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DIMENSION DROP FOR DIAGONALIZABLE FLOWS ON HOMOGENEOUS SPACES

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ABSTRACT. Let $X = G/\Gamma$, where G is a connected Lie group and Γ is a lattice in G. Let O be an open subset of X, and let $F = \{g_t : t \geq 0\}$ be a one-parameter subsemigroup of G. Consider the set of points in X whose F-orbit misses O; it has measure zero if the flow is ergodic. It has been conjectured that, assuming ergodicity, this set has Hausdorff dimension strictly smaller than the dimension of X. This conjecture has been proved when X is compact or when G is a simple Lie group of real rank 1, or, most recently, for certain flows on the space of lattices. In this paper we prove this conjecture for arbitrary Ad-diagonalizable flows on irreducible quotients of semisimple Lie groups. The proof uses exponential mixing of the flow together with the method of integral inequalities for height functions on G/Γ. We also derive an application to jointly Dirichlet-Improvable systems of linear forms.

1. INTRODUCTION

1.1. **The set-up.** Let G be a connected Lie group, and let Γ be a lattice in G. Denote by X the homogeneous space G/Γ and by μ the G-invariant probability measure on X. For an unbounded subset F of G and a non-empty open subset O of X define the sets $E(F,O)$ and $\widetilde{E}(F,O)$ as follows:

$$\begin{aligned} & E(F,O) := \big\{x \in X : gx \notin O \ \forall\, g \in F\big\} \\ (1.1)\qquad \subset\ & \widetilde{E}(F,O) := \big\{x \in X : \exists \text{ compact } Q \subset G \text{ such that } gx \notin O \ \forall\, g \in F \smallsetminus Q\big\} \\ & \qquad = \bigcup_{\text{compact } Q\subset G} E(F \smallsetminus Q, O) \end{aligned}$$

of points in X whose F-trajectory always (resp., eventually) stays away from O. If F is a subgroup or a subsemigroup of G acting ergodically on (X,μ), then the set $\{gx : g \in F\}$ is dense for μ-almost all $x \in X$, in particular $\mu\big(\widetilde{E}(F,O)\big) = 0$.

The present paper studies the following natural question, asked several years ago by Mirzakhani: for a subgroup or sub-semigroup $F \subset G$, if the set $E(F,O)$ has measure zero, does it necessarily have less than full Hausdorff dimension? It is

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reasonable to conjecture that the answer is always affirmative; in other words, that the following 'Dimension Drop Conjecture' (DDC) holds:

CONJECTURE (Dimension Drop Conjecture). *If $F \subset G$ is a subsemigroup and O is an open subset of X, then either $E(F,O)$ has positive measure, or its Hausdorff dimension is less than the dimension of X.*

When X is compact it follows from the variational principle for measure-theoretic entropy, as outlined in [23, §7]; an effective argument using exponential mixing was developed in [19]. See also [1, Theorem 1.1 and Corollary 1.3] which explores the dimension drop phenomenon in a different setting. We remark that in the case when the space is non-compact there exists a conjecture proposed by Cheung [5]: for any non-quasiunipotent flow on a finite volume homogeneous space, the set of points whose orbit is *divergent* has positive codimension.

A standard approach to this circle of problems is to use the phenomenon of *non-escape of mass* on homogeneous spaces, going back to the work of Eskin–Margulis–Mozes and Eskin–Margulis, see [11, 10] and also [15]. This is precisely how the aforementioned Cheung's conjecture has been recently verified by Guan and Shi [13]; see also [2, 25] for some related work. However combining the non-escape of mass argument with an additional construction taking care of the compact part of the space is more involved. Previously this was done in the case when G is a simple Lie group of real rank 1 [9], and then, in the most recent work of the authors [20], when

$$X = \mathrm{SL}_{m+n}(\mathbb{R})/\mathrm{SL}_{m+n}(\mathbb{Z}) \text{ and } F = \left\{\mathrm{diag}(e^{nt},\dots,e^{nt},e^{-mt},\dots,e^{-mt}) : t \geq 0\right\}. \tag{1.2}$$

In this paper we generalize the approach of [20] by exhibiting two abstract assumptions sufficient for the validity of DDC. One takes care of the compact part of the space, while the other deals with the non-escape of mass.

1.2. **Main results.** Let now $F = \{g_t : t \geq 0\}$ be an Ad-diagonalizable one-parameter subsemigroup of G. A key role in our approach will be played by the *unstable horospherical subgroup* with respect to F, defined as

$$H := \left\{g \in G : \mathrm{dist}(g_t g g_{-t}, e) \to 0 \text{ as } t \to -\infty\right\}. \tag{1.3}$$

Equivalently, H is the Lie group whose Lie algebra is a direct sum of eigenspaces of $\mathrm{Ad}\, g_1$ corresponding to eigenvalues with absolute value > 1, see the proof of Corollary 1.2 in § 2 for more detail. More generally, we will work with connected subgroups P of H normalized by F and will give conditions sufficient for 'dimension drop along P-orbits'; that is, ensuring a nontrivial upper estimate for

$$\dim\left(\left\{h \in P : hx \in \widetilde{E}(F,O)\right\}\right),$$

where $x \in X$ is arbitrary, and O is a non-empty open subset of X.

The following theorem, which is the main result of the paper, is phrased using two abstract conditions, which we call properties (EEP) and (ENDP) of the subgroup P relative to the flow (X,F). These will be defined and commented upon in the next subsections.

THEOREM 1.1. *Let G be a Lie group, Γ a lattice in G, $X = G/\Gamma$, F a one-parameter* Ad*-diagonalizable subsemigroup of G, H as in* (1.3)*, and P a subgroup of H which is normalized by F and has properties* (EEP) *and* (ENDP) *with respect to the flow (X,F). Then for any non-empty open subset O of X one has*

$$\inf_{x\in X} \operatorname{codim}\big(\{h \in P : hx \in \widetilde{E}(F,O)\}\big) > 0.$$

The next corollary provides a quite general situation when the above theorem can be applied:

COROLLARY 1.2. *Let*

$$G = \prod_{i=1}^{n} G_i \quad \textit{and} \quad \Gamma = \prod_{i=1}^{n} \Gamma_i,$$

where each G_i is a connected semisimple Lie group without compact factors, and each Γ_i is a non-uniform irreducible lattice in G_i. Let $X = G/\Gamma$, and let F be a one-parameter Ad*-diagonalizable subsemigroup of G such that the projection of F to each G_i is unbounded. Then for any non-empty open subset O of X one has* $\dim \widetilde{E}(F,O) < \dim X$*; that is, DDC holds in this generality.*

We remark that the main result of [20] in the case (1.2) actually contains an effective upper bound for the dimension of $E(F,O)$. In the more general set-up of this paper it is also possible to make our estimates effective. We have decided not to overcomplicate the exposition with the proof of the stronger result; however see §10.1 for some indications of the proof.

Another remark is that, similarly to [20], we could have considered cyclic semigroups F of the form $\{g^t : t \in \mathbb{Z}_+\}$, where g is an Ad-diagonalizable element of G. Then, after replacing g_t with g^t in (1.4), the conclusions of Theorem 1.1 and Corollary 1.2 can be established for discrete-time actions, with minor modifications of the proofs.

1.3. **Notation.** We start by introducing some notation which will be used throughout the paper. Fix a right-invariant Riemannian structure on G, and denote by ‘dist’ the corresponding Riemannian metric, using the same notation for the induced Riemannian metric on X. In what follows, if P is a subgroup of G, we will denote by $B^P(r)$ the open ball in P of radius r centered at the identity element with respect to the metric on P corresponding to the Riemannian structure induced from G. We will let ν stand for the Haar measure on P normalized so that $\nu\big(B^P(1)\big) = 1$. For simplicity, we use $B(r)$ instead of $B^G(r)$ to denote a ball of radius r in G centered at the identity element. Also, $B(x,\rho)$ will stand for the open ball in X centered at $x \in X$ of radius ρ.

For $x \in X$ denote by π_x the map $G \to X$ given by $\pi_x(g) := gx$, and by $r_0(x)$ the *injectivity radius* of x:

$$r_0(x) := \sup\big\{r > 0 : \pi_x \text{ is injective on } B(r)\big\}.$$

If K is a subset of X, let us denote by $r_0(K)$ the *injectivity radius* of K:

$$r_0(K) := \inf_{x\in K} r_0(x) = \sup\{r>0 : \pi_x \text{ is injective on } B(r) \ \ \forall\, x\in K\};$$

it is known that $r_0(K) > 0$ if and only if K is bounded.

The notation $A \gg B$ where A and B are quantities depending on certain parameters, will mean $A \geq CB$, where C is a constant independent of those parameters.

Throughout the proof we will pay close attention to translates of P-orbits in X by g_t. It will be convenient to use the following notation: if f is a function on P and $t \geq 0$, we will define the integral operator $I_{f,t}$ acting on functions ψ on X via

$$(I_{f,t}\psi)(x) := \int_P f(h)\psi(g_t hx)\, d\nu(h). \tag{1.4}$$

In other words, $(I_{f,t}\psi)(x)$ is the integral of ψ with respect to the g_t-translate of the π_x-pushforward of the signed measure $f\,d\nu$. When $f = 1_B$ for a subset B of P, we will write

$$I_{B,t} := \int_B \psi(g_t hx)\, d\nu(h)$$

in place of $I_{1_B,t}$.

1.4. **Exponential mixing and effective equidistribution.** The first ingredient of our proof is the effective equidistribution of g_t-translates of P-orbits on X. To introduce this property we will work with Sobolev spaces of functions on X. Let us define

$$C_2^\infty(X) := \{h \in C^\infty(X) : \|h\|_{\ell,2} < \infty \text{ for any } \ell \in \mathbb{Z}_+\},$$

where $\|\cdot\|_{\ell,2}$ is the "*L^2, order ℓ" Sobolev norm* (see §4 for more detail). Now let us introduce the following.

