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Analysis of a Mixed Finite Element Method for Stochastic Cahn-Hilliard Equation with Multiplicative Noise

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Abstract. This paper proposes and analyzes a novel fully discrete finite element scheme with an interpolation operator for stochastic Cahn-Hilliard equations with functional-type noise. The nonlinear term satisfies a one-sided Lipschitz condition and the diffusion term is globally Lipschitz continuous. The novelties of this paper are threefold. Firstly, the L^2-stability (L^∞ in time) and H^2-stability (L^2 in time) are proved for the proposed scheme. The idea is to utilize the special structure of the matrix assembled by the nonlinear term. None of these stability results has been proved for the fully implicit scheme in existing literature due to the difficulty arising from the interaction of the nonlinearity and the multiplicative noise. Secondly, higher moment stability in L^2-norm of the discrete solution is established based on the previous stability results. Thirdly, the Hölder continuity in time for the strong solution is established under the minimum assumption of the strong solution. Based on these findings, the strong convergence in H^{-1}-norm of the discrete solution is discussed. Several numerical experiments including stability and convergence are also presented to validate our theoretical results.

AMS subject classifications: 60H35, 65N12, 65N15, 65N30

Key words: Stochastic Cahn-Hilliard equations; multiplicative noise; higher moment stability; strong convergence.

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1 Introduction

Consider the following stochastic Cahn-Hilliard (SCH) problem:

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$$du=\Big[-\Delta\Big(\epsilon\Delta u-\frac{1}{\epsilon}f(u)\Big)\Big]dt+\delta g(u)dW_t \quad \text{in } \mathcal{D}_T:=\mathcal{D}\times(0,T], \tag{1.1}$$

$$\frac{\partial u}{\partial n}=\frac{\partial}{\partial n}\Big(\epsilon\Delta u-\frac{1}{\epsilon}f(u)\Big)=0 \quad \text{in } \partial\mathcal{D}_T:=\partial\mathcal{D}\times(0,T], \tag{1.2}$$

$$u=u_0 \quad \text{in } \mathcal{D}\times\{0\}, \tag{1.3}$$

where $\mathcal{D}\subset\mathbb{R}^d$ $(d=2,3)$ is a bounded domain, n is the unit outward normal, $\delta>0$ is a positive constant, and W_t is a standard real-valued Wiener process on a filtered probability space $(\Omega,\mathcal{F},\{\mathcal{F}_t:t\geq 0\},\mathbb{P})$. The analysis in this paper can be generalized to $[L^2(D)]^d$-valued Q-Wiener process. The function f is the derivative of a smooth double equal well potential function F given by

$$F(u)=\frac{1}{4}(u^2-1)^2. \tag{1.4}$$

The diffusion term g is assumed to have zero mean, is globally Lipschitz continuous (1.5), and satisfies the growth condition (1.6), i.e.,

$$|g(a)-g(b)|\leq C|a-b|, \tag{1.5}$$

$$|g(a)|^2\leq C(1+a^2). \tag{1.6}$$

The SCH problem (1.1)-(1.3) can be rewritten in the following mixed formulation by substituting in the so-called chemical potential $w:=-\epsilon\Delta u+\frac{1}{\epsilon}f(u)$:

$$du=\Delta w dt+\delta g(u)dW_t \quad \text{in } \mathcal{D}_T, \tag{1.7}$$

$$w=-\epsilon\Delta u+\frac{1}{\epsilon}f(u) \quad \text{in } \mathcal{D}_T, \tag{1.8}$$

$$\frac{\partial u}{\partial n}=\frac{\partial w}{\partial n}=0 \quad \text{on } \partial\mathcal{D}_T, \tag{1.9}$$

$$u=u_0 \quad \text{on } \mathcal{D}\times\{0\}. \tag{1.10}$$

The mixed formulation will be used to develop the fully discrete finite element scheme in this paper.

The deterministic Cahn-Hilliard equation was originally introduced in [12] to describe phase separation and coarsening processes in a melted alloy. It was proved in [2, 15, 41, 44] that the chemical potential approaches the Hele-Shaw problem as the interactive length ϵ decreases to 0. Numerical justification for this approximation can be found in [24,28,30,31,40,45]. We refer to some other references [17,25–27,33,43,46] about numerical approximation for the Cahn-Hilliard equation and the references therein. For stochastic cases, the Cahn-Hilliard-Cook equation with additive noise (with fixed ϵ) was studied in [14, 32, 35–38, 42]. The well-posedness of the stochastic Cahn-Hilliard equation was discussed in [18, 21, 23] for additive noise and in [5, 13, 39] for multiplicative noise. The stability and error estimates in the discrete H^{-1}-norm for the stochastic Cahn-Hilliard equation with gradient-type noise were derived in [29]. The convergence of the stochastic Cahn-Hilliard equation to the Hele-Shaw flow was considered in [3, 4]. The semigroup approach is used for the stochastic Cahn-Hilliard equation with additive noise in [7, 19, 20, 32], and the references therein.

The goal of the paper is to design a numerical scheme for stochastic Cahn-Hilliard equations with functional-type noise, and consequently, to investigate various numerical properties of the proposed scheme. Note that the nonlinear term is not globally Lipschitz continuous and only satisfies a one-sided Lipschitz continuity, which interacts with the diffusion term to add another layer of difficulty in designing numerical schemes. We are aiming to prove the existence of discrete solutions, derive some stability results and higher moment bounds, and finally establish convergence rates for the proposed numerical scheme.

The rest of the paper is organized as follows. In Section 2, the weak solution is defined and several Hölder continuity results are obtained. In Section 3, the scheme is designed, and several stability results and higher moment bounds are both derived for the proposed scheme. In Section 4, the strong convergence with discrete H^{-1}-norm is established for the scheme. In Section 5, numerical experiments are done to validate our theoretical results based on different initial conditions and diffusion terms.

2 Preliminary

Throughout this paper, we use C to denote a generic constant and adopt the standard Sobolev notations as described in [9]. Additionally, $(\cdot\,,\,\cdot)$ will denote the standard inner product of $L^2(\mathcal{D})$, and $\mathbb{E}[\cdot]$ denotes the expectation operator on the filtered probability space $(\Omega,\mathcal{F},\{\mathcal{F}_t : t\geq 0\},\mathbb{P})$.

In this section, based on references [5, 13, 39], we first define the following weak formulation for problem (1.1)-(1.3): seeking an $\mathcal{F}_t$-adapted and $H^1(\mathcal{D})\times H^1(\mathcal{D})$-valued process $(u(\cdot,t),w(\cdot,t))$ such that there hold $\mathbb{P}$-almost surely

$$
\begin{aligned}
(u(t),\phi) &= (u_0,\phi) - \int_0^t (\nabla w(s),\nabla\phi)\,ds \qquad (2.1)\\
&\quad + \delta\int_0^t (g(u)dW_s,\phi), \quad \forall \phi\in H^1(\mathcal{D}), \quad \forall t\in(0,T],\\
(w(t),\varphi) &= \epsilon(\nabla u(t),\nabla\varphi) + \frac{1}{\epsilon}(f(u(t)),\varphi), \quad \forall\varphi\in H^1(\mathcal{D}), \quad \forall t\in(0,T]. \qquad (2.2)
\end{aligned}
$$

Next, we derive several Hölder continuity results in time for u with respect to the spatial L^2-norm and for w with respect to the spatial H^1-seminorm. These results are crucial for the error analysis due to the low regularity of the time derivative of u.

Lemma 2.1. *Let u be the strong solution to problem* (1.7)-(1.10)*. Then for any $s,t\in[0,T]$ with $t<s$, we have*

$$\mathbb{E}\left[\|u(s)-u(t)\|_{L^2}^2\right]+\epsilon\mathbb{E}\left[\int_t^s\|\Delta(u(\xi)-u(t))\|_{L^2}^2d\xi\right]\le C_1(s-t),$$

where

$$C_1=C\sup_{t\le\xi\le s}\mathbb{E}\left[\|\Delta u(\xi)\|_{L^2}^2\right]+C\sup_{t\le\xi\le s}\mathbb{E}\left[\|u(\xi)\|_{H^1}^6\right].$$

Proof. Consider the functional ψ defined as:

$$\psi(u(s))=\|u(s)-u(t)\|_{L^2}^2.$$

The first two Gateaux derivatives are:

$$\begin{aligned}&D\psi(u(s))(v_1)=2(u(s)-u(t),v_1(s)),\\&D\psi(u(s))(v_1,v_2)=2(v_1(s),v_2(s)).\end{aligned}$$

Applying Itô's formula to $\psi(u(s))$ yields

$$\begin{aligned}\|u(s)-u(t)\|_{L^2}^2=&2\int_t^s\Big(u(\xi)-u(t),-\Delta(\epsilon\Delta u(\xi)-\frac{1}{\epsilon}f(u(\xi)))\Big)d\xi\\&+\delta^2\int_t^s(g(u(\xi)),g(u(\xi)))d\xi\\&+2\int_t^s(u(\xi)-u(t),\delta g(u(\xi)))dW_\xi.\end{aligned}$$

Now after using integration by parts twice on the first integral, we have

$$\begin{aligned}\|u(s)-u(t)\|_{L^2}^2=&2\int_t^s(\Delta(u(\xi)-u(t)),-\epsilon\Delta(u(\xi)-u(t)))d\xi\\&+2\int_t^s(\Delta(u(\xi)-u(t)),-\epsilon\Delta u(t))d\xi\\&+2\int_t^s\Big(\Delta(u(\xi)-u(t)),\frac{1}{\epsilon}f(u(\xi))\Big)d\xi\\&+\delta^2\int_t^s(g(u(\xi)),g(u(\xi)))d\xi\\&+2\int_t^s((u(\xi)-u(t)),\delta g(u(\xi)))dW_\xi.\end{aligned}$$

For the nonlinear term, by using the embedding theorem, we have

$$\begin{aligned}&\mathbb{E}\left[\int_t^s\|f(u(\xi))\|_{L^2}^2d\xi\right]\\=&\mathbb{E}\left[\int_t^s\int_{\mathcal{D}}u^6(\xi)-2u^4(\xi)+u^2(\xi)dxd\xi\right]\\\le&C\sup_{t\le\xi\le s}\mathbb{E}\left[\|u(\xi)\|_{H^1}^6\right](s-t).\end{aligned}$$

Then upon taking the expectation on both sides as well as applying the Cauchy-Schwarz inequality, the Young's inequality, and the Gronwall's inequality, we have

$$\begin{aligned}&\mathbb{E}\left[\|u(s)-u(t)\|_{L^2}^2\right]+\epsilon\mathbb{E}\left[\int_t^s \|\Delta(u(\xi)-u(t))\|_{L^2}^2 d\xi\right]\\ \le& C\sup_{t\le\xi\le s}\mathbb{E}\left[\|\Delta u(\xi)\|_{L^2}^2\right](s-t)+C\sup_{t\le\xi\le s}\mathbb{E}\left[\|u(\xi)\|_{H^1}^6\right](s-t).\end{aligned}$$

