Interaction Renormalization and Validity of Kinetic Equations for Turbulent States

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(Received 20 March 2024; revised 11 September 2024; accepted 6 November 2024; published 13 December 2024)

We consider turbulence of waves interacting weakly via four-wave scattering (sea waves, plasma waves, spin waves, etc.). In the first order in the interaction, a closed kinetic equation has stationary solutions describing turbulent cascades. We show that the higher-order terms generally diverge both at small (IR) and large (UV) wave numbers for direct cascades. The analysis up to the third order identifies the most UV-divergent terms. To gain qualitative analytic control, we sum a subset of the most UV divergent terms, to all orders, giving a perturbation theory free from UV divergence, showing that turbulence becomes independent of the dissipation scale when it goes to zero. On the contrary, the IR divergence (present in the majority of cases) makes the effective coupling parametrically larger than the naive estimate and growing with the pumping scale L (similar to anomalous scaling in fluid turbulence). In such cases, the kinetic equation does not describe wave turbulence even of arbitrarily small level at a given k if kL is large enough that is the cascade is sufficiently long. We show that the character of strong turbulence is determined by whether the effective four-wave interaction is enhanced or suppressed by collective effects. The enhancement possibly signals that strong turbulence is dominated by multiwave bound states (solitons, shocks, cusps), similar to confinement in quantum chromodynamics.

DOI: 10.1103/PhysRevLett.133.244002

Kinetic equations are workhorses of physics and engineering. They describe transport phenomena and turbulence, from gas pipes and oceans to plasmas in space and in thermonuclear reactors. These equations are so effective because they reduce the description of multiparticle or multiwave systems to a closed equation on a single-particle probability distribution or a single-wave vector occupation number. The equations have solutions that describe thermal equilibria, transport in weakly nonequilibrium states, and even turbulent states. The question is whether accounting for multiparticle and multimode correlations leads only to small modifications.

This question was first addressed for the density expansion beyond the Boltzmann equation describing binary collisions [1–6]. The higher-order terms involve subsequent collisions of the same particles and contain spatial integrals over their positions. The integrals have infrared (IR) divergences starting from the second order (in 2D) or from the third order (in 3D), reflecting long-distance multiparticle correlations. In thermal equilibrium, such divergences are canceled due to detailed balance, and the correlations are short so that the equations of state have a regular virial expansion. Spatial nonequilibrium prevents cancellation in transport states. Of course, the divergences appear because the "naive" virial expansion allows particles to travel arbitrarily long distances between collisions. Accounting for the collective effects imposes the mean free path as an IR cutoff. The renormalized expansion is singular as it involves noninteger powers of density. The corrections are small in dimensions exceeding 2. The divergences lead to logarithmic renormalization of the kinetic coefficients in 2D and to anomalous kinetics in 1D.

In contrast to transport states, turbulent states create fluxes (cascades) in momentum space rather than in real space. The cascade distributions were found as exact (Zakharov) solutions of the kinetic equations both for particles and waves assuming locality of interactions, which is the convergence of the collision integrals [7]. This means that the contribution of the lowest-order collisions and interactions is predominantly given by comparable momenta of colliding particles or wave numbers of interacting waves. The question is whether locality also holds in the higher-order corrections [8,9], which describe multiparticle collisions and multiwave interactions. Here, we answer this question in the negative, finding k-space divergences that bring a new four-wave coupling renormalized by multiwave interactions.

We consider quasiparticles interacting via four-wave scattering described by the Hamiltonian [7,12,13,23]

$$H = \sum_p \omega_p |a_p|^2 + \sum_{p_1,p_2,p_3,p_4} \lambda_{1234} a^*_{p_1} a^*_{p_2} a_{p_3} a_{p_4}. \quad (1)$$

A single wave propagating in an undisturbed medium has the frequency ω_k, and λ_{1234} defines the interaction energy of four waves without any other waves present. Both ω_k and λ_{1234} are renormalized in a multimode turbulence state, which is the subject of this work. Renormalization in turbulence theory is quite different from that in quantum field theory. There, one always deals with effective large-scale theories, describing how a (generally unknown) bare value at UV scale (Planck scale or lattice spacing) is getting renormalized as one increases the scale of measurements. In quantum field theory, one cannot switch off quantum fluctuations, so the meaning of renormalization is different.

