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DIMENSION BOUNDS FOR ESCAPE ON AVERAGE IN HOMOGENEOUS SPACES

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Abstract. Let $X = G/\Gamma$, where G is a Lie group and Γ is a uniform lattice in G, and let O be an open subset of X. We give an upper estimate for the Hausdorff dimension of the set of points whose trajectories escape O on average with frequency δ, where $0 < \delta \leq 1$.

1. **Introduction.** Throughout the paper we let G be a Lie group and Γ a lattice in G, denote by X the homogeneous space G/Γ and by μ the G-invariant probability measure on X. Let $F := (g_t)_{t\in\mathbb{R}}$ be a one-parameter subgroup of G, and let $F^+ := (g_t)_{t\geq 0}$. For a subset O of X, define

$$\widetilde{E}(F^+, O) := \big\{x \in X : \exists\, t_0 \text{ such that } g_t x \notin O \ \forall\, t \geq t_0\big\}$$

to be the set of $x \in X$ *eventually escaping* O under the action of F^+. If the F^+-action on (X, μ) is ergodic, then it follows from Birkhoff's Ergodic Theorem that $\mu\big(\widetilde{E}(F^+, O)\big) = 0$ as long as O has positive measure. Furthermore, under some additional assumptions one can obtain upper estimates for the Hausdorff dimension of $\widetilde{E}(F^+, O)$. More precisely, one has the following 'Dimension Drop Conjecture', originated from a question asked by Mirzakhani (private communication): *if $F^+ \subset G$ is a subsemigroup and O is an open subset of X, then either $\widetilde{E}(F^+, O)$ has positive measure, or its Hausdorff dimension is less than the dimension of X?* When X is compact, it follows from the variational principle for measure-theoretic entropy, as outlined in [13, §7]; an effective argument using exponential mixing was developed in [10]. See [4] (resp., [11, 12]) for the proof of this conjecture for Lie groups of rank one (resp., higher rank with some additional assumptions). Also note that when F^+ is generated by unipotent elements, the conjecture follows from Ratner's Measure Classification Theorem and the work of Dani and Margulis (see [14, Lemma 21.2] and [3, Proposition 2.1].

In this paper we consider trajectories which are allowed to enter O, but not as frequently as mandated by Birkhoff's Ergodic Theorem. Namely, for $0 < \delta \leq 1$

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let us say that a point $x \in X$ *δ-escapes O on average under the action of F^+* if it belongs to the set

$$E_\delta(F^+, O) := \left\{ x \in X : \limsup_{T\to\infty} \frac{1}{T}\int_0^T 1_{O^c}(g_t x)dt \geq \delta \right\}.$$

Again under the assumption of ergodicity for any measurable $O \subset X$ Birkhoff's Ergodic theorem implies that

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T 1_{O^c}(g_t x)\,dt = \mu(O^c).$$

Hence, the set $E_\delta(F^+, O)$ has full measure for any $0 < \delta \leq \mu(O^c)$, and has measure zero for any $\delta > \mu(O^c)$. Clearly for any δ as above one has

$$E_\delta(F^+, O) \supset E_1(F^+, O) \supset \widetilde{E}(F^+, O).$$

Thus one can ask the following questions: under some assumptions, can it be shown that the set $E_1(F^+, O)$ has less than full Hausdorff dimension? is the same true for $E_\delta(F^+, O)$ for some $\delta < 1$?

To state our results, we need to introduce some notation. If X is a metric space with metric 'dist', O is a subset of X and $r > 0$, we will denote by $\sigma_r O$ the *inner r-core* of O, defined as

$$\sigma_r O := \{x \in X : \mathrm{dist}(x, O^c) > r\},$$

and by $O^{(r)}$ the *r-neighborhood* of O, that is,

$$O^{(r)} := \{x \in X : \mathrm{dist}(x, O) < r\}.$$

The notation $A \gg B$ (resp., $A \ll B$), where A and B are quantities depending on certain parameters, will mean $A \leq CB$, (resp., $A \geq CB$), where C is a constant independent on those parameters but possibly dependent on G, Γ, F and a subgroup P of G that will appear later in Definition 1.4 and Theorem 1.5. Hausdorff dimension (see Definition 3.1) will be denoted by 'dim', and codim A will stand for the Hausdorff codimension of a set A, that is, the difference between the dimension of the ambient space and dim A.

Fix a right-invariant Riemannian structure on a Lie group G, and denote by 'dist' the corresponding Riemannian metric, using the same notation for the induced metric on homogeneous spaces of G. In what follows, if P is a subgroup of G, we will denote by $B^P(r)$ the open ball of radius r centered at the identity element with respect to the metric on P corresponding to the Riemannian structure induced from G $B(x, \rho)$ will stand for the open ball in X centered at $x \in X$ of radius ρ.

Throughout this paper we shall assume that Γ is a uniform lattice in G, that is, X is compact. We note that the methods and results of this paper can be extended to the noncompact case, however this would require an extra effort controlling the escape of mass and will be pursued in subsequent work.

For comparison, let us first state the effective dimension estimate from [10]. There the main tool was the exponential mixing of the action. Namely, let us say that the flow (X, F^+) is *exponentially mixing* if there exist $\gamma > 0$ and $\ell \in \mathbb{Z}_+$ such that for any $\varphi, \psi \in C^\infty(X)$ and for any $t \geq 0$ one has

$$\left| (g_t\varphi, \psi) - \int_X \varphi\, d\mu \int_X \psi\, d\mu \right| \ll e^{-\gamma t} \|\varphi\|_\ell \|\psi\|_\ell. \tag{1.1}$$

Here and hereafter $\|\cdot\|_\ell$ stands for the *"L^2, order ℓ"* Sobolev norm, see e.g. [10, §2] for a definition and basic facts. Note that the statement in (1.1) is nontrivial only if $\|\varphi\|_\ell$ and $\|\psi\|_\ell$ are finite.

The following is a special case of [10, Theorem 1.1]:

Theorem 1.1. *Let G be a Lie group, Γ a uniform*[1] *lattice in G, $X = G/\Gamma$, and let F be a one-parameter* Ad*-diagonalizable subgroup of G whose action on X is exponentially mixing. Then there exists $r_1 > 0$ such that for any $O \subset X$ and any $0 < r < r_1$ one has*[2]

$$\operatorname{codim} \widetilde{E}(F^+, O) \gg \frac{\mu(\sigma_r O)}{\log(1/r) + \log\big(1/\mu(\sigma_r O)\big)}\,. \tag{1.2}$$

An interesting feature of the above estimate is that its left hand side does not depend on r while the right hand side does, and tends to 0 when r either tends to 0 or becomes large enough. Thus for applications one is left to seek an optimal value of r. For example when $O = B(x,\rho)$ one can take $r = \rho/2$, producing the estimate

$$\operatorname{codim} \widetilde{E}\big(F^+, B(x,\rho)\big) \gg \frac{\rho^{\dim X}}{\log(1/\rho)}\,. \tag{1.3}$$

The main theme of the present paper is replacing the phenomenon of eventual escape by escape on average. In order to state our main result, we need to introduce the function $\phi : [0,1]\times[0,1] \to \mathbb{R}$ by

$$\phi(y,s) := (1-s)\log\frac{1}{1-\frac{y}{2}} - 2(1-s)\log\frac{1}{1-s} - s\cdot\log\frac{1+\frac{y}{4}}{s}. \tag{1.4}$$

We will employ the convention $0\cdot\log\frac{1}{0} = 0$; this way we can see that ϕ is a continuous function on $\mathbb{R}^2$, with

$$\phi(y,0) = \log\frac{1}{1-\frac{y}{2}} \text{ for all } y, \tag{1.5}$$

hence for any $y > 0$ one has $\phi(y,s) > 0$ for small enough (depending on y) $s > 0$.

The following is our first main theorem:

Theorem 1.2. *Let G, Γ, X and F be as in Theorem 1.1. Then there exists positive constant r_1 such that for any $O \subset X$, any $0 < r < r_1$ and any $\delta > 0$ one has*

$$\operatorname{codim} E_\delta(F^+, O) \gg \frac{\mu(\sigma_r O)\cdot\phi\big(\mu(\sigma_r O), \sqrt{1-\delta}\big)}{\log\frac{1}{r}}. \tag{1.6}$$

Thus, if given a subset O of X we define

$$\delta_O := \inf\Big\{0 < \delta < 1 : \phi\big(\mu(O), \sqrt{1-\delta}\big) > 0\Big\}, \tag{1.7}$$

[1]This is a simplified version of the theorem; in [10] instead of the compactness of X it was assumed that the complement of O was compact. The latter assumption has been removed in [11, 12] in many special cases.

