# ANNALS OF K-THEORY

vol. 9 no. 4 2024


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A JOURNAL OF THE K-THEORY FOUNDATION

ANNALS OF K-THEORY
Vol. 9 (2024), No. 4, pp. 583–612
DOI: 10.2140/akt.2024.9.583

# On Gromov's compactness question regarding positive scalar curvature

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We give both positive and negative answers to Gromov's compactness question regarding positive scalar curvature metrics on noncompact manifolds. First we construct examples that give a negative answer to Gromov's compactness question. These examples are based on the nonvanishing of certain index-theoretic invariants that arise at the infinity of the given underlying manifold. This is a $\varprojlim^1$ phenomenon and naturally leads one to conjecture that Gromov's compactness question has a positive answer provided that these $\varprojlim^1$ invariants also vanish. We prove this is indeed the case for a class of 1-tame manifolds.

## 1. Introduction

In the past several years, Gromov [2018; 2023] has formulated an extensive list of conjectures and open questions on scalar curvature. This has given rise to new perspectives on scalar curvature and inspired a wave of recent activity in this area (see, for example, [Gromov 2020; Lott 2021; Wang et al. 2021a; 2021b; 2024; Zhang 2017]). Among his many open questions, Gromov [2023, Section 3.6.1] proposed the following compactness question regarding positive scalar curvature on noncompact manifolds.

**Question** (Gromov's compactness question). *Let $X$ be a smooth manifold. Suppose for any given compact subset $V \subset X$ and any $\rho > 0$, there exists a (noncomplete) Riemannian metric on $X$ with scalar curvature $\geq 1$ such that the closed $\rho$-neighborhood $N_\rho(V)$ of $V$ in $X$ is compact. Then $X$ admits a complete Riemannian metric with scalar curvature $\geq 1$.*

In this paper, we construct both positive and negative examples for the above compactness question of Gromov. First, we construct negative examples to show that Gromov's compactness question, as stated in its complete generality, is false. These negative examples lead us to suggest a modification (Conjecture 4.1) of

Weinberger is partially supported by NSF 2105451. Xie is partially supported by NSF 1800737 and 1952693. Yu is partially supported by NSF 1700021, 2000082, and the Simons Fellows Program.
*MSC2020:* 58B34.
*Keywords:* Gromov's compactness conjecture on scalar curvature, index invariants at infinity.

Gromov's compactness question. We shall prove the conjecture for a class of 1-tame manifolds, and hence give a class of positive examples for which Gromov's compactness question holds true.

Our first main theorem of the paper is the following.

**Theorem 1.1** (Theorem 3.12). *There exists a noncompact smooth spin manifold $X$ such that for any given compact subset $V \subset X$ and any $\rho > 0$, there exists an incomplete Riemannian metric on $X$ with scalar curvature $\geq 1$ and the closed $\rho$-neighborhood $N_\rho(V)$ of $V$ in $X$ is compact, but $X$ itself does not admit a complete Riemannian metric with uniformly positive scalar curvature.*

Let us outline the key steps of our construction of negative examples. The idea behind this is conceptually simple. In the de Rham cohomology of a noncompact manifold $X$, a closed form $\omega$ which is exact on every compact submanifold of $X$ is itself exact, but this fails integrally for cochains. In other words, one can make a choice of cochains $\nu_i$ on larger and larger parts of $X$ with $\omega = d\nu_i$, but there is a global constraint for them to be compatible. This is governed by (the $\varprojlim^1$ of) the system of cohomology (in one dimension lower) of the submanifolds, which measures the indeterminacy of the forms $\nu_i$ showing that $\omega$ is exact. The analogous idea for the $K$-theory class associated to the Dirac operator on a spin manifold gives rise to the obstruction that we will use to construct our negative examples, where each positive scalar curvature metric on a compact submanifold of $X$ is an analogue of a cochain $\nu$ satisfying $d\nu = \omega$ on the submanifold above.

Recall that for any given noncompact complete metric spaces $Y$ equipped with some suitable exhaustion, Chang, Weinberger and Yu [Chang et al. 2020, Section 3] introduced the new index map

$$\sigma : \mathrm{KO}_*^{\mathrm{lf}}(Y) \to \mathrm{KO}_*(\mathcal{A}(Y)),$$

where $\mathrm{KO}_*^{\mathrm{lf}}(Y)$ are the locally finite KO groups of $Y$ and $\mathcal{A}(Y)$ is some geometric $C^*$-algebra constructed using the given exhaustion on $Y$ (see Section 2). The $C^*$-algebra $\mathcal{A}(Y)$ encodes information about the changing nature of the fundamental group as one moves to infinity. A key feature of the index map $\sigma$ is that it can be used to detect the geometry at infinity of $Y$. In particular, as an application of their index map, Chang, Weinberger and Yu constructed a noncompact spin manifold $M$ equipped with an exhaustion[1] consisting of codimension zero compact submanifolds $(M_i, \partial M_i)$ with boundary such that each $M_i$ has a metric of positive scalar curvature which is collared at the boundary, but $M$ itself does not have a complete metric of uniformly positive scalar curvature [Chang et al. 2020, Theorem 4.3]. Those examples of Chang–Weinberger–Yu are not quite sufficient to serve as negative

[1]Here we say $\{M_i\}$ is an exhaustion of $M$ if the $M_i$ are compact codimension zero submanifolds, $M_i \subset \mathring{M}_{i+1}$ and $M = \bigcup_i M_i$. Here $\mathring{M}_{i+1}$ is the interior of $M_{i+1}$.

examples to Gromov's compactness question. We shall improve upon the methods of Chang–Weinberger–Yu to construct a noncompact spin manifold $X$ equipped with an exhaustion consisting of codimension zero submanifolds $(X_i, \partial X_i)$ with boundary such that

(1) each $X_i$ has a metric of positive scalar curvature which is collared at the boundary,

(2) *additionally* each annulus region $A_i$ between $\partial X_i$ and $\partial X_{i+1}$ can be obtained from $\partial X_i \times [0, 1]$ by attaching handles[2] of index $\geq 2$,

(3) but $X$ itself does not have a complete metric of uniformly positive scalar curvature.

We claim that such an $X$ gives a negative answer to Gromov's compactness question. Indeed, for any given compact subset $V \subset X$, we have $V \subset \mathring{X}_i$ for some $i$, since $\{X_i\}$ is an exhaustion of $X$. Here $\mathring{X}_i$ is the interior of $X_i$. By construction, $(X_i, \partial X_i)$ admits a metric $g_i$ of scalar curvature $\geq 1$, which is collared at the boundary. For any $\rho > 0$, by stretching[3] the collar neighborhood of $\partial X_i$ if necessary, we can assume $N_\rho(V) \subset X_i$. Now it only remains to show that for each metric $g_i$ on $X_i$ as above, we can extend $g_i$ to an incomplete Riemannian metric on $X$ with scalar curvature $\geq 1$, which is a consequence of the following proposition.

**Proposition 1.2** (Proposition 3.8). *Let $Z$ be a cobordism between two closed smooth manifolds $\partial_- Z$ and $\partial_+ Z$ such that $Z$ is obtained from $\partial_+ Z \times [0, 1]$ by attaching handles of index $\geq 2$. Given any smooth Riemannian metric $h$ on $\partial_- Z$, for any $k > 0$ and $m \in \mathbb{R}$, there exists a smooth Riemannian metric $g$ on $Z$ such that*

(1) *$g$ extends $h$, that is,*
$$g|_{\partial_- Z} = h,$$

(2) *the scalar curvature $g$ satisfies $\mathrm{Sc}(g)_z \geq k$ for all $z \in Z$,*

(3) *the mean curvature of $\partial_- Z$ satisfies $H_g(\partial_- Z)_x \geq m$ for all $x \in \partial_- Z$.*

Indeed, assume by induction we have extended $g_i$ to a Riemannian metric $g_{i\to j}$ on $X_j$ with scalar curvature $\geq 1$. Since $\partial X_j$ is compact, the mean curvature of $\partial X_j$ (with respect to $g_{i\to j}$) is bounded below by some constant, say, $-m$. By Proposition 1.2, the metric $h = g_{i\to j}|_{\partial X_j}$ extends to a metric $g_{j\to j+1}$ on the annulus $A_j$ with scalar curvature $\geq 1$ such that the mean curvature of $\partial X_j$ (with respect to $g_{j\to j+1}$) is $\geq m + 1$. We glue the metrics $g_{i\to j}$ on $X_j$ and $g_{j\to j+1}$ on $A_j$ to obtain a *continuous* metric on $X_{j+1} = X_j \cup_{\partial X_j} A_j$. Now Miao's gluing lemma [2002, Section 3] implies that there exists a *smooth* Riemannian metric $g_{i\to j+1}$ on $X_{j+1}$ with scalar curvature $\geq 1$ such that $g_{i\to j+1}$ coincide with $g_{i\to j} \cup g_{j\to j+1}$ away

[2] This is equivalent to saying $\partial X_i$ can be obtained from $\partial X_{i-1}$ via surgeries of codimension $\geq 2$.

[3] Since $g_i$ has product structure near the boundary, such a stretching does not change the scalar curvature of $g_i$.

from an $\varepsilon$-neighborhood of $\partial X_j$ in $X_j \cup_{\partial X_j} A_j$. By repeating the above extension-and-gluing process inductively, we eventually extend $g_i$ on $X_i$ to an (incomplete) Riemannian metric on $X$ with scalar curvature $\geq 1$. This finishes the outline of our construction of negative examples to Gromov's compactness question. The full details will be given in Section 3.

Since writing the first version of this paper, it was pointed out to us that one can deduce the existence of such manifolds from [Chang et al. 2020] by using Gromov's $h$-principle for open Diff-invariant relations (applied to the positive scalar curvature relation); see [Eliashberg and Mishachev 2002] for more details on Gromov's $h$-principle. We prefer our original method because it gives us a clearer understanding of what the metrics "look like".

The above negative examples to Gromov's compactness question are based on the nonvanishing of certain $\varprojlim^1$ index-theoretic invariants that arise at infinity of the given underlying manifold. This naturally leads us to conjecture that Gromov's compactness question has a positive answer provided that these $\varprojlim^1$ invariants vanish. The precise statement of this conjecture (Conjecture 4.1), a modification of Gromov's compactness question, will be given in Section 4. As supporting evidence, we prove the conjecture for a class of 1-tame manifolds, hence give a class of positive examples for which Gromov's compactness question holds true. In particular, our second main theorem of the paper is the following.

**Theorem 1.3** (Theorem 4.4). *Let $M$ be a noncompact 1-tame spin manifold of dimension $n \geq 6$. Let*

$$\Gamma = \pi_1(M) \quad \textit{and} \quad G = \pi_1^\infty(M).$$

*Assume that the unstable relative Gromov–Lawson–Rosenberg conjecture holds for the pair $(\Gamma, G)$. Suppose for any given compact subset $V \subset M$ and any $\rho > 0$, there exists an (incomplete) Riemannian metric on $M$ with scalar curvature $\geq 1$ such that the closed $\rho$-neighborhood $N_\rho(V)$ of $V$ in $M$ is compact. Then $M$ admits a complete Riemannian metric of uniformly positive scalar curvature.*

Here $\pi_1^\infty(M)$ is the fundamental groupoid at infinity of $M$; see Definition 4.2 and the discussion after. The unstable relative Gromov–Lawson–Rosenberg conjecture will be reviewed in Conjecture 2.7.

The paper is organized as follows. In Section 2, we review the construction of some geometric $C^*$-algebras, the unstable relative Gromov–Lawson–Rosenberg conjecture for positive scalar curvature, and the construction of relative higher index. In Section 3, we construct negative examples to Gromov's compactness question. In Section 4, we suggest a modification of Gromov's compactness question, by imposing an extra vanishing condition on certain $\varprojlim^1$ invariants. We confirm the conjecture for a class of 1-tame manifolds, which in particular gives a class of positive examples for which Gromov's compactness question holds true.

## 2. Preliminaries

In this section, we review the construction from [Chang et al. 2020] of some geometric $C^*$-algebras associated with exhaustions of noncompact spaces. The $K$-theory groups of these $C^*$-algebras are the receptacle of index-theoretic invariants that detect the geometry at infinity of the underlying spaces, which constitute a key ingredient in our construction of counterexamples to Gromov's compactness question.

Let $\psi : A \to B$ be a homomorphism between two (real) $C^*$-algebras $A$ and $B$. The mapping cone $C^*$-algebra $C_\psi$ of $\psi$ is given by

$$C_\psi := \{(a, f) \mid a \in A,\ f \in C_0([0, 1), B) \text{ and } f(0) = \psi(a)\}.$$

It follows that we have the short exact sequence of $C^*$-algebras

$$0 \to \mathcal{S}B \to C_\psi \to A \to 0,$$

where $\mathcal{S}B = C_0((0, 1), B)$ is the suspension $C^*$-algebra of $B$.