The model—We assume $\omega_k = k^\alpha$ with $0 < \alpha \leq 2$. The real vertex $\lambda_{1234} \equiv \lambda_{p_1p_2p_3p_4}$ (which is nonvanishing only for $\mathbf{p_1} + \mathbf{p_2} = \mathbf{p_3} + \mathbf{p_4}$) can be of either sign and is a homogeneous function of degree β: $\lambda_{ap_1ap_2ap_3ap_4} = a^\beta \lambda_{p_1p_2p_3p_4}$. From the equations of motion on the complex amplitudes, $i da_k/dt = \partial H/\partial a^*_k$, we derive that for the occupation numbers $n_k = \langle |a_k|^2 \rangle$,

$$\frac{\partial n_k}{\partial t} = 4 \int d\mathbf{p_2} d\mathbf{p_3} d\mathbf{p_4} \lambda_{k234} \mathrm{Im} \langle a^*_k a^*_{p_2} a_{p_3} a_{p_4} \rangle. \quad (2)$$

The fourth moment in (2) can be found as a series in λ, assuming closeness to the Gaussian statistics of noncorrelated waves, $\mathcal{P}\{a_k\} \propto \exp[-\sum_k |a_k|^2/n_k]$, completely determined by n_k. The standard perturbation theory is described in the Supplemental Material [37]. It gives in the first nonvanishing order the Peierls kinetic equation (KE):

$$\frac{\partial n_k}{\partial t} = I_k = 16\pi \int d\mathbf{p_2} d\mathbf{p_3} d\mathbf{p_4} \lambda^2_{k234} n_k n_2 n_3 n_4 \times \delta(\mathbf{k} + \mathbf{p_2} - \mathbf{p_3} - \mathbf{p_4}) \delta(\omega_{k2;34}) \times \left(\frac{1}{n_k} + \frac{1}{n_2} - \frac{1}{n_3} - \frac{1}{n_4} \right), \quad (3)$$

where we defined $\omega_{k2;34} \equiv \omega_k + \omega_{p_2} - \omega_{p_3} - \omega_{p_4}$. What is traditionally required for the validity of (3) is to provide a dense enough set of such resonances, which requires taking the limit $kL_0 \to \infty$ (apart from $\lambda_{1234} \to 0$), where L_0 is the box size, see, e.g., [14–16]. This already requires a careful analysis of divergences in the kinetic equation and in the corrections to it.

The leading-order kinetic equation (3) conserves energy $\int \omega_k n_k d\vec{k}$ and wave action $\int n_k d\vec{k}$, and has two stationary solutions describing turbulent cascades. Here we focus on the direct energy cascade (to large wave numbers). Writing (3) as the energy continuity equation, $k^{d-1} \omega_k \partial n_k / \partial t = -\partial P_k / \partial k$, requiring the energy flux over wave numbers, $P_k = k^d \omega_k I_k = P$, to be k independent, and taking for the sake of power counting $\lambda_{kkkk} \simeq \lambda k^\beta$, we obtain

$$P \simeq \lambda^2 k^{2\beta + 3d} n_k^3 \Rightarrow n_k \simeq P^{1/3} \lambda^{-2/3} k^{-d-2\beta/3}. \quad (4)$$

One can also obtain (4) estimating P as the energy density $\omega_k n_k k^d$ divided by the nonlinear interaction time $1/\omega_k \epsilon_k^2$, where we introduced the dimensionless parameter of nonlinearity (coupling) at a given k:

$$\epsilon_k = \lambda n_k k^{\beta+d} / \omega_k = \epsilon_0 (k/k_p)^{\beta/3 - \alpha}. \quad (5)$$