DEFINITION 1.3 ([19])**.** Say that a subgroup P of G has *Effective Equidistribution Property* (EEP) with respect to the flow (X,F) if there exist constants $a,b,\lambda>0$ and $\ell\in\mathbb{N}$ such that for any $x\in X$ and $t>0$ with

$$t \geq a + b\log\frac{1}{r_0(x)}, \tag{1.5}$$

any $f \in C^\infty(P)$ with $\operatorname{supp} f \subset B^P(1)$ and any $\psi \in C_2^\infty(X)$ it holds that

$$\left|(I_{f,t}\psi)(x) - \int_P f\,d\nu \int_X \psi\,d\mu\right| \ll \max\left(\|\psi\|_{C^1}, \|\psi\|_{\ell,2}\right)\cdot \|f\|_{C^\ell}\cdot e^{-\lambda t}. \tag{1.6}$$

Note that the constants a,b in (1.5) and the implicit constant in (1.6) are allowed to depend on P and F, but not on x,t,f and ψ.

A widely used principle in homogeneous dynamics, which essentially originated from Margulis' doctoral thesis, see [24], is the concept that mixing implies the equidistribution of unstable leaves, that is, the orbits of H as defined in (1.3). Effective versions of this principle have been exploited in [17, 18]. Specifically, let us say that a flow (X,F) is *exponentially mixing* if there exist $\gamma>0$ and $\ell\in\mathbb{Z}_+$ such that for any $\varphi,\psi\in C_2^\infty(X)$ and for any $t\ge 0$ one has

$$\left|(g_t\varphi,\psi)-\int_X\varphi\,d\mu\int_X\psi\,d\mu\right|\ll e^{-\gamma t}\|\varphi\|_{\ell,2}\|\psi\|_{\ell,2}.$$

This property for non-quasiunipotent flows follows from the *strong spectral gap* of the regular representation of G, see [17]; the latter is known to hold for quotients of semisimple Lie groups without compact factors by irreducible lattices, see [16, p. 285].

The fact that property (EEP) for expanding horospherical subgroups follows from exponential mixing was established in [19] by a variation of the method developed in [17]:

THEOREM 1.4 ([19, Theorem 2.5])**.** *Let G be a Lie group, Γ a lattice in G, and let F be a one-parameter subsemigroup of G whose action on $X=G/\Gamma$ is exponentially mixing. Then H as in* (1.3) *satisfies property* (EEP) with respect to the flow (X,F).

1.5. **Height functions and integral inequalities.** The second ingredient of our proof is studying excursions of g_t-translates of P-orbits in X outside of compact subsets. For that it is helpful to have a family of positive functions on X which grow at infinity and behave nicely with respect to integral operators of type (1.4). This is done via the method of integral inequalities which goes back to [11] and [10], and has been recently applied by Guan and Shi [13] to upper estimates for the Hausdorff dimension of the set of points of X with divergent g_t-trajectories. To state their result, let us say that a non-negative continuous function u on X is a *height function* if it is proper, that is $u(x)\to\infty$ if and only if $x\to\infty$ in X, and *regular*, that is there exists a non-empty neighborhood B of identity in G and $C>0$ such that

$$u(hx)\le Cu(x)\text{ for every }h\in B\text{ and all }x\in X; \tag{1.7}$$

equivalently, if for any bounded $B\subset G$ there exists $C>0$ such that (1.7) holds. Also let us say that u satisfies the (c,d)-*Margulis inequality* with respect to an operator $I: C(X)\to C(X)$ if for all $x\in X$ one has

$$(Iu)(x)\le cu(x)+d.$$

See [12, 28], where functions u satisfying the (c,d)-Margulis inequality for some $c<1$ and $d\in\mathbb{R}$ are called *Margulis functions*. With this terminology, let us introduce the following definition.

DEFINITION 1.5. Say that a subgroup P of G has *Effective Non-Divergence Property* (ENDP) with respect to the flow (X,F) if there exists $0<c_0<1$ and $t_0>0$ such that for any $t\ge t_0$ one can find $d_t>0$ and a height function u_t such that u_t satisfies the (c_0,d_t)-Margulis inequality with respect to $I_{B^P(1),t}$.

In the course of proving the main result of [20] the above property was shown in the case (1.2), see [20, Proposition 3.4.]. The proof followed a construction from [15] and used functions on the space of lattices coming from the work of Eskin, Margulis and Mozes [11]. To get more examples, we will quote the following result from [13] (see also [29, Lemma 4.1]):

THEOREM 1.6 ([13, Lemma 4.3]). *Let Let G, Γ, X and F be as in Corollary 1.2, let H be as in (1.3), and let $B = B^H(1)$. Then there exist $\alpha, t_0 > 0$ and, for any $t \geq t_0$, a height function u_t on X and $d_t \in \mathbb{R}$ such that the function u_t satisfies an $(e^{-\alpha t}, d_t)$-Margulis inequality with respect to $I_{B,t}$. Consequently, H has property* (ENDP) *with respect to (X, F).*

We remark that, since the flow (X, F) as in the above theorem is exponentially mixing, the expanding horospherical subgroup H has property (EEP) with respect to (X, F) as well.

1.6. **The structure of the paper.** In the next section we state a technical theorem (Theorem 2.1) and show how it implies Theorem 1.1 and Corollary 1.2. The proof of Theorem 2.1 occupies the bulk of the paper. It has two main ingredients: one deals with orbits staying inside a fixed compact subset of X, which are handled in §4–5 with the help of the effective equidistribution assumption. The other one (§ 6–7) takes care of orbits venturing far away into the cusp of X; there we use the effective non-divergence property via the method of integral inequalities for height functions on X. The two ingredients are combined in §8–9. Some concluding remarks are presented in §10. In particular, §10.3 contains an application to joint Dirichlet improvement in Diophantine approximation. Namely, there we define the set $\mathbf{DI}_{m,n}^{(k)}(c)$ of c-Dirichlet improvable k-tuples of $m \times n$ matrices, and prove that the Hausdorff codimension of $\mathbf{DI}_{m,n}^{(k)}(c)$ is positive whenever $c < 1$. The case $k = 1$ was considered in [20] and was derived from a solution of DDC for the case (1.2).

2. THEOREM 2.1 ⇒ THEOREM 1.1 ⇒ COROLLARY 1.2

From this point and until the end of §9, we let G be a Lie group, Γ a lattice in G, $X = G/\Gamma$, $F = \{g_t : t \geq 0\}$ a one-parameter Ad-diagonalizable subsemigroup of G, H the unstable horospherical subgroup relative to F, and P a subgroup of H which is normalized by F. Throughout the argument we will be assuming that P satisfies either (EEP) or (ENDP) with respect to the flow (X, F), or, at the end of the proof, both of the properties.

Let us introduce the following notation: for a non-empty open subset O of X and $r > 0$ denote by $\sigma_r O$ the *inner r-core* of O, defined as

$$\sigma_r O := \left\{ x \in X : \operatorname{dist}(x, O^c) > r \right\}. \tag{2.1}$$

This is an open subset of O, whose measure is close to $\mu(O)$ for small enough values of r.

Furthermore, for a closed subset S of X denote by $\partial_r S$ the *r-neighborhood* of S, that is,

$$\partial_r S := \left\{ x \in X : \operatorname{dist}(x, S) < r \right\}.$$

In particular, for $z \in X$ we have $\partial_r\{z\} = B(z,r)$, the open ball of radius r centered at z. Note that we always have

$$\partial_r S \subset \left(\sigma_r(S^c)\right)^c \text{ for all } S \subset X,\ r > 0. \tag{2.2}$$

Also define $\mathfrak{p}$ to be the Lie algebra of P, and let

$$\lambda_{\min} := \min\left\{\lambda : \lambda \text{ is an eigenvalue of } \mathrm{ad}_{g_1}|_{\mathfrak{p}}\right\} \tag{2.3}$$

and

$$\lambda_{\max} := \max\left\{\lambda : \lambda \text{ is an eigenvalue of } \mathrm{ad}_{g_1}|_{\mathfrak{p}}\right\}. \tag{2.4}$$

Note that all eigenvalues of the restriction of ad_{g_1} to $\mathfrak{p}$, including $\lambda_{\min}$ and $\lambda_{\max}$, are positive.

In this section we derive Theorem 1.1 from the following crucial but technical theorem. It subdivides the argument into two cases: the first one deals with trajectories which venture out of some big compact set Q along certain arithmetic progressions, while the second, complementary case can be taken care of using recurrence of trajectories to this compact set Q.

THEOREM 2.1. *Let $G, \Gamma, X, F = \{g_t : t \geq 0\}$ and P be as in Theorem 1.1, and let $p = \dim P$. Then there exist*

$$r_*, C_1, C_2, a', b', \lambda > 0$$

such that the following holds:
For any $0 < c < 1$ there exist $t > 0$ and a compact subset Q of X such that:

1. *For all $x \in X$, and for all $2 \leq k \in \mathbb{N}$, the set*

$$S(k,t,x) := \left\{h \in P : g_{Nkt}hx \notin Q\ \forall N \in \mathbb{N}\right\} \tag{2.5}$$

   satisfies

$$\operatorname{codim} S(k,t,x) \geq \frac{1}{\lambda_{\max} kt} \log \frac{1-c}{4c}. \tag{2.6}$$

2. *For all $2 \leq k \in \mathbb{N}$, all r satisfying*

$$e^{\frac{a'-kt}{b'}} \leq r < \frac{1}{4}\min\left(r_0(\partial_1 Q), r_*\right), \tag{2.7}$$

   all $\theta \in \left[r, \frac{r_*}{2}\right]$, all $x \in X$, and for all open subsets O of X we have

$$\operatorname{codim}\left(\left\{h \in P \smallsetminus S(k,t,x) : hx \in \widetilde{E}(F,O)\right\}\right) \geq \frac{\mu\left(\sigma_{4\theta}O\right) - \frac{8C_1}{\theta^p}\frac{\sqrt{c}}{1-c} - \frac{C_2}{r^p}e^{-\lambda kt}}{\lambda_{\max} kt}. \tag{2.8}$$

We now show how the two estimates are put together.