**Definition 2.1.** A homomorphism $\varphi : G \to \Gamma$ between two discrete groups induces a homomorphism of real $C^*$-algebras $\varphi_* : C^*_{\max}(G) \to C^*_{\max}(\Gamma)$. We define $C^*_{\max}(\Gamma, G)$ to be the 7th suspension $\mathcal{S}^7 C_{\varphi_*} \cong C_0(\mathbb{R}^7) \otimes C_{\varphi_*}$ of the mapping cone $C^*$-algebra $C_{\varphi_*}$.

The $K$-theory of $C^*_{\max}(\Gamma, G)$ is the receptacle of relative higher indices. See [Baum and Connes 1988; Connes 1986; Connes and Moscovici 1990; Kasparov 1988; Rosenberg 1983] for related discussions.

**Definition 2.2.** Let $(Y, d)$ be a noncompact, complete metric space. Suppose $Y_1 \subseteq Y_2 \subseteq Y_3 \subseteq \cdots$ is a sequence of connected compact subsets of $Y$. We say $\{Y_i\}$ is an admissible exhaustion if the following are satisfied:

(1) $Y = \bigcup_{i=1}^{\infty} Y_i$;

(2) for all $j > i$, the subspace $Y_{i,j} = Y_j - \mathring{Y}_i$ is connected, where $\mathring{Y}_i$ is the interior of $Y_i$;

(3) $d(\partial Y_i, \partial Y_j) \to \infty$ as $|j - i| \to \infty$, where $\partial Y_i = Y_i - \mathring{Y}_i$.

Now suppose $\{Y_i; Y_{ij}\}$ is an admissible exhaustion of $Y$ as above. Define $D^*_i$ to be the $C^*$-algebra inductive limit

$$D^*_i := \varinjlim_{j>i} C^*_{\max}(\pi_1(Y_j), \pi_1(Y_{ij})) \otimes \mathcal{K} \tag{2.3}$$

of the directed system

$$\cdots \to C^*_{\max}(\pi_1(Y_j), \pi_1(Y_{i,j})) \otimes \mathcal{K} \xrightarrow{\iota_j} C^*_{\max}(\pi_1(Y_{j+1}), \pi_1(Y_{i,j+1})) \otimes \mathcal{K} \to \cdots,$$

where the homomorphism $\iota_j$ is induced by the inclusion $(Y_j, Y_{i,j}) \hookrightarrow (Y_{j+1}, Y_{i,j+1})$ and $\mathcal{K}$ is the real $C^*$-algebra of compact operators on a real Hilbert space.

Let
$$\prod_{i=1}^{\infty} D_i^* = \big\{(a_1, a_2, \ldots) \mid a_i \in D_i^* \text{ and } \sup_i \|a_i\| < \infty\big\}$$
be the $C^*$ product algebra of $D_i^*$. There is a natural homomorphism
$$\rho_{i+1} : D_{i+1}^* \to D_i^*$$
induced by the inclusions of the spaces
$$(Y_j, Y_{i+1,j}) \hookrightarrow (Y_j, Y_{i,j})$$
for $j > i+1$. Let $\rho : \prod_{i=1}^{\infty} D_i^* \to \prod_{i=1}^{\infty} D_i^*$ be the homomorphism that maps $(a_1, a_2, \ldots)$ to $(\rho_2(a_2), \rho_3(a_3), \ldots)$.

**Definition 2.4.** With the above notation, we define the $C^*$-algebra $\mathcal{A}(Y)$ by
$$\mathcal{A}(Y) := \left\{a \in C\left([0,1], \prod_{i=1}^{\infty} D_i^*\right) \;\middle|\; \rho(a(0)) = a(1)\right\}.$$

Consider the inverse system
$$\mathrm{KO}_n(D_1^*) \xleftarrow{(\rho_2)_*} \mathrm{KO}_n(D_2^*) \xleftarrow{(\rho_3)_*} \cdots,$$
where $(\rho_{i+1})_* : \mathrm{KO}_n(D_{i+1}^*) \to \mathrm{KO}_n(D_i^*)$ is the map induced by $\rho_{i+1} : D_{i+1}^* \to D_i^*$. If $\Phi : \prod_{i=1}^{\infty} \mathrm{KO}_n(D_i^*) \to \prod_{i=1}^{\infty} \mathrm{KO}_n(D_i^*)$ is defined by
$$\Phi(a_i) = (a_i - (\rho_{i+1})_*(a_{i+1})),$$
then by definition the inverse limit $\varprojlim \mathrm{KO}_n(D_i^*)$ of the above inverse system is simply the kernel $\ker(\Phi)$ of $\Phi$. We have the following Milnor exact sequence (see [Guentner and Yu 2012]):
$$0 \to {\varprojlim}^1 \mathrm{KO}_{n+1}(D_i^*) \to \mathrm{KO}_n(\mathcal{A}(Y)) \to \varprojlim \mathrm{KO}_n(D_i^*) \to 0, \tag{2.5}$$
where by definition ${\varprojlim}^1 \mathrm{KO}_n(D_i^*)$ is the cokernel $\mathrm{coker}(\Phi)$ of $\Phi$. This Milnor exact sequence can be derived from the KO-theory long exact sequence associated with the short exact sequence of $C^*$-algebras
$$0 \to \mathcal{S}\left(\prod_{i=1}^{\infty} D_i^*\right) \to \mathcal{A}(Y) \xrightarrow{\varpi} \prod_{i=1}^{\infty} D_i^* \to 0$$
together with the fact that
$$\mathrm{KO}_n\left(\prod_{i=1}^{\infty} D_i^*\right) \cong \prod_{i=1}^{\infty} \mathrm{KO}_n(D_i^*)$$

when the $D_i^*$ are stable, that is, $D_i^* \cong D_i^* \otimes \mathcal{K}$. Here $\varpi : \mathcal{A}(Y) \to \prod_{i=1}^{\infty} D_i^*$ is the evaluation map $\varpi(a) := a(0)$.

In [Chang et al. 2020, Section 3], Chang, Weinberger and Yu defined a natural index map

$$\sigma : \mathrm{KO}_*^{\mathrm{lf}}(Y) \to \mathrm{KO}_*(\mathcal{A}(Y)), \tag{2.6}$$

which can be used to detect the geometry at infinity of $Y$. As an application of their index map, they constructed a noncompact spin manifold $M$ equipped with an exhaustion consisting of codimension zero submanifolds $(M_i, \partial M_i)$ with boundary such that each $M_i$ has a metric of positive scalar curvature which is collared at the boundary, but $M$ itself does not have a complete metric of uniformly positive scalar curvature [Chang et al. 2020, Theorem 4.3]. We shall review the construction of this index map $\sigma$ in Section 2A.

Note that the Milnor exact sequence (2.5) gives rise to the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \varprojlim^1 \mathrm{KO}_{n+1}(Y_i, \partial Y_i) & \longrightarrow & \mathrm{KO}_n^{\mathrm{lf}}(Y) & \longrightarrow & \varprojlim \mathrm{KO}_n(Y_i, \partial Y_i) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \varprojlim^1 \mathrm{KO}_{n+1}(D_i^*) & \longrightarrow & \mathrm{KO}_n(\mathcal{A}(Y)) & \longrightarrow & \varprojlim \mathrm{KO}_n(D_i^*) & \longrightarrow & 0 \end{array}$$

Now let us briefly recall the following unstable relative Gromov–Lawson–Rosenberg conjecture; see [Chang et al. 2020, Conjecture 2.21].

**Conjecture 2.7** (unstable relative Gromov–Lawson–Rosenberg conjecture). *Let $(N, \partial N)$ be an $n$-dimensional compact spin manifold with boundary. If the relative higher index of the Dirac operator $D$ of $N$ is zero in $\mathrm{KO}_n(\pi_1(N), \pi_1(\partial N))$, then there is a metric of positive scalar curvature on $N$ that is collared near $\partial N$.*

Because of the failure of the ordinary unstable Gromov–Lawson–Rosenberg conjecture [Schick 1998, Example 2.2], in general, this statement cannot be true as stated. On the other hand, there are special cases where the unstable (relative) Gromov–Lawson–Rosenberg conjecture is true. For example, the $\pi$-$\pi$ case of the unstable relative Gromov–Lawson–Rosenberg conjecture is true for all manifolds $(N, \partial N)$ of dimension $\geq 6$, that is, the unstable relative Gromov–Lawson–Rosenberg conjecture holds if $\pi_1(\partial N) \to \pi_1(N)$ is an isomorphism. This follows from the surgery theory for positive scalar curvature of [Gromov and Lawson 1980, Theorem A; Schoen and Yau 1979, Corollary 6], as improved in [Gajer 1987]. Recall that the unstable Gromov–Lawson–Rosenberg conjecture holds when $\pi = \mathbb{Z}^k * F$, the free product of $\mathbb{Z}^k$ and $F$, where $\mathbb{Z}^k$ is the free abelian group of rank $k$ and $F$ is a finitely generated free group [Rosenberg and Stolz 1995, Corollary 3.8]. In particular, the unstable relative Gromov–Lawson–Rosenberg conjecture holds if $\pi_1(N) = \{e\}$ is the trivial group and $\pi_1(\partial N) = \mathbb{Z}^k * F$ with $F$ a finitely generated free group (see [Chang et al. 2020, Theorem 2.23 and Corollary 2.24] for more details). In our construction of negative examples to Gromov's compactness question, we shall exploit these special groups for which the unstable relative Gromov–Lawson–Rosenberg conjecture holds.

**2A.** ***Relative higher index.*** At the end of this section, let us review the construction of relative higher index for Dirac operators on spin manifolds with boundary and more generally relative higher index for complete spin manifolds (relative to complements of compact subsets); see [Chang et al. 2020, Section 2]. We also review the construction of this index map $\sigma$ from (2.6).

Let us first briefly recall the definition of some geometric $C^*$-algebras. For simplicity, let us assume $X$ is a complete Riemannian manifold and $\mathcal{S}$ is a Hermitian bundle over $X$. Let $H_X$ be the space $L^2(X, \mathcal{S})$ of $L^2$ sections of $\mathcal{S}$ over $X$.

**Definition 2.8.** Let $T$ be a bounded linear operator acting on $H_X$.

(i) The propagation of $T$ is defined to be the nonnegative real number

$$\sup\{d(x, y) \mid (x, y) \in \mathrm{Supp}(T)\},$$

where $\mathrm{Supp}(T)$ is the complement (in $X \times X$) of the set of points $(x, y) \in X \times X$ for which there exist $f, g \in C_0(X)$ such that $gTf = 0$ and $f(x) \neq 0$, $g(y) \neq 0$. Here $C_0(X)$ is the algebra of all continuous functions on $X$ which vanish at infinity.

(ii) $T$ is said to be locally compact if $fT$ and $Tf$ are compact for all $f \in C_0(X)$.

**Definition 2.9.** With the same notation as above, let $\mathcal{B}(H_X)$ be the algebra of all bounded linear operators on $H_X$.

(i) The Roe algebra of $X$, denoted by $C^*(X)$, is the $C^*$-algebra generated by all locally compact operators in $\mathcal{B}(H_X)$ with finite propagation.

(ii) If $Y$ is a subspace of $X$, then the $C^*$-algebra $C^*(Y; X)$ is defined to be the closed subalgebra of $C^*(X)$ generated by all elements $T$ such that $\mathrm{Supp}(T)$ is within finite distance of $Y \times Y$.

Now suppose $\widetilde{X}$ is a Galois covering space of $X$. Denote its deck transformation group by $\Gamma$. Lift the Riemannian metric of $X$ to a Riemannian metric on $\widetilde{X}$ so that the action of $\Gamma$ on $\widetilde{X}$ is an isometric action. Also lift the Hermitian bundle $\mathcal{S}$ over $X$ to a Hermitian bundle $\widetilde{\mathcal{S}}$ over $\widetilde{X}$.

**Definition 2.10.** With the above notation, let $H_{\widetilde{X}} = L^2(\widetilde{X}, \widetilde{\mathcal{S}})$. Denote by $\mathbb{C}[X]^\Gamma$ the $*$-algebra of all $\Gamma$-equivariant locally compact operators of finite propagation in $\mathcal{B}(H_{\widetilde{X}})$.