The standard claim is that (3) is valid and (4) is its solution for those k where $\epsilon_k \ll 1$ and under the conditions of convergence of the integrals in (3) upon substituting (4) [7,12,17]). That depends on the asymptotics:

$$\lim_{p_1,p_3 \ll p_2,p_4} \lambda_{1234} = \lambda (p_2 p_4)^{\beta_1/2} (p_1 p_3)^{(\beta - \beta_1)/2}, \quad (6)$$

$$\lim_{p_1,p_2,p_3 \gg p_4} \lambda_{1234} = \lambda_{12,1+2} p_4^{\beta_3}. \quad (7)$$

In some cases, $\beta_1 = \beta/2$, as for spin waves, where $2\beta_1 = \beta = 2$, $\alpha = 2$ so that $\lim_{p_2 \to 0} (\vec{p}_3 \cdot \vec{p}_4) \propto p_2$ and $\beta_3 = 1$. In some cases, $\beta_1 \leq \beta/2$ provides stronger convergence than might be expected on dimensional grounds.

Consider (3) with $n_k = k^{-\gamma}$ when $p_2, p_4 \to \infty$ while $\mathbf{q} = \mathbf{k} - \mathbf{p_3}$ remains finite. Then, $|\lambda_{1234}|^2 \sim p_2^{2\beta_1}$ and the UV behavior is determined by the integral $I_k \propto \int d\mathbf{p_2} p_2^{2\beta_1} \delta(\omega_{12;34})(n_4 - n_2)$, where $p_4 = |\mathbf{p_2} + \mathbf{q}|$. Expanding $n_4 - n_2 = \mathbf{q} \cdot \nabla n_2$ and accounting for the angular dependence in $\delta(\omega_{12;34})$ we obtain

$$I_k \propto p_2^{2\beta_1 + d - \gamma - \alpha - 1}. \quad (8)$$

Setting $\gamma = d + 2\beta/3$ gives the UV convergence if $2\beta_1 - 2\beta/3 < \alpha + 1$ [17].

IR convergence depends on $\alpha - 1$. If $\alpha > 1$, then a three-wave resonance is possible, and the main IR contribution is $\int_{k_p} d\mathbf{p_2} p_2^{2\beta_3 - \gamma} \propto k_p^{2\beta_3 - 2\beta/3}$. It converges for spin waves. The nonlinear Schrödinger equation (NSE) and plasmons in a nonisothermal plasma ($\alpha = 2, \beta = 0$) are IR borderline. For $\alpha < 1$, no three-wave resonance is possible and we must take both $p_2, p_3 \to 0$. Expanding $\delta(\omega_{12;34})$ up to $|\mathbf{p}_2 - \mathbf{p}_3|^2$ we obtain

$$\int_{k_p} d\mathbf{p_2}\, p_2^{\beta/2 - \gamma} \int_{k_p} d\mathbf{p_3}\, p_3^{\beta/2 - \gamma} \delta(\omega_{k2;34}) \times [(\omega_k + \omega_2 - \omega_3)^{\gamma/\alpha} - k^\gamma] \propto k_p^{2\beta/3 - 2\beta_1 - \alpha + 2}.$$

For $\alpha < 1$, the combination of the IR and UV conditions gives $2\beta_1 - \alpha - 1 + d < \gamma < d + \beta - \beta_1 + 1 - \alpha/2$ or $\beta_1 - \beta/3 - 1 < \alpha/2$ for $\gamma = d + 2\beta/3$.

The kinetic equation thus gives two cancellations in the IR and one in the UV, which provide a locality window for γ [7]. The locality is expected to guarantee that the solution does not depend on k_p (the IR cutoff) and k_m (the UV cutoff) in the limits $k_p/k \to 0$ and $k/k_m \to 0$. When $\beta > 3\alpha$ and ϵ_k grows along the cascade, on dimensional grounds one might have expected strong turbulence to appear at the k for which $\epsilon_k \simeq 1$.

Let us show that the locality window is generally absent for higher-order scattering processes and the effective dimensionless coupling parametrically exceeds ϵ_k, so that strong turbulence must start earlier than had been expected. For that we need to account for the renormalization of ω_k and λ_{1234} and the respective modification of the kinetic equation and its turbulence solution.

