Active Fault Detection in Static Systems

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Abstract—We study auxiliary signal-based fault detection in static linear systems with quadratic constraints. Auxiliary signals are perturbations that make faults easier to detect, at the cost of some disruption to normal operation. In this paper, we find minimally disruptive auxiliary signals that guarantee detection under set-based uncertainty. Our motivation is distance protection in inverter-dominated power systems, wherein small fault currents can go undetected by traditional schemes. We focus on static systems because distance protection is based on phasors.

We formulate a general auxiliary signal design problem with constraints imposed by system operational requirements, and additive and multiplicative noise. We use a relaxation and duality to reformulate the problem as a semidefinite bilinear program. In the special case of additive uncertainty and no constraints, we obtain an analytical lower bound on the magnitude an auxiliary signal must have to guarantee detection. We solve the optimizations in an example based on distance protection, in which the auxiliary signal is negative sequence current.

Index Terms—Fault detection; auxiliary signal; distance relay; negative sequence; inverter

I. Introduction

Faults can be difficult to detect in systems with high levels of uncertainty, even in the linear static case. Auxiliary signals are perturbations that make faults easier to detect in exchange for occasional disruptions to normal operation [1], [2]. In this paper, we optimize auxiliary signals to be minimally disruptive while guaranteeing fault detection.

We begin with a general static auxiliary signal design problem, which includes additive and multiplicative uncertainty and constraints. Constraints are beneficial in this setting because they

- allow for greater realism, e.g., limits on certain variables imposed by control or the physics of the problem; and
- reduce the magnitude of the optimal auxiliary signal. More precisely, the constraints make the detection problem easier by shrinking the range of possible observations, potentially mitigating the worst realizations.

The overall auxiliary signal design problem is a bilevel optimization. We use a semidefinite relaxation to convexify the inner minimization, which is bilinear when there is multiplicative uncertainty. As in our preliminary work [3], we then use duality to eliminate the inner minimization, which here results in a bilinear semidefinite program. The two main advantages of this approach are that

- the constraints fit naturally in the calculation of the dual program; and
- the resulting optimization, though not in a tractable class, is typically small in scale, and so is amenable to any nonlinear programming algorithm.

We also describe how the optimization can be equivalently constructed using Farkas' Lemma.

We now review the relevant literature. The textbook [1] covers auxiliary signal design for both static and linear time-invariant (LTI) systems. Our static system model differs in that it has constraints and a constant offset, without at least one of which the auxiliary signal can never be zero. Another literature stream is [4]–[8], which uses zonotopes to model additive uncertainty in LTI systems. This makes the reachable sets easier to characterize and leads to mixed-integer quadratic programs. Of particular relevance is [9], which, to our knowledge, is the only other formulation based on duality. Whereas we focus on static systems, they use Hahn-Banach duality to find hyperplanes that separate the limit sets of LTI systems.

Our main motivation is protection in power systems. During faults, synchronous machines produce currents with large magnitudes and predictable phase imbalances, which inform traditional detection schemes. On the other hand, fault currents from inverter-based resources (IBR) can be within a few percent of normal and, e.g., have no negative sequence content [10], [11]. As a result, conventional fault detection schemes can fail in inverter-dominated power systems, motivating the use of auxiliary signals [12]. Examples include adding harmonics [13] and negative sequence [11], [12] to the inverter's current, the latter of which is now stipulated by IEEE Standard 2800 [14]. Such schemes only lower power quality when a fault is suspected, which might be a small fraction of the time, and, because they require no additional hardware, are compatible with some existing protection setups [11]. Later, in Section VI, we give an example in which we optimize a negative sequence current injection to detect a phase-to-phase fault. Aside from our preliminary paper [3] and the first author's recent work in [15], none of the formalisms in the auxiliary signal literature have been applied to fault detection in power systems.

The original contributions of this papers are as follows.

C1. We use a relaxation and duality to reformulate fault detection in static systems as a bilinear semidefinite program. This allows for multiplicative uncertainty,

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constant offsets, inequality constraints in the system model, and solution via generic nonlinear programming algorithms.

C2. In the special case of only additive noise and no constraints, we obtain a necessary condition: a lower bound on the magnitude of the auxiliary signal, below which correct fault detection is not guaranteed.

C3. We apply the procedure to distance protection. Specifically, we optimize negative sequence current injection to assist detection of a phase-to-phase fault.

The remainder of the paper is organized as follows. In Section II, we cast the auxiliary signal design problem for static systems as a bilevel optimization. In Section III, we convexify the inner minimization via semidefinite relaxation, and in Section IV, use duality to reformulate the overall problem as a bilinear semidefinite program. In Section V, we look at the special case of additive uncertainty and no constraints. In Section VI, we apply the procedure to distance protection. We describe our intended future work in Section VII.

II. Problem statement

A linear static system has multiple potential models, which are indexed with the elements in the set $\mathcal{M} = \{0, 1, 2, \ldots, M\}$. We typically associate one of the models with normal system behavior, and the other M with behavior under different types of faults. Let θ and x denote real vectors associated with the auxiliary signal and known quantities (e.g., input and measured output signals), respectively. Let λ_k and ξ_k, $k \in \mathcal{M}$, denote vectors representing additive and multiplicative uncertainty, respectively. Let $\|\cdot\|$ denote the Euclidean norm for vectors and the Frobenius norm for matrices.

Model $k \in \mathcal{M}$ is written

$$([\Theta_k, X_k] + G_k(\xi_k)) \begin{bmatrix} \theta \\ x \end{bmatrix} = h_k + H_k \lambda_k \tag{1a}$$

$$x^\top Q_k^i x + A_k^{i\top} x + b_k^i \leq 0, \quad i = 1, ..., n_k, \tag{1b}$$

where the matrices Θ_k, X_k, $G_k(\xi_k)$, H_k, and vector h_k specify how the auxiliary, known, and noise signals enter model $k \in \mathcal{M}$. The noise terms must satisfy

$$\|\lambda_k\| < 1, \quad \|\xi_k\| < 1, \tag{1c}$$

for $k \in \mathcal{M}$. Each multiplicative noise matrix has the form

$$G_k(\xi_k) = \sum_{i=1}^{l_k} \xi_k^i \left[U_k^i, V_k^i\right],$$

where ξ_k^i is the i^{th} element of ξ_k. The matrices U_k^i and V_k^i respectively multiply θ and x in (1a). The symmetric matrix Q_k^i, vector A_k^i, and scalar b_k^i parameterize the inequality (1b), which specifies system limits, e.g., an inverter's maximum current magnitude. We only constrain x to streamline exposition and because the auxiliary signal should be small anyway, but could generalize (1b) to include θ.

