availability Statement: Restrictions apply to the data set: The dataset presented in this article is not readily available because of privacy. Permission was not sought from participants to share data beyond specific use for this study. Requests to access the datasets should be directed to the corresponding author.

Acknowledgments: This material is based upon work supported by the National Science Foundation under grant number 1908319. Our work is situated in a partnership of professional learning communities that includes Chicago, Buffalo, and New York City Public School teachers.

Conflicts of Interest: The authors declare no conflicts of interest. The funders had no role in the design of the study; in the collection, analyses, or interpretation of data; in the writing of the manuscript; or in the decision to publish the results.

References

Alles, M., Seidel, T., & Gröschner, A. (2018). Toward better goal clarity in instruction: How focus on content, social exchange and active learning supports teachers in improving dialogic teaching practices. *International Education Studies*, *11*(1), 11–24. [CrossRef]

Bayar, A. (2014). The components of effective professional development activities in terms of teachers' perspective. *International Online Journal of Educational Sciences*, *6*(2), 319–327. [CrossRef]

Black, P., & Wiliam, D. (1988). Assessment and classroom learning. *Assessment in Education: Principles, Policy & Practice*, *5*(1), 7–74. [CrossRef]

Boaler, J., & Greeno, J. G. (2000). Identity, agency and knowing in mathematics worlds. In J. Boaler (Ed.), *Multiple perspectives on mathematics teaching and learning* (pp. 171–200). Elsevier Science.

Bonaccorso, V. D., Leonard, H. S., Daniel, A., Kim, Y., & DiNapoli, J. (2023). Exploring changes in mathematics teacher practice from professional development rooted in the TRU framework. In *Proceedings of the forty-fifth annual meeting of the North American chapter of the international group for the psychology of mathematics education* (Vol. 1). University of Nevada. [CrossRef]

Borko, H., Jacobs, J., Eiteljorg, E., & Pittman, M. E. (2008). Video as a tool for fostering productive discussions in mathematics professional development. *Teaching and Teacher Education*, *24*(2), 417–436. [CrossRef]

Council of Chief State School Officers & National Governors Association. (2010). *CCSS; common core state standards—mathematics*. Council of Chief State School Officers and National Governors Association.

Darling-Hammond, L., Hyler, M. E., & Gardner, M. (2017). *Effective teacher professional development*. Learning Policy Institute. [CrossRef]

Desimone, L. M. (2011). A primer on effective professional development. *Phi Delta Kappan*, *92*(6), 68–71. [CrossRef]

Desimone, L. M. (2023). Rethinking teacher PD: A focus on how to improve student learning. *Professional Development in Education*, *49*(1), 1–3. [CrossRef]

Desimone, L. M., & Garet, M. S. (2015). Best practices in teacher's professional development in the United States. *Psychology, Society, & Education*, *7*(3), 252–263. [CrossRef]

DiNapoli, J., Daniel, A., Leonard, H. S., Kim, Y., Bonaccorso, V. D., & Murray, E. (2023). Characterizing mathematics teacher learning patterns through collegial conversation in a community of practice. *Journal of Mathematics Education Leadership*, *24*(2), 25–47.

Dyer, E. B., Jarry-Shore, M., Fong, A., Deutscher, R., Carlson, J., & Borko, H. (2023). Teachers' engagement with student mathematical agency and authority in school-based professional learning. *Teaching & Teacher Education*, *121*, 1–16. [CrossRef]

Garet, M. S., Porter, A. C., Desimone, L., Birman, B. F., & Yoon, K. S. (2001). What makes professional development effective? Results from a national sample of teachers. *American Educational Research Journal*, *38*(4), 915–945. [CrossRef]

Garet, M. S., Wayne, A. J., Stancavage, F., Taylor, J., Walters, K., Song, M., Brown, S., Hurlbutt, S., Zhu, P., Spanik, S., & Doolittle, F. (2010). Middle School Mathematics Professional Development Impact Study: Findings After the First Year of Implementation. NCEE 2010-4009. National Center for Education Evaluation and Regional Assistance. Available online: https://ies.ed.gov/ncee/pubs/20104009/pdf/20104010.pdf (accessed on 7 December 2022).

Goffman, E. (1974). *Frame analysis: An essay on the organization of experience*. Harper & Row.

Granberg, C., Palm, T., & Palmberg, B. (2021). A case study of a formative assessment practice and the effects on students' self-regulated learning. *Studies in Educational Evaluation*, *68*, 100955. [CrossRef]

Hill, H. C., Blunk, M. L., Charalambous, C. Y., Lewis, J. M., Phelps, G. C., Sleep, L., & Ball, D. L. (2008). Mathematical knowledge for teaching and the mathematical quality of instruction: An exploratory study. *Cognition and Instruction*, *26*, 430–511. [CrossRef]

Hodge, L. L., & Cobb, P. (2019). Two views of culture and their implications for mathematics teaching and learning. *Urban Education*, *54*(6), 860–884. [CrossRef]

Jackson, C., Taylor, C., & Buchheister, K. (2015, November 5–8). *What is equity? Ways of seeing* [Conference session]. 37th Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education, East Lansing, MI, USA.

Jacob, R., Hill, H., & Corey, D. (2017). The impact of a professional development program on teachers' mathematical knowledge for teaching, instruction, and student achievement. *Journal of Research on Educational Effectiveness*, *10*(2), 379–407. [CrossRef]

Kelly, P. (2006). What is teacher learning? A socio-cultural perspective. *Oxford Review of Education*, *32*(4), 505–519. [CrossRef]

Kennedy, M. M. (1998). *Form and substance in inservice teacher education*. University of Wisconsin National Institute for Science Education.