(i) We define the $\Gamma$-equivariant Roe algebra $C^*(\widetilde{X})^\Gamma$ to be the closure of $\mathbb{C}[X]^\Gamma$ in $\mathcal{B}(H_{\widetilde{X}})$.

(ii) If $Y$ is a subspace of $X$, then the $C^*$-algebra $C^*(\widetilde{Y};\widetilde{X})^\Gamma$ is defined to be the closed subalgebra of $C^*(\widetilde{X})^\Gamma$ generated by all elements $T$ in $C^*(\widetilde{X})^\Gamma$ such that $\mathrm{Supp}(T)$ is within finite distance of $\widetilde{Y}\times\widetilde{Y}$, where $\widetilde{Y}$ is the restriction of the covering space $\widetilde{X}$ on $Y\subset X$.

(iii) We define the maximal $\Gamma$-equivariant Roe algebra $C^*_{\max}(\widetilde{X})^\Gamma$ to be the completion of $\mathbb{C}[X]^\Gamma$ under the maximal norm:

$$\|a\|_{\max}=\sup_{\phi}\{\|\phi(a)\| : \text{all } *\text{-representations } \phi:\mathbb{C}[X]^\Gamma\to\mathcal{B}(H')\}.$$

For a subspace $Y$ of $X$, the maximal version $C^*_{\max}(\widetilde{Y};\widetilde{X})^\Gamma$ is defined similarly.

Now let us review the construction of relative higher index. If $(N,\partial N)$ is a compact spin manifold with boundary, then we attach an infinite cylinder $[0,\infty)\times\partial N$ to $N$ along $\partial N$, and extend the Riemannian metric of $N$ to a complete Riemannian metric on the resulting manifold. The relative higher index of the Dirac operator on $(N,\partial N)$ will in fact be constructed using this complete manifold (relative to the cylindrical end). So for brevity, let us now assume $X$ is a complete Riemannian spin manifold and $K$ is a codimension zero compact submanifold (with boundary) of $X$. Let $\Gamma=\pi_1(X)$ and $G=\pi_1(X-K)$. Here if $X-K$ has more than one connected component, then $\pi_1(X-K)$ should mean the fundamental groupoid of $X-K$, that is, $\pi_1(X-K)$ is the disjoint union $\coprod_{\alpha=1}^{\ell}\pi_1(Y_\alpha)$, where $Y_\alpha$ are the components of $X-\mathring{K}$. In this case, the maximal group $C^*$-algebra of $G=\pi_1(X-K)$ is defined to be

$$C^*_{\max}(G)=\bigoplus_{\alpha=1}^{\ell}C^*_{\max}(\pi_1(Y_\alpha)).$$

Let $\iota_\alpha:G_\alpha\to\Gamma$ be the group homomorphism induced by the inclusion of spaces $Y_\alpha\hookrightarrow X$. Let $C_{\iota_*}$ be the mapping cone $C^*$-algebra induced by the homomorphism

$$\iota_*:C^*_{\max}(G)\otimes\mathcal{K}\to M_\ell(\mathbb{C})\otimes C^*_{\max}(\Gamma)\otimes\mathcal{K} \tag{2.11}$$

given by

$$\iota_*(a_1\oplus\cdots\oplus a_\ell)=\begin{pmatrix}(\iota_1)_*a_1 & & \\ & \ddots & \\ & & (\iota_\ell)_*a_\ell\end{pmatrix},$$

where $M_\ell(\mathbb{C})$ is the algebra of $(\ell\times\ell)$ matrices. We denote by $C^*_{\max}(\Gamma,G)$ the 7th suspension $\mathcal{S}^7C_{\iota_*}\cong C_0(\mathbb{R}^7)\otimes C_{\iota_*}$ of the mapping cone $C^*$-algebra $C_{\iota_*}$.

In the following, we review the construction of the relative higher index of the Dirac operator on $X$ (relative to $K$). For simplicity, we assume

$$\dim X\equiv 0 \pmod 8,$$

while the other dimensions are completely similar by a standard suspension argument. Let $\widetilde{X}$ be the universal covering space of $X$ and $\widetilde{D}$ the associated Dirac operator on $\widetilde{X}$. Let $\widetilde{Y}_\alpha$ be the universal covering space of $Y_\alpha$ whose deck transformation group is $G_\alpha$.

Choose a normalizing function $f$, i.e., a continuous odd function $f:\mathbb{R}\to\mathbb{R}$ such that

$$\lim_{t\to\pm\infty} f(t)=\pm 1. \tag{2.12}$$

Throughout this section, assume without loss of generality that we have chosen the normalizing function $f$ so that its distributional Fourier transform has compact support. Let $F=f(\widetilde{D})$ be the operator obtained by applying functional calculus to $\widetilde{D}$. Since we are in the even-dimensional case, $\widetilde{D}$ has odd-degree with respect to the natural $\mathbb{Z}/2$-grading on the spinor bundle of $\widetilde{X}$, that is,

$$\widetilde{D}=\begin{pmatrix}0 & \widetilde{D}^- \\ \widetilde{D}^+ & 0\end{pmatrix}.$$

In particular, it follows that

$$F=\begin{pmatrix}0 & U\\ V & 0\end{pmatrix}$$

for some operators $U$ and $V$.

We define the invertible element

$$W:=\begin{pmatrix}1 & U\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ V & 1\end{pmatrix}\begin{pmatrix}1 & U\\ 0 & 1\end{pmatrix}\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}$$

and form the idempotent

$$p=W\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}W^{-1}=\begin{pmatrix}UV(2-UV) & (2-UV)(1-UV)U\\ V(1-UV) & (1-VU)^2\end{pmatrix}. \tag{2.13}$$

Let $\chi$ be the characteristic function on $K$ and denote its lift to $\widetilde{X}$ by $\widetilde{\chi}$. Let $u$ be an invertible element in the matrix algebra of $C_0(\mathbb{R}^7)^+$ representing a generator of $\mathrm{KO}_1(C_0(\mathbb{R}^7))\cong \mathrm{KO}_0(C_0(\mathbb{R}^8))=\mathbb{Z}$. Consider the invertible elements

$$\mathcal{U}=u\otimes p+1\otimes(1-p)\quad\text{and}\quad \mathcal{V}=u^{-1}\otimes p+1\otimes(1-p)$$

in $(C_0(\mathbb{R}^7)\otimes C^*_{\max}(\widetilde{X})^\Gamma)^+$, where $C^*_{\max}(\widetilde{X})^\Gamma$ is the $\Gamma$-equivariant maximal Roe algebra of $\widetilde{X}$ and $(C_0(\mathbb{R}^7)\otimes C^*_{\max}(\widetilde{X})^\Gamma)^+$ is the unitization of $C_0(\mathbb{R}^7)\otimes C^*_{\max}(\widetilde{X})^\Gamma$. For each $s\in[0,1]$, we define the invertible element

$$\mathcal{W}_s:=\begin{pmatrix}1 & (1-s)\widetilde{\chi}\mathcal{U}\widetilde{\chi}\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ -(1-s)\widetilde{\chi}\mathcal{V}\widetilde{\chi} & 1\end{pmatrix}\begin{pmatrix}1 & (1-s)\widetilde{\chi}\mathcal{U}\widetilde{\chi}\\ 0 & 1\end{pmatrix}\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix} \tag{2.14}$$

and form the idempotent

$$\mathfrak{p}_s=\mathcal{W}_s\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}\mathcal{W}_s^{-1}. \tag{2.15}$$

By construction, each element $\mathfrak{p}_s$ lies in $(C_0(\mathbb{R}^7) \otimes C^*_{\max}(\widetilde{K}; \widetilde{X})^\Gamma)^+$. In particular, we have

$$\mathrm{KO}_i(C^*_{\max}(\widetilde{K}; \widetilde{X})^\Gamma) \cong \mathrm{KO}_i(C^*_{\max}(\widetilde{K})^\Gamma) \cong \mathrm{KO}_i(C^*_{\max}(\Gamma)).$$

Now let $Z_\alpha = Y_\alpha \cup_{\partial Y_\alpha} (\partial Y_\alpha \times [0, \infty))$ be the manifold obtained from $Y_\alpha$ by attaching an infinite cylinder. We equip $Z_\alpha$ with a complete Riemannian metric that agrees with the Riemannian metric of $X$ in a small neighborhood of $Y_\alpha$. Let $\widetilde{Z}_\alpha$ be the universal covering space of $Z_\alpha$. Note that $\pi_1(Z_\alpha) = \pi_1(Y_\alpha)$. Denote by $\partial\widetilde{Y}_\alpha$ the restriction of the covering space $\widetilde{Z}_\alpha$ on $\partial Y_\alpha \subset Z_\alpha$. We apply the same construction above to the Dirac operator $\widetilde{D}_\alpha$ on $\widetilde{Z}_\alpha$ (i.e., by replacing $\widetilde{D}$ by $\widetilde{D}_\alpha$ and $\chi$ by the characteristic function of $Z_\alpha - Y_\alpha = \partial Y_\alpha \times [0, \infty)$ in the above construction) and denote by $\mathfrak{q}_\alpha$ the resulting idempotent for when $s = 0$. Since the normalizing function $f$ in (2.12) has compactly supported Fourier transform, it follows that the idempotent $\mathfrak{q}_\alpha$ lies in $(C_0(\mathbb{R}^7) \otimes C^*_{\max}(\partial\widetilde{Y}_\alpha; \widetilde{Z}_\alpha)^{G_\alpha})^+$.

The canonical $(\Gamma, G_\alpha)$-equivariant map $\widetilde{Z}_\alpha \to \widetilde{X}$ induces a natural $C^*$ homomorphism

$$\psi_\alpha : C_0(\mathbb{R}^7) \otimes C^*_{\max}(\partial\widetilde{Y}_\alpha, \widetilde{Z}_\alpha)^{G_\alpha} \to C_0(\mathbb{R}^7) \otimes C^*_{\max}(\widetilde{K}; \widetilde{X})^\Gamma.$$

Let us define the map

$$\psi : \bigoplus_\alpha C_0(\mathbb{R}^7) \otimes C^*_{\max}(\partial\widetilde{Y}_\alpha, \widetilde{Z}_\alpha)^{G_\alpha} \to M_\ell(\mathbb{C}) \otimes C_0(\mathbb{R}^7) \otimes C^*_{\max}(\widetilde{K}; \widetilde{X})^\Gamma$$

by setting

$$\psi(a_1 \oplus \cdots \oplus a_\ell) = \begin{pmatrix} \psi_1(a_1) & & \\ & \ddots & \\ & & \psi_\ell(a_\ell) \end{pmatrix}.$$

Recall that, if the Fourier transform $\hat{f}$ of $f$ is supported in $(-\varepsilon, \varepsilon)$, then the Fourier transform $\hat{f_\lambda}$ of $f_\lambda$ is supported in $(-\lambda\varepsilon, \lambda\varepsilon)$, where $f_\lambda$ is the normalizing function given by $f_\lambda(t) = f(\lambda t)$. Hence, by replacing the normalizing function $f$ in (2.12) by $f_\lambda$ for some sufficiently small $\lambda > 0$ if necessary, we can assume the propagations of $F = f(\widetilde{D})$ and $f(\widetilde{D}_\alpha)$ are very small. Then by a standard finite propagation argument, it follows from the above construction that

$$\psi\left(\bigoplus_{\alpha=1}^{\ell} \mathfrak{q}_\alpha\right) = \mathfrak{p}_0.$$

Also, the product formula of higher index implies that each $\mathfrak{q}_\alpha$ is a representative of the higher index class of the Dirac operator $D_{\partial Y_\alpha}$ in $\mathrm{KO}_0(C_0(\mathbb{R}^7) \otimes C^*_{\max}(\partial\widetilde{Y}_\alpha)^{G_\alpha})$. To summarize, we obtain the $K$-class

$$\left(\bigoplus_{\alpha=1}^{\ell} \mathfrak{q}_\alpha, \mathfrak{p}_s\right) \in C^*_{\max}(\Gamma, G)^+ \cong (C_0(\mathbb{R}^7) \otimes C_{\iota_*})^+,$$

where $C_{\iota_*}$ is the mapping cone $C^*$-algebra from (2.11).