[2]The result in [10] actually involved the slightly smaller set

$$E(F^+, O) := \{x \in X : g_t x \notin O \ \forall\, t \geq 0\}$$

of points whose trajectories avoid O, bit it is easy to see that $\dim E(F^+, O) = \dim \widetilde{E}(F^+, O)$.

which is strictly less than 1 whenever $\mu(O) > 0$, and if in addition O has non-empty interior, then for any $\delta > \delta_O$ one can choose $r > 0$ small enough to have $\phi\big(\mu(\sigma_r O), \sqrt{1-\delta}\big) > 0$. This implies

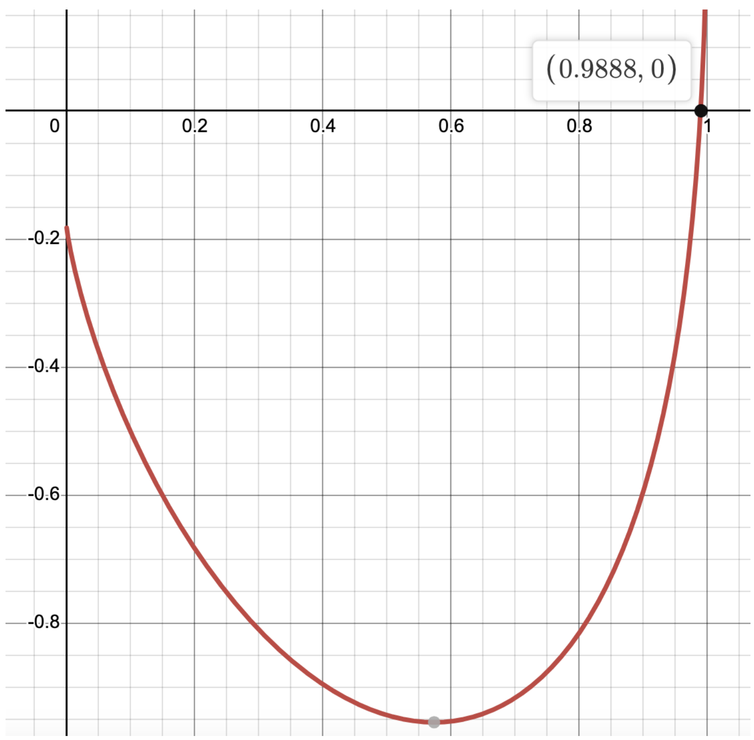
$$\operatorname{codim} E_\delta(F^+, O) > 0 \text{ whenever } \delta > \delta_O. \tag{1.8}$$

Also one can observe that

$$\phi\left(y, \frac{y}{2}\right) = -\left(1 - \frac{y}{2}\right)\log\frac{1}{1-\frac{y}{2}} - \frac{y}{2}\cdot\log\frac{1+\frac{y}{4}}{y/2} < 0$$

for any $0 < y < 1$; hence $\delta_O > 1 - \frac{\mu(O)^2}{4} > \mu(O^c)$.

As an example, here is a graph of the function $\delta \mapsto \phi(0.8, \sqrt{1-\delta})$; thus for $\mu(O) = \frac{4}{5}$ we get $\delta_O \approx 0.9888$.



In the special case $\delta = 1$, using (1.5) we obtain the following immediate corollary:

Corollary 1.3. *Let G, Γ, X, F^+ and r_1 be as in Theorem 1.2. Then for any $0 < r < r_1$ we have*

$$\operatorname{codim} E_1(F^+, O) \gg \frac{\mu(\sigma_r O)\cdot\log\frac{1}{1-\frac{\mu(\sigma_r O)}{2}}}{\log\frac{1}{r}} \gg \frac{\mu(\sigma_r O)^2}{\log\frac{1}{r}}.$$

One sees that the above corollary does not produce any improvement of (1.2); on the contrary, our new dimension bound for escape on average happens to be worse by a factor of $\mu(\sigma_r O)$ than the bound for the eventual escape. In the sequel to the present paper, using some ideas from the work [1] where a similar problem was studied for the Teichmüller geodesic flow, the authors plan to improve the existing estimates as well as extend the methods to treat non-compact homogeneous spaces.

We also remark that when X is not compact, one can consider the set

$$E_\delta^{comp}(F^+) := \bigcap_{C \subset X \text{ compact}} E_\delta(F^+, C)$$

of points in X with trajectories δ-escaping all compact subsets of X. Equivalently one can define $E_\delta^{comp}(F^+)$ as the set of $x \in X$ such that there exists a sequence $T_k \to \infty$ and a weak-$*$ limit μ^* of the sequence of probability measures $f \mapsto \frac{1}{T_k}\int_0^{T_k} f(g_t x)\,dt$ such that $\mu^*(X) \le 1 - \delta$. It is proved in [15, Theorem 1.3] that whenever G is a connected semisimple Lie group, Γ a lattice in G and F^+ a one-parameter semigroup contained in one of the simple factors of G, one has

$$\operatorname{codim}\left(E_\delta^{comp}(F^+)\right) \ge c\delta \text{ for all } 0 < \delta < 1,$$

where c depends only on G and Γ. See also [7, Remark 2.1] for a special case.

From now on let G, Γ, X and F^+ be as in Theorems 1.1 and 1.2. Similarly to the argument in [10, 11, 12], our main theorem is deduced from a result that estimates

$$\dim E_\delta(F^+, O) \cap Hx,$$

where $x \in X$ and H is the *unstable horospherical subgroup* with respect to F^+, defined as

$$H := \{g \in G : g_t h g_{-t} \to e \;\; as \; t \to +\infty\}. \tag{1.9}$$

More generally, in the theorem below we estimate

$$\dim E_\delta(F^+, O) \cap Px$$

for $x \in X$ and some proper subgroups P of H, namely those which have so-called Effective Equidistribution Property (EEP) with respect to the flow (X, F^+). The latter property was motivated by [9] and introduced in [10], where it was shown to hold for H as above under the assumption of exponential midefinitionxing. In what follows we shall denote by ν the Haar measure on P corresponding to the volume form induced by the Riemannian metric on G. Then we have the following

Definition 1.4. Say that a subgroup P of G has *Effective Equidistribution Property* (EEP) with respect to the flow (X, F^+) if P is normalized by F (that is, F is contained in the normalizer of P), and there exist $\lambda > 0$, $t_0 > 0$ and $\ell \in \mathbb{N}$ such that for any $x \in X$, $t \ge t_0$, $f \in C^\infty(P)$ with $\operatorname{supp} f \subset B^P(1)$ and $\psi \in C^\infty(X)$ it holds that

$$\left|\int_P f(h)\psi(g_t hx)\,d\nu(h) - \int_P f\,d\nu \int_X \psi\,d\mu\right| \ll \max\left(\|\psi\|_{C^1}, \|\psi\|_\ell\right) \cdot \|f\|_{C^\ell} \cdot e^{-\lambda t}. \tag{1.10}$$

We remark that in [10, 11, 12] this property was defined and used in the more general set-up of X being non-compact, with additional constraints on the *injectivity radius* at points $x \in X$ for which (1.10) is satisfied (see §3.2 for more detail). However when X is compact the injectivity radius is uniformly bounded from below, hence a possibility to simplify the definition.

Theorem 1.5. *Let G, Γ, X, F^+ be as above. Then there exists $r_2 > 0$ such that whenever a connected subgroup P of H has property* (EEP) *with respect to the flow*

(X, F^+), for any non-empty open subset O of X, any $0 < r \le r_2$, any $0 < \delta < 1$ and any $x \in X$ one has

$$\operatorname{codim}\big(E_\delta(F^+, O) \cap Px\big) \gg \frac{\mu(\sigma_r O)\cdot \phi\big(\mu(\sigma_r O), \sqrt{1-\delta}\big)}{\log \frac{1}{r}}.$$

In the next section we derive Theorem 1.2 from Theorem 1.5, and the rest of the paper is be dedicated to proving Theorem 1.5. Section 3 contains a discussion of basic technical constructions needed for the proof, such as Hausdorff dimension estimates for lim sup sets, tessellations of nilpotent Lie groups and Bowen boxes. In Section 4 we use property (EEP) to, given a subset S of X, a large $T > 0$ and $0 < \delta < 1$, write down a measure estimate for the set of $h \in P$ such that the Birkhoff average $\frac{1}{T}\int_0^T 1_S(g_t hx)\,dt \ge \delta$. In the subsequent section this estimate is used to bound the number of Bowen boxes that can cover certain exceptional sets. Finally, Section 6 contains the conclusion of the proof.

