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A SPACE-TIME MIXED FINITE ELEMENT METHOD FOR REDUCED FRACTURE FLOW MODELS ON NONMATCHING GRIDS

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Abstract. This paper is concerned with the numerical solution of the flow problem in a fractured porous medium where the fracture is treated as a lower dimensional object embedded in the rock matrix. We consider a space-time mixed variational formulation of such a reduced fracture model with mixed finite element approximations in space and discontinuous Galerkin discretization in time. Different spatial and temporal grids are used in the subdomains and in the fracture to adapt to the heterogeneity of the problem. Analysis of the numerical scheme, including well-posedness of the discrete problem, stability and a priori error estimates, is presented. Using substructuring techniques, the coupled subdomain and fracture system is reduced to a space-time interface problem which is solved iteratively by GMRES. Each GMRES iteration involves solution of time-dependent problems in the subdomains using the method of lines with local spatial and temporal discretizations. The convergence of GMRES is proved by using the field-of-values analysis and the properties of the discrete space-time interface operator. Numerical experiments are carried out to illustrate the performance of the proposed iterative algorithm and the accuracy of the numerical solution.

1. Introduction

Dimensionally reduced fracture models have been widely used for the modeling and simulations of fluid flow and transport in fractured porous media, where the fractures are represented as $(d-1)$-dimensional interfaces in a d-dimensional medium. These models are efficient as the width of the fractures is very small compared to the size of the surrounding medium and local mesh refinement around such fractures would be computationally expensive. In addition, the reduced models take into account the interactions between the flow in the fractures and in the rock matrix to provide approximations comparable to the ones obtained using the full dimensional approach [1,13,23,36,39]. Since the fractures can have much higher or much lower permeability than the surrounding medium, they can act as a conduit (i.e., allowing fluid flow much faster) or a geological barrier (i.e., blocking fluid flows across it). Consequently, the spatial and temporal scales may vary considerably across the domain of calculation, and it is desirable to develop numerical algorithms that can enforce different mesh sizes and time step sizes in the fractures and in the subdomains.

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Mathematically, reduced fracture models consist of systems of full dimensional Partial Differential Equations (PDEs) in the subdomains coupled with tangential PDEs in the lower dimensional fractures. The coupled problem can be solved directly as a monolithic system as in [6,7,10,14,16,20–22,26,28,33,35,41,44] for single-phase Darcy flow, [4,25,27,45] for transport problems, [11,24] for two-phase flow, and [13,34,42] for multiphysics problems in which different types of PDEs are considered in the fracture and in the subdomains. Solving a large coupled linear system as in the monolithic approach could be computationally costly, thus one can instead use nonoverlapping Domain Decomposition (DD) to decouple the system into subproblems of smaller sizes in the subdomains together with suitable transmission conditions on the fracture-interface as studied in [1–3, 19, 39]. We remark that most existing work uses the same time step in the subdomains and in the fracture. Local time discretization can be enforced by employing global-in-time DD methods as proposed in [29–31], where time-dependent problems are solved in the subdomains at each iteration and information is exchanged over the space-time fracture-interface. Though global-in-time DD allows local discretizations in both space and time, the aforementioned papers only consider matching spatial grids in the subdomains and in the fracture. Recently, a space-time DD method with nonmatching space-time grids has been analyzed in [32] for the case without fracture, in which mortar mixed finite elements are used for spatial discretization and discontinuous Galerkin for temporal discretization. It is well-known that the mortar spatial grid is required to be coarser than the grids in the subdomains to obtain stability [8,9]. However, for the reduced fracture model, it was shown in [22] that no mortars are needed and the mesh in the fracture can be much finer or coarser than in the subdomains. Note that only steady-state problems were considered in [22].

In this paper, we aim to develop and analyze space-time numerical approximations for the reduced fracture flow model with nonmatching space-time grids, i.e., different spatial mesh sizes and time step sizes in the subdomains and in the fracture. The problem is discretized in space by the mixed finite element method and in time by the discontinuous Galerkin method. As in the stationary case [22], there is no need to introduce a mortar finite element variable due to the tangential PDEs in the fracture. We remark that, unlike the case with artificial interfaces [32], the normal fluxes are not continuous across the fracture-interface. We carry out rigorous analysis for the well-posedness, stability and error estimates of the proposed numerical scheme for the monolithic fully discrete problem. In addition, improved error analysis is done by bounding the velocity divergence under the assumption of conforming time discretizations, which is similar to the case without fractures [32]. Based on global-in-time DD with the time-dependent Robin-to-Neumann interface operator (instead of the Dirichlet-to-Neumann operator for artificial interfaces [32]), we decouple the monolithic problem and reformulate it as an interface problem on the space-time fracture-interface. The interface problem is solved iteratively by GMRES, each iteration involving the solution of time-dependent problems in the subdomains using the method of lines with local spatial and temporal discretizations. The convergence of GMRES is proved by using the field-of-values analysis and the properties of the associated interface operator. The presented error estimates and convergence analysis of global-in-time DD for the reduced fracture model have not been done in the literature, even for the case with matching spatial meshes. Numerical results where the fracture is either a "fast path" or a geological barrier are presented to validate the theoretical error estimates as well as investigate the convergence of GMRES and the efficiency of nonmatching space-time grids. It should be noted that in this work, we have restricted our attention to the case with a single fracture and we require that the geometry of the fracture is respected by the meshes. Techniques from [5,10,26] could be used to extend the presented work to the case with intersecting fractures; however, this is beyond the scope of this article. We also refer to the review paper [20] (and the references therein) for more complex configurations with networks of fractures and for the case where some elements of the spatial grid may be cut by the fracture.

The rest of the paper is organized as follows: in the next section, we present the model problem and its weak formulation. In Section 3, the proposed numerical scheme using mixed finite element discretization in space and the discontinuous Galerkin method in time is introduced. The well-posedness and stability of the numerical solution are studied in Section 4, and a priori error estimates are derived in Section 5. In Section 6, we prove the boundedness of the velocity divergence and establish improved error estimates. In Section 7, global-in-time domain decomposition is utilized to decouple the system and reduce it to an interface problem; analysis of the interface operator is also presented. Finally, we discuss numerical results in Section 8.

2. Model problem

We consider a reduced fracture model in which the fracture is known a priori and is modeled as a hypersurface embedded in the porous medium. Let $\Omega \subset \mathbb{R}^d$ ($d = 2, 3$) be a bounded domain with Lipschitz boundary $\partial\Omega$. Suppose that the fracture Ω_f is a subdomain of Ω that separates Ω into two connected subdomains: $\Omega \backslash \overline{\Omega}_f = \Omega_1 \cup \Omega_2$, and $\Omega_1 \cap \Omega_2 = \emptyset$. We denote by γ_i the part of the boundary of Ω_i shared with the boundary of the fracture Ω_f: $\gamma_i = (\partial\Omega_i \cap \partial\Omega_f) \cap \Omega$, for $i = 1, 2$. Let $\boldsymbol{n}_i$ be the unit, outward pointing, normal vector field on $\partial\Omega_i$, $i = 1, 2$. We assume that Ω_f can be expressed as

$$\Omega_f = \left\{ \boldsymbol{x} \in \Omega : \boldsymbol{x} = \boldsymbol{x}_\gamma + \sigma \boldsymbol{n}, \text{ where } \boldsymbol{x}_\gamma \in \gamma \text{ and } \sigma \in \left(-\frac{\delta(\boldsymbol{x}_\gamma)}{2}, \frac{\delta(\boldsymbol{x}_\gamma)}{2} \right) \right\},$$

where γ is the intersection of a line ($d = 2$) or a plane ($d = 3$) with Ω, $\boldsymbol{n} = \boldsymbol{n}_1 = -\boldsymbol{n}_2$ is the unit normal vector to γ, and $\delta(\boldsymbol{x}_\gamma)$ is the width of the fracture at $\boldsymbol{x}_\gamma \in \gamma$. For $i = 1, 2, f$, and for any scalar, vector, or tensor valued function ϕ defined on Ω, we denote by ϕ_i the restriction of ϕ to Ω_i. The flow problem of a single phase, compressible fluid in the fractured porous medium Ω is given by

$$\begin{aligned}
s_i \partial_t p_i + \operatorname{div} \boldsymbol{u}_i &= q_i && \text{in } \Omega_i \times (0,T), && i = 1,2,f,\\
\boldsymbol{u}_i &= -\boldsymbol{K}_i \nabla p_i && \text{in } \Omega_i \times (0,T), && i = 1,2,f,\\
p_i &= p_f && \text{on } \gamma_i \times (0,T), && i = 1,2,\\
\boldsymbol{u}_i \cdot \boldsymbol{n}_i &= \boldsymbol{u}_f \cdot \boldsymbol{n}_i && \text{on } \gamma_i \times (0,T), && i = 1,2,\\
p_i &= 0 && \text{on } (\partial\Omega_i \cap \partial\Omega) \times (0,T), && i = 1,2,f,\\
p_i(\cdot, 0) &= p_{0,i} && \text{in } \Omega_i, && i = 1,2,f,
\end{aligned} \tag{2.1}$$

where, for $i = 1, 2, f$, p_i is the pressure, $\boldsymbol{u}_i$ the velocity, q_i the source term, $s_i > 0$ constant storage coefficients, and $\boldsymbol{K}_i$ a symmetric, time-independent, permeability tensor. We assume that initial condition $p_0 \in H_0^1(\Omega)$ is given and define $p_{0,i} = p_0|_{\Omega_i}$, for $i = 1, 2, f$. For simplicity, we have imposed homogeneous Dirichlet conditions on the external boundary.

As in the steady-state flow case [3, 22, 39], we treat the fracture as a domain of co-dimension 1, i.e., we collapse the fracture domain Ω_f onto its central axis γ. The reduced fracture model is then obtained by integrating over the cross sections of Ω_f the first two equations of (2.1) for the index f; more details can be found in [39]. In particular, we decompose $\boldsymbol{u}_f$ into its components normal and tangential to the fracture: $\boldsymbol{u}_f = \boldsymbol{u}_{f,n} + \boldsymbol{u}_{f,\tau}$, where $\boldsymbol{u}_{f,n} = (\boldsymbol{u}_f \cdot \boldsymbol{n})\boldsymbol{n}$. Define $\mathsf{T} = [\tau_1\ \tau_2 \dots \tau_{d-1}]$, where $\{\tau_l\}_{l=1}^{d-1}$ is an orthogonal system of unit tangent vectors on γ. The velocity and pressure on the $(d-1)$-dimensional fracture γ are defined as

$$\boldsymbol{u}_\gamma := \int_{-\frac{\delta}{2}}^{\frac{\delta}{2}} \mathsf{T}^T \boldsymbol{u}_{f,\tau}\, d\boldsymbol{n}, \quad p_\gamma := \frac{1}{\delta}\int_{-\frac{\delta}{2}}^{\frac{\delta}{2}} p_f d\boldsymbol{n}, \tag{2.2}$$

i.e., $\boldsymbol{u}_\gamma$ and p_γ are the average velocity and pressure, respectively, along the normal direction of the full dimensional fracture Ω_f. For simplicity, we assume that $\boldsymbol{K}_f$ is invariant in direction normal to the fracture and define:

$$\boldsymbol{K}_{f,n} := \boldsymbol{n}^T \boldsymbol{K}_f \boldsymbol{n}, \quad \boldsymbol{K}_{f,\tau} := \mathsf{T}^T \boldsymbol{K}_f \mathsf{T}.$$

Let $\boldsymbol{K}_\gamma := \delta \boldsymbol{K}_{f,\tau}$, $\kappa_\gamma := 2\boldsymbol{K}_{f,n}/\delta$ and $s_\gamma := \delta s_f$. In addition, we use the notation ∇_τ and div_τ for the tangential gradient and tangential divergence, respectively. The reduced model is given as an interface problem as follows, where we still denote the subdomains by Ω_i, $i = 1, 2$:

$$\begin{aligned}
s_i \partial_t p_i + \operatorname{div} \boldsymbol{u}_i &= q_i && \text{in } \Omega_i \times (0,T),\\
\boldsymbol{u}_i &= -\boldsymbol{K}_i \nabla p_i && \text{in } \Omega_i \times (0,T),\\
s_\gamma \partial_t p_\gamma + \operatorname{div}_\tau \boldsymbol{u}_\gamma &= q_\gamma + \textstyle\sum_{i=1}^2 (\boldsymbol{u}_i \cdot \boldsymbol{n}_i)_{|\gamma} && \text{in } \gamma \times (0,T),\\
\boldsymbol{u}_\gamma &= -\boldsymbol{K}_\gamma \nabla_\tau p_\gamma && \text{in } \gamma \times (0,T),\\
\kappa_\gamma (p_i - p_\gamma) &= \xi \boldsymbol{u}_i \cdot \boldsymbol{n}_i - (1-\xi)\boldsymbol{u}_j \cdot \boldsymbol{n}_j, && \text{in } \gamma \times (0,T),\\
p_i &= 0 && \text{on } (\partial\Omega_i \cap \partial\Omega) \times (0,T),\\
p_\gamma &= 0 && \text{on } \partial\gamma \times (0,T),\\
p_i(\cdot, 0) &= p_{0,i} && \text{in } \Omega_i,\\
p_\gamma(\cdot, 0) &= p_{0,\gamma} && \text{in } \gamma,
\end{aligned} \tag{2.3}$$

for $i = 1$, 2, and $j = (3-i)$, $\xi > \frac{1}{2}$ is a model parameter, $q_\gamma = \frac{1}{\delta}\int_{-\frac{\delta}{2}}^{\frac{\delta}{2}} q_f\, d\boldsymbol{n}$, and $p_{0,\gamma} = p_0|_\gamma$.

Throughout the paper, we assume that there exist positive constants s_- and s_+, K_- and K_+, $K_{\gamma-}$ and $K_{\gamma+}$, $\kappa_{\gamma-}$ and $\kappa_{\gamma+}$ such that

(A1) $s_- \le s_i \le s_+$, $i = 1, 2$,
(A2) $s_- \le s_\gamma \le s_+$,
(A3) $K_-|\varsigma|^2 \le \varsigma^T \boldsymbol{K}_i(\boldsymbol{x})\varsigma \le K_+|\varsigma|^2$, for a.e. $\boldsymbol{x} \in \Omega_i$ and $\forall \varsigma \in \mathbb{R}^d$, $i = 1, 2$,
(A4) $K_{\gamma-}|\eta|^2 \le \eta^T \boldsymbol{K}_\gamma(\boldsymbol{x})\eta \le K_{\gamma+}|\eta|^2$, for a.e. $\boldsymbol{x} \in \gamma$ and $\forall \eta \in \mathbb{R}^{d-1}$,
(A5) $\kappa_{\gamma-} \le \kappa_\gamma(\boldsymbol{x}) \le \kappa_{\gamma+}$ for a.e. $\boldsymbol{x} \in \gamma$.

We note that $\boldsymbol{K}_i$ and $\boldsymbol{K}_\gamma$ as well as their inverses are symmetric positive definite; moreover, it is implied from Assumptions (A3) and (A4) that the following inequalities hold:

$$
\begin{aligned}
&K_+^{-1}|\varsigma|^2 \le \varsigma^T \boldsymbol{K}_i^{-1}(\boldsymbol{x})\varsigma \le K_-^{-1}|\varsigma|^2, \text{ for a.e. } \boldsymbol{x}\in\Omega_i \text{ and } \forall\varsigma\in\mathbb{R}^d,\ i=1,2,\\
&K_{\gamma+}^{-1}|\eta|^2 \le \eta^T \boldsymbol{K}_\gamma^{-1}(\boldsymbol{x})\eta \le K_{\gamma-}^{-1}|\eta|^2, \text{ for a.e. } \boldsymbol{x}\in\gamma \text{ and } \forall\eta\in\mathbb{R}^{d-1}.
\end{aligned}
\tag{2.4}
$$

To write the weak formulation of (2.3), we use the convention that if V is a space of functions, then $\boldsymbol{V}$ is a space of vector functions having each component in V. For arbitrary domain $\mathcal{O}$, we denote by $(\cdot,\cdot)_{\mathcal{O}}$ the inner product in $L^2(\mathcal{O})$ or $\boldsymbol{L^2}(\mathcal{O})$ and by $\|\cdot\|_{0,\mathcal{O}}$ the norm in $L^2(\mathcal{O})$ or $\boldsymbol{L}^2(\mathcal{O})$. For $\mathcal{O}^T=\mathcal{O}\times(0,T)$, we write $(\cdot,\cdot)_{\mathcal{O}^T}=\int_0^T(\cdot,\cdot)_{\mathcal{O}}$. Denote by $M_i=L^2(\Omega_i)$ and $\boldsymbol{\Sigma}_i=H(\mathrm{div},\Omega_i)$ for $i=1,2$ with norms

$$
\|\mu_i\|_{M_i}=\|\mu_i\|_{0,\Omega_i} \quad\text{and}\quad \|\boldsymbol{v}_i\|^2_{\boldsymbol{\Sigma}_i}=\|\boldsymbol{v}_i\|^2_{0,\Omega_i}+\|\operatorname{div}\boldsymbol{v}_i\|^2_{0,\Omega_i}.
$$

Similarly, let $M_\gamma=L^2(\gamma)$ and $\boldsymbol{\Sigma}_\gamma=H(\mathrm{div}_\tau,\gamma)$ with norms

$$
\|\mu_\gamma\|_{M_\gamma}=\|\mu_\gamma\|_{0,\gamma} \quad\text{and}\quad \|\boldsymbol{v}_\gamma\|^2_{\boldsymbol{\Sigma}_\gamma}=\|\boldsymbol{v}_\gamma\|^2_{0,\gamma}+\|\operatorname{div}_\tau\boldsymbol{v}_\gamma\|^2_{0,\gamma}.
$$

We next define the following Hilbert spaces:

$$
\begin{aligned}
M&=\{\mu=(\mu_1,\mu_2,\mu_\gamma)\in M_1\times M_2\times M_\gamma\},\\
\boldsymbol{\Sigma}&=\left\{\boldsymbol{v}=(\boldsymbol{v}_1,\boldsymbol{v}_2,\boldsymbol{v}_\gamma)\in\boldsymbol{\Sigma}_1\times\boldsymbol{\Sigma}_2\times\boldsymbol{\Sigma}_\gamma:\boldsymbol{v}_i\cdot\boldsymbol{n}_{i|\gamma}\in L^2(\gamma),\ i=1,2\right\},
\end{aligned}
$$

which are equipped with the norms:

$$
\|\mu\|^2_M=\sum_{i=1,2,\gamma}\|\mu_i\|^2_{M_i},\quad \|\boldsymbol{v}\|^2_{\boldsymbol{\Sigma}}=\sum_{i=1,2,\gamma}\|\boldsymbol{v}_i\|^2_{\boldsymbol{\Sigma}_i}+\sum_{i=1}^2\|\boldsymbol{v}_i\cdot\boldsymbol{n}_i\|^2_{0,\gamma}.
$$

We define the bilinear forms $a(\cdot,\cdot)$, $b(\cdot,\cdot)$ and $c_s(\cdot,\cdot)$ on $\boldsymbol{\Sigma}\times\boldsymbol{\Sigma}$, $\boldsymbol{\Sigma}\times M$, and $M\times M$, respectively, and the linear form L_q on M by

$$
\begin{aligned}
a(\boldsymbol{u},\boldsymbol{v})&=\sum_{i=1}^2\left(\boldsymbol{K}_i^{-1}\boldsymbol{u}_i,\boldsymbol{v}_i\right)_{\Omega_i}+\left(\boldsymbol{K}_\gamma^{-1}\boldsymbol{u}_\gamma,\boldsymbol{v}_\gamma\right)_\gamma\\
&\qquad+\sum_{i=1}^2\left(\kappa_\gamma^{-1}\left(\xi\boldsymbol{u}_i\cdot\boldsymbol{n}_i+(1-\xi)\boldsymbol{u}_j\cdot\boldsymbol{n}_i\right),\boldsymbol{v}_i\cdot\boldsymbol{n}_i\right)_\gamma,\\
b(\boldsymbol{u},\mu)&=\sum_{i=1}^2(\operatorname{div}\boldsymbol{u}_i,\mu_i)_{\Omega_i}+(\operatorname{div}_\tau\boldsymbol{u}_\gamma,\mu_\gamma)_\gamma-([\![\boldsymbol{u}\cdot\boldsymbol{n}]\!],\mu_\gamma)_\gamma,\\
c_s(\eta,\mu)&=\sum_{i=1}^2(s_i\eta_i,\mu_i)_{\Omega_i}+(s_\gamma\eta_\gamma,\mu_\gamma)_\gamma,\quad L_q(\mu)=\sum_{i=1}^2(q_i,\mu_i)_{\Omega_i}+(q_\gamma,\mu_\gamma)_\gamma,
\end{aligned}
$$

where we have used the notation $[\![\boldsymbol{v}\cdot\boldsymbol{n}]\!]=\boldsymbol{v}_1\cdot\boldsymbol{n}_1|_\gamma+\boldsymbol{v}_2\cdot\boldsymbol{n}_2|_\gamma$. In addition, for any spatial bilinear form $\alpha(\cdot,\cdot)$ or linear form $l(\cdot)$, we denote by $\alpha^T(\cdot,\cdot)=\int_0^T\alpha(\cdot,\cdot)$ and $l^T(\cdot)=\int_0^T l(\cdot)$.

The weak form of (2.3) can be written as follows:

Find $\boldsymbol{u}\in L^2(0,T;\boldsymbol{\Sigma})$ and $p\in H^1(0,T;M)$ such that

$$
\begin{aligned}
a^T(\boldsymbol{u},\boldsymbol{v})-b^T(\boldsymbol{v},p)&=0 &&\forall\boldsymbol{v}\in L^2(0,T;\boldsymbol{\Sigma}),\\
c_s^T(\partial_t p,\mu)+b^T(\boldsymbol{u},\mu)&=L_q^T(\mu) &&\forall\mu\in L^2(0,T;M),
\end{aligned}
\tag{2.5}
$$

together with the initial conditions:

$$
p_i(\cdot,0)=p_{0,i},\ \text{in }\Omega_i,\ i=1,2,\quad\text{and}\quad p_\gamma(\cdot,0)=p_{0,\gamma},\ \text{in }\gamma.
\tag{2.6}
$$

Under assumptions (A1)–(A5), existence and uniqueness of a solution to the variational problem (2.5) can be proved using the well-posedness theory for abstract evolution problems in mixed form (cf. [29, Theorem 1.2]). The proof utilizes the coercivity of the bilinear form $a(\cdot,\cdot)$:

(2.7)

$$\begin{aligned} a(\boldsymbol{v},\boldsymbol{v}) = \sum_{i=1}^{2} \|\boldsymbol{K}_i^{-\frac{1}{2}}\boldsymbol{v}_i\|^2_{0,\Omega_i} + \|\boldsymbol{K}_\gamma^{-\frac{1}{2}}\boldsymbol{v}_\gamma\|^2_{0,\gamma} + \xi \sum_{i=1}^{2} \|\kappa_\gamma^{-\frac{1}{2}}\boldsymbol{v}_i\cdot\boldsymbol{n}\|^2_{0,\gamma} \\ + 2(1-\xi)\left(\kappa_\gamma^{-\frac{1}{2}}\boldsymbol{v}_1\cdot\boldsymbol{n}, \kappa_\gamma^{-\frac{1}{2}}\boldsymbol{v}_2\cdot\boldsymbol{n}\right)_\gamma \\ \geq \sum_{i=1}^{2} \|\boldsymbol{K}_i^{-\frac{1}{2}}\boldsymbol{v}_i\|^2_{0,\Omega_i} + \|\boldsymbol{K}_\gamma^{-\frac{1}{2}}\boldsymbol{v}_\gamma\|^2_{0,\gamma} + \min(1,2\xi-1)\sum_{i=1}^{2} \|\kappa_\gamma^{-\frac{1}{2}}\boldsymbol{v}_i\cdot\boldsymbol{n}\|^2_{0,\gamma}, \end{aligned}$$

where we have used the inequality $|ab| \leq \frac{a^2}{2}+\frac{b^2}{2}$. Note that $\xi > \frac{1}{2}$.

Finally, we introduce the following notation which shall be used in the analysis later:

$$\begin{aligned} &H = H_1 \times H_2 \times H_\gamma := H^1(\Omega_1)\times H^1(\Omega_2)\times H^1(\gamma) \subset M, \\ &\boldsymbol{H} = \boldsymbol{H}_1 \times \boldsymbol{H}_2 \times \boldsymbol{H}_\gamma := (H^1(\Omega_1))^d \times (H^1(\Omega_2))^d \times (H^1(\gamma))^{d-1} \subset \boldsymbol{\Sigma}, \\ &\boldsymbol{H}^\varepsilon = \boldsymbol{H}_1^\varepsilon \times \boldsymbol{H}_2^\varepsilon \times \boldsymbol{H}_\gamma^\varepsilon := (H^\varepsilon(\Omega_1))^d \times (H^\varepsilon(\Omega_2))^d \times (H^\varepsilon(\gamma))^{d-1}, \\ &\qquad\qquad\qquad\qquad\text{for any real number } \varepsilon > 0, \\ &\boldsymbol{M}^* = \big\{\boldsymbol{v}\in \boldsymbol{M}_1\times\boldsymbol{M}_2\times\boldsymbol{M}_\gamma := (L^2(\Omega_1))^d \times (L^2(\Omega_2))^d \times (L^2(\gamma))^{d-1} \text{ such that} \\ &\qquad\qquad\qquad\qquad \boldsymbol{v}_i\cdot\boldsymbol{n}_i|_\gamma \in L^2(\gamma)\big\} \supset \boldsymbol{\Sigma}. \end{aligned}$$

The space $\boldsymbol{M}^*$ is equipped with the norm

$$\|\boldsymbol{v}\|^2_{\boldsymbol{M}^*} = \sum_{i=1,2,\gamma} \|\boldsymbol{v}_i\|^2_{\boldsymbol{M}_i} + \sum_{i=1}^{2}\|\boldsymbol{v}_i\cdot\boldsymbol{n}_i\|^2_{0,\gamma}.$$

Also, to simplify notation, we shall write for $\boldsymbol{v}\in\boldsymbol{\Sigma}$, $\operatorname{div}\boldsymbol{v} = (\operatorname{div}\boldsymbol{v}_1, \operatorname{div}\boldsymbol{v}_2, \operatorname{div}_\tau \boldsymbol{v}_\gamma)$.