**Definition 2.16.** With the above notation, if $\dim X \equiv 0 \pmod 8$, the relative higher index $\mathrm{Ind}_{\Gamma,G}(D)$ of $D$ is defined to be

$$\mathrm{Ind}_{\Gamma,G}(\widetilde{D}) := \left[\left(\bigoplus_{\alpha=1}^{\ell} \mathfrak{q}_\alpha, \mathfrak{p}_s\right)\right] - \left[\left(\bigoplus_{\alpha=1}^{\ell} \begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}, \begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}\right)\right] \in \mathrm{KO}_0(C^*_{\max}(\Gamma, G)).$$

By [Chang et al. 2020, Theorem 2.18], if an $n$-dimensional spin manifold $X$ admits a complete Riemannian metric that has uniformly positive scalar curvature, then its relative higher index is zero in $\mathrm{KO}_n(C^*_{\max}(\Gamma, G))$, that is,

$$\mathrm{Ind}_{\Gamma,G}(D) = 0 \in \mathrm{KO}_n(C^*_{\max}(\Gamma, G)).$$

In fact, by a more careful finite propagation argument (as in [Guo et al. 2023; Xie 2023]), we have the following more refined quantitative vanishing theorem.

**Theorem 2.17** [Guo et al. 2023; Xie 2023]. *With the same notation as above, suppose $X$ admits a complete Riemannian metric $g$ such that the scalar curvature of $g$ is $\geq 1$ on the $\rho$-neighborhood $N_\rho(K)$ of $K$. Then there exists a universal constant $C > 0$ such that if $\rho > C$, then the relative higher index of $D$ is zero in* $\mathrm{KO}_n(C^*_{\max}(\Gamma, G))$, *that is,*

$$\mathrm{Ind}_{\Gamma,G}(D) = 0 \in \mathrm{KO}_n(C^*_{\max}(\Gamma, G)).$$

*Proof.* The essential ingredients of the proof have already been carried out in [Guo et al. 2023; Xie 2023]. For the convenience of the reader, we sketch a proof here.

Let $\chi$ be the characteristic function $\chi$ on $K$ that appeared in the above construction of the index class $\mathrm{Ind}_{\Gamma,G}(D)$. Observe that if we replace $\chi$ by the characteristic function $\chi_r$ of the $r$-neighborhood $N_r(K)$ of $K$, the resulting new element from the construction is also a representative of the index class $\mathrm{Ind}_{\Gamma,G}(D)$. This can be seen by replacing $\widetilde{\chi}\mathcal{U}\widetilde{\chi}$ and $\widetilde{\chi}\mathcal{V}\widetilde{\chi}$ by

$$t\widetilde{\chi}\mathcal{U}\widetilde{\chi} + (1-t)\widetilde{\chi}_r\mathcal{U}\widetilde{\chi}_r \quad \text{and} \quad t\widetilde{\chi}\mathcal{V}\widetilde{\chi} + (1-t)\widetilde{\chi}_r\mathcal{V}\widetilde{\chi}_r \tag{2.18}$$

in the definition of $\mathcal{W}_s$ from (2.14), with $t \in [0, 1]$, where $\widetilde{\chi}_r$ is the lift of $\chi_r$ to $\widetilde{X}$.

By assumption the scalar curvature of $g$ is $\geq 1$ on $N_\rho(K)$. It follows that

$$\|\widetilde{D}v\| \geq \tfrac{1}{4}\|v\|$$

for all smooth sections $v \in C_c^\infty(\mathring{N}_\rho(K), \widetilde{\mathcal{S}})$, where $\mathring{N}_\rho(K)$ is the interior of $N_\rho(K)$. A finite propagation argument shows that, as long as $\rho$ is sufficiently large, we can choose the characteristic function $\chi_{\rho/2}$ of $N_{\rho/2}(K)$ and an appropriate normalizing function $f$ in (2.12) such that $\mathfrak{p}_s$ in (2.15) becomes a trivial idempotent (cf. [Xie 2023, Lemma 3.2 and Appendix A]). The same argument applies to the construction of $\mathfrak{q}_\alpha$ above so that each $\mathfrak{q}_\alpha$ becomes a trivial idempotent (cf. [Guo et al. 2023, proof of Theorem 1.3]). This completes the proof. □

At the end of this section, let us review the construction of the index map

$$\sigma : \mathrm{KO}_*^{\mathrm{lf}}(Y) \to \mathrm{KO}_*(\mathcal{A}(Y))$$

(see [Chang et al. 2020, Section 3]) and also introduce a notion of $\varprojlim^1$ higher index.

An element in $\mathrm{KO}_*^{\mathrm{lf}}(Y)$ is represented by a smooth open spin manifold $X$ together with a proper continuous coarse map $\varphi : X \to Y$. Let $\{Y_i\}$ be an admissible exhaustion on $Y$ as in Definition 2.2. Without loss of generality, assume $X$ is equipped with an exhaustion $\{X_i\}$ such that the $X_i$ are compact connected codimension zero submanifolds with $X_i \subset \mathring{X}_{i+1}$ and $X = \bigcup_i X_i$. Here $\mathring{X}_{i+1}$ is the interior of $X_{i+1}$. Furthermore, without loss of generality, we assume $X$ is equipped with a complete Riemannian metric so that $Y_i \subset \varphi(X_i) \subset Y_{i+1}$.

Let us denote $\Gamma = \pi_1(X)$ and $G_i = \pi_1(X - X_i)$. Here, if $X - X_i$ has more than one connected component, then $\pi_1(X - X_i)$ should mean the fundamental groupoid of $X - X_i$, that is, $\pi_1(X - X_i)$ is the disjoint union $\coprod \pi_1(Y_{i\alpha})$, where $Y_{i\alpha}$ are the components of $X - X_i$. In this case, the maximal group $C^*$-algebra of $G_i = \pi_1(X - X_i)$ is defined to be

$$C^*_{\max}(G_i) = \bigoplus_{\alpha} C^*_{\max}(\pi_1(Y_{i\alpha})).$$

Let $\iota_\alpha : \pi_1(Y_{i\alpha}) \to \Gamma$ be the group homomorphism induced by the inclusion $Y_{i\alpha} \hookrightarrow X$. Let $C_{\iota_*}$ be the mapping cone $C^*$-algebra induced by the homomorphism

$$\iota_* : C^*_{\max}(G_i) \otimes \mathcal{K} \to M_\ell(\mathbb{C}) \otimes C^*_{\max}(\Gamma) \otimes \mathcal{K} \tag{2.19}$$

given by

$$\iota_*(a_1 \oplus \cdots \oplus a_\ell) = \begin{pmatrix} (\iota_1)_* a_1 & & \\ & \ddots & \\ & & (\iota_\ell)_* a_\ell \end{pmatrix},$$

where $M_\ell(\mathbb{C})$ is the algebra of $(\ell \times \ell)$ matrices. We denote by $C^*_{\max}(\Gamma, G_i)$ the 7th suspension $\mathcal{S}^7 C_{\iota_*} \cong C_0(\mathbb{R}^7) \otimes C_{\iota_*}$ of the mapping cone $C^*$-algebra $C_{\iota_*}$. The inclusions $(X, X - X_{i+1}) \hookrightarrow (X, X - X_i)$ induce $C^*$ homomorphisms

$$\rho_{i+1} : C^*_{\max}(\Gamma, G_{i+1}) \to C^*_{\max}(\Gamma, G_i).$$

Let $\rho : \prod_{i=1}^\infty C^*_{\max}(\Gamma, G_i) \to \prod_{i=1}^\infty C^*_{\max}(\Gamma, G_i)$ be the homomorphism that maps $(a_1, a_2, \ldots)$ to $(\rho_2(a_2), \rho_3(a_3), \ldots)$. By definition, we have

$$\mathcal{A}(X) := \left\{ a \in C\left([0,1], \prod_{i=1}^\infty C^*_{\max}(\Gamma, G_i)\right) \;\middle|\; \rho(a(0)) = a(1) \right\}.$$

Now suppose $\widetilde{D}$ is the Dirac operator on the universal covering space of $\widetilde{X}$. By the construction of relative higher index (Definition 2.16), for each $(X, X - X_i)$, we have the relative higher index of $\widetilde{D}$ represented by

$$\Big[\Big(\bigoplus_{\alpha}\mathfrak{q}_{i\alpha},(\mathfrak{p}_i)_s\Big)\Big]-\Big[\Big(\bigoplus_{\alpha}\begin{pmatrix}1&0\\0&0\end{pmatrix},\begin{pmatrix}1&0\\0&0\end{pmatrix}\Big)\Big]\in \mathrm{KO}_n(C^*_{\max}(\Gamma,G_i)).$$

For simplicity, let us write

$$\mathfrak{a}_i:=\Big(\bigoplus_{\alpha}\mathfrak{q}_{i\alpha},(\mathfrak{p}_i)_s\Big)\quad\text{and}\quad \mathfrak{b}_i:=\Big(\bigoplus_{\alpha}\begin{pmatrix}1&0\\0&0\end{pmatrix},\begin{pmatrix}1&0\\0&0\end{pmatrix}\Big)$$

in $C^*_{\max}(\Gamma,G_i)$.

Consider the characteristic functions $\chi_i$ of $X_i$ and $\chi_{i+1}$ of $X_{i+1}$. By applying a linear path similar to that from (2.18) in the construction of relative higher index, we obtain a continuous path of elements $\mathfrak{a}_i(t)$ with $t\in[0,1]$ such that

$$\mathfrak{a}_i(0)=\mathfrak{a}_i\quad\text{and}\quad \mathfrak{a}_i(1)=\rho_{i+1}(\mathfrak{a}_{i+1}),$$

which in particular defines a $K$-theory class of $\mathcal{A}(X)$.

**Definition 2.20.** The index map $\sigma:\mathrm{KO}^{\mathrm{lf}}_*(Y)\to\mathrm{KO}_*(\mathcal{A}(Y))$ is defined by setting

$$\sigma([D,\varphi])=\varphi_*[(\mathfrak{a}_1(t),\mathfrak{a}_2(t),\dots)]-\varphi_*[(\mathfrak{b}_1,\mathfrak{b}_2,\dots)]\in\mathrm{KO}_n(\mathcal{A}(Y)),$$

where $\varphi_*:\mathrm{KO}_n(\mathcal{A}(X))\to\mathrm{KO}_n(\mathcal{A}(Y))$ is the homomorphism induced by $\varphi:X\to Y$.

Theorem 3.3 of [Chang et al. 2020] showed that if $X$ admits a complete Riemannian metric that has uniformly positive scalar curvature on the whole $X$, then the above index $\sigma([D,\varphi])$ vanishes in $\mathrm{KO}_n(\mathcal{A}(Y))$. This for example follows by an argument that is similar to that used in the proof of Theorem 2.17.

Note that we have the following Milnor exact sequence (cf. (2.5)):

$$0\to\varprojlim{}^1\mathrm{KO}_{n+1}(C^*_{\max}(\Gamma,G_i))\xrightarrow{\zeta}\mathrm{KO}_n(\mathcal{A}(X))\xrightarrow{\theta}\varprojlim\mathrm{KO}_n(C^*_{\max}(\Gamma,G_i))\to 0.$$

Let $\{X_i\}$ be the exhaustion of $X$ as above. Suppose we are in a special case where the relative higher index of $D$ on $(X,X_i)$ vanishes in $\mathrm{KO}_n(C^*_{\max}(\Gamma,G_i))$. For example, by Theorem 2.17, such a condition is satisfied if for each $X_i\subset X$, there is a complete Riemannian metric $g_i$ on $X$ such that the scalar curvature of $g_i$ is $\geq 1$ on the $\rho$-neighborhood $N_\rho(X_i)$ of $X_i$ for some sufficiently large $\rho>0$. Let $\sigma(D)$ be the index of $D$ in $\mathrm{KO}_n(\mathcal{A}(X))$. Then in this case, the image of $\sigma(D)$ under the map $\theta$ is zero in $\varprojlim\mathrm{KO}_n(C^*_{\max}(\Gamma,G_i))$. It follows $\sigma(D)=\zeta(c)$ for some unique element $c\in\varprojlim{}^1\mathrm{KO}_{n+1}(C^*_{\max}(\Gamma,G_i))$ in this case.

**Definition 2.21.** When $\theta(\sigma(D))$ vanishes in $\varprojlim\mathrm{KO}_n(C^*_{\max}(\Gamma,G_i))$, we define the $\varprojlim{}^1$ higher index of $D$ to be

$$\varprojlim{}^1\mathrm{Ind}_{\Gamma,G}(D):=c\in\varprojlim{}^1\mathrm{KO}_{n+1}(C^*_{\max}(\Gamma,G_i)),$$

where $c$ is the unique element in $\varprojlim{}^1\mathrm{KO}_{n+1}(C^*_{\max}(\Gamma,G_i))$ such that $\zeta(c)=\sigma(D)$.