The conclusion is proved. □

Lemma 2.2. *For any $s,t\in[0,T]$ with $t<s$, the chemical potential w satisfies*

$$\mathbb{E}\left[\|\nabla w(s)-\nabla w(t)\|_{L^2}^2\right]\le C_2(s-t),$$

where

$$C_2=C\sup_{t\le\zeta\le s}\mathbb{E}\left[\|u(\zeta)\|_{H^6}^6\right].$$

Proof. First define $g(u(s))=g_1(u(s))+g_2(u(s))$ where

$$\begin{aligned}g_1(u(s))&=\|\epsilon\nabla\Delta u(s)-\epsilon\nabla\Delta u(t)\|_{L^2}^2,\\ g_2(u(s))&=\left\|\frac{1}{\epsilon}\nabla f(u(s))-\frac{1}{\epsilon}\nabla f(u(t))\right\|_{L^2}^2.\end{aligned}$$

The first two Gateaux derivatives of g_1 are:

$$\begin{aligned}Dg_1(u(s))(\nu_1)&=2\epsilon^2\int_{\mathcal{D}}(\nabla\Delta u(s)-\nabla\Delta u(t))\cdot\nabla\Delta\nu_1(s)dx,\\ Dg_1(u(s))(\nu_1,\nu_2)&=2\epsilon^2\int_{\mathcal{D}}\nabla\Delta\nu_1\cdot\nabla\Delta\nu_2 dx.\end{aligned}$$

The first two Gateaux derivatives of g_2 are:

$$\begin{aligned}Dg_2(u(s))(\nu_1)=&\frac{2}{\epsilon^2}\int_{\mathcal{D}}[3u^2(s)\nabla u(s)-\nabla u(s)-\nabla f(u(t))]\\ &\cdot[6u(s)\nu_1(s)\nabla u(s)+3u^2(s)\nabla\nu_1(s)-\nabla\nu_1(s)]dx,\\ Dg_2(u(s))(\nu_1,\nu_2)=&\frac{2}{\epsilon^2}\int_{\mathcal{D}}[3u^2(s)\nabla u(s)-\nabla u(s)-\nabla f(u(t))]\\ &\cdot[6\nu_2(s)\nu_1(s)\nabla u(s)+6u(s)\nu_1(s)\nabla\nu_2(s)+6u(s)\nu_2(s)\nabla\nu_1(s)]dx\\ &+\frac{2}{\epsilon^2}\int_{\mathcal{D}}[6u(s)\nu_1(s)\nabla u(s)+3u^2(s)\nabla\nu_1(s)-\nabla\nu_1(s)]\\ &\cdot[3u^2(s)\nabla\nu_2(s)+6u(s)\nu_2(s)\nabla u(s)-\nabla\nu_2(s)]dx.\end{aligned}$$

Then applying the Itô's formula to $g(w(s)) = \|\nabla w(s) - \nabla w(t)\|_{L^2}^2$ yields

$$
\begin{aligned}
&\|\nabla w(s)-\nabla w(t)\|_{L^2}^2\\
=&2\epsilon^2\int_t^s(\nabla\Delta(u(\xi)-u(t)),\nabla\Delta G_1(\xi))d\xi\\
&+\epsilon^2\int_t^s(\nabla\Delta G_2(\xi),\nabla\Delta G_2(\xi))d\xi+2\epsilon^2\int_t^s(\nabla\Delta(u(\xi)-u(t)),\nabla\Delta G_2(\xi))dW_\xi\\
&+\frac{2}{\epsilon^2}\int_t^s\int_{\mathcal{D}}[3u^2(\xi)\nabla u(\xi)-\nabla u(\xi)-\nabla f(u(t))]\\
&\qquad\cdot[6u(\xi)G_1(\xi)\nabla u(\xi)+3u^2(\xi)\nabla G_1(\xi)-\nabla G_1(\xi)]dxd\xi\\
&+\frac{2}{\epsilon^2}\int_t^s\int_{\mathcal{D}}[3u^2(\xi)\nabla u(\xi)-\nabla u(\xi)-\nabla f(u(t))]\\
&\qquad\cdot[6u(\xi)G_2(\xi)\nabla u(\xi)+3u^2(\xi)\nabla G_2(\xi)-\nabla G_2(\xi)]dxdW_\xi\\
&+\frac{\delta^2}{\epsilon^2}\int_t^s\int_{\mathcal{D}}[3u^2(\xi)\nabla u(\xi)-\nabla u(\xi)-\nabla f(u(t))]\\
&\qquad\cdot[6G_2^2(\xi)\nabla u(\xi)+6u(\xi)G_2(\xi)\nabla G_2(\xi)+6u(\xi)G_2(\xi)\nabla G_2(\xi)]dxd\xi\\
&+\frac{\delta^2}{\epsilon^2}\int_t^s\int_{\mathcal{D}}[6u(\xi)G_2(\xi)\nabla u(\xi)+3u^2(\xi)\nabla G_2(\xi)-\nabla G_2(\xi)]\\
&\qquad\cdot[3u^2(\xi)\nabla G_2(\xi)+6u(\xi)G_2(\xi)\nabla u(\xi)-\nabla G_2(\xi)]dxd\xi,
\end{aligned}
$$

where

$$
\begin{aligned}
&G_1(\xi)=-\Delta\left(\epsilon\Delta u(\xi)-\frac{1}{\epsilon}f(u(\xi))\right),\\
&G_2(\xi)=\delta g(u(\xi)).
\end{aligned}
$$

Taking the expectation on both sides of the above equation, and using the Young's inequality and the embedding theorem, we get

$$
\|\nabla w(s)-\nabla w(t)\|_{L^2}^2\le C\sup_{t\le\xi\le s}\mathbb{E}\left[\|u(\xi)\|_{H^6}^6\right](s-t), \tag{2.3}
$$

where the term H^6 norm is from the term $\frac{2}{\epsilon^2}\int_t^s\int_{\mathcal{D}}9u^4(\xi)\nabla u(\xi)\nabla G_1(\xi)dxd\xi$. □

3 Stability and convergence

Let $t_n = n\tau$ $(n=0,1,\cdots,N)$ be a uniform partition of $[0,T]$ and $\mathcal{T}_h$ be a quasi-uniform triangulation of $\mathcal{D}$. Define V_h to be the finite element space given by

$$
V_h := \{v_h\in H^1(\mathcal{D}) : v_h|_K\in\mathcal{P}_1(K),\ \forall K\in\mathcal{T}_h\}, \tag{3.1}
$$

where $\mathcal{P}_1(K)$ denotes the space of linear polynomials on the element K. Define $\mathring{V}_h$ be the subspace of V_h with zero mean, i.e.,

$$\mathring{V}_h := \{v_h \in V_h : (v_h, 1) = 0\}. \tag{3.2}$$

The fully discrete mixed finite element methods for (1.7)-(1.10) is to seek $\mathcal{F}_{t_n}$-adapted and $V_h \times V_h$-valued process $\{(u_h^n, w_h^n)\}$ $(n = 1, \cdots, N)$ such that $\mathbb{P}$-almost surely

$$(u_h^n - u_h^{n-1}, \eta_h) + \tau(\nabla w_h^n, \nabla \eta_h) = \delta(g(u_h^{n-1}), \eta_h)\Delta W^n, \quad \forall \eta_h \in V_h, \tag{3.3}$$

$$\epsilon(\nabla u_h^n, \nabla v_h) + \frac{1}{\epsilon}(I_h f^n, v_h) = (w_h^n, v_h), \quad \forall v_h \in V_h, \tag{3.4}$$

where $\Delta W^n := W^n - W^{n-1}$ satisfies the normal distribution $\mathcal{N}(0,1)$, $f^n := (u_h^n)^3 - u_h^n$, and $I_h : C(\bar{\mathcal{D}}) \to V_h$ is the standard nodal value interpolation operator defined by

$$I_h v := \sum_{i=1}^{N_h} v(a_i)\psi_i.$$

Here N_h denotes the number of vertices of $\mathcal{T}_h$ and ψ_i denotes the nodal basis for V_h corresponding to the vertex a_i. The initial conditions are chosen to be $u_h^0 = P_h u_0$ and $w_h^0 = P_h w_0$, where $P_h : L^2(\mathcal{D}) \to V_h$ is the L^2-projection operator defined by

$$(P_h v, v_h) = (v, v_h), \quad \forall v_h \in V_h.$$

The following properties of the L^2-projection can be found in [9,16]:

$$\|v - P_h v\|_{L^2} + h\|\nabla(v - P_h v)\|_{L^2} \le C h^{\min\{2,s\}} \|v\|_{H^s}, \tag{3.5}$$

$$\|v - P_h v\|_{L^\infty} \le C h^{2-\frac{d}{2}} \|v\|_{H^2}, \tag{3.6}$$

for all $v \in H^s(\mathcal{D})$ such that $s > \frac{3}{2}$. Furthermore, the inverse discrete Laplace operator $\Delta_h^{-1} : \mathring{V}_h \to \mathring{V}_h$ is defined by

$$\big(\nabla(-\Delta_h^{-1}\zeta_h), \nabla v_h\big) = \big(\zeta_h, v_h\big), \quad \forall v_h \in \mathring{V}_h. \tag{3.7}$$

Then for $\zeta_h, \Phi_h \in \mathring{V}_h$ the discrete H^{-1} inner product is defined as

$$(\zeta_h, \Phi_h)_{-1,h} = \big(\nabla(-\Delta_h^{-1}\zeta_h), \nabla(-\Delta_h^{-1}\Phi_h)\big) = \big(\zeta_h, -\Delta_h^{-1}\Phi_h\big) = \big(-\Delta_h^{-1}\zeta_h, \Phi_h\big). \tag{3.8}$$

It is easy to show that the discrete H^{-1}-norm satisfies the following two properties

$$|(\zeta_h, \Phi_h)| \le \|\zeta_h\|_{-1,h} |\Phi_h|_{H^1}, \quad \forall \zeta_h, \Phi_h \in \mathring{V}_h, \tag{3.9}$$

$$\|\zeta_h\|_{-1,h} \le C\|\zeta_h\|_{L^2}, \quad \forall \zeta_h \in \mathring{V}_h. \tag{3.10}$$

Lastly, define the discrete Laplace operator $\Delta_h : V_h \to V_h$ by

$$(\Delta_h \zeta_h, v_h) = -(\nabla \zeta_h, \nabla v_h), \quad \forall v_h \in V_h. \tag{3.11}$$

In the following we prove the solvability of the fully discrete scheme.

Theorem 3.1. *There exists a solution to the scheme* (3.3)-(3.4).