Next-to-leading order—The first correction renormalizes the frequency $\tilde{\omega}_k = \omega_k + \sum_{\mathbf{q}} \lambda_{kqkq} n_q$. It is zero for spin waves. For NSE, $\beta = 0$, $n_q = q^{-d}$, the sum diverges logarithmically; this case will be analyzed elsewhere. For the rest of the cases, the sum always converges in the UV. For water waves, $\lambda_{kqkq} \propto q(\vec{k} \cdot \vec{q})$ for $q \ll k$—angular integration cancels the IR divergence. A power-law IR divergence takes place for plasma turbulence with $\beta = 2 = \alpha$ but the frequency renormalization grows slower with k. Even when the one-loop frequency renormalization is comparable to ω_k, the replacement $\omega_k \to \tilde{\omega}_k$ does not change the energy cascade spectrum. Higher-order corrections are complex and have increasingly higher powers of divergence, which will be addressed elsewhere. Here we conclude that the lowest-order frequency renormalization does not substantially change the weak-turbulence spectra.

The next-order renormalization of the quartic interaction gives the contribution to $\partial n_k/\partial t$ which is KE (3) multiplied by the sum of two loop integrals $\mathcal{L}_a + \mathcal{L}_b$ [8,9]:

$$\mathcal{L}_a = 4 \sum_{p_5, p_6} (n_5 + n_6) \lambda_{k256} \lambda_{5634} / \lambda_{k234} \omega_{k256}, \tag{9}$$

$$\mathcal{L}_b = 16 \sum_{p_5, p_6} (n_6 - n_5) \lambda_{k635} \lambda_{2546} / \lambda_{k234} \omega_{4625}. \tag{10}$$

The integrals of $1/\omega$ are the principal value part. For thermal equilibrium, $n_k \sim 1/(\omega_k + \mu)$, the corrections vanish, at all orders. Let us now substitute the weak-turbulence spectrum $n_k = k^{-d-2\beta/3}$ into the new equation and compare convergence with that of (3). There are two issues here: an extra (loop) integration over p_5 and extra powers of p_2, p_3, p_4 in the external integration. UV convergence of the loop integration is the same as for the bare KE (except for spin waves described in the Supplemental Material [37], which includes [10,11]). Here we assume the loop integration is UV convergent.

IR divergence of the loop integration is determined by one wave number going to zero, which gives $k_p^{2\beta_3+d-\gamma}$ for any α since p_5 is not bound to the resonance surface. Since in all cases (except spin waves) $\beta_3 = \beta/4$, the divergence scales as $k_p^{-\beta/6}$ for the direct cascade with $\gamma = d + 2\beta/3$. Setting in (9), (10) $p_5 = k_p$ and $\mathbf{p_6} = \mathbf{k_1} + \mathbf{k_2}$, we obtain the addition to the vertex λ_{1234}:

$$\delta\lambda_{1234} = \lambda_{1234}(\mathcal{L}_a + \mathcal{L}_b) = \frac{96\Omega_d k_p^{-\beta/6}}{\beta(2\pi)^d} \left(\frac{\lambda_{1,2;1+2}\lambda_{3,4;3+4}}{\omega_1 + \omega_2 - \omega_{1+2}} + \frac{2\lambda_{1,4-1;4}\lambda_{2;3,2-3}}{\omega_4 + \omega_{4-1} - \omega_1} + \frac{2\lambda_{4,1-4;1}\lambda_{3;2,3-2}}{\omega_4 + \omega_{1-4} - \omega_1} \right). \tag{11}$$

We see that the corrections to the vertex are determined not by ϵ_k from (5) but by the loop integrals (9), (10), (11) (the true dimensionless couplings) estimated as

$$\mathcal{L}_a \simeq \mathcal{L}_b \simeq \epsilon_0 (k/k_p)^{\beta/2-\alpha} = \epsilon_k (k/k_p)^{\beta/6} \gg \epsilon_k, \tag{12}$$

This is the main result of our work: Deviations from weak turbulence are $\simeq \mathcal{L}$, which grows faster (or decay slower) along the direct cascade. When we keep ϵ_k small and fixed, decreasing k_p eventually violates the validity of the kinetic equation (3) and its Zakharov solution. For the inverse cascade, $\gamma = d + 2\beta/3 - \alpha/3$ and the divergence condition, $\beta \geq 2\alpha$, is more restrictive than $\beta \geq 0$ yet still admits physical systems of interest.

