ON THE BERNOULLI PROBLEM WITH UNBOUNDED JUMPS

STANLEY SNELSON & EDUARDO V. TEIXEIRA

Abstract. We investigate Bernoulli free boundary problems prescribing infinite jump conditions. The mathematical set-up leads to the analysis of non-differentiable minimization problems of the form $\int \left(\nabla u \cdot (A(x)\nabla u) + \varphi(x)1_{\{u>0\}}\right) \mathrm{d}x \to \min$, where $A(x)$ is an elliptic matrix with bounded, measurable coefficients and φ is not necessarily locally bounded. We prove universal Hölder continuity of minimizers for the one- and two-phase problems. Sharp regularity estimates along the free boundary are also obtained. Furthermore, we perform a thorough analysis of the geometry of the free boundary around a point ξ of infinite jump, $\xi \in \varphi^{-1}(\infty)$. We show that it is determined by the blow-up rate of φ near ξ and we obtain an analytical description of such cusp geometries.

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Contents

1. Introduction

For a bounded domain $\Omega \subset \mathbb{R}^d$, $d \geq 2$, with Lipschitz boundary, let

$$M_{\lambda,\Lambda}(\Omega) := \left\{A : \Omega \to \mathbb{R}^{d\times d},\ A_{ij} = A_{ji}, \lambda|e|^2 \leq A_{ij}(x)e_ie_j \leq \Lambda|e|^2 \text{ for all } x \in \Omega, e \in \mathbb{R}^d\right\},$$

be the space of symmetric $d \times d$ matrix-valued functions on Ω with ellipticity constants $0 < \lambda \leq \Lambda$. For fixed $A \in M_{\lambda,\Lambda}(\Omega)$ and $\varphi\colon \Omega \to \mathbb{R}$, we are interested in minimizers of the energy functional

$$\mathcal{J}_{A,\varphi}(u) := \int_\Omega \left(\nabla u \cdot (A(x)\nabla u) + \varphi(x)1_{\{u>0\}}\right) \mathrm{d}x, \tag{1.1}$$

over $H^1_g(\Omega)$, for some boundary condition g in the trace space $H^{1/2}(\partial\Omega)$. Hereafter, $1_{\{u>0\}}$ denotes the indicator function of the set $\{u > 0\}$, and φ is a nonnegative function that we will place further conditions on below.

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The energy functional described in (1.1) is part of a large class of free boundary models describing cavity or jet flows, and is also related to overdetermined Bernoulli-type problems. The analysis of such free boundary problems was launched by the epoch-marking works of Alt-Caffarelli [1] and Alt-Caffarelli-Friedman [2], and since then has promoted major knowledge leverage across pure and applied sciences, viz. [4, 7, 10, 19, 20, 17, 18, 8, 9, 13, 12, 14, 32] to cite a few.

The main key novelty in this work is that the function φ is only required to belong to a weak L^q space, and thus it may become unbounded. More importantly, the minimal assumption $\varphi \in L^q_{\text{weak}}(\Omega)$ leads to multiple free boundary geometries. That is, the condition $\varphi \in L^q_{\text{weak}}$ sets a *maximum* blow-up rate near a generic point ξ, viz. $|x-\xi|^{-n/q}$; however it is not granted that $\varphi(x)$ blows up at the same rate for all free boundary points. In turn, the geometry of the free boundary is modulated by the blow-up rate of φ around a free boundary point, which may change point-by-point. A decisive new approach we discuss here concerns precise analytical quantities that allow one to classify such free boundary geometries; see Theorem 3 below.

The free boundary model investigated in this article should be thought of as the dual of the Stokes conjecture, as investigated in [33]. By allowing $\varphi(x)$ to vanish at a precise rate – in the case of the Stokes conjecture, $\varphi(x, y) = -y$ and $(0, 0)$ is a free boundary point – one can offer a variational treatment of the Stokes conjecture. See also the series [22, 23, 24] for a related setting in which φ vanishes in part of the domain. In this article, we treat the complementary case, when the Bernoulli cost function, φ, is allowed to become infinite. Heuristically, in the degenerate case, i.e. when the Bernoulli cost function φ vanishes at a free boundary point, a gain of smoothness is observed; for the Stokes conjecture, minimizers presents a sharp $C^{3/2}$ behavior. In our case, the cost function blows up, leading a loss of regularity, which is precisely quantified by Theorem 1.

We further comment that we do not impose any continuity condition on the coefficients $x \mapsto A(x)$. Our regularity results are of universal nature, and hence applicable to a plethora of other models, e.g. free transmission problems, homogenization issues, etc.

Recall that weak solutions to the homogeneous elliptic equation

$$\nabla \cdot (A(x)\nabla u) = 0, \quad x \in \Omega, A \in M_{\lambda,\Lambda}(\Omega), \tag{1.2}$$

are locally Hölder continuous, with universal estimates. This is the content of the celebrated De Giorgi-Nash-Moser regularity theory, [11, 26, 27]. That is, solutions to equation (1.2) satisfy the estimate

$$\|u\|_{C^{\alpha_0}(K)} \leq C\|u\|_{L^2(\Omega)}, \quad K \Subset \Omega, \tag{1.3}$$

for some *maximal* Hölder exponent $\alpha_0 \in (0, 1)$, depending on d, λ, and Λ, but not on K or Ω. Hereafter we say a constant is *universal* if it depends only on d, q, λ, and Λ. The constant $C > 1$ in (1.3) depends on universal parameters, K, and Ω, but it is independent of the solution u.

Let us now discuss the main results proven in this article. The first key observation is that local minimizers of (1.1) should satisfy an elliptic PDE like (1.2) in each phase, $\{u > 0\}$ and $\{u < 0\}$. Hence, any (universal) regularity estimate for local minimizers of (1.1) must conform to the maximum regularity imposed by (1.3). Our first main theorem yields local regularity estimates for minimizers in the Hölder space and captures the settlement described above, in a sharp fashion.

Theorem 1 (**Hölder estimate**). *Let u be a minimizer of $\mathcal{J}_{A,\varphi}(\Omega)$ over $H^1_g(\Omega)$, with $g \in H^{1/2}(\partial\Omega)$ and $A \in M_{\lambda,\Lambda}(\Omega)$. Let α_0 be as in (1.3). For any $\Omega' \subseteq \Omega$, if $\varphi \in L^q_{\text{weak}}(\Omega')$ for some $q > d/2$, then for any α satisfying*

$$\alpha \in (0, 1 - d/(2q)] \cap (0, \alpha_0), \tag{1.4}$$

the minimizer u is locally α-Hölder continuous in Ω'. One also has the estimate

$$\|u\|_{C^\alpha(K)} \le C\|u\|_{L^2(\Omega')}, \quad K \subset\subset \Omega'.$$

The constant C depends only on d, λ, Λ, q, α, $\|\varphi\|_{L^q_{\text{weak}}(\Omega')}$, and K.

Upon extra oscillation control of the function $x \mapsto A(x)$, higher regularity estimates for the homogeneous equation (1.2) become available, and one may take $\alpha_0 = 1$ in (1.4). We could then take $\alpha = 1 - d/(2q)$ in Theorem 1. In other words, if the coefficient matrix A is "continuous enough" as to allow Lipchitz estimates for the homogenous PDE (1.2), then minimizers display the sharp Hölder regularity with exponent $1 - d/(2q)$. See [5] for optimal conditions yielding Lipschitz regularity of solutions in great generality.

We highlight that Theorem 1 is for the two-phase problem, i.e., it does not carry any sign restriction on u. Our next main result gives improved regularity for the one-phase problem. It says that *at free boundary points*, the sharp Hölder exponent $1 - d/(2q)$ is achieved, regardless of the value of α_0 in (1.3).

Theorem 2 (**Improved regularity at free boundary points**). *With u, g, and φ as in Theorem 1, suppose that the boundary data g, and therefore u, are nonnegative. Let x_0 be a boundary point of $\{u > 0\}$, and suppose $\varphi \in L^q_{\text{weak}}(\Omega')$ for some open $\Omega' \subseteq \Omega$ containing x_0. Then $u \in C^{1-d/(2q)}$ at x_0, in the sense that*

$$|u(x)| \le C|x - x_0|^{1-d/(2q)}\|u\|_{L^2(\Omega)},$$

for a constant $C > 0$ that depends only on universal parameters, Ω', and Ω.

We also obtain nondegeneracy estimates that say u must grow at least at a certain rate near free boundary points. The asymptotics of these lower bounds are determined by the local blowup behavior of φ—more specifically, if φ satisfies an average lower bound like

$$|\{\varphi > t\} \cap B_r(x_0)| \ge \min\left(ct^{-p}, |B_r(x_0)|\right),$$

for some $c > 0$, all $t > 0$, and sufficiently small $r > 0$, then u satisfies

$$|u(x)| \ge c|x - x_0|^{1-d/(2p)}, \tag{1.5}$$

for x near x_0 such that $u(x) > 0$ and $|x - x_0| = \operatorname{dist}(x, \{u = 0\})$. See Lemma 10 for a more precise (and more general) statement. In particular, this lower bound implies $|\nabla u| \to \infty$ as $x \to x_0$ from inside $\{u > 0\}$. Note that for $\varphi \in L^q_{\text{weak}}$, the exponent $p \ge q$ may be strictly greater than q, and this causes extra subtleties in our analysis.

Next, we address the geometry of the free boundary $\partial\{u > 0\}$. The key point is to describe how the rate at which φ blows up near a free boundary point impacts the free boundary configuration.