Proof. For all $v_h \in V_h$, set $\eta_h = -\Delta_h^{-1} v_h$ in (3.3), we have

$$(u_h^n, v_h)_{-1,h} + \tau\epsilon(\nabla u_h^n, \nabla v_h) + \frac{\tau}{\epsilon}(I_h(u_h^n)^3, v_h) - \frac{\tau}{\epsilon}(u_h^n, v_h) \\ -\delta(g(u_h^{n-1}), -\Delta_h^{-1} v_h)\Delta W^n - (u_h^{n-1}, v_h)_{-1,h} = 0. \tag{3.12}$$

Consider the following iterative scheme for $k \geq 1$

$$(u_h^{n,k}, v_h)_{-1,h} + \tau\epsilon(\nabla u_h^{n,k}, w_h) + \frac{\tau}{\epsilon}(I_h(u_h^{n,k-1})^3, v_h) - \frac{\tau}{\epsilon}(u_h^{n,k-1}, v_h) \\ -\delta(g(u_h^{n-1}), -\Delta_h^{-1} v_h)\Delta W^n - (u_h^{n-1}, v_h)_{-1,h} = 0, \tag{3.13}$$

where $u_h^{n,0} = u_h^{n-1}$. The scheme (3.13) is uniquely solvable due to the explicitly treatment of the nonlinear term. By choosing $v_h = u_h^{n,k}$ in (3.13), we can prove by an induction argument that

$$\|u_h^{n,k}\|_{H^1} \leq C(\tau, h), \quad \forall\, k \geq 1. \tag{3.14}$$

Therefore, the solution sequence $\{u_h^{n,k}\}_{k=1}^{\infty}$ is bounded in $V_h \subset H^1(\mathcal{D})$. Applying the weak compact theorem and the Rellich-Kondrachov Theorem [1, 34], we can extract a subsequence $\{u_h^{n,k_i}\}_{i=1}^{\infty}$ such that u_h^{n,k_i} converges to u_h^* weakly in $H^1(\mathcal{D})$ and u_h^{n,k_i} converges to u_h^* strongly in $L^6(\mathcal{D})$ as $i \to \infty$. Letting $k = k_i$ and $i \to \infty$ in (3.13), we thus obtain

$$(u_h^*, v_h)_{-1,h} + \tau\epsilon(\nabla u_h^*, \nabla v_h) + \frac{\tau}{\epsilon}(I_h(u_h^*)^3, v_h) - \frac{\tau}{\epsilon}(u_h^*, v_h) \\ -\delta(g(u_h^{n-1}), -\Delta_h^{-1} v_h)\Delta W^n - (u_h^{n-1}, v_h)_{-1,h} = 0. \tag{3.15}$$

Note that the following inequality is the key to establish Eq. (3.15):

$$\begin{aligned} &(I_h(u_h^{n,k_{i-1}})^3 - I_h(u_h^*)^3, v_h) \\ \leq &\|((u_h^{n,k_{i-1}})^3 - (u_h^*)^3)\|_{L^2}\|v_h\|_{L^2} \\ \leq &C\|u_h^{n,k_{i-1}} - u_h^*\|_{L^6}(\|u_h^{n,k_{i-1}}\|_{L^6}^2 + \|u_h^*\|_{L^6}^2)\|v_h\|_{L^2}. \end{aligned} \tag{3.16}$$

The right-hand side of (3.16) approaches 0 as $i \to \infty$. Eq. (3.15) establishes the existence of the solution to (3.12). The solution u_h^n together with w_h^n determined by (3.4) provides a solution (u_h^n, w_h^n) to the scheme (3.3)-(3.4). □

Theorem 3.2. *Let u_h^n $(n=1,2,\cdots N)$ be the solution of* (3.3) *and* (3.4)*, then there exists a positive constant C independent of h and τ that*

$$\max_{1\leq n\leq N} \mathbb{E}\left[\|u_h^n\|_{L^2}^2\right] + \sum_{n=1}^{N} \mathbb{E}\left[\|u_h^n - u_h^{n-1}\|_{L^2}^2\right] + \tau\sum_{n=1}^{N} \mathbb{E}\left[\|\Delta_h u_h^n\|_{L^2}^2\right] \leq C. \tag{3.17}$$

Proof. Testing (3.3) and (3.4) by $\eta_h = u_h^n$ and $v_h = -\Delta_h u_h^n$, respectively, we get

$$(u_h^n - u_h^{n-1}, u_h^n) + \tau(\nabla w_h^n, \nabla u_h^n) = \delta(g(u_h^{n-1}), u_h^n)\Delta W^n, \tag{3.18}$$

$$\epsilon(\nabla u_h^n, -\nabla\Delta_h u_h^n) + \frac{1}{\epsilon}(I_h f^n, -\Delta_h u_h^n) = (w_h^n, -\Delta_h u_h^n). \tag{3.19}$$

Thus,

$$\frac{1}{2}\|u_h^n\|_{L^2}^2 - \frac{1}{2}\|u_h^{n-1}\|_{L^2}^2 + \frac{1}{2}\|u_h^n - u_h^{n-1}\|_{L^2}^2 + \tau(\nabla w_h^n, \nabla u_h^n) = \delta(g(u_h^{n-1}), u_h^n)\Delta W^n, \tag{3.20}$$

$$\epsilon\|\Delta_h u_h^n\|_{L^2}^2 + \frac{1}{\epsilon}(\nabla I_h f^n, \nabla u_h^n) = (\nabla w_h^n, \nabla u_h^n). \tag{3.21}$$

Multiplying the second equation by τ and substituting into the first equation, and then rearranging, we get

$$\begin{aligned}&\frac{1}{2}\|u_h^n\|_{L^2}^2 - \frac{1}{2}\|u_h^{n-1}\|_{L^2}^2 + \frac{1}{2}\|u_h^n - u_h^{n-1}\|_{L^2}^2 + \epsilon\tau\|\Delta_h u_h^n\|_{L^2}^2 \\ =&\delta(g(u_h^{n-1}), u_h^n)\Delta W^n - \frac{\tau}{\epsilon}(\nabla I_h f^n, \nabla u_h^n).\end{aligned} \tag{3.22}$$

One key role of the interpolation operator is to bound the nonlinear term. Denote $u_i = u_h^n(a_i)$, and then

$$\begin{aligned}-\frac{\tau}{\epsilon}(\nabla I_h f^n, \nabla u_h^n) &= \frac{\tau}{\epsilon}\|\nabla u_h^n\|_{L^2}^2 - \frac{\tau}{\epsilon}\Big(\nabla\sum_{i=1}^{N_h} u_i^3\varphi_i, \nabla\sum_{i=j}^{N_h} u_j\varphi_j\Big) \\ &= \frac{\tau}{\epsilon}\|\nabla u_h^n\|_{L^2}^2 - \frac{\tau}{\epsilon}\sum_{i,j=1}^{N_h} b_{ij}(\nabla\varphi_i, \nabla\varphi_j),\end{aligned}$$

where $b_{ij} = u_i^3 u_j$. Note when $i \neq j$, we have

$$b_{ij} \leq \frac{3}{4}u_i^4 + \frac{1}{4}u_j^4. \tag{3.23}$$

The stiffness matrix is diagonally dominant, and then

$$\begin{aligned}&-\frac{\tau}{\epsilon}\sum_{i,j=1}^{N_h} b_{ij}(\nabla\varphi_i, \nabla\varphi_j) \\ \leq& -\frac{\tau}{\epsilon}\sum_{k=1}^{N_h} b_{kk}\Big[(\nabla\varphi_k, \nabla\varphi_k) - \frac{3}{4}\sum_{\substack{i=1\\ i\neq k}}^{N_h}|(\nabla\varphi_i, \nabla\varphi_k)| - \frac{1}{4}\sum_{\substack{j=1\\ j\neq k}}^{N_h}|(\nabla\varphi_k, \nabla\varphi_j)|\Big] \\ \leq& -\frac{\tau}{\epsilon}\sum_{k=1}^{N_h} b_{kk}\Big[(\nabla\varphi_k, \nabla\varphi_k) - \sum_{\substack{i=1\\ i\neq k}}^{N_h}|(\nabla\varphi_i, \nabla\varphi_k)|\Big] \\ \leq& 0.\end{aligned} \tag{3.24}$$

Based on this inequality, we have

$$
\begin{aligned}
-\frac{\tau}{\epsilon}(\nabla I_h f^n, \nabla u_h^n) &\le \frac{\tau}{\epsilon}\|\nabla u_h^n\|_{L^2}^2, \\
&\le \frac{C\tau}{\epsilon^3}\|u_h^n\|_{L^2}^2 + \frac{\epsilon\tau}{2}\|\Delta_h u_h^n\|_{L^2}^2.
\end{aligned} \tag{3.25}
$$

For the diffusion term, apply the growth property of $g(\cdot)$ and the martingale property to obtain

$$
\begin{aligned}
&\delta\mathbb{E}\left[(g(u_h^{n-1}), u_h^n)\Delta W^n\right] \\
=&\delta\mathbb{E}\left[(g(u_h^{n-1}), u_h^n - u_h^{n-1})\Delta W^n\right] \\
\le&\frac{1}{4}\mathbb{E}\left[\|u_h^n - u_h^{n-1}\|_{L^2}^2\right] + \delta^2\tau\mathbb{E}\left[\|g(u_h^{n-1})\|_{L^2}^2\right] \\
\le&\frac{1}{4}\mathbb{E}\left[\|u_h^n - u_h^{n-1}\|_{L^2}^2\right] + C\delta^2\tau + C\delta^2\tau\mathbb{E}\left[\|u_h^{n-1}\|_{L^2}^2\right].
\end{aligned} \tag{3.26}
$$

Taking the expectation and taking the summation over n from 1 to ℓ in (3.22), we have

$$
\begin{aligned}
&\mathbb{E}\left[\frac{1}{2}\|u_h^\ell\|_{L^2}^2\right] + \frac{1}{4}\sum_{n=1}^{\ell}\mathbb{E}\left[\|u_h^n - u_h^{n-1}\|_{L^2}^2\right] + \frac{\epsilon\tau}{2}\sum_{n=1}^{\ell}\mathbb{E}\left[\|\Delta_h u_h^n\|_{L^2}^2\right] \\
\le&\frac{C\tau}{\epsilon^3}\sum_{n=1}^{\ell}\mathbb{E}\left[\|u_h^n\|_{L^2}^2\right] + C\delta^2 + C\delta^2\tau\sum_{n=1}^{\ell}\mathbb{E}\left[\|u_h^{n-1}\|_{L^2}^2\right] + \mathbb{E}\left[\frac{1}{2}\|u_h^0\|_{L^2}^2\right].
\end{aligned} \tag{3.27}
$$

By the discrete Gronwall inequality, we obtain

$$
\mathbb{E}\left[\frac{1}{2}\|u_h^\ell\|_{L^2}^2\right] + \frac{1}{4}\sum_{n=1}^{\ell}\mathbb{E}\left[\|u_h^n - u_h^{n-1}\|_{L^2}^2\right] + \frac{\epsilon\tau}{2}\sum_{n=1}^{\ell}\mathbb{E}\left[\|\Delta_h u_h^n\|_{L^2}^2\right] \le C, \tag{3.28}
$$

where C depends on δ and ϵ. Finally, the estimate (3.17) follows from (3.28). □

By Eq. (3.11), the Cauchy-Schwartz inequality, and Theorem 3.2, we could directly obtain the following Corollary.