We assume that one of the models is correct. Once x is known, i.e., measurements are taken, we want to know which one it is. Given x and θ, we check if each model $k \in \mathcal{M}$ could be correct by solving

$$\begin{aligned} \mathcal{T}_k: \quad & \min_{\omega, \lambda_k, \xi_k} \ \omega \quad \text{s.t.} \\ & (1a), (1b) \\ & \|\lambda_k\|^2 \leq \omega \qquad \text{(2a)} \\ & \|\xi_k\|^2 \leq \omega. \qquad \text{(2b)} \end{aligned}$$

Problem $\mathcal{T}_k$ is a convex quadratically constrained program, and so fairly easy to solve. Let $\tau_k(\theta, x)$ denote the optimal objective of $\mathcal{T}_k$. If $\mathcal{T}_k$ is infeasible because x does not satisfy (1b), we follow the convention that $\tau_k(\theta, x) = \infty$.

$\mathcal{T}_k$ simplifies in the following two cases.

- If there is only additive uncertainty and H_k has full column rank, then we may solve (1a) directly for λ_k, and $\tau_k(\theta, x) = \|\lambda_k\|^2$.
- If there is only multiplicative uncertainty and the matrix

$$\left[U_k^1 \theta + V_k^1 x, \ldots, U_k^{l_k} \theta + V_k^{l_k} x\right]$$

has full column rank, then we may solve (1a) directly for ζ_k, and $\tau_k(\theta, x) = \|\zeta_k\|^2$.

We will see in Section V that when there is only additive uncertainty, $\mathcal{T}_k$ simplifies to evaluating (1a).

Definition 1: x is consistent with model k if $\tau_k(\theta, x) < 1$. If x is consistent with model k, then it is a feasible solution of (1) for k, which is to say that model k could be correct.

Due to the uncertainty, x could be consistent with multiple models. The role of the auxiliary signal, θ, is to ensure that x is only consistent with the correct model. To do so, we constrain θ so that no realization of x is consistent with more than one model. Let $\mathcal{S} \subseteq \mathcal{M}$ be a pair of models. Consider the optimization

$$\begin{aligned} \mathcal{P}_{\mathcal{S}}^0: \quad & \min_{x, \omega, \lambda, \xi} \ \omega \quad \text{s.t. for } k \in \mathcal{S} \\ & (1a), (1b) \\ & \|\lambda_k\|^2 \leq \omega \qquad \text{(3a)} \\ & \|\xi_k\|^2 \leq \omega. \qquad \text{(3b)} \end{aligned}$$

For concision, we omit subscripts from variables under the min to indicate the full vectors for both $k \in \mathcal{S}$. Let $\sigma_{\mathcal{S}}(\theta)$ denote the optimal objective of $\mathcal{P}_{\mathcal{S}}^0$.

Definition 2: θ separates $\mathcal{S}$ if $\sigma_{\mathcal{S}}(\theta) \geq 1$.

Observe that unlike $\mathcal{T}_k$, x is an optimization variable in $\mathcal{P}_{\mathcal{S}}^0$. This means that if θ separates $\mathcal{S}$, then there is no x that is consistent with both models. If one of the models is correct, then any realization of x must be consistent with it.

Note that if the offsets, h_k and b_k, are not present (as in [1]) and $\theta = 0$, then $\sigma_{\mathcal{S}}(\theta) = 0$. This implies that some auxiliary signal is always needed to achieve separation, which is not realistic. The offsets reflect the fact that most systems do not operate at zero, and allow for systems in which auxiliary signals might not be necessary for fault detection.

Let C be a positive semidefinite matrix. We aim to solve

$$\mathcal{P}^1: \qquad \min_{\theta}\ \theta^\top C\theta \quad \text{s.t.}$$
$$1 \le \sigma_{\mathcal{S}}(\theta) \text{ for all } \mathcal{S} \subseteq \mathcal{M},\ |\mathcal{S}| = 2. \tag{4}$$

The objective is a proxy for the disruption caused by the auxiliary signal. The constraint ensures that the auxiliary signal separates all pairs of models $\mathcal{S} \subseteq \mathcal{M}$, of which there are $\binom{|\mathcal{M}|}{2}$. This implies that x will only be consistent with the correct model. We identify the correct model by evaluating $\tau_k(\theta, x)$ for all $k \in \mathcal{M}$.

As the two models in a pair, $\mathcal{S}$, approach equivalence, the objectives of $\mathcal{P}^0_{\mathcal{S}}$ and $\mathcal{P}^1$ approach zero and infinity, and a larger and larger auxiliary signal is needed for separation. Problem $\mathcal{P}^1$ is infeasible if no auxiliary signal can separate the models, i.e., because two of the models are identical. Observe that the constraints in (1b) restrict the values x can take on in $\mathcal{P}^0_{\mathcal{S}}$, increasing its objective. The presence of such constraints can therefore only increase $\sigma_{\mathcal{S}}(\theta)$, shrinking the size of auxiliary signal needed to attain separation.

Problem $\mathcal{P}^1$ is difficult to solve due to nonconvexity and the minimum embedded in (4). Over the next few sections, we develop a general computational strategy and analyze special cases.