Theorem 3 (**Control on the severity of cusps in the free boundary**). *Let u be a minimizer of $\mathcal{J}_{A,\varphi}$ over $H^1_g(\Omega)$ as above, with $g \ge 0$. Let $x_0 \in \partial\{u > 0\}$, and assume that for some*

$r_0 > 0$, $q_1 > d/2$, and $q_2 \geq q_1$,

$$\begin{aligned} &\varphi \in L^{q_1}(B_{r_0}(x_0)), \quad \textit{and} \\ &|\{\varphi(x) > t\} \cap B_r(x_0)| \geq \min\left(c_0 t^{-q_2}, |B_r(x_0)|\right) \quad \textit{for all } r \in (0, r_0), t > 0. \end{aligned} \tag{1.6}$$

Then, with $\alpha_1 = 1 - d/(2q_1)$, $\alpha_2 = 1 - d/(2q_2)$, and for any $\alpha \in (0, \alpha_1] \cap (0, \alpha_0)$, with α_0 as in Theorem 1, there exists a constant $c > 0$ such that for all $r \in (0, r_0)$,

$$\frac{|\{u > 0\} \cap B_r(x_0)|}{|B_r(x_0)|^{\alpha_2/\alpha}} \geq c, \quad \textit{and} \quad \frac{|\{u = 0\} \cap B_r(x_0)|}{|B_r(x_0)|^{P(q_1,q_2)}} \geq c, \tag{1.7}$$

where

$$P(q_1, q_2) = \left(\frac{\alpha_2}{\alpha_1} - \frac{1}{q_2}\right)\frac{q_1}{q_1 - 1} \geq 1.$$

In particular, if $\alpha_0 > \alpha_1$ (so that we can take $\alpha = \alpha_1$) and φ is such that $q_1 = q_2$ in (1.6), then there exists $c \in (0, 1)$ such that

$$c \leq \frac{|\{u > 0\} \cap B_r(x_0)|}{|B_r(x_0)|} \leq 1 - c,$$

for all $r \in (0, r_0)$.

Theorem 3 serves as a sort of cusp classification of the free boundary based on the blow-up rate of the cost function φ around a given free boundary point x_0. More precisely, if φ blows up at x_0 in an uneven (in measure) fashion — that is the case when $q_1 < q_2$ — then one deduces from the density estimates of Theorem 3 that

$$c|B_r(x_0)|^{\frac{\alpha_2-\alpha}{\alpha}} \leq \frac{|\{u > 0\} \cap B_r(x_0)|}{|B_r(x_0)|} \leq 1 - c|B_r(x_0)|^{P(q_1,q_2)-1}.$$

The geometric-measure estimates established in this case limits the regularity of $\partial\{u > 0\}$ at x_0. Note that, since u solves (1.2) in $\{u > 0\}$ and is not Lipschitz at a boundary point x_0 where (1.5) holds, we conclude that $\{u > 0\}$ cannot satisfy the exterior sphere condition. Even further, the exponent $1 - d/(2p)$ in (1.5) yields an upper limit (depending on d, p, λ, and Λ) on the aperture of any exterior cone touching the free boundary at x_0.

The case when the cost function φ blows-up at a steady rate, i.e. for $q_1 = q_2$, then

$$P(q_1, q_2) = 1 = \frac{\alpha_2}{\alpha_1},$$

and we recover the classical theory. In particular, in this case one obtains the porosity of the free boundary and the local finiteness of the $\mathcal{H}^{d-\sigma}$-Hausdorff measure of the free boundary, for a universal constant $0 < \sigma \leq 1$.

Finally, we note that the exponents α_2/α and $P(q_1, q_2)$ in Theorem 3 may eventually not be sharp, except in the case $q_1 = q_2$ and $\alpha_0 > \alpha_1$, where both exponents equal 1.

When dealing with free boundary minimization problems involving an L^q_{weak} Bernoulli cost function φ, a natural question would be whether the free boundary eventually meets the set $\varphi^{-1}(\infty)$. This question is addressed in the last Section 7. We state the conclusion as a theorem:

Theorem 4. *Let u be a minimizer of $\mathcal{J}_{A,\varphi}$ with $\varphi \in L^q_{\text{weak}}$. The set of free boundary points intersecting $\varphi^{-1}(\infty)$ is, in principle, unavoidable. More precisely, for any $d \geq 3$ and $q > d/2$, there exists $\varphi \in L^q_{\text{weak}}(B_1)$ and a constant $m > 0$ such that the minimizer u of $\mathcal{J}_{I,\varphi}(B_1)$ over $H^1_m(B_1)$ satisfies $\varphi^{-1}(\infty) \cap \partial\{u > 0\} \neq \emptyset$. Here, B_1 is the unit ball in $\mathbb{R}^d$,*

$\mathcal{J}_{I,\varphi}$ is the functional (1.1) with $A(x) \equiv I$, and $H^1_m(B_1)$ is the set of $H^1(B_1)$ functions with constant trace equal to m on ∂B_1.

1.1. **Outline of the proofs and related open questions.** In this subsection we discuss the nuances and heuristics of the proofs, leading to a plethora of challenging open questions for future collaborative endeavors. While unraveling these concepts, we might occasionally prioritize fluency over strict mathematical precision.

Variational free boundary problems frequently find representation through the minimization of discontinuous functionals, defined over appropriate functional spaces, expressed as follows:

$$\mathscr{J}(\Omega, u) = \int_\Omega F(x, Du)dx + \int_\Omega f(x,u)dx. \tag{1.8}$$

The term $F(x, Du)$ encodes diffusion as well as physical properties of the medium in which the system takes place. It can be local or nonlocal, linear or nonlinear, with constant or varying coefficients, etc. Solutions to the corresponding homogeneous Euler-Lagrange equation, $\operatorname{div}\left(D_pF(x, Du)\right) = 0$, are bound to a specific (maximal) regularity theory. When such regularity estimates are below the expected free boundary geometry, the problem becomes much more challenging. This is the case of Theorem 1, when $\alpha_0 \leq 1 - d/(2q)$ as well as of Theorem 2.

The second term, $f(x,u)$, specifies the ruptures along the free interfaces. Typically, either $u \mapsto f(x,u)$ and/or $u \mapsto \frac{\partial f}{\partial u}(x,u)$ are discontinuous functions. In heuristic terms, the more singular $f(x,u)$ is, the less smooth solutions are *along the interfaces*. Such a drop in the smoothness of solutions is captured by their precise geometric behavior across the free boundary. However, this is often a rather delicate issue.

A way to interpret Theorem 1 is then as follows: if the (maximal) regularity theory available for A-harmonic functions "*permits*" $(1 - d/(2q))$–cones, then minimizers will attain the level of regularity dictated by their geometric behavior along the free boundary, viz. locally of class $C^{0,1-\frac{d}{2q}}$. In the complementary case, i.e., when A-harmonic functions are in general less smooth than the expected free boundary geometric behavior, then the maximal regularity theory available for A-harmonic functions restricts the smoothness of minimizers. In this case, however, Theorem 1 is an asymptotically optimal result, i.e. minimizers are *almost* as regular as A-harmonic functions. Whether or not one can achieve $\alpha = \alpha_0$ in Theorem 1 (in the case that $\alpha_0 \leq 1 - d/(2q)$) remains an interesting open question.

Similarly, one could ask what happens in the case $q < \frac{d}{2}$. This is also an interesting line of investigation, as we expect based on the scaling properties that only some local L^μ estimate could be available (at least if the boundary datum is not bounded).

The discussion above also serves to highlight the remarkable nature of Theorem 2. In essence, it asserts that in the one-phase case, the behavior of minimizers along the free boundary disregards the roughness of the medium. The reason why one can attain an improved regularity along the free boundary is subtle and we explain it here in heuristic terms. The proof of both Theorem 1 and Theorem 2 are motivated by the rather powerful compactness method, originally introduced by Caffarelli, in [6], in the context of fully nonlinear elliptic equations. In the two-phase problem, the *tangential* space, i.e. the set of functions one arrives at after the compactness procedure, are A-harmonic functions. The method then seizes the regularity available within the *tangential* space and imports back to the original problem adjusted through the caliber of the tangential path. In the one-phase case, however, the *tangential* space is only formed by hyperplanes. This is a consequence of the Harnack inequality. This is why we are able to bypass the restrictions arising from the maximal regularity theory of A-harmonic functions. A very interesting, though, difficult question is whether a similar result can be attained in the two-phase case. Our impression is that, if no further conditions are imposed (say one-sided control, or higher medium organization), then minimizers of the two-phase problem will not present an improved regularity along the free boundary.

1.2. **Outline.** The rest of the paper is organized as follows. In Section 2 we discuss some preliminary results needed for the proofs of the main Theorems. In particular we discuss the scaling feature of the problem and establish a Caccioppoli-type estimate. In Section 3 we prove Theorem 1, by means of a careful approximation analysis. In the intermediary Section 4 we discuss the limiting case when $\varphi \in L^{d/2}_{\text{weak}}$, and obtain universal BMO estimates. In Section 5 we establish the sharp $C^{1-d/(2q)}$ regularity at one-phase free boundary points. The interesting feature here is that such an estimate is not limited by the universal regularity theory of A-harmonic functions. Section 6 is devoted to the geometric analysis of the free boundary, where we prove Theorem 3. Finally, in Section 7 we prove Theorem 4, which contains an example showing the free boundary may indeed intersect the infinite points of the Bernoulli function φ in a non-trivial subregion of the domain.

2. Preliminaries

We begin by recalling that for $q \geq 1$, a function f lies in the space $L^q_{\text{weak}}(\Omega)$ if

$$\|f\|_{L^q_{\text{weak}}(\Omega)} := \sup_{t>0} t\,|\{x \in \Omega : f(x) > t\}|^{1/q} < \infty.$$

Note that existence of minimizers for $\mathcal{J}_{A,\varphi}$ over $H^1_g(\Omega)$ follows as in [1], so we omit the details.

Next, let us investigate how the minimization problem for $\mathcal{J}_{A,\varphi}$ transforms under translation and rescaling around a point. For $\gamma \in \mathbb{R}$, define the more general functional

$$\mathcal{J}_{A,\varphi,\gamma}(u) := \int_\Omega \left(\nabla u \cdot (A(x)\nabla u) + \varphi(x)1_{\{u>\gamma\}}\right) \mathrm{d}x.$$

Clearly, $\mathcal{J}_{A,\varphi} = \mathcal{J}_{A,\varphi,0}$. We suppress the dependence of $\mathcal{J}_{A,\varphi,\gamma}$ on the domain Ω, which will always be clear from context.