2. **Theorem 1.5 $\Rightarrow$ Theorem 1.2.** The reduction of Theorem 1.2 to Theorem 1.5 is fairly standard. Let $\mathfrak{g}$ be a Lie algebra of G, $\mathfrak{g}_{\mathbb{C}}$ its complexification, and for $\lambda \in \mathbb{C}$, let E_λ be the eigenspace of $\operatorname{Ad} g_1$ corresponding to λ. Let $\mathfrak{h}$, $\mathfrak{h}^{\circ}$, $\mathfrak{h}^-$ be the subalgebras of $\mathfrak{g}$ with complexifications:

$$\mathfrak{h}_{\mathbb{C}} = \operatorname{span}(E_\lambda : |\lambda| > 1),\ \mathfrak{h}^0_{\mathbb{C}} = \operatorname{span}(E_\lambda : |\lambda| = 1),\ \mathfrak{h}^-_{\mathbb{C}} = \operatorname{span}(E_\lambda : |\lambda| < 1).$$

Let H, H^0, H^- be the corresponding subgroups of G. Note that H is precisely the unstable horospherical subgroup with respect to F^+ (defined in (1.9)) and H^- is the stable horospherical subgroup defined by:

$$H^- = \{h \in G : g_t h g_{-t} \to e \ \textit{as}\ t \to +\infty\}.$$

Recall that in Theorems 1.1 and 1.2 $\operatorname{Ad} g_1$ is assumed to be diagonalizable over $\mathbb{C}$. This implies that $\mathfrak{g}_{\mathbb{C}}$ is the direct sum of $\mathfrak{h}_{\mathbb{C}}$, $\mathfrak{h}^0_{\mathbb{C}}$ and $\mathfrak{h}^-_{\mathbb{C}}$, and each of the latter three subalgebras of $\mathfrak{g}_{\mathbb{C}}$ is a direct sum of their real and imaginary components. Hence $\mathfrak{g}$ is the direct sum of $\mathfrak{h}$, $\mathfrak{h}^0$ and $\mathfrak{h}^-$, and therefore G is locally (at a neighborhood of identity) a direct product of H, H^0 and H^- (in any order).

Denote the group H^-H^0 by $\tilde{H}$, and fix $0 < \rho < 1$ with the following properties:

$$\text{the multiplication map } \tilde{H} \times H \to G \text{ is one to one on } B^{\tilde{H}}(\rho) \times B^H(\rho), \tag{2.1}$$

and

$$g_t B^{\tilde{H}}(r) g_{-t} \subset B^{\tilde{H}}(2r) \text{ for any } 0 < r \le \rho \text{ and } t \ge 0 \tag{2.2}$$

(the latter can be done since F is Ad-diagonalizable and the restriction of the map $g \to g_t g g_{-t}$, $t > 0$, to the subgroup $\tilde{P}$ is non-expanding).

Proof of Theorem 1.2 assuming Theorem 1.5. Let ρ be as in (2.1), (2.2) and define $r_1 := \min(\rho, r_2)$, where r_2 is as in Theorem 1.5. For any $0 < r < r_1$ choose s such that

$$B^G(s) \subset B^{\tilde{H}}(r/4) B^H(r/4). \tag{2.3}$$

Now take $O \subset X$, and for $x \in X$ denote

$$E_{\delta,x,s} := \left\{g \in B^G(s) : gx \in E_\delta(F^+, O)\right\}. \tag{2.4}$$

In view of the countable stability of Hausdorff dimension, in order to prove the theorem it suffices to show that for any $x \in X$,

$$\operatorname{codim} E_{\delta,x,s} \gg \frac{\mu(\sigma_r O)\cdot \phi\big(\mu(\sigma_r O), \sqrt{1-\delta}\big)}{\log \frac{1}{r}} \tag{2.5}$$

Indeed, $E_\delta(F^+,O)$ can be covered by countably many sets $\{gx : g \in E_{\delta,x,s}\}$, with the quotient maps $\pi_x : B^G(s) \to X$ being Lipschitz and one-to-one. Since, in view of (2.3), every $g \in B^G(s)$ can be written as $g = \tilde{h}h$, where $\tilde{h} \in B^{\tilde{H}}(r/4)$ and $h \in B^H(r/4)$, for any $y \in X$ we can write

$$\begin{aligned} \operatorname{dist}(g_t g x, y) &\le \operatorname{dist}(g_t \tilde{h} h x, g_t h x) + \operatorname{dist}(g_t h x, y) \\ &= \operatorname{dist}\big(g_t \tilde{h} g_{-t} g_t h x, g_t h x\big) + \operatorname{dist}(g_t h x, y). \end{aligned} \tag{2.6}$$

Hence in view of (2.2), $g \in E_{\delta,x,s}$ implies that hx belongs to $E_\delta(F^+, \sigma_{r/2}O)$, and by using Wegmann's Product Theorem [17], which implies that for two metric spaces X_1, X_2 one has $\dim(X_1 \times X_2) = \dim(X_1) + \dim(X_2)$ whenever the Hausdorff dimension of X_2 is equal to its box dimension, we conclude that:

$$\begin{aligned} \dim E_{\delta,x,s} &\le \dim\Big(\{h \in B^H(r/4) : hx \in E_\delta(F^+, \sigma_{r/2}O)\} \times B^{\tilde{H}}(r/4)\Big) \\ &= \dim\big(\{h \in B^H(r/4) : hx \in E_\delta(F^+, \sigma_{r/2}O)\}\big) + \dim \tilde{H}. \end{aligned} \tag{2.7}$$

Since (X, F^+) is exponentially mixing, by [10, Theorem 2.6] H has (EEP) with respect to (X, F^+). Therefore, by Theorem 1.5 applied to O replaced by $\sigma_{r/2}O$ and r replaced with $r/4$ we have for any $0 < r < r_1$

$$\begin{aligned} &\operatorname{codim}\{h \in B^H(r/4) : hx \in E_\delta(F^+, \sigma_{r/2}O)\} \\ &\gg \frac{\mu(\sigma_{r/4}\sigma_{r/2}O) \cdot \phi\big(\mu(\sigma_{r/4}\sigma_{r/2}O), \sqrt{1-\delta}\big)}{\log \frac{4}{r}} \\ &\gg \frac{\mu(\sigma_r O) \cdot \phi\big(\mu(\sigma_r O), \sqrt{1-\delta}\big)}{\log \frac{4}{r}} \gg \frac{\mu(\sigma_r O) \cdot \phi\big(\mu(\sigma_r O), \sqrt{1-\delta}\big)}{\log \frac{1}{r}}. \end{aligned} \tag{2.8}$$

Now from (2.7) and (2.8) we conclude that (2.5) is satisfied, which finishes the proof. □

3. Auxillary facts.

3.1. Hausdorff dimension of limsup sets.

The exceptional sets we study in this paper are of the form $A = \limsup_{N\to\infty} A_N$, that is

$$A = \bigcap_{N \ge 1} \bigcup_{n \ge N} A_n$$

for a sequence of subsets A_N.