Proof of Theorem 1.1 assuming Theorem 2.1. Recall that we are given the constants $r_*, C_1, C_2, a', b', \lambda > 0$ such that statements (1), (2) of Theorem 2.1 hold. Let O be an open subset of X. Define

$$\theta_O := \sup\left\{0 < \theta \leq 1 : \mu(\sigma_{4\theta}O) \geq \frac{1}{2}\mu(O)\right\}, \tag{2.9}$$

then put $\theta := \min(\theta_O, \frac{r_*}{2})$ and

$$c := \min\left(\frac{1}{4e^{1/2}+1}, \left(\frac{\mu(O)}{128C_1}\cdot\theta^p\right)^2\right). \tag{2.10}$$

Now choose t and Q as in the assumption of Theorem 2.1. Then in view of (2.10), statement (1) of Theorem 2.1 readily implies that for any $2 \le k \in \mathbb{N}$ one has

$$\operatorname{codim} S(k,t,x) \ge \frac{1}{2\lambda_{\max} k t}. \tag{2.11}$$

Next, let

$$r := \frac{1}{4}\min\big(r_0(\partial_1 Q), r_*, \theta_O\big). \tag{2.12}$$

Clearly the second inequality in (2.7) is then satisfied. Now take $2 \le k \in \mathbb{N}$ sufficiently large so that

$$e^{\frac{a'-kt}{b'}} \le r \quad \text{and} \quad \frac{C_2}{r^p}e^{-\lambda k t} \le \frac{\mu(O)}{8}; \tag{2.13}$$

this will imply the first inequality in (2.7). Also it is easy to see from (2.12) and the definition of θ that $\theta \in \left[r, \frac{r_*}{2}\right]$; hence (2.8) holds.

Observe that since $\theta \le \theta_O$, in view of (2.9) we have

$$\mu\big(\sigma_{4\theta}O\big) \ge \frac{\mu(O)}{2}. \tag{2.14}$$

The definition of c implies

$$\frac{8C_1}{\theta^p}\frac{\sqrt{c}}{1-c} \underset{c<1/2}{<} \frac{8C_1}{\theta^p}\cdot 2\sqrt{c} \underset{(2.10)}{\le} \frac{\mu(O)}{8}. \tag{2.15}$$

Hence, by combining (2.13), (2.14) and (2.15), we conclude that the numerator in the right hand side of (2.8) is not less than $\mu(O)/4$. Thus (2.8) implies

$$\operatorname{codim}\big(\{h \in P \smallsetminus S(k,t,x) : hx \in \widetilde{E}(F,O)\}\big) \ge \frac{\mu(O)}{4\lambda_{\max} k t}.$$

Combining it with (2.11), we obtain

$$\operatorname{codim}\big(\widetilde{E}(F,O) \cap Px\big) \ge \frac{1}{4\lambda_{\max} k t}\min\big(2, \mu(O)\big) = \frac{\mu(O)}{4\lambda_{\max} k t},$$

which is a positive number independent of x. This finishes the proof. □

Proof of Corollary 1.2. Let G, Γ, X and F be as in Corollary 1.2, and let H be as in (1.3). Let $\mathfrak{g}$ be the Lie algebra of G, $\mathfrak{g}_{\mathbb{C}}$ its complexification, and for $\lambda \in \mathbb{C}$, let E_λ be the eigenspace of $\operatorname{Ad} g_1$ corresponding to λ. Let $\mathfrak{h}$, $\mathfrak{h}^0$, $\mathfrak{h}^-$ be the subalgebras of $\mathfrak{g}$ with complexifications:

$$\mathfrak{h}_{\mathbb{C}} = \operatorname{span}(E_\lambda : |\lambda| > 1),\ \mathfrak{h}^0_{\mathbb{C}} = \operatorname{span}(E_\lambda : |\lambda| = 1),\ \mathfrak{h}^-_{\mathbb{C}} = \operatorname{span}(E_\lambda : |\lambda| < 1).$$

Note that $\mathfrak{h}$ is the Lie algebra of H. Moreover, $\mathfrak{h}^-$ is the Lie algebra of the *stable horospherical subgroup* defined by

$$H^- := \{h \in G : g_t h g_{-t} \to e \ as\ t \to +\infty\}.$$

Since $\mathrm{Ad}\, g_1$ is assumed to be diagonalizable, $\mathfrak{g}$ is the direct sum of $\mathfrak{h}$, $\mathfrak{h}^0$ and $\mathfrak{h}^-$. Hence, if we denote the group H^-H^0 by $\widetilde{H}$, G is locally (at a neighborhood of identity) a direct product of H and $\widetilde{H}$.

Now let O be a non-empty open subset of X, and fix $0<\rho<1$ such that the following properties are satisfied: the multiplication map $\widetilde{H}\times H\to G$ is one to one on $B^{\widetilde{H}}(\rho)\times B^H(\rho)$,

$$g_t B^{\widetilde{H}}(\rho) g_{-t} \subset B^{\widetilde{H}}(2\rho) \text{ for any } t\ge 0, \tag{2.16}$$

and

$$\sigma_{2\rho} O \neq \varnothing. \tag{2.17}$$

Note that (2.16) can be satisfied since F is Ad-diagonalizable and the restriction of the map $g\to g_t g g_{-t}$, $t\ge 0$, to $\widetilde{H}$ is non-expanding. Also (2.17) can be achieved, since, in view of (2.1), $\sigma_r O$ is non-empty when $r>0$ is sufficiently small.

Now in view of (2.17), we can apply Theorem 1.1 with O replaced with $\sigma_{2\rho}O$ and conclude that there exists $\varepsilon>0$ such that

$$\dim\left(\left\{h\in H : hx\in \widetilde{E}(F,\sigma_{2\rho}O)\right\}\right) = \dim H-\varepsilon < \dim H. \tag{2.18}$$

Choose $s>0$ such that $B(s)$ is contained in the product $B^{\widetilde{H}}(\rho)B^H(\rho)$, and for $x\in X$ denote

$$E_x := \left\{g\in B(s) : gx\in \widetilde{E}(F,O)\right\}.$$

In view of the countable stability of Hausdorff dimension, in order to prove the corollary it suffices to prove that for any $x\in X$,

$$\dim E_x \le \dim X-\varepsilon,$$

where ε is as in (2.18); note that $\widetilde{E}(F,O)$ can be covered by countably many sets $\{gx : g\in E_x\}$, with the maps $\pi_x : E_x\to X$ being Lipschitz and at most finite-to-one. Since every $g\in B(s)$ can be written as $g=h'h$, where $h'\in B^{\widetilde{H}}(\rho)$ and $h\in B^H(\rho)$, for any $y\in X$ we can write

$$\begin{aligned}\mathrm{dist}(g_t gx, y) &\le \mathrm{dist}(g_t h' hx, g_t hx) + \mathrm{dist}(g_t hx, y)\\ &= \mathrm{dist}\big(g_t h' g_{-t} g_t hx, g_t hx\big) + \mathrm{dist}(g_t hx, y).\end{aligned}$$

Hence in view of (2.16), $g\in E_x$ implies that hx belongs to $E(F,\sigma_{2\rho}O)$, and by using Wegmann's Product Theorem [31] we have

$$\begin{aligned}\dim E_x &\le \dim\Big(\{h\in B^H(\rho) : hx\in E(F,\sigma_{2\rho}O)\}\times B^{\widetilde{H}}(\rho)\Big)\\ &\le \dim\big(\{h\in B^H(\rho) : hx\in \widetilde{E}(F,\sigma_{2\rho}O)\}\big) + \dim \widetilde{H}\\ &\underset{(2.18)}{\le} \dim H-\varepsilon+\dim\widetilde{H} = \dim X-\varepsilon.\end{aligned}$$

This ends the proof of the corollary. □

3. TESSELLATIONS AND BOWEN BOXES

Following [17], say that an open subset V of P is a *tessellation domain* relative to a countable subset Λ of P if

- $\nu(\partial V) = 0$;
- $V\gamma_1 \cap V\gamma_2 = \varnothing$ for different $\gamma_1, \gamma_2 \in \Lambda$;
- $P = \bigcup_{\gamma\in\Lambda} \overline{V}\gamma$.

Note that P is a connected simply connected nilpotent Lie group. Denote $\mathfrak{p} := \mathrm{Lie}(P)$ and $p := \dim P$. As shown in [17, Proposition 3.3], one can choose a basis of $\mathfrak{p}$ such that for any $r > 0$, $\exp\big(rI_{\mathfrak{p}}\big)$, where $I_{\mathfrak{p}} \subset \mathfrak{p}$ is the cube centered at 0 with side length 1 with respect to that basis, is a tessellation domain relative to some discrete subset of P. Let us denote

$$V_r := \exp\left(\frac{r}{4\sqrt{p}} I_{\mathfrak{p}}\right) \tag{3.1}$$

and choose a countable $\Lambda_r \subset P$ such that V_r is a tessellation domain relative to Λ_r.