In addition, the new effective vertex affects the integration over p_2, p_3, p_4. We denote $\kappa = \max\{0, \alpha - 1\} \geq 0$. When $p_2, p_4, p_6 \to \infty$ we have $\omega_{k2;56} \sim p_2^{-\kappa}$, $\lambda_{1256}\lambda_{5634}/\lambda_{1234} \sim p_2^{\beta_1}$, and similarly for the b term. Thus the extra factor relative to (3) brings an extra $p_2^{\beta_1-\kappa}$ into the integrand. The power $\beta_1 - \kappa$ is non-negative for all cases with $\alpha < 1$ and for some cases with $\alpha > 1$ (plasma turbulence). This power counting suggests that if $\beta_1 - \kappa \geq 0$, then starting from the mth order, where m is such that $\beta/3 - \alpha - \kappa + m(\beta_1 - \kappa) \geq 0$, the perturbation theory brings terms whose degrees of UV divergences grow linearly with m.

This is supported by a lengthy computation of the two-loop contributions is presented in the Supplemental Material [37]. All diagrams, as expected, have doubled IR divergence $k_p^{-\beta/3}$. Yet the power of the UV divergences allows us to sort the two-loop diagrams. Each one adds to the vertex schematically $\lambda_{12ij}\lambda_{ijkl}\lambda_{kl34}(n_i + n_j)(n_k + n_l)/\omega_{12ij}\omega_{kl34}$. When $p_i, p_k \to \infty$ the vertices give the power $2\beta_1$, while every ω_k is expected to give $-\kappa$ as in the KE and one-loop above. This is indeed so for the bubble diagrams adding to KE the UV factor $k_m^{2\beta_1-2\kappa}$. They are the squares of the first two terms from the last bracket in (11), so they add rather than cancel. Nor are they canceled by the nonbubbles (mixtures of a and b terms) which are subleading due to internal cancellations similar to (8); they add to KE factors like $k_m^{2\beta_1-\alpha-\kappa}$, $k_m^{2\beta_1-2\alpha}$. That means that UV divergences at higher orders are real and need to be taken care of.

Since we cannot yet sum all the most UV divergent diagrams, we sum the bubble diagrams to all orders (which are iterations of the one loop diagram). This will cure the UV divergence, but may not give a quantitatively correct answer in general. The sum satisfies an integral Schwinger-Dyson equation in the momentum-frequency domain, see the Supplemental Material [37], which includes [18–24]. We were able to solve this equation explicitly for two particular classes of the bare vertex. The simplest case is when the ratio $\lambda_{1256}\lambda_{5634}/\lambda_{1234}$ is only a function of p_5 and p_6. The sum of bubble diagrams then turns into a geometric series, allowing us to write the "renormalized" kinetic equation as the sum of two terms, with $\mathcal{L} = \mathcal{L}_a$ and $\mathcal{L} = \mathcal{L}_b$:

$$\frac{\partial n_k}{\partial t} = -16\mathrm{Im}\sum_{a,b}\sum_{p_2,p_3,p_4}\lambda_{k234}^2\left\{\hat{\omega}[34;k2]\left[\frac{(n_k+n_2)n_3n_4}{1-\mathcal{L}} - \frac{(n_3+n_4)n_kn_2}{1-\mathcal{L}^*}\right] + \mathcal{M}\frac{(n_k+n_2)(n_3+n_4)}{|1-\mathcal{L}|^2}\right\} = \tilde{I}_k,$$

$$\mathcal{M} = 2\sum_{p_5}\frac{\lambda_{k256}\lambda_{5634}}{\lambda_{k234}}n_5n_6\hat{\omega}[12;56]\hat{\omega}[34;56]. \tag{13}$$