**Remark 2.22.** The definition of $\mathcal{A}(X)$ depends on the particular choice of an exhaustion $\{X_i\}$ and may vary if we choose a different exhaustion of $X$. However, the KO-theory of $\mathcal{A}(X)$ is in fact independent of the choice of exhaustion. Indeed, if $\{X'_k\}$ is another exhaustion of $X$, then there exists a subsequence $\{X_{i_k}\}$ of the exhaustion $\{X_i\}$ such that $X'_k \subset X_{i_k}$ for each $k \geq 1$. If we denote by $\mathcal{A}(X; \{X_{i_k}\})$ (resp. $\mathcal{A}(X; \{X'_k\})$) the $C^*$-algebra $\mathcal{A}(X)$ determined by the exhaustion $\{X_{i_k}\}$ (resp. $\{X'_k\}$), then there is a natural $C^*$ homomorphism

$$\mathcal{A}(X; \{X_{i_k}\}) \to \mathcal{A}(X; \{X'_k\}) \tag{2.23}$$

induced by the canonical inclusions of spaces $(X, X - X_{i_k}) \hookrightarrow (X, X - X'_k)$. Similarly, there is a subsequence $\{X'_{k_i}\}$ of the exhaustion $\{X'_k\}$ such that $X_i \subset X'_{k_i}$ for each $i \geq 1$, which gives a $C^*$ homomorphism

$$\mathcal{A}(X; \{X'_{k_i}\}) \to \mathcal{A}(X; \{X_i\}). \tag{2.24}$$

Consider the Milnor exact sequence

$$0 \to \varprojlim{}^1 \mathrm{KO}_{n+1}(C^*_{\max}(\Gamma, G_i)) \\ \to \mathrm{KO}_n(\mathcal{A}(X; \{X_i\})) \to \varprojlim \mathrm{KO}_n(C^*_{\max}(\Gamma, G_i)) \to 0.$$

As both $\varprojlim \mathrm{KO}_n(C^*_{\max}(\Gamma, G_i))$ and $\varprojlim^1 \mathrm{KO}_{n+1}(C^*_{\max}(\Gamma, G_i))$ remain unchanged when passing to (cofinal) subsequences, it follows that $\mathrm{KO}_n(\mathcal{A}(X; \{X_i\}))$ remains unchanged when passing to subsequences of the exhaustion $\{X_i\}$. Now by passing further to subsequences of $\{X_{i_k}\}$ and $\{X'_{k_i}\}$, it is not difficult to see that the $C^*$ homomorphisms from (2.23) and (2.24) induce isomorphisms at the level of KO-theory. This shows that $\mathrm{KO}_*(\mathcal{A}(X))$ is independent of the choice of exhaustion.

## 3. Negative answers to Gromov's compactness question

In this section, we prove our main theorem (Theorem 1.1). First, let us fix some notation. Consider an inverse system of groups

$$G_0 \xleftarrow{\varphi_1} G_1 \xleftarrow{\varphi_2} G_2 \xleftarrow{\varphi_3} \cdots,$$

where each $\varphi_i$ is surjective. Let $Y_0$ be the cone over $BG_0$, where $BG_0$ is the classifying space of $G_0$. We define $Y_i$ inductively as follows. Note that the group homomorphism $\varphi_i$ induces a continuous map $\Phi_i : BG_i \to BG_{i-1}$, where $BG_i$ is the classifying space of $G_i$. Let $Y_i$ be the mapping cylinder obtained by gluing $BG_i \times I = BG_i \times [0, 1]$ to $Y_{i-1}$ along $\partial Y_{i-1} = BG_{i-1}$ via the map $\Phi_i$.

**Definition 3.1.** We define $B_G$ to be the resulting mapping cylinder of the above infinite composite. Sometimes we say $B_G$ is the classifying space associated to the inverse system $\{G_i, \varphi_i\}$.

Throughout the rest of the paper, we only use groups $G_i$ of the form $G_i = \mathbb{Z}^2 * F_i$, where $F_i$ is a finitely generated free group and $\mathbb{Z}^2 * F_i$ is the free product of $\mathbb{Z}^2$ with $F_i$. So from now on, for simplicity, let us assume each $BG_i$ is compact and has been equipped with a complete metric. Clearly, by appropriately stretching each cylinder $BG_i \times I$, we can equip $B_G$ with a complete metric such that the sequence $\{Y_i\}$ becomes an admissible exhaustion of $B_G$ in the sense of Definition 2.2. Let $\psi_i : (Y_i, \partial Y_i) \to (Y_{i-1}, \partial Y_{i-1})$ be the obvious collapse map, that is, $\psi_i$ is the identity map on $Y_{i-1} \subset Y_i$ and $\psi_i$ collapses $BG_i \times I$ to $\partial Y_{i-1} = BG_{i-1}$ via the map $\Phi_i$. Let us write $Y_{i,i+1}$ for the annulus region $Y_{i+1} - \mathring{Y}_i$ between $\partial Y_i$ and $\partial Y_{i+1}$. Clearly, $Y_{i,i+1}$ is just the mapping cylinder obtained by gluing $BG_i \times I$ to $BG_{i-1}$ via the map $\Phi_i : BG_i \to BG_{i-1}$.

For the above space $B_G$ with the exhaustion $\{Y_i\}$, there is the Milnor exact sequence

$$0 \to \varprojlim{}^1 \mathrm{KO}_{n+1}(Y_j, \partial Y_j) \to \mathrm{KO}_n^{\mathrm{lf}}(B_G) \to \varprojlim \mathrm{KO}_n(Y_j, \partial Y_j) \to 0,$$

where $\mathrm{KO}_n^{\mathrm{lf}}(B_G)$ is the $n$-th locally finite KO-homology of $B_G$. In the following, we construct an inverse system of groups

$$G_0 \xleftarrow{\varphi_1} G_1 \xleftarrow{\varphi_2} G_2 \xleftarrow{\varphi_3} \cdots$$

such that each $\varphi_i$ is surjective and $\varprojlim{}^1 \mathrm{KO}_{n+1}(Y_j, \partial Y_j)$ is nontrivial for the associated classifying space $B_G$ equipped with the exhaustion $\{Y_i\}$.

Let us define $G_0 = \mathbb{Z}^2$ and $G_i = \mathbb{Z}^2 * F_i$, where each $F_i$ is a finitely generated free group. Let

$$\varphi_i : \mathbb{Z}^2 * F_i \to \mathbb{Z}^2 * F_{i-1}$$

be a surjective group homomorphism such that $\varphi_i$ maps the subgroup $\mathbb{Z}^2 * \{e\}$ of $\mathbb{Z}^2 * F_i$ to the subgroup $\mathbb{Z}^2 * \{e\}$ of $\mathbb{Z}^2 * F_{i-1}$ via the $\times 3$ map, that is,

$$\mathbb{Z}^2 \to \mathbb{Z}^2, \quad (a, b) \mapsto (3a, 3b).$$

Clearly, $F_i$ and $\varphi_i$ with the above properties exist. Consider the resulting inverse system of groups

$$G_0 \xleftarrow{\varphi_1} G_1 \xleftarrow{\varphi_2} G_2 \xleftarrow{\varphi_3} \cdots . \tag{3.2}$$

We have the following nonvanishing result for the $\varprojlim{}^1$ term.

**Proposition 3.3.** *Let $B_G$ be the classifying space associated to the inverse system given in* (3.2) *equipped with the exhaustion $\{Y_i\}$. Then the group $\varprojlim{}^1 \mathrm{KO}_3(Y_j, \partial Y_j)$ is nontrivial.*

*Proof.* Note that each $Y_i$ is contractible. It follows that

$$\mathrm{KO}_n(Y_i, \partial Y_i) \cong \widetilde{\mathrm{KO}}_{n-1}(\partial Y_i) = \widetilde{\mathrm{KO}}_{n-1}(BG_i).$$

The classifying space $BG_i = B\mathbb{Z}^2 \vee BF_i$ is the wedge sum of $B\mathbb{Z}^2$ and $BF_i$, where $B\mathbb{Z}^2$ is a 2-dimensional torus $\mathbb{T}^2$ and $BF_i$ is a wedge sum of circles. In particular, we have

$$H_2(\partial Y_i) = H_2(BG_i) = H_2(B\mathbb{Z}^2) \cong \mathbb{Z}$$

in this case. It follows that the group homomorphism $\varphi_i : \mathbb{Z}^2 * F_i \to \mathbb{Z}^2 * F_{i-1}$ induces the following homomorphism on homology:

$$H_2(BG_i) \cong \mathbb{Z} \xrightarrow{\times 9} H_2(BG_i) \cong \mathbb{Z}.$$

It follows that $\varprojlim^1 \mathrm{KO}_3(Y_j, \partial Y_j)$ contains a copy of $\widehat{\mathbb{Z}}_3/\mathbb{Z}$ as a subgroup, where $\widehat{\mathbb{Z}}_3$ is the group of 3-adic integers.[4] In particular, $\varprojlim^1 \mathrm{KO}_3(Y_j, \partial Y_j)$ is nontrivial. □

The next few results use some basic constructions from surgery theory. Let us first recall some standard terminology from surgery theory. Let $M$ be a manifold of dimension $n$. Observe that

$$\partial(\mathbb{S}^p \times \mathbb{D}^q) = \mathbb{S}^p \times \mathbb{S}^{q-1} = \partial(\mathbb{D}^{p+1} \times \mathbb{S}^{q-1}).$$

Given an embedded $\mathbb{S}^p \times \mathbb{D}^q \subset M$ with $p + q = n$, let $M'$ be the manifold obtained by removing the interior of $\mathbb{S}^p \times \mathbb{D}^q$ and gluing in a copy of $\mathbb{D}^{p+1} \times \mathbb{S}^{q-1}$ along $\mathbb{S}^p \times \mathbb{S}^{q-1}$. In this case, we say $M'$ is obtained from $M$ by a surgery of dimension $p$ (or codimension $q$).

The trace of a $p$-surgery is given by

$$W := (M \times I) \cup_{\mathbb{S}^p \times \mathbb{D}^q} (\mathbb{D}^{p+1} \times \mathbb{D}^q),$$

which is a cobordism between $M$ and $M'$. In this case, we say $W$ is obtained from $M \times I$ by attaching a handle of index $(p + 1)$.

**Lemma 3.4.** *Let $W$ be a spin cobordism between two closed spin manifolds $\partial_- W$ and $\partial_+ W$. Assume both $\partial_+ W$ and $W$ are connected, and $\pi_1(\partial_+ W) \to \pi_1(W)$ is an isomorphism. If* $\dim W \geq 6$, *then there exists a spin cobordism $W'$ between $\partial_- W$ and $\partial_+ W$ such that*

(1) *$\pi_1(\partial_+ W) \to \pi_1(W')$ is an isomorphism,*

(2) *$W'$ is obtained from $(\partial_+ W) \times I$ by attaching handles of index $\geq 3$, or equivalently $\partial_+ W$ is obtained from $\partial_- W$ via surgeries of codimension $\geq 3$. Moreover, $W$ can be obtained from $W'$ by finitely many surgeries of codimension $\geq 3$ away from the boundary.*

*Proof.* A proof of this lemma can be found for example in the proof of [Rosenberg 1986, Theorem 2.2]. See also [Schick and Zenobi 2020, Proposition 3.1]. For the

[4] Recall that $\widehat{\mathbb{Z}}_3$ is the inverse limit of the inverse system $\cdots \to \mathbb{Z}/3^3\mathbb{Z} \to \mathbb{Z}/3^2\mathbb{Z} \to Z/3\mathbb{Z}$. Equivalently, $\widehat{\mathbb{Z}}_3$ can also be viewed as the inverse limit of the inverse system $\cdots \to \mathbb{Z}/9^3\mathbb{Z} \to \mathbb{Z}/9^2\mathbb{Z} \to Z/9\mathbb{Z}$.