Take $0 < r_* < 1/4$ such that the exponential map from $\mathfrak{p}$ to P is 2-bi-Lipschitz on $B^{\mathfrak{p}}(r_*)$. The latter implies that

$$B^P\left(\frac{r}{16\sqrt{p}}\right) \subset V_r \subset B^P\left(\frac{r}{4}\right) \quad \text{for any } 0 < r \le r_*. \tag{3.2}$$

Also, the measure ν and the pushforward of the Lebesgue measure Leb on $\mathfrak{p}$ are absolutely continuous with respect to each other with locally bounded Radon–Nikodym derivative. This implies that there exist $0 < c_1 < c_2$ such that

$$c_1 \mathrm{Leb}(A) \le \nu\big(\exp(A)\big) \le c_2 \mathrm{Leb}(A) \quad \forall \text{ measurable } A \subset B^{\mathfrak{p}}(1). \tag{3.3}$$

In what follows we will be taking $\theta \ge r$ and approximating V_θ by the union of Λ_r-translates of V_r. The following estimate will be helpful:

LEMMA 3.1. *For any* $0 < r \le \theta \le r_*/2$

$$\#\{\gamma \in \Lambda_r : V_r\gamma \cap V_\theta \ne \varnothing\} \le \frac{c_2}{c_1}\left(\frac{\theta}{r} + 8\sqrt{p}\right)^p.$$

Proof. Note that if $V_r\gamma$ intersects V_θ, then in view of (3.2) we must have $V_r\gamma \subset \partial_{r/2} V_\theta$. Hence, using the fact that V_r is a tessellation domain relative to Λ_r, we have

$$\begin{aligned}\#\{\gamma \in \Lambda_r : V_r\gamma \cap V_\theta \ne \varnothing\} &\le \frac{\nu(\partial_{r/2} V_\theta)}{\nu(V_r)} \underset{(3.3)}{\le} \frac{c_2}{c_1} \cdot \frac{\mathrm{Leb}\Big(\partial_r\big(\frac{\theta}{4\sqrt{p}} I_{\mathfrak{p}}\big)\Big)}{\mathrm{Leb}\Big(\frac{r}{4\sqrt{p}} I_{\mathfrak{p}}\Big)} \\ &= \frac{c_2}{c_1} \cdot \frac{\Big(\frac{\theta}{4\sqrt{p}} + 2r\Big)^p}{\Big(\frac{r}{4\sqrt{p}}\Big)^p} = \frac{c_2}{c_1}\left(\frac{\theta}{r} + 8\sqrt{p}\right)^p,\end{aligned}$$

where in the second inequality above we were able to use the bi-Lipschitz property of exp since

$$\partial_r\big(\frac{\theta}{4\sqrt{p}}I_{\mathfrak{p}}\big)\subset B^{\mathfrak{p}}\left(\frac{\theta}{8}+r\right)\subset B^{\mathfrak{p}}(r_*).$$

This finishes the proof. □

Recall that all eigenvalues of the restriction of ad_{g_1} to $\mathfrak{p}$ are positive. Using the bi-Lipschitz property of exp, one can conclude that

$$\begin{aligned}\operatorname{diam}(g_{-t}V_r g_t)&\le 2\cdot\operatorname{diam}\left(\exp\left(\frac{re^{-\lambda_{\min}t}}{4\sqrt{p}}I_{\mathfrak{p}}\right)\right)\\ &\le\frac{re^{-\lambda_{\min}t}}{2}\quad\text{for any } 0<r\le r_* \text{ and any } t\ge 0,\end{aligned}\tag{3.4}$$

where $\lambda_{\min}$ is as in (2.3). Also let $\delta:=\mathrm{Tr}\,\mathrm{ad}_{g_1}|_{\mathfrak{p}}$; clearly one then has

$$\nu(g_{-t}Ag_t)=e^{-\delta t}\nu(A)\text{ for any measurable } A\subset P.\tag{3.5}$$

Let us now define a *Bowen* (t,r)*-box* in P to be a set of the form $g_{-t}\overline{V_r}\gamma g_t$ for some $\gamma\in P$ and $t\ge 0$. The following lemma, analogous to [17, Proposition 3.4] and [19, Lemma 6.1], gives an upper bound for the number of $\gamma\in\Lambda_r$ such that the Bowen box $g_{-t}\overline{V_r}\gamma g_t$ has non-empty intersection with $\overline{V_r}$:

LEMMA 3.2. *For any* $0<r\le r_*/2$ *and*

$$t\ge\frac{\log(8\sqrt{p})}{\lambda_{\min}},\tag{3.6}$$

one has

$$\#\{\gamma\in\Lambda_r : g_{-t}\overline{V_r}\gamma g_t\cap\overline{V_r}\ne\varnothing\}\le e^{\delta t}\left(1+C_0e^{-\lambda_{\min}t}\right),$$

where

$$C_0:=\frac{2^{p+3}p^{3/2}c_2}{c_1}.\tag{3.7}$$

Proof. Let $0<r\le r_*/2$. One has:

$$\begin{aligned}&\#\left\{\gamma\in\Lambda_r : g_{-t}\overline{V_r}\gamma g_t\cap\overline{V_r}\ne\varnothing\right\}\\ &\quad=\#\left\{\gamma\in\Lambda_r : g_{-t}V_r\gamma g_t\subset V_r\right\}+\#\left\{\gamma\in\Lambda_r : g_{-t}\overline{V_r}\gamma g_t\cap\partial V_r\ne\varnothing\right\}.\end{aligned}$$

Since V_r is a tessellation domain of P relative to Λ_r, the first term in the above sum is not greater than $\frac{\nu(V_r)}{\nu(g_{-t}V_rg_t)}=e^{\delta t}$, while, in view of (3.4), the second term is not greater than

$$\frac{\nu\left(\partial_{\frac{re^{-\lambda_{\min}t}}{2}}(\partial V_r)\right)}{\nu(g_{-t}V_rg_t)}\underset{\substack{(3.3)\\(3.5)}}{\le}c_2e^{\delta t}\,\frac{\mathrm{Leb}\left(\partial_{re^{-\lambda_{\min}t}}\left(\partial\left(\frac{r}{4\sqrt{p}}I_{\mathfrak{p}}\right)\right)\right)}{\nu(V_r)}.\tag{3.8}$$

(Here we used the fact that

$$\partial_{re^{-\lambda_{\min}t}}\left(\frac{r}{4\sqrt{p}}I_{\mathfrak{p}}\right)\subset B^{\mathfrak{p}}\left(\frac{r}{8}+re^{-\lambda_{\min}t}\right)\subset B^{\mathfrak{p}}\left(\frac{9r}{8}\right)\subset B^{\mathfrak{p}}\left(\frac{9r_*}{16}\right)\subset B^{\mathfrak{p}}(1),$$

hence we can use the 2-bi-Lipschitz property of exp to conclude that

$$\exp\left(\partial_{re^{-\lambda_{\min} t}}\left(\partial(\tfrac{r}{4\sqrt{p}} I_{\mathfrak{p}})\right)\right) \supset \partial_{\frac{re^{-\lambda_{\min} t}}{2}}(\partial V_r),$$

and the estimate (3.3) is applicable.) It is easy to see that the numerator in the right hand side of (3.8) is not greater than

$$\begin{aligned}
&\left(\frac{r}{4\sqrt{p}} + 2re^{-\lambda_{\min} t}\right)^p - \left(\frac{r}{2\sqrt{p}} - 2re^{-\lambda_{\min} t}\right)^p \\
&\quad \underset{\text{(MVT)}}{\le} 4re^{-\lambda_{\min} t} p \left(\frac{r}{4\sqrt{p}} + 2re^{-\lambda_{\min} t}\right)^{p-1} \\
&\quad \underset{(3.6)}{\le} 4pre^{-\lambda_{\min} t}\left(\frac{r}{2\sqrt{p}}\right)^{p-1} = 2^{p+3} p^{3/2} \left(\frac{r}{4\sqrt{p}}\right)^p e^{-\lambda_{\min} t} \\
&\quad \underset{\substack{(3.1)\\(3.3)}}{\le} \frac{2^{p+3} p^{3/2}}{c_1} \nu(V_r) e^{-\lambda_{\min} t},
\end{aligned}$$

which finishes the proof. □

We conclude the section with a lemma, which is a slight modification of [19, Lemma 6.4], to be used at the last stage of the proof for switching from coverings by Bowen boxes to coverings by balls.

LEMMA 3.3. *For any $t > 0$ and any $0 < r \le r_*$, any Bowen (t, r)-box in P can be covered with at most $e^{(p\lambda_{\max} - \delta)t}$ balls of radius $re^{-\lambda_{\max} t}$, where $\lambda_{\max}$ is as in (2.4).*

Proof. Using the 2-bi-Lipschitz property of exp again, one can cover $g_{-t}\overline{V_r}g_t$ by at most as many balls of radius $re^{-\lambda_{\max} t}$, as the number of translates of $\frac{re^{-\lambda_{\max} t}}{\sqrt{p}} I_{\mathfrak{p}}$ needed to cover $\mathrm{Ad}(g_{-t})\left(\frac{r}{4\sqrt{p}}\overline{I_{\mathfrak{p}}}\right)$. The latter can be written as the direct product of intervals $I_1, \dots, I_p$, where $\min_i \mathrm{Leb}(I_i) = \frac{re^{-\lambda_{\max} t}}{4\sqrt{p}}$. Clearly each I_i can be covered by the union of intervals of length $\frac{re^{-\lambda_{\max} t}}{\sqrt{p}}$ whose total measure is at most $4\mathrm{Leb}(I_i)$. Hence $\mathrm{Ad}(g_{-t})\left(\frac{r}{4\sqrt{p}}\overline{I_{\mathfrak{p}}}\right)$ can be covered by at most

$$\frac{4^p \mathrm{Leb}\left(\mathrm{Ad}(g_{-t})\left(\frac{r}{4\sqrt{p}} I_{\mathfrak{p}}\right)\right)}{\mathrm{Leb}\left(\frac{re^{-\lambda_{\max} t}}{\sqrt{p}} I_{\mathfrak{p}}\right)} = \frac{4^p e^{-\delta t}\left(\frac{r}{4\sqrt{p}}\right)^p}{\left(\frac{re^{-\lambda_{\max} t}}{\sqrt{p}}\right)^p} = e^{(p\lambda_{\max} - \delta)t}$$

translates of $\frac{re^{-\lambda_{\max} t}}{\sqrt{p}} I_{\mathfrak{p}}$, which finished the proof of the lemma. □

4. PROPERTY (EEP) AND A MEASURE ESTIMATE

Our goal in this section is to use property (EEP) of P to find a lower bound for the measure of the sets of the type

$$\{h \in V_r : g_t h x \in O\}, \tag{4.1}$$

where $x \in X$, O is a subset of X, $r > 0$ is small enough, and $t > 0$ is large enough. This step is similar to [19, Theorem 4.1], where balls in P were used in place of tessellation domains V_r. For our new proof the use of tessellations is crucial; to make the paper self-contained we present a complete argument.