First, we recall the definition of the Hausdorff dimension. Let A be a subset of a metric space Y. For any $\rho, \beta > 0$, we define

$$\mathcal{H}^\beta_\rho(A) = \inf\left\{\sum_{I \in \mathcal{U}} \operatorname{diam}(I)^\beta : \mathcal{U} \text{ is a cover of A by balls of diameter } < \rho\right\}.$$

Then, the β-dimensional Hausdorff measure of A is defined to be

$$\mathcal{H}^\beta(A) = \lim_{\rho \to 0} \mathcal{H}^\beta_\rho(A).$$

Definition 3.1. The Hausdorff dimension of a subset A of a metric space Y is equal to

$$\dim(A) = \inf\left\{\beta \ge 0 : \mathcal{H}^\beta(A) = 0\right\} = \sup\left\{\beta \ge 0 : \mathcal{H}^\beta(A) = \infty\right\}.$$

Lemma 3.2. *Let $\{A_N\}_{N\geq 1}$ be a collection of subsets of Y. Suppose there exist constants $C>0$, $0<\rho<1$, $N_0\in\mathbb{N}$, and a sequence $\{\alpha_N\}_{N\geq 1}$ such that for each $N\geq N_0$, the set A_N can be covered with at most $\rho^{-\alpha_N N}$ balls of radius $C\rho^N$. Then*

$$\dim\left(\limsup_{N\to\infty} A_N\right)\leq \liminf_{N\to\infty}\alpha_N.$$

Proof. Let $A:=\limsup_{N\to\infty} A_N$ and $\alpha:=\liminf_{N\to\infty}\alpha_N$; without loss of generality we can assume that $\alpha<\infty$ and $\alpha_N\to\alpha$ as $N\to\infty$. Take $\beta>\alpha$; we will show that $\mathcal{H}^\beta(A)=0$, which will imply the lemma. For any $\xi\in(0,1)$, let $N_\xi\geq N_0$ be a natural number such that $\rho^{-N}<\xi$ and $|\alpha_N-\alpha|<\frac{\beta-\alpha}{2}$ for all $N\geq N_\xi$. Notice that N_ξ tends to infinity as ξ goes to 0. Take $N\geq N_\xi$ and denote by $\mathcal{O}_N$ a cover of the set A_N by balls of radius $C\rho^N$ such that $\#\mathcal{O}_N$, the number of balls in the cover $\mathcal{O}_N$, is at most $\rho^{-\alpha_N N}$. Then $\mathcal{O}=\bigcup_{N\geq N_\xi}\mathcal{O}_N$ is a cover of A for which the following holds:

$$\begin{aligned}\sum_{B\in\mathcal{O}}\operatorname{diam}(B)^\beta &= \sum_{N\geq N_\xi}\sum_{B\in\mathcal{O}_N}\operatorname{diam}(B)^\beta \leq (2C)^\beta\sum_{N\geq N_\xi}\#\mathcal{O}_N\cdot\rho^{\beta N}\\ &\leq (2C)^\beta\sum_{N\geq N_\xi}\rho^{(\beta-\alpha_N)N}\leq (2C)^\beta\sum_{N\geq N_\xi}\rho^{\frac{\beta-\alpha}{2}N}\\ &\leq (2C)^\beta\frac{\rho^{\frac{\beta-\alpha}{2}N_\xi}}{1-\rho^{\frac{\beta-\alpha}{2}}}\xrightarrow{\xi\to 0} 0.\end{aligned}$$

This implies that $\mathcal{H}^\beta(A)=0$, and the conclusion of the lemma follows. □

3.2. **Choosing r_2.** Recall that as part of the proof of Theorem 1.5 we need to define a bound r_2 for possible values of r. This bound will come from two ingredients. Namely, we define

$$r_2:=\frac{1}{2}\min(r_0,r'),\tag{3.1}$$

where

- $0<r'<1/4$ is chosen so that for any Lie subalgebra $\mathfrak{p}$ of $\mathfrak{g}$ the exponential map from $\mathfrak{p}$ to $P=\exp(\mathfrak{p})$ is 2-bi-Lipischitz on $B^{\mathfrak{p}}(r')$; in particular, we will have
$$B^P(2r)\supset\exp\big(B^{\mathfrak{p}}(r)\big)\supset B^P(r/2)\text{ for any }0<r\leq r'.\tag{3.2}$$
- $r_0:=r_0(X)=\inf\{r_0(x):x\in X\}$, where
$$r_0(x):=\sup\{r>0:\text{the map }\pi_x:g\mapsto gx\text{ is injective on }B(r)\}$$
(the *injectivity radius* of x). Note that $r_0>0$ since X is assumed to be compact.

3.3. **Tessellations of P.** Now let P be a connected subgroup of H normalized by F^+, and let ν be a Haar measure on P.

Definition 3.3. Following [8], say that an open subset V of P is a *tessellation domain* for P relative to a countable subset Λ of P if

- $\nu(\partial V)=0$;
- $V\gamma_1\cap V\gamma_2=\varnothing$ for different $\gamma_1,\gamma_2\in\Lambda$;
- $P=\bigcup_{\gamma\in\Lambda}\overline{V}\gamma$.

Note that P is a connected simply connected nilpotent Lie group. Denote $\mathfrak{p} := \mathrm{Lie}(P)$ and $L := \dim P$, and fix a Haar measure ν on P. As shown in [8, Proposition 3.3], one can choose a basis of $\mathfrak{p}$ such that for any $r > 0$, $\exp(rI_{\mathfrak{p}})$, where $I_{\mathfrak{p}} \subset \mathfrak{p}$ is the cube centered at 0 with side length 1 with respect to that basis, is a tessellation domain relative to some discrete subset Λ of H. Let us denote

$$V_r := \exp\left(\frac{r}{4\sqrt{L}} I_{\mathfrak{p}}\right), \tag{3.3}$$

and choose a countable $\Lambda_r \subset P$ such that V_r is a tessellation domain relative to Λ_r.

Then, since $L \geq 1$, it follows from (3.2) that for any $0 < r \leq r'$ one has

$$B^P\left(\frac{r}{16\sqrt{L}}\right) \subset V_r \subset B^P\left(\frac{r}{4}\right). \tag{3.4}$$

3.4. **Bowen boxes.** Note that the measure ν and the pushforward of the Lebesgue measure Leb on $\mathfrak{p}$ by the exponential map are absolutely continuous with respect to each other with locally bounded Radon–Nikodym derivative. This implies that there exists $0 < c_1 < c_2$ such that

$$c_1 \,\mathrm{Leb}(A) \leq \nu\big(\exp(A)\big) \leq c_2 \,\mathrm{Leb}(A) \quad \forall \text{ measurable } A \subset B^{\mathfrak{p}}(1) \tag{3.5}$$

(note that since P is nilpotent, the map $\exp : \mathfrak{p} \to P$ is a diffeomorphism).

Define

$$\lambda_{\min} := \min\{|\lambda| : \lambda \text{ is an eigenvalue of } \mathrm{ad}_{g_1}|_{\mathfrak{p}}\} \tag{3.6}$$

and

$$\lambda_{\max} := \max\{|\lambda| : \lambda \text{ is an eigenvalue of } \mathrm{ad}_{g_1}|_{\mathfrak{p}}\}. \tag{3.7}$$

Note that $\lambda_{\min} > 0$ since $P \subset H$. Using the bi-Lipschitz property of exp, we can conclude that

$$\begin{aligned} \mathrm{diam}(g_{-t}V_r g_t) &\leq 2 \cdot \mathrm{diam}\left(\exp\left(\frac{re^{-\lambda_{\min}t}}{4\sqrt{p}} I_{\mathfrak{p}}\right)\right) \\ &\leq \frac{re^{-\lambda_{\min}t}}{2} \quad \text{for any } 0 < r \leq r' \text{ and any } t \geq 0. \end{aligned} \tag{3.8}$$

Also let $\eta := \mathrm{Tr}\,\mathrm{ad}_{g_1}|_{\mathfrak{p}}$; clearly one then has

$$\nu(g_{-t}Bg_t) = e^{-\eta t}\nu(B) \text{ for any measurable } B \subset P. \tag{3.9}$$

Let us now define a *Bowen (t,r)-box* in P to be a set of the form $g_{-t}V_r\gamma g_t$ for some $\gamma \in P$ and $t > 0$. Our approach to estimating Hausdorff dimension of various subsets of P will be through covering them by Bowen boxes. We are going to need three results proved in [12]. The first one, a slight modification of [8, Proposition 3.4], gives an upper bound for the number of $\gamma \in \Lambda_r$ such that the Bowen box $g_{-t}V_r\gamma g_t$ has non-empty intersection with V_r:

Lemma 3.4. [12, Lemma 4.1] *There exists $c_0 > 0$ such that for any $0 < r \leq r'$ and for any $t > \frac{\log 8}{\lambda_{\min}}$*

$$\#\{\gamma \in \Lambda_r : g_{-t}\overline{V_r}\gamma g_t \cap V_r \neq \varnothing\} \leq e^{\eta t}\left(1 + c_0 e^{-\lambda_{\min}t}\right).$$

The second one gives us an upper bound for the number of balls of radius $re^{-\lambda_{\max}t}$ needed to cover a Bowen box.

Lemma 3.5. [12, Lemma 7.4] *There exists $C_1 > 0$ such that for any $0 < r < 1$ and any $t > 0$, any Bowen (t,r)-box in P can be covered with at most $C_1 e^{(\lambda_{\max}L - \eta)t}$ balls in P of radius $re^{-\lambda_{\max}t}$.*

The third result is a direct consequence of property (EEP); it is a simplified version of [12, Proposition 4.4].