Corollary 3.1. *Let u_h^n $(n=1,2,\cdots N)$ be the solution of* (3.3) *and* (3.4), *then there holds*

$$
\tau\sum_{n=1}^{N}\mathbb{E}\left[\|\nabla u_h^n\|_{L^2}^2\right] \le C. \tag{3.29}
$$

Theorem 3.3. *Suppose the mesh constraint $\tau \le C\epsilon^3$ holds, and let u_h^n $(n=1,2,\cdots N)$ be the solution of* (3.3) *and* (3.4), *then there exists a positive constant independent of h and τ such that there holds for any $p \ge 2$ that*

$$
\sup_{0\le n\le N}\mathbb{E}\left[\|u_h^n\|_{L^2}^p\right] \le C. \tag{3.30}
$$

Proof. The proof is divided into three steps. In Step 1, we prove the bound for $\mathbb{E}\left\|u_h^n\right\|_{L^2}^4$. In Step 2, we establish the bound for $\mathbb{E}\left\|u_h^n\right\|_{L^2}^p$, where $p=2^r$ and r is an arbitrary positive integer. In Step 3, using Step 1 and Step 2, we could obtain the bound for $\mathbb{E}\left\|u_h^n\right\|_{L^2}^p$, where p is an arbitrary real number and $p\geq 2$.

Step 1. Based on (3.22)-(3.25), we have

$$\begin{aligned}&\frac{1}{2}\|u_h^n\|_{L^2}^2-\frac{1}{2}\|u_h^{n-1}\|_{L^2}^2+\frac{1}{2}\|u_h^n-u_h^{n-1}\|_{L^2}^2+\epsilon\tau\|\Delta_h u_h^n\|_{L^2}^2\\ \leq&\frac{\tau}{\epsilon}\|\nabla u_h^n\|_{L^2}^2+\delta(g(u_h^{n-1}),u_h^n)\Delta W^n.\end{aligned}\tag{3.31}$$

Multiplying the quantity

$$\|u_h^n\|_{L^2}^2+\frac{1}{2}\|u_h^{n-1}\|_{L^2}^2=\frac{3}{4}\left(\|u_h^n\|_{L^2}^2+\|u_h^{n-1}\|_{L^2}^2\right)+\frac{1}{4}\left(\|u_h^n\|_{L^2}^2-\|u_h^{n-1}\|_{L^2}^2\right)$$

on both sides of (3.31) gives us

$$\begin{aligned}&\frac{3}{8}(\|u_h^n\|_{L^2}^4-\|u_h^{n-1}\|_{L^2}^4)+\frac{1}{8}(\|u_h^n\|_{L^2}^2-\|u_h^{n-1}\|_{L^2}^2)^2\\ &\quad+\left(\frac{1}{2}\|u_h^n-u_h^{n-1}\|_{L^2}^2+\epsilon\tau\|\Delta_h u_h^n\|_{L^2}^2\right)\left(\|u_h^n\|_{L^2}^2+\frac{1}{2}\|u_h^{n-1}\|_{L^2}^2\right)\\ \leq&\frac{\tau}{\epsilon}\|\nabla u_h^n\|_{L^2}^2\left(\|u_h^n\|_{L^2}^2+\frac{1}{2}\|u_h^{n-1}\|_{L^2}^2\right)\\ &\quad+\delta(g(u_h^{n-1}),u_h^n)\Delta W^n\left(\|u_h^n\|_{L^2}^2+\frac{1}{2}\|u_h^{n-1}\|_{L^2}^2\right).\end{aligned}\tag{3.32}$$

The first term on the right-hand side of (3.32) can be estimated as

$$\begin{aligned}&\frac{\tau}{\epsilon}\|\nabla u_h^n\|_{L^2}^2\left(\|u_h^n\|_{L^2}^2+\frac{1}{2}\|u_h^{n-1}\|_{L^2}^2\right)\\ \leq&\frac{\epsilon}{2}\tau\|\Delta_h u_h^n\|_{L^2}^2\left(\|u_h^n\|_{L^2}^2+\frac{1}{2}\|u_h^{n-1}\|_{L^2}^2\right)\\ &\quad+\frac{C\tau}{\epsilon^3}\|u_h^n\|_{L^2}^2\left(\frac{3}{2}\|u_h^n\|_{L^2}^2-\frac{1}{2}\|u_h^n\|_{L^2}^2-\|u_h^{n-1}\|_{L^2}^2\right)\\ \leq&\frac{\epsilon}{2}\tau\|\Delta_h u_h^n\|_{L^2}^2\left(\|u_h^n\|_{L^2}^2+\frac{1}{2}\|u_h^{n-1}\|_{L^2}^2\right)\\ &\quad+\frac{C\tau}{\epsilon^3}\|u_h^n\|_{L^2}^4+\frac{C\tau}{\epsilon^3}\left(\|u_h^n\|_{L^2}^2-\|u_h^{n-1}\|_{L^2}^2\right)^2.\end{aligned}\tag{3.33}$$

The second term can be estimated by the Cauchy-Schwarz inequality and the growth property of $g(\cdot)$ as

$$
\begin{aligned}
&\delta(g(u_h^{n-1}),u_h^n)\Delta W^n\left(\|u_h^n\|_{L^2}^2+\frac{1}{2}\|u_h^{n-1}\|_{L^2}^2\right)\\
=&\delta(g(u_h^{n-1}),u_h^n-u_h^{n-1}+u_h^{n-1})\Delta W^n\left(\|u_h^n\|_{L^2}^2+\frac{1}{2}\|u_h^{n-1}\|_{L^2}^2\right)\\
\leq&\left(\frac{1}{4}\|u_h^n-u_h^{n-1}\|_{L^2}^2+C\delta^2(1+\|u_h^{n-1}\|_{L^2}^2)(\Delta W^n)^2\right)\left(\|u_h^n\|_{L^2}^2+\frac{1}{2}\|u_h^{n-1}\|_{L^2}^2\right)\\
&+\delta(g(u_h^{n-1}),u_h^{n-1})\Delta W^n\left(\|u_h^n\|_{L^2}^2+\frac{1}{2}\|u_h^{n-1}\|_{L^2}^2\right).
\end{aligned}
\tag{3.34}
$$

By the Cauchy-Schwarz inequality, we have

$$
\begin{aligned}
&C\delta^2\left(1+\|u_h^{n-1}\|_{L^2}^2\right)(\Delta W^n)^2\left(\|u_h^n\|_{L^2}^2+\frac{1}{2}\|u_h^{n-1}\|_{L^2}^2\right)\\
=&C\delta^2\left(1+\|u_h^{n-1}\|_{L^2}^2\right)(\Delta W^n)^2\left(\|u_h^n\|_{L^2}^2-\|u_h^{n-1}\|_{L^2}^2+\frac{3}{2}\|u_h^{n-1}\|_{L^2}^2\right)\\
\leq&\theta_1\left(\|u_h^n\|_{L^2}^2-\|u_h^{n-1}\|_{L^2}^2\right)^2+C\delta^4\left(1+\|u_h^{n-1}\|_{L^2}^4\right)(\Delta W^n)^4\\
&+C\delta^2\|u_h^{n-1}\|_{L^2}^4(\Delta W^n)^2+C\delta^2\|u_h^{n-1}\|_{L^2}^2(\Delta W^n)^2,
\end{aligned}
\tag{3.35}
$$

where $\theta_1>0$ will be specified later. Furthermore, we have

$$
\begin{aligned}
&\delta(g(u_h^{n-1}),u_h^{n-1})\Delta W^n\left(\|u_h^n\|_{L^2}^2+\frac{1}{2}\|u_h^{n-1}\|_{L^2}^2\right)\\
=&\delta(g(u_h^{n-1}),u_h^{n-1})\Delta W^n\left(\|u_h^n\|_{L^2}^2-\|u_h^{n-1}\|_{L^2}^2+\frac{3}{2}\|u_h^{n-1}\|_{L^2}^2\right)\\
\leq&\theta_2\left(\|u_h^n\|_{L^2}^2-\|u_h^{n-1}\|_{L^2}^2\right)^2+C\delta^2\left(1+\|u_h^{n-1}\|_{L^2}^4\right)(\Delta W^n)^2\\
&+\frac{3}{2}\delta(g(u_h^{n-1}),u_h^{n-1})\|u_h^{n-1}\|_{L^2}^2\Delta W^n,
\end{aligned}
\tag{3.36}
$$

where τ, θ_1, and θ_2 are chosen to be small enough such that

$$
\frac{C\tau}{\epsilon^3}+\theta_1+\theta_2\leq\frac{1}{16}. \tag{3.37}
$$

Then by the martingale property and properties of the Wiener process, after taking the summation from $n=1$ to $n=\ell$ and the expectation on both sides of (3.32), we get

$$\begin{aligned}&\left(\frac{3}{8}-\frac{C\tau}{\epsilon^3}\right)\mathbb{E}\left[\|u_h^\ell\|_{L^2}^4\right]+\frac{1}{16}\sum_{n=1}^{\ell}\mathbb{E}\left[\left(\|u_h^n\|_{L^2}^2-\|u_h^{n-1}\|_{L^2}^2\right)^2\right]\\&\quad+\sum_{n=1}^{\ell}\mathbb{E}\left[\left(\frac{1}{4}\|u_h^n-u_h^{n-1}\|_{L^2}^2+\frac{\epsilon}{2}\tau\|\Delta_h u_h^n\|_{L^2}^2\right)\left(\|u_h^n\|_{L^2}^2+\frac{1}{2}\|u_h^{n-1}\|_{L^2}^2\right)\right]\\&\le\frac{C\tau}{\epsilon^3}\sum_{n=0}^{\ell-1}\mathbb{E}\left[\|u_h^n\|_{L^2}^4\right]+\frac{3}{8}\mathbb{E}\left[\|u_h^0\|_{L^2}^4\right]+C(\delta^2+\delta^4)\tau^2\sum_{n=0}^{\ell-1}\mathbb{E}\left[\|u_h^n\|_{L^2}^4\right]\\&\quad+C(\delta^2+\delta^4)\tau\sum_{n=0}^{\ell-1}\mathbb{E}\left[\|u_h^n\|_{L^2}^4\right]+C(\delta^2+\delta^4\tau).\end{aligned}\tag{3.38}$$

Under the mesh constraint $\tau\le C\epsilon^3$, we have

$$\begin{aligned}&\frac{1}{4}\mathbb{E}\left[\|u_h^\ell\|_{L^2}^4\right]+\frac{1}{16}\sum_{n=1}^{\ell}\mathbb{E}\left[\left(\|u_h^n\|_{L^2}^2-\|u_h^{n-1}\|_{L^2}^2\right)^2\right]\\&\quad+\sum_{n=1}^{\ell}\mathbb{E}\left[\left(\frac{1}{4}\|u_h^n-u_h^{n-1}\|_{L^2}^2+\frac{\epsilon}{2}\tau\|\Delta_h u_h^n\|_{L^2}^2\right)\left(\|u_h^n\|_{L^2}^2+\frac{1}{2}\|u_h^{n-1}\|_{L^2}^2\right)\right]\\&\le C\left(\frac{1}{\epsilon^3}+\delta^2+\delta^4\right)\tau\sum_{n=0}^{\ell-1}\mathbb{E}\left[\|u_h^n\|_{L^2}^4\right]+\frac{3}{8}\mathbb{E}\left[\|u_h^0\|_{L^2}^4\right]+C(\delta^2+\delta^4).\end{aligned}\tag{3.39}$$