We start with the definition of Sobolev spaces. Let L be a Lie group and Δ a discrete subgroup of L such that L/Δ admits an L-invariant measure. Fix a basis $\{Y_1,\ldots,Y_N\}$ for the Lie algebra of L, and, given $h \in C^\infty(L/\Delta)$, $k \in \mathbb{N}$ and $\ell \in \mathbb{Z}_+$, define the "*L^k, order ℓ" Sobolev norm* $\|h\|_{\ell,k}$ of h by

$$\|h\|_{\ell,k} \overset{\text{def}}{=} \sum_{|\alpha|\le\ell} \|D^\alpha h\|_k,$$

where $\|\cdot\|_k$ stands for the L^k-norm, $\alpha = (\alpha_1,\ldots,\alpha_N)$ is a multiindex, $|\alpha| = \sum_{i=1}^N \alpha_i$, and D^α is a differential operator of order $|\alpha|$ which is a monomial in $Y_1,\ldots,Y_N$, namely $D^\alpha = Y_1^{\alpha_1}\cdots Y_N^{\alpha_N}$. This definition depends on the basis, however, a change of basis would only distort $\|\cdot\|_{\ell,k}$ by a bounded factor. We will also use the operators D^α to define C^ℓ norms of smooth functions f on L/Δ:

$$\|f\|_{C^\ell} := \sup_{x\in L/\Delta,\ |\alpha|\le\ell} |D^\alpha f(x)|.$$

We will work with Sobolev spaces of functions on G (letting $L = G$ and $\Delta = \{e\}$), on P (letting $G = P$ and $\Delta = \{e\}$), as well on $X = G/\Gamma$.

The next two lemmas provide a way to approximate subsets of G and X respectively by smooth functions. We start with a basic lemma constructing test functions supported inside small neighborhoods of identity in G. It is an immediate corollary of [14, Lemma 2.6], see also [17, Lemma 2.4.7(b)].

LEMMA 4.1. *For each $\ell \in \mathbb{Z}_+$ there exists $M_{G,\ell} \ge 1$ with the following property: for any $0<\varepsilon<1$ there exists a nonnegative smooth function φ_ε on G such that*

1. *the support of φ_ε is contained in $B(\varepsilon)$;*
2. $\|\varphi_\varepsilon\|_1 = 1$*;*
3. $\|\varphi_\varepsilon\|_{\ell,1} \le M_{G,\ell}\cdot\varepsilon^{-\ell}$.

The next lemma is a slightly easier version of [19, Lemma 5.2]; we provide the proof for the sake of completeness. Before stating the lemma, we introduce the notation 1_E to denote the characteristic function of a set E.

LEMMA 4.2. *For any $\ell \in \mathbb{Z}_+$ there exists a constant $M_\ell > 0$ (depending only on ℓ and G) such that for any nonempty open subset O of X and any $0<\varepsilon<1$ one can find a nonnegative function $\psi_\varepsilon \in C^\infty(X)$ such that*

1. $1_{\sigma_\varepsilon O} \le \psi_\varepsilon \le 1_O$*;*
2. $\max\left(\|\psi_\varepsilon\|_{\ell,2}, \|\psi_\varepsilon\|_{C^\ell}\right) \le M_\ell\varepsilon^{-\ell}$.

Proof. Let O be a nonempty open subset of X, and let $0<\varepsilon<1$. Now take $\psi_\varepsilon = \varphi_{\varepsilon/2} * 1_{\sigma_{\varepsilon/2}O}$, where $\varphi_{\varepsilon/2}$ is as in Lemma 4.1. It follows from the definition of ψ_ε and the normalization $\|\varphi_\varepsilon\|_1 = 1$ that $\psi_\varepsilon(x) \le 1$ for all x. Also, since $\varphi_{\varepsilon/2}$ is supported on $B(\varepsilon/2)$, the support of the function ψ_ε is contained in $\partial_{\varepsilon/2}\sigma_{\varepsilon/2}O \subset O$, which implies $\psi_\varepsilon \le 1_O$. Furthermore, if $x \in \sigma_\varepsilon O$ and $g \in B(\varepsilon/2)$, then $gx \in \partial_{\varepsilon/2}\sigma_\varepsilon O \subset \sigma_{\varepsilon/2}O$, i.e., $1_{\sigma_{\varepsilon/2}O}(gx) = 1$. Therefore,

$$\psi_\varepsilon(x) = \int_G \varphi_{\varepsilon/2}(g) 1_{\sigma_{\varepsilon/2}O}(gx)\, d\mu_G = \int_G \varphi_{\varepsilon/2}(g)\, d\mu_G = 1.$$

Hence property (1) holds.

Let $\alpha = (\alpha_1, \dots, \alpha_N)$ be such that $|\alpha| \le \ell$. For any $x \in X$ we have

$$\begin{aligned} \left|D^\alpha \psi_\varepsilon(x)\right| &= \left|D^\alpha (\varphi_{\varepsilon/2} * 1_{\sigma_{\varepsilon/2}O})(x)\right| = \left|D^\alpha \varphi_{\varepsilon/2} * 1_{\sigma_{\varepsilon/2}O}(x)\right| \\ &\le \left\|D^\alpha \varphi_{\varepsilon/2}\right\|_1 \le \left\|\varphi_{\varepsilon/2}\right\|_{\ell,1} \le M_{G,\ell}(\tfrac{\varepsilon}{2})^{-\ell}, \end{aligned}$$

where $M_{G,\ell}$ is as in Lemma 4.1. Likewise, by Young's inequality,

$$\begin{aligned} \left\|D^\alpha \psi_\varepsilon\right\|_2 &\le \|D^\alpha \varphi_{\varepsilon/2} * 1_{\sigma_{\varepsilon/2}O}\|_2 \le \left\|D^\alpha \varphi_{\varepsilon/2}\right\|_1 \cdot \left\|1_{\sigma_{\varepsilon/2}O}\right\|_2 \\ &\le \left\|D^\alpha \varphi_{\varepsilon/2}\right\|_1 \le M_{G,\ell}(\tfrac{\varepsilon}{2})^{-\ell}, \end{aligned}$$

which implies (2) with $M_\ell = 2^\ell M_{G,\ell}$. □

The next lemma is a modification of [19, Lemma 5.3] where we replace balls of radius r in P with V_r; we omit the proof.

LEMMA 4.3. *Let r^* be as in (3.2). For any $\ell \in \mathbb{Z}_+$ there exist constants $M'_\ell \ge 1$ (depending only on ℓ and P) such that the following holds: for any $0 < \varepsilon, r \le r_*/2$ there exists a function $f_\varepsilon : P \to [0,1]$ such that*

1. $f_\varepsilon = 1$ *on* V_r;
2. $f_\varepsilon = 0$ *on* $\left(V_{r+\varepsilon}\right)^c$;
3. $\max\left(\left\|f_\varepsilon\right\|_{\ell,2}, \left\|f_\varepsilon\right\|_{C^\ell}\right) \le M'_\ell \varepsilon^{-\ell}$.

Here is the main result of the section, which is a modified and improved version of [19, Proposition 5.1]. Roughly speaking, it states that for large enough t the relative measure of $h \in V_r$ such that $g_t hx \in O$ is not much less than $\mu(O)$.

PROPOSITION 4.4. *Suppose that P has property* (EEP) *with respect to the flow (X, F). Then for any open $O \subset X$, any $x \in X$, any*

$$0 < r < \frac{1}{2}\min\left(r_0(x), r_*\right), \tag{4.2}$$

and any t satisfying

$$t \ge a' + b \log \frac{1}{r_0(x)} \tag{4.3}$$

one has

$$\nu\left(\{h \in V_r : g_t hx \in O\}\right) \ge \nu(V_r)\, \mu(\sigma_{e^{-\lambda' t}}O) - e^{-\lambda' t}.$$

Here

$$\lambda' := \frac{\lambda}{4\ell + 2} \tag{4.4}$$

and

$$a' := \max\left(a, \frac{1}{\lambda'}\log\left(M_\ell M'_\ell E + pc_2\right), \frac{\log\frac{2}{r_*}}{2\lambda'}\right), \tag{4.5}$$

where ℓ, λ, a, b are as in Definition 1.3, E is an implicit constant from (1.6), c_2 is as in (3.3), and M_ℓ, M'_ℓ are as in Lemmas 4.2 and 4.3.