Lemma 1. *For some domain Ω, $\varphi \in L^q_{\text{weak}}(\Omega)$, and boundary data $g \in H^{1/2}(\partial\Omega)$, let u be a minimizer of $\mathcal{J}_{A,\varphi}$ over $H^1_g(\Omega)$. Then, for any $x_0 \in \Omega$, $\kappa \geq 0$, $0 < r < \operatorname{dist}(x_0, \partial\Omega)$, and $\gamma \in \mathbb{R}$, the function*

$$v(x) := \kappa(u(x_0 + rx) - \gamma)$$

is a minimizer of $\mathcal{J}_{\tilde{A},\tilde{\varphi},\tilde{\gamma}}$ over $H^1_{\tilde{g}}(B_1)$, with

$$\begin{aligned}
\tilde{\varphi}(x) &= \kappa^2 r^2 \varphi(x_0 + rx),\\
\tilde{\gamma} &= -\kappa\gamma,\\
\tilde{A}(x) &= A(x_0 + rx),\\
\tilde{g}(x) &= \kappa(u(x_0 + rx) - \gamma), \quad x \in \partial B_1.
\end{aligned}$$

Proof. Direct calculation. □

Next, by a standard argument, we can show that minimizers of $\mathcal{J}_{A,\varphi,\gamma}$ are subsolutions of the homogeneous equation (1.2):

Lemma 2. *Let u be a minimizer of $\mathcal{J}_{A,\varphi,\gamma}$ over*

$$H^1_g(\Omega) := \{w \in H^1(\Omega), w = g \text{ on } \partial\Omega \text{ in trace sense}\},$$

for some $g \in H^{1/2}(\partial\Omega)$. Then

$$\int_\Omega (A(x)\nabla u)\cdot\nabla v \geq 0, \quad v \in C_0^\infty(\Omega), v \geq 0.$$

Proof. This lemma follows by noting that $\mathcal{J}_{A,\varphi,\gamma}(u) \leq \mathcal{J}_{A,\varphi,\gamma}(u - \varepsilon v)$ and $1_{\{u-\varepsilon v>\gamma\}} \leq 1_{\{u>\gamma\}}$ for all $\varepsilon > 0$ and all nonnegative $v \in C_0^\infty(\Omega)$. □

From Lemma 2 and the maximum principle, we conclude that minimizers are bounded whenever $g \in L^\infty(\partial\Omega)$, with the estimate $\|u\|_{L^\infty(\Omega)} \leq \|g\|_{L^\infty(\partial\Omega)}$.

Our next lemma is a Caccioppoli-type estimate that will be needed in our approximation argument:

Lemma 3. *There exists a constant $C > 0$ depending on d, q, λ, and Λ, such that any minimizer u of $\mathcal{J}_{A,\varphi,\gamma}$ over $H^1_g(B_1)$ satisfies*

$$\int_{B_{1/2}} |\nabla u|^2 \,\mathrm{d}x \leq C\left(\int_{B_1} |u|^2 \,\mathrm{d}x + \|\varphi\|_{L^q_{\text{weak}}(B_1)}\right).$$

Note that u is allowed to change sign in this lemma. If we were concerned only with the one-phase problem, u would be a nonnegative subsolution of (1.2), and we could apply an existing Caccioppoli estimate such as [16, Lemma 3.27].

Proof. For any $\zeta \in C_0^\infty(B_1)$ with $0 \leq \zeta \leq 1$, we let $w = u(1-\zeta^2)$ so that $w = u$ on ∂B_1. The minimizing property $\mathcal{J}_{A,\varphi,\gamma}(u) \leq \mathcal{J}_{A,\varphi,\gamma}(w)$ implies

$$\int_{B_1} \nabla u \cdot (A(x)\nabla u)\,\mathrm{d}x \leq \int_{B_1} \left[\nabla w \cdot (A(x)\nabla w) + \varphi(x)\left(1_{\{w>\gamma\}} - 1_{\{u>\gamma\}}\right)\right] \mathrm{d}x. \tag{2.1}$$

With $w = u(1-\zeta^2)$, straightforward calculations imply

$$\begin{aligned}\int_{B_1} \nabla u \cdot (A(x)\nabla u)\zeta^2(2-\zeta^2)\,\mathrm{d}x \leq 4\int_{B_1} \left(u^2\zeta^2\nabla\zeta\cdot(A(x)\nabla\zeta) - u\zeta(1-\zeta^2)\nabla\zeta\cdot(A(x)\nabla u)\right)\mathrm{d}x \\ + C_{d,q}\|\varphi\|_{L^q_{\text{weak}}(B_1)}.\end{aligned}$$

In the last term on the right, we used Hölder's inequality together with the fact that $\|\varphi\|_{L^r(B_1)} \lesssim \|\varphi\|_{L^q_{\text{weak}}(B_1)}$ for $1 < r < q$. With $0 \leq 1-\zeta^2 \leq 1$ and Young's inequality, we have

$$\begin{aligned}\lambda\int_{B_1} \zeta^2|\nabla u|^2\,\mathrm{d}x &\leq 4\Lambda\int_{B_1}\left(u^2|\nabla\zeta|^2 + |u|\zeta|\nabla\zeta||\nabla u|\right)\mathrm{d}x + C_{d,q}\|\varphi\|_{L^q_{\text{weak}}(B_1)} \\ &\leq \frac{\lambda}{2}\int_{B_1}\zeta^2|\nabla u|^2\,\mathrm{d}x + \frac{2\Lambda}{\lambda}\int_{B_1} u^2|\nabla\zeta|^2\,\mathrm{d}x + C_{d,q}\|\varphi\|_{L^q_{\text{weak}}(B_1)}.\end{aligned}$$

Choosing ζ equal to 1 in $B_{1/2}$ and 0 outside $B_{3/4}$, with $|\nabla\zeta|$ bounded by a constant, the proof is complete. □

3. Hölder continuity

In this section, we establish a universal Hölder estimate for the one- and two-phase problems. In Section 5, we will further improve such an estimate (at the free boundary) to the sharp exponent $1 - d/(2q)$, regardless of the regularity of A-harmonic functions.

The analysis will be based on the following key approximation lemma, which says minimizers of $\mathcal{J}_{A,\varphi,\gamma}$ are close to minimizers of $\int_\Omega \nabla u \cdot (A(x)\nabla u)\,\mathrm{d}x$, if the norm of φ is sufficiently small in L^q_{weak}, c.f. [29, 30, 31] for related analysis employed in "non-free boundary" PDE models.

Lemma 4. *Given $\tau > 0$, there exists $\varepsilon = \varepsilon(d, q, \lambda, \Lambda, \tau) > 0$ such that for any $\varphi \in L^q_{\text{weak}}(B_1)$ with $q \in (1, \infty)$ and $\|\varphi\|_{L^q_{\text{weak}}(B_1)} \le \varepsilon$, any $A \in M_{\lambda,\Lambda}(B_1)$, any $g \in H^{1/2}(\partial B_1)$, any $\gamma \in \mathbb{R}$, and any minimizer u of $\mathcal{J}_{A,\varphi,\gamma}$ over $H^1_g(B_1)$ such that $⨍_{B_1} u^2\,\mathrm{d}x \le 1$, there holds*

$$\int_{B_{1/2}} |u - h|^2\,\mathrm{d}x \le \tau,$$

where $h \in H^1(B_{1/2})$ satisfies $⨍_{B_{1/2}} h^2\,\mathrm{d}x \le 2^{d+2}$ and is a weak solution to

$$\nabla \cdot (A(x)\nabla h) = 0, \quad x \in B_{1/2}. \tag{3.1}$$

In fact, we can take h to be the A-harmonic lifting of u in $B_{1/2}$.

Proof. With u as in the statement of the lemma, define $h : B_{1/2} \to \mathbb{R}$ as the minimizer of

$$F(w) := \int_{B_{1/2}} \nabla w \cdot (A(x)\nabla w)\,\mathrm{d}x,$$

over

$$\{w \in H^1(B_{1/2}) : w - u \in H^1_0(B_{1/2})\}.$$

Let us also define $h(x) = u(x)$ for $x \in B_1 \setminus B_{1/2}$. By standard arguments, this h is the weak solution to

$$\begin{cases} \nabla \cdot (A(x)\nabla h) = 0, & x \in B_{1/2}, \\ h - u \in H^1_0(B_{1/2}), \end{cases}$$

which in particular implies the identity $\int_{B_{1/2}} \nabla(u-h)\cdot(A(x)\nabla h)\,\mathrm{d}x = 0$. Using this identity, Poincaré's inequality, and the minimizing property of u, we have

$$\begin{aligned} \int_{B_{1/2}} |u - h|^2\,\mathrm{d}x &\le C \int_{B_{1/2}} |\nabla(u - h)|^2\,\mathrm{d}x \\ &\le \frac{C}{\lambda} \int_{B_{1/2}} \nabla(u - h)\cdot(A(x)\nabla(u - h))\,\mathrm{d}x \\ &= \frac{C}{\lambda}\left(\int_{B_{1/2}} \nabla u \cdot (A(x)\nabla u)\,\mathrm{d}x - \int_{B_{1/2}} \nabla h \cdot (A(x)\nabla h)\,\mathrm{d}x\right) \\ &\le \frac{C}{\lambda}\left(\int_{B_{1/2}} \varphi(x)(1_{\{h>\gamma\}} - 1_{\{u>\gamma\}})\,\mathrm{d}x\right) \\ &\le \frac{C}{\lambda}\int_{B_{1/2}} \varphi\,\mathrm{d}x \\ &\le \frac{C}{\lambda}\|\varphi\|_{L^q_{\text{weak}}(B_1)} \le \frac{C}{\lambda}\varepsilon, \end{aligned} \tag{3.2}$$

for some constant $C > 0$ depending on q and d. Choosing $\varepsilon \le \tau\lambda/C$, we have

$$\int_{B_{1/2}} |u-h|^2 \,\mathrm{d}x \le \tau,$$

as desired. By choosing ε smaller if necessary, depending only on d, C, and λ, we also ensure that

$$\fint_{B_{1/2}} h^2 \,\mathrm{d}x \le 2\fint_{B_{1/2}} (u-h)^2 \,\mathrm{d}x + 2\fint_{B_{1/2}} u^2 \,\mathrm{d}x \le \frac{2^{d+1}C\varepsilon}{\lambda\omega_d} + 2^{d+1} \le 2^{d+2},$$

where ω_d is the volume of the d-dimensional unit ball, and we have used $\fint_{B_1} u^2 \,\mathrm{d}x \le 1$. This completes the proof. □

Next, we show that when $\|\varphi\|_{L^q_{\text{weak}}}$ is small, since minimizers u are close to solutions h of the homogeneous equation (3.1), the Hölder regularity of h implies some local integrability estimates for u:

Lemma 5. *Let $\alpha_0 = \alpha_0(d,\lambda,\Lambda)$ be the exponent from (1.3). For any $\alpha \in (0,\alpha_0)$, there exist $\varepsilon > 0$, $r_0 \in (0,\frac{1}{4})$, $K > 0$ and $C_0 > 0$, depending only on universal quantities and α, such that for any minimizer u of $\mathcal{J}_{A,\varphi,\gamma}$ over $H^1_g(B_1)$ with $\|\varphi\|_{L^q_{\text{weak}}(B_1)} \le \varepsilon$, $\gamma \in \mathbb{R}$, $A \in M_{\lambda,\Lambda}(B_1)$, and $\fint_{B_1} u^2 \,\mathrm{d}x \le 1$, there holds*

$$\fint_{B_{r_0}} |u-\mu|^2 \,\mathrm{d}x \le r_0^{2\alpha}, \tag{3.3}$$

for some constant μ with $|\mu| \le K$.