Proposition 3.6. *Let F^+ be a one-parameter subsemigroup of G and P a subgroup of G with property* (EEP) *with respect to F^+. Then there exist $t_1 \geq 1$ and $\lambda' > 0$ such that for any open $O \subset X$, any $x \in X$, any $r \leq r_2$ and any $t \geq t_1$ one has*

$$\nu\left(\{h \in V_r : g_t hx \in O\}\right) \geq \nu\left(V_r\right) \mu(\sigma_{e^{-\lambda' t}} O) - e^{-\lambda' t}.$$

4. **Effective equidistribution of translates and a measure estimate.** From now on we will work with F^+ and P as in Theorem 1.5, and for the rest of the paper fix a positive $r \leq r_2$, where r_2 is as in (3.1). Note that by the countable stability of Hausdorff dimension, in order to prove Theorem 1.5 it suffices, for a subset O of X and $0 < \delta < 1$, to get a uniform (in $x \in X$) upper bound for the Hausdorff dimension of the set

$$\begin{aligned} S_{x,\delta}(O) &:= \{h \in V_r : hx \in E_\delta(F^+, O)\} \\ &= \left\{h \in V_r : \limsup_{T \to \infty} \frac{1}{T} \int_0^T 1_{O^c}(g_t hx)\, dt \geq \delta \right\}. \end{aligned} \tag{4.1}$$

For this it will be convenient to discretize the above definition. Namely let us introduce the following notation: given $T > 0$, a subset S of X, $x \in X$ and $0 < \delta < 1$, let us define

$$A_{x,\delta}(T, S) := \left\{h \in V_r : \frac{1}{T} \int_0^T 1_S(g_t hx)\, dt \geq \delta \right\}. \tag{4.2}$$

In the following proposition we find the relation between the set $S_{x,\delta}(O)$ and the family of sets $\{A_{x,\delta}(NT, O^c)\}_{N\in\mathbb{N}}$.

Proposition 4.1. *For any $T > 0$ and any $x \in X$ and we have*

$$S_{x,\delta}(O) = \limsup_{N\in\mathbb{N},\, N\to\infty} A_{x,\delta}(NT, O^c)$$

Proof. In view of definition of $S_{x,\delta}(O)$, it suffices, for a fixed $T > 0$, to prove that

$$\begin{aligned} \limsup_{R\to\infty} \frac{1}{R} \int_0^R 1_{O^c}(g_t hx)\, dt &= \limsup_{N\in\mathbb{N},\, N\to\infty} \frac{1}{NT} \int_0^{NT} 1_{O^c}(g_t hx)\, dt \\ &= \lim_{N_0\to\infty} \sup_{N \geq N_0} \frac{1}{NT} \int_0^{NT} 1_{O^c}(g_t hx)\, dt. \end{aligned}$$

Let $N_0 \in \mathbb{N}$, and let $R \geq N_0 T$. Then one can find $N \geq N_0$ and $0 \leq R' < T$ such that $R = NT + R'$. Hence

$$\begin{aligned} \frac{1}{R} \int_0^R 1_{O^c}(g_t hx)\, dt =& \frac{1}{NT + R'} \int_0^{NT+R'} 1_{O^c}(g_t hx)\, dt \\ \leq& \frac{1}{NT} \int_0^{NT+R'} 1_{O^c}(g_t hx)\, dt \\ =& \frac{1}{NT} \left(\int_0^{NT} 1_{O^c}(g_t hx)\, dt + \int_{NT}^{NT+R'} 1_{O^c}(g_t hx)\, dt \right) \\ \underset{(1_{O^c} \leq 1,\, R' \leq T)}{\leq}& \frac{1}{NT} \int_0^{NT} 1_{O^c}(g_t hx)\, dt + \frac{1}{N} \\ \underset{(N \geq N_0)}{\leq}& \frac{1}{NT} \int_0^{NT} 1_{O^c}(g_t hx)\, dt + \frac{1}{N_0}. \end{aligned}$$

Therefore, we get

$$\limsup_{R\to\infty} \frac{1}{R}\int_0^R 1_{O^c}(g_t hx)\,dt \le \lim_{N_0\to\infty}\sup_{N\ge N_0}\frac{1}{NT}\int_0^{NT} 1_{O^c}(g_t hx)\,dt.$$

The reverse inequality is obvious. □

The next proposition gives an upper estimate for the measure of $A_{x,\delta}(T, O^c)$. We will use it in §5 to obtain an upper bound for the number of Bowen (r, T)-boxes in P needed to cover this set (see Corollary 5.2).

Proposition 4.2. *For all $x \in X$, $0 < \varepsilon < 1$, for any open $O \subset X$ and for all*

$$T > T_r := \max\left(t_1,\ \frac{1}{\lambda'}\log\frac{2}{r},\ \frac{1}{\lambda'}\log\frac{(4\sqrt{L})^L}{c_1\lambda' r^L},\ \frac{\log 8}{\lambda_{\min}}\right), \tag{4.3}$$

where c_1 is as in (3.5) and t_1, λ' are as in Proposition 3.6, one has

$$\nu\big(A_{x,1-\varepsilon}(T,O)\big) \ge \nu\left(V_r\right)\left(1 - \frac{1}{\varepsilon}\Big(1 - \mu(\sigma_{r/2}O) + \frac{T_r+1}{T}\Big)\right). \tag{4.4}$$

Proof. Let x, ε, O and T be as above. Then by definition we have:

$$\begin{aligned}\nu\big(A_{x,\,1-\varepsilon}(T,O)\big) &= \nu\left(\left\{h \in V_r : \frac{1}{T}\int_0^T 1_O(g_t hx)\,dt \ge 1-\varepsilon\right\}\right)\\ &= \nu\left(V_r\right) - \nu\left(\left\{h \in V_r : \frac{1}{T}\int_0^T 1_{O^c}(g_t hx)\,dt > \varepsilon\right\}\right)\\ &\ge \nu\left(V_r\right) - \nu\big(A_{x,\varepsilon}(T,O^c)\big).\end{aligned} \tag{4.5}$$

Our goal is to estimate the right-hand side of (4.5) from below. We have:

$$\begin{aligned}\nu\big(A_{x,\varepsilon}(T,O^c)\big) &= \nu\left(\left\{h \in V_r : \frac{1}{T}\int_0^T 1_{O^c}(g_t hx)\,dt \ge \varepsilon\right\}\right)\\ &\underset{\text{(Markov's inequality)}}{\le} \frac{1}{\varepsilon T}\int_0^T\int_{V_r} 1_{O^c}(g_t hx)\,d\nu(h)\,dt\\ &= \frac{1}{\varepsilon T}\int_0^T \nu\left(\{h\in V_r : g_t hx \in O^c\}\right)dt\\ &= \frac{1}{\varepsilon T}\left(\int_0^{T_r} \nu\left(\{h\in V_r : g_t hx \in O^c\}\right)dt\right.\\ &\qquad\left. + \int_{T_r}^T \nu\left(\{h\in V_r : g_t hx \in O^c\}\right)dt\right).\end{aligned} \tag{4.6}$$

Note that

$$\int_0^{T_r} \nu\left(\{h\in V_r : g_t hx \in O^c\}\right)dt \le T_r\cdot\nu\left(V_r\right), \tag{4.7}$$

and, since $T_r \geq t_1$, by applying Proposition 3.6 we get

$$\begin{aligned}
&\int_{T_r}^{T} \nu\left(\{h \in V_r : g_t hx \in O^c\}\right) dt \\
&= \int_{T_r}^{T} \big(\nu(V_r) - \nu\left(\{h \in V_r : g_t hx \in O\}\right)\big)\, dt \\
&\leq \int_{T_r}^{T} \Big(\nu\left(V_r\right)\left(1 - \mu(\sigma_{e^{-\lambda' T_r}} O)\right) + e^{-\lambda' t}\Big)\, dt \qquad (4.8)\\
&\leq T \cdot \nu\left(V_r\right)\left(1 - \mu(\sigma_{e^{-\lambda' T_r}} O)\right) + \frac{e^{-\lambda' T_r}}{\lambda'} \\
&\underset{(4.3)}{\leq} T \cdot \nu\left(V_r\right)\big(1 - \mu(\sigma_{r/2} O)\big) + \frac{e^{-\lambda' T_r}}{\lambda'}.
\end{aligned}$$

It remains to observe that (4.3), in combination with (3.3) and (3.5), also implies that $\frac{e^{-\lambda' T_r}}{\lambda'} \leq \nu(V_r)$. Hence (4.4) follows from (4.5), (4.6), (4.7) and (4.8). □

5. **A covering result.** In this section we will prove a covering result for the sets of the form $A_{x,\delta}(NT, O^c)$. Then, by using Proposition 4.1 and Lemma 3.2, in the next section we will obtain an upper estimate for the Hausdorff dimension of $S_{x,\delta}(O)$.