3. Space-time mixed finite element method

For $i = 1, 2$, let $\mathcal{T}_{h,i}$ be a partition of Ω_i into d-dimensional simplicial or rectangular elements, and let $\mathcal{T}_{h,\gamma}$ be a partition of γ into $(d-1)$-dimensional simplicial or rectangular elements. There are no matching requirements between any of these partitions. Let $h_i = \max_{E\in\mathcal{T}_{h,i}} \operatorname{diam}(E)$, $i = 1,2,\gamma$ and $h = \max_{i=1,2,\gamma} h_i$. In time, for $i = 1,2,\gamma$, let $\mathcal{T}_i^{\Delta t}: 0 = t_i^0 < t_i^1 < \cdots < t_i^{N_i} = T$ be a partition of the time interval $(0,T)$ in each subdomain and the fracture. Let $\Delta t_i = \max_{1\leq k\leq N_i}(t_i^k - t_i^{k-1})$, $i = 1,2,\gamma$, and $\Delta t = \max_{i=1,2,\gamma}\Delta t_i$. A space-time partition of $\Omega_i\times(0,T)$ is given by $\mathcal{T}_{h,i}^{\Delta t} := \mathcal{T}_{h,i}\times\mathcal{T}_i^{\Delta t}$, for $i = 1,2$. Similarly for the space-time fracture-interface $\gamma\times(0,T)$, its space-time partition is denoted by $\mathcal{T}_{h,\gamma}^{\Delta t} := \mathcal{T}_{h,\gamma}\times\mathcal{T}_\gamma^{\Delta t}$.

For spatial discretization, we consider for simplicity the Raviart-Thomas space on d or $(d-1)$ dimensional simplices or rectangles [12]:

$$M_h = M_{h,1}\times M_{h,2}\times M_{h,\gamma} \subset M, \text{ and } \boldsymbol{\Sigma}_h = \boldsymbol{\Sigma}_{h,1}\times\boldsymbol{\Sigma}_{h,2}\times\boldsymbol{\Sigma}_{h,\gamma}\subset\boldsymbol{\Sigma}.$$

The results can be easily extended to other stable mixed finite element spaces, such as the BDM spaces [12]. For time discretization, we apply the discontinuous

Galerkin (DG) method where discontinuous piecewise polynomials are used to approximate the solution on the time grid $\mathcal{T}_i^{\Delta t}$. Denote by $W_i^{\Delta t}$, $i = 1, 2, \gamma$, the time discretization of the pressure and velocity in each subdomain and the fracture. The space-time discretizations are then given by

$$M_h^{\Delta t} = M_{h,1}^{\Delta t} \times M_{h,2}^{\Delta t} \times M_{h,\gamma}^{\Delta t}, \text{ and } \boldsymbol{\Sigma}_h^{\Delta t} = \boldsymbol{\Sigma}_{h,1}^{\Delta t} \times \boldsymbol{\Sigma}_{h,2}^{\Delta t} \times \boldsymbol{\Sigma}_{h,\gamma}^{\Delta t},$$

where

$$M_{h,i}^{\Delta t} = M_{h,i} \otimes W_i^{\Delta t}, \text{ and } \boldsymbol{\Sigma}_{h,i}^{\Delta t} = \boldsymbol{\Sigma}_{h,i} \otimes W_i^{\Delta t}, \text{ for } i = 1, 2, \gamma.$$

To write the weak formulation with DG time discretization, we introduce the following notation for functions $\varphi(\boldsymbol{x}, \cdot)$ and $\psi(\boldsymbol{x}, \cdot)$ in $L^2(0, T)$:

$$\forall \boldsymbol{x} \in \Omega_i,\ i = 1, 2, \text{ or } \boldsymbol{x} \in \gamma, \quad \int_0^T \widetilde{\partial}_t \varphi\, \psi = \sum_{n=1}^{N_i} \int_{t_i^{n-1}}^{t_i^n} \partial_t \varphi\, \psi + \sum_{n=1}^{N_i} [\varphi]_{n-1} \psi_{n-1}^+, \tag{3.1}$$

for $i = 1, 2, \gamma$, where $[\varphi]_n = \varphi_n^+ - \varphi_n^-$, with $\varphi_n^+ = \lim_{t \to t_i^{n},+} \varphi$ and $\varphi_n^- = \lim_{t \to t_i^{n},-} \varphi$.

The space-time mixed finite element method for (2.3) reads as:

Find $\boldsymbol{u}_h^{\Delta t} \in \boldsymbol{\Sigma}_h^{\Delta t}$ and $p_h^{\Delta t} \in M_h^{\Delta t}$ such that

$$\begin{aligned} a^T\left(\boldsymbol{u}_h^{\Delta t}, \boldsymbol{v}_h^{\Delta t}\right) - b^T\left(\boldsymbol{v}_h^{\Delta t}, p_h^{\Delta t}\right) &= 0 && \forall \boldsymbol{v}_h^{\Delta t} \in \boldsymbol{\Sigma}_h^{\Delta t}, \\ c_s^T\left(\widetilde{\partial}_t p_h^{\Delta t}, \mu_h^{\Delta t}\right) + b^T\left(\boldsymbol{u}_h^{\Delta t}, \mu_h^{\Delta t}\right) &= L_q^T(\mu_h^{\Delta t}) && \forall \mu_h^{\Delta t} \in M_h^{\Delta t}. \end{aligned} \tag{3.2}$$

Note that we will use the initial condition to determine $(p_h^{\Delta t})^-$ needed in the evaluation of $c_s^T\left(\widetilde{\partial}_t p_h^{\Delta t}, \mu_h^{\Delta t}\right)$. Further details will be discussed in Subsection 4.3.

4. Well-posedness analysis

We study the existence and uniqueness of the solution to the fully discrete problem (3.2) with nonconforming space-time grids. Throughout the paper, we use C to denote a generic constant that is independent of the spatial mesh sizes and time step sizes.

4.1. **Space-time interpolants.** For $i = 1, 2, \gamma$, let $\mathcal{P}_{h,i}$ be the L^2-orthogonal projection onto $M_{h,i}$ and $\mathcal{P}_i^{\Delta t}$ be the L^2-orthogonal projection onto $W_i^{\Delta t}$. The L^2-orthogonal projections in space and time are then defined on the subdomains as

$$\mathcal{P}_{h,i}^{\Delta t} = \mathcal{P}_i^{\Delta t} \circ \mathcal{P}_{h,i} :\ L^2\left(0, T; M_i\right) \to M_{h,i}^{\Delta t}, \quad i = 1, 2,$$

and on the fracture as

$$\mathcal{P}_{h,\gamma}^{\Delta t} = \mathcal{P}_\gamma^{\Delta t} \circ \mathcal{P}_{h,\gamma} :\ L^2\left(0, T; M_\gamma\right) \to M_{h,\gamma}^{\Delta t}.$$

From that, we define the global space-time L^2-orthogonal projection

$$\mathcal{P}_h^{\Delta t} : L^2\left(0, T; M\right) \to M_h^{\Delta t}, \quad \mathcal{P}_h^{\Delta t}|_{\Omega_i^T} = \mathcal{P}_{h,i}^{\Delta t}, \text{ for } i = 1, 2, \text{ and } \mathcal{P}_h^{\Delta t}|_{\gamma^T} = \mathcal{P}_{h,\gamma}^{\Delta t}.$$

For $i = 1, 2, \gamma$, let $\boldsymbol{\Pi}_{h,i} : \boldsymbol{H}_i^\varepsilon \cap \boldsymbol{\Sigma}_i \to \boldsymbol{\Sigma}_{h,i}$ be the Raviart-Thomas interpolant [12, 43], and let

$$\boldsymbol{\Pi}_{h,i}^{\Delta t} = \mathcal{P}_i^{\Delta t} \circ \boldsymbol{\Pi}_{h,i} :\ L^2\left(0, T; \boldsymbol{H}_i^\varepsilon \cap \boldsymbol{\Sigma}_i\right) \to \boldsymbol{\Sigma}_{h,i}^{\Delta t}, \quad i = 1, 2, \tag{4.1}$$

$$\boldsymbol{\Pi}_{h,\gamma}^{\Delta t} = \mathcal{P}_\gamma^{\Delta t} \circ \boldsymbol{\Pi}_{h,\gamma} :\ L^2\left(0, T; \boldsymbol{H}_\gamma^\varepsilon \cap \boldsymbol{\Sigma}_\gamma\right) \to \boldsymbol{\Sigma}_{h,\gamma}^{\Delta t}. \tag{4.2}$$

The space-time interpolants $\boldsymbol{\Pi}_{h,i}^{\Delta t}$, $i = 1, 2$, satisfy the following properties [12, 40], for all $\boldsymbol{v} \in L^2(0,T; \boldsymbol{H}_i^\varepsilon \cap \boldsymbol{\Sigma}_i)$:

$$\left(\operatorname{div}(\boldsymbol{\Pi}_{h,i}^{\Delta t}\boldsymbol{v}_i - \boldsymbol{v}_i), \mu_{h,i}^{\Delta t}\right)_{\Omega_i^T} = 0, \quad \forall \mu_{h,i}^{\Delta t} \in M_{h,i}^{\Delta t}, \tag{4.3}$$

$$\left((\boldsymbol{\Pi}_{h,i}^{\Delta t}\boldsymbol{v}_i - \boldsymbol{v}_i)\cdot \boldsymbol{n}_i, \boldsymbol{w}_{h,i}^{\Delta t}\cdot \boldsymbol{n}_i\right)_{\partial\Omega_i^T} = 0, \quad \forall \boldsymbol{w}_{h,i}^{\Delta t} \in \boldsymbol{\Sigma}_{h,i}^{\Delta t}, \tag{4.4}$$

$$\|\boldsymbol{\Pi}_{h,i}^{\Delta t}\boldsymbol{v}_i\|_{L^2(0,T;\boldsymbol{\Sigma}_i)} \le C\left(\|\boldsymbol{v}_i\|_{L^2(0,T;\boldsymbol{H}_i^\varepsilon)} + \|\operatorname{div}\boldsymbol{v}_i\|_{L^2(0,T;M_i)}\right), \tag{4.5}$$

$$\|\boldsymbol{\Pi}_{h,i}^{\Delta t}\boldsymbol{v}_i\cdot\boldsymbol{n}_i\|_{L^2(0,T;M_\gamma)} \le \|\boldsymbol{v}_i\cdot\boldsymbol{n}_i\|_{L^2(0,T;M_\gamma)}. \tag{4.6}$$

Similarly, $\boldsymbol{\Pi}_{h,\gamma}^{\Delta t}$ satisfies, for all $\boldsymbol{v}_\gamma \in L^2(0,T;\boldsymbol{H}_\gamma^\varepsilon \cap \boldsymbol{\Sigma}_\gamma)$:

$$\left(\operatorname{div}_\tau(\boldsymbol{\Pi}_{h,\gamma}^{\Delta t}\boldsymbol{v}_\gamma - \boldsymbol{v}_\gamma), \mu_{h,\gamma}^{\Delta t}\right)_{\gamma^T} = 0, \quad \forall \mu_{h,\gamma}^{\Delta t} \in M_{h,\gamma}^{\Delta t}, \tag{4.7}$$

$$\left((\boldsymbol{\Pi}_{h,\gamma}^{\Delta t}\boldsymbol{v}_\gamma - \boldsymbol{v}_\gamma)\cdot \boldsymbol{n}_\gamma, \boldsymbol{w}_{h,\gamma}^{\Delta t}\cdot \boldsymbol{n}_\gamma\right)_{\partial\gamma^T} = 0, \quad \forall \boldsymbol{w}_{h,\gamma}^{\Delta t} \in \boldsymbol{\Sigma}_{h,\gamma}^{\Delta t}, \tag{4.8}$$

$$\|\boldsymbol{\Pi}_{h,\gamma}^{\Delta t}\boldsymbol{v}_\gamma\|_{L^2(0,T;\boldsymbol{\Sigma}_\gamma)} \le C\left(\|\boldsymbol{v}_\gamma\|_{L^2(0,T;\boldsymbol{H}_\gamma^\varepsilon)} + \|\operatorname{div}_\tau\boldsymbol{v}_\gamma\|_{L^2(0,T;M_\gamma)}\right). \tag{4.9}$$

4.2. **Discrete inf-sup condition.**

Lemma 4.1 (Discrete inf-sup condition)**.** *There exists a constant $\beta > 0$ independent of h and Δt such that*

$$\forall \mu_h^{\Delta t} \in M_h^{\Delta t}, \qquad \sup_{\boldsymbol{0}\neq \boldsymbol{v}_h^{\Delta t}\in \boldsymbol{\Sigma}_h^{\Delta t}} \frac{b^T(\boldsymbol{v}_h^{\Delta t}, \mu_h^{\Delta t})}{\|\boldsymbol{v}_h^{\Delta t}\|_{L^2(0,T;\boldsymbol{\Sigma})}} \ge \beta \|\mu_h^{\Delta t}\|_{L^2(0,T;M)}. \tag{4.10}$$

Proof. We generalize the arguments used for the steady-state problem in [22] and adopt the techniques in [32] to handle the nonmatching grids in both space and time. Note that, unlike [32], no mortars are needed here and the grid in the fracture can be finer or coarser than those in the subdomains. Let $\mu_h^{\Delta t} = \left(\mu_{h,1}^{\Delta t}, \mu_{h,2}^{\Delta t}, \mu_{h,\gamma}^{\Delta t}\right) \in M_h^{\Delta t}$ be given, we construct an element $\boldsymbol{v}_h^{\Delta t} \in \boldsymbol{\Sigma}_h^{\Delta t}$ such that $b^T(\boldsymbol{v}_h^{\Delta t}, \mu_h^{\Delta t}) = \|\mu_h^{\Delta t}\|^2_{L^2(0,T;M)}$ and $\|\boldsymbol{v}_h^{\Delta t}\|_{L^2(0,T;\boldsymbol{\Sigma})} \le C\|\mu_h^{\Delta t}\|_{L^2(0,T;M)}$ where C is independent of h and Δt.

For a.e. $t \in (0,T)$, consider the auxiliary problem, for $i = 1, 2$:

$$\begin{aligned} -\Delta\varphi_i(\cdot,t) &= \mu_{h,i}^{\Delta t}(\cdot,t), && \text{on } \Omega, \\ \varphi_i(\cdot,t) &= 0, && \text{on } \partial\Omega_i \setminus \gamma, \\ -\nabla\varphi_i(\cdot,t)\cdot\boldsymbol{n}_i &= \mu_{h,\gamma}^{\Delta t}(\cdot,t) && \text{on } \gamma. \end{aligned} \tag{4.11}$$

Then, for a.e. $t \in (0,T)$, there exists a unique weak solution $\varphi_i(t) \in H^{1+\varepsilon}(\Omega_i)$ to (4.11), for $i = 1, 2$. Let

$$\boldsymbol{v}_i(t) := -\nabla\varphi_i(t), \text{ for a.e. } t \in (0,T), \quad \text{and } \boldsymbol{v}_{h,i}^{\Delta t} = \boldsymbol{\Pi}_{h,i}^{\Delta t}\boldsymbol{v}_i, \quad i = 1, 2. \tag{4.12}$$

As $\operatorname{div}\boldsymbol{\Sigma}_{h,i} = M_{h,i}$, we deduce that

$$\operatorname{div}\boldsymbol{v}_{h,i}^{\Delta t} = \operatorname{div}\boldsymbol{\Pi}_{h,i}^{\Delta t}\boldsymbol{v}_i = \mathcal{P}_{h,i}^{\Delta t}\operatorname{div}\boldsymbol{v}_i = \mathcal{P}_{h,i}^{\Delta t}\mu_{h,i}^{\Delta t} = \mu_{h,i}^{\Delta t}. \tag{4.13}$$

Similarly, we let $\varphi_\gamma \in H^{1+\varepsilon}(\gamma)$ be the solution of the following problem

$$\begin{aligned} -\Delta_\tau\varphi_\gamma(\cdot,t) &= \mu_{h,\gamma}^{\Delta t}(\cdot,t) + [\![\boldsymbol{v}_h^{\Delta t}\cdot\boldsymbol{n}]\!](\cdot,t), && \text{on } \gamma, \\ \varphi_\gamma(\cdot,t) &= 0, && \text{on } \partial\gamma, \end{aligned} \tag{4.14}$$

where $[\![\boldsymbol{v}_h^{\Delta t}\cdot\boldsymbol{n}]\!] = \boldsymbol{v}_{h,1}^{\Delta t}\cdot\boldsymbol{n}_1|_\gamma + \boldsymbol{v}_{h,2}^{\Delta t}\cdot\boldsymbol{n}_2|_\gamma$ and $\boldsymbol{v}_{h,i}^{\Delta t}$ $(i=1,2)$ are given by (4.12). Note that because of the nonmatching grids, $[\![\boldsymbol{v}_h^{\Delta t}(t)\cdot\boldsymbol{n}]\!] \neq 0$ even though $[\![\boldsymbol{v}(t)\cdot\boldsymbol{n}]\!] = 0$. Let

$$\boldsymbol{v}_\gamma(t) := -\nabla_\tau\varphi_\gamma(t), \text{ for a.e. } t\in(0,T), \quad \text{and } \boldsymbol{v}_{h,\gamma}^{\Delta t} = \boldsymbol{\Pi}_{h,\gamma}^{\Delta t}\boldsymbol{v}_\gamma. \tag{4.15}$$

As $\operatorname{div}\boldsymbol{\Sigma}_{h,\gamma} = M_{h,\gamma}$, we deduce that

$$\begin{aligned}\operatorname{div}_\tau \boldsymbol{v}_{h,\gamma}^{\Delta t} &= \operatorname{div}_\tau \boldsymbol{\Pi}_{h,\gamma}^{\Delta t}\boldsymbol{v}_\gamma = \mathcal{P}_{h,\gamma}^{\Delta t}\operatorname{div}_\tau \boldsymbol{v}_\gamma\\ &= \mathcal{P}_{h,\gamma}^{\Delta t}\left(\mu_{h,\gamma}^{\Delta t} + [\![\boldsymbol{v}_h^{\Delta t}\cdot\boldsymbol{n}]\!]\right) = \mu_{h,\gamma}^{\Delta t} + \mathcal{P}_{h,\gamma}^{\Delta t}[\![\boldsymbol{v}_h^{\Delta t}\cdot\boldsymbol{n}]\!].\end{aligned} \tag{4.16}$$

We have constructed $\boldsymbol{v}_h^{\Delta t} = \left(\boldsymbol{v}_{h,1}^{\Delta t}, \boldsymbol{v}_{h,2}^{\Delta t}, \boldsymbol{v}_{h,\gamma}^{\Delta t}\right) \in \boldsymbol{\Sigma}_h^{\Delta t}$ such that

$$\begin{aligned}b^T(\boldsymbol{v}_h^{\Delta t}, \mu_h^{\Delta t}) &= \int_0^T \Bigg(\sum_{i=1}^2 (\mu_{h,i}^{\Delta t}, \mu_{h,i}^{\Delta t})_{\Omega_i} + (\mu_{h,\gamma}^{\Delta t} + \mathcal{P}_{h,\gamma}^{\Delta t}[\![\boldsymbol{v}_h^{\Delta t}\cdot\boldsymbol{n}]\!], \mu_{h,\gamma}^{\Delta t})_\gamma\\ &\qquad - ([\![\boldsymbol{v}_h^{\Delta t}\cdot\boldsymbol{n}]\!], \mu_{h,\gamma}^{\Delta t})_\gamma\Bigg)\\ &= \|\mu_h^{\Delta t}\|_{L^2(0,T;M)}^2,\end{aligned}$$

where we have used the property of the L^2 projection $\mathcal{P}_{h,\gamma}^{\Delta t}$ from $L^2(0,T;L^2(\gamma))$ onto $M_{h,\gamma}^{\Delta t}$ to obtain

$$\left(\mathcal{P}_{h,\gamma}^{\Delta t}[\![\boldsymbol{v}_h^{\Delta t}\cdot\boldsymbol{n}]\!] - [\![\boldsymbol{v}_h^{\Delta t}\cdot\boldsymbol{n}]\!], \mu_{h,\gamma}^{\Delta t}\right)_\gamma = 0, \quad \text{for } \mu_{h,\gamma}^{\Delta t}\in M_{h,\gamma}^{\Delta t}.$$

Next we show that $\|\boldsymbol{v}_h^{\Delta t}\|_{L^2(0,T;\boldsymbol{\Sigma})} \leq C\|\mu_h^{\Delta t}\|_{L^2(0,T;M)}$. We have

(4.17)

$$\begin{aligned}\|\boldsymbol{v}_h^{\Delta t}\|_{L^2(0,T;\boldsymbol{\Sigma})}^2 &= \int_0^T \Bigg(\sum_{i=1}^2\left(\|\boldsymbol{v}_{h,i}^{\Delta t}\|_{0,\Omega_i}^2 + \|\operatorname{div}\boldsymbol{v}_{h,i}^{\Delta t}\|_{0,\Omega_i}^2\right) + \|\boldsymbol{v}_{h,\gamma}^{\Delta t}\|_{0,\gamma}^2\\ &\qquad + \|\operatorname{div}_\tau \boldsymbol{v}_{h,\gamma}^{\Delta t}\|_{0,\gamma}^2 + \sum_{i=1}^2\|\boldsymbol{v}_{h,i}^{\Delta t}\cdot\boldsymbol{n}_i\|_{0,\gamma}^2\Bigg)\\ &= \sum_{i=1}^2\left(\|\boldsymbol{\Pi}_{h,i}^{\Delta t}\boldsymbol{v}_i\|_{L^2(0,T;\boldsymbol{M}_i)}^2 + \|\mu_{h,i}^{\Delta t}\|_{L^2(0,T;M_i)}^2\right) + \|\boldsymbol{\Pi}_{h,\gamma}^{\Delta t}\boldsymbol{v}_\gamma\|_{L^2(0,T;\boldsymbol{M}_\gamma)}^2\\ &\qquad + \|\mu_{h,\gamma}^{\Delta t} + \mathcal{P}_{h,\gamma}^{\Delta t}[\![\boldsymbol{v}_h^{\Delta t}\cdot\boldsymbol{n}]\!]\|_{L^2(0,T;M_\gamma)}^2 + \sum_{i=1}^2\|\boldsymbol{\Pi}_{h,i}^{\Delta t}\boldsymbol{v}_i\cdot\boldsymbol{n}_i\|_{L^2(0,T;M_\gamma)}^2.\end{aligned}$$

We now control all terms on the right above. Using (4.5) we deduce, for $i=1,\,2$, that

$$\begin{aligned}\|\boldsymbol{\Pi}_{h,i}^{\Delta t}\boldsymbol{v}_i\|_{L^2(0,T;\boldsymbol{M}_i)}^2 &\leq C\left(\|\boldsymbol{v}_i\|_{L^2(0,T;\boldsymbol{H}_i^\varepsilon)}^2 + \|\operatorname{div}\boldsymbol{v}_i\|_{L^2(0,T;M_i)}^2\right)\\ &= C\left(\|\nabla\varphi_i\|_{L^2(0,T;\boldsymbol{H}_i^\varepsilon)}^2 + \|\mu_{h,i}^{\Delta t}\|_{L^2(0,T;M_i)}^2\right).\end{aligned} \tag{4.18}$$

By the elliptic regularity for the auxiliary problem (4.11) for a.e. $t\in(0,T)$, we have

$$\begin{aligned}\|\nabla\varphi_i\|_{L^2(0,T;\boldsymbol{H}_i^\varepsilon)}^2 &\leq C\|\varphi_i\|_{L^2(0,T;H^{1+\varepsilon}(\Omega_i))}^2\\ &\leq C_{\Omega_i}\left(\|\mu_{h,i}^{\Delta t}\|_{L^2(0,T;M_i)}^2 + \|\mu_{h,\gamma}^{\Delta t}\|_{L^2(0,T;M_\gamma)}^2\right).\end{aligned}$$

This and (4.18) imply that

$$\|\boldsymbol{\Pi}^{\Delta t}_{h,i}\boldsymbol{v}_i\|^2_{L^2(0,T;\boldsymbol{M}_i)} \le C\left(\|\mu^{\Delta t}_{h,i}\|^2_{L^2(0,T;M_i)} + \|\mu^{\Delta t}_{h,\gamma}\|^2_{L^2(0,T;M_\gamma)}\right),\ i=1,2. \tag{4.19}$$

Similarly, using (4.9), elliptic regularity for the auxiliary problem (4.14), (4.6) and (4.11), we obtain:

$$\begin{aligned}\|\boldsymbol{\Pi}^{\Delta t}_{h,\gamma}\boldsymbol{v}_\gamma\|^2_{L^2(0,T;\boldsymbol{M}_\gamma)} &\le C\left(\|\boldsymbol{v}_\gamma\|^2_{L^2(0,T;\boldsymbol{H}^\varepsilon_\gamma)} + \|\operatorname{div}_\tau \boldsymbol{v}_\gamma\|^2_{L^2(0,T;M_\gamma)}\right)\\ &= C\left(\|\nabla_\tau\varphi_\gamma\|^2_{L^2(0,T;\boldsymbol{H}^\varepsilon_\gamma)} + \|\mu^{\Delta t}_{h,\gamma} + [\![\boldsymbol{v}^{\Delta t}_h\cdot\boldsymbol{n}]\!]\|^2_{L^2(0,T;M_\gamma)}\right)\\ &\le C\left(\|\varphi_\gamma\|^2_{L^2(0,T;H^{1+\varepsilon}(\gamma))} + \|\mu^{\Delta t}_{h,\gamma} + [\![\boldsymbol{v}^{\Delta t}_h\cdot\boldsymbol{n}]\!]\|^2_{L^2(0,T;M_\gamma)}\right)\\ &\le C(C_\gamma+1)\,\|\mu^{\Delta t}_{h,\gamma} + [\![\boldsymbol{v}^{\Delta t}_h\cdot\boldsymbol{n}]\!]\|^2_{L^2(0,T;M_\gamma)}\\ &\le C\left(\|\mu^{\Delta t}_{h,\gamma}\|^2_{L^2(0,T;M_\gamma)} + \sum_{i=1}^2\|\boldsymbol{v}_i\cdot\boldsymbol{n}_i\|^2_{L^2(0,T;M_\gamma)}\right)\\ &\le C\|\mu^{\Delta t}_{h,\gamma}\|^2_{L^2(0,T;M_\gamma)}.\end{aligned} \tag{4.20}$$

As $\mathcal{P}^{\Delta t}_{h,\gamma}$ is an L^2 projection, and using the same argument as in (4.20), we have

$$\|\mu^{\Delta t}_{h,\gamma} + \mathcal{P}^{\Delta t}_{h,\gamma}[\![\boldsymbol{v}^{\Delta t}_h\cdot\boldsymbol{n}]\!]\|^2_{L^2(0,T;M_\gamma)} \le C\|\mu^{\Delta t}_{h,\gamma}\|^2_{L^2(0,T;M_\gamma)}. \tag{4.21}$$

Finally for the last term in (4.17), using (4.6) and (4.11) we deduce, for $i=1,\ 2$, that

$$\|\boldsymbol{\Pi}^{\Delta t}_{h,i}\boldsymbol{v}_i\cdot\boldsymbol{n}_i\|^2_{L^2(0,T;M_\gamma)} \le \|\boldsymbol{v}_i\cdot\boldsymbol{n}_i\|^2_{L^2(0,T;M_\gamma)} = \|\mu^{\Delta t}_{h,\gamma}\|^2_{L^2(0,T;M_\gamma)}. \tag{4.22}$$

It follows from (4.17) and (4.19)–(4.22) that $\|\boldsymbol{v}^{\Delta t}_h\|_{L^2(0,T;\boldsymbol{\Sigma})} \le C\|\mu^{\Delta t}_h\|_{L^2(0,T;M)}$, which completes the proof. □

4.3. **Discrete initial data.** Recall that $p_0 \in H^1_0(\Omega)$ and $p_{0,i} = p_0|_{\Omega_i}$, $i=1,\ 2$, $p_{0,\gamma} = p_0|_\gamma$. Define $u_0 \in \boldsymbol{M}^*$ such that

$$a(\boldsymbol{u}_0,\boldsymbol{v}) - b^*(\boldsymbol{v},p_0) = 0, \quad \forall \boldsymbol{v}\in\boldsymbol{M}^*, \tag{4.23}$$

where $b^*(\boldsymbol{v},\mu) = -\sum_{i=1}^2(\boldsymbol{v}_i,\nabla\mu_i)_{\Omega_i} - (\boldsymbol{v}_\gamma,\nabla_\tau\mu_\gamma)_\gamma$. The problem has a unique solution due to the Lax-Milgram Theorem, cf. (2.7). It clearly holds that

$$a(\boldsymbol{u}_0,\boldsymbol{v}) - b(\boldsymbol{v},p_0) = 0, \quad \forall \boldsymbol{v}\in\boldsymbol{\Sigma}. \tag{4.24}$$

Assume further that p_0 is sufficiently smooth, so that $\boldsymbol{u}_0\in\boldsymbol{\Sigma}$. It is easy to see that the solution to the continuous problem (2.5) satisfies $\boldsymbol{u}_i(0) = \boldsymbol{u}_{0,i}$ for $i=1,2$, $\boldsymbol{u}_\gamma(0) = \boldsymbol{u}_{0,\gamma}$.