Proof. Let $\tau > 0$ be a constant to be chosen later. With u as in the statement of the lemma, let $h : B \to \mathbb{R}$ be the solution to (3.1) in $B_{1/2}$ with

$$\int_{B_{1/2}} |u-h|^2 \,\mathrm{d}x < \tau,$$

whose existence is granted by Lemma 4. The choice of τ will determine ε. Since $\fint_{B_{1/2}} h^2 \,\mathrm{d}x \le 2^{d+2}$, the interior regularity theory of the equation satisfied by h (e.g. [15, Theorem 8.24]) implies there is a $C_0 > 0$ with

$$|h(x) - h(0)| \le C_0|x|^{\alpha_0},$$

for any $x \in B_{1/4}$. For any $r_0 \in (0,\frac{1}{4})$, we then have

$$\begin{aligned} \fint_{B_{r_0}} |u(x)-h(0)|^2 \,\mathrm{d}x &\le 2\left(\fint_{B_{r_0}} |u(x)-h(x)|^2 \,\mathrm{d}x + \fint_{B_{r_0}} |h(x)-h(0)|^2 \,\mathrm{d}x\right) \\ &\le 2\omega_d^{-1} r_0^{-d}\tau + 2C_0 r_0^{2\alpha_0}. \end{aligned} \tag{3.4}$$

where ω_d is the volume of the d-dimensional unit ball. Choosing

$$r_0 = \left(\frac{1}{4C_0}\right)^{1/[2(\alpha_0-\alpha)]}, \quad \tau = \frac{1}{4}\omega_d r_0^{d+2\alpha},$$

we have

$$\fint_{B_{r_0}} |u(x)-h(0)|^2 \,\mathrm{d}x \le r_0^{2\alpha}.$$

The estimate (3.3) now follows by choosing $\mu = h(0)$. The bound $|\mu| \le K$ is a result of the interior L^2-to-L^∞ estimate satisfied by h [15, Theorem 8.17]. □

Next, by iterating Lemma 5, we show that under similar conditions, minimizers are Hölder continuous at the origin.

Lemma 6. *For $\frac{d}{2} < q < \infty$, let $\alpha > 0$ satisfy*

$$\alpha \in (0, 1 - d/(2q)] \cap (0, \alpha_0),$$

where α_0 is as in Lemma 5. Then there exist $\varepsilon, C > 0$ and $r_0 \in (0, \frac{1}{4})$, depending on universal quantities and α, such that for any minimizer u of $\mathcal{J}_{A,\varphi}$ over $H^1_g(B_1)$ with $\|\varphi\|_{L^q_{\text{weak}}(B_1)} \leq \varepsilon$ and $\fint_{B_1} u^2 \,\mathrm{d}x \leq 1$, there holds

$$|u(x)| \leq C \quad \text{and} \quad |u(x) - u(0)| \leq C|x|^\alpha, \quad \text{if } |x| < r_0.$$

Proof. With α as in the statement of the lemma, let ε and r_0 be the corresponding constants from Lemma 5, and let u be a minimizer of $\mathcal{J}_{A,\varphi}$ over $H^1_g(B_1)$. Our goal is to show by induction that

$$\fint_{B_{r_0^k}} (u - \mu_k)^2 \,\mathrm{d}x \leq r_0^{2k\alpha}, \quad k = 1, 2, \ldots, \tag{3.5}$$

for some convergent sequence μ_k.

The base case $k = 1$ follows directly from Lemma 5, for some constant μ_1 with $|\mu_1| \leq K$. Here, K depends only on universal quantities and δ. Now, assume (3.5) holds for some $k \geq 1$ and some constant μ_k. Define

$$v(x) = \frac{u(r_0^k x) - \mu_k}{r_0^{k\alpha}}, \quad x \in B_1.$$

By Lemma 1, v is a minimizer of $\mathcal{J}_{A_k,\varphi_k,\gamma_k}$ over $H^1_{g_k}(B_1)$, where

$$\begin{aligned}
\varphi_k(x) &= r_0^{2k(1-\alpha)} \varphi(r_0^k x), \\
\gamma_k &= -r_0^{k\alpha} \mu_k, \\
A_k(x) &= A(r_0^k x), \\
g_k(x) &= r_0^{-k\alpha}(u(r_0^k x) - \mu_k), \quad x \in \partial B_1.
\end{aligned}$$

The inductive hypothesis (3.5) implies

$$\fint_{B_1} v^2 \,\mathrm{d}x = r_0^{-2k\alpha} \fint_{B_{r_0^k}} (u - \mu_k)^2 \,\mathrm{d}x \leq 1.$$

Our choice of α implies $2 - 2\alpha - d/q \geq 0$, and

$$\|\varphi_k\|_{L^q_{\text{weak}}(B_1)} = r_0^{k(2-2\alpha-d/q)} \|\varphi\|_{L^q_{\text{weak}}(B_{r_0^k})} \leq \|\varphi\|_{L^q_{\text{weak}}(B_1)} < \varepsilon.$$

Therefore, we can apply Lemma 5 to conclude

$$\fint_{B_{r_0}} |v - \mu|^2 \,\mathrm{d}x \leq r_0^{2\alpha},$$

for some constant μ with $|\mu| \leq K$. Translating back to u with the change of variables $x \mapsto r_0^k x$, the previous inequality becomes

$$\fint_{B_{r_0^{k+1}}} |u - \mu_k - r_0^{k\alpha}\mu|^2 \,\mathrm{d}x \leq r_0^{2\alpha(k+1)}.$$

We have established (3.5) with $\mu_{k+1} = \mu_k + r_0^{k\alpha}\mu$. To show that the sequence $\{\mu_k\}$ is convergent, note that $|\mu_{k+1} - \mu_k| = |r_0^{k\alpha}\mu| \le K r_0^{k\alpha}$, and for any $j > k$,

$$|\mu_j - \mu_k| \le K(1 - r_0^{\alpha})^{-1} r_0^{k\alpha}, \tag{3.6}$$

which demonstrates that $\{\mu_k\}$ is Cauchy. In fact, letting $\mu_0 = \lim_{k\to\infty}\mu_k$, taking $j \to \infty$ in (3.6) yields $|\mu_k - \mu_0| \le K(1 - r_0^{\alpha})^{-1} r_0^{k\alpha}$ for all k.

Finally, for $0 < r \le r_0$, choose k such that $r_0^{k+1} < r \le r_0^k$. Then, by (3.5),

$$⨍_{B_r} |u - \mu_0|^2 \,\mathrm{d}x \le 2 ⨍_{B_r} |u - \mu_k|^2 \,\mathrm{d}x + 2|\mu_k - \mu_0|^2 \le r_0^{-2\alpha}\left(\frac{2}{r_0^d} + 2K^2(1 - r_0^{\alpha})^{-2}\right) r^{2\alpha},$$

which implies the Hölder estimate $|u(x) - u(0)| \le C|x|^{\alpha}$, with C as in the statement of the theorem. Since $|u(0)| = |\mu_0| \le K(1 - r_0^{\alpha})^{-1} + K$, the triangle inequality implies $|u(x)| \le C$ in B_{r_0}, with C as above. □

We are now ready to prove the main result of this section:

Theorem 5. *Let u be a minimizer of $\mathcal{J}_{A,\varphi}$ over $H^1_g(\Omega)$, with $g \in H^{1/2}(\partial\Omega)$. For any $\Omega' \subset\subset \Omega$, if $\varphi \in L^q_{\mathrm{weak}}(\Omega')$ with $d/2 < q < \infty$, then u is Hölder continuous in Ω', with*

$$\|u\|_{C^{\alpha}(\Omega')} \le C\|u\|_{L^2(\Omega)},$$

where α is as in Lemma 6, and C depends only on universal quantities, α, Ω', Ω, and $\|\varphi\|_{L^q_{\mathrm{weak}}(\Omega)}$.