We start with the following lemma:

Lemma 5.1. *Let O be an open subset of X, and let $0 < \varepsilon < 1$. Then for any $T > 0$, $\gamma \in \Lambda_r$, $x \in X$, any $0 < \alpha < \varepsilon$ and for any Bowen (r, T)-box $B = g_{-T} V_r \gamma g_T$ which has non-empty intersection with the set $A_{x,1-\alpha}(T, \sigma_{4r} O)$, we have*

$$B \cap A_{x,\varepsilon}(T, O^c) = \varnothing.$$

Proof. Let O be an open subset of X and let $\gamma \in \Lambda_r$. Consider the Bowen (r, T)-box $B = g_{-T} V_r \gamma g_T$. Let $x \in X$, take $p \in B$, and assume that $g_t px \in \sigma_{4r} O$ for some $0 \leq t \leq T$. Any $p' \in B$ is of the form $p' = hp$ where $h \in g_{-T}(V_r \cdot V_r) g_T$. (Here we use the fact that $V_r = V_r^{-1}$, which is a simple consequence of (3.3).) Thus we have

$$\begin{aligned}
g_t p' x = g_t h g_{-t} g_t px &\in g_{-(T-t)}(V_r \cdot V_r) g_{(T-t)} g_t px \\
&\underset{(3.8)}{\in} B^G(2re^{-\lambda_{\min}(T-t)}) \cdot B^G(2re^{-\lambda_{\min}(T-t)}) g_t px \\
&\in B^G(4r) g_t px
\end{aligned}$$

which, in view of (3.4), implies that $\operatorname{dist}(g_t p' x, g_t px) \leq 4r$. Hence $g_t p' x \in O$. Now assume in addition that $0 < \alpha < \varepsilon < 1$ and $p \in A_{x,1-\alpha}(T, \sigma_{4r} O)$; then

$$\begin{aligned}
\frac{1}{T}\int_0^T 1_{O^c}(g_t p' x)\, dt &= 1 - \frac{1}{T}\int_0^T 1_O(g_t p' x)\, dt \\
&\leq 1 - \frac{1}{T}\int_0^T 1_{\sigma_{4r} O}(g_t px)\, dt \leq 1 - (1 - \alpha) = \alpha < \varepsilon.
\end{aligned}$$

Therefore, if B has non-empty intersection with $A_{x,1-\alpha}(T, \sigma_{4r} O)$, then for any $p' \in B$ we have $p' \notin A_{x,\varepsilon}(T, O^c)$. This ends the proof. □

Now, by combining the previous lemma with Lemma 3.4 and Proposition 4.2 we obtain the following corollary which is a covering result for the sets of type $A_{x,\varepsilon}(T, O^c)$:

Corollary 5.2. *Let T_r be as in (4.3). Then for any $0 < \varepsilon < 1$, any $T > T_r$, any $x \in X$, and for any open subset O of X, the set $A_{x,\varepsilon}(T, O^c)$ can be covered with at most $\frac{e^{\eta T}}{\varepsilon} C(T,O)$ Bowen (T, r)-boxes in P, where*

$$C(T,O) := 1 - \mu(\sigma_{5r}O) + \frac{T_r + 1}{T} + c_0 e^{-\lambda_{\min} T}. \tag{5.1}$$

Proof. Let $0 < \varepsilon < 1$, $0 < \alpha < \varepsilon$, $T > T_r$, $x \in X$, and let O be an open subset of X. Then we have:

$$\begin{aligned}
& \#\{\gamma \in \Lambda_r : g_{-T} V_r \gamma g_T \cap A_{x,\varepsilon}(T, O^c) \neq \varnothing\} \\
\underset{\text{Lemma 5.1}}{\leq}\ & \#\{\gamma \in \Lambda_r : g_{-T} V_r \gamma g_T \cap V_r \neq \varnothing\} \\
& - \#\{\gamma \in \Lambda_r : g_{-T} V_r \gamma g_T \cap A_{x,1-\alpha}(T, \sigma_{4r} O) \neq \varnothing\} \\
\underset{\text{Lemma 3.4}}{\leq}\ & e^{\eta T}\left(1 + c_0 e^{-\lambda_{\min} T}\right) - \#\{\gamma \in \Lambda_r : g_{-T} V_r \gamma g_T \cap A_{x,1-\alpha}(T, \sigma_{4r} O) \neq \varnothing\} \\
\underset{\text{Defintion 3.3}}{\leq}\ & e^{\eta T}\left(1 + c_0 e^{-\lambda_{\min} T}\right) - \frac{\nu\big(A_{x,1-\alpha}(T, \sigma_{4r} O)\big)}{\nu(g_{-T} V_r g_T)} \\
\underset{\text{Proposition 4.2}}{\leq}\ & e^{\eta T}\left(1 + c_0 e^{-\lambda_{\min} T}\right) - e^{\eta T}\left(1 - \alpha^{-1} \cdot \left(1 - \mu(\sigma_{r/2}\sigma_{4r} O) + \frac{T_r + 1}{T}\right)\right) \\
\underset{(5.1)}{<}\ & \frac{e^{\eta T}}{\alpha} C(T, O).
\end{aligned}$$

Now since $0 < \alpha < \varepsilon$ was arbitrary, by letting α approach ε we get that $A_{x,\varepsilon}(T, O^c)$ can be covered with at most $\frac{e^{\eta T}}{\varepsilon} C(T,O)$ Bowen (T, r)-boxes in P, as desired. □

Next we will need a generalized version of the definition (4.2) of sets $A_{x,\delta}(T, S)$. Namely, given $S \subset X$, $x \in X$, $T > 0$, $0 < \delta < 1$ and $J \subset \mathbb{N}$, let us define

$$A_{x,\,\delta}(T, S, J) := \left\{ h \in V_r : \frac{1}{T} \int_{(i-1)T}^{iT} 1_S(g_t h x)\, dt \geq \delta \quad \forall\, i \in J \right\}. \tag{5.2}$$

Clearly $A_{x,\delta}(T, S) = A_{x,\,\delta}\big(T, S, \{1\}\big)$. Using the above corollary inductively, in the following proposition we obtain a covering result for the sets of the form (5.2).

Proposition 5.3. *Let O be a non-empty open subset of X, and let T_r be as in (4.3). Then for all $0 < \varepsilon < 1$, $T > T_r$, $N \in \mathbb{Z}_+$, $J \subset \{1, \dots, N\}$, and for all $x \in X$, the set $A_{x,\varepsilon}(T, O^c, J)$ can be covered with at most*

$$e^{\eta N T} \left(\frac{C(T,O)}{\varepsilon} \right)^{|J|} \left(1 + c_0 e^{-\lambda_{\min} T}\right)^{N - |J|} \tag{5.3}$$

Bowen (NT, r)-boxes in P.

Proof. Let $0 < \varepsilon < 1$, let $T > T_r$, and let $x \in X$. We argue by induction on N; the base case is given by $N = 0$ and $J = \varnothing$, which makes the quantity (5.3) equal to 1. This makes sense, since $A_{x,\varepsilon}(T, O^c, J) = V_r$, which is precisely a $(0, r)$-box.

Now take an arbitrary $N \in \mathbb{N}$ and let $J' := J \smallsetminus \{N\}$. By the induction assumption, the set $A_{x,\varepsilon}(T, O^c, J')$ can be covered with at most

$$e^{\eta (N-1) T} \cdot \left(\frac{C(T,O)}{\varepsilon} \right)^{|J'|} \left(1 + c_0 e^{-\lambda_{\min} T}\right)^{N - 1 - |J'|} \tag{5.4}$$

Bowen $\big((N-1)T,r\big)$-boxes in P. Now let $g_{-(N-1)T}V_r\gamma g_{(N-1)T}$ be one of the Bowen $\big((N-1)T,r\big)$-boxes in the above cover which has non-empty intersection with $A_{x,\varepsilon}\big((N-1)T,O^c,J'\big)$. Take any $q=g_{-(N-1)T}h\gamma g_{(N-1)T}\in g_{-(N-1)T}V_r\gamma g_{(N-1)T}$, and consider two cases.