We now define the discrete initial solution $(\boldsymbol{u}_{h,0},p_{h,0})\in\boldsymbol{\Sigma}_h\times M_h$ as the elliptic projection of $(\boldsymbol{u}_0,p_0)$:

$$\begin{aligned}&a(\boldsymbol{u}_{h,0},\boldsymbol{v}_h) - b(\boldsymbol{v}_h,p_{h,0}) = a(\boldsymbol{u}_0,\boldsymbol{v}_h) - b(\boldsymbol{v}_h,p_0) = 0, \quad \forall \boldsymbol{v}_h\in\boldsymbol{\Sigma}_h,\\ &b(\boldsymbol{u}_{h,0},\mu_h) = b(\boldsymbol{u}_0,\mu_h) \quad \forall \mu_h\in M_h.\end{aligned} \tag{4.25}$$

The well-posedness of (4.25) (with nonmatching meshes) is shown in [22], and the following estimates hold

$$\|\boldsymbol{u}_{h,0}\|_{\boldsymbol{\Sigma}} + \|p_{h,0}\|_M \le C\|\boldsymbol{u}_0\|_{\boldsymbol{\Sigma}}, \tag{4.26}$$

$$\|\boldsymbol{u}_0 - \boldsymbol{u}_{h,0}\|_{\boldsymbol{M}^*} + \|p_0 - p_{h,0}\|_M \le C\left(\|\boldsymbol{u}_0 - \boldsymbol{\Pi}_h\boldsymbol{u}_0\|_{\boldsymbol{M}^*} + \|p_0 - \mathcal{P}_h p_0\|_M\right), \tag{4.27}$$

where $\mathbf{\Pi}_h \boldsymbol{v} = (\mathbf{\Pi}_{h,1}\boldsymbol{v}_1, \mathbf{\Pi}_{h,2}\boldsymbol{v}_2, \mathbf{\Pi}_{h,\gamma}\boldsymbol{v}_\gamma)$ and $\mathcal{P}_h\mu = (\mathcal{P}_{h,1}\mu_1, \mathcal{P}_{h,2}\mu_2, \mathcal{P}_{h,\gamma}\mu_\gamma)$.

In the subsequent analysis, we set

$$(p_h^{\Delta t})_0^- = p_{h,0}, \quad (\boldsymbol{u}_h^{\Delta t})^- = \boldsymbol{u}_{h,0}. \tag{4.28}$$

4.4. **Existence, uniqueness, and stability with respect to data.**

Lemma 4.2 (Summation in time)**.** *The following holds for any $\varphi(\boldsymbol{x},\cdot)$ in $W_i^{\Delta t}$, $i = 1,\ 2,\ \gamma$, where $\boldsymbol{x} \in \Omega_i$ if $i = 1,\ 2$, or $\boldsymbol{x} \in \gamma$ if $i = \gamma$:*

$$\int_0^T \widetilde{\partial}_t\varphi\varphi = \frac{1}{2}\left((\varphi_{N_i}^-)^2 - (\varphi_0^-)^2\right) + \frac{1}{2}\sum_{n=1}^{N_i}([\varphi]_{n-1})^2. \tag{4.29}$$

Lemma 4.2 is proved by using the definition of $\widetilde{\partial}_t\varphi$ (3.1). We refer to [32, Lemma 4.3] for more details of the proof.

For convenience of the presentation, for $\mu = (\mu_1, \mu_2, \mu_\gamma) \in M$, we denote

$$\|\mu\|^2_{M,DG} = \sum_{i=1,2,\gamma}\|\mu_i\|^2_{M_i,DG}, \quad \text{where} \quad \|\mu_i\|^2_{M_i,DG} = \|(\mu_i)^-_{N_i}\|^2_{M_i} + \sum_{n=1}^{N_i}\|[\mu_i]_{n-1}\|^2_{M_i}. \tag{4.30}$$

Similarly, for $\boldsymbol{v} = (\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_\gamma) \in \boldsymbol{M}^*$, let

$$\|\boldsymbol{v}\|^2_{\boldsymbol{M}^*,DG} = \sum_{i=1,2,\gamma}\|\boldsymbol{v}_i\|^2_{\boldsymbol{M}_i,DG} + \sum_{i=1}^{2}\|\boldsymbol{v}_i\cdot\boldsymbol{n}_i\|^2_{0,\gamma,DG}, \tag{4.31}$$

where $\|\boldsymbol{v}_i\|^2_{\boldsymbol{M}_i,DG} = \|(\boldsymbol{v}_i)^-_{N_i}\|^2_{\boldsymbol{M}_i} + \sum_{n=1}^{N_i}\|[\boldsymbol{v}_i]_{n-1}\|^2_{\boldsymbol{M}_i}$, and

$$\|\boldsymbol{v}_i\cdot\boldsymbol{n}_i\|^2_{0,\gamma,DG} = \|(\boldsymbol{v}_i\cdot\boldsymbol{n}_i)^-_{N_i}\|^2_{0,\gamma} + \sum_{n=1}^{N_i}\|[\boldsymbol{v}_i\cdot\boldsymbol{n}_i]_{n-1}\|^2_{0,\gamma}.$$

Theorem 4.1. *The space-time mixed finite element method for (2.3) has a unique solution and the following estimate holds for some constant $C > 0$ independent of h and Δt:*

$$\|p_h^{\Delta t}\|_{M,DG} + \|\boldsymbol{u}_h^{\Delta t}\|_{L^2(0,T;\boldsymbol{M}^*)} + \|p_h^{\Delta t}\|_{L^2(0,T;M)} \le C\left(\|q\|_{L^2(0,T;M)} + \|\boldsymbol{u}_0\|_{\boldsymbol{\Sigma}}\right). \tag{4.32}$$

Proof. We choose $\boldsymbol{v}_h^{\Delta t} = \boldsymbol{u}_h^{\Delta t}$ in $(3.2)_1$ and $\mu_h^{\Delta t} = p_h^{\Delta t}$ in in $(3.2)_2$, then by adding the two resulting equations, we obtain

$$a^T(\boldsymbol{u}_h^{\Delta t}, \boldsymbol{u}_h^{\Delta t}) + c_s^T(\widetilde{\partial}_t p_h^{\Delta t}, p_h^{\Delta t}) = L_q^T(p_h^{\Delta t}). \tag{4.33}$$

Using (2.7) we have

$$a^T(\boldsymbol{u}_h^{\Delta t}, \boldsymbol{u}_h^{\Delta t}) \ge \sum_{i=1}^{2}\|\boldsymbol{K}_i^{-\frac{1}{2}}\boldsymbol{u}_{h,i}^{\Delta t}\|^2_{L^2(0,T;\boldsymbol{M}_i)} + \|\boldsymbol{K}_\gamma^{-\frac{1}{2}}\boldsymbol{u}_{h,\gamma}^{\Delta t}\|^2_{L^2(0,T;\boldsymbol{M}_\gamma)} + \min(1, 2\xi - 1)\sum_{i=1}^{2}\|\kappa_\gamma^{-\frac{1}{2}}\boldsymbol{u}_{h,i}^{\Delta t}\cdot\boldsymbol{n}\|^2_{L^2(0,T;M_\gamma)}. \tag{4.34}$$

Next, using Lemma 4.2, (A1)–(A2), and the notation (4.30), we deduce that

$$c_s^T(\widetilde{\partial}_t p_h^{\Delta t}, p_h^{\Delta t}) \ge \frac{s_-}{2}\|p_h^{\Delta t}\|^2_{M,DG} - \frac{s_+}{2}\|p_{h,0}\|^2_M, \tag{4.35}$$

where $p_{h,0}$ is constructed in (4.25) and we have used $(p_{h,i}^{\Delta t})^- = p_{h,0,i}$, for $i = 1, 2, \gamma$. For the term on the right-hand side of (4.33), applying Young's inequality for $\varepsilon > 0$ yields

$$L_q(p_h^{\Delta t}) \leq \varepsilon \|p_h^{\Delta t}\|^2_{L^2(0,T;M)} + \frac{1}{2\varepsilon}\|q\|^2_{L^2(0,T;M)}. \tag{4.36}$$

To bound $\|p_h^{\Delta t}\|^2_{L^2(0,T;M)}$, we use $(3.2)_1$, (A5), and (2.4) to obtain

$$\begin{aligned} b^T(\boldsymbol{v}_h^{\Delta t}, p_h^{\Delta t}) &= a^T(\boldsymbol{u}_h^{\Delta t}, \boldsymbol{v}_h^{\Delta t}) \\ &\leq C\left(K_-^{-1}, K_{\gamma-}^{-1}, \kappa_{\gamma-}^{-1}, \xi\right) \|\boldsymbol{u}_h^{\Delta t}\|_{L^2(0,T;\boldsymbol{M}^*)} \|\boldsymbol{v}_h^{\Delta t}\|_{L^2(0,T;\boldsymbol{M}^*)}, \end{aligned}$$

which, combined with Lemma 4.1, implies

$$\|p_h^{\Delta t}\|_{L^2(0,T;M)} \leq C\left(K_-^{-1}, K_{\gamma-}^{-1}, \kappa_{\gamma-}^{-1}, \xi, \beta^{-1}\right) \|\boldsymbol{u}_h^{\Delta t}\|_{L^2(0,T;\boldsymbol{M}^*)}. \tag{4.37}$$

Finally, (4.32) is obtained by combining (4.33)–(4.37), choosing $\varepsilon > 0$ sufficiently small and using (4.26). Existence and uniqueness of a solution follows from (4.32) by taking $q = 0$ and $p_0 = 0$ and concluding that the homogeneous system has only the zero solution. □

5. A priori error analysis

We establish a priori error estimates for the solution of the discrete problem (3.2). We first recall some properties of the space-time interpolants.

5.1. **Interpolation estimates.** In the following analysis, we use the same order of space-time approximation spaces for the subdomains and for the fracture. In space, let $\rho \geq 0$ be the order of the Raviart-Thomas space $M_{h,i} \times \boldsymbol{\Sigma}_{h,i}$, while in time, we denote by $k \geq 0$ the order of the polynomials in $W_i^{\Delta t}$, for $i = 1, 2, \gamma$. The space-time projection operators $\mathcal{P}_{h,i}^{\Delta t}$ and $\boldsymbol{\Pi}_{h,i}^{\Delta}$, $i = 1, 2, \gamma$, defined in Subsection 4.1, have the following approximation properties: for $1 \leq r_\rho \leq \rho + 1, 1 \leq r_k \leq k + 1$,

$$\|\mu_i - \mathcal{P}_{h,i}^{\Delta t}\mu_i\|_{L^2(0,T;M_i)} \leq C\|\mu_i\|_{H^{r_k}(0,T;H^{r_\rho}(\Omega_i))}(h^{r_\rho} + \Delta t^{r_k}), \tag{5.1}$$

$$\|\mu_\gamma - \mathcal{P}_{h,\gamma}^{\Delta t}\mu_\gamma\|_{L^2(0,T;M_\gamma)} \leq C\|\mu_\gamma\|_{H^{r_k}(0,T;H^{r_\rho}(\gamma))}(h^{r_\rho} + \Delta t^{r_k}), \tag{5.2}$$

$$\|\boldsymbol{v}_i - \boldsymbol{\Pi}_{h,i}^{\Delta t}\boldsymbol{v}_i\|_{L^2(0,T;\boldsymbol{M}_i)} \leq C\|\boldsymbol{v}_i\|_{H^{r_k}(0,T;\boldsymbol{H}^{r_\rho}(\Omega_i))}(h^{r_\rho} + \Delta t^{r_k}), \tag{5.3}$$

$$\|\boldsymbol{v}_\gamma - \boldsymbol{\Pi}_{h,\gamma}^{\Delta t}\boldsymbol{v}_\gamma\|_{L^2(0,T;\boldsymbol{M}_\gamma)} \leq C\|\boldsymbol{v}_\gamma\|_{H^{r_k}(0,T;\boldsymbol{H}^{r_\rho}(\gamma))}(h^{r_\rho} + \Delta t^{r_k}), \tag{5.4}$$

$$\|\operatorname{div}(\boldsymbol{v}_i - \boldsymbol{\Pi}_{h,i}^{\Delta t}\boldsymbol{v}_i)\|_{L^2(0,T;M_i)} \leq C\|\operatorname{div}\boldsymbol{v}_i\|_{H^{r_k}(0,T;H^{r_\rho}(\Omega_i))}(h^{r_\rho} + \Delta t^{r_k}), \tag{5.5}$$

$$\|\operatorname{div}_\tau(\boldsymbol{v}_\gamma - \boldsymbol{\Pi}_{h,\gamma}^{\Delta t}\boldsymbol{v}_\gamma)\|_{L^2(0,T;M_\gamma)} \leq C\|\operatorname{div}_\tau \boldsymbol{v}_\gamma\|_{H^{r_k}(0,T;H^{r_\rho}(\gamma))}(h^{r_\rho} + \Delta t^{r_k}), \tag{5.6}$$

$$\|(\boldsymbol{v}_i - \boldsymbol{\Pi}_{h,i}^{\Delta t}\boldsymbol{v}_i)\cdot\boldsymbol{n}_i\|_{L^2(0,T;M_\gamma)} \leq C\|\boldsymbol{v}_i\cdot\boldsymbol{n}_i\|_{H^{r_k}(0,T;H^{r_\rho}(\gamma))}(h^{r_\rho} + \Delta t^{r_k}). \tag{5.7}$$

From the stability of L^2 projection in L^∞ [15], we also have the following properties:

$$\|\mu_i - \mathcal{P}_{h,i}^{\Delta t}\mu_i\|_{L^\infty(0,T;M_i)} \leq C\|\mu_i\|_{W^{r_k,\infty}(0,T;H^{r_\rho}(\Omega_i))}(h^{r_\rho} + \Delta t^{r_k}), \tag{5.8}$$

$$\|\mu_\gamma - \mathcal{P}_{h,\gamma}^{\Delta t}\mu_\gamma\|_{L^\infty(0,T;M_\gamma)} \leq C\|\mu_\gamma\|_{W^{r_k,\infty}(0,T;H^{r_\rho}(\gamma))}(h^{r_\rho} + \Delta t^{r_k}). \tag{5.9}$$

5.2. **A priori error estimates.**

Theorem 5.1. *Assume that* $\Delta t \leq C\Delta t_i$, *for* $i = 1, 2, \gamma$. *For* $(\boldsymbol{u}, p)$ *the solution of problem* (2.5) *and* $(\boldsymbol{u}_h^{\Delta t}, p_h^{\Delta t})$ *the solution of problem* (3.2), *if* $\boldsymbol{u}$ *and* p *are sufficiently smooth, then, for* $1 \leq r_\rho \leq \rho + 1, 1 \leq r_k \leq k + 1$,

(5.10)

$$\begin{aligned}
&\|p - p_h^{\Delta t}\|_{M,DG} + \|\boldsymbol{u} - \boldsymbol{u}_h^{\Delta t}\|_{L^2(0,T;\boldsymbol{M}^*)} + \|p - p_h^{\Delta t}\|_{L^2(0,T;M)} \\
&\quad \leq C\Bigg((h^{r_\rho} + \Delta t^{r_k}) \Big(\|\boldsymbol{u}\|_{H^{r_k}(0,T;\boldsymbol{H}^{r_\rho})} + \sum_{i=1}^{2} \|\boldsymbol{u}_i \cdot \boldsymbol{n}_i\|_{H^{r_k}(0,T;H^{r_\rho}(\gamma))} \\
&\qquad\qquad + \|p_\gamma\|_{H^{r_k}(0,T;H^{r_\rho}(\gamma))} + \Delta t^{-\frac{1}{2}} \|p\|_{W^{r_k,\infty}(0,T;H^{r_\rho})} \Big) \\
&\qquad\qquad + h^{r_\rho} \left(\|\boldsymbol{u}_0\|_{\boldsymbol{H}^{r_\rho}} + \sum_{i=1}^{2} \|\boldsymbol{u}_{0,i} \cdot \boldsymbol{n}_i\|_{H^{r_\rho}(\gamma)} + \|p_0\|_{H^{r_\rho}} \right) \Bigg),
\end{aligned}$$

where $C > 0$ *is a constant independent of* h *and* Δt.

Proof. By subtracting (3.2) from (2.5), we obtain the error equations:

$$\begin{aligned}
a^T\left(\boldsymbol{u} - \boldsymbol{u}_h^{\Delta t}, \boldsymbol{v}_h^{\Delta t}\right) - b^T\left(\boldsymbol{v}_h^{\Delta t}, p - p_h^{\Delta t}\right) &= 0 \quad \forall \boldsymbol{v}_h^{\Delta t} \in \boldsymbol{\Sigma}_h^{\Delta t}, \\
c_s^T\left(\partial_t p - \widetilde{\partial}_t p_h^{\Delta t}, \mu_h^{\Delta t}\right) + b^T\left(\boldsymbol{u} - \boldsymbol{u}_h^{\Delta t}, \mu_h^{\Delta t}\right) &= 0 \quad \forall \mu_h^{\Delta t} \in M_h^{\Delta t}.
\end{aligned} \tag{5.11}$$

Since $\operatorname{div} \boldsymbol{\Sigma}_{h,i} = M_{h,i}$ for $i = 1$, 2, and $\operatorname{div}_\tau \boldsymbol{\Sigma}_{h,\gamma} = M_{h,\gamma}$, the L^2- projection $\mathcal{P}_h^{\Delta t}$ satisfies:

$$\begin{aligned}
\left(\mathcal{P}_{h,i}^{\Delta t}\mu_i - \mu_i, \operatorname{div} \boldsymbol{v}_{h,i}^{\Delta t}\right)_{\Omega_i^T} &= 0, \quad \forall \boldsymbol{v}_{h,i}^{\Delta t} \in \boldsymbol{\Sigma}_{h,i}^{\Delta t},\ i = 1, 2, \\
\left(\mathcal{P}_{h,\gamma}^{\Delta t}\mu_\gamma - \mu_\gamma, \operatorname{div}_\tau \boldsymbol{v}_{h,\gamma}^{\Delta t}\right)_{\gamma^T} &= 0, \quad \forall \boldsymbol{v}_{h,\gamma}^{\Delta t} \in \boldsymbol{\Sigma}_{h,\gamma}^{\Delta t}.
\end{aligned}$$

Using these equations and the properties (4.3) and (4.7) of the interpolant $\boldsymbol{\Pi}_h^{\Delta t}$, we rewrite (5.11) equivalently as

(5.12)

$$\begin{aligned}
a^T\left(\boldsymbol{u} - \boldsymbol{u}_h^{\Delta t}, \boldsymbol{v}_h^{\Delta t}\right) - b^T\left(\boldsymbol{v}_h^{\Delta t}, \mathcal{P}_h^{\Delta t} p - p_h^{\Delta t}\right) + \left([\![\boldsymbol{v}_h^{\Delta t} \cdot \boldsymbol{n}]\!], p_\gamma - \mathcal{P}_{h,\gamma}^{\Delta t} p_\gamma\right)_{\gamma^T} &= 0, \\
\forall \boldsymbol{v}_h^{\Delta t} &\in \boldsymbol{\Sigma}_h^{\Delta t}, \\
c_s^T\left(\partial_t p - \widetilde{\partial}_t p_h^{\Delta t}, \mu_h^{\Delta t}\right) + b^T\left(\boldsymbol{\Pi}_h^{\Delta t}\boldsymbol{u} - \boldsymbol{u}_h^{\Delta t}, \mu_h^{\Delta t}\right) + \left([\![(\boldsymbol{\Pi}_h^{\Delta t}\boldsymbol{u} - \boldsymbol{u}) \cdot \boldsymbol{n}]\!], \mu_{h,\gamma}^{\Delta t}\right)_{\gamma^T} &= 0, \\
\forall \mu_h^{\Delta t} &\in M_h^{\Delta t}.
\end{aligned}$$

Choosing $\boldsymbol{v}_h^{\Delta t} = \boldsymbol{\Pi}_h^{\Delta t}\boldsymbol{u} - \boldsymbol{u}_h^{\Delta t}$ and $\mu_h^{\Delta t} = \mathcal{P}_h^{\Delta t} p - p_h^{\Delta t}$ and adding the resulting equations, we obtain:

(5.13)

$$\begin{aligned}
&a^T\left(\boldsymbol{\Pi}_h^{\Delta t}\boldsymbol{u} - \boldsymbol{u}_h^{\Delta t}, \boldsymbol{\Pi}_h^{\Delta t}\boldsymbol{u} - \boldsymbol{u}_h^{\Delta t}\right) + c_s^T\left(\partial_t p - \widetilde{\partial}_t p_h^{\Delta t}, \mathcal{P}_h^{\Delta t} p - p_h^{\Delta t}\right) \\
&\quad = a^T\left(\boldsymbol{\Pi}_h^{\Delta t}\boldsymbol{u} - \boldsymbol{u}, \boldsymbol{\Pi}_h^{\Delta t}\boldsymbol{u} - \boldsymbol{u}_h^{\Delta t}\right) - \left([\![(\boldsymbol{\Pi}_h^{\Delta t}\boldsymbol{u} - \boldsymbol{u}_h^{\Delta t}) \cdot \boldsymbol{n}]\!], p_\gamma - \mathcal{P}_{h,\gamma}^{\Delta t} p_\gamma\right)_{\gamma^T} \\
&\qquad - \left([\![(\boldsymbol{\Pi}_h^{\Delta t}\boldsymbol{u} - \boldsymbol{u}) \cdot \boldsymbol{n}]\!], \mathcal{P}_{h,\gamma}^{\Delta t} p_\gamma - p_{h,\gamma}^{\Delta t}\right)_{\gamma^T}.
\end{aligned}$$

To bound the second term on the left-hand side of (5.13), we follow the techniques in [32]. In particular, using the notation (3.1), we have

$$
\begin{aligned}
c_s^T\left(\partial_t p-\widetilde{\partial}_t p_h^{\Delta t}, \mathcal{P}_h^{\Delta t} p-p_h^{\Delta t}\right) &= c_s^T\left(\widetilde{\partial}_t(p-p_h^{\Delta t}), \mathcal{P}_h^{\Delta t} p-p_h^{\Delta t}\right) \\
&= c_s^T\left(\widetilde{\partial}_t(p-p_h^{\Delta t}), p-p_h^{\Delta t}\right)+c_s^T\left(\widetilde{\partial}_t(p-p_h^{\Delta t}), \mathcal{P}_h^{\Delta t} p-p\right):=I_1+I_2.
\end{aligned} \tag{5.14}
$$

Using Lemma 4.2 and (4.30), we have

$$
I_1 \geq \frac{s_-}{2}\|p-p_h^{\Delta t}\|_{M,DG}-\frac{s_+}{2}\|p_0-p_{h,0}\|_M^2. \tag{5.15}
$$

The term I_2 has contributions from Ω_i, $i=1,\,2$, and γ: $I_2=I_2^1+I_2^2+I_2^\gamma$. For I_2^i, $i=1,\,2$, using (3.1), we write

$$
\begin{aligned}
I_2^i = &\sum_{n=1}^{N_i}\int_{t_i^{n-1}}^{t_i^n} s_i\left(\partial_t(p_i-p_{h,i}^{\Delta t}), \mathcal{P}_{h,i}^{\Delta t}p_i-p_i\right)_{\Omega_i} \\
&+\sum_{n=1}^{N_i} s_i\left([p_i-p_{h,i}^{\Delta t}]_{n-1}, (\mathcal{P}_{h,i}^{\Delta t}p_i-p_i)_{n-1}^+\right)_{\Omega_i}.
\end{aligned} \tag{5.16}
$$