III. Semidefinite relaxation

Problem $\mathcal{P}^0_{\mathcal{S}}$ is nonconvex due to the bilinearities $\xi^i_k x$, $i = 1, ..., l_k$, $k \in \mathcal{S}$. We convexify them using a standard semidefinite relaxation (see, e.g., [16]). We first introduce the matrices.

$$\begin{bmatrix} 1 & \xi^i_k & x^\top \\ \xi^i_k & z^i_k & F^{i\top}_k \\ x & F^i_k & W \end{bmatrix},$$

which we substitute, element-wise, for the outer products

$$\begin{bmatrix} 1 \\ \xi^i_k \\ x \end{bmatrix} \begin{bmatrix} 1 \\ \xi^i_k \\ x \end{bmatrix}^\top.$$

The product of a real vector with its transpose is positive semidefinite and has rank one. We obtain the following relaxation, $\hat{\mathcal{P}}^0_{\mathcal{S}}$, by only enforcing positive semidefiniteness on the new matrices:

$$\hat{\mathcal{P}}^0_{\mathcal{S}}: \qquad \min_{x,\omega,\lambda,\xi,z,F,W}\ \omega \quad \text{s.t. for } k \in \mathcal{S}$$
$$[\Theta_k, X_k]\begin{bmatrix}\theta \\ x\end{bmatrix} + \sum_{i=1}^{l_k} \xi^i_k U^i_k \theta + V^i_k F^i_k = h_k + H_k \lambda_k \tag{5a}$$
$$\operatorname{tr}\left(Q^i_k W\right) + A^{i\top}_k x + b^i_k \le 0, \quad i = 1, ..., n_k \tag{5b}$$
$$\|\lambda_k\|^2 \le \omega \tag{5c}$$
$$\sum_{i=1}^{l_k} z^i_k \le \omega \tag{5d}$$
$$\begin{bmatrix} 1 & \xi^i_k & x^\top \\ \xi^i_k & z^i_k & F^{i\top}_k \\ x & F^i_k & W \end{bmatrix} \succeq 0, \quad i = 1, ..., l_k. \tag{5e}$$

Note that if (1b) has no quadratic part, then we can add the valid inequalities[1]

$$b^i_k b^j_k + \left(b^i_k A^j_k + b^j_k A^i_k\right) x + A^{i\top}_k W A^j_k \ge 0,\ i, j = 1, ..., n_k.$$

However, if (1b) is not present at all, then the eigenvalues of W are unbounded, and the relaxation may be loose.

Let $\hat{\sigma}_{\mathcal{S}}(\theta)$ denote the optimal objective of $\hat{\mathcal{P}}^0_{\mathcal{S}}$. Because it is a relaxation, $\hat{\sigma}_{\mathcal{S}}(\theta) \le \sigma_{\mathcal{S}}(\theta)$. Therefore, $\hat{\sigma}_{\mathcal{S}}(\theta) \ge 1$ implies that θ separates $\mathcal{S}$. Then, instead of $\mathcal{P}^1$, we solve

$$\hat{\mathcal{P}}^1: \quad \min_{\theta}\ \theta^\top C\theta \quad \text{s.t.}$$
$$1 \le \hat{\sigma}_{\mathcal{S}}(\theta) \text{ for all } \mathcal{S} \subseteq \mathcal{M},\ |\mathcal{S}| = 2. \tag{6}$$

Because (6) is a sufficient condition for (4), it is a tighter constraint. Therefore, the optimal objective of $\hat{\mathcal{P}}^1$ will be greater than that of $\mathcal{P}^1$, and any θ that is feasible for $\hat{\mathcal{P}}^1$ will be feasible for $\mathcal{P}^1$.

IV. Duality

The constraint (6) is awkward because evaluating $\hat{\sigma}_{\mathcal{S}}(\theta)$ means solving $\hat{\mathcal{P}}^0_{\mathcal{S}}$, a minimization. We can eliminate the minimization by replacing $\hat{\mathcal{P}}^0_{\mathcal{S}}$ with its dual, a maximization; this is a standard technique in robust optimization [17]. In doing so, we convert a bilevel optimization to one with a single-level, which is amenable to a broader range of analytical and algorithmic tools.

Because $\hat{\mathcal{P}}^0_{\mathcal{S}}$ is convex, strong duality ensures that it has the same objective as its dual (assuming some constraint qualification holds). Let $\Omega_{\mathcal{S}}(\theta)$ denote the constraints in the dual of $\hat{\mathcal{P}}^0_{\mathcal{S}}$, with the additional constraint that the objective be no less than one; this is shown in the Appendix. The additional constraint ensures that the maximum objective is greater than one, as in (6). This yields the following optimization:

$$\mathcal{P}^2: \qquad \min_{\theta}\ \theta^\top C\theta \quad \text{s.t.}$$
$$\Omega_{\mathcal{S}}(\theta) \text{ is feasible for all } \mathcal{S} \subseteq \mathcal{M},\ |\mathcal{S}| = 2.$$

Note that the dual of a quadratic program like $\mathcal{P}^0_{\mathcal{S}}$ is equivalent to the dual of its semidefinite relaxation [16]. Therefore, were we to dualize $\mathcal{P}^0_{\mathcal{S}}$ in $\mathcal{P}^1$, we would also obtain $\mathcal{P}^2$.

Lemma 1: If strong duality holds for $\hat{\mathcal{P}}^0_{\mathcal{S}}$ for all pairs $\mathcal{S} \subseteq \mathcal{M}$, then $\hat{\mathcal{P}}^1$ is equivalent to $\mathcal{P}^2$.

Solving $\mathcal{P}^2$ produces an auxiliary signal, θ, which is feasible if potentially suboptimal for $\mathcal{P}^1$. Suboptimality results if $\hat{\mathcal{P}}^0_{\mathcal{S}}$ has a lower objective than $\mathcal{P}^0_{\mathcal{S}}$, i.e., because the semidefinite relaxation is not tight. In this case, feasibility still ensures separation of all models, and thus correct detection, albeit with a larger than necessary auxiliary signal. A smaller auxiliary signal could be obtained by solving $\mathcal{P}^1$ directly, e.g., using grid search or techniques from global optimization [18].

[1] We form valid inequalities by taking the product of (1b) with itself for each i and j, $\left(A^{i\top}_k x + b^i_k\right)\left(x^\top A^j_k + b^j_k\right) \ge 0$, and then substituting the matrix W for the outer product $xx^\top$. These valid inequalities tighten the relaxation by further constraining the lifted variable W, thus making $\hat{\mathcal{P}}^0_{\mathcal{S}}$ a better approximation of $\mathcal{P}^0_{\mathcal{S}}$.

A. Farkas' Lemma

We could alternatively construct $\mathcal{P}^2$ using Farkas' Lemma [19].[2] We now informally describe how to do so. Consider the system $\hat{\sigma}_{\mathcal{S}}(\theta) < 1$. Because it is based on a relaxation, it is feasible for values of θ that both do and do not separate $\mathcal{S}$.[3] To remove the minimum embedded in $\hat{\sigma}_{\mathcal{S}}(\theta) < 1$, we define the system $\Pi_{\mathcal{S}}(\theta)$ as below.

$$\begin{aligned} &\text{Variables: } x, \omega, \lambda_k, \xi_k, z_k, F_k, W, k \in \mathcal{S} \\ &\text{Constraints: } \omega < 1, (5). \end{aligned}$$

$\Pi_{\mathcal{S}}(\theta)$ is feasible for the same values of θ as $\hat{\sigma}_{\mathcal{S}}(\theta) < 1$.