Koellner, K., & Jacobs, J. (2015). Distinguishing Models of Professional Development. *Journal of Teacher Education*, *66*(1), 51–67. [CrossRef]

Lave, J., & Wenger, E. (1991). *Situated learning: Legitimate peripheral participation*. Cambridge University Press. [CrossRef]

Leonard, H. S., DiNapoli, J., Murray, E., & Bonaccorso, V. D. (2022). Collegial frame processes supporting mathematics teacher learning in a community of practice. In *The American education research association online paper repository*. American Educational Research Association. [CrossRef]

Merriam, S. B. (1998). *Qualitative research and case study applications in education* (2nd ed.). Jossey-Bass Publishers.

Murray, E. C. (2011). *Implementing higher-order thinking in middle school mathematics classrooms* [Doctoral dissertation, University of Georgia].

Murray, E. C. (2015). Improving Teaching through Collaborative Reflective Teaching Cycles. *Investigations in Mathematics Learning*, *7*(3), 23–29. [CrossRef]

Otten, S., de Araujo, Z., Candela, A. G., Vahle, C., Stewart, M. E., & Wonsavage, F. P. (2022, November 17–20). *Incremental change as an alternative to ambitious professional development*. Forty-Fourth Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education, Nashville, TN, USA. [CrossRef]

Prast, E. J., Van de Weijer-Bergsma, E., Kroesbergen, E. H., & Van Luit, J. E. H. (2018). Differentiated instruction in primary mathematics: Effects of teacher professional development on student achievement. *Learning and Instruction*, *54*, 22–34. [CrossRef]

Roth, W. M., McGinn, M. K., Woszczyna, C., & Boutonne, S. (1999). Differential participation during science conversations: The interaction of focal artifacts, social configurations, and physical arrangements. *Journal of the Learning Sciences*, *8*(3–4), 293–347. [CrossRef]

Rüede, C., Mok, S. Y., & Staub, F. C. (2023). Fostering Flexibility Using Comparing Solution Strategies and Classroom Discussion: Effects of Two Professional Development Programs. *Journal for Research in Mathematics Education*, *54*(1), 43–63. [CrossRef]

Santagata, R., & Bray, W. (2016). Professional development processes that promote teacher change: The case of a video-based program focused on leveraging students' mathematical errors. *Professional Development in Education*, *42*(4), 547–568. [CrossRef]

Schoenfeld, A. H. (2015). Thoughts on scale. *ZDM Mathematics Education*, *47*(1), 161–169. [CrossRef]

Schoenfeld, A. H., Floden, R. E., & the Algebra Teaching Study and Mathematics Assessment Project. (2014). *The TRU math scoring rubric*. Graduate School of Education, University of California, Berkeley & College of Education, Michigan State University. Available online: http://ats.berkeley.edu/tools.html (accessed on 11 December 2023).

Schoenfeld, A., Fink, H., Sayavedra, A., Weltman, A., & Zuñiga-Ruiz, S. (2023). *Mathematics teaching on target: A guide to teaching for robust understanding at all grade levels*. Routledge. [CrossRef]

Selkrig, M., & Keamy, K. (2015). Promoting a willingness to wonder: Moving from congenial to collegial conversations that encourage deep and critical reflection for teacher educators. *Teachers and Teaching*, *21*(4), 421–436. [CrossRef]

Smith, M. S. (2001). *Practice-based professional development for teachers of mathematics*. National Council of Teachers of Mathematics.

Smith, R., Ralston, N. C., Naegele, Z., & Waggoner, J. (2020). Team teaching and learning: A model of effective professional development for teachers. *Professional Educator*, *43*(1), 80–90.

Star, J. R. (2016). Improve math teaching with incremental improvements. *Phi Delta Kappan*, *97*(7), 58–62. [CrossRef]

Thurm, D., & Barzel, B. (2020). Effects of a professional development program for teaching mathematics with technology on teachers' beliefs, self-efficacy and practices. *ZDM*, *52*(7), 1411–1422. [CrossRef]

Timperley, H., Wilson, A., Barrar, H., & Fung, I. (2007). *Teacher professional learning and development. best evidence synthesis iteration (BES)*. Ministry of Education. [CrossRef]

Van Es, E. A., & Sherin, M. G. (2008). Mathematics teachers' "learning to notice" in the context of a video club. *Teaching and Teacher Education*, *24*(2), 244–276. [CrossRef]

Voss, T., Kleickmann, T., Kunter, M., & Hachfeld, A. (2013). Mathematics teachers' beliefs. In *Cognitive activation in the mathematics classroom and professional competence of teachers: Results from the COACTIV project* (pp. 249–271). Springer. [CrossRef]

Walkoe, J. (2014). Exploring teacher noticing of student algebraic thinking in a video club. *Journal of Mathematics Teacher Education*, *18*, 523–550. [CrossRef]

Wenger, E. (1998a). Communities of practice: Learning as a social system. *Systems Thinker*, *9*(5), 1–8. [CrossRef]

Wenger, E. (1998b). *Communities of practice: Learning, meaning, and identity*. Cambridge University Press. [CrossRef]

Wenger-Trayner, E., & Wenger-Trayner, B. (2015). An introduction to communities of practice: A brief overview of the concept and its uses. Available online: www.wenger-trayner.com/introduction-to-communities-of-practice (accessed on 23 October 2023).

Yin, R. (2018). *Case study research and applications design and methods* (6th ed.). Sage Publications.