Proof. Let u be as in the statement, and let $x_0 \in \Omega$ be arbitrary. To recenter around x_0 and ensure the hypotheses of Lemma 6 are satisfied, we define

$$w(x) := \kappa u(x_0 + rx), \quad x \in B_1,$$

where

$$r = \min\left\{\frac{1}{2}\mathrm{dist}(x_0, \partial\Omega), \left(\frac{\varepsilon}{\|\varphi\|_{L^q_{\mathrm{weak}}(\Omega)}}\right)^{q/(2q-d)}\right\}, \quad \kappa = \left(\frac{1}{⨍_{B_r(x_0)} u^2 \,\mathrm{d}x}\right)^{1/2},$$

and ε is the constant from Lemma 5. Then $⨍_{B_1} w^p \,\mathrm{d}x \le 1$, and by Lemma 1, w minimizes $\mathcal{J}_{\tilde{A},\tilde{\varphi}}$ over $H^1_{\tilde{g}}(B_1)$, with $\tilde{A}(x) = A(x_0 + rx)$, $\tilde{\varphi}(x) = r^2\kappa^2\varphi(x_0 + rx)$, and $\tilde{g}(x) = \kappa g(x_0 + rx)$ for $x \in \partial B_1$. Since $\|\tilde{\varphi}\|_{L^q_{\mathrm{weak}}(B_1)} \le \kappa^2 r^{2-d/q}\|\varphi\|_{L^q_{\mathrm{weak}}(\Omega)} < \varepsilon$, we may apply Lemma 6 and obtain $|w(x) - w(0)| \le C|x|^{\alpha}$ for $|x| \le r_0$. Since C and r_0 depend only on d, q, λ, and Λ, when we translate back to u, we conclude that

$$|u(x) - u(x_0)| \le C\|u\|_{L^2(\Omega)}|x - x_0|^{\alpha}, \quad |x - x_0| \le r',$$

with C and r' depending only on d, q, λ, Λ, and $\mathrm{dist}(x_0, \partial\Omega)$. The L^{∞} norm of u is also bounded uniformly in any compact subset of Ω, by Lemma 6. □

4. The borderline case $q = d/2$

In the limiting case $q \searrow d/2$, the corresponding exponent α in the iteration of Lemma 6 becomes zero. As usual, one should not expect to obtain continuity. In this section we obtain a (sharp) estimate in the space of bounded mean oscillation (BMO) functions. Recall that a function f on Ω is BMO if

$$\|f\|_{BMO(\Omega)} := \sup\left\{N : ⨍_B |f - f_B| \,\mathrm{d}x \le N \text{ for every ball } B \subset \Omega\right\} < \infty,$$

where f_B is the average of f over the ball B.

First, we need a corresponding version of Lemma 5 for a BMO-type estimate:

Lemma 7. *There exist $\varepsilon > 0$ and $r_0 \in (0, \frac{1}{4})$, depending only on d, λ, and Λ, such that for any minimizer u of $\mathcal{J}_{A,\varphi,\gamma}$ over $H^1_g(B_1)$ with $\|\varphi\|_{L^{d/2}_{\text{weak}}(B_1)} \le \varepsilon$, $\gamma \in \mathbb{R}$, and $⨍_{B_1} u^2\,\mathrm{d}x \le 1$, there holds*

$$⨍_{B_{r_0}} |u - u_{r_0}|^2\,\mathrm{d}x \le 1, \tag{4.1}$$

where $u_{r_0} = ⨍_{B_{r_0}} u\,\mathrm{d}x$.

Proof. First, we recall a general inequality: for any $\mu \in \mathbb{R}$ and $r_0 > 0$,

$$⨍_{B_{r_0}} |u - u_{r_0}|^2\,\mathrm{d}x \le 4 ⨍_{B_{r_0}} |u - \mu|^2\,\mathrm{d}x. \tag{4.2}$$

This inequality follows by writing

$$\begin{aligned}\left(⨍_{B_{r_0}} \left|u - ⨍_{B_{r_0}} u\,\mathrm{d}x\right|^2 \mathrm{d}x\right)^{1/2} &\le \left(⨍_{B_{r_0}} |u-\mu|^2\,\mathrm{d}x\right)^{1/2} + \left(⨍_{B_{r_0}} |\mu - ⨍_{B_{r_0}} u\,\mathrm{d}x|^2\,\mathrm{d}x\right)^{1/2}\\ &\le \left(⨍_{B_{r_0}} |u-\mu|^2\,\mathrm{d}x\right)^{1/2} + ⨍_{B_{r_0}} |u-\mu|\,\mathrm{d}x\\ &\le 2\left(⨍_{B_{r_0}} |u-\mu|^2\,\mathrm{d}x\right)^{1/2}.\end{aligned}$$

Now, let $\tau > 0$ be a constant to be chosen later. As in the proof of Lemma 5, let $h : B_{1/2} \to \mathbb{R}$ be the solution of the homogeneous equation (3.1) with

$$\int_{B_{1/2}} |u - h|^2\,\mathrm{d}x < \tau,$$

given by Lemma 4. As above, we have

$$|h(x) - h(0)| \le C|x|^\beta,$$

for any $x \in B_{1/4}$, for some $C > 0$ and $\beta \in (0, 1)$ depending only on d, λ, and Λ. For $r_0 \in (0, \frac{1}{4})$, we then have

$$\begin{aligned}⨍_{B_{r_0}} |u(x) - h(0)|^2\,\mathrm{d}x &\le 2\left(⨍_{B_{r_0}} |u(x) - h(x)|^2\,\mathrm{d}x + ⨍_{B_{r_0}} |h(x) - h(0)|^2\,\mathrm{d}x\right)\\ &\le 2\omega_d^{-1} r_0^{-d}\tau + 2Cr_0^{2\beta}.\end{aligned} \tag{4.3}$$

Choosing

$$r_0 = \left(\frac{1}{16C}\right)^{1/(2\beta)}, \quad \tau = \frac{\omega_d r_0^d}{16},$$

we have

$$⨍_{B_{r_0}} |u(x) - h(0)|^2\,\mathrm{d}x \le \frac{1}{4}.$$

Applying (4.2) with $\mu = h(0)$, the conclusion of the lemma follows, with $\varepsilon > 0$ determined from our choice of τ via Lemma 4. ◻

Lemma 8. *There exist $\varepsilon > 0$, $r_0 \in (0, \frac{1}{4})$ and $C > 0$, depending only on d, λ, and Λ, such that if u is a minimizer of $\mathcal{J}_{A,\varphi}$ over $H^1_g(B_1)$ with $\|\varphi\|_{L^{d/2}_{\text{weak}}(B_1)} \leq \varepsilon$ and $⨍_{B_1} u^2 \, \mathrm{d}x \leq 1$, then for any $r \in (0, r_0]$,*

$$⨍_{B_r} |u - u_r|^2 \, \mathrm{d}x \leq C.$$

Proof. Let ε and r_0 be the constants from Lemma 7, and let u be a minimizer of $\mathcal{J}_{A,\varphi}$ over $H^1_g(B_1)$. Our goal is to show by induction that

$$⨍_{B_{r_0^k}} |u - u_{r_0^k}|^2 \, \mathrm{d}x \leq 1, \quad k = 1, 2, \dots. \tag{4.4}$$

The base case $k = 1$ follows directly from Lemma 7. Assuming that (4.4) holds for some $k \geq 1$, define

$$v(x) = u(r_0^k x) - u_{r_0^k}, \quad x \in B_1.$$

By Lemma 1, v is a minimizer of $\mathcal{J}_{A_k,\varphi_k,\gamma_k}$ over $H^1_{g_k}(B_1)$, where

$$\begin{aligned} A_k(x) &= A(x_0 + r_0 x), \\ \varphi_k(x) &= r_0^{2k} \varphi(r_0^k x), \\ \gamma_k &= -u_{r_0^k}, \\ g_k(x) &= u(r_0^k x) - u_{r_0^k}, \quad x \in \partial B_1. \end{aligned}$$

The inductive hypothesis (4.4) implies

$$⨍_{B_1} v^2 \, \mathrm{d}x = ⨍_{B_{r_0^k}} |u - u_{r_0^k}|^2 \, \mathrm{d}x \leq 1.$$

With the choice $q = d/2$, the scaling of $\|\varphi_k\|_{L^q_{\text{weak}}}$ is as follows:

$$\|\varphi_k\|_{L^{d/2}_{\text{weak}}(B_1)} = \|\varphi\|_{L^{d/2}_{\text{weak}}(B_{r_0^k})} \leq \|\varphi\|_{L^{d/2}_{\text{weak}}(B_1)} < \varepsilon,$$

and we can apply Lemma 7 to conclude

$$⨍_{B_{r_0}} |v - v_{r_0}|^2 \, \mathrm{d}x \leq 1, \tag{4.5}$$

A quick calculation shows $v_{r_0} = u_{r_0^{k+1}} - u_{r_0^k}$. Therefore, (4.5) implies

$$⨍_{B_{r_0^{k+1}}} |u - u_{r_0^{k+1}}|^2 \, \mathrm{d}x \leq 1,$$

and we conclude that (4.4) holds for all k.

Now, for $0 < r \leq r_0$, choose k such that $r_0^{k+1} < r \leq r_0^k$. By (4.4),

$$⨍_{B_r} |u - u_{r_0^k}|^2 \, \mathrm{d}x \leq \left(\frac{r_0^k}{r} \right)^d ⨍_{B_{r_0^k}} |u - u_{r_0^k}|^2 \, \mathrm{d}x \leq r_0^{-d},$$

and the proof is complete. ◻

Now we apply a scaled version of Lemma 8 to conclude interior BMO regularity for u:

Theorem 6. *For any $\varphi \in L^{d/2}(\Omega)$ and $g \in H^{1/2}(\partial\Omega)$, and any minimizer u of $\mathcal{J}_{A,\varphi}$ over $H^1_g(\Omega)$, there holds for any $\Omega' \subset\subset \Omega$,*

$$\|u\|_{BMO(\Omega')} \leq C\|u\|_{L^2(\Omega)},$$

where the constant C depends only on universal constants, $\|\varphi\|_{L^{d/2}_{\text{weak}}(\Omega)}$, and Ω'.

Proof. Letting ε be the constant from Lemma 8, for any $x_0 \in \Omega$, we define

$$w(x) := \kappa u(x_0 + rx),$$

where

$$r = \min\left\{\frac{1}{2}\text{dist}(x_0, \partial B), 1\right\}, \quad \kappa^2 = \min\left\{\frac{1}{⨍_{B_r(x_0)} u^2\, \mathrm{d}x}, \frac{\varepsilon}{\|\varphi(x_0 + rx)\|_{L^{d/2}_{\text{weak}}(B_1)}}\right\}$$

Then, by Lemma 1, w minimizes $\mathcal{J}_{\tilde{A},\tilde{\varphi}}$ with $\tilde{A}(x) = A(x_0 + rx)$, $\tilde{\varphi}(x) = r^2\kappa^2\varphi(x_0 + rx)$, $⨍_{B_1} w^2 \leq 1$, and $\|\tilde{\varphi}\|_{L^{d/2}_{\text{weak}}(B_1)} \leq \varepsilon$. Applying Lemma 8 and translating back to u, we conclude that

$$⨍_{B_r(x_0)} |u - u_{B_r(x_0)}|^2\, \mathrm{d}x \leq C \int_\Omega u^2\, \mathrm{d}x,$$

for a constant C depending only on universal constants and $\text{dist}(x_0, \partial\Omega)$. □

5. Improved Hölder exponent

For the one-phase problem, at free boundary points, we are able to improve the Hölder estimate to obtain the sharp exponent $\alpha = 1 - d/(2q)$, regardless of the regularity theory available for A-harmonic functions.