Proof of Proposition 4.4. Let $O \subset X$ be an open subset of X, and take $x \in X$ and r as in (4.2). Now set $f = 1_{V_r}$, take t as in (4.3) and put $\varepsilon := e^{-2\lambda' t}$. Note that (4.3) and (4.5) give

$$\varepsilon \le \frac{r_*}{2}. \tag{4.6}$$

Now let ψ_ε and f_ε be the functions constructed in Lemmas 4.2 and 4.3 respectively. Then we have

$$\begin{aligned}\max(\|\psi_\varepsilon\|_{C^1}, \|\psi_\varepsilon\|_{\ell,2}) \cdot \|f_\varepsilon\|_{C^\ell} \cdot e^{-\lambda t} &\le \max(\|\psi_\varepsilon\|_{C^\ell}, \|\psi_\varepsilon\|_{\ell,2}) \cdot \|f_\varepsilon\|_{C^\ell} \cdot e^{-\lambda t} \\ &\le M_\ell \varepsilon^{-\ell} M'_\ell \varepsilon^{-\ell} e^{-\lambda t} \\ &= M_\ell M'_\ell e^{4\ell\lambda' t - \lambda t} \underset{(4.4)}{\le} M_\ell M'_\ell e^{-2\lambda' t}.\end{aligned} \tag{4.7}$$

Furthermore, by (3.2) and (4.6),

$$\operatorname{supp} f_\varepsilon \subset V_{r+e^{-2\lambda' t}} \subset V_{r+\frac{r_*}{2}} \subset V_{r_*} \subset B^P(1). \tag{4.8}$$

Also, in view of (4.5) we have $a' \ge a$; hence, inequality (1.5) is satisfied for any x, t satisfying (4.3). Hence the estimate (1.6) can be applied to ψ_ε, f_ε, x and t, and, in view of (4.7), yields

$$\int_P f_\varepsilon(h)\psi_\varepsilon(g_t hx)\, d\nu(h) \ge \int_P f_\varepsilon\, d\nu \int_X \psi_\varepsilon\, d\mu - M_\ell M'_\ell E e^{-2\lambda' t}.$$

Thus we have

$$\begin{aligned}\nu\left(\{h \in V_r : g_t hx \in O\}\right) &= \int_P f(h) 1_O(g_t hx)\, d\nu(h) \\ &\ge \int_P f(h)\psi_\varepsilon(g_t hx) d\nu(h) \\ &\ge \int_P f_\varepsilon(h)\psi_\varepsilon(g_t hx)\, d\nu(h) - \int_P |f_\varepsilon - f|\, d\nu \\ &\ge \int_P f_\varepsilon(h)\psi_\varepsilon(g_t hx)\, d\nu(h) - \nu\left(V_{r+e^{-2\lambda' t}} \smallsetminus V_r\right).\end{aligned}$$

Since by (4.2) and (4.6) we have $r + e^{-2\lambda' t} \le r_*$, it follows that $\frac{r+e^{-2\lambda' t}}{4\sqrt{p}} I_{\mathfrak{p}} \subset B^{\mathfrak{p}}(1)$. So in view of (3.3),

$$\begin{aligned}\nu\left(V_{r+e^{-2\lambda' t}} \smallsetminus V_r\right) &\le c_2 \mathrm{Leb}\left(\frac{r+e^{-2\lambda' t}}{4\sqrt{p}} I_{\mathfrak{p}} \smallsetminus \frac{r}{4\sqrt{p}} I_{\mathfrak{p}}\right) \\ &\underset{(\mathrm{MVT})}{\le} c_2 \left(\frac{1}{\sqrt{4p}}\right)^p e^{-2\lambda' t} p\left(r + e^{-2\lambda' t}\right)^{p-1} \le c_2 p e^{2\lambda' t}.\end{aligned}$$

Combining the above computations, we obtain

$$\begin{aligned}\nu\left(\{h \in V_r : g_t hx \in O\}\right) &\geq \int_P f_\varepsilon(h)\psi_\varepsilon(g_t hx)\,d\nu(h) - c_2 p e^{-2\lambda' t} \\ &\geq \int_P f_\varepsilon\,d\nu \int_X \psi_\varepsilon\,d\mu - M_\ell M'_\ell E e^{-2\lambda' t} - c_2 p e^{-2\lambda' t} \\ &\geq \nu(V_r)\,\mu(\sigma_\varepsilon O) - (M_\ell M'_\ell E + c_2 p) e^{-2\lambda' t} \\ &= \nu(V_r)\,\mu(\sigma_{e^{-2\lambda' t}} O) - (M_\ell M'_\ell E + c_2 p) e^{-2\lambda' t} \\ &\underset{(4.5)}{\geq} \nu(V_r)\,\mu(\sigma_{e^{-\lambda' t}} O) - e^{-\lambda' t}. \qquad \square\end{aligned}$$

5. COVERINGS BY BOWEN BOXES

For $x \in X$, $t > 0, N \in \mathbb{N}$ and a subset S of X let us define

$$A_x^N(t,r,S) := \left\{h \in \overline{V_r} : g_{\ell t} hx \in S \ \forall\, \ell \in \{1,\dots,N\}\right\}. \tag{5.1}$$

Clearly the set (4.1) studied in the previous section has the same measure as $A_x^1(t,r,O)$. Our goal in this section will be to inductively use Proposition 4.4 to find an effective covering result for the set $A_x^N(t,r,O^c)$. We start with the following theorem, which is a modified and improved version of [19, Proposition 5.1]:

THEOREM 5.1. *Let P be a subgroup of G that has property* (EEP) *relative to the flow* (X,F). *Then there exist positive constants $a', b' \geq \frac{\log(8\sqrt{p})}{\lambda_{\min}}, C_2, \lambda$ such that for any open $O \subset X$, any*

$$0 < r < \frac{1}{2}\min\left(r_0\left(\partial_{1/2}(O^c)\right), r_*\right), \tag{5.2}$$

any $x \in \partial_r(O^c)$, any t satisfying

$$t \geq a' + b' \log\frac{1}{r}, \tag{5.3}$$

and any $N \in \mathbb{N}$, the set $A_x^N(t,r,O^c)$ can be covered with at most

$$e^{\delta N t}\left(1 - \mu(\sigma_r O) + \frac{C_2}{r^p} e^{-\lambda t}\right)^N$$

Bowen (Nt, r)-boxes in P.

We remark that the above theorem, as well as Theorem 5.3 proved later in this section, is applicable only to the situations when the complement of O is compact: indeed, otherwise $r_0\left(\partial_{1/2}(O^c)\right) = 0$ and (5.2) is never satisfied.

Before we prove the theorem, we need the following lemma:

LEMMA 5.2. *For any $x \in X$, any $O \subset X$, any $0 < r \leq r_*$ and any $t > 0$ we have*

$$\#\left\{\gamma \in \Lambda_r : \overline{V_r}\gamma g_t x \subset O\right\} \geq \frac{\nu\left(A_x^1(t,r,\sigma_{r/2} O)\right)}{\nu(g_{-t} V_r g_t)}. \tag{5.4}$$

Proof. For any $\gamma \in P$ and any $h_1, h_2 \in \overline{V_r}$ we have

$$\operatorname{dist}\big(h_1\gamma g_t x, h_2\gamma g_t x\big) \le \operatorname{dist}(h_1, h_2) \le \operatorname{diam}(\overline{V_r}) \le r/2. \tag{5.5}$$

Hence, if

$$A_x^1(t, r, \sigma_{r/2}O) \cap g_{-t}\overline{V_r}\gamma g_t \neq \varnothing$$

for $\gamma \in \Lambda_r$, then for some $h \in \overline{V_r}$ one has $g_t h x \in \sigma_{r/2}O \cap \overline{V_r}\gamma g_t x$, and, in view of (5.5) and $\partial_{r/2}(\sigma_{r/2}O) \subset O$, we can conclude that $\overline{V_r}\gamma g_t x \subset O$. Thus

$$A_x^1(t, r, \sigma_{r/2}O) \subset \bigcup_{\substack{\gamma \in \Lambda_r \\ \overline{V_r}\gamma g_t x \subset O}} g_{-t}\overline{V_r}\gamma g_t,$$

and (5.4) follows from the definition of V_r being a tessellation domain relative to Λ_r. □

Proof of Theorem 5.1. Take a', b, λ' be as in Proposition 4.4, and $\lambda_{\min}$ as in (2.3). Also set

$$b' := \max\left(b, \frac{1}{\lambda'}, \frac{\log(16\sqrt{p})}{\lambda_{\min}}\right). \tag{5.6}$$

Fix an open $O \subset X$, and take r as in (5.2). Also take $x \in \partial_r(O^c)$ and t as in (5.3).

First let us show how to derive the desired result for $N = 1$ from Proposition 4.4. Observe that

$$t \underset{(5.3)}{\ge} a' + b' \log\frac{1}{r} \underset{(5.2)}{\ge} b' \log\frac{2}{r_*} \underset{(r_* < \frac{1}{4})}{>} b' \underset{(5.6)}{\ge} \frac{\log(8\sqrt{p})}{\lambda_{\min}}.$$

So, by combining Lemma 3.2 with Lemma 5.2, we conclude that $A_x^1(t, r, O^c)$ can be covered with at most

$$\begin{aligned} &\#\left\{\gamma \in \Lambda_r : g_{-t}\overline{V_r}\gamma g_t \cap \overline{V_r} \neq \varnothing\right\} - \#\left\{\gamma \in \Lambda_r : \overline{V_r}\gamma g_t x \subset O\right\} \\ &\quad \le e^{\delta t}\left(1 + C_0 e^{-\lambda_{\min} t}\right) - \frac{\nu\left(A_x^1(t, r, \sigma_{r/2}O)\right)}{\nu(g_{-t}V_r g_t)} \end{aligned}$$

Bowen (t, r)-boxes in P, where C_0 is as in (3.7). Note that whenever $x \in \partial_r(O^c)$, (4.2) and (4.3) follow from (5.2), (5.3) and (5.6). Moreover, we have

$$\lambda' t \underset{(5.3)}{\ge} \lambda' a' + \lambda' b' \log\frac{1}{r} \underset{\substack{(4.5)\\(5.6)}}{\ge} \frac{\log\frac{2}{r_*}}{2} + \log\frac{1}{r} \underset{(r_* < \frac{1}{4})}{\ge} \log\frac{2}{r}. \tag{5.7}$$