convenience of the reader, let us repeat the argument here. Note that there is an exact sequence of homotopy groups

$$\pi_2(\partial_+ W) \to \pi_2(W) \to \pi_2(W, \partial_+ W) \to \pi_1(\partial_+ W) \to \pi_1(W),$$

which reduces to

$$\pi_2(\partial_+ W) \to \pi_2(W) \to \pi_2(W, \partial_+ W) \to 0$$

since $\pi_1(\partial_+ W) \to \pi_1(W)$ is an isomorphism. As both $\partial_+ W$ and $W$ are connected and $\pi_1(\partial_+ W) \to \pi_1(W)$ is an isomorphism, it follows that $W$ is obtained from $\partial_+ W$ by attaching finitely many handles of index $\geq 2$. In particular, it follows that $\pi_2(W, \partial_+ W)$ is finitely generated as a $\mathbb{Z}[\pi_1(\partial_+ W)]$-module. Since $\dim W \geq 6$ (in fact, $\geq 5$ would suffice here), a set of elements of $\pi_2(W)$ that generate $\pi_2(W, \partial_+ W) = \pi_2(W)/\pi_2(\partial_+ W)$ can be represented by smoothly embedded 2-spheres which do not intersect the boundary of $W$. Since $W$ is a spin manifold, these 2-spheres have trivial normal bundles, and can be removed by surgeries preserving the spin structure and fundamental group. Since $\pi_2(W, \partial_+ W)$ is finitely generated as a $\mathbb{Z}[\pi_1(\partial_+ W)]$-module, only finitely many surgeries are needed in order to annihilate $\pi_2(W, \partial_+ W)$. Denote the resulting new cobordism by $W'$. Note that $(W', \partial_+ W)$ is 2-connected. Now choose a handle decomposition of $W'$, and proceed to eliminate 0-, 1- and 2-handles as in [Kervaire 1965, Lemma 1]. Note that the proof of [Kervaire 1965, Lemma 1] shows that such a process does not require $W'$ to be an $h$-cobordism, only that $\pi_1(\partial_+ W) \cong \pi_1(W')$ and that $\pi_2(W', \partial_+ W) = 0$. Thus we conclude that $W'$ can be obtained from $(\partial_+ W) \times I$ by attaching handles of index $\geq 3$. Turning this handle decomposition upside down, we see that $(\partial_+ W) \times I$ is obtained from $W'$ by attaching handles of index $\leq n-2$, i.e., $\partial_+ W$ is obtained from $\partial_- W$ by performing surgeries of codimension $\geq 3$. Furthermore, since by construction $W'$ is obtained from $W$ by surgeries of dimension 2, we see that $W$ can be obtained from $W'$ by surgeries of codimension 3. □

**Lemma 3.5.** *Let $W$ be a cobordism between two closed manifolds $\partial_- W$ and $\partial_+ W$ such that both $\partial_+ W$ and $W$ are connected, and $\pi_1(\partial_+ W) \to \pi_1(W)$ is surjective. If* $\dim W \geq 6$, *then $W$ is obtained from $(\partial_+ W) \times I$ by attaching handles of index $\geq 2$ (in other words, $\partial_+ W$ is obtained from $\partial_- W$ via surgeries of codimension $\geq 2$).*

*Proof.* Recall that given any finitely generated group $G$, if $H$ is a normal subgroup of $G$ such that $G/H$ is a finitely presented group, then $H$ is finitely normally generated[5] in $G$. In particular, since both $\pi_1(\partial_+ W)$ and $\pi_1(W)$ are finitely presented and $\pi_1(\partial_+ W) \to \pi_1(W)$ is surjective, it follows that the kernel of $\pi_1(\partial_+ W) \to \pi_1(W)$ is finitely normally generated. We present these finitely many elements by disjoint

[5] A normal subgroup $H$ of $G$ is finitely normally generated if $H$ is the normal closure of a subgroup generated by finitely many elements.

circles, and consider disjoint 2-dimensional discs $\{\mathbb{D}_i^2\}_{1\leq i\leq \ell}$ in $W$ which intersect $\partial_+ W$ in these circles. A small regular neighborhood $V$ of $(\partial_+ W \cup \bigcup_{i=1}^{\ell} \mathbb{D}_i^2)$ gives a cobordism between $\partial_+ W$ and a new closed manifold, denoted by $N$. By construction, we may view $\partial_+ W$ as obtained from $N$ via surgeries of codimension 2.

Let $\mathring{V}$ be the interior of $V$. Consider the cobordism $Z := W - \mathring{V}$ between $N$ and $\partial_- W$, where the map $\pi_1(N) \to \pi_1(Z)$ is an isomorphism. As both $N$ and $Z$ are connected and $\pi_1(N) \to \pi_1(Z)$ is an isomorphism, it follows that $Z$ is obtained from $N$ by attaching finitely many handles of index $\geq 2$.

To summarize, we see that $W$ is obtained from $(\partial_+ W) \times I$ by attaching handles of index $\geq 2$. This finishes the proof. □

The following theorem is a consequence of Lemma 3.5 and [Chang et al. 2020, Theorem 4.2].

**Theorem 3.6.** *Let $B_G$ be the space from Definition 3.1 equipped with the exhaustion $\{Y_i\}$. Then given any $c \in \mathrm{KO}_n^{\mathrm{lf}}(B_G)$, there is an element $(M, f) \in \Omega_*^{\mathrm{spin,lf}}(B_G)$ such that $M$ is a noncompact spin manifold and $f$ is a proper map from $M$ to $B_G$ satisfying the following*:

(1) $f_*[D_M] = c$;

(2) *the inverse images $(M_i, \partial M_i) = f^{-1}(Y_i, \partial Y_i)$ are compact manifolds with boundary such that the induced maps $\pi_1(M_i) \to \pi_1(Y_i)$ and $\pi_1(\partial M_i) \to \pi_1(\partial Y_i)$ are all isomorphisms*;

(3) *the induced maps $\pi_1(A_i) \to \pi_1(Y_{i,i+1})$ are isomorphisms for all $i \geq 0$, where $A_i = M_{i+1} - \mathring{M}_i$ is the annulus region between $\partial M_i$ and $\partial M_{i+1}$*;

(4) *each $A_i$ is obtained from $(\partial M_{i+1}) \times I$ by attaching handles of index $\geq 2$. Equivalently, $\partial M_{i+1}$ is obtained from $\partial M_i$ via surgeries of codimension $\geq 2$.*

*Proof.* By [Chang et al. 2020, Theorem 4.2], for any $c \in \mathrm{KO}_n^{\mathrm{lf}}(B_G)$, there is an element $(M, f) \in \Omega_*^{\mathrm{spin,lf}}(B_G)$ such that $M$ is a noncompact spin manifold and $f$ is a proper map from $M$ to $B_G$ satisfying the following:

(a) $f_*[D_M] = c$, where $D_M$ is the Dirac operator on $M$;

(b) the inverse images $(M_i, \partial M_i) = f^{-1}(Y_i, \partial Y_i)$ are compact manifolds with boundary such that the induced maps $\pi_1(M_i) \to \pi_1(Y_i)$ and $\pi_1(\partial M_i) \to \pi_1(\partial Y_i)$ are all isomorphisms;

(c) the induced maps $\pi_1(A_i) \to \pi_1(Y_{i,i+1})$ are isomorphisms for all $i \geq 0$, where $A_i = M_{i+1} - \mathring{M}_i$ is the annulus region between $\partial M_i$ and $\partial M_{i+1}$.

By Lemma 3.5, these $A_i$'s satisfy condition (4). This finishes the proof. □

We also need an extension result for Riemannian metrics on certain types of cobordisms (Proposition 3.8). First let us recall the following extension lemma of Shi, Wang and Wei.

**Lemma 3.7** [Shi et al. 2022, Lemma 2.1]. *Let $\Sigma$ be a closed smooth manifold. Suppose $h_1$ and $h_0$ are two smooth Riemannian metrics on $\Sigma$ such that $h_1 < h_0$. Then for any constant $k > 0$ and $m \in \mathbb{R}$, there exists a smooth Riemannian metric $g$ on the cylinder $\Sigma \times [0, 1]$ such that*

(1) *$g$ extends $h_0$ and $h_1$, that is,*

$$g|_{\Sigma\times\{0\}} = h_0 \quad \textit{and} \quad g|_{\Sigma\times\{1\}} = h_1,$$

(2) $\mathrm{Sc}(g) \geq k$,

(3) *the mean curvature at 0-end of the cylinder is bounded below by $m$, that is,*

$$H_g(\Sigma \times \{0\})_x \geq m$$

*for all $x \in \Sigma \times \{0\}$.*

Here the mean curvature $H_g(\Sigma \times \{0\})_x$ is calculated with respect to the outer normal vector. In particular, our convention is that the mean curvature of the standard $n$-dimensional sphere is positive when viewed as the boundary of the standard $(n+1)$-dimensional Euclidean ball.

Combining the above extension lemma of Shi, Wang and Wei with the surgery theory for positive scalar curvature metrics of [Gromov and Lawson 1980; Schoen and Yau 1979], we have the following proposition.

**Proposition 3.8.** *Let $Z$ be a cobordism between two closed smooth manifolds $\Sigma_1$ and $\Sigma_2$ such that $Z$ is obtained from $\Sigma_2 \times I$ by attaching handles of index $\geq 2$. Given any smooth Riemannian metric $h$ on $\Sigma_1$, for any constants $k > 0$ and $m \in \mathbb{R}$, there exists a smooth Riemannian metric $g$ on $Z$ such that*

(1) *$g$ extends $h$, that is,*

$$g|_{\Sigma_1} = h,$$

(2) $\mathrm{Sc}(g) \geq k$,

(3) *$H_g(\Sigma_1)_x \geq m$ for all $x \in \Sigma_1$.*

*Proof.* Consider the cylinder $\Sigma_1 \times I$. Let us equip $\Sigma_1 \times \{0\}$ with the metric $h$, and $\Sigma_1 \times \{1\}$ with any Riemannian metric $h_1$ such that $h_1 < h$ (e.g., $h_1 = h/2$). Let $g_0$ be a Riemannian metric on $\Sigma_1 \times I$ delivered by Lemma 3.7 for the constants $(k+1)$ and $m$.

Now let us consider the cobordism

$$W = Z \cup_{\Sigma_2} (-Z)$$

obtained by gluing $Z$ with its opposite $-Z$ along the boundary component $\Sigma_2$.

**Claim 3.9.** *$W$ is obtained from the cylinder $\Sigma_1 \times I$ via surgeries of codimension $\geq 3$.*

Indeed, the space $Z \times [0,1]$ is a cobordism between

$$Z \times \{0\} \cup \Sigma_2 \times [0,1] \cup Z \times \{1\} \cong W \quad \text{and} \quad \Sigma_1 \times \left[-\tfrac{1}{2}, \tfrac{1}{2}\right] \cong \Sigma_1 \times I.$$

Since $Z$ is obtained from $\Sigma_2 \times I$ by attaching handles of index $\geq 2$, it is not difficult to see that $W$ is obtained from the cylinder $\Sigma_1 \times I$ via surgeries of codimension $\geq 3$. More precisely, recall that attaching a handle of index $(p+1)$ to $\Sigma_1 \times I$ is given by

$$(\Sigma_1 \times I) \cup_{\mathbb{S}^p \times \mathbb{D}^q} (\mathbb{D}^{p+1} \times \mathbb{D}^q),$$

where $\mathbb{S}^p$ is a $p$-dimensional sphere in $\Sigma_1 \times \{0\} = \Sigma_1$ and $\mathbb{S}^p \times \mathbb{D}^q$ is a tubular neighborhood of $\mathbb{S}^p$ in $\Sigma_1$. For brevity, let us denote the resulting space from the above handle attaching construction by $Y$. Note that $Y$ has two boundary components: one of them is $\Sigma_1$ and the other one is denoted by $\Sigma'$. Let $Y \cup_{\Sigma'} Y$ be the space obtained from two copies of $Y$ glued along $\Sigma'$. We claim that $Y \cup_{\Sigma'} Y$ is obtained from $(\Sigma_1 \times I) \cup_{(\Sigma_1 \times \{0\})} (\Sigma_1 \times I)$ by performing a codimension $(q+1)$ surgery. Indeed, let $\mathbb{S}^p$ be the $p$-dimensional sphere in $\Sigma_1 \times \{0\}$ from above. Its tubular neighborhood in $(\Sigma_1 \times I) \cup_{(\Sigma_1 \times \{0\})} (\Sigma_1 \times I)$ is $\mathbb{S}^p \times \mathbb{D}^{q+1}$. It is not difficult to see that the space $Y \cup_{\Sigma'} Y$ is obtained from $(\Sigma_1 \times I) \cup_{(\Sigma_1 \times \{0\})} (\Sigma_1 \times I)$ by removing the interior of $\mathbb{S}^p \times \mathbb{D}^{q+1}$ and gluing a copy of

$$\mathbb{D}^{p+1} \times (\mathbb{D}^q \cup_{\partial \mathbb{D}^q} \mathbb{D}^q) \cong \mathbb{D}^{p+1} \times \mathbb{S}^q$$

to $\mathbb{S}^p \times \mathbb{S}^q = \partial(\mathbb{S}^p \times \mathbb{D}^{q+1})$. By assumption $Z$ is obtained from $\Sigma_2 \times I$ by attaching handles of index $\geq 2$, or equivalently $Z$ is obtained from $\Sigma_1 \times I$ by attaching handles of index $\leq (\dim Z - 2)$. It follows from the above discussion that $W$ is obtained from $(\Sigma_1 \times I) \cup_{(\Sigma_1 \times \{0\})} (\Sigma_1 \times I) \cong \Sigma_1 \times I$ via surgeries of codimension $\geq 3$. This proves the claim.