Farkas' Lemma states that $\Pi_{\mathcal{S}}(\theta)$ has a dual system, the feasibility of which implies the infeasibility of $\Pi_{\mathcal{S}}(\theta)$. The dual system is also parameterized by θ, and is only feasible for values of θ that separate $\mathcal{S}$. This is because if the dual system is feasible for θ, then $\Pi_{\mathcal{S}}(\theta)$ must be infeasible; enforcing the dual system and (5) then implies that $\omega \geq 1$, i.e., separation of $\mathcal{S}$. The dual system of $\Pi_{\mathcal{S}}(\theta)$ is in fact $\Omega_{\mathcal{S}}(\theta)$ in the Appendix. Optimizing the auxiliary signal, θ, over the dual system, $\Omega_{\mathcal{S}}(\theta)$, leads us precisely to $\mathcal{P}^2$.

B. Numerical solution

We now discuss how to solve $\mathcal{P}^2$. The main difficulties are the bilinear inequality (15a) and the bilinear matrix inequality (15f). While these put $\mathcal{P}^2$ outside of tractable optimization classes, auxiliary signal design problems are not typically large in scale. If the auxiliary signal is under three or so dimensions, it is viable to grid the space and evaluate $\Omega_{\mathcal{S}}(\theta)$ for all values of θ, similar to Procedure 1 in [9].

If θ is not low-dimensional, it is more practical to treat $\mathcal{P}^2$ as a nonlinear program. For example, in our preliminary work on static systems with only additive noise [3], we used the convex-concave procedure [22], [23]. Other options include global optimization [18] and generic nonlinear solvers.

V. Purely additive uncertainty

Several aspects of the problem simplify when there is no multiplicative uncertainty, i.e., $G_k(\xi_k) = 0$ for all $k \in \mathcal{M}$. Below are two straightforward simplifications.

- As mentioned in Section II, if H_k also has full column rank, solving $\mathcal{T}_k$ (to check the correctness of model k), reduces to evaluating (1a).
- The constraint (1a) is linear, not bilinear. If also $Q_k^i \succeq 0$ for all $i = 1, ..., n_k$, $k \in \mathcal{M}$, then (1b) is a convex quadratic constraint. In this case $\mathcal{P}^0_{\mathcal{S}}$ is a convex quadratic program. There is no need for the semidefinite relaxation of Section III, as $\mathcal{P}^0_{\mathcal{S}}$ has strong duality.

[2] Farkas' Lemma technically does not apply here due to the nonlinear constraints in (5). There are, however, extensions to convex and semidefinite systems [20], [21], which allow us to proceed in the same fashion.

[3] Whereas $\sigma_{\mathcal{S}}(\theta) < 1$, which we do not work with due to nonconvexity, is feasible only for values of θ that separate $\mathcal{S}$.

With a few more assumptions, we can derive a lower bound on $\|\theta\|$ that depends analytically on the parameters. The bound serves as a necessary condition—if the magnitude of the auxiliary signal is any lower, the models will not be separated.

For convenience, let $\mathcal{S} = \{0, 1\}$ for the rest of this section. Suppose there are no inequality constraints and that H_0 and H_1 have full column rank, so that they can be incorporated into the other system matrices via the pseudoinverse. Assume also that $\begin{bmatrix} X_0 \\ X_1 \end{bmatrix}$ has full column rank. Then, $\mathcal{P}^0_{\mathcal{S}}$ then takes the form

$$\min_{x,\omega,\lambda_0,\lambda_1} \ \omega \quad \text{s.t. for } k \in \mathcal{S}$$

$$[\Theta_k, X_k] \begin{bmatrix} \theta \\ x \end{bmatrix} = h_k + \lambda_k \tag{7a}$$

$$\|\lambda_k\|^2 \leq \omega. \tag{7b}$$

Denote by $(x^*, \omega^*, \lambda_0^*, \lambda_1^*)$ the optimal solution (7). Let $\omega_0^* = \|\lambda_0^*\|^2$ and $\omega_1^* = \|\lambda_1^*\|^2$, so that $\omega^* = \max\{\omega_0^*, \omega_1^*\}$.

Consider the below related problem.

$$\min_{x,\omega_0,\omega_1,\lambda_0,\lambda_1} \ \frac{1}{2}(\omega_0 + \omega_1) \quad \text{s.t. for } k \in \mathcal{S}$$

$$[\Theta_k, X_k] \begin{bmatrix} \theta \\ x \end{bmatrix} = h_k + \lambda_k \tag{8a}$$

$$\|\lambda_k\|^2 \leq \omega_k. \tag{8b}$$

In (7), we minimize the maximum uncertainty, and in (8), we minimize the mean uncertainty. Both have the same set of feasible x, λ_0, and λ_1.

Problem (8) may be written as an equality constrained quadratic program. Its analytical solution is

$$x = \Lambda_x^{-1}\left(X_0^\top h_0 + X_1^\top h_1 - \Lambda_\theta \theta\right),$$

where

$$\begin{aligned} \Lambda_x &= X_0^\top X_0 + X_1^\top X_1 \\ \Lambda_\theta &= X_0^\top \Theta_0 + X_1^\top \Theta_1. \end{aligned}$$

The matrix Λ_x is invertible because $\begin{bmatrix} X_0 \\ X_1 \end{bmatrix}$ has full column rank. The optimal objective is

$$\hat{\omega} = \frac{1}{2}(\hat{\omega}_0 + \hat{\omega}_1),$$

where

$$\begin{aligned} \hat{\omega}_0 = \big\| &\left(\Theta_0 - X_0\Lambda_x^{-1}\Lambda_\theta\right)\theta + \\ &\left(X_0\Lambda_x^{-1}X_0^\top - \boldsymbol{I}\right)h_0 + X_0\Lambda_x^{-1}X_1^\top h_1\big\|^2 \\ \hat{\omega}_1 = \big\| &\left(\Theta_1 - X_1\Lambda_x^{-1}\Lambda_\theta\right)\theta + \\ &X_1\Lambda_x^{-1}X_0^\top h_0 + \left(X_1\Lambda_x^{-1}X_1^\top - \boldsymbol{I}\right)h_1\big\|^2. \end{aligned}$$

Lemma 2: $\hat{\omega} \leq \omega^* \leq \max\{\hat{\omega}_0, \hat{\omega}_1\}$.