Due to the orthogonality property of $\mathcal{P}_{h,i}^{\Delta t}$,

$$
\left(\partial_t p_{h,i}^{\Delta t}, \mathcal{P}_{h,i}^{\Delta t}p_i-p_i\right)_{\Omega_i} = \left(\partial_t \mathcal{P}_h^{\Delta t}p_i, \mathcal{P}_{h,i}^{\Delta t}p_i-p_i\right)_{\Omega_i}=0,
$$

thus

$$
\begin{aligned}
\int_{t_i^{n-1}}^{t_i^n} s_i\left(\partial_t(p_i-p_{h,i}^{\Delta t}), \mathcal{P}_{h,i}^{\Delta t}p_i-p_i\right)_{\Omega_i} &= \int_{t_i^{n-1}}^{t_i^n} s_i\left(\partial_t(p_i-\mathcal{P}_{h,i}^{\Delta t}p_i), \mathcal{P}_{h,i}^{\Delta t}p_i-p_i\right)_{\Omega_i} \\
&= -\frac{s_i}{2}\int_{t_i^{n-1}}^{t_i^n}\partial_t\|\mathcal{P}_{h,i}^{\Delta t}p_i-p_i\|_{\Omega_i}^2 = -\frac{s_i}{2}\|\mathcal{P}_{h,i}^{\Delta t}p_i-p_i\|_{\Omega_i}^2\Big|_{t_i^{n-1}}^{t_i^n}.
\end{aligned} \tag{5.17}
$$

We perform similar calculations for I_2^γ and combine with (5.14)–(5.17) to deduce that

$$
\begin{aligned}
c_s^T&\left(\partial_t p-\widetilde{\partial}_t p_h^{\Delta t}, \mathcal{P}_h^{\Delta t}p-p_h^{\Delta t}\right) \geq \frac{s_-}{2}\|p-p_h^{\Delta t}\|_{M,DG}-\frac{s_+}{2}\|p_0-p_{h,0}\|_M^2 \\
&-\frac{1}{2}\sum_{i=1}^{2}\sum_{n=1}^{N_i} s_i\|\mathcal{P}_{h,i}^{\Delta t}p_i-p_i\|_{\Omega_i}^2\Big|_{t_i^{n-1}}^{t_i^n}+\sum_{i=1}^{2}\sum_{n=1}^{N_i} s_i\left([p_i-p_{h,i}^{\Delta t}]_{n-1}, (\mathcal{P}_{h,i}^{\Delta t}p_i-p_i)_{n-1}^+\right)_{M_i} \\
&-\frac{s_\gamma}{2}\sum_{n=1}^{N_\gamma}\|\mathcal{P}_{h,\gamma}^{\Delta t}p_\gamma-p_\gamma\|_{M_\gamma}^2\Big|_{t_\gamma^{n-1}}^{t_\gamma^n}+\sum_{n=1}^{N_\gamma} s_\gamma\left([p_\gamma-p_{h,\gamma}^{\Delta t}]_{n-1}, (\mathcal{P}_{h,\gamma}^{\Delta t}p_\gamma-p_\gamma)_{n-1}^+\right)_\gamma.
\end{aligned} \tag{5.18}
$$

Using (5.18), (2.7), (4.4), Assumptions (A1)–(A5), and the Cauchy-Schwarz and Young inequalities, we obtain from (5.13) that

(5.19)

$$
\begin{aligned}
&\sum_{i=1,2,\gamma} \|\boldsymbol{K}_i^{-\frac{1}{2}}(\boldsymbol{\Pi}_{h,i}^{\Delta t}\boldsymbol{u} - \boldsymbol{u}_{h,i}^{\Delta t})\|^2_{L^2(0,T;\boldsymbol{M}_i)} + \sum_{i=1}^{2} \|\kappa_\gamma^{-\frac{1}{2}}(\boldsymbol{\Pi}_{h,i}^{\Delta t}\boldsymbol{u}_i - \boldsymbol{u}_{h,i}^{\Delta t})\cdot \boldsymbol{n}_i\|^2_{L^2(0,T;M_\gamma)} \\
&\quad + \|p - p_h^{\Delta t}\|^2_{M,DG} \\
&\le C\Bigg(\|\boldsymbol{\Pi}_h^{\Delta t}\boldsymbol{u} - \boldsymbol{u}\|_{L^2(0,T;\boldsymbol{M}^*)} \, \|\boldsymbol{\Pi}_h^{\Delta t}\boldsymbol{u} - \boldsymbol{u}_h^{\Delta t}\|_{L^2(0,T;\boldsymbol{M}^*)} \\
&\qquad + \sum_{i=1}^{2} \|(\boldsymbol{\Pi}_{h,i}^{\Delta t}\boldsymbol{u}_i - \boldsymbol{u}_{h,i}^{\Delta t})\cdot \boldsymbol{n}_i\|_{L^2(0,T;M_\gamma)} \, \|p_\gamma - \mathcal{P}_{h,\gamma}^{\Delta t} p_\gamma\|_{L^2(0,T;M_\gamma)} \\
&\qquad + \sum_{i=1}^{2} \|(\boldsymbol{\Pi}_{h,i}^{\Delta t}\boldsymbol{u}_i - \boldsymbol{u}_i)\cdot \boldsymbol{n}_i\|_{L^2(0,T;M_\gamma)} \, \|\mathcal{P}_{h,\gamma}^{\Delta t} p_\gamma - p_{h,\gamma}^{\Delta t}\|_{L^2(0,T;M_\gamma)} \\
&\qquad + \sum_{i=1,2,\gamma} \sum_{n=1}^{N_i} \left(\|[p_i - p_{h,i}^{\Delta t}]_{n-1}\|_{M_i} \, \|(\mathcal{P}_{h,i}^{\Delta t} p_i - p_i)^+_{n-1}\|_{M_i} + \|\mathcal{P}_{h,i}^{\Delta t} p_i - p_i\|^2_{M_i}\Big|_{t_i^{n-1}}^{t_i^n} \right) \\
&\qquad + \|p_0 - p_{h,0}\|^2_M \Bigg) \\
&\le \epsilon \left(\|\boldsymbol{\Pi}_h^{\Delta t}\boldsymbol{u} - \boldsymbol{u}_h^{\Delta t}\|^2_{L^2(0,T;\boldsymbol{M}^*)} + \|\mathcal{P}_{h,\gamma}^{\Delta t} p_\gamma - p_{h,\gamma}^{\Delta t}\|^2_{L^2(0,T;M_\gamma)} + \|p - p_h^{\Delta t}\|^2_{M,DG} \right) \\
&\quad + C_\epsilon \Bigg(\|\boldsymbol{\Pi}_h^{\Delta t}\boldsymbol{u} - \boldsymbol{u}\|^2_{L^2(0,T;\boldsymbol{M}^*)} + \|p_\gamma - \mathcal{P}_{h,\gamma}^{\Delta t} p_\gamma\|^2_{L^2(0,T;M_\gamma)} \\
&\qquad\quad + \sum_{i=1,2,\gamma} \sum_{n=1}^{N_i} \|(\mathcal{P}_{h,i}^{\Delta t} p_i - p_i)^+_{n-1}\|^2_{M_i} \Bigg) \\
&\quad + C\left(\sum_{i=1,2,\gamma} \sum_{n=1}^{N_i} \|\mathcal{P}_{h,i}^{\Delta t} p_i - p_i\|^2_{M_i}\Big|_{t_i^{n-1}}^{t_i^n} + \|p_0 - p_{h,0}\|^2_M \right).
\end{aligned}
$$

From the first equation in (5.12), we have:

$$
\begin{aligned}
&b^T(\boldsymbol{v}_h^{\Delta t}, \mathcal{P}_h^{\Delta t} p - p_h^{\Delta t}) \\
&\quad = a^T(\boldsymbol{u} - \boldsymbol{u}_h^{\Delta t}, \boldsymbol{v}_h^{\Delta t}) + \left([\![\boldsymbol{v}_h^{\Delta t}\cdot \boldsymbol{n}]\!], p_\gamma - \mathcal{P}_{h,\gamma}^{\Delta t} p_\gamma \right)_{\gamma^T} \\
&\quad \le C\left(K_-^{-1}, K_{\gamma-}^{-1}, \kappa_{\gamma-}^{-1}, \xi\right) \\
&\qquad \cdot \left(\|\boldsymbol{u} - \boldsymbol{u}_h^{\Delta t}\|_{L^2(0,T;\boldsymbol{M}^*)} + \|p_\gamma - \mathcal{P}_{h,\gamma}^{\Delta t} p_\gamma\|_{L^2(0,T;M_\gamma)} \right) \|\boldsymbol{v}_h^{\Delta t}\|_{L^2(0,T;\boldsymbol{M}^*)}.
\end{aligned}
$$

By the inf-sup condition (4.10), we obtain

(5.20)

$$
\begin{aligned}
\|\mathcal{P}_h^{\Delta t} p - p_h^{\Delta t}\|_{L^2(0,T;M)} &\le C\left(K_-^{-1}, K_{\gamma-}^{-1}, \kappa_{\gamma-}^{-1}, \xi, \beta^{-1}\right) \\
&\qquad \left(\|\boldsymbol{u} - \boldsymbol{u}_h^{\Delta t}\|_{L^2(0,T;\boldsymbol{M}^*)} + \|p_\gamma - \mathcal{P}_{h,\gamma}^{\Delta t} p_\gamma\|_{L^2(0,T;M_\gamma)} \right).
\end{aligned}
$$

Combining (5.19) and (5.20) and choosing ϵ sufficiently small yields

$$\begin{aligned}\|\mathbf{\Pi}_h^{\Delta t}\boldsymbol{u} - \boldsymbol{u}_h^{\Delta t}\|_{L^2(0,T;\boldsymbol{M}^*)} + \|p - p_h^{\Delta t}\|_{M,DG} + \|\mathcal{P}_h^{\Delta t}p - p_h^{\Delta t}\|_{L^2(0,T;M)} \\ \leq C\Big(\|\mathbf{\Pi}_h^{\Delta t}\boldsymbol{u} - \boldsymbol{u}\|_{L^2(0,T;\boldsymbol{M}^*)} + \|\mathcal{P}_{h,\gamma}^{\Delta t}p_\gamma - p_\gamma\|_{L^2(0,T;M_\gamma)} \\ + \Delta t^{-\frac{1}{2}}\|\mathcal{P}_h^{\Delta t}p - p\|_{L^\infty(0,T;M)} + \|p_0 - p_{h,0}\|_M\Big),\end{aligned}$$

where the factor $\Delta t^{-\frac{1}{2}}$ comes from the fact that $N_i \leq (CT)/\Delta t_i$ and $\Delta t \leq C\Delta t_i$.

The proof is completed by using the triangle inequality, (4.27), and the interpolation estimates in Subsection 5.1, where for the initial error we used the version of (5.1)–(5.7) for space only. □

Remark 5.1. The error of the normal flux in the term

$$\Big([\![(\mathbf{\Pi}_h^{\Delta t}\boldsymbol{u} - \boldsymbol{u}_h^{\Delta t})\cdot\boldsymbol{n}]\!], p_\gamma - \mathcal{P}_{h,\gamma}^{\Delta t}p_\gamma\Big)_{\gamma^T}$$

on the right-hand side of (5.13) is controlled by the term $\sum_{i=1}^2 \|\kappa_\gamma^{-\frac{1}{2}}(\mathbf{\Pi}_{h,i}^{\Delta t}\boldsymbol{u}_i - \boldsymbol{u}_{h,i}^{\Delta t})\cdot\boldsymbol{n}_i\|_{L^2(0,T;M_\gamma)}^2$ on the left-hand side of (5.19). When κ_γ^{-1} is negligible, the error $\|(\mathbf{\Pi}_{h,i}^{\Delta t}\boldsymbol{u}_i - \boldsymbol{u}_{h,i}^{\Delta t})\cdot\boldsymbol{n}_i\|_{L^2(0,T;M_\gamma)}$ on the right-hand side of (5.19) can be bounded instead by applying the inverse trace inequality, which leads to a factor $h^{-\frac{1}{2}}$ in the error estimates. However, if the spaces of the normal traces of the subdomain velocities are contained in the space of the fracture pressure, the term $\Big([\![(\mathbf{\Pi}_h^{\Delta t}\boldsymbol{u} - \boldsymbol{u}_h^{\Delta t})\cdot\boldsymbol{n}]\!], p_\gamma - \mathcal{P}_{h,\gamma}^{\Delta t}p_\gamma\Big)_{\gamma^T}$ in (5.13) becomes zero and the factor $h^{-\frac{1}{2}}$ can be avoided, even when κ_γ^{-1} approaches zero. This is in particular the case for Raviart-Thomas approximations with each subdomain grid being a coarsening of the fracture grid. In fact, we observe in Subsection 8.1 that with a finer fracture grid, which is not a refinement of the subdomain grids, optimal velocity convergence is obtained even for small κ_γ^{-1}. This is likely due to the fact that some orthogonality occurs in the term in question when the fracture grid is finer, hence the term is small and it can be controlled by the term $\sum_{i=1}^2 \|\kappa_\gamma^{-\frac{1}{2}}(\mathbf{\Pi}_{h,i}^{\Delta t}\boldsymbol{u}_i - \boldsymbol{u}_{h,i}^{\Delta t})\cdot\boldsymbol{n}_i\|_{L^2(0,T;M_\gamma)}^2$ even for small values of κ_γ^{-1}. However, we observe in Subsection 8.2 that in the case of a coarse fracture grid, the velocity rate of convergence is reduced by a factor of $h^{-\frac{1}{2}}$ for small κ_γ^{-1}.

6. Control of the velocity divergence and improved error estimates

We aim to improve the estimates in Theorem 5.1 by removing the factor $\Delta t^{-\frac{1}{2}}$. We first control the velocity divergence and then use a different time interpolant as done in [32] for the case without fractures. For the analysis in this section, we assume that the time steps in the subdomains and in the fracture are the same, i.e. $\mathcal{T}_1^{\Delta t} = \mathcal{T}_2^{\Delta t} = \mathcal{T}_\gamma^{\Delta t} := \mathcal{T}^{\Delta t} : 0 = t^0 < t^1 < \cdots < t^N$, so that

$$W_1^{\Delta t} = W_2^{\Delta t} = W_\gamma^{\Delta t} =: W^{\Delta t}, \tag{6.1}$$

where $W^{\Delta t}$ consists of discontinuous piecewise polynomials of degree k on the time grid $\mathcal{T}^{\Delta t}$.

6.1. **Radau reconstruction operator.** For each time interval $I_n = [t^{n-1}, t^n]$, denote by L_r^n the rth Legendre polynomial on I_n. Then

$$L_r^n(t^n) = 1, \quad L_r^n(t^{n-1}) = (-1)^r, \tag{6.2}$$

and $\{L_r^n\}_{r\geq 0}$ are L^2-orthogonal on I_n. We utilize the Radau reconstruction operator $\mathcal{I}$ [18, 38] defined as

$$\begin{aligned} &\mathcal{I} : W^{\Delta t} \longrightarrow H^1(0,T), \\ &\mu^{\Delta t} \longmapsto \mathcal{I}\mu^{\Delta t}|_{I_n} := \mu^{\Delta t}|_{I_n} - \frac{(-1)^k}{2}(L_k^n - L_{k+1}^n)[\mu^{\Delta t}]_{n-1}. \end{aligned} \tag{6.3}$$

Note that

$$\mathcal{I}\mu^{\Delta t}|_{I_n}(t_n) = \mu_n^-, \quad \mathcal{I}\mu^{\Delta t}|_{I_n}(t_{n-1}) = \mu_{n-1}^-. \tag{6.4}$$

Lemma 6.1. *For any $\mu^{\Delta t}, \phi^{\Delta t} \in W^{\Delta t}$ and for all $1 \leq n \leq N$:*

$$\int_{t^{n-1}}^{t^n} \partial_t \mathcal{I}\mu^{\Delta t}\phi^{\Delta t} = \int_{t^{n-1}}^{t^n} \partial_t \mu^{\Delta t}\phi^{\Delta t} + [\mu^{\Delta t}]_{n-1}(\phi^{\Delta t})_{n-1}^+. \tag{6.5}$$

$$\int_{t^{n-1}}^{t^n} \left(\partial_t \mathcal{I}\mu^{\Delta t}\,\phi^{\Delta t} + \partial_t \mathcal{I}\phi^{\Delta t}\,\mu^{\Delta t}\right) = \int_{t^{n-1}}^{t^n} \partial_t(\mathcal{I}\mu^{\Delta t}\,\mathcal{I}\phi^{\Delta t}) + [\mu^{\Delta t}]_{n-1}[\phi^{\Delta t}]_{n-1}. \tag{6.6}$$

$$\int_0^T \partial_t \mathcal{I}\mu^{\Delta t}\phi^{\Delta t} = \int_0^T \widetilde{\partial}_t \mu^{\Delta t}\phi^{\Delta t}. \tag{6.7}$$

Proof. The properties (6.5) and (6.6) are deduced by using integration by parts, the orthogonality of the polynomials L_k^n and L_{k+1}^n to all polynomials of degree strictly less than k on the time interval I_n, and (6.2). In particular, to obtain (6.6), we have

$$\begin{aligned} \int_{t^{n-1}}^{t^n} \partial_t \mathcal{I}\mu^{\Delta t}\,\mathcal{I}\phi^{\Delta t} &= \int_{t^{n-1}}^{t^n} \partial_t \left(\mu^{\Delta t} - \frac{(-1)^k}{2}(L_k^n - L_{k+1}^n)[\mu^{\Delta t}]_{n-1}\right) \\ &\qquad \cdot \left(\phi^{\Delta t} - \frac{(-1)^k}{2}(L_k^n - L_{k+1}^n)[\phi^{\Delta t}]_{n-1}\right) \\ &= \int_{t^{n-1}}^{t^n} \partial_t \mathcal{I}\mu^{\Delta t}\phi^{\Delta t} - \frac{(-1)^k}{2}[\phi^{\Delta t}]_{n-1}\int_{t^{n-1}}^{t^n} \partial_t \mu^{\Delta t}(L_k^n - L_{k+1}^n) \\ &\quad + \frac{(-1)^k}{2}[\mu^{\Delta t}]_{n-1}\frac{(-1)^k}{2}[\phi^{\Delta t}]_{n-1}\int_{t^{n-1}}^{t^n} \partial_t(L_k^n - L_{k+1}^n)(L_k^n - L_{k+1}^n) \\ &= \int_{t^{n-1}}^{t^n} \partial_t \mathcal{I}\mu^{\Delta t}\phi^{\Delta t} + \frac{1}{4}[\mu^{\Delta t}]_{n-1}[\phi^{\Delta t}]_{n-1}\left.\frac{(L_k^n - L_{k+1}^n)^2}{2}\right|_{t^{n-1}}^{t^n} \\ &= \int_{t^{n-1}}^{t^n} \partial_t \mathcal{I}\mu^{\Delta t}\phi^{\Delta t} - \frac{1}{2}[\mu^{\Delta t}]_{n-1}[\phi^{\Delta t}]_{n-1}. \end{aligned}$$

The identity (6.7) is a direct result of (6.5) and (3.1). □

6.2. **Control of the velocity divergence.** Under the assumption (6.1), we obtain the following stability bound for the velocity divergence

$$\operatorname{div} \boldsymbol{u}_h^{\Delta t} := \left(\operatorname{div} \boldsymbol{u}_{h,1}^{\Delta t}, \operatorname{div} \boldsymbol{u}_{h,2}^{\Delta t}, \operatorname{div}_\tau \boldsymbol{u}_{h,\gamma}^{\Delta t}\right).$$

Theorem 6.1. *Suppose that (6.1) holds, then there exists a constant $C > 0$ independent of h and Δt such that*

$$\begin{aligned}\|\partial_t \mathcal{I} p_h^{\Delta t}\|_{L^2(0,T;M)} + \|\operatorname{div} \boldsymbol{u}_h^{\Delta t}\|_{L^2(0,T;M)} + \|\boldsymbol{u}_h^{\Delta t}\|_{\boldsymbol{M}^*,DG} \\ \leq C\left(\|q\|_{L^2(0,T;M)} + \|\boldsymbol{u}_0\|_{\boldsymbol{\Sigma}}\right).\end{aligned} \tag{6.8}$$

Proof. Using (6.7), the second equation of (3.2) can be rewritten as

$$c_s^T\left(\partial_t \mathcal{I} p_h^{\Delta t}, \mu_h^{\Delta t}\right) + b^T\left(\boldsymbol{u}_h^{\Delta t}, \mu_h^{\Delta t}\right) = L_q^T(\mu_h^{\Delta t}),\ \forall \mu_h^{\Delta t} \in M_h^{\Delta t}. \tag{6.9}$$

Due to (6.1), the first equation of (3.2) implies that for each $n = 1, \ldots, N$, and every $t \in [t^{n-1}, t^n]$:

$$a(\boldsymbol{u}_h^{\Delta t}, \boldsymbol{v}_h) - b(\boldsymbol{v}_h, p_h^{\Delta t}) = 0, \quad \forall \boldsymbol{v}_h \in \boldsymbol{\Sigma}_h.$$

As the initial data also satisfy this equation (cf. (4.25)), we deduce that

$$a^T\left(\partial_t \mathcal{I} \boldsymbol{u}_h^{\Delta t}, \boldsymbol{v}_h^{\Delta t}\right) - b^T\left(\boldsymbol{v}_h^{\Delta t}, \partial_t \mathcal{I} p_h^{\Delta t}\right) = 0, \quad \forall \boldsymbol{v}_h^{\Delta h} \in \boldsymbol{\Sigma}_h^{\Delta t}. \tag{6.10}$$

Choosing $\mu_h^{\Delta t} = \partial_t \mathcal{I} p_h^{\Delta t}$ in (6.9) and $\boldsymbol{v}_h^{\Delta t} = \boldsymbol{u}_h^{\Delta}$ in (6.10), and adding the resulting equations, we obtain

$$c_s^T\left(\partial_t \mathcal{I} p_h^{\Delta t}, \partial_t \mathcal{I} p_h^{\Delta t}\right) + a^T\left(\partial_t \mathcal{I} \boldsymbol{u}_h^{\Delta t}, \boldsymbol{u}_h^{\Delta t}\right) = L_q^T(\partial_t \mathcal{I} p_h^{\Delta t}), \tag{6.11}$$

which, using (4.31), leads to

(6.12)

$$\begin{aligned}
L_q^T(\partial_t \mathcal{I} p_h^{\Delta t}) = {} & \sum_{i=1}^{2}\left(s_i \partial_t \mathcal{I} p_{h,i}^{\Delta t}, \mathcal{I} p_{h,i}^{\Delta t}\right)_{\Omega_i^T} + \left(s_\gamma \partial_t \mathcal{I} p_{h,\gamma}^{\Delta t}, \mathcal{I} p_{h,\gamma}^{\Delta t}\right)_{\gamma^T} \\
& + \sum_{i=1}^{2}\left(\boldsymbol{K}_i^{-1} \partial_t \mathcal{I} \boldsymbol{u}_{h,i}^{\Delta t}, \boldsymbol{u}_{h,i}^{\Delta t}\right)_{\Omega_i^T} + \left(\boldsymbol{K}_\gamma^{-1} \partial_t \mathcal{I} \boldsymbol{u}_{h,\gamma}^{\Delta t}, \boldsymbol{u}_{h,\gamma}^{\Delta t}\right)_{\gamma^T} \\
& + \sum_{i=1}^{2}\left(\kappa_\gamma^{-1}(\xi \partial_t \mathcal{I} \boldsymbol{u}_{h,i}^{\Delta t} \cdot \boldsymbol{n}_i + (1-\xi)\partial_t \mathcal{I} \boldsymbol{u}_{h,j}^{\Delta t} \cdot \boldsymbol{n}_i), \boldsymbol{u}_{h,i}^{\Delta t} \cdot \boldsymbol{n}_i\right)_{\gamma^T} \\
\geq {} & \frac{s_-}{2}\|\partial_t \mathcal{I} p_h^{\Delta t}\|_{L^2(0,T;M)}^2 \\
& + \frac{1}{2}\sum_{i=1,2,\gamma}\left(\|\boldsymbol{K}_i^{-\frac{1}{2}} \boldsymbol{u}_{h,i}^{\Delta t}\|_{M_i,DG}^2 - \|\boldsymbol{K}_i^{-\frac{1}{2}}(\boldsymbol{u}_{h,i}^{\Delta t})_0^-\|_{M_i}^2\right) \\
& + \frac{\xi}{2}\sum_{i=1}^{2}\left(\|\kappa_\gamma^{-\frac{1}{2}} \boldsymbol{u}_{h,i}^{\Delta t} \cdot \boldsymbol{n}_i\|_{0,\gamma,DG}^2 - \|\kappa_\gamma^{-\frac{1}{2}}(\boldsymbol{u}_{h,i}^{\Delta t})_0^- \cdot \boldsymbol{n}_i\|_{0,\gamma}^2\right) \\
& + (1-\xi)\Big((\kappa_\gamma^{-\frac{1}{2}} \partial_t \mathcal{I} \boldsymbol{u}_{h,1}^{\Delta t} \cdot \boldsymbol{n}_1, \kappa_\gamma^{-\frac{1}{2}} \boldsymbol{u}_{h,2}^{\Delta t} \cdot \boldsymbol{n}_1)_{\gamma^T} \\
& \qquad + (\kappa_\gamma^{-\frac{1}{2}} \partial_t \mathcal{I} \boldsymbol{u}_{h,2}^{\Delta t} \cdot \boldsymbol{n}_2, \kappa_\gamma^{-\frac{1}{2}} \boldsymbol{u}_{h,1}^{\Delta t} \cdot \boldsymbol{n}_2)_{\gamma^T}\Big).
\end{aligned}$$

Letting $\boldsymbol{n} = \boldsymbol{n}_1$ and using the properties (6.6) and (6.4) of the Radau operator, we have

$$\begin{aligned}
(6.13)\quad & \Big((\kappa_\gamma^{-\frac{1}{2}}\partial_t\mathcal{I}\boldsymbol{u}_{h,1}^{\Delta t}\cdot\boldsymbol{n}_1, \kappa_\gamma^{-\frac{1}{2}}\boldsymbol{u}_{h,2}^{\Delta t}\cdot\boldsymbol{n}_1)_{\gamma^T} + (\kappa_\gamma^{-\frac{1}{2}}\partial_t\mathcal{I}\boldsymbol{u}_{h,2}^{\Delta t}\cdot\boldsymbol{n}_2, \kappa_\gamma^{-\frac{1}{2}}\boldsymbol{u}_{h,1}^{\Delta t}\cdot\boldsymbol{n}_2)_{\gamma^T}\Big) \\
&= \sum_{n=1}^N \int_{t^{n-1}}^{t^n} \partial_t\left(\kappa_\gamma^{-\frac{1}{2}}\mathcal{I}\boldsymbol{u}_{h,1}^{\Delta t}\cdot\boldsymbol{n}, \kappa_\gamma^{-\frac{1}{2}}\mathcal{I}\boldsymbol{u}_{h,2}^{\Delta t}\cdot\boldsymbol{n}\right)_\gamma \\
&\quad + \sum_{n=1}^N [\kappa_\gamma^{-\frac{1}{2}}\boldsymbol{u}_{h,1}^{\Delta t}\cdot\boldsymbol{n}]_{n-1}\,[\kappa_\gamma^{-\frac{1}{2}}\boldsymbol{u}_{h,2}^{\Delta t}\cdot\boldsymbol{n}]_{n-1} \\
&= (\kappa_\gamma^{-\frac{1}{2}}(\boldsymbol{u}_{h,1}^{\Delta t})_N^-\cdot\boldsymbol{n}, \kappa_\gamma^{-\frac{1}{2}}(\boldsymbol{u}_{h,2}^{\Delta t})_N^-\cdot\boldsymbol{n})_\gamma \\
&\quad - (\kappa_\gamma^{-\frac{1}{2}}(\boldsymbol{u}_{h,1}^{\Delta t})_0^-\cdot\boldsymbol{n}, \kappa_\gamma^{-\frac{1}{2}}(\boldsymbol{u}_{h,2}^{\Delta t})_0^-\cdot\boldsymbol{n})_\gamma \\
&\quad + \sum_{n=1}^N [\kappa_\gamma^{-\frac{1}{2}}\boldsymbol{u}_{h,1}^{\Delta t}\cdot\boldsymbol{n}]_{n-1}\,[\kappa_\gamma^{-\frac{1}{2}}\boldsymbol{u}_{h,2}^{\Delta t}\cdot\boldsymbol{n}]_{n-1}.
\end{aligned}$$