Now by surgery theory for positive scalar curvature metrics of [Gromov and Lawson 1980; Schoen and Yau 1979], it follows that there exists a positive scalar curvature metric $g_1$ on $W$. Furthermore, the construction of $g_1$ on $W$ (as in [Gromov and Lawson 1980]) in fact shows that for any $\varepsilon > 0$, there exists a Riemannian metric $\bar{g}$ on $W$ such that $\mathrm{Sc}(\bar{g}) \geq k + 1 - \varepsilon$. Therefore, without loss of generality, let us assume $\mathrm{Sc}(g_1) \geq k$. Moreover, as all surgeries are performed away from the boundary, $g_1$ and $g_0$ coincide near the boundary. In particular, we still have

$$g_1|_{\Sigma_1 \times \{0\}} = h \quad \text{and} \quad H_{g_1}(\Sigma_1 \times \{0\})_x \geq m \text{ for all } x \in \Sigma_1.$$

Let $g = g_1|_Z$ be the restriction of the Riemannian metric $g_1$ on $Z$. Then $g$ satisfies all the required properties. This finishes the proof. □

**Remark 3.10.** Note that in Proposition 3.8 above, we do not have too much control of the Riemannian metric at $\Sigma_2$, nor the mean curvature of $\Sigma_2$.

Another ingredient needed for the proof of our main theorem is the following gluing lemma of Miao [2002, Section 3]. See also [Gromov 2018, Section 11.5].

**Lemma 3.11.** *Let $(X_1, g_1)$ and $(X_2, g_2)$ be two Riemannian manifolds with compact boundary. Suppose $\varphi : \partial X_1 \to \partial X_2$ is an isometry with respect to the induced metrics on $X_1$ and $X_2$. If*

$$H_{g_1}(\partial X_1)_x + H_{g_2}(\partial X_2)_{\varphi(x)} > 0$$

*for all $x \in \partial X_1$, then the natural continuous Riemannian metric $g_1 \cup g_2$ on $X_1 \cup_\varphi X_2$ can be approximated by smooth Riemannian metrics $g_\varepsilon$ with their scalar curvatures bounded from below by the scalar curvature[6] of $g_1 \cup g_2$. Furthermore, for any $\varepsilon > 0$, $g_\varepsilon$ can be chosen to coincide with $g_1 \cup g_2$ away from $\varepsilon$-neighborhood of $\partial X_1$ in $X_1 \cup_\varphi X_2$.*

We now combine the above ingredients to give a negative answer to Gromov's compactness question when interpreted in its complete generality.

**Theorem 3.12.** *There exists a noncompact smooth spin manifold $M$ such that for any given compact subset $V \subset M$ and any $\rho > 0$, there exists a (noncomplete) Riemannian metric on $X$ with scalar curvature $\geq 1$ and the closed $\rho$-neighborhood $N_\rho(V)$ of $V$ in $M$ is compact, but $M$ itself does not admit a complete Riemannian metric with uniformly positive scalar curvature.*

*Proof.* Let $\{G_i, \varphi_i\}$ be the inverse system from (3.2). Let $B_G$ be the corresponding space equipped with the exhaustion $\{Y_i\}$ as before.

By Proposition 3.3, $\varprojlim^1 \mathrm{KO}_3(Y_j, \partial Y_j)$ is nontrivial. Let $c$ be a nonzero element in $\varprojlim^1 \mathrm{KO}_3(Y_j, \partial Y_j) \subset \mathrm{KO}_3^{\mathrm{lf}}(B_G)$. By Theorem 3.6, there is an element $(M, f) \in \Omega_*^{\mathrm{spin,lf}}(B_G)$ such that $M$ is a noncompact spin manifold of dimension $\geq 6$ and $f$ is a proper map from $M$ to $B_G$ satisfying the following:

(1) $f_*[D_M] = c$;

(2) the inverse images $(M_i, \partial M_i) = f^{-1}(Y_i, \partial Y_i)$ are compact manifolds with boundary such that the induced maps $\pi_1(M_i) \to \pi_1(Y_i)$ and $\pi_1(\partial M_i) \to \pi_1(\partial Y_i)$ are all isomorphisms;

(3) the induced maps $\pi_1(A_i) \to \pi_1(Y_{i,i+1})$ are isomorphisms for all $i \geq 0$, where $A_i = M_{i+1} - \mathring{M}_i$ is the annulus region between $\partial M_i$ and $\partial M_{i+1}$;

(4) each $A_i$ is obtained from $(\partial M_{i+1}) \times I$ by attaching handles of index $\geq 2$, or equivalently $\partial M_{i+1}$ is obtained from $\partial M_i$ via surgeries of codimension $\geq 2$.

[6]For $x \in \partial X_1 \cong \partial X_2$, this means $\mathrm{Sc}(g_\varepsilon)_x \geq \max\{\mathrm{Sc}(g_1)_x, \mathrm{Sc}(g_2)_{\varphi(x)}\}$.

We have the following commutative diagram (cf. [Guentner and Yu 2012]):

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \varprojlim^1 \mathrm{KO}_{n+1}(M_i, \partial M_i) & \longrightarrow & \mathrm{KO}_n^{\mathrm{lf}}(M) & \longrightarrow & \varprojlim \mathrm{KO}_n(M_i, \partial M_i) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \varprojlim^1 \mathrm{KO}_{n+1}(Y_i, \partial Y_i) & \longrightarrow & \mathrm{KO}_n^{\mathrm{lf}}(B_G) & \longrightarrow & \varprojlim \mathrm{KO}_n(Y_i, \partial Y_i) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \sigma} & & \downarrow{\scriptstyle \beta} & & \\
0 & \longrightarrow & \varprojlim^1 \mathrm{KO}_{n+1}(D_i^*) & \longrightarrow & \mathrm{KO}_n(\mathcal{A}(B_G)) & \longrightarrow & \varprojlim \mathrm{KO}_n(D_i^*) & \longrightarrow & 0
\end{array}$$

where $\mathcal{A}(B_G)$ is defined as in Definition 2.4 and

$$D_i^* \coloneqq \varinjlim_{j>i} C^*_{\max}(\pi_1(Y_j), \pi_1(Y_{ij})) \otimes \mathcal{K}$$

with $Y_{ij} = Y_j - \mathring{Y}_i$. By construction, $Y_{ij}$ deformation retracts to $\partial Y_i = BG_i$ for all $j > i$. By the homotopy invariance of the fundamental group, it follows that

$$D_i^* \coloneqq C^*_{\max}(\pi_1(Y_i), \pi_1(\partial Y_i)) \otimes \mathcal{K}.$$

By construction, each $Y_i$ is contractible. Moreover, $\partial Y_i = BG_i$ with $G_i = \mathbb{Z}^2 * F_i$, where $F_i$ is a finitely generated free group. Consequently, the (relative) Baum–Connes assembly map

$$\beta : \mathrm{KO}_n(Y_i, \partial Y_i) = \mathrm{KO}_n(\{pt\}, BG_i) \to \mathrm{KO}_n(D_i^*) = \mathrm{KO}_n(C^*_{\max}(\{e\}, G_i))$$

is an isomorphism. In particular, it follows that the maps $\alpha$, $\beta$ and $\sigma$ in the above commutative diagram are isomorphisms.

Since $c$ is an element in $\varprojlim^1 \mathrm{KO}_3(Y_j, \partial Y_j)$, it follows from the above commutative diagram that its image in $\varprojlim \mathrm{KO}_2(Y_i, \partial Y_i)$ equals 0, hence 0 in $\varprojlim \mathrm{KO}_2(D_i^*)$. In particular, it is 0 in each $\mathrm{KO}_2(D_i^*)$. By the discussion following the unstable relative Gromov–Lawson–Rosenberg conjecture (Conjecture 2.7), since $G_i = \mathbb{Z}^2 * F_i$ with $F_i$ a finitely generated free group, each $M_i$ admits a positive scalar curvature metric that is collared near $\partial M_i$.

Note that any compact subset $V \subset M$ is contained in some $M_i$. Since each $M_i$ has a metric of positive scalar curvature which has product structure near the boundary, by stretching the collar neighborhood of the boundary (which does not change the scalar curvature) if necessary, it follows that for any compact subset $V \subset M$ and any $\rho > 0$, there exists $M_i$ equipped with a metric $g_i$ of positive scalar curvature such that $\mathrm{Sc}(g_i) \geq 1$ and the closed $\rho$-neighborhood $N_\rho(V)$ of $V$ is contained in $M_i$, and hence $N_\rho(V)$ is compact.

Now we show that the Riemannian metric $g_i$ on $M_i$ can be extended to a (noncomplete) Riemannian metric $g$ on $M$ with $\mathrm{Sc}(g) \geq 1$. Let us denote by $h_i$ the

restriction of $g_i$ on $\partial M_i$. Let $m_i$ be a real number such that the mean curvature of $\partial M_i$ satisfies
$$H_{g_i}(\partial M_i)_x > -m_i$$
for all $x \in \partial M_i$. By our choice of $M$, the annulus region $A_i = M_{i+1} - \mathring{M}_i$ obtained from $\partial M_{i+1} \times I$ by attaching handles of index $\geq 2$. Then it follows from Proposition 3.8 that there exists a smooth Riemannian metric $\tilde{g}_i$ on $A_i$ such that

(1) $\tilde{g}_i$ extends $h_i$, that is,
$$\tilde{g}_i|_{\partial M_i} = h_i,$$
(2) $\mathrm{Sc}(\tilde{g}_i) \geq 1$,

(3) $H_g(\partial M_i)_x \geq m_i + 1$ for all $x \in \partial M_i$.

By applying Miao's gluing lemma (Lemma 3.11), we obtain a smooth Riemannian metric $g_{i+1}$ on $M_{i+1} = M_i \cup_{\partial M_i} A_i$ such that

(i) $g_{i+1} = g_i$ on $M_i - N_{\varepsilon_i}(\partial M_i)$, where $\varepsilon_i$ is a sufficiently small[7] positive number,

(ii) $\mathrm{Sc}(\tilde{g}_{i+1}) \geq 1$.

Now repeat the above argument inductively on each $A_j$ for $j > i+1$. In the end, we obtain a (noncomplete) Riemannian metric on $M$ with scalar curvature $\geq 1$ such that the closed $\rho$-neighborhood $N_\rho(V)$ of $V$ in $M$ is compact.

Now we complete the proof by showing that $M$ itself does not have a complete metric of uniformly positive scalar curvature. Let us prove this by contradiction. Assume to the contrary that $M$ admits a complete metric of uniformly positive scalar curvature. We can choose a metric on $B_G$ under which $\{Y_i, Y_{ij}\}$ is an admissible exhaustion of $B_G$ and the proper map $f : M \to B_G$ is a continuous proper coarse map. By [Chang et al. 2020, Theorem 3.3], the higher index of $D_M$ under the index map
$$\sigma : \mathrm{KO}_*^{\mathrm{lf}}(B_G) \to \mathrm{KO}_*(\mathcal{A}(B_G))$$
vanishes, that is, $\sigma(f_*[D_M]) = 0 \in \mathrm{KO}_*(\mathcal{A}(B_G))$. This contradicts the fact that $\sigma$ is an isomorphism and the assumption that $f_*[D_M] = c \neq 0$. This finishes the proof. □

**Remark 3.13.** In the above proof, we used Proposition 3.8 and Miao's gluing lemma (Lemma 3.11) to construct an incomplete Riemannian metric $g_\rho$ on $M$ with scalar curvature $\geq 1$ such that the closed $\rho$-neighborhood $N_\rho(V)$ of $V$ in $M$ is compact. A key ingredient in the proof of Proposition 3.8 is Claim 3.9, which roughly says that if $\Sigma_2$ is obtained from $\Sigma_1$ via surgeries of codimension $\geq 2$ and $Z$ is the cobordism representing the trace of the surgeries, then $Z \cup_{\Sigma_2} Z$ is obtained from $\Sigma_1 \times I$ via surgeries of codimension $\geq 3$. In fact, one can also construct such an incomplete metric $g_\rho$ on $M$ by directly applying Claim 3.9 together with the

[7]Here "sufficiently small" means that the $\varepsilon_i$-neighborhood $N_{\varepsilon_i}(\partial M_i)$ is disjoint from $M_{i-1}$.