Proof: We show the left side by noting that

$$\begin{aligned} \hat{\omega} &= \frac{1}{2}(\hat{\omega}_0 + \hat{\omega}_1) \\ &\leq \frac{1}{2}(\omega_0^* + \omega_1^*) \quad \text{by the optimality of (8)} \\ &\leq \max\{\omega_0^*, \omega_1^*\} \\ &= \omega^*. \end{aligned}$$

We show the right side using a similar argument:

$$\begin{aligned}\omega^* &= \max\{\omega_0^*, \omega_1^*\} \\ &\le \max\{\hat{\omega}_0, \hat{\omega}_1\} \quad \text{by the optimality of (7).}\end{aligned}$$

■

Corollary 1: If $\hat{\omega}_0 = \hat{\omega}_1$, then $\omega^* = \hat{\omega}$.

Recall that $\omega^* \ge 1$ means that θ separates $\mathcal{S}$. By Lemma 2, $\hat{\omega} \ge 1$ is a sufficient condition, and $\max\{\hat{\omega}_0, \hat{\omega}_1\} \ge 1$ is a necessary condition for separation of $\mathcal{S}$. We now use Lemma 2 to obtain an analytical necessary condition. Let

$$\begin{aligned}\chi &= \max\left\{\left\|\Theta_0 - X_0\Lambda_x^{-1}\Lambda_\theta\right\|, \left\|\Theta_1 - X_1\Lambda_x^{-1}\Lambda_\theta\right\|\right\} \\ \zeta &= \max\left\{\left\|\left(X_0\Lambda_x^{-1}X_0^\top - \boldsymbol{I}\right)h_0 + X_0\Lambda_x^{-1}X_1^\top h_1\right\|,\right. \\ &\qquad \left.\left\|\left(X_1\Lambda_x^{-1}X_1^\top - \boldsymbol{I}\right)h_1 + X_1\Lambda_x^{-1}X_0^\top h_0\right\|\right\}.\end{aligned}$$

Note that we can evaluate χ and ζ directly from the system parameters.

Theorem 1: If $\|\theta\| < (1-\zeta)/\chi$, then θ does not separate $\mathcal{S}$.

Proof: First, observe that $\max\{\hat{\omega}_0, \hat{\omega}_1\} \ge 1$ if and only if $\max\left\{\sqrt{\hat{\omega}_0}, \sqrt{\hat{\omega}_1}\right\} \ge 1$. Working with the latter allows us to use the triangle inequality. We have that

$$\begin{aligned}&\max\left\{\sqrt{\hat{\omega}_0}, \sqrt{\hat{\omega}_1}\right\} \\ &\le \max\left\{\left\|\left(\Theta_0 - X_0\Lambda_x^{-1}\Lambda_\theta\right)\theta\right\| + \right. \\ &\qquad \left\|\left(X_0\Lambda_x^{-1}X_0^\top - \boldsymbol{I}\right)h_0 + X_0\Lambda_x^{-1}X_1^\top h_1\right\|, \\ &\qquad \left\|\left(\Theta_1 - X_1\Lambda_x^{-1}\Lambda_\theta\right)\theta\right\| + \\ &\qquad \left.\left\|X_1\Lambda_x^{-1}X_0^\top h_0 + \left(X_1\Lambda_x^{-1}X_1^\top - \boldsymbol{I}\right)h_1\right\|\right\} \\ &\qquad\qquad \text{(by the triangle inequality)} \\ &\le \max\left\{\left\|\left(\Theta_0 - X_0\Lambda_x^{-1}\Lambda_\theta\right)\right\|\|\theta\| + \right. \\ &\qquad \left\|\left(X_0\Lambda_x^{-1}X_0^\top - \boldsymbol{I}\right)h_0 + X_0\Lambda_x^{-1}X_1^\top h_1\right\|, \\ &\qquad \left\|\left(\Theta_1 - X_1\Lambda_x^{-1}\Lambda_\theta\right)\right\|\|\theta\| + \\ &\qquad \left.\left\|X_1\Lambda_x^{-1}X_0^\top h_0 + \left(X_1\Lambda_x^{-1}X_1^\top - \boldsymbol{I}\right)h_1\right\|\right\} \\ &\qquad\qquad \text{(by the Cauchy-Schwarz inequality)} \\ &\le \max\left\{\left\|\left(\Theta_0 - X_0\Lambda_x^{-1}\Lambda_\theta\right)\right\|, \left\|\left(\Theta_1 - X_1\Lambda_x^{-1}\Lambda_\theta\right)\right\|\right\}\|\theta\| \\ &\quad + \max\left\{\left\|\left(X_0\Lambda_x^{-1}X_0^\top - \boldsymbol{I}\right)h_0 + X_0\Lambda_x^{-1}X_1^\top h_1\right\|,\right. \\ &\qquad \left.\left\|X_1\Lambda_x^{-1}X_0^\top h_0 + \left(X_1\Lambda_x^{-1}X_1^\top - \boldsymbol{I}\right)h_1\right\|\right\} \\ &\qquad\qquad \text{(by the properties of the maximum)} \\ &= \chi\|\theta\| + \zeta.\end{aligned}$$

Therefore, if $\chi\|\theta\| + \zeta < 1$, then, by the above inequalities and Lemma 2, $\omega^* < 1$, and $\mathcal{S}$ is not separated. ■

We expect the bound to be tighter when the models are more similar, in which case the third inequality is closer to equality. A simple implication of Theorem 1 that if $(1-\zeta)/\chi > 0$, then an auxiliary signal is necessary to separate $\mathcal{S}$. We could straightforwardly obtain bounds on the cost, $\theta^\top C\theta$, by incorporating a factor of $\sqrt{C^{-1}}$ into χ.

Corollary 2: If the models are identical, then $\chi = \zeta = 0$. In this case, we could use a limiting argument to say that the lower bound is infinite. As expected, no auxiliary signal of any magnitude can separate $\mathcal{S}$.