By the Gronwall's inequality, we have

$$\begin{aligned}&\frac{1}{4}\mathbb{E}\left[\|u_h^\ell\|_{L^2}^4\right]+\frac{1}{16}\sum_{n=1}^{\ell}\mathbb{E}\left[\left(\|u_h^n\|_{L^2}^2-\|u_h^{n-1}\|_{L^2}^2\right)^2\right]\\&\quad+\sum_{n=1}^{\ell}\mathbb{E}\left[\left(\frac{1}{4}\|u_h^n-u_h^{n-1}\|_{L^2}^2+\frac{\epsilon}{2}\tau\|\Delta_h u_h^n\|_{L^2}^2\right)\left(\|u_h^n\|_{L^2}^2+\frac{1}{2}\|u_h^{n-1}\|_{L^2}^2\right)\right]\\&\le C.\end{aligned}\tag{3.40}$$

Step 2. From (3.31)-(3.36), we have

$$\begin{aligned}&\frac{1}{4}\left(\|u_h^n\|_{L^2}^4-\|u_h^{n-1}\|_{L^2}^4\right)+\frac{1}{16}\left(\|u_h^n\|_{L^2}^2-\|u_h^{n-1}\|_{L^2}^2\right)^2\\&\quad+\left(\frac{1}{4}\|(u_h^n-u_h^n)\|_{L^2}^2+\frac{\epsilon}{2}\tau\|\Delta_h u_h^n\|_{L^2}^2\right)\left(\|u_h^n\|_{L^2}^2+\frac{1}{2}\|u_h^{n-1}\|_{L^2}^2\right)\\&\le C\frac{\tau}{\epsilon^3}\|u_h^n\|_{L^2}^4+C\delta^4(1+\|u_h^{n-1}\|_{L^2}^4)(\Delta W^n)^4\\&\quad+C\delta^2(1+\|u_h^{n-1}\|_{L^2}^4)(\Delta W_n)^2+C\delta\|u_h^{n-1}\|_{L^2}^4\Delta W^n.\end{aligned}\tag{3.41}$$

Similar to Step 1, by multiplying (3.41) by $\|u_h^n\|_{L^2}^4+\frac{1}{2}\|u_h^{n-1}\|_{L^2}^4$, we could get the 8-th moment of the L^2-stability of the numerical solution. Then repeat this process, we could obtain the 2^r-th moment of the L^2-stability of the numerical solution.

Step 3. Suppose $2^{r-1}\le p\le 2^r$, and then by Young's inequality, we have

$$\mathbb{E}\left[\|u_h^\ell\|_{L^2}^p\right]\le \mathbb{E}\left[\|u_h^\ell\|_{L^2}^{2^r}\right]+C<\infty, \tag{3.42}$$

where the second inequality follows from the results in Step 2. The proof is complete. □

4 Error estimates

Define a sequence of subsets as below

$$\widetilde{\Omega}_{\kappa,n}=\Big\{\omega\in\Omega:\ \max_{1\le i\le n}\|u_h^i\|_{H^1}^2\le\kappa\Big\}, \tag{4.1}$$

where κ will be specified later. Clearly, it holds that $\widetilde{\Omega}_{\kappa,0}\supset\widetilde{\Omega}_{\kappa,1}\supset\cdots\supset\widetilde{\Omega}_{\kappa,\ell}$.

Next, for each $n=0,1,\cdots,N$, define

$$\begin{aligned}
&E^n:=u(t_n)-u_h^n, && G^n:=w(t_n)-w_h^n,\\
&\Theta^n:=u(t_n)-P_hu(t_n), && \Lambda^n:=w(t_n)-P_hw(t_n),\\
&\Phi^n:=P_hu(t_n)-u_h^n, && \Psi^n:=P_hw(t_n)-w_h^n.
\end{aligned}$$

Theorem 4.1. *The following error estimate holds for any $\ell=1,2,\cdots,N$:*

$$\mathbb{E}\left[\mathbb{1}_{\widetilde{\Omega}_{\kappa,\ell}}\|E^\ell\|_{-1,h}^2\right]+\tau\sum_{n=1}^{\ell}\mathbb{E}\left[\mathbb{1}_{\widetilde{\Omega}_{\kappa,n}}\|\nabla E^n\|_{L^2(\mathcal{D})}^2\right]\le C\tau+Ch^2+Ch^4\big(\ln h\big)^6\tau^{-2}, \tag{4.2}$$

where C depends on

$$\sup_{t\in[0,T]}\mathbb{E}\left[\|u(t)\|_{H^6}^6\right]\quad and\quad \sup_{t\in[0,T]}\mathbb{E}\left[\|w(t)\|_{H^2}^2\right].$$

Proof. From the weak formulation (2.1)-(2.2), we get

$$\begin{aligned}
(u(t_n),\eta_h)=(u(t_{n-1}),\eta_h)-\int_{t_{n-1}}^{t_n}(\nabla w(s),\nabla\eta_h)ds\\
+\delta\int_{t_{n-1}}^{t_n}(g(u(s)),\eta_h)dW_s,\quad\forall\eta_h\in V_h,
\end{aligned} \tag{4.3}$$

$$(w(t_n),v_h)=\epsilon(\nabla u(t_n),\nabla v_h)+\frac{1}{\epsilon}(f(u(t_n)),v_h),\quad\forall v_h\in V_h. \tag{4.4}$$

hold $\mathbb{P}$-almost surely. Subtracting (3.3)-(3.4) from (4.3)-(4.4), we get

$$\begin{aligned}
(E^n,\eta_h)=(E^{n-1},\eta_h)-\int_{t_{n-1}}^{t_n}(\nabla w(s)-\nabla w_h^n,\nabla\eta_h)ds\\
+\delta\int_{t_{n-1}}^{t_n}(g(u(s))-g(u_h^{n-1}),\eta_h)dW_s,\quad\forall\eta_h\in V_h,
\end{aligned} \tag{4.5}$$

$$(G^n,v_h)=\epsilon(\nabla E^n,\nabla v_h)+\frac{1}{\epsilon}(f(u(t_n))-I_hf^n,v_h),\quad\forall v_h\in V_h. \tag{4.6}$$

Choosing

$$\eta_h = -\Delta_h^{-1}\Phi^n \in \mathring{V}_h \quad \text{and} \quad v_h = \tau\Phi^n \in \mathring{V}_h$$

yields

$$\begin{aligned}
(E^n - E^{n-1}, -\Delta_h^{-1}\Phi^n) = & -\int_{t_{n-1}}^{t_n} (\nabla(w(s) - w_h^n), \nabla(-\Delta_h^{-1}\Phi^n))ds \\
& + \delta\int_{t_{n-1}}^{t_n} (g(u(s)) - g(u_h^n), -\Delta_h^{-1}\Phi^n)dW_s, \\
(G^n, \tau\Phi^n) = & \epsilon(\nabla E^n, \nabla(\tau\Phi^n)) + \frac{1}{\epsilon}(f(u(t_n)) - I_h f^n, \tau\Phi^n).
\end{aligned}$$

Thus we have

$$\begin{aligned}
(\nabla\Delta_h^{-1}(\Phi^n - \Phi^{n-1}), \nabla\Delta_h^{-1}\Phi^n) = & (\Theta^n - \Theta^{n-1}, \Delta_h^{-1}\Phi^n) \\
& + \tau(\nabla\Lambda^n, \nabla\Delta_h^{-1}\Phi^n) + \tau(\nabla\Psi^n, \nabla\Delta_h^{-1}\Phi^n) \\
& + \int_{t_{n-1}}^{t_n} (\nabla w(s) - \nabla w(t_n), \nabla\Delta_h^{-1}\Phi^n)ds \\
& + \delta\int_{t_{n-1}}^{t_n} (g(u(s)) - g(u_h^n), -\Delta_h^{-1}\Phi^n)dW_s, \qquad (4.7) \\
\tau(\Lambda^n + \Psi^n, \Phi^n) = & \epsilon\tau(\nabla\Theta^n, \nabla\Phi^n) + \epsilon\tau(\nabla\Phi^n, \nabla\Phi^n) \\
& + \frac{\tau}{\epsilon}(f(u(t_n)) - I_h f^n, \Phi^n). \qquad (4.8)
\end{aligned}$$

Taking expectation on both sides over the set $\widetilde{\Omega}_{\kappa,n}$ and substituting the second equation into the first yield

$$\begin{aligned}
& \mathbb{E}\left[\mathbb{1}_{\widetilde{\Omega}_{\kappa,n}}(\Phi^n - \Phi^{n-1}, \Phi^n)_{-1,h}\right] + \epsilon\tau\mathbb{E}\left[\mathbb{1}_{\widetilde{\Omega}_{\kappa,n}}(\nabla\Phi^n, \nabla\Phi^n)\right] \\
= & \mathbb{E}\left[\mathbb{1}_{\widetilde{\Omega}_{\kappa,n}}(\Theta^n - \Theta^{n-1}, \Delta_h^{-1}\Phi^n)\right] - \epsilon\tau\mathbb{E}\left[\mathbb{1}_{\widetilde{\Omega}_{\kappa,n}}(\nabla\Theta^n, \nabla\Phi^n)\right] \\
& - \frac{\tau}{\epsilon}\mathbb{E}\left[\mathbb{1}_{\widetilde{\Omega}_{\kappa,n}}(f(u(t_n)) - I_h f^n, \Phi^n)\right] + \mathbb{E}\left[\mathbb{1}_{\widetilde{\Omega}_{\kappa,n}}\int_{t_{n-1}}^{t_n}(\nabla w(s) - \nabla w(t_n), \nabla\Delta_h^{-1}\Phi^n)ds\right] \\
& + \delta\mathbb{E}\left[\mathbb{1}_{\widetilde{\Omega}_{\kappa,n}}\int_{t_{n-1}}^{t_n}(g(u(s)) - g(u_h^n), -\Delta_h^{-1}\Phi^n dW_s)\right] + \tau\mathbb{E}\left[\mathbb{1}_{\widetilde{\Omega}_{\kappa,n}}(\nabla\Lambda^n, \nabla\Delta_h^{-1}\Phi^n)\right] \\
:= & \sum_{i=1}^{6} T_i. \qquad (4.9)
\end{aligned}$$

The first term on the left-hand side of (4.9) can be bounded by

$$
\begin{aligned}
&\mathbb{E}\left[\mathbb{1}_{\widetilde{\Omega}_{\kappa,n}}(\Phi^n-\Phi^{n-1},\Phi^n)_{-1,h}\right]\\
=&\frac{1}{2}\mathbb{E}\left[\mathbb{1}_{\widetilde{\Omega}_{\kappa,n}}\|\Phi^n\|_{-1,h}^2\right]-\frac{1}{2}\mathbb{E}\left[\mathbb{1}_{\widetilde{\Omega}_{\kappa,n}}\|\Phi^{n-1}\|_{-1,h}^2\right]+\frac{1}{2}\mathbb{E}\left[\mathbb{1}_{\widetilde{\Omega}_{\kappa,n}}\|\Phi^n-\Phi^{n-1}\|_{-1,h}^2\right]\\
=&\frac{1}{2}\mathbb{E}\left[\mathbb{1}_{\widetilde{\Omega}_{\kappa,n}}\|\Phi^n\|_{-1,h}^2\right]-\frac{1}{2}\mathbb{E}\left[\mathbb{1}_{\widetilde{\Omega}_{\kappa,n-1}}\|\Phi^{n-1}\|_{-1,h}^2\right]+\frac{1}{2}\mathbb{E}\left[\mathbb{1}_{\widetilde{\Omega}_{\kappa,n}}\|\Phi^n-\Phi^{n-1}\|_{-1,h}^2\right]\\
&+\frac{1}{2}\mathbb{E}\left[(\mathbb{1}_{\widetilde{\Omega}_{\kappa,n-1}}-\mathbb{1}_{\widetilde{\Omega}_{\kappa,n}})\|\Phi^{n-1}\|_{-1,h}^2\right]\\
\geq&\frac{1}{2}\mathbb{E}\left[\mathbb{1}_{\widetilde{\Omega}_{\kappa,n}}\|\Phi^n\|_{-1,h}^2\right]-\frac{1}{2}\mathbb{E}\left[\mathbb{1}_{\widetilde{\Omega}_{\kappa,n-1}}\|\Phi^{n-1}\|_{-1,h}^2\right]+\frac{1}{2}\mathbb{E}\left[\mathbb{1}_{\widetilde{\Omega}_{\kappa,n}}\|\Phi^n-\Phi^{n-1}\|_{-1,h}^2\right].
\end{aligned}
\tag{4.10}
$$

The first term T_1 on the right-hand side of (4.9) is zero by the definition of the P_h operator.