First, we revisit the approximation argument of Lemma 4, equipped with the extra regularity provided by Lemma 6.

Lemma 9. *Given $\tau > 0$, there exists $\varepsilon = \varepsilon(d, \lambda, \Lambda, q, \tau) > 0$ such that for any $g \in H^{1/2}(\partial B_1)$ with $g \geq 0$, any $A \in M_{\lambda,\Lambda}(B_1)$, any φ such that $\|\varphi\|_{L^q_{\text{weak}}(B_1)} < \varepsilon$, and any minimizer u of $\mathcal{J}_{A,\varphi}$ over $H^1_g(B_1)$ with $⨍_{B_1} u^2\, \mathrm{d}x \leq 1$ and $u(0) = 0$, there holds*

$$\sup_{B_{1/10}} u \leq \tau.$$

Proof. Let h be the A-harmonic lifting of u in $B_{1/2}$, i.e. the weak solution of

$$\begin{aligned} \nabla\cdot(A(x)\nabla h) &= 0, \quad x \in B_{1/2}, \\ h - u &\in H^1_0(B_{1/2}). \end{aligned} \tag{5.1}$$

Let us choose ε small enough so that (i) the Hölder estimate of Lemma 6 applies, and (ii) from Lemma 4, the inequality

$$\int_{B_{1/2}} |u - h|^2\, \mathrm{d}x \leq \tau' \tag{5.2}$$

holds, for some τ' to be determined below. Recall from Lemma 2 that u is a subsolution of (5.1). Therefore, the comparison principle and $g \geq 0$ imply $0 \leq u \leq h$ in $B_{1/2}$, and since h satisfies the Harnack inequality [15, Theorem 8.20] as well as an L^2-to-L^∞ estimate [15, Theorem 8.17], we have for any $r \in (0, \frac{1}{2})$,

$$\sup_{B_{1/10}} u \leq \sup_{B_{1/10}} h \leq C_1 h(0) \leq C_2 r^{-d/2}\|h\|_{L^2(B_r)}, \tag{5.3}$$

where C_1 and C_2 depend only on d, λ, and Λ. Next, proceeding in a similar manner to the proof of Lemma 5, we have

$$\begin{aligned} r^{-d}\|h\|^2_{L^2(B_r)} &= \omega_d \fint_{B_r} h^2 \,\mathrm{d}x \\ &\le 2\omega_d \left(\fint_{B_r} |u-h|^2 \,\mathrm{d}x + \fint_{B_r} u^2 \,\mathrm{d}x \right) \\ &\le 2r^{-d}\tau' + 2\omega_d \fint_{B_r} u^2 \,\mathrm{d}x, \end{aligned} \tag{5.4}$$

by (5.2). For the last term on the right, we use the fact that $u(0) = 0$ and the Hölder estimate of Lemma 6 to obtain

$$\fint_{B_r} u^2 \,\mathrm{d}x = \fint_{B_r} |u(x) - u(0)|^2 \,\mathrm{d}x \le C^2 r^{2\alpha},$$

if $r < r_0$, where C, α, and r_0 are the constants from Lemma 6. Choosing

$$r = \min\left\{ \frac{r_0}{2}, \left[\frac{1}{4\omega_d C^2} \left(\frac{\tau}{C_2}\right)^2 \right]^{1/(2\alpha)} \right\}, \quad \tau' = \frac{r^d}{4}\left(\frac{\tau}{C_2}\right)^2,$$

then (5.3) and (5.4) imply $\sup_{B_{1/10}} u \le \tau$, as desired. □

Now we iterate Lemma 9 to prove Hölder regularity at boundary points, with the optimal exponent:

Proof of Theorem 2. With $\alpha = 1 - d/(2q)$, take $\tau = 1/10^\alpha$, and let ε be the corresponding constant from Lemma 9. As in the proof of Theorem 5, we choose κ and r such that, after replacing u with

$$w(x) := \kappa u(x_0 + rx),$$

we can ensure $w(0) = 0$, $\fint_{B_1} w^2 \,\mathrm{d}x \le 1$, and that w minimizes $\mathcal{J}_{A,\varphi}$ over $H^1_g(B_1)$ with $\|\varphi\|_{L^q_{\mathrm{weak}}(B_1)} \le \varepsilon$ and $g \ge 0$.

Applying Lemma 9, we have

$$\sup_{B_{1/10}} w \le \frac{1}{10^\alpha}.$$

Next, assume by induction that for some positive integer k,

$$\sup_{B_{10^{-k}}} w \le \frac{1}{10^{k\alpha}}. \tag{5.5}$$

By Lemma 1, the function

$$w_k(x) := 10^{k\alpha} w\left(\frac{x}{10^k}\right)$$

minimizes $\mathcal{J}_{A_k,\varphi_k}$ over $H^1_{g_k}(B_1)$ with

$$\begin{aligned} \varphi_k(x) &= 10^{2k(1-\alpha)}\varphi(10^{-k}x), \\ A_k(x) &= A(10^{-k}x), \\ g_k(x) &= 10^{k\alpha} w(10^{-k}x), \quad x \in \partial B_1. \end{aligned}$$

The inductive hypothesis gives

$$\fint_{B_1} w_k^2 \,\mathrm{d}x = 10^{2k\alpha} \fint_{B_{10^{-k}}} w^2 \,\mathrm{d}x \le 1,$$

and our choice of $\alpha = 1 - d/(2q)$ gives

$$\|\varphi_k\|_{L^q_{\text{weak}}(B_1)} = 10^{2k(1-\alpha)-kd/q}\|\varphi\|_{L^q_{\text{weak}}(B_{10^{-k}})} \leq \varepsilon.$$

Lemma 9 now implies $\sup_{B_{1/10}} w_k \leq 1/10$, or

$$\sup_{B_{10^{-(k+1)}}} w \leq \frac{1}{10^{(k+1)\alpha}},$$

and we have established that (5.5) holds for all $k = 1, 2, \ldots$.

Now, for any $x \in B_{1/10}$, there is some k such that $10^{-(k+1)} < |x| \leq 10^{-k}$. By (5.5), there holds

$$|w(x)| \leq \sup_{B_{10^{-k}}} \leq \frac{1}{10^{k\alpha}} \leq 10^{\alpha}|x|^{\alpha}.$$

After translating from w to u, the proof is complete. □

6. Geometry of the free boundary

The following lemma is a nondegeneracy estimate for u near free boundary points.

Lemma 10. *Let u be a minimizer of $\mathcal{J}_{A,\varphi}$ over $H^1_g(\Omega)$ with g and u nonnegative. Let $x_0 \in \partial\{u > 0\}$ be a free boundary point, and for some $p > 1$ and $\sigma \in [0, d)$, assume that there exist $c_0, r_0 > 0$ such that*

$$|\{\varphi(x) > t\} \cap B_r(x_0)| \geq \min\left(c_0 r^{\sigma} t^{-p}, |B_r(x_0)|\right), \quad \text{for all } t > 0, r \in (0, r_0). \tag{6.1}$$

Then, for any $x \in \Omega$ with $u(x) > 0$ such that $r := |x - x_0| = \operatorname{dist}(x, \partial\{u > 0\}) \leq r_0/2$, the estimate

$$u(x) \geq C|x - x_0|^{1-(d-\sigma)/(2p)}.$$

holds, with $C > 0$ depending only on d, p, λ, Λ, and c_0.

In (6.1), the case $\sigma = 0$ corresponds, for example, to an inverse power function $\varphi(x) = |x - x_0|^{-d/p} \in L^p_{\text{weak}}(\Omega)$. In such a case, Lemma 10 gives a lower bound for u with exponent $1 - d/(2p)$, which matches the asymptotics of the Hölder estimate of Theorem 2. The cases with $\sigma > 0$ include choices such as $\varphi(x) = \operatorname{dist}(x, \Gamma)^{(m-d)/p}$ for a smooth m-dimensional submanifold Γ, with $1 \leq m \leq d - 1$. This example is also in $L^p_{\text{weak}}(\Omega)$.

Proof of Lemma 10. Let $x = x_\varepsilon$ satisfy $u(x_\varepsilon) = \varepsilon > 0$. Our goal is to bound ε from below in terms of the proper power of r.

Since u satisfies the homogeneous equation (1.2) in $B_{3r/4}(x_\varepsilon) \subset \{u > 0\}$, the Harnack inequality, see for instance [28, Theorem 5], gives a constant $C_0 > 0$, independent of r, with

$$u(x) \leq C_0\varepsilon, \quad x \in \partial B_{r/2}(x_\varepsilon).$$

Let ζ be a smooth cutoff function equal to 0 in $B_{r/4}(x_\varepsilon)$ and equal to 1 outside $B_{r/2}(x_\varepsilon)$, with $|\nabla\zeta| \leq Cr^{-1}$. Define the test function

$$v(x) := \min\{u(x), C_0\varepsilon\zeta(x)\}.$$

By construction, $v = u$ on $\partial B_{r/2}(x_\varepsilon)$, so the minimizing property $\mathcal{J}_{A,\varphi}(u) \le \mathcal{J}_{A,\varphi}(v)$ implies

$$\begin{aligned}\int_{B_{r/2}(x_\varepsilon)} \varphi(x)\left(1_{\{u>0\}} - 1_{\{v>0\}}\right) \mathrm{d}x &\le \frac{1}{2}\int_{B_{r/2}(x_\varepsilon)} \left(\nabla v\cdot(A(x)\nabla v) - \nabla u\cdot(A(x)\nabla u)\right) \mathrm{d}x \\ &\le \frac{1}{2}\Lambda C_0^2\varepsilon^2 \int_{B_{r/2}(x_\varepsilon)\cap\{v>u\}} |\nabla\zeta|^2\,\mathrm{d}x \\ &\le \frac{1}{2}\Lambda C_0^2\varepsilon^2\omega_d(r/2)^d C r^{-2}.\end{aligned} \tag{6.2}$$