- If $N\in J$, so that $|J'|=|J|-1$ and $N-1-|J'|=N-|J|$, write
$$\frac{1}{T}\int_{(N-1)T}^{NT}1_{O^c}(g_tqx)\,dt=\frac{1}{T}\int_{(N-1)T}^{NT}1_{O^c}\big(g_t(g_{-(N-1)T}h\gamma g_{(N-1)T})x\big)\,dt$$
$$=\frac{1}{T}\int_0^T 1_{O^c}\big(g_th(\gamma g_{(N-1)T}x)\big)\,dt.$$
Consequently,
$$\begin{aligned}&\left\{q\in g_{-(N-1)T}V_r\gamma g_{(N-1)T}:\frac{1}{T}\int_{(N-1)T}^{NT}1_{O^c}(g_tqx)\,dt\geq\varepsilon\right\}\\ &=g_{(N-1)T}\left\{h\in V_r:\frac{1}{T}\int_0^T 1_{O^c}\big(g_th(\gamma g_{(N-1)T}x)\big)\,dt\geq\varepsilon\right\}\gamma g_{(N-1)T}\\ &=g_{(N-1)T}\,A_{\gamma g_{(N-1)T}x,\varepsilon}(T,r,O^c)\gamma g_{(N-1)T}.\end{aligned}\tag{5.5}$$
Hence, by applying Corollary 5.2 with $\gamma g_{(N-1)T}x$ in place of x, we can cover the set in the left hand side of (5.5) with at most $e^{\eta T}\left(\frac{C(T,O)}{\varepsilon}\right)^{|J|}$ Bowen (NT,r)-boxes in P. Therefore the number of Bowen (NT,r)-boxes needed to cover $A_{x,\varepsilon}(T,O^c,J)$ is at most $e^{\eta T}\left(\frac{C(T,O)}{\varepsilon}\right)^{|J|}$ times the quantity in (5.3), which is precisely (5.4).
- If $N\notin J$, so that $|J'|=|J|$ and $N-1-|J'|=N-1-|J|$, the argument is even clearer. By Lemma 3.4, $g_{-(N-1)T}V_r\gamma g_{(N-1)T}$ can be covered by at most $e^{\eta T}\left(1+c_0e^{-\lambda_{\min}T}\right)$ Bowen (NT,r)-boxes. Hence the number of Bowen (NT,r)-boxes needed to cover $A_{x,\varepsilon}(T,O^c,J)$ is at most $e^{\eta T}\left(1+c_0e^{-\lambda_{\min}T}\right)$ times the quantity in (5.3), which again is precisely (5.4).

□

Recall that our goal in this section is to find a covering result for the sets of the form $A_{x,\delta}(NT,O^c)$. The following lemma reduces this task to a covering result for the sets of the form $A_{x,\varepsilon}(T,O^c,J)$ for $0<\varepsilon<\delta$ and $J\subset\{1,\dots,N\}$.

Lemma 5.4. *For any $S\subset X$, $N\in\mathbb{N}$, $T>0$, $x\in X$, $0<\delta<1$, and for any $0<\varepsilon<\delta$*
$$A_{x,\delta}(NT,S)\subset\bigcup_{J\subset\{1,\dots,N\}:|J|\geq\lceil\left(1-\frac{1-\delta}{1-\varepsilon}\right)N\rceil}A_{x,\varepsilon}(T,S,J)$$
.

Proof. Let $N\in\mathbb{N}$, $r>0$, $T>0$, $x\in X$ and $0<\varepsilon<\delta<1$. Also let $h\in A_{x,\delta}(NT,S)$, and define
$$E:=\big\{j\in\{1,\dots,N\}:h\notin A_{x,\varepsilon}(T,S,\{j\})\big\}.$$
Then
$$\delta\leq\frac{1}{NT}\int_0^{NT}1_S(g_thx)\,dt$$

$$= \frac{1}{N}\sum_{j\in E}\frac{1}{T}\int_{(j-1)T}^{jT} 1_S(g_t hx)\,dt + \frac{1}{N}\sum_{j\in\{1,\ldots,N\}\smallsetminus E}\frac{1}{T}\int_{(j-1)T}^{jT} 1_S(g_t hx)\,dt$$
$$\leq \frac{1}{N}\cdot|E|\cdot\varepsilon + \frac{1}{N}\cdot\big|\{1,\ldots,N\}\smallsetminus E\big| = \frac{1}{N}\big(|E|\cdot\varepsilon + N - |E|\big).$$

This implies

$$|E| \leq \frac{1-\delta}{1-\varepsilon}N.$$

Note that it follows immediately from the definition of E that h is an element of $A_{x,\varepsilon}\big(T,S,\{1,\ldots,N\}\smallsetminus E\big)$. Hence, in view of the above inequality we conclude that

$$h \in \bigcup_{J\subset\{1,\ldots,N\}:|J|\geq\lceil\left(1-\frac{1-\delta}{1-\varepsilon}\right)N\rceil} A_{x,\varepsilon}(T,S,J),$$

finishing the proof of the lemma. □

From the above lemma combined with Proposition 5.3 we get the following crucial covering result:

Corollary 5.5. *Let C_1 be as in Lemma 3.5, $0<\delta<1$, $0<r\leq r_2$, and let T_r be as in (4.3). Then for any $x\in X$, any $N\in\mathbb{N}$, any $T>T_r$, and for any $0<\varepsilon<\delta$ the set $A_{x,\delta}(NT,O^c)$ can be covered with at most*

$$C_1 e^{\lambda_{\max}LNT}\binom{N}{\lceil\left(1-\frac{1-\delta}{1-\varepsilon}\right)N\rceil}\cdot\left(\frac{C(T,O)}{\varepsilon}\right)^{\lceil\left(1-\frac{1-\delta}{1-\varepsilon}\right)N\rceil}\cdot\left(1+c_0e^{-\lambda_{\min}T}\right)^{N-\lceil\left(1-\frac{1-\delta}{1-\varepsilon}\right)N\rceil} \tag{5.6}$$

balls in P of radius $re^{-\lambda_{\max}NT}$.

Proof. Let $x\in X$, $N\in\mathbb{N}$, $T>T_r$, and let $0<\varepsilon<\delta$. By the above lemma we have:

$$A_{x,\delta}(NT,O^c) \subset \bigcup_{J\subset\{1,\ldots,N\}:|J|\geq\lceil\left(1-\frac{1-\delta}{1-\varepsilon}\right)N\rceil} A_{x,\varepsilon}(T,O^c,J).$$

Now note that for any $J\subset\{1,\ldots,N\}$, if we take any subset J' of J, then it follows immediately that $A_{x,\varepsilon}(T,O^c,J)\subset A_{x,\varepsilon}(T,O^c,J')$. Therefore, the above inclusion yields the following inclusion:

$$A_{x,\delta}(NT,O^c) \subset \bigcup_{J\subset\{1,\ldots,N\}:|J|=\lceil\left(1-\frac{1-\delta}{1-\varepsilon}\right)N\rceil} A_{x,\varepsilon}(T,O^c,J).$$

Also, by Lemma 3.5, every Bowen (NT,r)-boxes in P can be covered with at most $C_1e^{(\lambda_{\max}L-\eta)NT}$ balls in P of radius $re^{-\lambda_{\max}NT}$. From this, combined with Proposition 5.3 and the above inclusion we can conclude the proof. □

6. **Proof of Theorem 1.5.**

Proof of Theorem 1.5. Let O be an open subset of X, $x\in X$, and let $\delta>0$. In view of countable stability of Hausdorff dimension, it suffices to show that for any $0<r\leq r_2$, we have

$$\operatorname{codim} S_{x,\delta}(O) \gg \frac{\mu(\sigma_r O)\cdot\phi\left(\mu(\sigma_r O),\sqrt{1-\delta}\right)}{\log\frac{1}{r}}$$

where $S_{x,\delta}(O)$ is as in (4.1) and ϕ is as in (1.4). In order to prove the above statement, it is evident that it suffices to demonstrate that for any $0 < r \le r_2/5$ we have

$$\operatorname{codim} S_{x,\delta}(O) \gg \frac{\mu(\sigma_{5r}O) \cdot \phi\left(\mu(\sigma_{5r}O), \sqrt{1-\delta}\right)}{\log \frac{1}{5r}}. \tag{6.1}$$

If $\mu(\sigma_{5r}O) = 0$, then the above statement follows immediately. So in this proof we assume always that $\mu(\sigma_{5r}O) > 0$. We start with the following combinatorial lemma:

Lemma 6.1. *Let $m = m(n) \le n$ be a function of n such that $\lim_{n\to\infty} \frac{m}{n} = z$ for some fixed constant $0 < z < 1$. Then*

$$\binom{n}{m} = o(1) B\left(\tfrac{m}{n}\right)^n,$$

where

$$B(z) := \left(\frac{1}{z}\right)^z \left(\frac{1}{1-z}\right)^{1-z}. \tag{6.2}$$

Proof. Note that $\lim_{n\to\infty} \frac{m}{n} = z < 1$ implies that both m and $n-m$ tend to infinity as n goes to infinity; moreover, $\lim_{n\to\infty} \frac{n-m}{n} = 1 - z$. Hence, by using Stirling's approximation we have:

$$\begin{aligned}
\binom{n}{m} = \frac{n!}{m!(n-m)!} &= \big(1 + o(1)\big) \frac{\sqrt{2\pi n}(\frac{n}{e})^n}{\sqrt{2\pi m}(\frac{m}{e})^m + \sqrt{2\pi(n-m)}(\frac{n-m}{e})^{n-m}} \\
&= \big(1 + o(1)\big) \sqrt{\frac{n}{2\pi m(n-m)}} \left(\frac{n}{m}\right)^m \left(\frac{n}{n-m}\right)^{n-m} \\
&= o(1) \left(\frac{n}{m}\right)^m \left(\frac{n}{n-m}\right)^{n-m} \\
&= o(1) B\left(\tfrac{m}{n}\right)^n,
\end{aligned}$$

where the third equality above follows from the fact that

$$\lim_{n\to\infty} \frac{n}{2\pi m(n-m)} = \lim_{n\to\infty} \frac{1}{2\pi n} \cdot \lim_{n\to\infty} \frac{n}{m} \cdot \lim_{n\to\infty} \frac{n}{n-m} = 0 \cdot \frac{1}{z} \cdot \frac{1}{1-z} = 0.$$

□

Now given $0 < \varepsilon < \delta$ set

$$z := 1 - \frac{1-\delta}{1-\varepsilon} \quad \Longleftrightarrow \quad \varepsilon = 1 - \frac{1-\delta}{1-z} = \frac{\delta - z}{1-z}. \tag{6.3}$$

Lemma 6.1, applied with n replaced with N and m replaced with $\lceil zN \rceil$, then implies that there exists $N_0 = N_0(z) \in \mathbb{N}$ such that

$$\binom{N}{\lceil zN \rceil} \le B\left(\frac{\lceil zN \rceil}{N}\right)^N \qquad \text{for all } N \ge N_0. \tag{6.4}$$

Take $0 < r \le r_2/5$ and T_r as in (4.3), and let $T > T_r$. By combining Corollary 5.5 with (6.4) we get that for any $N \ge N_0$ and any $0 < \varepsilon < \delta$,

$$\begin{gathered} A_{x,\delta}(NT, O^c) \text{ can be covered with at most} \\ C_1 e^{LN\lambda_{\max} T} \cdot \beta_N^N \text{ balls in } P \text{ of radius } r e^{-\lambda_{\max} NT}, \end{gathered} \tag{6.5}$$

where

$$\beta_N := B\left(\frac{\lceil zN\rceil}{N}\right)\cdot\left(\frac{C(T,O)}{\varepsilon}\right)^{\frac{\lceil zN\rceil}{N}}\left(1+c_0e^{-\lambda_{\min}T}\right)^{1-\frac{\lceil zN\rceil}{N}}.$$

Note that we have

$$\lim_{N\to\infty}\beta_N = B(z)\cdot\left(\frac{C(T,O)}{\varepsilon}\right)^{z}\left(1+c_0e^{-\lambda_{\min}T}\right)^{1-z} \tag{6.6}$$

In view of (6.3), (6.5), (6.6) and Proposition 4.1, by applying Lemma 3.2 with $e^{-\lambda_{\max}T}$ in place of ρ, $\frac{\log C_1}{\lambda_{\max}NT}+L+\frac{\log\beta_N}{\lambda_{\max}T}$ in place of α_N and r in place of C, we conclude that for any $0<z<\delta$ the Hausdorff dimension of the set $S_{x,\delta}(O)$ is bounded from above by

$$\begin{aligned}
&L+\frac{1}{\lambda_{\max}T}\log\left(B(z)\left(\frac{C(T,O)(1-z)}{\delta-z}\right)^{z}\left(1+c_0e^{-\lambda_{\min}T}\right)^{1-z}\right)\\
&\underset{(6.2)}{=} L+\frac{1}{\lambda_{\max}T}\log\left(\left(\frac{C(T,O)(1-z)}{z(\delta-z)}\right)^{z}\left(\frac{1+c_0e^{-\lambda_{\min}T}}{1-z}\right)^{1-z}\right).
\end{aligned}$$

This shows that our objective should be to find $z\in(0,\delta)$ and $T>T_r$ such that the value of

$$\frac{1}{T}\left(z\log\left(\frac{z(\delta-z)}{C(T,O)(1-z)}\right)+(1-z)\log\left(\frac{1-z}{1+c_0e^{-\lambda_{\min}T}}\right)\right)$$

is the largest possible. We are going to approximate the maximum by first choosing T in a convenient way. Take $T_0\geq 1$ sufficiently large so that for any $T\geq T_0$ one has

$$c_0e^{-\lambda_{\min}T}<\frac{1}{T} \tag{6.7}$$

(note that T_0 depends only on G, F_+ and P), and set

$$T:=\max\left(\frac{8T_r}{\mu(\sigma_{5r}O)},T_0\right). \tag{6.8}$$

Then

$$\frac{1}{T}<\frac{1+T_r}{T}\leq\frac{2T_r}{T}\leq\frac{\mu(\sigma_{5r}O)}{4},$$

which, in combination with (6.7), yields

$$1+c_0e^{-\lambda_{\min}T}\leq 1+\frac{1}{T}<1+\frac{\mu(\sigma_{5r}O)}{4}$$

and

$$\begin{aligned}
C(T,O)&=1-\mu(\sigma_{5r}O)+\frac{T_r+1}{T}+c_0e^{-\lambda_{\min}T}\\
&\leq 1-\mu(\sigma_{5r}O)+\frac{\mu(\sigma_{5r}O)}{4}+\frac{\mu(\sigma_{5r}O)}{4}=1-\frac{\mu(\sigma_{5r}O)}{2}.
\end{aligned}$$

Therefore for T as in (6.8) the codimension of $S_{x,\delta}(O)$ is for any $0<z<\delta$ bounded from below by

$$\frac{1}{\lambda_{\max}T}\left(z\log\left(\frac{z(\delta-z)}{\left(1-\frac{\mu(\sigma_{5r}O)}{2}\right)(1-z)}\right)+(1-z)\log\left(\frac{1-z}{1+\frac{\mu(\sigma_{5r}O)}{4}}\right)\right).$$

Note that the second summand in the above expression is always negative; thus it makes sense to try choosing $0<z<\delta$ in a way that ensures that the first summand is maximized and is positive if possible (this condition could prove to be unsuccessful for any $0 < z < \delta$, contingent upon $\mu(\sigma_{5r}O)$ and δ; in this case we will not achieve a dimension drop).

We will solve the latter problem approximately by finding $0 < z < \delta$ which maximizes the ratio $\frac{z(\delta-z)}{1-z}$. An elementary calculus exercise shows that for that one should take $z = 1-\sqrt{1-\delta}$, so that $\frac{z(\delta-z)}{1-z} = (1-\sqrt{1-\delta})^2$. Denoting $s = \sqrt{1-\delta}$ and $y = \mu(\sigma_{5r}O)$, we get an estimate

$$\begin{aligned}\operatorname{codim} S_{x,\delta}(O) \geq\ & \frac{1}{\lambda_{\max}T}\left((1-s)\log\left(\frac{(1-s)^2}{1-\frac{y}{2}}\right) + s\log\left(\frac{s}{1+\frac{y}{4}}\right)\right) \\ \underset{(1.4)}{=}\ & \frac{1}{\lambda_{\max}T}\phi(y,s).\end{aligned}$$

It remains to observe that

$$T \underset{(6.8)}{\ll} \frac{8T_r}{\mu(\sigma_{5r}O)} \underset{(4.3)}{\ll} \frac{\log\frac{1}{r}}{\mu(\sigma_{5r}O)} \underset{r\leq\frac{r_2}{5}<1/40}{\ll} \frac{\log\frac{1}{5r}}{\mu(\sigma_{5r}O)};$$

thus (6.1) follows, which ends the proof of Theorem 1.5. □

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