Using the Young's inequality and properties of the projection, the second term T_2 can be bounded by

$$
\begin{aligned}
T_2\leq&\epsilon\tau\mathbb{E}\left[\mathbb{1}_{\widetilde{\Omega}_{\kappa,n}}\|\nabla\Theta^n\|_{L^2}^2\right]+\frac{\epsilon\tau}{4}\mathbb{E}\left[\mathbb{1}_{\widetilde{\Omega}_{\kappa,n}}\|\nabla\Phi^n\|_{L^2}^2\right]\\
\leq&C\tau h^2\mathbb{E}\left[\mathbb{1}_{\widetilde{\Omega}_{\kappa,n}}|u(t_n)|_{H^2}^2\right]+\frac{\epsilon\tau}{4}\mathbb{E}\left[\mathbb{1}_{\widetilde{\Omega}_{\kappa,n}}\|\nabla\Phi^n\|_{L^2}^2\right].
\end{aligned}
\tag{4.11}
$$

In order to estimate the third term T_3, we write

$$
\begin{aligned}
(f(u(t_n))-I_hf^n,\Phi^n)=&(f(u(t_n)-f(P_hu(t_n)),\Phi^n)\\
&+(f(P_hu(t_n))-f^n,\Phi^n)+(f^n-I_hf^n,\Phi^n).
\end{aligned}
\tag{4.12}
$$

Using the properties of the projection and the embedding theorem, we have

$$
\begin{aligned}
&-\frac{\tau}{\epsilon}\mathbb{E}\left[\mathbb{1}_{\widetilde{\Omega}_{\kappa,n}}\big(f(u(t_n))-f(P_hu(t_n)),\Phi^n\big)\right]\\
=&-\frac{\tau}{\epsilon}\mathbb{E}\left[\mathbb{1}_{\widetilde{\Omega}_{\kappa,n}}\big(\Theta^n(\sum_{i=0}^{2}(u(t_n))^i(P_hu(t_n))^{2-i}-1\big),\Phi^n\big)\right]\\
\leq&C\tau\mathbb{E}\left[\mathbb{1}_{\widetilde{\Omega}_{\kappa,n}}\|\sum_{i=0}^{2}(u(t_n))^i(P_hu(t_n))^{2-i}-1\|_{L^\infty}^2\|\Theta^n\|_{L^2}^2\right]+\tau\mathbb{E}\left[\mathbb{1}_{\widetilde{\Omega}_{\kappa,n}}\|\Phi^n\|_{L^2}^2\right]\\
\leq&C\tau\Big(\mathbb{E}\left[\mathbb{1}_{\widetilde{\Omega}_{\kappa,n}}\big(\|P_hu(t_n)\|_{L^\infty}^6+\|u(t_n)\|_{L^\infty}^6+|D|^3\big)\right]\Big)^{\frac{2}{3}}\Big(\mathbb{E}\left[\mathbb{1}_{\widetilde{\Omega}_{\kappa,n}}\|\Theta^n\|_{L^2}^6\right]\Big)^{\frac{1}{3}}\\
&+\tau\mathbb{E}\left[\mathbb{1}_{\widetilde{\Omega}_{\kappa,n}}\|\Phi^n\|_{L^2}^2\right]\\
\leq&C\tau h^2+C\tau h^2\mathbb{E}\left[\mathbb{1}_{\widetilde{\Omega}_{\kappa,n}}\|u(t_n)\|_{H^2}^6\right]+\tau\mathbb{E}\left[\mathbb{1}_{\widetilde{\Omega}_{\kappa,n}}\|\Phi^n\|_{L^2}^2\right].
\end{aligned}
\tag{4.13}
$$

The second term on the right-hand side of (4.12) can be bounded by

$$
-\frac{\tau}{\epsilon}\mathbb{E}\left[\mathbb{1}_{\widetilde{\Omega}_{\kappa,n}}\big(f(P_hu(t_n))-f^n,\Phi^n\big)\right]\leq\frac{\tau}{\epsilon}\mathbb{E}\left[\mathbb{1}_{\widetilde{\Omega}_{\kappa,n}}\|\Phi^n\|_{L^2}^2\right].
\tag{4.14}
$$

Using the inverse inequality and Eq. (4.1), we have

$$\begin{aligned}\mathbb{1}_{\widetilde{\Omega}_{\kappa,n}} \max_{1\le n\le \ell} \|u_h^n\|_{L^\infty}^2 &\le C(\ln h)^2 \mathbb{1}_{\widetilde{\Omega}_{\kappa,n}} \left(\max_{1\le n\le \ell} \|u_h^n\|_{L^2}^2 + \max_{1\le n\le \ell} \|\nabla u_h^n\|_{L^2}^2 \right) \\ &\le C(\ln h)^2 \kappa. \end{aligned} \tag{4.15}$$

Then by the properties of the interpolation operator, the inverse inequality, and Eq. (4.15), the third term on the right-hand side of (4.12) can be handled by

$$\begin{aligned} &-\frac{\tau}{\epsilon}\mathbb{E}\left[\mathbb{1}_{\widetilde{\Omega}_{\kappa,n}}\left(f^n - I_h f^n, \Phi^n\right)\right] \\ &\le C\tau\mathbb{E}\left[\mathbb{1}_{\widetilde{\Omega}_{\kappa,n}}\|f^n - I_h f^n\|_{H^{-1}}^2\right] + \frac{\epsilon}{8}\tau\mathbb{E}\left[\mathbb{1}_{\widetilde{\Omega}_{\kappa,n}}\|\nabla\Phi^n\|_{L^2}^2\right] \\ &\le C\tau h^4\mathbb{E}\left[\mathbb{1}_{\widetilde{\Omega}_{\kappa,n}}\sum_K |(u_h^n)^3|_{H^1(K)}^2\right] + \frac{\epsilon}{8}\tau\mathbb{E}\left[\mathbb{1}_{\widetilde{\Omega}_{\kappa,n}}\|\nabla\Phi^n\|_{L^2}^2\right] \\ &\le C\tau h^4\mathbb{E}\left[\mathbb{1}_{\widetilde{\Omega}_{\kappa,n}}\sum_K \|(u_h^n)^2\nabla u_h^n\|_{L^2(K)}^2\right] + \frac{\epsilon}{8}\tau\mathbb{E}\left[\mathbb{1}_{\widetilde{\Omega}_{\kappa,n}}\|\nabla\Phi^n\|_{L^2}^2\right] \\ &\le C\tau h^4\mathbb{E}\left[\mathbb{1}_{\widetilde{\Omega}_{\kappa,n}}\|\nabla u_h^n\|_{L^2}^2 \max_{1\le n\le m}\|u_h^n\|_{L^\infty}^4\right] + \frac{\epsilon}{8}\tau\mathbb{E}\left[\mathbb{1}_{\widetilde{\Omega}_{\kappa,n}}\|\nabla\Phi^n\|_{L^2}^2\right] \\ &\le Ch^4\left(\ln h\right)^4\kappa^2\tau\mathbb{E}\left[\mathbb{1}_{\widetilde{\Omega}_{\kappa,n}}\|\nabla u_h^n\|_{L^2}^2\right] + \frac{\epsilon}{8}\tau\mathbb{E}\left[\mathbb{1}_{\widetilde{\Omega}_{\kappa,n}}\|\nabla\Phi^n\|_{L^2}^2\right]. \end{aligned} \tag{4.16}$$

Combine (4.12)-(4.16) to obtain

$$\begin{aligned} T_3 \le & C\tau\mathbb{E}\left[\mathbb{1}_{\widetilde{\Omega}_{\kappa,n}}\|\Phi^n\|_{L^2}^2\right] + \frac{\epsilon}{8}\tau\mathbb{E}\left[\mathbb{1}_{\widetilde{\Omega}_{\kappa,n}}\|\nabla\Phi^n\|_{L^2}^2\right] + Ch^2\tau \\ &+ Ch^2\tau\mathbb{E}\left[\mathbb{1}_{\widetilde{\Omega}_{\kappa,n}}\|u(t_n)\|_{H^2}^6\right] + Ch^{6-d}\left(\ln(h^{-\beta})\right)^8\tau^{-4}\tau\mathbb{E}\left[\mathbb{1}_{\widetilde{\Omega}_{\kappa,n}}\|\nabla u_h^n\|_{L^2}^2\right] \\ \le & C\tau\mathbb{E}\left[\mathbb{1}_{\widetilde{\Omega}_{\kappa,n}}\|\Phi^n\|_{-1,h}^2\right] + \frac{\epsilon}{4}\tau\mathbb{E}\left[\mathbb{1}_{\widetilde{\Omega}_{\kappa,n}}\|\nabla\Phi^n\|_{L^2}^2\right] + Ch^2\tau \\ &+ Ch^2\tau\sup_{t\in[0,T]}\mathbb{E}\left[\mathbb{1}_{\widetilde{\Omega}_{\kappa,n}}\|u(t)\|_{H^2}^6\right] + Ch^4\left(\ln h\right)^4\kappa^2\tau\mathbb{E}\left[\mathbb{1}_{\widetilde{\Omega}_{\kappa,n}}\|\nabla u_h^n\|_{L^2}^2\right]. \end{aligned} \tag{4.17}$$

Note that the last term can be bounded by Corollary 3.1 after taking the summation.