surgery theory of positive scalar curvature [Gromov and Lawson 1980, Theorem A; Schoen and Yau 1979, Corollary 6]. We thank Bernhard Hanke for pointing this out to us. Here is a sketch of this alternative argument. The first step is the same as in the proof. For any compact subset $V \subset M$ and any $\rho > 0$, there exists $M_i$ equipped with a metric $g_i$ of positive scalar curvature such that $\mathrm{Sc}(g_i) \geq \frac{3}{2}$ and the closed $\rho$-neighborhood $N_\rho(V)$ of $V$ is contained in $M_i$, hence $N_\rho(V)$ is compact. We need to extend the Riemannian metric $g_i$ on $M_i$ to an (incomplete) Riemannian metric $g_\rho$ on $M$ with $\mathrm{Sc}(g_\rho) \geq 1$. By our choice of $M$, the annulus region $A_i = M_{i+1} - \mathring{M}_i$ is obtained from $\partial M_{i+1} \times I$ by attaching handles of index $\geq 2$. Consider a new manifold $X = M_{i+1} \cup_{\partial M_{i+1}} (-A_i)$ obtained by gluing another copy of $A_i$ to $M_{i+1}$ along the common boundary $\partial M_{i+1}$. By Claim 3.9, $A_i \cup_{\partial M_{i+1}} (-A_i)$ is obtained from $\partial M_i \times I$ via surgeries of codimension $\geq 3$. Since $g_i$ is a positive scalar curvature metric on $M_i$ such that $\mathrm{Sc}(g_i) \geq 1$, if we view $M_i \cong M_i \cup_{\partial M_i} (\partial M_i \times I)$, it follows from the surgery theory of positive scalar curvature that $X$ admits a positive scalar curvature metric $g_{i+1}$ such that $g_{i+1} = g_i$ away from $A_i \cup_{\partial M_{i+1}} (-A_i) \subset X$ and $\mathrm{Sc}(g_{i+1}) \geq \frac{3}{2} - \varepsilon_i$ for some arbitrarily small $\varepsilon_i$. We denote the restriction of $g_{i+1}$ on $M_{i+1} \subset X$ still by $g_{i+1}$. Now we repeat this argument for each $A_k = M_{k+1} - \mathring{M}_k$ and finally obtain an incomplete Riemannian metric $g_\rho$ on $M$ with scalar curvature $\geq 1$ such that the closed $\rho$-neighborhood $N_\rho(V)$ of $V$ in $M$ is compact.

## 4. Positive answers to Gromov's compactness question

In our construction of negative examples to Gromov's compactness question (see Theorem 3.12), we have seen that the nonvanishing of a certain $\varprojlim^1$ index is an obstruction to the existence of complete Riemannian metrics of uniformly positive scalar curvature on some noncompact spin manifolds. This suggests that Gromov's compactness question has a positive answer if in addition one assumes that an appropriate $\varprojlim^1$ index vanishes. More precisely, we have the following conjecture based on Gromov's compactness question.

**Conjecture 4.1.** *Let $X$ be a smooth spin manifold. Suppose for any given compact subset $V \subset X$ and any $\rho > 0$, there exists a (noncomplete) Riemannian metric on $X$ with scalar curvature $\geq 1$ such that the closed $\rho$-neighborhood $N_\rho(V)$ of $V$ in $X$ is compact. If the $\varprojlim^1$ higher index of the Dirac operator of $X$ vanishes, then $X$ admits a complete Riemannian metric with scalar curvature $\geq 1$.*

The $\varprojlim^1$ higher index was introduced in Definition 2.21. The assumption "for any given compact subset $V \subset X$ and any $\rho > 0$, there exists a (noncomplete) Riemannian metric on $X$ with scalar curvature $\geq 1$ such that the closed $\rho$-neighborhood $N_\rho(V)$ of $V$ in $X$ is compact" implies that the $\varprojlim^1$ higher index of the Dirac operator of $X$ is well-defined (see the discussion before Definition 2.21).

Due to the failure of the unstable Gromov–Lawson–Rosenberg conjecture [Schick 1998, Example 2.2], the failure of the unstable relative Gromov–Lawson–Rosenberg conjecture, Conjecture 4.1 is most likely not true in its complete generality. Rather one should interpret it as a guide towards the correct compactness statement for positive scalar curvature metrics on open manifolds. On the other hand, as supporting evidence for the philosophy behind Conjecture 4.1, we verify the conjecture for a class of 1-tame spin manifolds.[8]

**Definition 4.2.** A manifold $M$ is said to be 1-tame if there is a sequence of codimension zero compact submanifolds $\{K_i\}_{i\geq 1}$ (with boundary) such that:

(1) $K_i \subset K_{i+1}$ and $M = \bigcup_{i\geq 1} K_i$.

(2) All $M - K_i$ have the same finite number of connected components. If we denote the connected components of $M - K_i$ by $\{P_{i,j}\}_{1\leq j\leq m}$ with $P_{i+1,j} \subset P_{i,j}$, then the natural homomorphism $\pi_1(P_{i+1,j}) \to \pi_1(P_{i,j})$ induced by the inclusion is an isomorphism for all $i$ and $j$.

For simplicity, we define $\pi_1(M - K_i)$ to be the disjoint union $\coprod_j \pi_1(P_{i,j})$. We say the map $\pi_1(M - K_{i+1}) \to \pi_1(M - K_i)$ is an isomorphism if the condition (2) above is satisfied. In this case, we call $\pi_1(M - K_i)$ the fundamental groupoid at infinity of $M$, denoted by $\pi_1^\infty(M) = \coprod_{\alpha=1}^{\ell} G_\alpha$. Throughout this section, we assume each $G_\alpha$ is finitely presented.

**Remark 4.3.** By [Siebenmann 1965, Theorem 3.10], for any smooth open manifold $M$ with $\dim M \geq 5$, if $M$ is 1-tame, then we can choose $\{K_i\}_{i\geq 1}$ in Definition 4.2 so that the inclusion $\partial K_i \to (M - \mathring{K}_i)$ induces an isomorphism $\pi_1(\partial K_i) \to \pi_1(M - \mathring{K}_i)$ for each $i \geq 1$. As a consequence, if we denote the annulus region between $\partial K_i$ and $\partial K_{i+1}$ by $A_i$, then the inclusions $\partial K_i \to A_i$ and $\partial K_{i+1} \to A_i$ induce isomorphisms (componentwise) on $\pi_1$ [Siebenmann 1965, Lemma 3.12].

The following theorem gives a positive answer to Gromov's compactness question for 1-tame spin manifolds, provided that the unstable relative Gromov–Lawson–Rosenberg conjecture holds for the relevant fundamental groups.

**Theorem 4.4.** *Let $M$ be a noncompact* 1*-tame spin manifold of dimension $n \geq 6$. Let $\Gamma = \pi_1(M)$ and $G = \pi_1^\infty(M)$. Assume that the unstable relative Gromov–Lawson–Rosenberg conjecture holds for the pair* $(\Gamma, G)$. *Suppose for any given compact subset $V \subset M$ and any $\rho > 0$, there exists a* (*noncomplete*) *Riemannian metric on $M$ with scalar curvature $\geq 1$ such that the closed $\rho$-neighborhood $N_\rho(V)$ of $V$ in $M$ is compact. Then $M$ admits a complete Riemannian metric of uniformly positive scalar curvature.*

[8]With extra geometric conditions, Theorem 4.4 of this section still holds for nonspin manifolds.

*Proof.* Let $\{K_i\}$ be a sequence of codimension zero compact submanifolds of $M$ satisfying the properties given in Definition 4.2. By Remark 4.3, without loss of generality, we assume the inclusion $\partial K_i \to (M - \mathring{K}_i)$ induces an isomorphism $\pi_1(\partial K_i) \to \pi_1(M - \mathring{K}_i)$ for each $i \geq 1$. If we denote the annulus region between $\partial K_i$ and $\partial K_{i+1}$ by $A_i$, then the inclusions $\partial K_i \to A_i$ and $\partial K_{i+1} \to A_i$ induce isomorphisms (componentwise) on $\pi_1$ [Siebenmann 1965, Lemma 3.12].

Choose $\rho > 0$ to be sufficiently large and $V = K_1$. By the assumption, there exists a Riemannian metric $g_0$ on $M$ such that $\mathrm{Sc}(g_0) \geq 1$ and the closed $\rho$-neighborhood $N_\rho(V)$ of $V$ in $M$ is compact. The Riemannian metric $g_0$ is incomplete in general. But we can always modify the metric on $M - N_\rho(V)$ while keeping the metric on $N_\rho(V)$ fixed so that the resulting new metric $h$ is a complete Riemannian metric on $M$ such that $\mathrm{Sc}(h) \geq 1$ on $N_\rho(V)$.

Let us write $M - V$ as a disjoint union of connected components

$$M - V = \coprod_{\alpha=1}^{\ell} Y_\alpha$$

and similarly $\partial V = \coprod_{\alpha=1}^{\ell} \partial Y_\alpha$. By the discussion above, $\pi_1(\partial Y_\alpha) \to \pi_1(Y_\alpha)$ is an isomorphism. For simplicity, we write $G = \coprod_{\alpha=1}^{\ell} G_\alpha$ with $G_\alpha = \pi_1(\partial Y_\alpha)$. Let us still denote by $h$ the restriction of the metric $h$ on $Y_\alpha$. Each inclusion $\iota_\alpha : Y_\alpha \to M$ induces a homomorphism $\iota_\alpha : G_\alpha \to \Gamma$. Consequently, we have a $C^*$-algebra homomorphism

$$\iota_* = \bigoplus_{\alpha=1}^{\ell} (\iota_\alpha)_* : \bigoplus_{\alpha=1}^{\ell} C^*_{\max}(G_\alpha) \to C^*_{\max}(\Gamma).$$

We define $C^*_{\max}(\Gamma, G)$ to be the 7th suspension $\mathcal{S}^7 C_{\iota_*} \cong C_0(\mathbb{R}^7) \otimes C_{\iota_*}$ of the mapping cone $C^*$-algebra $C_{\iota_*}$. With the above setup, the relative higher index $\mathrm{Ind}_{\Gamma,G}(D_V)$ of the Dirac operator $D_V$ of $V$ is an element in $\mathrm{KO}_n(C^*_{\max}(\Gamma, G))$. See Section 2A for more details on relative higher index. As long as $\rho$ is sufficiently large, it follows from Theorem 2.17 that $\mathrm{Ind}_{\Gamma,G}(D_V)$ vanishes in $\mathrm{KO}_n(C^*_{\max}(\Gamma, G))$.

By assumption, the unstable relative Gromov–Lawson–Rosenberg conjecture holds for the pair $(\Gamma, G)$. It follows that[9] $V$ admits a Riemannian metric $g_V$ of positive scalar curvature $\geq 2$ that is collared near the boundary. Now it only remains to show that the metric $g_V$ on $V$ extends to a complete Riemannian metric on $M$ with scalar curvature $\geq 1$.

Consider the cobordism $A_i$ between $\partial K_i$ and $\partial K_{i+1}$. Since the inclusions $\partial K_i \to A_i$ and $\partial K_{i+1} \to A_i$ induce isomorphisms (componentwise) on $\pi_1$. It follows from Lemma 3.4 that $A_i$ can be viewed as the trace of surgeries of codimension $\geq 3$ from $\partial K_i$ to $\partial K_{i+1}$. By the surgery theory for positive scalar curvature [Gromov and

[9] Here we have implicitly used the fact that $\pi_1(\partial Y_\alpha) \to \pi_1(Y_\alpha)$ is an isomorphism.

Lawson 1980, Theorem A; Schoen and Yau 1979, Corollary 6], we can extend the metric $g_V$ on $V = K_1$ to a Riemannian metric $g_2$ on $K_2$ such that $\mathrm{Sc}(g_2) \geq 2 - \frac{1}{2}$ and $g_2$ is collared near $\partial K_2$. By stretching out a small tubular neighborhood of $\partial K_2$ if necessary,[10] we can assume that $\mathrm{dist}(\partial K_1, \partial K_2) \geq 2$. Now by the surgery theory for positive scalar curvature, we can inductively extend the metric $g_i$ on $K_i$ to a Riemannian metric on $g_{i+1}$ on $K_{i+1}$ such that $\mathrm{Sc}(g_{i+1}) \geq 2 - \sum_{\beta=1}^{i} 1/2^\beta$, the metric $g_{i+1}$ is collared near $\partial K_{i+1}$ and $\mathrm{dist}(\partial K_i, \partial K_{i+1}) \geq i$. Hence, by induction, we eventually obtain a complete Riemannian metric on $M$ with scalar curvature $\geq 1$. This finishes the proof. □

## Acknowledgment

We would like to thank Bernhard Hanke for helpful comments.

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Received 1 May 2024. Accepted 20 Sep 2024.

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