Using this equation, we can bound the last two terms in (6.12) by

(6.14)

$$\begin{aligned}
&\frac{\xi}{2}\sum_{i=1}^2\left(\|\kappa_\gamma^{-\frac{1}{2}}\boldsymbol{u}_{h,i}^{\Delta t}\cdot\boldsymbol{n}_i\|_{0,\gamma,DG}^2 - \|\kappa_\gamma^{-\frac{1}{2}}(\boldsymbol{u}_{h,i}^{\Delta t})_0^-\cdot\boldsymbol{n}_i\|_{0,\gamma}^2\right) \\
&+ (1-\xi)\Big((\kappa_\gamma^{-\frac{1}{2}}\partial_t\mathcal{I}\boldsymbol{u}_{h,1}^{\Delta t}\cdot\boldsymbol{n}_1, \kappa_\gamma^{-\frac{1}{2}}\boldsymbol{u}_{h,2}^{\Delta t}\cdot\boldsymbol{n}_1)_{\gamma^T} + (\kappa_\gamma^{-\frac{1}{2}}\partial_t\mathcal{I}\boldsymbol{u}_{h,2}^{\Delta t}\cdot\boldsymbol{n}_2, \kappa_\gamma^{-\frac{1}{2}}\boldsymbol{u}_{h,1}^{\Delta t}\cdot\boldsymbol{n}_2)_{\gamma^T}\Big) \\
&\geq \frac{1}{2}\min\{1, 2\xi-1\}\sum_{i=1}^2\|\kappa_\gamma^{-\frac{1}{2}}\boldsymbol{u}_{h,i}^{\Delta t}\cdot\boldsymbol{n}_i\|_{0,\gamma,DG}^2 - \sum_{i=1}^2\|\kappa_\gamma^{-\frac{1}{2}}(\boldsymbol{u}_{h,i}^{\Delta t})_0^-\cdot\boldsymbol{n}_i\|_{0,\gamma}^2.
\end{aligned}$$

Using this, the Young inequality as in (4.36), the uniform boundedness of $\boldsymbol{K}_i$ and κ_γ (cf. Assumptions (A3)–(A5)), the discrete initial data (4.28), and (4.26), we deduce from (6.12) that

$$(6.15)\qquad \|\partial_t\mathcal{I}p_h^{\Delta t}\|_{L^2(0,T;M)} + \|\boldsymbol{u}_h^{\Delta t}\|_{\boldsymbol{M}^*,DG} \leq C\left(\|q\|_{L^2(0,T;M)} + \|\boldsymbol{u}_0\|_{\boldsymbol{\Sigma}}\right).$$

To bound the divergence, we choose $\mu_h^{\Delta t} = (\operatorname{div}\boldsymbol{u}_{h,1}^{\Delta t}, \operatorname{div}\boldsymbol{u}_{h,2}^{\Delta t}, \operatorname{div}_\tau\boldsymbol{u}_{h,\gamma}^{\Delta t})$ in (6.9) to get

$$\begin{aligned}
&\|\operatorname{div}\boldsymbol{u}_h^{\Delta t}\|_{L^2(0,T;M)} \\
(6.16)\quad &\leq C\left(\|\partial_t\mathcal{I}p_h^{\Delta t}\|_{L^2(0,T;M)} + \sum_{i=1}^2\|\boldsymbol{u}_{h,i}^{\Delta t}\cdot\boldsymbol{n}_i\|_{L^2(0,T;M_\gamma)} + \|q\|_{L^2(0,T;M)}\right).
\end{aligned}$$

The proof is completed by combining (6.15), (4.32) and (6.16). □

6.3. **Improved a priori error estimates.** To avoid the factor $\Delta t^{-\frac{1}{2}}$ appeared in (5.10), we follow the idea in [32] and apply a different time interpolant $\widetilde{\mathcal{P}}^{\Delta t} : H^1(0,T) \to W^{\Delta t}$ such that, for $\varphi \in H^1(0,T)$:

$$(6.17)\qquad \begin{aligned} \int_{t^{n-1}}^{t^n}(\widetilde{\mathcal{P}}^{\Delta t}\varphi - \varphi)w = 0, \quad &\forall w \in P_{k-1}, \\ (\widetilde{\mathcal{P}}^{\Delta t}\varphi)_n^- = \varphi(t^n), & \end{aligned}\qquad \forall n = 1,\ldots,N.$$

In addition, we set $(\widetilde{\mathcal{P}}^{\Delta t}\varphi)_0^- = \varphi(0)$. Next we define the space-time interpolants:

$$\widetilde{\mathcal{P}}_{h,i}^{\Delta t} := \widetilde{\mathcal{P}}^{\Delta t} \circ \mathcal{P}_{h,i},\ i = 1, 2, \quad \widetilde{\mathcal{P}}_{h,\gamma}^{\Delta t} := \widetilde{\mathcal{P}}^{\Delta t} \circ \mathcal{P}_{h,\gamma}, \tag{6.18}$$

$$\widetilde{\mathbf{\Pi}}_{h,i}^{\Delta t} := \widetilde{\mathcal{P}}^{\Delta t} \circ \widetilde{\mathbf{\Pi}}_{h,i},\ i = 1, 2, \quad \widetilde{\mathbf{\Pi}}_{h,\gamma}^{\Delta t} := \widetilde{\mathcal{P}}^{\Delta t} \circ \mathcal{P}_{h,\gamma}. \tag{6.19}$$

In addition, let $\widetilde{\mathcal{P}}_h^{\Delta t} : H^1(0,T;M) \to M_h^{\Delta t}$, such that $\widetilde{\mathcal{P}}_h^{\Delta t}|_{\Omega_i^T} = \widetilde{\mathcal{P}}_{h,i}^{\Delta t}$, for $i = 1,\ 2$, and $\widetilde{\mathcal{P}}_h^{\Delta t}|_{\gamma^T} = \widetilde{\mathcal{P}}_{h,\gamma}^{\Delta t}$. with a similar definition for $\widetilde{\mathbf{\Pi}}_h^{\Delta t} : H^1(0,T;\mathbf{\Sigma}) \to \mathbf{\Sigma}_h^{\Delta t}$. We have the following properties of $\widetilde{\mathcal{P}}^{\Delta t}$ and $\widetilde{\mathcal{P}}_{h,i}^{\Delta t}$ (cf. [32, Lemma 6.2]):

$$\int_0^T \partial_t\varphi\, w^{\Delta t} = \int_0^T \widetilde{\partial}_t\widetilde{\mathcal{P}}^{\Delta t}\varphi\, w^{\Delta t}, \quad \forall\varphi \in H^1(0,T), \forall w^{\Delta t} \in W^{\Delta t}, \tag{6.20}$$

$$\int_0^T (\partial_t\mu_i, \phi_{h,i}^{\Delta t})_{\Omega_i} = \int_0^T (\widetilde{\partial}_t\widetilde{\mathcal{P}}_{h,i}^{\Delta t}\mu_i, \phi_{h,i}^{\Delta t})_{\Omega_i}, \quad \forall\mu_i \in H^1(0,T;M_i),\ \forall\phi_{h,i}^{\Delta t} \in W_{h,i}^{\Delta t},\ i = 1, 2, \tag{6.21}$$

$$\int_0^T (\partial_t\mu_\gamma, \phi_{h,\gamma}^{\Delta t})_\gamma = \int_0^T (\widetilde{\partial}_t\widetilde{\mathcal{P}}_{h,\gamma}^{\Delta t}\mu_\gamma, \phi_{h,\gamma}^{\Delta t})_\gamma,\ \forall\mu_\gamma \in H^1(0,T;M_\gamma),\ \forall\phi_{h,\gamma}^{\Delta t} \in W_{h,\gamma}^{\Delta t}. \tag{6.22}$$

Moreover, the space-time interpolants $\widetilde{\mathcal{P}}_h^{\Delta t}$ and $\widetilde{\mathbf{\Pi}}_h^{\Delta t}$ satisfy, for $1 \le r_\rho \le \rho+1, 1 \le r_k \le k+1$,

$$\|\mu_i - \widetilde{\mathcal{P}}_{h,i}^{\Delta t}\mu_i\|_{L^2(0,T;M_i)} \le C\|\mu_i\|_{H^{r_k}(0,T;H^{r_\rho}(\Omega_i))}(h^{r_\rho} + \Delta t^{r_k}), \tag{6.23}$$

$$\|\mu_\gamma - \widetilde{\mathcal{P}}_{h,\gamma}^{\Delta t}\mu_\gamma\|_{L^2(0,T;M_\gamma)} \le C\|\mu_\gamma\|_{H^{r_k}(0,T;H^{r_\rho}(\gamma))}(h^{r_\rho} + \Delta t^{r_k}), \tag{6.24}$$

$$\|\boldsymbol{v}_i - \widetilde{\mathbf{\Pi}}_{h,i}^{\Delta t}\boldsymbol{v}_i\|_{L^2(0,T;M_i)} \le C\|\boldsymbol{v}_i\|_{H^{r_k}(0,T;H^{r_\rho}(\Omega_i))}(h^{r_\rho} + \Delta t^{r_k}), \tag{6.25}$$

$$\|\boldsymbol{v}_\gamma - \widetilde{\mathbf{\Pi}}_{h,\gamma}^{\Delta t}\boldsymbol{v}_\gamma\|_{L^2(0,T;\boldsymbol{M}_\gamma)} \le C\|\boldsymbol{v}_\gamma\|_{H^{r_k}(0,T;H^{r_\rho}(\gamma))}(h^{r_\rho} + \Delta t^{r_k}), \tag{6.26}$$

$$\|\operatorname{div}(\boldsymbol{v}_i - \widetilde{\mathbf{\Pi}}_{h,i}^{\Delta t}\boldsymbol{v}_i)\|_{L^2(0,T;M_i)} \le C\|\operatorname{div}\boldsymbol{v}_i\|_{H^{r_k}(0,T;H^{r_\rho}(\Omega_i))}(h^{r_\rho} + \Delta t^{r_k}), \tag{6.27}$$

$$\|\operatorname{div}_\tau(\boldsymbol{v}_\gamma - \widetilde{\mathbf{\Pi}}_{h,\gamma}^{\Delta t}\boldsymbol{v}_\gamma)\|_{L^2(0,T;M_\gamma)} \le C\|\operatorname{div}_\tau\boldsymbol{v}_\gamma\|_{H^{r_k}(0,T;H^{r_\rho}(\gamma))}(h^{r_\rho} + \Delta t^{r_k}), \tag{6.28}$$

$$\|(\boldsymbol{v}_i - \widetilde{\mathbf{\Pi}}_{h,i}^{\Delta t}\boldsymbol{v}_i)\cdot\boldsymbol{n}_i\|_{L^2(0,T;M_\gamma)} \le C\|\boldsymbol{v}_i\cdot\boldsymbol{n}_i\|_{H^{r_k}(0,T;H^{r_\rho}(\gamma))}(h^{r_\rho} + \Delta t^{r_k}). \tag{6.29}$$

Theorem 6.2. *Suppose that (6.1) holds, then there exists a constant $C > 0$ independent of h and Δt such that*

$$\begin{aligned}
&\|\boldsymbol{u}(t^N)-(\boldsymbol{u}_h^{\Delta t})_N^-\|_{\boldsymbol{M}^*}+\|\boldsymbol{u}-\boldsymbol{u}_h^{\Delta t}\|_{L^2(0,T;\boldsymbol{\Sigma})}+\|p(t^N)-(p_h^{\Delta t})_N^-\|_M+\|p-p_h^{\Delta t}\|_{L^2(0,T;M)}\\
&\quad\le C\bigg((h^{r_\rho}+\Delta t^{r_k})\Big(\|\boldsymbol{u}\|_{H^{r_k}(0,T;\boldsymbol{H}^{r_\rho})}+\|\operatorname{div}\boldsymbol{u}\|_{H^{r_k}(0,T;\boldsymbol{H}^{r_\rho})}\\
&\qquad+\sum_{i=1}^2\|\boldsymbol{u}_i\cdot\boldsymbol{n}_i\|_{H^{r_k}(0,T;H^{r_\rho}(\gamma))}+\|p\|_{H^{r_k}(0,T;H^{r_\rho})}\Big)\\
&\qquad+h^{r_\rho}\Big(\|\boldsymbol{u}\|_{H^1(0,T;\boldsymbol{H}^{r_\rho})}+\sum_{i=1}^2\|\boldsymbol{u}_i\cdot\boldsymbol{n}_i\|_{H^1(0,T;H^{r_\rho}(\gamma))}+\|p_\gamma\|_{H^1(0,T;H^{r_\rho}(\gamma))}\\
&\qquad+\|\boldsymbol{u}_0\|_{\boldsymbol{H}^{r_\rho}}+\sum_{i=1}^2\|\boldsymbol{u}_{0,i}\cdot\boldsymbol{n}_i\|_{H^{r_\rho}(\gamma)}+\|p_0\|_{H^{r_\rho}}\Big)\bigg).
\end{aligned}\tag{6.30}$$

Proof. By subtracting (3.2) from (2.5), we obtain the error equations:

$$\begin{aligned}
&a^T\left(\boldsymbol{u}-\boldsymbol{u}_h^{\Delta t},\boldsymbol{v}_h^{\Delta t}\right)-b^T\left(\boldsymbol{v}_h^{\Delta t},\widetilde{\mathcal{P}}_h^{\Delta t}p-p_h^{\Delta t}\right)-b^T\left(\boldsymbol{v}_h^{\Delta t},p-\widetilde{\mathcal{P}}_h^{\Delta t}p\right)=0,\\
&\hspace{22em}\forall\boldsymbol{v}_h^{\Delta t}\in\boldsymbol{\Sigma}_h^{\Delta t},\\
&c_s^T\left(\partial_t p-\widetilde{\partial}_t p_h^{\Delta t},\mu_h^{\Delta t}\right)+b^T\left(\widetilde{\boldsymbol{\Pi}}_h^{\Delta t}\boldsymbol{u}-\boldsymbol{u}_h^{\Delta t},\mu_h^{\Delta t}\right)+b^T\left(\boldsymbol{u}-\widetilde{\boldsymbol{\Pi}}_h^{\Delta t}\boldsymbol{u},\mu_h^{\Delta t}\right)=0,\\
&\hspace{22em}\forall\mu_h^{\Delta t}\in M_h^{\Delta t}.
\end{aligned}\tag{6.31}$$

We choose $\boldsymbol{v}_h^{\Delta t}=\widetilde{\boldsymbol{\Pi}}_h^{\Delta t}\boldsymbol{u}-\boldsymbol{u}_h^{\Delta t}$ and $\mu_h^{\Delta t}=\widetilde{\mathcal{P}}_h^{\Delta t}p-p_h^{\Delta t}$ and sum the resulting equations to get

$$\begin{aligned}
&a^T\left(\widetilde{\boldsymbol{\Pi}}_h^{\Delta t}\boldsymbol{u}-\boldsymbol{u}_h^{\Delta t},\widetilde{\boldsymbol{\Pi}}_h^{\Delta t}\boldsymbol{u}-\boldsymbol{u}_h^{\Delta t},\right)+c_s^T\left(\partial_t p-\widetilde{\partial}_t p_h^{\Delta t},\widetilde{\mathcal{P}}_h^{\Delta t}p-p_h^{\Delta t}\right)\\
&\quad=a^T\left(\widetilde{\boldsymbol{\Pi}}_h^{\Delta t}\boldsymbol{u}-\boldsymbol{u},\widetilde{\boldsymbol{\Pi}}_h^{\Delta t}\boldsymbol{u}-\boldsymbol{u}_h^{\Delta t},\right)-b^T\left(\widetilde{\boldsymbol{\Pi}}_h^{\Delta t}\boldsymbol{u}-\boldsymbol{u}_h^{\Delta t},p-\widetilde{\mathcal{P}}_h^{\Delta t}p\right)\\
&\qquad-b^T\left(\boldsymbol{u}-\widetilde{\boldsymbol{\Pi}}_h^{\Delta t}\boldsymbol{u},\widetilde{\mathcal{P}}_h^{\Delta t}p-p_h^{\Delta t}\right).
\end{aligned}\tag{6.32}$$

Using (6.20), Lemma 4.2, (4.30), and (A1)–(A2), we deduce that

$$\begin{aligned}
c_s^T\left(\partial_t p-\widetilde{\partial}_t p_h^{\Delta t},\widetilde{\mathcal{P}}_h^{\Delta t}p-p_h^{\Delta t}\right)&=\sum_{i=1}^2 s_i\left(\widetilde{\partial}_t(\widetilde{\mathcal{P}}_{h,i}^{\Delta t}p_i-p_{h,i}^{\Delta t}),\widetilde{\mathcal{P}}_{h,i}^{\Delta t}p_i-p_{h,i}^{\Delta t}\right)_{\Omega_i^T}\\
&\quad+s_\gamma\left(\widetilde{\partial}_t(\widetilde{\mathcal{P}}_{h,\gamma}p_\gamma-p_{h,\gamma}^{\Delta t}),\widetilde{\mathcal{P}}_{h,\gamma}^{\Delta t}p_\gamma-p_{h,\gamma}^{\Delta t}\right)_{\gamma^T}\\
&\ge\frac{s_-}{2}\|\widetilde{\mathcal{P}}_h^{\Delta t}p-p_h^{\Delta t}\|_{M,DG}^2-\frac{s_+}{2}\|\mathcal{P}_hp_0-p_{h,0}\|_M^2.
\end{aligned}\tag{6.33}$$

From this and (6.32), we obtain:

$$\begin{aligned}
(6.34)\qquad &\|\widetilde{\mathbf{\Pi}}_h^{\Delta t}\boldsymbol{u} - \boldsymbol{u}_h^{\Delta t}\|^2_{L^2(0,T;\boldsymbol{M}^*)} + \|\widetilde{\mathcal{P}}_h^{\Delta t}p - p_h^{\Delta t}\|^2_{M,DG} \\
&\leq C\Big(\|\widetilde{\mathbf{\Pi}}_h^{\Delta t}\boldsymbol{u} - \boldsymbol{u}\|_{L^2(0,T;\boldsymbol{M}^*)}\,\|\widetilde{\mathbf{\Pi}}_h^{\Delta t}\boldsymbol{u} - \boldsymbol{u}_h^{\Delta t}\|_{L^2(0,T;\boldsymbol{M}^*)} \\
&\quad + \|\operatorname{div}(\widetilde{\mathbf{\Pi}}_h^{\Delta t}\boldsymbol{u} - \boldsymbol{u}_h^{\Delta t})\|_{L^2(0,T;M)}\,\|p - \widetilde{\mathcal{P}}_h^{\Delta t}p\|_{L^2(0,T;M)} \\
&\quad + \sum_i \|(\widetilde{\mathbf{\Pi}}_{h,i}^{\Delta t}\boldsymbol{u}_i - \boldsymbol{u}_{h,i}^{\Delta t})\cdot\boldsymbol{n}_i\|_{L^2(0,T;M_\gamma)}\,\|p_\gamma - \widetilde{\mathcal{P}}_{h,\gamma}^{\Delta t}p_\gamma\|_{L^2(0,T;M_\gamma)} \\
&\quad + \|\operatorname{div}(\boldsymbol{u} - \widetilde{\mathbf{\Pi}}_h^{\Delta t}\boldsymbol{u})\|_{L^2(0,T;M)}\,\|\widetilde{\mathcal{P}}_h^{\Delta t}p - p_h^{\Delta t}\|_{L^2(0,T;M)} \\
&\quad + \sum_i \|(\boldsymbol{u}_i - \widetilde{\mathbf{\Pi}}_{h,i}^{\Delta t}\boldsymbol{u}_i)\cdot\boldsymbol{n}_i\|_{L^2(0,T;M_\gamma)}\,\|\widetilde{\mathcal{P}}_{h,\gamma}^{\Delta t}p_\gamma - p_{h,\gamma}^{\Delta t}\|_{L^2(0,T;M_\gamma)} \\
&\quad + \|\mathcal{P}_h p_0 - p_{h,0}\|^2_M \Big) \\
&\leq \epsilon\Big(\|\widetilde{\mathbf{\Pi}}_h^{\Delta t}\boldsymbol{u} - \boldsymbol{u}_h^{\Delta t}\|^2_{L^2(0,T;\boldsymbol{M}^*)} + \|\operatorname{div}(\widetilde{\mathbf{\Pi}}_h^{\Delta t}\boldsymbol{u} - \boldsymbol{u}_h^{\Delta t})\|^2_{L^2(0,T;M)} \\
&\quad + \|\widetilde{\mathcal{P}}_h^{\Delta t}p - p_h^{\Delta t}\|^2_{L^2(0,T;M)}\Big) \\
&\quad + C_\epsilon\Big(\|\widetilde{\mathbf{\Pi}}_h^{\Delta t}\boldsymbol{u} - \boldsymbol{u}\|^2_{L^2(0,T;\boldsymbol{M}^*)} + \|p - \widetilde{\mathcal{P}}_h^{\Delta t}p\|^2_{L^2(0,T;M)} \\
&\qquad + \|\operatorname{div}(\boldsymbol{u} - \widetilde{\mathbf{\Pi}}_h^{\Delta t}\boldsymbol{u})\|^2_{L^2(0,T;M)}\Big) \\
&\quad + C\|\mathcal{P}_h p_0 - p_{h,0}\|^2_M.
\end{aligned}$$

There remains to bound the terms

$$\|\widetilde{\mathcal{P}}_h^{\Delta t}p - p_h^{\Delta t}\|_{L^2(0,T;M)} \quad\text{and}\quad \|\operatorname{div}(\boldsymbol{u} - \widetilde{\mathbf{\Pi}}_h^{\Delta t}\boldsymbol{u})\|_{L^2(0,T;M)}$$

in (6.34). For the pressure error, similarly to (5.20), from the first equation of (6.31) and the inf-sup condition (4.10), we deduce that

$$\begin{aligned}
(6.35)\qquad &\|\widetilde{\mathcal{P}}_h^{\Delta t}p - p_h^{\Delta t}\|_{L^2(0,T;M)} \leq C\left(K_-^{-1}, K_{\gamma-}^{-1}, \kappa_{\gamma-}^{-1}, \xi, \beta^{-1}\right) \\
&\Big(\|\widetilde{\mathbf{\Pi}}_h^{\Delta t}\boldsymbol{u} - \boldsymbol{u}_h^{\Delta t}\|_{L^2(0,T;\boldsymbol{M}^*)} + \|\boldsymbol{u} - \widetilde{\mathbf{\Pi}}_h^{\Delta t}\boldsymbol{u}\|_{L^2(0,T;\boldsymbol{M}^*)} + \|p - \widetilde{\mathcal{P}}_h^{\Delta t}p\|_{L^2(0,T;M)}\Big),
\end{aligned}$$

where we have used the triangle inequality

$$\|\boldsymbol{u} - \boldsymbol{u}_h^{\Delta t}\|_{L^2(0,T;\boldsymbol{M}^*)} \leq \|\widetilde{\mathbf{\Pi}}_h^{\Delta t}\boldsymbol{u} - \boldsymbol{u}_h^{\Delta t}\|_{L^2(0,T;\boldsymbol{M}^*)} + \|\boldsymbol{u} - \widetilde{\mathbf{\Pi}}_h^{\Delta t}\boldsymbol{u}\|_{L^2(0,T;\boldsymbol{M}^*)}.$$

We proceed with bounding the divergence error. Recall the assumption (6.1), which implies that (6.10) holds. Combining it with the time-differentiated first equation in (2.5), we obtain

$$(6.36)\qquad a^T\left(\partial_t(\boldsymbol{u} - \mathcal{I}\boldsymbol{u}_h^{\Delta t}), \boldsymbol{v}_h^{\Delta t}\right) - b^T\left(\boldsymbol{v}_h^{\Delta t}, \partial_t(p - \mathcal{I}p_h^{\Delta t})\right) = 0, \quad \forall \boldsymbol{v}_h^{\Delta t} \in \boldsymbol{\Sigma}_h^{\Delta t}.$$

The first term can be rewritten as, using (6.20) and (6.7),

$$
\begin{aligned}
a^T\left(\partial_t(\boldsymbol{u}-\mathcal{I}\boldsymbol{u}_h^{\Delta t}),\boldsymbol{v}_h^{\Delta t}\right) &= a^T\left(\partial_t(\boldsymbol{u}-\boldsymbol{\Pi}_h\boldsymbol{u}),\boldsymbol{v}_h^{\Delta t}\right)+a^T\left(\partial_t(\boldsymbol{\Pi}_h\boldsymbol{u}-\mathcal{I}\boldsymbol{u}_h^{\Delta t}),\boldsymbol{v}_h^{\Delta t}\right)\\
&= a^T\left(\partial_t(\boldsymbol{u}-\boldsymbol{\Pi}_h\boldsymbol{u}),\boldsymbol{v}_h^{\Delta t}\right)+a^T\left(\widetilde{\partial}_t(\widetilde{\boldsymbol{\Pi}}_h^{\Delta t}\boldsymbol{u}-\boldsymbol{u}_h^{\Delta t}),\boldsymbol{v}_h^{\Delta t}\right).
\end{aligned}
\tag{6.37}
$$