Hence one can apply Proposition 4.4 and conclude that $A_x^1(t, r, O^c)$ can be covered with at most

$$\begin{aligned} &e^{\delta t}\left(1 + C_0 e^{-\lambda_{\min} t} - \mu(\sigma_{e^{-\lambda' t}}\sigma_{r/2}O) + \frac{e^{-\lambda' t}}{\nu(V_r)}\right) \\ &\quad \underset{(5.7)}{\le} e^{\delta t}\left(1 + C_0 e^{-\lambda_{\min} t} - \mu(\sigma_r O) + \frac{e^{-\lambda' t}}{\nu(V_r)}\right) \\ &\quad \le e^{\delta t}\left(1 - \mu(\sigma_r O) + \frac{C_2}{r^p} e^{-\lambda t}\right) =: N(r, t) \end{aligned}$$

Bowen (t,r)-boxes in P, where $\lambda := \min(\lambda_{\min}, \lambda')$ and

$$C_2 := C_0 + \frac{(4\sqrt{p})^p}{c_1}. \tag{5.8}$$

Now let $g_{-t}\overline{V_r}\gamma g_t$ be one of the Bowen (t,r)-boxes in the above cover which has non-empty intersection with $A^1_x(t,r,O^c)$. Take any $q = g_{-t}h\gamma g_t \in g_{-t}\overline{V_r}\gamma g_t$; then $g_t q x = h\gamma g_t x$, hence

$$\left\{g_t q x : q \in g_{-t}\overline{V_r}\gamma g_t\right\} = \left\{h\gamma g_t x : h \in \overline{V_r}\right\}.$$

Consequently,

$$\{q \in g_{-t}\overline{V_r}\gamma g_t : g_{2t} q x \notin O\} = g_{-t} A^1_x(t,r,O^c)\gamma g_t. \tag{5.9}$$

Note that since $\operatorname{diam}(\overline{V_r}\gamma) < r$ and $g_{-t}\overline{V_r}\gamma g_t \cap A^1_x(t,r,O^c)$ is non-empty, we have $\gamma g_t x \in \partial_r(O^c)$. Hence, by going through the same procedure, this time using $\gamma g_t x$ in place of x, we can cover the set in the left hand side of (5.9) with at most $N(r,t)$ Bowen $(2t,r)$-boxes in P. Therefore, we conclude that the set $A^2_x(t,r,O^c)$ can be covered with at most $N(r,t)^2$ Bowen $(2t,r)$-boxes in P. By doing this procedure inductively, we can see that for any $N \in \mathbb{N}$, the set $A^N_x(t,r,O^c)$ can be covered with at most

$$N(r,t)^N = e^{\delta N t}\left(1 - \mu(\sigma_r O) + \frac{C_2}{r^p}e^{-\lambda t}\right)^N$$

Bowen (Nt,r)-boxes in P. This finishes the proof. □

Next we are going to apply Theorem 5.1 to cover $A^N_x(t,r,O^c)$ with Bowen (Nt,θ)-boxes, where $r \le \theta \le \frac{r_*}{2}$.

THEOREM 5.3. *Let $P \subset G$ have property* (EEP) *relative to the flow* (X,F)*. Then, with a', b', C_2, λ as in Theorem 5.1, for any open $O \subset X$, any t as in* (5.3)*, any r such that*

$$0 < r < \frac{1}{4}\min\Big(r_0\big(\partial_1(O^c)\big), r_*\Big), \tag{5.10}$$

any $x \in \partial_r(O^c)$, any $N \in \mathbb{N}$, and any $\theta \in \left[r, \frac{r_*}{2}\right]$, the set $A^N_x(t,r,O^c)$ can be covered with at most

$$\frac{c_2}{c_1}\left(\frac{2r}{\theta}\right)^p e^{\delta N t}\left(1 - \mu\big(\sigma_{4\theta} O\big) + \frac{C_2}{r^p}e^{-\lambda t}\right)^N$$

Bowen (Nt,θ)-boxes in P.

Proof. Consider the covering of $A^N_x(t,r,O^c)$ by Bowen boxes

$$\left\{g_{-Nt}\overline{V_\theta}\gamma g_{Nt} : \gamma \in \Lambda_\theta\right\}.$$

Let R be one of those boxes, so that

$$R \cap A^N_x\left(t,r,O^c\right) \ne \varnothing. \tag{5.11}$$

Since $\theta < r_*$, in view of (3.4) we have $\operatorname{diam}(R) \le \frac{\theta}{2} e^{-\lambda_{\min} N t}$; furthermore,

$$\theta e^{-\lambda_{\min} t} \underset{(5.3)}{\le} \theta e^{-\lambda_{\min} b' \log \frac{1}{r}} \le \theta e^{-\log(8\sqrt{p}) \log \frac{1}{r}} \underset{(\theta \le 1)}{\le} \frac{r}{8\sqrt{p}}. \tag{5.12}$$

Since $R \cap V_r \neq \varnothing$, it follows that

$$R \subset \partial_{\frac{\theta}{2} e^{-\lambda_{\min} N t}} \overline{V_r} \underset{(5.12)}{\subset} \partial_{\frac{r}{16\sqrt{p}}} \overline{V_r} \subset V_{2r},$$

where in the last inclusion we again use the 2-bi-Lipschitz property of exp.

We now claim that R is contained in $A_x^N\big(t, 2r, \partial_{2\theta}(O^c)\big)$. Indeed, in view of (5.11) we can find $h_1' := g_{-Nt} h_1 \gamma g_{Nt} \in R$ such that $g_{it} h_1' x \in O^c$ for all $i \in \{1, \dots, N\}$ (here $h_1 \in \overline{V_\theta}$). Then take any $h_2' := g_{-Nt} h_2 \gamma g_{Nt} \in R$, where again $h_2 \in \overline{V_\theta}$, and for any $i \in \{1, \dots, N\}$ write

$$\begin{aligned} g_{it} h_2' x &= (g_{-(N-i)t} h_2 h_1^{-1} g_{(N-i)t}) g_{it} h_1' x \\ &\in (g_{-(N-i)t} h_2 h_1^{-1} g_{(N-i)t}) O^c \underset{(3.4)}{\subset} \partial_{2\theta e^{-\lambda_{\min}(N-i)t}}(O^c) \subset \partial_{2\theta}(O^c). \end{aligned}$$

Note that since $\theta \le r_*/2 < 1/8$, we have $\partial_{1/2}\big(\partial_{2\theta}(O^c)\big) \subset \partial_1 O^c$, which implies $r_0\Big(\partial_{1/2}\big(\partial_{2\theta}(O^c)\big)\Big) \ge r_0(\partial_1 O^c)$. Thus, since (5.10) is satisfied, the following is satisfied as well:

$$0 < 2r < \frac{1}{2} \min\Big(r_0\big(\partial_{1/2}\big(\partial_{2\theta}(O^c)\big), r_*\big)\Big).$$

Consequently, Theorem 5.1, applied to O replaced with $\sigma_{2\theta} O$ and r replaced with $2r$, implies that

$$\begin{aligned} \nu\Big(A_x^N\big(t, 2r, \partial_{2\theta}(O^c)\big)\Big) &\underset{(2.2)}{\le} \nu\Big(A_x^N\big(t, 2r, (\sigma_{2\theta} O)^c\big)\Big) \\ &\le \nu(g_{-Nt} V_{2r} g_{Nt}) \cdot e^{\delta N t} \left(1 - \mu\big(\sigma_{2r}(\sigma_{2\theta} O)\big) + \frac{C_2}{(2r)^p} e^{-\lambda t}\right)^N \\ &\underset{(\theta \ge r)}{\le} \nu(V_{2r}) \left(1 - \mu\big(\sigma_{4\theta} O\big) + \frac{C_2}{r^p} e^{-\lambda t}\right)^N \end{aligned}$$

for any $x \in \partial_r(O^c) \subset \partial_{2r}\big((\sigma_{2\theta} O)^c\big)$. This forces the number of $\gamma \in \Lambda_\theta$ such that

$$g_{-Nt} \overline{V_\theta} \gamma g_{Nt} \cap A_x^N\big(t, r, O^c\big) \neq \varnothing$$

to be not greater than $\Big(1 - \mu\big(\sigma_{4\theta} O\big) + \frac{C_2}{r^p} e^{-\lambda t}\Big)^N$ multiplied by

$$\frac{\nu(V_{2r})}{\nu(g_{-Nt} V_\theta g_{Nt})} \underset{(3.3)}{\le} \frac{c_2 \big(\frac{2r}{4\sqrt{p}}\big)^p}{e^{-\delta N t} c_1 \big(\frac{\theta}{4\sqrt{p}}\big)^p} = \frac{c_2}{c_1} e^{\delta N t} \left(\frac{2r}{\theta}\right)^p.$$

This finishes the proof of the theorem. □

6. ENDP AND ITERATIONS OF MARGULIS INEQUALITY

In the next two sections we let P be a subgroup of G which has property (ENDP) relative to the F-action on X. Then by the definition one can find $0 < c_0 < 1$ and $t_0 > 0$ such that the following holds: for any $t \geq t_0$ one can find a height function u_t and $d_t > 0$ such that u_t satisfies the (c_0, d_t)-Margulis inequality with respect to $I_{B^P(1),t} u_t$; that is,

$$(I_{B^P(1),t} u_t)(x) \leq c_0 u_t(x) + d_t. \tag{6.1}$$

Let $t_1 > 0$ be sufficiently large so that

$$g_{-t} B^P(r) g_t \subset B^P(r/4) \text{ for all } 0 < r \leq 1,\ t \geq t_1, \tag{6.2}$$

and set

$$t_* := \max(t_0, t_1) \tag{6.3}$$

In the following proposition, by using inequality (6.1) N times for t sufficiently large, we prove that u_t satisfies the $\left(c_0^N, \frac{d_t}{1-c_0}\right)$-Margulis inequality with respect to $I_{B^P(1/2),Nt}$. The argument is similar to the proof of [28, Theorem 15].