Corollary 3: If models are very different, then $\zeta \ge 1$, and the lower bound is nonpositive.
In this case, an auxiliary signal might not be needed, i.e., $\theta = \mathbf{0}$ might separate $\mathcal{S}$. Note that this could not occur without the offsets, h_k, in which case $\zeta = 0$.

We have not obtained a corresponding lower bound on $\|\theta\|$, and we do not expect one to exist in the general case. This is because such a bound would enable us to guarantee separation knowing only the magnitude of the auxiliary signal. This is not realistic because if θ is not in the right direction, then no magnitude will be sufficient.

VI. Application to distance protection

A distance relay detects faults on transmission lines by measuring local currents, i, and voltages, e (from each phase to ground or between phases). Let $\bar{e}$ and $\bar{i}$ denote complex numbers, referred to as phasors, associated with e and i, respectively. The relay computes several quantities from $\bar{e}$ and $\bar{i}$, including their positive and negative sequence components, and an apparent impedance, defined as $\bar{z} = \bar{e}/\bar{i}$. Faults affect the voltage and current, and therefore the impedance computed by the relay. The relay opens a circuit breaker if the impedance is in its zone of operation, e.g., a circle or quadrilateral on the complex plane. This zone is based on estimates of the impedance the relay will see under normal conditions and during a fault [24]. A distance relay can misdiagnose an IBR-fed fault, in part because the current might differ little from normal. We remark that this is a simplistic description of distance relaying [25], and is meant to provide context for our examples.

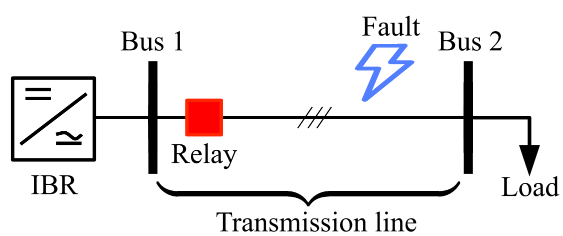

Fig. 1. Single-IBR single-load system. A fault occurs somewhere along the length of the transmission line.

A. Single-IBR single-load system

An IBR supplies power to a load, as shown in Figure 1. A phase a-to-phase b fault occurs on the transmission line. A distance relay on the line determines if a fault has occurred from local voltage and current measurements. All quantities are in per unit. Denote the positive and negative sequence components of $\bar{e}$ and $\bar{i}$ by $\bar{e}_+$, $\bar{e}_-$, $\bar{i}_+$, and $\bar{i}_-$, respectively.[4] When there is no fault, the measurements

[4]Let $e_\mathrm{a}(t)$, $e_\mathrm{b}(t)$, and $e_\mathrm{c}(t)$ denote three sinusoidal signals with identical frequency, and let $\bar{e}_\mathrm{a}$, $\bar{e}_\mathrm{b}$, and $\bar{e}_\mathrm{c}$ denote the phasors associated with said signals. Then, $\bar{e}_\mathrm{a}$, $\bar{e}_\mathrm{b}$, and $\bar{e}_\mathrm{c}$ can be decomposed into the sum of their so-called sequence components as follows: $\bar{e}_\mathrm{a} = \bar{e}_{+,\mathrm{a}} + \bar{e}_{-,\mathrm{a}} + \bar{e}_{0,\mathrm{a}}$, $\bar{e}_\mathrm{b} = \bar{e}_{+,\mathrm{b}} + \bar{e}_{-,\mathrm{b}} + \bar{e}_{0,\mathrm{b}}$, and $\bar{e}_\mathrm{c} = \bar{e}_{+,\mathrm{c}} + \bar{e}_{-,\mathrm{c}} + \bar{e}_{0,\mathrm{c}}$, where $\bar{e}_{+,\mathrm{b}} = e^{-j2\pi/3}\bar{e}_{+,\mathrm{a}}$, $\bar{e}_{+,\mathrm{c}} = e^{j2\pi/3}\bar{e}_{+,\mathrm{a}}$, $\bar{e}_{-,\mathrm{b}} = e^{j2\pi/3}\bar{e}_{-,\mathrm{a}}$, $\bar{e}_{-,\mathrm{c}} = e^{-j2\pi/3}\bar{e}_{-,\mathrm{a}}$, and $\bar{e}_{0,\mathrm{a}} = \bar{e}_{0,\mathrm{b}} = \bar{e}_{0,\mathrm{c}}$.

will satisfy

$$\bar{e}_- \approx \bar{z}_- \bar{i}_- \approx 0, \quad \bar{e}_+ \approx \bar{z}_+ \bar{i}_+ \approx 1,$$

where $\bar{z}_+$ and $\bar{z}_-$ are estimates of the apparent positive and negative sequence impedances of the line and load. Suppose now there is a fault between phases a and b. Let $\bar{z}_\text{f}$ is an estimate of the impedance to the fault. Then the measurements will satisfy

$$\bar{e}_- - \bar{z}_\text{f} \bar{i}_- \approx \bar{u}_+, \quad \bar{e}_+ - \bar{z}_\text{f} \bar{i}_+ \approx \bar{u}_-,$$

where $\bar{u}_+$ is the positive and $\bar{u}_-$ the negative sequence components of the voltage phasor at the fault point. $\bar{u}_+$ and $\bar{u}_-$ are uncertain but satisfy $\bar{u}_+ = \bar{u}_-$ (see, e.g., Section 5.4.1 of [24]). We represent the uncertainty in these relations explicitly below.

During faults, synchronous machines act like voltage sources, injecting large currents with negative sequence content. IBRs, on the other hand, behave more like current sources, preventing large currents and keeping $\bar{i}_- \approx 0$ even during faults. This can lead the relay to malfunction due to inaccurate estimation of fault impedances [12]. As in IEEE Standard 2800 [14], we remedy this by having the inverter inject negative sequence current. The auxiliary signal, $\bar{\theta}$, thus takes the place of the negative sequence current phasor, $\bar{i}_-$.

We assume that during normal operation, after appropriately rotating all phasors, the positive sequence voltage will be roughly one per unit. To approximate the voltage drop typical of IBRs during faults [26], we set the positive sequence voltage to $e_{+,1}$. We make no such assumptions about the negative sequence voltage because it depends on $\bar{\theta}$, and so does not help separate models. In both cases, the magnitude of the positive sequence current, $\bar{i}_+$, stays within i_max.