Regarding the second term in (6.36), we have

$$
\begin{aligned}
& b^T\left(\boldsymbol{v}_h^{\Delta t},\partial_t(p-\mathcal{I}p_h^{\Delta t})\right)\\
&\quad=\sum_i\left(\operatorname{div}\boldsymbol{v}_{h,i}^{\Delta t},\partial_t(p_i-\mathcal{I}p_{h,i}^{\Delta t})\right)_{\Omega_i^T}+\left(\operatorname{div}_\tau\boldsymbol{v}_{h,\gamma}^{\Delta t},\partial_t(p_\gamma-\mathcal{I}p_{h,\gamma}^{\Delta t})\right)_{\gamma^T}\\
&\qquad-\left([\![\boldsymbol{v}_h^{\Delta t}\cdot\boldsymbol{n}]\!],\partial_t(p_\gamma-\mathcal{P}_{h,\gamma}p_\gamma)\right)_{\gamma^T}-\left([\![\boldsymbol{v}_h^{\Delta t}\cdot\boldsymbol{n}]\!],\partial_t(\mathcal{P}_{h,\gamma}p_\gamma-\mathcal{I}p_{h,\gamma}^{\Delta t})\right)_{\gamma^T}\\
&\quad=b^T\left(\boldsymbol{v}_h^{\Delta t},\partial_t(\mathcal{I}\widetilde{\mathcal{P}}_h^{\Delta t}p-\mathcal{I}p_h^{\Delta t})\right)-\left([\![\boldsymbol{v}_h^{\Delta t}\cdot\boldsymbol{n}]\!],\partial_t(p_\gamma-\mathcal{P}_{h,\gamma}p_\gamma)\right)_{\gamma^T},
\end{aligned}
\tag{6.38}
$$

where we used (6.20)–(6.22) and (6.7) in the last equality. Substituting (6.37) and (6.38) into (6.36) and choosing $\boldsymbol{v}_h^{\Delta t}=\widetilde{\boldsymbol{\Pi}}_h^{\Delta t}\boldsymbol{u}-\boldsymbol{u}_h^{\Delta t}$, we get

$$
\begin{aligned}
& a^T\left(\partial_t(\boldsymbol{u}-\boldsymbol{\Pi}_h\boldsymbol{u}),\widetilde{\boldsymbol{\Pi}}_h^{\Delta t}\boldsymbol{u}-\boldsymbol{u}_h^{\Delta t}\right)+a^T\left(\widetilde{\partial}_t(\widetilde{\boldsymbol{\Pi}}_h^{\Delta t}\boldsymbol{u}-\boldsymbol{u}_h^{\Delta t}),\widetilde{\boldsymbol{\Pi}}_h^{\Delta t}\boldsymbol{u}-\boldsymbol{u}_h^{\Delta t}\right)\\
&\quad-b^T\left(\widetilde{\boldsymbol{\Pi}}_h^{\Delta t}\boldsymbol{u}-\boldsymbol{u}_h^{\Delta t},\partial_t(\mathcal{I}\widetilde{\mathcal{P}}_h^{\Delta t}p-\mathcal{I}p_h^{\Delta t})\right)\\
&\quad-\left([\![(\widetilde{\boldsymbol{\Pi}}_h^{\Delta t}\boldsymbol{u}-\boldsymbol{u}_h^{\Delta t})\cdot\boldsymbol{n}]\!],\partial_t(p_\gamma-\mathcal{P}_{h,\gamma}p_\gamma)\right)_{\gamma^T}\\
&=0.
\end{aligned}
\tag{6.39}
$$

Using (6.7) and (6.21)–(6.22), we rewrite the second equation of (6.31) as

$$
c_s^T\left(\partial_t(\mathcal{I}\widetilde{\mathcal{P}}_h^{\Delta t}p-\mathcal{I}p_h^{\Delta t}),\mu_h^{\Delta t}\right)+b^T\left(\widetilde{\boldsymbol{\Pi}}_h^{\Delta t}\boldsymbol{u}-\boldsymbol{u}_h^{\Delta t},\mu_h^{\Delta t}\right)+b^T\left(\boldsymbol{u}-\widetilde{\boldsymbol{\Pi}}_h^{\Delta t}\boldsymbol{u},\mu_h^{\Delta t}\right)=0,
\tag{6.40}
$$

for all $\mu_h^{\Delta t}\in M_h^{\Delta t}$. Choose $\mu_h^{\Delta t}=\partial_t(\mathcal{I}\widetilde{\mathcal{P}}_h^{\Delta t}p-\mathcal{I}p_h^{\Delta t})$ in (6.40) and sum the resulting equation with (6.39) to obtain

$$
\begin{aligned}
& c_s^T\left(\partial_t(\mathcal{I}\widetilde{\mathcal{P}}_h^{\Delta t}p-\mathcal{I}p_h^{\Delta t}),\partial_t(\mathcal{I}\widetilde{\mathcal{P}}_h^{\Delta t}p-\mathcal{I}p_h^{\Delta t})\right)+a^T\left(\widetilde{\partial}_t(\widetilde{\boldsymbol{\Pi}}_h^{\Delta t}\boldsymbol{u}-\boldsymbol{u}_h^{\Delta t}),\widetilde{\boldsymbol{\Pi}}_h^{\Delta t}\boldsymbol{u}-\boldsymbol{u}_h^{\Delta t}\right)\\
&\quad=-a^T\left(\partial_t(\boldsymbol{u}-\boldsymbol{\Pi}_h^{\Delta t}\boldsymbol{u}),\widetilde{\boldsymbol{\Pi}}_h^{\Delta t}\boldsymbol{u}-\boldsymbol{u}_h^{\Delta t}\right)-b^T\left(\boldsymbol{u}-\widetilde{\boldsymbol{\Pi}}_h^{\Delta t}\boldsymbol{u},\partial_t(\mathcal{I}\widetilde{\mathcal{P}}_h^{\Delta t}p-\mathcal{I}p_h^{\Delta t})\right)\\
&\qquad-\left([\![(\widetilde{\boldsymbol{\Pi}}_h^{\Delta t}\boldsymbol{u}-\boldsymbol{u}_h^{\Delta t})\cdot\boldsymbol{n}]\!],\partial_t(p_\gamma-\mathcal{P}_{h,\gamma}p_\gamma)\right)_{\gamma^T}.
\end{aligned}
$$

Similarly to (6.15), from this we obtain

$$
\begin{aligned}
&\|\partial_t(\mathcal{I}\widetilde{\mathcal{P}}_h^{\Delta t}p-\mathcal{I}p_h^{\Delta t})\|^2_{L^2(0,T;M)}+\|\widetilde{\boldsymbol{\Pi}}_h^{\Delta t}\boldsymbol{u}-\boldsymbol{u}_h^{\Delta t}\|^2_{\boldsymbol{M}^*,DG}\\
&\quad\le\epsilon\Big(\|\widetilde{\boldsymbol{\Pi}}_h^{\Delta t}\boldsymbol{u}-\boldsymbol{u}_h^{\Delta t}\|^2_{L^2(0,T;\boldsymbol{M}^*)}+\|\partial_t(\mathcal{I}\widetilde{\mathcal{P}}_h^{\Delta t}p-\mathcal{I}p_h^{\Delta t})\|^2_{L^2(0,T;M)}\Big)\\
&\qquad+C_\epsilon\Big(\|\partial_t(\boldsymbol{u}-\boldsymbol{\Pi}_h\boldsymbol{u})\|^2_{L^2(0,T;\boldsymbol{M}^*)}+\|\operatorname{div}(\boldsymbol{u}-\widetilde{\boldsymbol{\Pi}}_h^{\Delta t}\boldsymbol{u})\|^2_{L^2(0,T;M)}\\
&\qquad+\sum_{i=1}^2\|(\boldsymbol{u}_i-\widetilde{\boldsymbol{\Pi}}_{h,i}^{\Delta t}\boldsymbol{u}_i)\cdot\boldsymbol{n}_i\|_{L^2(0,T;M_\gamma)}+\|\partial_t(p_\gamma-\mathcal{P}_{h,\gamma}p_\gamma)\|^2_{L^2(0,T;M_\gamma)}\Big)\\
&\qquad+C\|\boldsymbol{\Pi}_h\boldsymbol{u}_0-\boldsymbol{u}_{h,0}\|^2_{\boldsymbol{M}^*},
\end{aligned}
\tag{6.41}
$$

where we used that $\boldsymbol{u}(0)=\boldsymbol{u}_0$. Combining (6.34), (6.35), and (6.41) and choosing ϵ small enough, we obtain

$$
\begin{aligned}
(6.42)\quad & \|\widetilde{\boldsymbol{\Pi}}_h^{\Delta t}\boldsymbol{u}-\boldsymbol{u}_h^{\Delta t}\|^2_{L^2(0,T;\boldsymbol{M}^*)}+\|\widetilde{\mathcal{P}}_h^{\Delta t}p-p_h^{\Delta t}\|^2_{M,DG}+\|\widetilde{\mathcal{P}}_h^{\Delta t}p-p_h^{\Delta t}\|_{L^2(0,T;M)}\\
&\quad+\|\partial_t(\mathcal{I}\widetilde{\mathcal{P}}_h^{\Delta t}p-\mathcal{I}p_h^{\Delta t})\|^2_{L^2(0,T;M)}+\|\widetilde{\boldsymbol{\Pi}}_h^{\Delta t}\boldsymbol{u}-\boldsymbol{u}_h^{\Delta t}\|^2_{\boldsymbol{M}^*,DG}\\
&\le \epsilon\|\operatorname{div}(\widetilde{\boldsymbol{\Pi}}_h^{\Delta t}\boldsymbol{u}-\boldsymbol{u}_h^{\Delta t})\|^2_{L^2(0,T;M)}\\
&\quad+C\Big(\|\widetilde{\boldsymbol{\Pi}}_h^{\Delta t}\boldsymbol{u}-\boldsymbol{u}\|^2_{L^2(0,T;\boldsymbol{\Sigma})}+\|p-\widetilde{\mathcal{P}}_h^{\Delta t}p\|^2_{L^2(0,T;M)}\\
&\qquad+\|\partial_t(\boldsymbol{u}-\boldsymbol{\Pi}_h\boldsymbol{u})\|^2_{L^2(0,T;\boldsymbol{M}^*)}+\|\partial_t(p_\gamma-\mathcal{P}_{h,\gamma}p_\gamma)\|^2_{L^2(0,T;M_\gamma)}\\
&\qquad+\|\mathcal{P}_hp_0-p_{h,0}\|^2_M\\
&\qquad+\|\boldsymbol{\Pi}_h\boldsymbol{u}_0-\boldsymbol{u}_{h,0}\|^2_{\boldsymbol{M}^*}\Big).
\end{aligned}
$$

Finally, we choose $\mu_h^{\Delta t}=\operatorname{div}(\widetilde{\boldsymbol{\Pi}}_h^{\Delta t}\boldsymbol{u}-\boldsymbol{u}_h^{\Delta t})$ in (6.40) and obtain

$$
\begin{aligned}
(6.43)\quad & \|\operatorname{div}(\widetilde{\boldsymbol{\Pi}}_h^{\Delta t}\boldsymbol{u}-\boldsymbol{u}_h^{\Delta t})\|_{L^2(0,T;M)}\le C\Bigg(\|\partial_t(\mathcal{I}\widetilde{\mathcal{P}}_h^{\Delta t}p-\mathcal{I}p_h^{\Delta t})\|_{L^2(0,T;M)}\\
&\quad+\sum_{i=1}^{2}\|(\widetilde{\boldsymbol{\Pi}}_{h,i}^{\Delta t}\boldsymbol{u}_i-\boldsymbol{u}_{h,i}^{\Delta t})\cdot\boldsymbol{n}_i\|_{L^2(0,T;M_\gamma)}+\|\boldsymbol{u}-\widetilde{\boldsymbol{\Pi}}_h^{\Delta t}\boldsymbol{u}\|_{L^2(0,T;\boldsymbol{\Sigma})}\Bigg).
\end{aligned}
$$

The proof is completed by using (6.42) and (6.43), the triangle inequality, the second equation in (6.17), the approximation properties (6.23)–(6.29) of the space-time interpolants $\widetilde{\mathcal{P}}_h^{\Delta t}$ and $\widetilde{\boldsymbol{\Pi}}_h^{\Delta t}$, the bounds for discrete initial data (4.27), and the spatial-only version of (5.1)–(5.7). □

7. Global-in-time domain decomposition

As in the steady-state flow case [39], when $\xi=1$, we can use DD to solve the coupled problem (2.3) efficiently. Moreover, here we consider the global-in-time DD approach where the monolithic system (2.3) is reformulated through the use of interface operators as a problem on the space-time fracture-interface.

7.1. **Reduction to an interface problem.** For $\xi=1$, the fifth equation of (2.3) takes a simpler form as follows:

$$
(7.1)\qquad -\boldsymbol{u}_i\cdot\boldsymbol{n}_i+\kappa_\gamma p_i=\kappa_\gamma p_\gamma,\quad \text{on } \gamma\times(0,T),\quad i=1,2.
$$

We impose this equation as Robin boundary conditions for the subdomain problems:

$$
(7.2)\qquad
\begin{aligned}
s_i\partial_t p_i+\operatorname{div}\boldsymbol{u}_i&=q_i && \text{in } \Omega_i\times(0,T),\\
\boldsymbol{u}_i&=-\boldsymbol{K}_i\nabla p_i && \text{in } \Omega_i\times(0,T),\\
-\boldsymbol{u}_i\cdot\boldsymbol{n}_i+\kappa_\gamma p_i&=\kappa_\gamma p_\gamma && \text{in } \gamma\times(0,T),\\
p_i&=0 && \text{on } (\partial\Omega_i\cap\partial\Omega)\times(0,T),\\
p_i(\cdot,0)&=p_{0,i} && \text{in } \Omega_i,
\end{aligned}
$$

for $i=1,2$. We denote by $(\boldsymbol{u}_i(p_\gamma,q_i,p_{0,i}),p_i(p_\gamma,q_i,p_{0,i}))$ the solution to (7.2) and define the time-dependent Robin-to-Neumann operator as follows:

$$
\begin{aligned}
\mathcal{S}_i^{\mathrm{RtN}}:L^2(0,T;M_\gamma)\times L^2(0,T;M_i)\times H_*^1(\Omega_i)\to L^2(0,T;M_\gamma),&\\
\mathcal{S}_i^{\mathrm{RtN}}(p_\gamma,q_i,p_{0,i})\mapsto-\boldsymbol{u}_i(p_\gamma,q_i,p_{0,i})\cdot\boldsymbol{n}_i|_{\gamma^T}.&
\end{aligned}
$$

Problem (2.3) is reduced to an interface problem with unknowns p_γ and $\boldsymbol{u}_\gamma$:

$$\begin{aligned} s_\gamma \partial_t p_\gamma + \operatorname{div}_\tau \boldsymbol{u}_\gamma &= q_\gamma - \textstyle\sum_{i=1}^2 \mathcal{S}_i^{\mathrm{RtN}}(p_\gamma, q_i, p_{0,i}) && \text{in } \gamma \times (0,T), \\ \boldsymbol{u}_\gamma &= -\boldsymbol{K}_\gamma \nabla_\tau p_\gamma && \text{in } \gamma \times (0,T), \\ p_\gamma &= 0 && \text{on } \partial\gamma \times (0,T), \\ p_\gamma(\cdot, 0) &= p_{0,\gamma} && \text{in } \gamma, \end{aligned} \tag{7.3}$$

or equivalently

$$\begin{aligned} s_\gamma \partial_t p_\gamma + \operatorname{div}_\tau \boldsymbol{u}_\gamma + \textstyle\sum_{i=1}^2 \mathcal{S}_i^{\mathrm{RtN}}(p_\gamma, 0, 0) &= -\textstyle\sum_{i=1}^2 \mathcal{S}_i^{\mathrm{RtN}}(0, q_i, p_{0,i}) && \text{in } \gamma \times (0,T), \\ \boldsymbol{u}_\gamma &= -\boldsymbol{K}_\gamma \nabla_\tau p_\gamma && \text{in } \gamma \times (0,T), \\ p_\gamma &= 0 && \text{on } \partial\gamma \times (0,T), \\ p_\gamma(\cdot, 0) &= p_{0,\gamma} && \text{in } \gamma. \end{aligned} \tag{7.4}$$

Define

$$\begin{aligned} \mathcal{S} : L^2(0,T;M_\gamma) &\to L^2(0,T;M_\gamma), \\ p_\gamma &\mapsto \mathcal{S} p_\gamma = \sum_i \mathcal{S}_i^{\mathrm{RtN}}(p_\gamma, 0, 0). \end{aligned} \tag{7.5}$$

The weak form of the interface problem (7.4) is given by

$$\begin{aligned} (s_\gamma \partial_t p_\gamma, \mu_\gamma)_{\gamma^T} + (\operatorname{div}_\tau \boldsymbol{u}_\gamma, \mu_\gamma)_{\gamma^T} + (\mathcal{S} p_\gamma, \mu_\gamma)_{\gamma^T} &= (q_\gamma, \mu_\gamma)_{\gamma^T} + (\chi, \mu_\gamma)_{\gamma^T}, \\ & \qquad \forall \mu_\gamma \in L^2(0,T;M_\gamma), \\ \left(\boldsymbol{K}_\gamma^{-1} \boldsymbol{u}_\gamma, \boldsymbol{v}_\gamma\right)_{\gamma^T} - (\operatorname{div}_\tau \boldsymbol{v}_\gamma, p_\gamma)_{\gamma^T} &= 0, \ \forall \boldsymbol{v}_\gamma \in L^2(0,T;\boldsymbol{\Sigma}_\gamma), \end{aligned} \tag{7.6}$$

where $\chi \in L^2(0,T;M_\gamma)$ is defined as

$$\chi = -\sum_{i=1}^2 \mathcal{S}_i^{\mathrm{RtN}}(0, q_i, p_{0,i}).$$

Let $a_\gamma(\cdot,\cdot)$, $b_\gamma(\cdot,\cdot)$ and $c_{s,\gamma}(\cdot,\cdot)$ be bilinear forms on $\boldsymbol{\Sigma}_\gamma \times \boldsymbol{\Sigma}_\gamma$, $\boldsymbol{\Sigma}_\gamma \times M_\gamma$, and $M_\gamma \times M_\gamma$, respectively:

$$\begin{aligned} a_\gamma(\boldsymbol{u}_\gamma, \boldsymbol{v}_\gamma) = \left(\boldsymbol{K}_\gamma^{-1} \boldsymbol{u}_\gamma, \boldsymbol{v}_\gamma\right)_\gamma, \quad b_\gamma(\boldsymbol{v}_\gamma, p_\gamma) = (\operatorname{div}_\tau \boldsymbol{v}_\gamma, p_\gamma)_\gamma, \\ c_{s,\gamma}(p_\gamma, \mu_\gamma) = (s_\gamma p_\gamma, \mu_\gamma)_\gamma. \end{aligned} \tag{7.7}$$

Problem (7.6) can be rewritten as: find $(\boldsymbol{u}_\gamma, p_\gamma) \in L^2(0,T;\boldsymbol{\Sigma}_\gamma) \times L^2(0,T;M_\gamma)$ such that

$$\begin{aligned} a_\gamma^T(\boldsymbol{u}_\gamma, \boldsymbol{v}_\gamma) - b_\gamma^T(\boldsymbol{v}_\gamma, p_\gamma) &= 0, \ \forall \boldsymbol{v}_\gamma \in L^2(0,T;\boldsymbol{\Sigma}_\gamma), \\ c_{s,\gamma}^T(\partial_t p_\gamma, \mu_\gamma) + (\mathcal{S} p_\gamma, \mu_\gamma)_{\gamma^T} + b_\gamma^T(\boldsymbol{u}_\gamma, \mu_\gamma) &= (q_\gamma, \mu_\gamma)_{\gamma^T} + (\chi, \mu_\gamma)_{\gamma^T}, \\ & \qquad \forall \mu_\gamma \in L^2(0,T;M_\gamma). \end{aligned} \tag{7.8}$$

7.2. **Discrete interface problem and GMRES convergence.** Under the discretization by mixed finite elements in space and discontinuous Galerkin in time as presented in Section 3, the discrete counterpart of the Robin-to-Neumann operator $\mathcal{S}$ defined in (7.5) is given by

$$\mathcal{S}_h^{\Delta t} : M_{h,\gamma}^{\Delta t} \to M_{h,\gamma}^{\Delta t}, \quad \left(\mathcal{S}_h^{\Delta t} p_{h,\gamma}^{\Delta t}, \mu_{h,\gamma}^{\Delta t}\right)_{\gamma^T} = \sum_i \left(-\boldsymbol{u}_{h,i}^{\Delta t}(p_{h,\gamma}^{\Delta t}) \cdot \boldsymbol{n}_i, \mu_{h,\gamma}^{\Delta t}\right)_{\gamma^T}, \tag{7.9}$$

where $\left(\boldsymbol{u}_{h,i}^{\Delta t}(p_{h,\gamma}^{\Delta t}), p_{h,i}^{\Delta t}(p_{h,\gamma}^{\Delta t})\right) \in \boldsymbol{\Sigma}_{h,i}^{\Delta t} \times M_{h,i}^{\Delta t}$ is the solution to

$$\begin{aligned} a_i^T\left(\boldsymbol{u}_{h,i}^{\Delta t}, \boldsymbol{v}_{h,i}^{\Delta t}\right) - b_i^T\left(\boldsymbol{v}_{h,i}^{\Delta t}, p_{h,i}^{\Delta t}\right) &= -\left(p_{h,\gamma}^{\Delta t}, \boldsymbol{v}_{h,i}^{\Delta t}\cdot\boldsymbol{n}_i\right)_{\gamma^T} && \forall \boldsymbol{v}_{h,i}^{\Delta t} \in \boldsymbol{\Sigma}_{h,i}^{\Delta}, \\ c_{s,i}^T\left(\widetilde{\partial}_t p_{h,i}^{\Delta t}, \mu_{h,i}^{\Delta t}\right) + b_i^T\left(\boldsymbol{u}_{h,i}^{\Delta t}, \mu_{h,i}^{\Delta t}\right) &= 0 && \forall \mu_{h,i}^{\Delta t} \in M_{h,i}^{\Delta t}, \end{aligned} \tag{7.10}$$

with a zero initial condition, i.e.,

$$(p_{h,i}^{\Delta t})_0^- = 0, \tag{7.11}$$

where $a_i(\cdot,\cdot)$, $b_i(\cdot,\cdot)$ and $c_{s,i}(\cdot,\cdot)$ are bilinear forms on $\boldsymbol{\Sigma}_i \times \boldsymbol{\Sigma}_i$, $\boldsymbol{\Sigma}_i \times M_i$, and $M_i \times M_i$, respectively, and are given by

$$\begin{aligned} a_i\left(\boldsymbol{u}_i, \boldsymbol{v}_i\right) &= \left(\boldsymbol{K}_i^{-1}\boldsymbol{u}_i, \boldsymbol{v}_i\right)_{\Omega_i} + \left(\kappa_\gamma^{-1}\boldsymbol{u}_i\cdot\boldsymbol{n}_i, \boldsymbol{v}_i\cdot\boldsymbol{n}_i\right)_\gamma, \\ b_i\left(\boldsymbol{u}_i, \mu_i\right) &= \left(\operatorname{div}\boldsymbol{u}_i, \mu_i\right)_{\Omega_i}, \quad c_{s,i}\left(\eta_i, \mu_i\right) = \left(s_i\eta_i, \mu_i\right)_{\Omega_i}. \end{aligned}$$

Similarly, $\chi_h^{\Delta t} \in M_{h,\gamma}^{\Delta t}$ is defined as

$$(\chi_h^{\Delta t}, \mu_{h,\gamma}^{\Delta t})_{\gamma^T} = \sum_i \left(\overline{\boldsymbol{u}}_{h,i}^{\Delta t}\cdot\boldsymbol{n}_i\mu_{h,\gamma}^{\Delta t}\right)_{\gamma^T},$$

where $\left(\overline{\boldsymbol{u}}_{h,i}^{\Delta t}, \overline{p}_{h,i}^{\Delta t}\right) \in \boldsymbol{\Sigma}_{h,i}^{\Delta t} \times M_{h,i}^{\Delta t}$ is the solution to

$$\begin{aligned} a_i^T\left(\overline{\boldsymbol{u}}_{h,i}^{\Delta t}, \boldsymbol{v}_{h,i}^{\Delta t}\right) - b_i^T\left(\boldsymbol{v}_{h,i}^{\Delta t}, \overline{p}_{h,i}^{\Delta t}\right) &= 0 && \forall \boldsymbol{v}_{h,}^{\Delta t} \in \boldsymbol{\Sigma}_{h,i}^{\Delta}, \\ c_{s,i}^T\left(\widetilde{\partial}_t \overline{p}_{h,i}^{\Delta t}, \mu_{h,i}^{\Delta t}\right) + b_i^T\left(\overline{\boldsymbol{u}}_{h,i}^{\Delta t}, \mu_{h,i}^{\Delta t}\right) &= (q_i, \mu_{h,i}^{\Delta t})_{\Omega_i^T} && \forall \mu_{h,i}^{\Delta t} \in M_{h,i}^{\Delta t}, \end{aligned} \tag{7.12}$$

with the initial condition $(\overline{p}_{h,i}^{\Delta t})_0^- = p_{h,0,i}$.

The interface problem (7.6) after discretization becomes:

Find $(\boldsymbol{u}_{h,\gamma}^{\Delta t}, p_{h,\gamma}^{\Delta t}) \in \boldsymbol{\Sigma}_{h,\gamma}^{\Delta t} \times M_{h,\gamma}^{\Delta t}$ such that for all $\boldsymbol{v}_{h,\gamma}^{\Delta t} \in \boldsymbol{\Sigma}_{h,\gamma}^{\Delta t}$ and $\mu_{h,\gamma}^{\Delta t} \in M_{h,\gamma}^{\Delta t}$,

$$\begin{aligned} &a_\gamma^T(\boldsymbol{u}_{h,\gamma}^{\Delta t}, \boldsymbol{v}_{h,\gamma}^{\Delta t}) - b_\gamma^T(\boldsymbol{v}_{h,\gamma}^{\Delta t}, p_{h,\gamma}^{\Delta t}) = 0, \\ &c_{s,\gamma}^T(\widetilde{\partial}_t p_{h,\gamma}^{\Delta t}, \mu_{h,\gamma}^{\Delta t}) + \left(\mathcal{S}_h^{\Delta t} p_{h,\gamma}^{\Delta t}, \mu_{h,\gamma}^{\Delta t}\right)_{\gamma^T} + b_\gamma^T(\boldsymbol{u}_{h,\gamma}^{\Delta t}, \mu_{h,\gamma}^{\Delta t}) = (q_\gamma, \mu_{h,\gamma}^{\Delta t})_{\gamma^T} + (\chi_h^{\Delta t}, \mu_{h,\gamma}^{\Delta t})_{\gamma^T}, \end{aligned} \tag{7.13}$$

subject to the initial condition $(p_{h,\gamma}^{\Delta t})_0^- = p_{h,\gamma,0}$. The corresponding algebraic system for (7.13) takes the form:

$$\begin{bmatrix} \boldsymbol{A}_\gamma & \boldsymbol{B}_\gamma^T \\ \boldsymbol{B}_\gamma & \boldsymbol{C}_\gamma + \boldsymbol{S} \end{bmatrix} \begin{bmatrix} \boldsymbol{U}_\gamma \\ \boldsymbol{P}_\gamma \end{bmatrix} = \begin{bmatrix} 0 \\ * \end{bmatrix}, \tag{7.14}$$

where $\boldsymbol{U}_\gamma$ and $\boldsymbol{P}_\gamma$ represent the space-time velocity and pressure unknowns on the fracture over the time interval $[0,T]$; here for simplicity, we do not write the right-hand side explicitly. It should be noted that the matrix $\boldsymbol{S}$ associated with the discrete Robin-to-Neumann operator $\mathcal{S}_h^{\Delta t}$ is not computed explicitly. Its action $\boldsymbol{SP}_\gamma$ is computed by solving the subdomain problems (7.10) over $[0,T]$.