where $\hat{\omega}[ij;mn] \equiv \lim_{\epsilon\to 0}(\omega_{ij;mn} + i\epsilon)^{-1}$. The equation allows one to obtain several fundamental conclusions and far-reaching assumptions about weak and strong wave turbulence. It is UV convergent, that is valid at arbitrary k_m. Generally, corrections of higher orders have UV divergences cut off by the denominators $|1-\mathcal{L}_a|^{-2}$ and $|1-\mathcal{L}_b|^{-1}$. The cutoff then depends on λ, which will generally make the corrections proportional to noninteger power of λ, signaling singular renormalized perturbation theory, like for transport states. In any case, the cascade solution is independent of k_m when $k_m \to \infty$. This conclusion is supported by numerics, [25,26].

Let us now use (13) to discuss strong turbulence. The most salient issue is whether the spectrum is getting more or less steep when the direct cascade turns from weak to strong at large k. Two alternative scenarios of strong turbulence are often discussed (without any criterion for choosing between them for a given system). The first is a qualitatively similar cascade, just with the weak-turbulence time, $1/\omega_k\epsilon_k^2$ replaced by the nonlinear time $1/\omega_k\epsilon_k = 1/\lambda_k n_k k^d$, so that the spectral energy flux P is estimated as the spectral density $\tilde{\omega}_k n_k k^d$ divided by the nonlinear time, $P \simeq \tilde{\omega}_k n_k k^d \lambda_k n_k k^d$, which gives $n_k \simeq (P/\lambda_k\tilde{\omega}_k)^{1/2}k^{-d}$. We shall show below that the true answer is different since it is independent of λ. The second scenario is the hypothesis of critical balance $\epsilon_k = \mathrm{const}$, which gives the universal (flux-independent) spectrum $n_k \simeq k^{-d}\omega_k/\lambda_k$, see, e.g., [27–30]. From the below consideration, we will see that the critical balance is $\mathcal{L}(k) = \mathrm{const}$ instead.

Let us now apply the renormalized kinetic equation (13) to the classes of turbulence with $\beta \geq 2\alpha$, where the effective coupling grows along the energy cascade. Now, instead of (4), the flux constancy requires $k^d\tilde{\omega}_k\tilde{I}_k = P$, where $\mathcal{L}$ must be determined self-consistently by the new n_k. The factor $|1-\mathcal{L}|^{-2}$ determines the deviations from weak turbulence and the character of strong turbulence. Like in the field theory, the sign of $\mathcal{L}$ chooses the scenario.

Negative $\mathcal{L}$ means that the multimode correlations suppress interactions like screening in quantum electrodynamics. Increase of $|\mathcal{L}|$ (with increasing k or decreasing k_p) must lead to an increase of n_k relative to (4) to keep the same flux. When $|\mathcal{L}| > 1$, we put $k^d\tilde{\omega}_k\tilde{I}_k \simeq \lambda^2 k^{2\beta+3d}n_k^3/|\mathcal{L}|^2 = P$, which predicts the strong-turbulence spectrum independent of λ. If the integral for $\mathcal{L}$ converges with the new spectrum, then $\mathcal{L} \simeq \lambda n_k k^{\beta+d}/\tilde{\omega}_k$ and $n_k \simeq Pk^{-d}/\tilde{\omega}_k$.

Positive effective coupling $\mathcal{L}$ corresponds to interaction enhancement like in quantum chromodynamics. Similarly, we expect no smooth passing through $\mathcal{L} \simeq 1$, but a (confinement) transition to strong turbulence dominated by bound states. Formally, the equation $k^d\tilde{\omega}_k\tilde{I}_k = P$ could have a solution approaching at large k the critical balance $|\mathcal{L}| = 1$. Finding such solutions for specific systems will be attempted elsewhere.