To solve $\mathcal{P}_2$ numerically, we must specify the values of $\bar{z}_-, \bar{z}_+$, and $\bar{z}_\text{f}$. We set

$$\begin{aligned}\bar{z}_- &= 0.1 + 0.9j,\\ \bar{z}_+ &= 0.1 + j,\\ \bar{z}_\text{f} &= 0.05 + jx_\text{f},\end{aligned}$$

where x_f is the reactance of the fault path. We vary the parameters $e_{+,1}$, i_max, and x_f in the simulations below. All optimizations were carried out in Python using CVXPy [27] and the solver MOSEK [28].

B. Additive uncertainty and no constraints

We first assume there is only additive uncertainty and neglect the current limit, so that we can compare the lower bound of Theorem 1 with the magnitude of the auxiliary signal. We use the convex-concave procedure [22], [23], as described in [3], to solve for the optimal auxiliary signal.

When there is no fault, the system model is

$$\bar{e}_- = \bar{z}_- \bar{\theta} + 0.1 \bar{\lambda}_{-,0} \tag{9a}$$
$$\bar{e}_+ = \bar{z}_+ \bar{i}_+ + 0.1 \bar{\lambda}_{+,0} \tag{9b}$$
$$\bar{e}_+ = 1 + 0.1 \lambda_{e,0}. \tag{9c}$$

This corresponds to (1a) when $k = 0$. When there is a fault, the system model is

$$\bar{e}_- = \bar{z}_\text{f} \bar{\theta} + 0.7 \bar{\lambda}_{-,1} \tag{10a}$$
$$\bar{e}_+ = \bar{z}_\text{f} \bar{i}_+ + 0.7 \bar{\lambda}_{+,1} \tag{10b}$$
$$\bar{e}_+ = e_{+,1} + 0.7 \lambda_{e,1}. \tag{10c}$$

This corresponds to (1a) when $k = 1$. The noises must satisfy

$$\left\| \begin{bmatrix} \bar{\lambda}_{-,k} \\ \bar{\lambda}_{+,k} \\ \lambda_{e,k} \end{bmatrix} \right\| \leq 1, \ k = 0, 1. \tag{11}$$

Writing (9)-(11) in the form of (1) entails splitting all complex quantities into real and imaginary parts, and then arranging the coefficients and variables into vectors and matrices. We omit the details here because they does not provide further insight and refer the reader to our earlier work [3], wherein the exercise is carried out for a similar example.

The coefficients in H_1 are larger than in H_0, which represents the high level of uncertainty during faulty operation, e.g., about the fault impedance and location. We want to find an auxiliary signal, $\bar{\theta}$, that separates $\mathcal{S} = \{0, 1\}$, i.e., so that either (9) or (10) is feasible, but not both.

Figure 2 shows the magnitude of the optimal auxiliary signal and the lower bound of Theorem 1 as $e_{+,1}$ and x_f are varied; recall that if the magnitude of the auxiliary signal is less than the lower bound, then the auxiliary signal does not achieve separation. As expected, both the auxiliary signal and lower bound are largest when the models are similar. When $e_{+,1}$ is very small or large, the normal and faulty models differ enough that no auxiliary signal is needed. The lower bound is useful in this case because it is positive over a slightly smaller range than the auxiliary signal magnitude, meaning that it accurately indicates when an auxiliary signal is needed. In the bottom plot, the lower bound estimates the magnitude of the optimal auxiliary signal reasonably well for all values of x_f.

1) Multiplicative and additive uncertainty: We now use both kinds of uncertainty, which enables us to treat the impedances as uncertain. We also include the magnitude constraint on $\bar{i}_+$. We search for auxiliary signals by testing the feasibility of $\Omega_\mathcal{S}\left(\bar{\theta}\right)$, which amounts to solving a semidefinite program for each value of $\bar{\theta}$.

When there is no fault, the system model is

$$\bar{e}_- = \bar{z}_- (1 + 0.1 \xi_{z,0}) \bar{\theta} \tag{12a}$$
$$\bar{e}_+ = \bar{z}_+ (1 + 0.1 \xi_{z,0}) \bar{i}_+ \tag{12b}$$
$$\bar{e}_+ = 1 + 0.1 \lambda_{e,0} \tag{12c}$$
$$\|\bar{i}_+\| \leq 1.2. \tag{12d}$$

This corresponds to (1a) and (1b) when $k = 0$. When auxiliary signal becomes necessary only when constraints are removed.

If, e.g., $\bar{\theta} = \bar{i}_+$, then we'll have $\bar{e}_- = \bar{e}_+$ if there is a fault, and $\bar{e}_- \neq \bar{e}_+$ if not, regardless of the realization of the uncertainty. A similar rule can be derived for any auxiliary signal satisfying (14). By including constraints on both the current and voltage, we shrink the sets of possible relay measurements, so that the two models are separated even without an auxiliary signal.

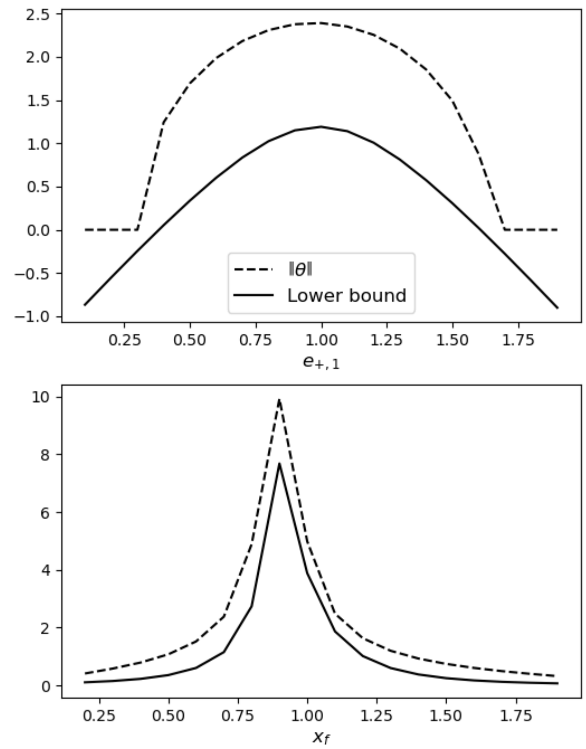

Fig. 2. Lower bound and $\|\theta\|$ versus $e_{+,1}$ and x_{f}. In the top plot, $x_{\mathrm{f}} = 0.7$, and in the bottom, $e_{+,1} = 0.9$.