By Lemma 2.2, the fourth term T_4 on the right-hand side of (4.9) can be bounded by

$$\begin{aligned} T_4 \le & \mathbb{E}\left[\mathbb{1}_{\widetilde{\Omega}_{\kappa,n}}\int_{t_{n-1}}^{t_n} 2\|\nabla w(s) - \nabla w(t_n)\|_{L^2}^2 + C\|\Phi^n\|_{-1,h}^2 ds\right] \\ \le & C\tau^2\sup_{t\in[0,T]}\mathbb{E}\left[\|u(t)\|_{H^6}^6\right] + \tau\mathbb{E}\left[\mathbb{1}_{\widetilde{\Omega}_{\kappa,n}}\|\Phi^n\|_{-1,h}^2\right]. \end{aligned} \tag{4.18}$$

By Lemma 2.1, the fifth term T_5 on the right-hand side of (4.9) can be estimated as:

$$\begin{aligned}
T_5 =& \delta\mathbb{E}\left[\mathbb{1}_{\widetilde{\Omega}_{\kappa,n}}\int_{t_{n-1}}^{t_n}(g(u(s))-g(u_h^n),-\Delta_h^{-1}\Phi^n dW_s)\right]\\
\leq& C\tau^2\left(\sup_{t\in[0,T]}\mathbb{E}\left[\|\Delta u(t)\|_{L^2}^2\right]+C\sup_{t\in[0,T]}\mathbb{E}\left[\|u(t)\|_{H^1}^6\right]\right)+C\tau\mathbb{E}\left[\mathbb{1}_{\widetilde{\Omega}_{\kappa,n}}\|\Theta^n\|_{L^2}^2\right]\\
&+C\tau\mathbb{E}\left[\mathbb{1}_{\widetilde{\Omega}_{\kappa,n}}\|\Phi^n\|_{L^2}^2\right]+\frac{1}{4}\mathbb{E}\left[\mathbb{1}_{\widetilde{\Omega}_{\kappa,n}}\|\Delta_h^{-1}\Phi^n-\Delta_h^{-1}\Phi^{n-1}\|_{L^2}^2\right]\\
\leq& C\tau^2\left(\sup_{t\in[0,T]}\mathbb{E}\left[\|\Delta u(t)\|_{L^2}^2\right]+C\sup_{t\in[0,T]}\mathbb{E}\left[\|u(t)\|_{H^1}^6\right]\right)+C\tau h^4\sup_{t\in[0,T]}\mathbb{E}\left[\|u(t)\|_{H^2}^2\right]\\
&+\frac{\epsilon\tau}{4}\mathbb{E}\left[\mathbb{1}_{\widetilde{\Omega}_{\kappa,n}}\|\nabla\Phi^n\|_{L^2}^2\right]+C\tau\mathbb{E}\left[\mathbb{1}_{\widetilde{\Omega}_{\kappa,n}}\|\Phi^n\|_{-1,h}^2\right]+\frac{1}{4}\mathbb{E}\left[\mathbb{1}_{\widetilde{\Omega}_{\kappa,n}}\|\Phi^n-\Phi^{n-1}\|_{-1,h}^2\right]. \quad (4.19)
\end{aligned}$$

The sixth term T_6 on the right-hand side of (4.9) can be bounded by

$$T_6\leq C\tau h^2\sup_{t\in[0,T]}\mathbb{E}\left[\|w(t)\|_{H^2}^2\right]+\tau\mathbb{E}\left[\|\Phi^n\|_{-1,h}^2\right]. \quad (4.20)$$

By Theorem 3.2 and Corollary 3.1, we have

$$\max_{1\leq n\leq N}\mathbb{E}\left[\|\nabla u_h^n\|_{H^1}^2\right]\leq C\tau^{-1}. \quad (4.21)$$

Choosing $\kappa := C|\ln h|\tau^{-1}$, using Eq. (4.21), the Markov's inequality, and the discrete Burkholder–Davis–Gundy inequalities [8,10,11,22], we have

$$\begin{aligned}
\mathbb{P}\left[\widetilde{\Omega}_{\kappa,\ell}\right]\geq& 1-\frac{\mathbb{E}\left[\max_{1\leq n\leq \ell}\|u_h^n\|_{H^1}^2\right]}{C|\ln h|\tau^{-1}}\\
\geq& 1-\frac{C}{|\ln h|}\to 1 \quad \text{as} \quad h\to 0. \quad (4.22)
\end{aligned}$$

Combining (4.9)-(4.22) and using the discrete Gronwall inequality and Corollary (3.1), we obtain the conclusion. □

Remark 4.1. 1. The sequence of subsets defined in (4.1) is used to handle the nonlinearity in (4.16). This is caused by the low regularity of the numerical solution.

2. If the following stability result holds:

$$\max_{1\leq n\leq N}\mathbb{E}\left[\|\nabla u_h^n\|_{H^1}^2\right]\leq C,$$

then the last term on the right-hand side of (4.2) will be $Ch^4(\ln h)^6$ by choosing $\kappa := C|\ln h|$. Furthermore, if there exists any $p>2$ such that

$$\max_{1\leq n\leq N}\mathbb{E}\left[\|\nabla u_h^n\|_{H^1}^p\right]\leq C,$$

we can remove the probability one set $\widetilde{\Omega}_{\kappa,n}$ in the error estimates, i.e.,

$$\mathbb{E}\left[\|E^\ell\|_{-1,h}^2\right]+\tau\sum_{n=1}^{\ell}\mathbb{E}\left[\|\nabla E^n\|_{L^2(\mathcal{D})}^2\right]\leq C\tau+Ch^2.$$

This can be shown by using the analysis in (4.16) and Theorem 3.3.

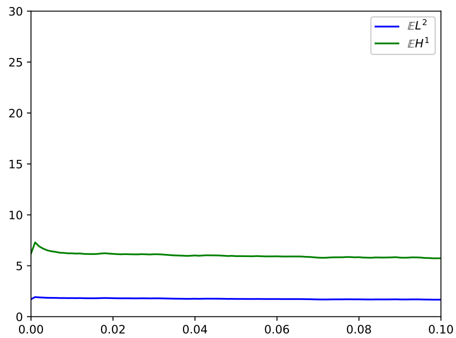
5 Numerical experiments

This section presents the results from three numerical tests. In all tests, the domain chosen is $\mathcal{D}=[-1,1]\times[-1,1]$. The purpose of Tests 1 and 2 is to check the stability and evolution of the numerical approximations from (3.3) under two noise intensities, $\delta=0.1,1$. Test 1 is based on the initial condition with a circular zero-level set. Test 2 is based on the initial condition with an ellipse as its zero-level set. In Test 3, we compute the $L^\infty\mathbb{E}L^2$ and $\mathbb{E}L^2H^1$ errors to check for the orders of convergence in h. In Test 4, we compute the $L^\infty\mathbb{E}L^2$ and $\mathbb{E}L^2H^1$ errors to check for the orders of convergence in τ.

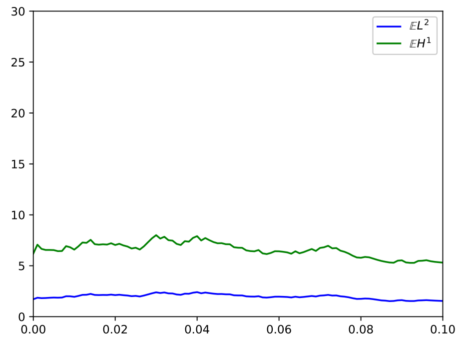
Test 1: For this test, we use the initial condition $u_h^0=P_hu_0$ where

$$u_0(x,y)=\tanh\left(\frac{x^2+y^2-0.6^2}{\sqrt{2}\epsilon}\right)$$

with $\epsilon=0.1$. The nonlinear term is $f(u)=u^3-u$ and the diffusion term is $g(u)=u$. Figs. 1 and 2 show the $\mathbb{E}L^2$ and $\mathbb{E}H^1$ stability results of the numerical solution in one sample and the average of samples, respectively. Note that the stability curves are bounded under both diffusion intensities. The shaded regions in Fig. 2 represent all possible trajectories of the simulated solutions.

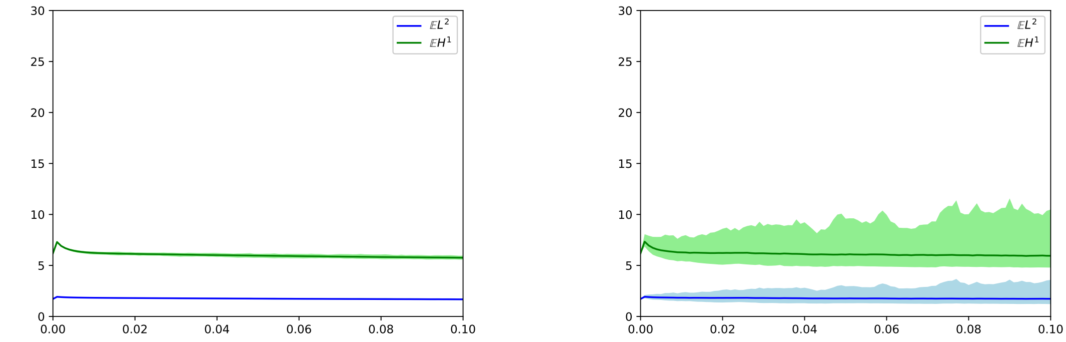


(a) $\delta=0.1$



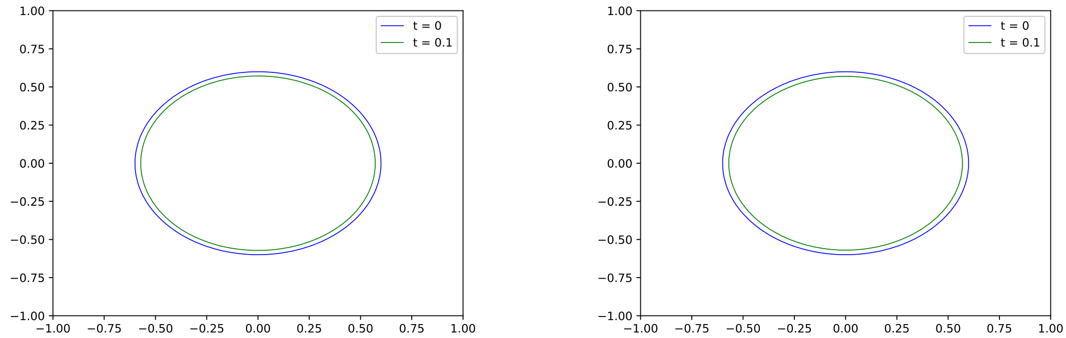
(b) $\delta=1$

Figure 1: $\mathbb{E}L^2$ and $\mathbb{E}H^1$ stability curves (one sample): $\epsilon=0.1$, $h\approx0.044$, and $\tau=0.001$.



(a) $\delta = 0.1$ (b) $\delta = 1$

Figure 2: $\mathbb{E}L^2$ and $\mathbb{E}H^1$ stability curves (average): $\epsilon = 0.1$, $h \approx 0.044$, and $\tau = 0.001$.