The space-time interface problem (7.13) is solved iteratively using GMRES, each iteration involves solution of time-dependent problems in the subdomains using the method of lines with local mesh sizes and time step sizes. To study the convergence of GMRES iterations, we rewrite the second equation in (7.13) as

$$\int_\gamma s_\gamma\Bigg(\sum_{n=1}^{N_\gamma}\int_{t_\gamma^{n-1}}^{t_\gamma^n}\partial_t p_{h,\gamma}^{\Delta t}\,\mu_{h,\gamma}^{\Delta t}+\sum_{n=2}^{N_\gamma}[p_{h,\gamma}^{\Delta t}]_{n-1}(\mu_{h,\gamma}^{\Delta t})_{n-1}^{+}+(p_{h,\gamma}^{\Delta t})_0^{+}(\mu_{h,\gamma}^{\Delta t})_0^{+}\Bigg)$$
$$+\big(\mathcal{S}_h^{\Delta t}p_{h,\gamma}^{\Delta t},\mu_{h,\gamma}^{\Delta t}\big)_{\gamma^T}+b_\gamma^T(\boldsymbol{u}_{h,\gamma}^{\Delta},\mu_{h,\gamma}^{\Delta t})$$
$$=\int_\gamma s_\gamma(p_{h,\gamma}^{\Delta t})_0^{-}(\mu_{h,\gamma}^{\Delta t})_0^{+}+\big(q_\gamma,\mu_{h,\gamma}^{\Delta t}\big)_{\gamma^T}+\big(\chi_h^{\Delta t},\mu_{h,\gamma}^{\Delta t}\big)_{\gamma^T},\quad\forall\mu_{h,\gamma}^{\Delta t}\in M_{h,\gamma}^{\Delta t},$$

and define the interface operator $\mathcal{B}_h^{\Delta t}:\boldsymbol{\Sigma}_{h,\gamma}^{\Delta t}\times M_{h,\gamma}^{\Delta t}\to\boldsymbol{\Sigma}_{h,\gamma}^{\Delta t}\times M_{h,\gamma}^{\Delta t}$ as follows:
(7.15)

$$\begin{aligned}&\big\langle\mathcal{B}_h^{\Delta t}(\boldsymbol{u}_{h,\gamma}^{\Delta t},p_{h,\gamma}^{\Delta t}),(\boldsymbol{v}_{h,\gamma}^{\Delta t},\mu_{h,\gamma}^{\Delta t})\big\rangle\\&\quad=a_\gamma^T(\boldsymbol{u}_{h,\gamma}^{\Delta t},\boldsymbol{v}_{h,\gamma}^{\Delta t})-b_\gamma^T(\boldsymbol{v}_{h,\gamma}^{\Delta t},p_{h,\gamma}^{\Delta t})+\big(\mathcal{S}_h^{\Delta t}p_{h,\gamma}^{\Delta t},\mu_{h,\gamma}^{\Delta t}\big)_{\gamma^T}+b_\gamma^T(\boldsymbol{u}_{h,\gamma}^{\Delta},\mu_{h,\gamma}^{\Delta t})\\&\qquad+\int_\gamma s_\gamma\Bigg(\sum_{n=1}^{N_\gamma}\int_{t_\gamma^{n-1}}^{t_\gamma^n}\partial_t p_{h,\gamma}^{\Delta t}\,\mu_{h,\gamma}^{\Delta t}+\sum_{n=2}^{N_\gamma}[p_{h,\gamma}^{\Delta t}]_{n-1}(\mu_{h,\gamma}^{\Delta t})_{n-1}^{+}+(p_{h,\gamma}^{\Delta t})_0^{+}(\mu_{h,\gamma}^{\Delta t})_0^{+}\Bigg),\end{aligned}$$

for $(\boldsymbol{v}_{h,\gamma}^{\Delta t},\mu_{h,\gamma}^{\Delta t})\in\boldsymbol{\Sigma}_{h,\gamma}^{\Delta t}\times M_{h,\gamma}^{\Delta t}$. Then the fully discrete interface problem (7.13) can be rewritten in the compact form:
(7.16)

$$\big\langle\mathcal{B}_h^{\Delta t}(\boldsymbol{u}_{h,\gamma}^{\Delta t},p_{h,\gamma}^{\Delta t}),(\boldsymbol{v}_{h,\gamma}^{\Delta t},\mu_{h,\gamma}^{\Delta t})\big\rangle=\int_\gamma s_\gamma(p_{h,\gamma}^{\Delta t})_0^{-}(\mu_{h,\gamma}^{\Delta t})_0^{+}+\big(q_\gamma,\mu_{h,\gamma}^{\Delta t}\big)_{\gamma^T}+\big(\chi_h^{\Delta t},\mu_{h,\gamma}^{\Delta t}\big)_{\gamma^T},$$
$$\forall(\boldsymbol{v}_{h,\gamma}^{\Delta t},\mu_{h,\gamma}^{\Delta t})\in\boldsymbol{\Sigma}_{h,\gamma}^{\Delta t}\times M_{h,\gamma}^{\Delta t}.$$

In the following, we use the framework in [37, Chapter 3] based on the field-of-values analysis [17] to analyze the convergence of GMRES applied to the interface problem (7.13).

Lemma 7.1 ([37, Corollary 3.3.1])**.** *Let $\mathcal{O}$ be a finite dimensional Hilbert space equipped with an inner product $(\cdot,\cdot)$ and a norm $\|\cdot\|$ and $\boldsymbol{\mathcal{A}}:\mathcal{O}\to\mathcal{O}$ an invertible linear operator. Consider solving the linear system $\boldsymbol{\mathcal{A}}\boldsymbol{x}=\boldsymbol{b}$ by GMRES, where $\boldsymbol{0}\neq\boldsymbol{b}\in\boldsymbol{H}$. The m-th residual $\boldsymbol{r}_m:=\boldsymbol{b}-\boldsymbol{\mathcal{A}}\boldsymbol{x}_m$ of GMRES is bounded by*

$$\frac{\|\boldsymbol{r}_m\|}{\|\boldsymbol{r}_0\|}\le\left(1-\frac{\theta^2}{\Theta^2}\right)^{m/2},\tag{7.17}$$

where

$$\theta\le\frac{|(\boldsymbol{v},\boldsymbol{\mathcal{A}}\boldsymbol{v})|}{\|\boldsymbol{v}\|^2},\qquad\frac{\|\boldsymbol{\mathcal{A}}\boldsymbol{v}\|}{\|\boldsymbol{v}\|}\le\Theta,\quad\forall\boldsymbol{0}\neq\boldsymbol{v}\in\boldsymbol{H}.\tag{7.18}$$

We shall apply Lemma 7.1 for the interface problem (7.16) with the linear operator $\boldsymbol{\mathcal{A}}:=\mathcal{B}_h^{\Delta t}$ and the Hilbert space $\mathcal{O}:=\boldsymbol{\Sigma}_{h,\gamma}^{\Delta t}\times M_{h,\gamma}^{\Delta t}$ equipped with the norm

$$\|(\boldsymbol{u}_{h,\gamma}^{\Delta t},p_{h,\gamma}^{\Delta t})\|_{\mathcal{O}}^2=\|\boldsymbol{u}_{h,\gamma}^{\Delta t}\|_{L^2(0,T;\boldsymbol{M}_\gamma)}^2+\|p_{h,\gamma}^{\Delta t}\|_{L^2(0,T;M_\gamma)}^2+\|(p_{h,\gamma}^{\Delta t})_0^{+}\|_{M_\gamma}^2,\tag{7.19}$$

for $(\boldsymbol{u}_{h,\gamma}^{\Delta t},p_{h,\gamma}^{\Delta t})\in\boldsymbol{\Sigma}_{h,\gamma}^{\Delta t}\times M_{h,\gamma}^{\Delta t}$. We begin with the properties of the discrete Robin-to-Neumann operator $\mathcal{S}_h^{\Delta t}$.

Lemma 7.2. *Assume that the subdomain meshes $\mathcal{T}_{h,i}$ are quasi-uniform and $h\le Ch_i$, for $i=1,2$. The discrete operator $\mathcal{S}_h^{\Delta t}$ is nonnegative and continuous.*

Proof. Choosing $\boldsymbol{v}_{h,}^{\Delta t} = \boldsymbol{u}_{h,i}^{\Delta t}(\mu_{h,\gamma}^{\Delta t})$ and $\mu_{h,i}^{\Delta t} = p_{h,i}^{\Delta t}(\mu_{h,\gamma}^{\Delta t})$ in (7.10), for $\mu_{h,\gamma}^{\Delta t} \in M_{h,\gamma}^{\Delta t}$, we obtain

(7.20)
$$a_i^T\left(\boldsymbol{u}_{h,i}^{\Delta t}(p_{h,\gamma}^{\Delta t}), \boldsymbol{u}_{h,i}^{\Delta t}(\mu_{h\gamma}^{\Delta t})\right) - b_i^T\left(\boldsymbol{u}_{h,i}^{\Delta t}(\mu_{h,\gamma}^{\Delta t}), p_{h,i}^{\Delta t}(p_{h,\gamma}^{\Delta t})\right) = -\left(p_{h,\gamma}^{\Delta t}, \boldsymbol{u}_{h,i}(\mu_{h,\gamma}^{\Delta t})\cdot\boldsymbol{n}_i\right)_\gamma,$$
$$c_{s,i}^T\left(\widetilde{\partial}_t p_{h,i}^{\Delta t}(p_{h,\gamma}^{\Delta t}), p_{h,i}^{\Delta t}(\mu_{h,\gamma}^{\Delta t})\right) + b_i^T\left(\boldsymbol{u}_{h,i}^{\Delta t}(p_{h,\gamma}^{\Delta t}), p_{h,i}^{\Delta t}(\mu_{h,\gamma}^{\Delta t})\right) = 0.$$

Adding the two equations where the roles of $p_{h,\gamma}^{\Delta t}$ and $\mu_{h,\gamma}^{\Delta t}$ have been interchanged in the first equation, we obtain

$$\begin{aligned} &c_{s,i}^T\left(\widetilde{\partial}_t p_{h,i}^{\Delta t}(p_{h,\gamma}^{\Delta t}), p_{h,i}^{\Delta t}(\mu_{h,\gamma}^{\Delta t})\right) + a_i^T\left(\boldsymbol{u}_{h,i}^{\Delta t}(p_{h,\gamma}^{\Delta t}), \boldsymbol{u}_{h,i}^{\Delta t}(\mu_{h,\gamma}^{\Delta t})\right) \\ &\quad = -\left(\boldsymbol{u}_{h,i}^{\Delta t}(p_{h,\gamma}^{\Delta t})\cdot\boldsymbol{n}_i, \mu_{h,\gamma}^{\Delta t}\right)_\gamma. \end{aligned}$$

From this equation, (7.9) and (7.7), we have

$$\begin{aligned} &\left(\mathcal{S}_h^{\Delta t} p_{h,\gamma}^{\Delta t}, \mu_{h,\gamma}^{\Delta t}\right)_{\gamma^T} \\ &\quad = \sum_{i=1}^2 \left(-\boldsymbol{u}_{h,i}^{\Delta t}(p_{h,\gamma}^{\Delta t})\cdot\boldsymbol{n}_i, \mu_{h,\gamma}^{\Delta t}\right)_{\gamma^T} \\ &\quad = \sum_{i=1}^2 c_{s,i}^T\left(\widetilde{\partial}_t p_{h,i}^{\Delta t}(p_{h,\gamma}^{\Delta t}), p_{h,i}^{\Delta t}(\mu_{h,\gamma}^{\Delta t})\right) + a_i^T\left(\boldsymbol{u}_{h,i}^{\Delta t}(p_{h,\gamma}^{\Delta t}), \boldsymbol{u}_{h,i}^{\Delta t}(\mu_{h,\gamma}^{\Delta t})\right) \\ &\quad = \sum_{i=1}^2 \left(s_i\widetilde{\partial}_t p_{h,i}^{\Delta t}(p_{h,\gamma}^{\Delta t}), p_{h,i}^{\Delta t}(\mu_{h,\gamma}^{\Delta t})\right)_{\Omega_i^T} + \left(\boldsymbol{K}_i^{-1}\boldsymbol{u}_{h,i}^{\Delta t}(p_{h,\gamma}^{\Delta t}), \boldsymbol{u}_{h,i}^{\Delta t}(\mu_{h,\gamma}^{\Delta t})\right)_{\Omega_i^T} \\ &\qquad + \left(\kappa_\gamma^{-1}\boldsymbol{u}_{h,i}^{\Delta t}(p_{h,\gamma}^{\Delta t})\cdot\boldsymbol{n}_i, \boldsymbol{u}_{h,i}^{\Delta t}(\mu_{h,\gamma}^{\Delta t})\cdot\boldsymbol{n}_i\right)_{\gamma^T}. \end{aligned} \tag{7.21}$$

By using Lemma 4.2 and (7.11), we obtain

$$\begin{aligned} \left(\mathcal{S}_h^{\Delta t} p_{h,\gamma}^{\Delta t}, p_{h,\gamma}^{\Delta t}\right)_{\gamma^T} \geq \sum_{i=1}^2 &\left(\boldsymbol{K}_i^{-1}\boldsymbol{u}_{h,i}^{\Delta t}(p_{h,\gamma}^{\Delta t}), \boldsymbol{u}_{h,i}^{\Delta t}(p_{h,\gamma}^{\Delta t})\right)_{\Omega_i^T} \\ &+ \left(\kappa_\gamma^{-1}\boldsymbol{u}_{h,i}^{\Delta t}(p_{h,\gamma}^{\Delta t})\cdot\boldsymbol{n}_i, \boldsymbol{u}_{h,i}^{\Delta t}(p_{h,\gamma}^{\Delta t})\cdot\boldsymbol{n}_i\right)_{\gamma^T} \geq 0, \end{aligned} \tag{7.22}$$

hence $\mathcal{S}_h^{\Delta t}$ is nonnegative.

We next show that $\mathcal{S}_h^{\Delta t}$ is continuous. First, using its definition (7.9), the discrete trace (inverse) inequality [32] for quasi-uniform meshes $\mathcal{T}_{h,i}$, $i = 1, 2$ and $h \leq Ch_i$, $i = 1$, 2, we obtain

$$\begin{aligned} \left(\mathcal{S}_h^{\Delta t} p_{h,\gamma}^{\Delta t}, p_{h,\gamma}^{\Delta t}\right)_{\gamma^T} &\leq \sum_{i=1}^2 \|\boldsymbol{u}_{h,i}^{\Delta t}(p_{h,\gamma}^{\Delta t})\cdot\boldsymbol{n}_i\|_{L^2(0,T;M_\gamma)} \|p_{h,\gamma}^{\Delta t}\|_{L^2(0,T;M_\gamma)} \\ &\leq Ch^{-\frac{1}{2}} \sum_{i=1}^2 \|\boldsymbol{u}_{h,i}^{\Delta t}(p_{h,\gamma}^{\Delta t})\|_{L^2(0,T;M_i)} \|p_{h,\gamma}^{\Delta t}\|_{L^2(0,T;M_\gamma)}. \end{aligned} \tag{7.23}$$

On the other hand, we deduce from (7.22) that

$$\begin{aligned} \left(\mathcal{S}_h^{\Delta t} p_{h,\gamma}^{\Delta t}, p_{h,\gamma}^{\Delta t}\right)_{\gamma^T} &\geq \sum_{i=1}^2 \left(\boldsymbol{K}_i^{-1}\boldsymbol{u}_{h,i}^{\Delta t}(p_{h,\gamma}^{\Delta t}), \boldsymbol{u}_{h,i}^{\Delta t}(p_{h,\gamma}^{\Delta t})\right)_{\Omega_i^T} \\ &\geq K_+^{-1} \sum_{i=1}^2 \|\boldsymbol{u}_{h,i}^{\Delta t}(p_{h,\gamma}^{\Delta t})\|_{L^2(0,T;M_i)}^2, \end{aligned} \tag{7.24}$$

which, combined with (7.23), implies

$$\left(\mathcal{S}_h^{\Delta t} p_{h,\gamma}^{\Delta t}, p_{h,\gamma}^{\Delta t}\right)_{\gamma^T} \leq CK_+^{\frac{1}{2}} h^{-\frac{1}{2}} \left(\mathcal{S}_h^{\Delta t} p_{h,\gamma}^{\Delta t}, p_{h,\gamma}^{\Delta t}\right)_{\gamma^T}^{\frac{1}{2}} \|p_{h,\gamma}^{\Delta t}\|_{L^2(0,T;M_\gamma)},$$

or

$$\left(\mathcal{S}_h^{\Delta t} p_{h,\gamma}^{\Delta t}, p_{h,\gamma}^{\Delta t}\right)_{\gamma^T} \leq CK_+ h^{-1} \|p_{h,\gamma}^{\Delta t}\|_{L^2(0,T;M_\gamma)}^2. \tag{7.25}$$

Now, (7.22) implies that

$$\sum_{i=1}^{2} \kappa_{\gamma+}^{-1} \|\boldsymbol{u}_{h,i}^{\Delta t}(p_{h,\gamma}^{\Delta t}) \cdot \boldsymbol{n}_i\|_{L^2(0,T;M_\gamma)}^2 \leq \left(\mathcal{S}_h^{\Delta t} p_{h,\gamma}^{\Delta t}, p_{h,\gamma}^{\Delta t}\right)_{\gamma^T}.$$

Thus

$$\begin{aligned}
\left(\mathcal{S}_h^{\Delta t} p_{h,\gamma}^{\Delta t}, \mu_{h,\gamma}^{\Delta t}\right)_{\gamma^T} &\leq \textstyle\sum_{i=1}^{2} \|\boldsymbol{u}_{h,i}^{\Delta t}(p_{h,\gamma}^{\Delta t}) \cdot \boldsymbol{n}_i\|_{L^2(0,T;M_\gamma)} \|\mu_{h,\gamma}^{\Delta t}\|_{L^2(0,T;M_\gamma)} \\
&\leq \kappa_{\gamma+}^{\frac{1}{2}} \left(\mathcal{S}_h^{\Delta t} p_{h,\gamma}^{\Delta t}, p_{h,\gamma}^{\Delta t}\right)_{\gamma^T}^{\frac{1}{2}} \|\mu_{h,\gamma}^{\Delta t}\|_{L^2(0,T;M_\gamma)} \qquad (7.26)\\
&\leq C \kappa_{\gamma+}^{\frac{1}{2}} K_{+}^{\frac{1}{2}} h^{-\frac{1}{2}} \|p_{h,\gamma}^{\Delta t}\|_{L^2(0,T;M_\gamma)} \|\mu_{h,\gamma}^{\Delta t}\|_{L^2(0,T;M_\gamma)},
\end{aligned}$$

where we have used (7.25) in the last inequality. □

We next establish the bounds of the interface operator $\mathcal{B}_h^{\Delta t}$ needed to apply Lemma 7.1.

Lemma 7.3. *Assume that the meshes $\mathcal{T}_{h,i}$ are quasi-uniform and $h \leq Ch_i$, for $i = 1, 2, \gamma$. There exist positive constants C_0 and C_1 independent of the mesh size h and time step size Δt such that*

$$\begin{aligned}
&\left\langle \mathcal{B}_h^{\Delta t}(\boldsymbol{u}_{h,\gamma}^{\Delta t}, p_{h,\gamma}^{\Delta t}), (\boldsymbol{u}_{h,\gamma}^{\Delta t}, p_{h,\gamma}^{\Delta t}) \right\rangle \\
&\qquad \geq C_0 \|(\boldsymbol{u}_{h,\gamma}^{\Delta t}, p_{h,\gamma}^{\Delta t})\|_{\mathcal{O}}^2, \quad \forall (\boldsymbol{u}_{h,\gamma}^{\Delta t}, p_{h,\gamma}^{\Delta t}) \in \boldsymbol{\Sigma}_{h,\gamma}^{\Delta t} \times M_{h,\gamma}^{\Delta t}, \qquad (7.27)\\
&\left| \left\langle \mathcal{B}_h^{\Delta t}(\boldsymbol{u}_{h,\gamma}^{\Delta t}, p_{h,\gamma}^{\Delta t}), (\boldsymbol{v}_{h,\gamma}^{\Delta t}, \mu_{h,\gamma}^{\Delta t}) \right\rangle \right| \\
&\qquad \leq C_1 \max\{h^{-1}, \Delta t^{-1}\} h^{-1} \|(\boldsymbol{u}_{h,\gamma}^{\Delta t}, p_{h,\gamma}^{\Delta t})\|_{\mathcal{O}} \|(\boldsymbol{v}_{h,\gamma}^{\Delta t}, \mu_{h,\gamma}^{\Delta t})\|_{\mathcal{O}}. \qquad (7.28)
\end{aligned}$$

Proof. We start with the proof of (7.27). Recalling the definition of $\mathcal{B}_h^{\Delta t}$ (7.15), we note that, similarly to (4.29),

$$\begin{aligned}
&\int_\gamma s_\gamma \left(\sum_{n=1}^{N_\gamma} \int_{t_\gamma^{n-1}}^{t_\gamma^n} \partial_t p_{h,\gamma}^{\Delta t}\, p_{h,\gamma}^{\Delta t} + \sum_{n=2}^{N_\gamma} [p_{h,\gamma}^{\Delta t}]_{n-1} (p_{h,\gamma}^{\Delta t})_{n-1}^{+} + (p_{h,\gamma}^{\Delta t})_0^{+} (p_{h,\gamma}^{\Delta t})_0^{+} \right) \\
&\quad = \frac{1}{2} \left(\|s_\gamma (p_{h,\gamma}^{\Delta t})_{N_\gamma}^{-}\|_{M_\gamma}^2 + \sum_{n=2}^{N_\gamma} \|s_\gamma [(p_{h,\gamma}^{\Delta t})]_{n-1}\|_{M_\gamma}^2 + \|s_\gamma (p_{h,\gamma}^{\Delta t})_0^{+}\|_{M_\gamma}^2 \right) \geq 0.
\end{aligned}$$

Then, using (7.15) and (7.22), we have

$$\left\langle \mathcal{B}_h^{\Delta t}(\boldsymbol{u}_{h,\gamma}^{\Delta t}, p_{h,\gamma}^{\Delta t}), (\boldsymbol{u}_{h,\gamma}^{\Delta t}, p_{h,\gamma}^{\Delta t}) \right\rangle \geq K_{\gamma+}^{-1} \|\boldsymbol{u}_{h,\gamma}^{\Delta t}\|_{L^2(0,T;M_\gamma)}^2 + s_{-} \|(p_{h,\gamma}^{\Delta t})_0^{+}\|_{M_\gamma}^2. \qquad (7.29)$$

In addition, the following discrete inf-sup condition holds for the interface problem (7.13) [12]:

$$\forall \mu_{h,\gamma}^{\Delta t} \in M_{h,\gamma}^{\Delta t}, \qquad \sup_{\boldsymbol{0} \neq \boldsymbol{v}_{h,\gamma}^{\Delta t} \in \boldsymbol{\Sigma}_{h,\gamma}^{\Delta t}} \frac{b_\gamma^T(\boldsymbol{v}_{h,\gamma}^{\Delta t}, \mu_{h,\gamma}^{\Delta t})}{\|\boldsymbol{v}_{h,\gamma}^{\Delta t}\|_{L^2(0,T;\boldsymbol{\Sigma}_\gamma)}} \geq \beta_\gamma \|\mu_{h,\gamma}^{\Delta t}\|_{L^2(0,T;M_\gamma)}. \qquad (7.30)$$

Using this and the first equation of (7.13), we obtain

$$\beta_\gamma \|p_{h,\gamma}^{\Delta t}\|_{L^2(0,T;M_\gamma)} \leq K_{\gamma-}^{-1} \|\boldsymbol{u}_{h,\gamma}^{\Delta t}\|_{L^2(0,T;M_\gamma)}. \qquad (7.31)$$

Combining (7.29) and (7.31), we arrive at (7.27).

We continue with the upper bound (7.28). From the definition of $\mathcal{B}_h^{\Delta t}$ in (7.15), using the Cauchy-Schwarz inequality, bound (7.26), property (6.5), and the discrete Cauchy-Schwarz inequality, we obtain:

(7.32)

$$
\begin{aligned}
&\left|\left\langle \mathcal{B}_h^{\Delta t}(\boldsymbol{u}_{h,\gamma}^{\Delta t}, p_{h,\gamma}^{\Delta t}), (\boldsymbol{v}_{h,\gamma}^{\Delta t}, \mu_{h,\gamma}^{\Delta t})\right\rangle\right|^2 \\
&\quad \le C\Bigg(\|\boldsymbol{u}_{h,\gamma}^{\Delta t}\|^2_{L^2(0,T;\boldsymbol{M}_\gamma)} + \|\operatorname{div}_\tau \boldsymbol{u}_{h,\gamma}^{\Delta t}\|^2_{L^2(0,T;M_\gamma)} + h^{-1}\|p_{h,\gamma}^{\Delta t}\|^2_{L^2(0,T;M_\gamma)} \\
&\qquad\quad + \|\partial_t \mathcal{I} p_{h,\gamma}^{\Delta t}\|^2_{L^2(0,T;M_\gamma)} + \|(p_{h,\gamma}^{\Delta t})_0^+\|^2_{M_\gamma}\Bigg) \\
&\qquad \cdot \Bigg(\|\boldsymbol{v}_{h,\gamma}^{\Delta t}\|^2_{L^2(0,T;\boldsymbol{M}_\gamma)} + \|\operatorname{div}_\tau \boldsymbol{v}_{h,\gamma}^{\Delta t}\|^2_{L^2(0,T;M_\gamma)} + \|\mu_{h,\gamma}^{\Delta t}\|^2_{L^2(0,T;M_\gamma)} + \|(\mu_{h,\gamma}^{\Delta t})_0^+\|^2_{M_\gamma}\Bigg),
\end{aligned}
$$

where the constant C depends on $K_{\gamma-}^{-1}$, $\kappa_{\gamma+}$, K_+ and s_+. Bound (7.28) follows from the inverse inequalities

$$
\begin{aligned}
\|\operatorname{div}_\tau \boldsymbol{u}_{h,\gamma}^{\Delta t}\|_{L^2(0,T;M_\gamma)} &\le C h^{-1}\|\boldsymbol{u}_{h,\gamma}^{\Delta t}\|_{L^2(0,T;\boldsymbol{M}_\gamma)}, \quad \boldsymbol{u}_{h,\gamma}^{\Delta t} \in \boldsymbol{\Sigma}_{h,\gamma}^{\Delta t}, \\
\|\partial_t \mathcal{I} p_{h,\gamma}^{\Delta t}\|_{L^2(0,T;M_\gamma)} &\le C\Delta t^{-1}\|p_{h,\gamma}^{\Delta t}\|_{L^2(0,T;M_\gamma)}, \quad p_{h,\gamma}^{\Delta t} \in M_{h,\gamma}^{\Delta t}.
\end{aligned}
$$
□

We are now ready to establish the convergence rate of GMRES for (7.16).