The left-hand side of this inequality can be bounded from below as follows, using our choice of ζ:

$$\int_{B_{r/2}(x_\varepsilon)} \varphi(x)\left(1_{\{u>0\}} - 1_{\{v>0\}}\right) \mathrm{d}x \ge \int_{B_{r/2}(x_\varepsilon)\setminus B_{r/4}(x_\varepsilon)} \varphi(x)\,\mathrm{d}x. \tag{6.3}$$

To bound the last expression from below, we choose a suitable $t > 0$ in the blow-up condition (6.1) so that $\varphi > t$ in a large percentage of $B_{2r}(x_0)$, which must include at least half of $B_{r/2}(x_\varepsilon) \setminus B_{r/4}(x_\varepsilon)$. In more detail, let

$$\mu_d = \frac{|B_{r/2}(x_\varepsilon)\setminus B_{r/4}(x_\varepsilon)|}{|B_{2r}(x_0)|} \in (0,1),$$
$$k = \left[\frac{1}{c_0} 2^d\omega_d(1-\mu_d/2)\right]^{-1/p},$$

where $\omega_d = |B_1|$. Then, since $2r \le r_0$, (6.1) with $t = kr^{(\sigma-d)/p}$ implies

$$|\{\varphi(x) > kr^{(\sigma-d)/p}\}\cap B_{2r}(x_0)| \ge c_0 k^{-p} r^d = (1-\mu_d/2)|B_{2r}(x_0)|.$$

Therefore,

$$|\{\varphi(x) > kr^{(\sigma-d)/p}\}\cap (B_{r/2}(x_\varepsilon)\setminus B_{r/4}(x_\varepsilon))| \ge \frac{1}{2}|B_{r/2}(x_\varepsilon)\setminus B_{r/4}(x_\varepsilon)|,$$

and

$$\int_{B_{r/2}(x_\varepsilon)\setminus B_{r/4}(x_\varepsilon)} \varphi(x)\,\mathrm{d}x \ge \frac{1}{2}|B_{r/2}(x_\varepsilon)\setminus B_{r/4}(x_\varepsilon)| k r^{(\sigma-d)/p} = c_d k r^{d(1-1/p)+\sigma/q}.$$

Note that k depends only on d, p, and c_0. Combining this with (6.2), we finally have

$$\varepsilon^2 \ge C r^{2-(d-\sigma)/p},$$

for a constant $C > 0$ as in the statement of the lemma, and the proof is complete. □

Next, we have a generalization of Lemma 10 for Bernoulli functions φ that are singular as $x \to x_0$ only from certain directions. A typical example would be a φ that blows up on one side of a hypersurface but is bounded on the other side, such as $\varphi(x) = 1 + 1_{\{x_1>0\}} x_1^{-1/p}$. We are mainly interested in this generalization so that we can rigorously prove our example in Section 7 has a free boundary point where $|\nabla u|$ is infinite.

Lemma 11. *Let u be a minimizer of $\mathcal{J}_{A,\varphi}$ over $H^1_g(\Omega)$ with g and u nonnegative. Let $x_0 \in \partial\{u > 0\}$ be a free boundary point, and for some $p > 1$, $\sigma \in [0, d)$, and some cone $\Xi \subset \mathbb{R}^d$ with vertex at x_0, assume that there exist $c_0, r_0 > 0$ such that*

$$|\{\varphi(x) > t\} \cap B_r(x_0) \cap \Xi| \ge \min\left(c_0 r^\sigma t^{-p}, |B_r(x_0)\cap\Xi|\right), \quad \textit{for all } t > 0, r \in (0, r_0). \tag{6.4}$$

Then, for any $x \in \Omega \cap \Xi$ with $u(x) > 0$ such that $r := |x - x_0| = \operatorname{dist}(x, \partial\{u > 0\}) \le r_0/2$, the estimate

$$u(x) \ge C|x - x_0|^{1-(d-\sigma)/(2p)}.$$

holds, with $C > 0$ depending only on d, p, λ, Λ, c_0, and Ξ.

Proof. This lemma is proven by the same method as Lemma 10, with the following alteration: to obtain a lower bound for the integral on the right in (6.3), one applies the condition (6.4) with $t = kr^{(\sigma-d)/p}$ and k chosen depending on Ξ so that

$$c_0 k^{-p} r^d = |B_{2r}(x_0) \cap \Xi| - \frac{1}{2}\left|(B_{r/2}(x_\varepsilon) \setminus B_{r/4}(x_\varepsilon)) \cap \Xi\right|.$$

This implies

$$\left|\{\varphi(x) > kr^{(\sigma-d)/p}\} \cap (B_{r/2}(x_\varepsilon) \setminus B_{r/4}(x_\varepsilon)) \cap \Xi\right| \geq \frac{1}{2}\left|(B_{r/2}(x_\varepsilon) \setminus B_{r/4}(x_\varepsilon)) \cap \Xi\right|,$$

and the remainder of the argument proceeds as in the proof of Lemma 10. □

Now we are ready to prove our last main result, which allows us to control the severity of cusps along the free boundary:

Proof of Theorem 3. For a free boundary point x_0, let q_1 and q_2 be as in the statement of the Theorem, i.e. $\varphi \in L^{q_1}(B_{r_0}(x_0))$ and for $r \in (0, r_0)$,

$$|\{\varphi(x) > t\} \cap B_r(x_0)| \geq \min\left(c_0 t^{-q_2}, |B_r(x_0)|\right),$$

Then Theorem 2 implies $u(x) \leq C|x - x_0|^{1-d/(2q_1)}$ in $B_r(x_0)$, and Lemma 10 implies $u(x) \geq c_{q_2}|x - x_0|^{1-d/(2q_2)}$ in $B_r(x_0)$.

To keep the notation brief, we define $\alpha_1 = 1 - d/(2q_1)$ and $\alpha_2 = 1 - d/(2q_2)$. Note that $\alpha_1 \leq \alpha_2$.

For fixed $r \in (0, r_0)$, let $x_1 \in \partial B_r(x_0)$ be such that $u(x_1) \geq \frac{c_{q_2}}{2} r^{\alpha_2}$. By Theorem 1, u is Hölder continuous of order α at x_1. Define $\rho = r^{\alpha_2/\alpha}$. For $\kappa > 0$ and $y \in B_{\kappa\rho}(x_1)$, we have

$$u(y) \geq \frac{c_{q_2}}{2} r^{\alpha_2} - C(\kappa\rho)^\alpha = \left(\frac{c_{q_2}}{2} - C\kappa^\alpha\right) r^{\alpha_2},$$

which is strictly positive if we choose $\kappa = [c_{q_2}/(4C)]^{1/\alpha}$. We conclude that $B_{\kappa\rho}(x_1) \subset \{u > 0\}$. A simple geometric argument gives $|B_r(x_0) \cap B_{\kappa\rho}(x_1)| \geq c(\kappa)\rho^d$, which implies

$$\frac{|B_r(x_0) \cap \{u > 0\}|}{r^{d\alpha_2/\alpha}} \geq c, \tag{6.5}$$

for a constant $c > 0$ depending on q_2 and q_1 but independent of r.

In the case $\alpha_0 > \alpha_1$, we can take $\alpha = \alpha_1$ in (6.5). If in addition $q_1 = q_2$, we can in fact take $\alpha = \alpha_1 = \alpha_2$, yielding a corner-like estimate.

For the lower bound on $|B_r(x_0) \cap \{u = 0\}|$, arguing as in [3], we compare u to the function v defined by

$$\begin{cases} \nabla \cdot (A(x)\nabla v) = 0 & \text{in } B_r(x_0), \\ v - u \in H_0^1(B_r(x_0)). \end{cases}$$

Since $v - u \geq 0$ on $\partial B_r(x_0)$, the maximum principle and Lemma 2 imply $v \geq u$ in the interior of $B_r(x_0)$.

Using the Poincaré inequality, the identity $\int_{B_r(x_0)} \nabla(u-v)\cdot(A(x)\nabla v)\,\mathrm{d}x = 0$, and the minimizing property of u, we have

$$
\begin{aligned}
\frac{1}{r^2}\int_{B_r(x_0)} |u-v|^2\,\mathrm{d}x &\le C\int_{B_r(x_0)} |\nabla(u-v)|^2\,\mathrm{d}x \\
&\le C\lambda^{-1}\int_{B_r(x_0)} [\nabla u\cdot(A(x)\nabla u) - \nabla v\cdot(A(x)\nabla v)]\,\mathrm{d}x \\
&\le C\lambda^{-1}\int_{B_r(x_0)} \varphi(x)1_{\{u=0\}}\,\mathrm{d}x \\
&\le C\lambda^{-1}\|\varphi\|_{L^{q_1}(B_r(x_0))}|B_r(x_0)\cap\{u=0\}|^{1-1/q_1}.
\end{aligned}
\tag{6.6}
$$

As above, let $\rho = r^{\alpha_2/\alpha_1}$. The Harnack inequality (applied to v) and the nondegeneracy of u (Lemma 10) imply, for $\kappa \in (0,1)$ sufficiently small,

$$v(x) \ge c \sup_{B_{r/2}(x_0)} u \ge c r^{\alpha_2}, \quad x \in B_{\kappa\rho(x_0)}.$$

Since $u(x_0) = 0$, the Hölder continuity of u implies $u \le C(\kappa\rho)^{\alpha_1}$ in $B_{\kappa\rho}(x_0)$. We therefore have

$$v(x) - u(x) \ge c r^{\alpha_2} - C(\kappa\rho)^{\alpha_1} = (c - C\kappa^{\alpha_1})r^{\alpha_2} \ge \frac{c}{2} r^{\alpha_2},$$

if we choose $\kappa = [c/(2C)]^{1/\alpha_1}$. With (6.6), we now have

$$|B_r(x_0)\cap\{u=0\}|^{1-1/q_1} \ge K r^{-2+2\alpha_2}\rho^d = K r^{2(\alpha_2-1)+d\alpha_2/\alpha_1},$$

or

$$|B_r(x_0)\cap\{u=0\}| \ge K r^{d(\alpha_2/\alpha_1 - 1/q_2)q_1/(q_1-1)}.$$

Finally, in the case $q_1 = q_2$, we obtain $|B_r(x_0)\cap\{u=0\}| \ge K r^d$, as in the classical theory. ◻

7. An example of a free boundary point with infinite jump

In this final section, we prove Theorem 4 by constructing an example that shows the free boundary can indeed intersect the infinite set of φ.