(a) $\delta = 0.1$ (b) $\delta = 1$

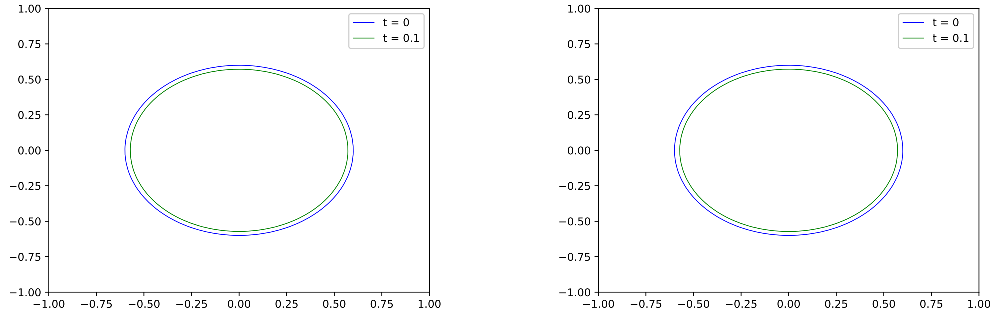
Figure 3: Zero-level sets (one sample): $\epsilon = 0.1$, $h \approx 0.044$, and $\tau = 0.001$.

Figs. 3 and 4 show the zero-level sets of one sample and the average of samples, respectively. We can see that the evolution is of a shrinking circle that stabilizes at the final time $T = 0.1$.

Test 2: For this test, we use the initial condition $u_h^0 = P_h u_0$ where

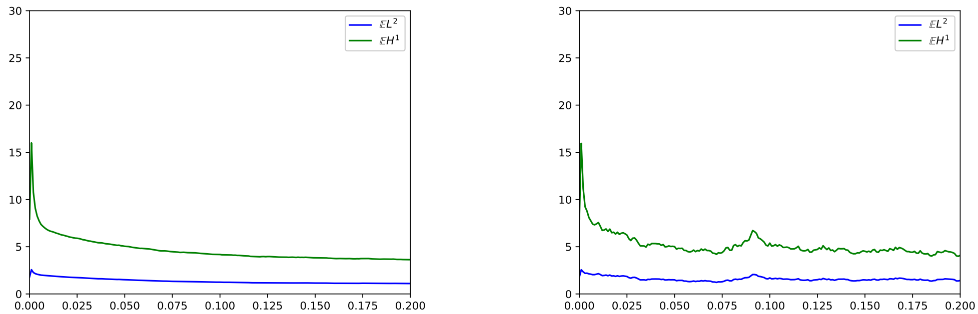
$$u_0(x,y) = \tanh\left(\frac{\sqrt{(x/0.7)^2 + (y/0.65)^2} - 1}{\sqrt{2}\epsilon}\right)$$

with $\epsilon = 0.1$. The nonlinear term is $f(u) = u^3 - u$ and the diffusion term is $g(u) = u$. Figs. 5 and 6 show the $\mathbb{E}L^2$ and $\mathbb{E}H^1$ stability results of the numerical solution in one sample



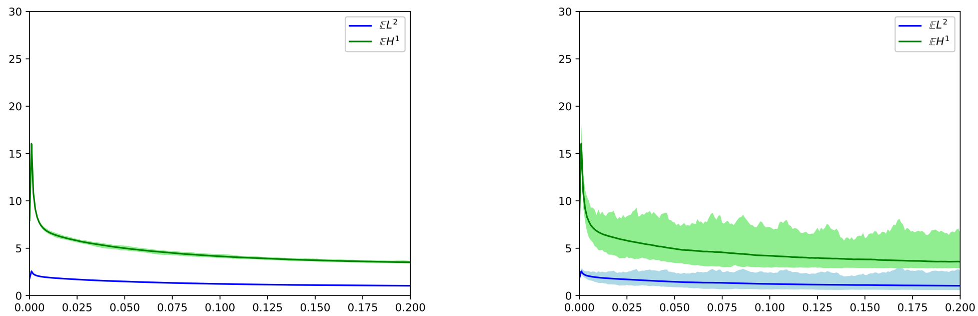
(a) $\delta=0.1$ (b) $\delta=1$

Figure 4: Zero-level sets (average): $\epsilon=0.1$, $h\approx0.044$, and $\tau=0.001$.



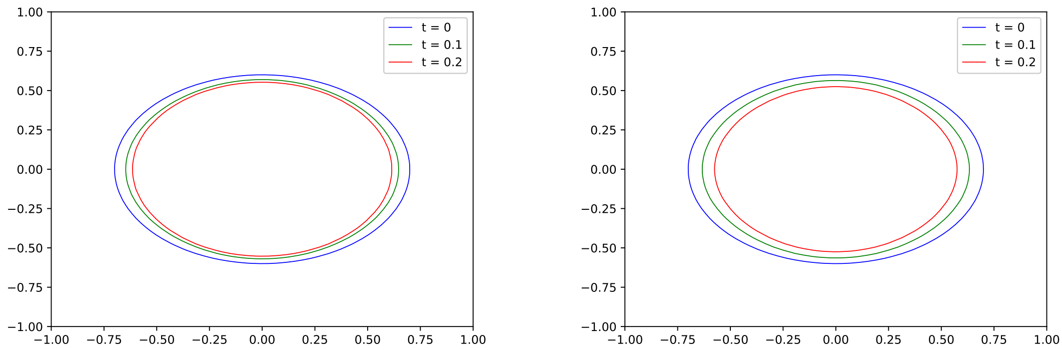
(a) $\delta=0.1$ (b) $\delta=1$

Figure 5: $\mathbb{E}L^2$ and $\mathbb{E}H^1$ stability curves (one sample): $\epsilon=0.1$, $h\approx0.044$, and $\tau=0.001$.



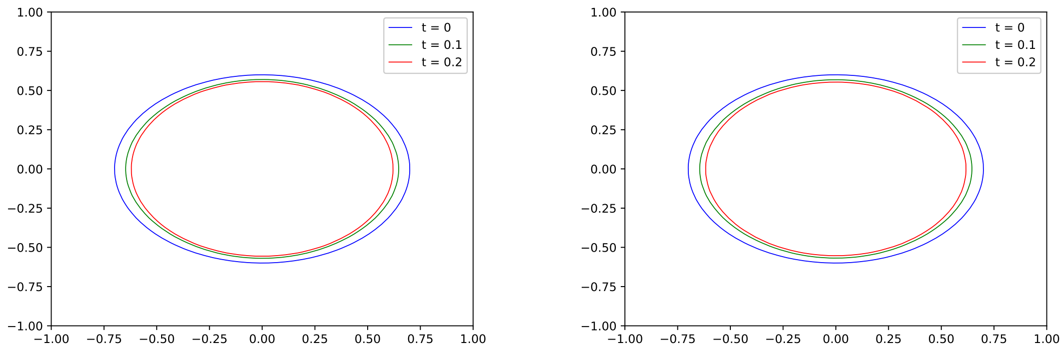
(a) $\delta=0.1$ (b) $\delta=1$

Figure 6: $\mathbb{E}L^2$ and $\mathbb{E}H^1$ stability curves (average): $\epsilon=0.1$, $h\approx0.044$, and $\tau=0.001$.



(a) $\delta = 0.1$ (b) $\delta = 1$

Figure 7: Zero-level sets (one sample): $\epsilon = 0.1$, $h \approx 0.044$, and $\tau = 0.001$.



(a) $\delta = 0.1$ (b) $\delta = 1$

Figure 8: Zero-level sets (average): $\epsilon = 0.1$, $h \approx 0.044$, and $\tau = 0.001$.

and the average of samples, respectively. Since the initial condition takes time to become a circle, the curves in these figures require a larger final time than in the previous test to stabilize. The shaded regions in Fig. 6 represent all possible trajectories of the simulated solutions.

Figs. 7 and 8 show the zero-level sets of one sample and the average of samples, respectively. The numerical solutions approach a stable circle, which occurs faster for larger diffusion intensities.

Test 3: For this test, we use the initial condition $u_h^0 = P_h u_0$ where

$$u_0(x,y) = \tanh\left(\frac{(\sqrt{x^2/0.7+y^2/0.1}-1)(\sqrt{x^2/0.1+y^2/0.7}-1)}{\sqrt{2}\epsilon}\right)$$

with $\epsilon = 0.05$. The nonlinear term is $f(u) = u^3 - u$ and the diffusion term is $g(u) = \sqrt{u^2+1}$. We compute the spatial $L^\infty \mathbb{E} L^2$ and $\mathbb{E} L^2 H^1$ errors which denote

$$\left(\max_{0\le n\le N} \mathbb{E}\left[\|E^n\|^2_{L^2(\mathcal{D})}\right]\right)^{1/2} \quad \text{and} \quad \left(\mathbb{E}\left[\tau\sum_{n=1}^{N}\|\nabla E^n\|^2_{L^2(\mathcal{D})}\right]\right)^{1/2}, \tag{5.1}$$

respectively. Table 1 below contains these errors. The final time is $T = 1/10^4$.

Table 1: Strong spatial errors and error orders: $\epsilon = 0.05$, $\delta = 1$, $\tau = 1/10^6$.

h	$L^\infty \mathbb{E} L^2$	Order	$\mathbb{E} L^2 H^1$	Order
$0.2\sqrt{2}$	0.89614673		13.81865669	
$0.1\sqrt{2}$	0.23059268	1.95838825	7.212195952	0.93810688
$0.05\sqrt{2}$	0.06355413	1.85928886	3.674838827	0.97275762
$0.025\sqrt{2}$	0.01715789	1.88911367	1.946229545	0.91699910

Test 4: For this test, we use the initial condition $u_h^0 = P_h u_0$ where

$$u_0(x,y) = \tanh\left(\frac{(\sqrt{x^2/0.64+y^2/0.16}-1)(\sqrt{x^2/0.16+y^2/0.64}-1)}{\sqrt{2}\epsilon}\right)$$

with $\epsilon = 0.1$. The nonlinear term is $f(u) = u^3 - u$ and the diffusion term is $g(u) = \sqrt{u^2+1}$. We compute the temporal $L^\infty \mathbb{E} L^2$, and $\mathbb{E} L^2 H^1$ errors. Table 2 below contains these errors. The final time is $T = 0.1$.

Table 2: Strong temporal errors and error orders: $\epsilon = 0.1$, $\delta = 1$, $h = 2\sqrt{2}/32$.

τ	$L^\infty \mathbb{E} L^2$	Order	$\mathbb{E} L^2 H^1$	Order
0.1/8	0.01402384		0.03299978	
0.1/16	0.00954442	0.55515189	0.02326743	0.50414445
0.1/32	0.00643871	0.56788606	0.01654998	0.49148196
0.1/64	0.00437326	0.55808025	0.01092142	0.59966935

6 Conclusion

In this manuscript, we design a numerical scheme for the stochastic Cahn-Hilliard equation with multiplicative noise. The scheme utilizes the interpolation operator to handle the interaction between the drift term and the diffusion term. The existence of the discrete solutions is proven and the discrete solutions are proven to maintain some stability results and higher moment results. Based on these stability results, we construct a probability one set such that the error estimates in the discrete H^{-1}-norm hold on this set.

Future work will remove the probability one set, i.e., establishing the error estimates in the entire probability space. A crucial step in this direction involves proving the higher moment bounds for the H^1-norm. This remains an open question for this class of stochastic Cahn-Hilliard equations and some new techniques need to be brought in.

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