Theorem 7.1 (Convergence rate of GMRES)**.** *Assume that the meshes $\mathcal{T}_{h,i}$ are quasi-uniform and $h \le Ch_i$, for $i = 1, 2, \gamma$. The m-th residual of GMRES for solving the interface problem* (7.16) *is bounded by*

$$
\|\boldsymbol{r}_m\|_{\mathcal{O}} \le \left(1 - \frac{\theta^2}{\Theta^2}\right)^{m/2} \|\boldsymbol{r}_0\|_{\mathcal{O}}, \tag{7.33}
$$

where $\theta = C_0$ and $\Theta = C_1 \max\{h^{-1}, \Delta t^{-1}\}h^{-1}$ are given in Lemma 7.3.

Proof. The proof is completed by applying Lemma 7.1 for the operator $\mathcal{B}_h^{\Delta t}$ with the lower bound (7.27) and the upper bound (7.28). □

Remark 7.1. From Theorem 7.1, the convergence of GMRES depends on the mesh size and time step size. For a normal matrix, the number of GMRES iterations behaves like $\sqrt{\frac{\Theta}{\theta}}$, i.e. $\max\{h^{-\frac{1}{2}}, \Delta t^{-\frac{1}{2}}\}h^{-\frac{1}{2}}$. We shall verify in the next section this dependence when either h or Δt is fixed.

8. Numerical results

In this section, we consider different test cases to investigate the convergence of the decoupled algorithm with global-in-time DD and verify convergence rates in space and in time with nonmatching space-time grids. Three examples are presented where the fracture's permeability is either much higher or much lower than the surrounding medium. Unless otherwise specified, the domain Ω is a rectangle of dimension 2×1 and is divided into two equally sized subdomains by a fracture of width $\delta = 0.001$ parallel to the y axis. The permeability tensors in the subdomains and in the fracture are constant and isotropic: $\boldsymbol{K} = \mathbf{I}$, $i = 1, 2$, and $\boldsymbol{K}_f = K_f\mathbf{I}$, where $\mathbf{I}$ is the identity matrix and K_f is a scalar to be specified later. On the external boundaries of the subdomains, a Dirichlet condition is imposed: on the right ($p = 1$) and on the left ($p = 0$), and no flow boundary condition is imposed on the top and bottom. There are no source terms in the subdomains and in the fracture, and zero initial conditions are imposed on the whole domain. We use the lowest Raviart-Thomas mixed finite element RT_0 space for spatial discretization and the backward Euler method for time stepping. The value of the parameter ξ in (2.3) is taken to be $\xi = 1$ (cf. Section 7). The discrete space-time interface problem is solved by GMRES with a zero initial guess; the iteration is stopped when the relative residual is smaller than 10^{-7}. All tests are implemented using MATLAB on Linux servers with Intel Xeon Gold 6242 at 2.8GHz and 512 GB memory.

8.1. **Example 1: A highly permeable fracture with a fine grid.** For this test case, the fracture has much higher permeability than the subdomains. In particular, $K_f = 2000$, thus $\boldsymbol{K}_\gamma = \delta \boldsymbol{K}_{f,\tau} = 2$ and $\kappa_\gamma = 2\boldsymbol{K}_{f,n}/\delta = 4\,10^6$. A pressure drop of 1 from the top to the bottom of the fracture is imposed. The final time is fixed to be $T = 0.5$. As the fluid flows faster in the fracture, we use a finer space-time grid in the fracture than in the subdomains. Results with a coarsely meshed fracture will be discussed in the next example (cf. Subsection 8.2). Figure 1 depicts the evolution of the pressure over the whole time interval $[0, T]$ with nonmatching space-time grids, and Figure 2 shows the snapshots of the pressure and velocity fields at the final time.

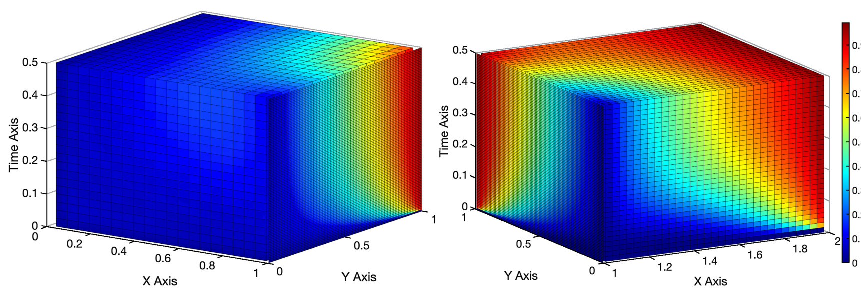

FIGURE 1. [Example 1] Evolution of the pressure over the time interval $[0, T] = [0, 0.5]$ with nonmatching space-time grids: $h_1 = 2\Delta t_1 = 1/20$, $h_2 = 2\Delta t_2 = 1/32$ and $h_\gamma = 2\Delta t_\gamma = 1/100$

We first verify the convergence rates in space and time of the monolithic solver and the decoupled algorithm. We use nonmatching spatial meshes in the subdomains and in the fracture with $h_i = 1/N_i$ for $i = 1, 2, \gamma$, where $N_1 < N_2$ and $N_\gamma = 5N_1$. In time, we consider both conforming (fine) time steps, $\Delta t_i = T/N_\gamma$, and nonconforming ones, $\Delta t_i = T/N_i$, for $i = 1, 2, \gamma$. For the latter, we only report the results with the decoupled algorithm as it would require the (full) space-time discretization with the monolithic approach. To calculate the errors of the approximate solution, we compute the reference solution on a conforming mesh of size $h_{\mathrm{ref}} = 1/800$ with a fine time step $\Delta t_{\mathrm{ref}} = T/2000$. Tables 1 and 2 show the L^2 errors of the pressure and velocity, respectively, at the final time $T = 0.5$ by the monolithic and DD solvers, with the mesh sizes and time step sizes decreased by half at each refinement. First order convergence rates in space and time of the pressure and velocity are observed for both monolithic and DD solvers, with either conforming or nonconforming time steps. The solution computed by the decoupled

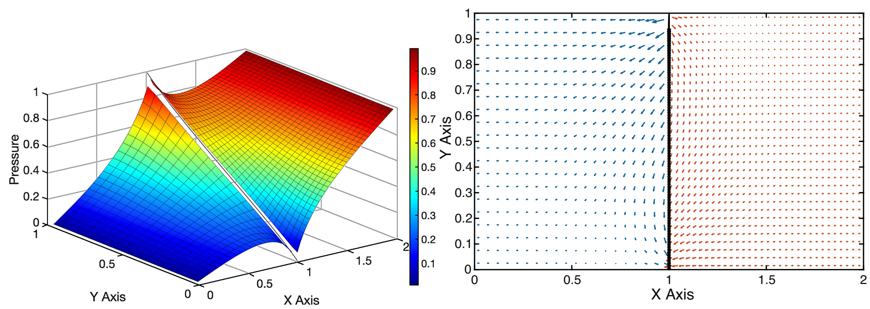

FIGURE 2. [Example 1] Snapshots of the pressure (left) and velocity (right) at the final time $T = 0.5$ with nonmatching space-time grids: $h_1 = 2\Delta t_1 = 1/20$, $h_2 = 2\Delta t_2 = 1/32$ and $h_\gamma = 2\Delta t_\gamma = 1/100$

TABLE 1. [Example 1] L^2 errors of the pressure at $T = 0.5$. Corresponding convergence rates are shown in square brackets.

N_1	N_2	N_γ	Conforming in time: $\Delta t_i = T/N_\gamma$		Nonconforming in time: $\Delta t_i = T/N_i$
			Monolithic	DD	
5	8	25	4.97e-02	4.97e-02	5.25e-02
10	16	50	2.50e-02 [0.99]	2.50e-02 [0.99]	2.62e-02 [1.00]
20	32	100	1.25e-02 [1.00]	1.25e-02 [1.00]	1.31e-02 [1.00]
40	64	200	6.24e-03 [1.00]	6.25e-03 [1.00]	6.52e-03 [1.00]

TABLE 2. [Example 1] L^2 errors of the velocity at $T = 0.5$. Corresponding convergence rates are shown in square brackets.

N_1	N_2	N_γ	Conforming in time: $\Delta t_i = T/N_\gamma$		Nonconforming in time: $\Delta t_i = T/N_i$
			Monolithic	DD	
5	8	25	3.98e-02	3.98e-02	4.56e-02
10	16	50	2.00e-02 [0.99]	2.00e-02 [0.99]	2.25e-02 [1.02]
20	32	100	9.98e-03 [1.00]	9.98e-03 [1.00]	1.11e-02 [1.02]
40	64	200	4.98e-03 [1.00]	5.00e-03 [1.00]	5.57e-03 [1.00]

scheme is almost the same as the one given by the monolithic scheme when a uniform time step is used across the domain. In addition, using nonconforming time grids gives similar errors as using conforming fine time steps.

Next we investigate the convergence of GMRES to solve the space-time interface problem iteratively, i.e., the decoupled scheme. In Table 3, we report the numbers of iterations required to reach the relative residual 10^{-7} and the corresponding computer running times (in seconds). We observe that GMRES converges at a similar speed when using either conforming or nonconforming time grids; importantly, using different time steps is more efficient as it significantly reduces the running times (approximately by a factor of 2.13) compared to using uniformly fine time steps on the whole domain. We also notice the increase of the number of iterations as the

TABLE 3. [Example 1] Number of GMRES iterations with conforming and nonconforming time steps for a tolerance of 10^{-7} and corresponding running times (in seconds)

N_1	N_2	N_γ	# GMRES		Running times	
			Conforming in time	Nonconforming in time	Conforming in time	Nonconforming in time
5	8	25	285	292	2s	1s
10	16	50	490	494	17s	8s
20	32	100	927	928	189s	85s
40	64	200	2198	2177	3640s	1681s

TABLE 4. [Example 1] Number of GMRES iterations for a tolerance of 10^{-7} when either Δt is fixed (a) or h is fixed (b)

(a) With $\Delta t = T/100$

h	1/8	1/16	1/32	1/64
# GMRES	154	210	334	578

(b) With $h = 1/20$

Δt	$T/4$	$T/8$	$T/16$	$T/32$
# GMRES	147	175	196	204

time step sizes and mesh sizes decrease as predicted in Theorem 7.1. To further investigate such dependence, we show in Table 4 the number of GMRES iterations when only h or Δt is refined; here uniform grids are used for simplicity, the results with nonmatching grids are similar. We see that the iterations grow like $O(h^{-1})$ (with fixed Δt) or $O\left(\Delta t^{-\frac{1}{2}}\right)$ (with fixed h), which is consistent with our theoretical rates in Remark 7.1. Note that with the use of suitable preconditioners as proposed in [30], the convergence of GMRES is significantly accelerated and almost independent of the discretization parameters. Such preconditioners will be studied in our future work.

8.2. **Example 2: A highly permeable fracture with a coarse grid.** As pointed out in Remark 5.1, the rates of convergence may reduce when κ_γ^{-1} is close to zero. In Example 1, we observed that when the fracture grid is finer than the subdomain, optimal order convergence is obtained, even for a small value of κ_γ^{-1}. We now consider the case where the fracture grid is coarser. We shall vary κ_γ as shown in Table 5; note that in all cases, the permeability in the fracture is higher than the subdomains. Figure 3 depicts the evolution of the pressure over the whole time interval $[0, T]$ with a coarsely meshed fracture, for $\kappa_\gamma = 4\,10^6$ (Case 4) and $T = 0.5$. The corresponding snapshots of the pressure and velocity fields at the final time are shown in Figure 4. We observe oscillations of the pressure and velocity in the vicinity of the fracture.

Next we investigate the convergence rates with different values of κ_γ when the fracture mesh is coarser than the subdomain meshes. For simplicity, we consider matching grids in the subdomains. The spatial mesh sizes are given by $h_1 = h_2 =$

TABLE 5. Parameters for Example 2

Case	1	2	3	4
δ	0.1	0.1	0.01	0.001
K_f	2	20	200	2000
κ_γ	40	400	$4\,10^4$	$4\,10^6$

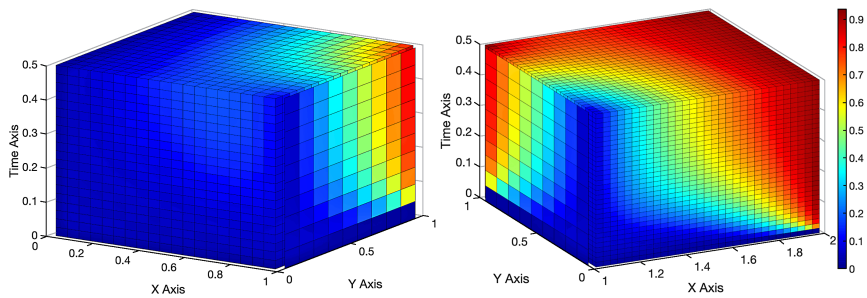

FIGURE 3. [Example 2, Case 4] Evolution of the pressure over the time interval $[0,T]=[0,0.5]$ with nonmatching space-time grids: $h_1=2\Delta t_1=1/20$, $h_2=2\Delta t_2=1/32$ and $h_\gamma=2\Delta t_\gamma=1/10$

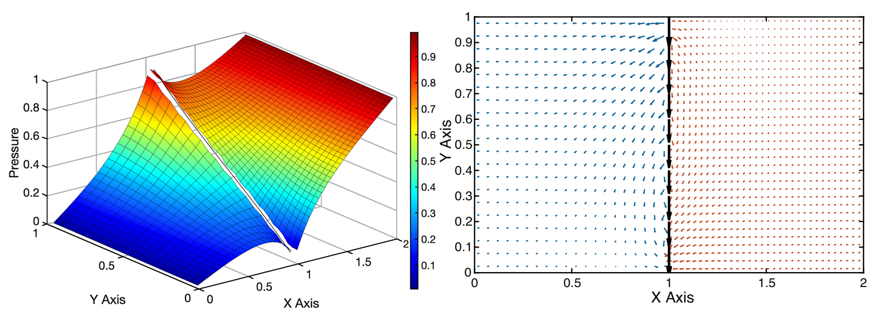

FIGURE 4. [Example 2, Case 4] Snapshots of the pressure (left) and velocity (right) at the final time $T=0.5$ with nonmatching space-time grids: $h_1=2\Delta t_1=1/20$, $h_2=2\Delta t_2=1/32$ and $h_\gamma=2\Delta t_\gamma=1/10$

$1/N$ and $h_\gamma=1/N_\gamma$ with $N_\gamma=N/2$. We present the results with conforming time steps on the whole domain $\Delta t_i=T/N$ for $i=1,2,\gamma$, where $T=0.5$; note that using finer or coarse time steps in the fracture gives similar results. We compute the errors of the numerical solution (obtained by the monolithic or DD solver) by comparing with a reference solution on a conforming mesh of size $h_{\text{ref}}=1/512$ and time step $\Delta t_{\text{ref}}=1/1280$. Tables 6 and 7 report the L^2 errors of the pressure and velocity, respectively, at $T=0.5$ with increasing values of κ_γ. The convergence rates for the pressure are linear in all cases while the rates for the velocity reduce to 0.5 as κ_γ increases. This is due to the factor $h^{-\frac{1}{2}}$ as pointed out in Remark 5.1. Note

TABLE 6. [Example 2] L^2 errors of the pressure at $T = 0.5$ for different values of κ_γ. Corresponding convergence rates are shown in square brackets.

N	N_γ	$\kappa_\gamma = 40$	$\kappa_\gamma = 400$	$\kappa_\gamma = 4\,10^4$	$\kappa_\gamma = 4\,10^6$
16	8	5.72e-02	4.14e-02	4.15e-02	4.15e-02
32	16	2.77e-02 [1.05]	2.06e-02 [1.01]	2.06e-02 [1.01]	2.06e-02 [1.01]
64	32	1.36e-02 [1.03]	1.02e-02 [1.01]	1.02e-02 [1.01]	1.02e-02 [1.01]
128	64	6.68e-03 [1.03]	5.03e-03 [1.02]	5.04e-03 [1.02]	5.04e-03 [1.02]
256	128	3.20e-03 [1.06]	2.41e-03 [1.06]	2.41e-03 [1.06]	2.41e-03 [1.06]

TABLE 7. [Example 2] L^2 errors of the velocity at $T = 0.5$ for different values of κ_γ. Corresponding convergence rates are shown in square brackets.

N	N_γ	$\kappa_\gamma = 40$	$\kappa_\gamma = 400$	$\kappa_\gamma = 4\,10^4$	$\kappa_\gamma = 4\,10^6$
16	8	1.78e-01	1.01e-01	1.10e-01	1.10e-01
32	16	8.36e-02 [1.09]	6.30e-02 [0.68]	7.47e-02 [0.56]	7.49e-02 [0.55]
64	32	3.45e-02 [1.28]	3.74e-02 [0.75]	5.15e-02 [0.54]	5.17e-02 [0.54]
128	64	1.31e-02 [1.40]	2.04e-02 [0.88]	3.58e-02 [0.53]	3.61e-02 [0.52]
256	128	4.75e-03 [1.46]	9.96e-03 [1.03]	2.50e-02 [0.52]	2.54e-02 [0.51]

that for a highly permeable fracture, one should use a finer grid there, which would result in optimal convergence rates for both pressure and velocity as presented in Example 1.

8.3. **Example 3: A geological barrier.** We consider a similar setting as proposed in [22] for the steady-state flow, where the central part of the fracture is a barrier with much lower permeability $K_f = 0.002$, thus $\boldsymbol{K}_\gamma = \delta \boldsymbol{K}_{f,\tau} = 2\,10^{-6}$ and $\kappa_\gamma = 4$. In the upper and lower quarters of the fracture, the permeability $\boldsymbol{K}_f$ is the same as in the surrounding subdomains, i.e. , $K_f = 1$. Unlike Examples 1 and 2, here homogeneous Neumann conditions are imposed on the fracture boundaries. The final time is fixed to be $T = 2$. The pressure field over the whole time interval $[0, T]$ with nonmatching space-time grids is shown in Figure 5, and the snapshots of the pressure and velocity fields at the final time are depicted in Figure 6.

We remark that the convergence of GMRES depends on the bounds $K_{\gamma+}^{-1}$ and $K_{\gamma-}^{-1}$ (cf. the constant C in (7.32) and C_0 in Lemma 7.3), which are considerably large in this example: $K_{\gamma+}^{-1} = K_{\gamma-}^{-1} = 2\,10^6$. Consequently, if we directly apply GMRES to solve the interface problem (7.13), the algorithm is very slow to converge. Instead, we scale the first equation of (7.13) by $K_\gamma = 2\,10^{-6}$ before applying GMRES (equivalently, we precondition GMRES with a diagonal matrix whose diagonal entries are either 1 or K_γ).

To verify the convergence rates, we consider nonmatching spatial meshes with $h_i = 1/N_i$ for $i = 1, 2, \gamma$ with $N_\gamma < N_1 < N_2$. Note that a coarse fracture mesh is considered in this case as there is almost no fluid flowing in this barrier; moreover, using a finer mesh in the fracture could result in oscillations in the pressure as observed in [22]. In time, we investigate both conforming (fine) time steps, $\Delta t_i = T/N_2$, and nonconforming ones, $\Delta t_i = T/N_i$, for $i = 1, 2, \gamma$. A reference solution

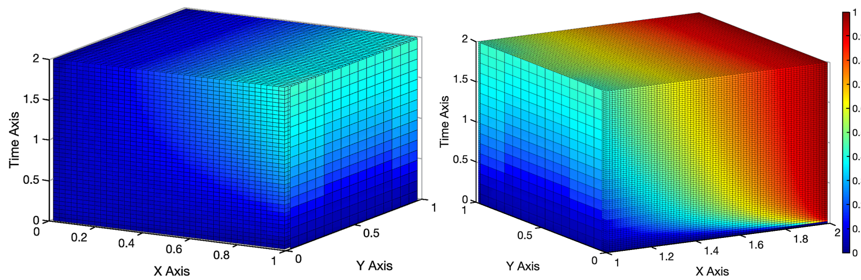

FIGURE 5. [Example 3] Evolution of the pressure over the time interval $[0, T] = [0, 2]$ with nonmatching space-time grids: $h_1 = \frac{1}{2}\Delta t_1 = 1/40$, $h_2 = \frac{1}{2}\Delta t_2 = 1/80$ and $h_\gamma = \frac{1}{2}\Delta t_\gamma = 1/16$

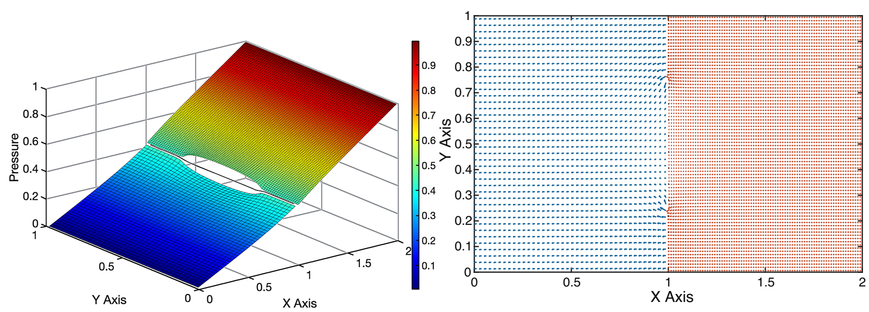

FIGURE 6. [Example 3] Snapshots of the pressure (left) and velocity (right) at the final time $T = 2$ with nonmatching space-time grids: $h_1 = \frac{1}{2}\Delta t_1 = 1/40$, $h_2 = \frac{1}{2}\Delta t_2 = 1/80$ and $h_\gamma = \frac{1}{2}\Delta t_\gamma = 1/16$. The velocity in the fracture is very small and can hardly be seen.

is computed on a conforming mesh of size $h_{\text{ref}} = 1/800$ with a fine time step $\Delta t_{\text{ref}} = T/2000$. Tables 8 and 9 show the L^2 errors of the pressure and velocity, respectively, as the spatial and temporal mesh sizes are refined. The sublinear convergence behavior of the velocity is expected due to the lack of regularity of the solution in the fracture - similar results were observed for the steady-state case in [22].

Table 10 reports the numbers of GMRES iterations required to reach the tolerance 10^{-7} with conforming and nonconforming time grids, and the corresponding running times. Unlike Example 1, here GMRES converges much faster with nonconforming time steps; in particular, compared to using uniform time steps across the domain, the number of GMRES iterations is reduced approximately by a factor of 2.47 and the running time by a factor of 2.66. We also observe linear growth of the iterations as the mesh size and time step size are refined.

TABLE 8. [Example 3] L^2 errors of the pressure at $T = 0.5$. Corresponding convergence rates are shown in square brackets.

N_1	N_2	N_γ	Conforming in time: $\Delta t_i = T/N_2$		Nonconforming in time: $\Delta t_i = T/N_i$
			Monolithic	DD	
20	40	8	9.95e-03	9.95e-03	2.34e-02
40	80	16	4.95e-03 [1.00]	4.95e-03 [1.01]	1.15e-02 [1.03]
80	160	32	2.44e-03 [1.02]	2.44e-03 [1.02]	5.60e-03 [1.04]
160	320	64	1.21e-03 [1.01]	1.21e-03 [1.01]	2.72e-03 [1.04]

TABLE 9. [Example 3] L^2 errors of the velocity at $T = 0.5$. Corresponding convergence rates are shown in square brackets.

N_1	N_2	N_γ	Conforming in time: $\Delta t_i = T/N_2$		Nonconforming in time: $\Delta t_i = T/N_i$
			Monolithic	DD	
20	40	8	3.90e-02	3.90e-02	4.92e-02
40	80	16	2.65e-02 [0.56]	2.65e-02 [0.56]	3.05e-02 [0.69]
80	160	32	1.71e-02 [0.63]	1.71e-02 [0.63]	1.87e-02 [0.71]
160	320	64	1.01e-02 [0.76]	1.01e-02 [0.76]	1.08e-02 [0.79]

TABLE 10. [Example 3] Number of GMRES iterations with conforming and nonconforming time steps for a tolerance of 10^{-7} and corresponding running times (in seconds)

N_1	N_2	N_γ	# GMRES		Running times	
			Conforming in time	Nonconforming in time	Conforming in time	Nonconforming in time
20	40	8	67	25	3s	1s
40	80	16	127	54	49s	20s
80	160	32	248	110	840s	351s
160	320	64	454	175	13193s	4737s

9. Conclusions

We developed and analyzed a space-time mixed finite element method for the reduced fracture flow model in which local spatial and temporal discretizations can be used in the subdomains and on the fracture. Well-posedness and a priori error estimates of the numerical solution were demonstrated. Due to the tangential PDEs imposed on the fracture-interface, we can use either a coarser or very finer mesh on the fracture without affecting optimal order convergence, as opposed to the mortar methods for nonfractured domains [32]. To efficiently solve the coupled algebraic system, a domain decomposition algorithm was constructed by decoupling the subdomain problems and formulating a space-time interface problem on the fracture. Convergence of GMRES for solving the interface problem was established via field-of-values analysis. Numerical experiments were carried out to illustrate theoretical results on test cases where the fracture represents either a fast path or a geological barrier.

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