The above picture with a single sign of $\mathcal{L}$ is the simplest case; (11) shows that the loop integral may have different signs for different configurations, depending on the product of the bare vertex and the frequency difference. As was noticed in [31], it is a turbulent analog of the Lighthill criterium for modulational instability $\lambda d^2\omega_k/dk^2 < 0$. For example, all the denominators in (11) are positive for $\alpha < 1$ when $d^2\omega_k/dk^2 < 0$. For $\alpha > 1$, the correction changes sign at the resonant surfaces $\omega(\vec{k}_i) + \omega(\vec{k}_j) = \omega(\vec{k}_i + \vec{k}_j)$. Dependence on ω_k is in stark contrast with thermal equilibrium being determined by the sign of λ alone—the Gibbs state is non-normalizable for attraction between waves when $\lambda < 0$. Our derivation shows that the bound states in turbulence could dominate not when there is attraction but when the signs of nonlinearity and dispersion are opposite, which is also the condition for solitons and collapses. When the interaction is enhanced, one may expect turbulence dominated by solitons or collapse events. The analysis of specific configurations and of the integral effect of the vertex renormalization needs to be done for each system specifically. Such analysis must also account for higher-order terms in the Hamiltonian whose contributions must be compared with (9), (10), this is beyond the general approach accepted here and will be attempted elsewhere.

Discussion—Here we described how nonlocality enhances or suppresses nonlinearity in nonequilibrium: deviations from weak turbulence are magnified by IR divergences. The main technical result is the renormalized kinetic equation (13) where effective couplings are (9), (10). Their signs determine enhancement or suppression of four-wave scattering by multiwave correlations and the character of the transition from weak to strong turbulence. It is likely that universal (flux-independent) spectra dominated by bound states are possible when there is an enhancement. The hypothesis of structure-dominated universal spectra has, up to the present work, remained generic speculation, with neither a proof nor any criteria specifying when it is possible. The interaction renormalization suggests such a criterium, which is an important step towards identifying universality classes of turbulence. Even more fascinating is the hypothetic possibility that the new kinetic equation (13) could describe both weak and strong turbulence (for instance, Zakharov and Phillips spectra of sea turbulence) as two opposite asymptotics of the steady cascade solution.

In any turbulence, scale invariance is broken both by the pumping $k_p = 2\pi/L$ and dissipation k_m. In incompressible fluid turbulence, the velocity moments are finite when $k_m \to \infty$. We identified here wide classes of wave turbulence with the spectrum independent of k_m (the example of spin waves in the Supplemental Material [37] shows that wave turbulence allows for richer opportunities). The anomalous scaling in fluid turbulence makes the effect of the pumping scale felt at arbitrarily short scales [32–34]: When one fixes the energy flux (the third velocity moment in incompressible fluid turbulence) at a given k, the spectral energy (the second moment) at that scale goes to zero when $kL \to \infty$, while the moments higher than the third diverge. Similarly, our IR divergence makes the deviations from weak turbulence to grow with the pumping scale which is a direct analog of intermittency and anomalous scaling. Even when naive ϵ_k is small, one makes turbulence strong by increasing L.

Recall that rigorous proofs of the validity of the kinetic equation (3) are done in the double scaling limit $\lambda \to 0$ and the box size $L_0 \to \infty$, while keeping some combination finite [14–16]. Our work shows which combination of λ and $k_p = 2\pi/L$ defines corrections to the weak-turbulence spectra where L is the pumping scale for a direct cascade or sink scale for an inverse cascade. One should also bear in mind possible influence of condensation at the box size [35,36].

Acknowledgments—We thank M. Smolkin for collaboration on related work, A. Zamolodchikov, J. Shatah, D. Schubring, and C. Cheung for helpful discussions. The work of V. R. is supported by NSF Grant No. 2209116 and by the ITS through a Simons grant. V. R. thanks the Aspen Center for Physics (NSF Grant No. PHY-2210452) and G. F. thanks NYU and the Simons Center for Geometry and Physics where part of this work was completed. The work of G. F. is supported by the Excellence Center at WIS, by the Grants No. 662962 and No. 617006 of the Simons Foundation, and by the EU Horizon Grants No. 873028 and No. 823937.

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[37] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevLett.133.244002 for additional technical details.