For $d \ge 3$ fixed, let $m > 0$ be a constant to be chosen later, and define

$$\tau(r) := \frac{m(d-2)}{r - r^{d-1}}, \quad r \in (0,1).$$

The function τ arises in the analysis of radially symmetric harmonic functions, which are involved in our proof below. The minimum of τ is achieved at

$$r_* = \left(\frac{1}{d-1}\right)^{\frac{1}{d-2}} \in (0,1),$$

and $\tau(r_*) = \left(m r_*^{d-1}\right)^2$. Next, let $r^* = \dfrac{1+r_*}{2}$, and for fixed $q > 1$, define $\varphi : B_1 \to (0,+\infty]$ by

$$\varphi(x) = \begin{cases} \left(m r_*^{d-1}\right)^2, & |x| < r_*, \\ (|x| - r_*)^{-1/q}, & r_* \le |x| < r^*, \\ \left(m r_*^{d-1}\right)^2, & r^* \le |x| \le 1, \end{cases}$$

and note that $\varphi \in L^q_{\text{weak}}(B_1)$. We consider the minimization problem for

$$\mathcal{J}_{I,\varphi}(v) = \int_{B_1} (|\nabla v|^2 + \varphi(x)1_{\{v>0\}})\,\mathrm{d}x, \quad v \in H^1_g(B_1),$$

with $g \equiv m$ on ∂B_1 for some constant m. We claim that for $m > 0$ sufficiently small, depending only on d and q, there is a minimizer u such that $\partial\{u > 0\}$ intersects $\{\varphi(x) = \infty\}$.

Since φ is not a monotonic function of $|x|$, we cannot apply rearrangement methods to conclude minimizers of $\mathcal{J}_{I,\varphi}$ are radially symmetric. In fact, we will show that a non-symmetric minimizer exists for certain choices of m.

On a technical note, our argument below assumes u is differentiable at free boundary points where $\varphi < \infty$. This is not always true in the pointwise sense, but by understanding solutions to the Bernoulli problem in the viscosity sense, our argument (which uses nothing more than the comparison principle) can be made rigorous (see, for instance, [25, Section 6] for a detailed discussion of the meaning of the free boundary condition $|\nabla u|^2 = \varphi$ in the viscosity sense). We omit the details about this issue because it concerns the boundary condition at points where φ is finite, which is suitably explained by existing theory.

First, we define a useful class of comparison functions: radially symmetric functions that are zero in B_r for some $r \in (0, 1)$, and harmonic in $B_1 \setminus B_r$, with boundary values equal to m on ∂B_1. Explicitly, these functions are given by

$$u_r(x) = \begin{cases} 0, & |x| < r, \\ \frac{m}{r^{2-d}-1}(r^{2-d} - |x|^{2-d}), & |x| \geq r, \end{cases} \tag{7.1}$$

and they have energy

$$\begin{aligned}\mathcal{J}_{I,\varphi}(u_r) &= \omega_d \int_r^1 \left(\frac{m^2(d-2)^2}{(r^{2-d}-1)^2}\rho^{2-2d} + \varphi(\rho) \right) \rho^{d-1}\,\mathrm{d}\rho \\ &= \omega_d \left(\frac{m^2(d-2)}{r^{2-d}-1} + \int_r^1 \varphi(\rho)\rho^{d-1}\,\mathrm{d}\rho \right),\end{aligned}$$

where ω_d is the measure of $\mathbb{S}^{d-1}$, and we have written $\varphi(\rho) = \varphi(|x|)$. For any $r \in (0, 1)$, the function u_r is admissible for the minimization problem.

Now, let u be a minimizer. We claim that $\{u = 0\}$ cannot be empty, if m is chosen sufficiently small. Indeed, if u is positive in all of B_1, then it is harmonic in B_1 and must be identically equal to m. Then $\mathcal{J}_{I,\varphi}(u) = \int_{B_1} \varphi\,\mathrm{d}x$. To rule out this case, we would like to find $r \in (0, 1)$ such that u_r defined by (7.1) has energy less than $\int_{B_1} \varphi\,\mathrm{d}x$, or in other words,

$$m^2 \frac{d-2}{r^{2-d}-1} < \int_0^r \varphi(\rho)\rho^{d-1}\,\mathrm{d}\rho. \tag{7.2}$$

Choosing $r = r^*$, we see that

$$\int_0^{r^*} \varphi(\rho)\rho^{d-1}\,\mathrm{d}\rho \geq \int_{r_*}^{r^*} \varphi(\rho)\rho^{d-1}\,\mathrm{d}\rho = \int_{r_*}^{r^*} (\rho - r_*)^{-1/q}\rho^{d-1}\,\mathrm{d}\rho.$$

Therefore, by choosing $m > 0$ small enough that

$$m^2 < \frac{(r^*)^{2-d} - 1}{d-2} \int_{r_*}^{r^*} (\rho - r_*)^{-1/q}\rho^{d-1}\,\mathrm{d}\rho,$$

we ensure (7.2) is satisfied when $r = r^*$, so $u \equiv m$ cannot be a minimizer. Note that such m can be chosen depending only on d and q.

Now, since $\{u = 0\}$ is not empty, let x_1 and x_2 be the points on $\partial\{u = 0\}$ of smallest and largest magnitude, respectively. Define $r_1 = |x_1|$ and $r_2 = |x_2|$. For $i = 1, 2$, let $u_i := u_{r_i}$ denote the function defined in (7.1) with $r = r_i$.

By our choice of x_1 and x_2, we have $u_1 \geq u \geq u_2$ on $\partial\{u > 0\}$. The comparison principle implies $u_1 \geq u \geq u_2$ in all of B_1, and

$$\partial_r u_1(x_1) \geq \partial_r u(x_1), \quad \partial_r u_2(x_2) \leq \partial_r u(x_2), \tag{7.3}$$

where ∂_r is the derivative in the radial direction. These two inequalities will imply useful bounds on r_1 and r_2.

Starting with r_2, the definition of u_2 implies, with (7.3),

$$\partial_r u(x_2) \geq \frac{(d-2)m}{r_2 - r_2^{d-1}} = \tau(r_2).$$

On the other hand, since u is a minimizer, we have $\partial_r u(x_2) = \sqrt{\varphi(x_2)}$, so that

$$\sqrt{\varphi(x_2)} \geq \tau(r_2). \tag{7.4}$$

This implies $r_2 = |x_2|$ must lie in the part of $[0, 1]$ where $\sqrt{\varphi} \geq \tau$. Choosing $m > 0$ smaller if necessary (depending only on d and q) we can ensure that

$$\lim_{r \to r^*-} \varphi(r) = (r^* - r_*)^{-1/q} > \frac{(d-2)m}{r^* - (r^*)^{d-1}} = \tau(r^*).$$

Since τ is increasing on $(r_*, 1)$, we also have

$$\tau(r^*) > \tau(r_*) = \lim_{r \to r^*+} \varphi(r).$$

Therefore, the inequality $\sqrt{\varphi(x_2)} \geq \tau(r_2)$ implies $r_2 \in [r_*, r^*]$.

Regarding r_1, we similarly have from (7.3) that

$$\tau(r_1) = \frac{(d-2)m}{r_1 - r_1^{d-1}} \geq \partial_r u(x_1) = \sqrt{\varphi(x_1)} \tag{7.5}$$

Since $\sqrt{\varphi(r)} > \tau(r)$ for $r \in (r_*, r^*)$, inequality (7.5) implies $r_1 \in [0, r_*] \cup [r_*, 1]$. Since $r_1 \leq r_2$ by definition and $r_2 \in [r_*, r^*]$, we in fact have $r_1 \in [0, r_*] \cup \{r^*\}$.

Next, we would like to rule out the case $r_1 = r_2 = r^*$. In this case, the inequalities $u_1 \geq u \geq u_2$ imply $u_1 = u = u_2$, and therefore u is given by (7.1) with $r = r^*$. In particular, $\partial_r u(x_2) = \tau(r^*)$. But in the set $\{u > 0\} = B_1 \setminus B_{r^*}$, there holds $\varphi \equiv \left(m r_*^{d-1}\right)^2$, so one should have $\partial_r u(x_2) = m r_*^{d-1} = \tau(r_*) < \tau(r^*)$ for any solution of the Bernoulli problem, which is a contradiction. We conclude that

$$r_1 \leq r_* \leq r_2.$$

Since $u_1 \geq u \geq u_2$, we clearly have $B_{r_1} \subset \{u = 0\} \subset B_{r_2}$, and with $r_1 \leq r_* \leq r_2$, this implies the existence of at least one point $x_0 \in \partial\{u > 0\}$ with $|x_0| = r_*$, and such that $\{\varphi(x) = +\infty\} = \partial B_{r_*}$ intersects $\{u > 0\} \cap B_\rho(x_0)$ for any $\rho > 0$, as claimed.

For any $x_0 \in \partial B_{r_*}$, there is a cone Ξ with vertex at x_0 and aperture π (i.e. Ξ is a half-plane), such that

$$|\{\varphi(x) > t\} \cap B_r(x_0) \cap \Xi| \geq \min\left(c r^{d-1} t^{-q}, |B_r(x_0) \cap \Xi|\right),$$

for $r > 0$ sufficiently small. From Lemma 11 with $p = q$ and $\sigma = d - 1$, we have $u(x) \geq C|x - x_0|^{1-1/(2q)}$ for $x \in \Xi$ near x_0. We conclude that $|\nabla u(x)| \to \infty$ as x approaches x_0 from inside $\Xi \cap \{u > 0\}$.

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Department of Mathematical Sciences, Florida Institute of Technology, Melbourne, FL, USA
Email address: `ssnelson@fit.edu`

Department of Mathematics, University of Central Florida, Orlando, FL, USA
Email address: `eduardo.teixeira@ucf.edu`