there is a fault, the system model is

$$\bar{e}_- = \bar{z}_{\mathrm{f}}(1 + \xi_{z,1})\bar{\theta} \tag{13a}$$

$$\bar{e}_+ = \bar{z}_{\mathrm{f}}(1 + \xi_{z,1})\bar{i}_+ \tag{13b}$$

$$\bar{e}_+ = e_{+,1} + \lambda_{e,1} \tag{13c}$$

$$\|\bar{i}_+\| \leq 1.2. \tag{13d}$$

This corresponds to (1a) and (1b) when $k = 1$. We set $x_{\mathrm{f}} = 0.7$ and $e_{+,1} = 0.9$. The noises must satisfy

$$|\xi_{z,k}| \leq 1, \ |\lambda_{e,k}| \leq 1, \ k = 0, 1. \tag{13e}$$

As written, no auxiliary signal is needed, which can be verified by solving $\mathcal{P}^2$. If we omit either the positive sequence voltage constraints, (12c) and (13c), or the current magnitude constraints, (12d) and (13d), then an auxiliary signal is necessary and must satisfy

$$\mathcal{R}\left[\bar{\theta}\right]\mathcal{I}\left[\bar{\theta}\right] \neq 0, \tag{14}$$

which is to say is not on the real or imaginary axis of the complex plane. This means the auxiliary signal can be quite small so long as it is not below the relay's sensitivity.

We can see why this is by examining the two models. First, removing a constraint will always make the detection problem harder because it expands the set of possible measurements the relay must prepare for. In this problem, an auxiliary signal becomes necessary only when constraints are removed.

VII. Conclusion

We have formulated an auxiliary signal design problem for static systems with constraints and both additive and multiplicative uncertainty. We used a relaxation and duality to reformulate the problem as a bilinear semidefinite program. Two benefits of this approach are modeling flexibility and amenability to generic nonlinear programming algorithms. In the special case of additive noise and no constraints, we obtained a lower bound, which serves as a necessary condition on the magnitude of the auxiliary signal.

Our motivation is protection in inverter-dominated power systems, in which faults can be difficult to detect because the resulting currents do not resemble those of synchronous machines. A fairly recent remedy is to perturb the inverter's output so that faults are more apparent to relays. We have posed the design of these perturbations as an auxiliary signal problem. We believe this is a systematic and general strategy for fault detection when the inverter's default behavior is not sufficiently informative.

Two immediate directions of future work are systems with multiple detectors (relays) and auxiliary signal injection points (inverters), and application to the full set of fault types encountered in unbalanced power systems.

Appendix

We here describe the system $\Omega_{\mathcal{S}}(\theta)$, which contains the constraints in $\mathcal{P}^2$. $\Omega_{\mathcal{S}}(\theta)$ consists of the constraints in the dual of $\hat{\mathcal{P}}^0_{\mathcal{S}}$, plus a constraint in which the objective is greater than or equal to one. $\Omega_{\mathcal{S}}(\theta)$ is written below.

Variables:

$$\alpha_k, \beta_k, \delta_k, \epsilon_k^j \quad \text{(nonnegative)}$$
$$\rho_k, \tilde{\psi}_k^i, \tilde{\theta}_k^i, \tilde{x}_k^i, \tilde{W}_k^i$$
$$j = 1, ..., n_k,\ i = 1, ..., l_k,\ k \in \mathcal{S}.$$

Constraints:

$$1 \leq \sum_{k \in \mathcal{S}} \rho_k^\top (\Theta_k \theta - h_k) - \delta_k + \sum_{i=1}^{n_k} \epsilon_k^i b_k^i - \sum_{i=1}^{l_k} \tilde{\psi}_k^i \tag{15a}$$

$$\sum_{k \in \mathcal{S}} X_k^\top \rho_k + \sum_{i=1}^{n_k} \epsilon_k^i A_k^i - 2 \sum_{i=1}^{l_k} \tilde{x}_k^i = 0 \tag{15b}$$

$$1 = \sum_{k \in \mathcal{S}} \alpha_k + \beta_k \tag{15c}$$

$$\sum_{k \in \mathcal{S}} \sum_{i=1}^{n_k} \epsilon_k^i Q_k^i - \sum_{i=1}^{l_k} \tilde{W}_k^i = 0 \tag{15d}$$

$$4\alpha_k \delta_k \geq \left\| H_k^\top \rho_k \right\|^2, \quad k \in \mathcal{S} \tag{15e}$$

$$\begin{bmatrix} 2\tilde{\psi}_k^i & \rho_k^\top U_k^i \theta & 2\tilde{x}_k^{i\top} \\ \rho_k^\top U_k^i \theta & 2\beta_k & \rho_k^\top V_k^i \\ 2\tilde{x}_k^i & V_k^{i\top} \rho_k & 2\tilde{W}_k^i \end{bmatrix} \succeq 0,\ i = 1, ..., l_k,\ k \in \mathcal{S}. \tag{15f}$$

In the dual of $\hat{\mathcal{P}}_{\mathcal{S}}^0$, the variables are as in $\Omega_{\mathcal{S}}(\theta)$, the objective is the right side of (15a), and the constraints are (15b)-(15f). For $k \in \mathcal{S}$, ρ_k is the dual multiplier of (5a); ϵ_k^i of (5b), $i = 1, ..., n_k$; α_k of (5c); β_k of (5d). δ_k is a dummy variable that simplifies (15a). The rest of the variables are multipliers of the entries of the matrix in (5e).

In the special case of additive uncertainty and no constraints, $\Omega_{\mathcal{S}}(\theta)$ simplifies to the below.

Variables:

$$\alpha_k, \delta_k \quad \text{(nonnegative)}$$
$$\rho_k, k \in \mathcal{S}.$$

Constraints:

$$1 \leq \sum_{k \in \mathcal{S}} \rho_k^\top (\Theta_k \theta - h_k) - \delta_k$$
$$X_0^\top \rho_0 + X_1^\top \rho_1 = 0$$
$$\alpha_0 + \alpha_1 = 1$$
$$4\alpha_k \delta_k \geq \left\| H_k^\top \rho_k \right\|^2, \quad k \in \mathcal{S}.$$

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