PROPOSITION 6.1. *Let $\{u_t\}_{t>0}$ be the family of height functions in Definition 1.5, and let t_* be as in (6.3). Then for any $t \geq t_*$ and any $N \in \mathbb{N}$, the function u_t satisfies the $\left(c_0^N, \frac{d_t}{1-c_0}\right)$-Margulis inequality with respect to $I_{B^P(1/2),Nt}$. In other words, for any $t \geq t_*$, any $N \in \mathbb{N}$ and any $x \in X$ one has*

$$(I_{B^P(1/2),Nt} u_t)(x) \leq c_0^N u_t(x) + \frac{d_t}{1-c_0}. \tag{6.4}$$

As a corollary, we get the following crucial statement which will be useful in later sections:

COROLLARY 6.2. *Let t_1 be as in (6.2). Then there exists a height function u and $d > 0$ such that for any $0 < c < 1$ one can find positive $t_c \geq t_1$ such that for any $t \in \mathbb{N}t_c$, the function u satisfies the (c,d)-Margulis inequality with respect to $I_{B^P(1/2),t}$. In other words, for all $x \in X$ we have*

$$(I_{B^P(1/2),t} u)(x) \leq c u(x) + d. \tag{6.5}$$

Proof. Let $0 < c < 1$, and take c_0 as in Proposition 6.1. Choose N sufficiently large so that $c_0^N \leq c$, and set

$$u := u_{t_1},\ t_c := N t_* \underset{(6.3)}{\geq} t_1,\ d := \frac{d_{t_1}}{1-c_0}.$$

Now let $t = n t_c = n N t_*$ be an element in $\mathbb{N}t_c$. Then, by Proposition 6.1 applied with N replaced by nN, we have

$$(I_{B^P(1/2),t} u)(x) = (I_{B^P(1/2),nNt_*} u)(x) \leq c_0^{nN} u(x) + d \leq c_0^N u(x) + d \leq c u(x) + d.$$

This finishes the proof. $\square$

Proof of Proposition 6.1. Given $n \in \mathbb{N}$ and $t > 0$, define $\eta_{n,t} : B^P(1)^n \to P$ by

$$\begin{aligned}\eta_{n,t}(h_1,\dots,h_n) &:= g_{-(n-1)t} h_n g_t \cdots h_2 g_t h_1 \\ &= \tilde{h}_n \cdots \tilde{h}_1, \qquad \text{where } \tilde{h}_i = g_{-(i-1)t} h_i g_{(i-1)t}.\end{aligned} \tag{6.6}$$

For any $n \in \mathbb{N}$ and $t > 0$, let $\tilde{\nu}_{n,t}$ be the pushforward of $\nu|_{B^P(1)}$ via the conjugation by g_{nt}, that is, defined by

$$\int_P \phi(h)\, d\tilde{\nu}_{n,t}(h) = \int_{B^P(1)} \phi(g_{-nt} h g_{nt})\, d\nu(h) \tag{6.7}$$

for all $\phi \in C_c(P)$. For any positive integer n let

$$\nu_{n,t} := \tilde{\nu}_{n-1,t} * \cdots * \tilde{\nu}_{1,t} * \tilde{\nu}_{0,t}$$

be the measure on P defined by the n convolutions. It is easy to see that $\nu_{n,t}$ is absolutely continuous with respect to ν, and $\nu_{n,t}$ is the pushforward of $(\nu|_{B^P(1)})^{\otimes n}$ by the map $\eta_{n,t}$. These measures were considered in [13], and the following was shown:

LEMMA 6.3. [13, Lemma 5.5] *For all $t \geq t_1$ as in (6.2), all $h \in B^P(1/2)$, and for all $n \in \mathbb{N}$ we have $\frac{d\nu_{n,t}}{d\nu}(h) \geq 1$.*

Using Lemma 6.3, we have for all $N \in \mathbb{N}$ and all $t \geq t_1$:

$$\begin{aligned}(I_{B^P(1/2),Nt}u)(x) &= \int_{B^P(1/2)} u(g_{Nt}hx)\, d\nu(h) \leq \int_{B^P(1/2)} u(g_{Nt}hx)\, d\nu_{N,t}(h) \\ &\leq \int_{B^P(1)^N} u(g_t h_N \cdots g_t h_1 x)\, d\nu^{\otimes N}(h_1,\dots,h_N)\end{aligned} \tag{6.8}$$

Take $0 < c_0 < 1$ and $t_0 > 0$ as in the definition of (ENDP), and let $t \geq t_* = \max(t_0, t_1)$. Recall that $\nu\big(B^P(1)\big) = 1$. Since $t \geq t_0$, we can apply (6.1) and for any $i = 2, \dots$ conclude that

$$\begin{aligned}&\int_{B^P(1)^i} u(g_t h_i \cdots g_t h_1 x)\, d\nu^{\otimes i}(h_1,\dots,h_i) \\ &\quad \leq \int_{B^P(1)^{i-1}} \big(c_0 \cdot u(g_t h_{i-1} \cdots g_t h_1 x) + d_t\big)\, d\nu^{\otimes i-1}(h_1,\dots,h_{i-1}) \\ &\quad = c_0 \int_{B^P(1)^{i-1}} u(g_t h_{i-1} \cdots g_t h_1 x)\, d\nu^{\otimes i-1}(h_1,\dots,h_{i-1}) + d_t \cdot \nu\Big(B^P(1)^{i-1}\Big) \\ &\quad = c_0 \int_{B^P(1)^{i-1}} u(g_t h_{i-1} \cdots g_t h_1 x)\, d\nu^{\otimes i-1}(h_1,\dots,h_{i-1}) + d_t.\end{aligned} \tag{6.9}$$

Let $N \in \mathbb{N}$. If $N = 1$, then (6.4) follows immediately from the combination of (6.1) and (6.8). If $N \geq 2$, then by using (6.9) repeatedly and combining with (6.8) we obtain

$$
\begin{aligned}
(I_{B^P(1/2),Nt}u)(x) &\le \int_{B^P(1)^N} u(g_t h_N \cdots g_t h_1 x)\, d\nu^{\otimes n}(h_1,\dots,h_N) \\
&\le c_0^{N-1}\int_{B^P(1)} u(g_t h_1 x)\, d\nu(h_1) + c_0^{N-2} d_t + \cdots + c_0 d_t + d_t \\
&\underset{(6.1)}{\le} c_0^N u(x) + c_0^{N-1} d_t + c_0^{N-2} d_t + \cdots + c_0 d_t + d_t \\
&< c_0^N u(x) + d_t(1 + c_0 + c_0^2 + \cdots) = c_0^N u(x) + \frac{d_t}{1-c_0}.
\end{aligned}
\tag{6.10}
$$

This finishes the proof. □

7. ENDP AND ESCAPE OF MASS

Fix a height function u on X as in in Corollary 6.2. For $M > 0$ define the following sets:

$$X_{>M} := \{x \in X : u(x) > M\}, \quad X_{\le M} := \{x \in X : u(x) \le M\}.$$

Since u is proper, the sets $X_{\le M}$ are compact. Furthermore, since u is regular, by definition there exists $C \ge 1$ such that

$$C^{-1}u(x) \le u(gx) \le Cu(x) \text{ for all } g \in B(2) \text{ and } x \in X. \tag{7.1}$$

Moreover, it is easy to see from (7.1) that there exists $\alpha > 0$ such that for any $t > 0$ we have

$$e^{-\alpha t}u(x) \le u(g_t x) \le e^{\alpha t}u(x). \tag{7.2}$$

Now let $0 < c < 1$, take d and $t_c \ge t_1$ as in Corollary 6.2, and let $t \in \mathbb{N}t_c$. Note that (6.5) immediately implies that if $u(x) \ge \frac{d}{c}$, then

$$(I_{B^P(1/2),t}u)(x) \le 2c \cdot u(x). \tag{7.3}$$

Now define

$$\ell_{c,t} := \max\left(\frac{d}{c}, e^{\alpha t}\right). \tag{7.4}$$

In the following key proposition, we obtain an upper bound for the measure of the sets of type $A_x^N\left(kt,\theta,X_{>C^2\ell^2_{c,t}}\right)$, where $2 \le k \in \mathbb{N}$, $\theta \in (0,r_*]$, and C is as in (7.1). We will use this measure estimate to derive a covering result for the sets of type $A_x^N\left(kt,\theta,X_{>C^3\ell^2_{c,t}}\right)$ in Corollary 7.3.

PROPOSITION 7.1. *For any $2 \le k \in \mathbb{N}$, any $\theta \in (0,r_*]$, any $N \in \mathbb{N}$, and for any $x \in X$ we have*

$$\nu\left(A_x^N\left(kt,\theta,X_{>C^2\ell^2_{c,t}}\right)\right) \le \left(\frac{4c}{1-c}\right)^N \frac{\max\left(u(x),d\right)}{\ell^2_{c,t}}. \tag{7.5}$$

Proof of Proposition 7.1. Let $2 \le k \in \mathbb{N}$, $N \in \mathbb{N}$ and $x \in X$. Define

$$Z_x(k,N) := \left\{(h_1,\dots,h_{Nk}) \in B^P(1/2)^{Nk} : u(g_t h_{nk}\cdots g_t h_1 x) > C\ell^2_{c,t}\ \forall n \in \{1,\dots,N\}\right\}.$$

We need the following lemma:

LEMMA 7.2. *For all $\theta \in (0, r_*]$ and for all $h \in A_x^N\big(kt, \theta, X_{>C^2\ell_{c,t}^2}\big)$ one has $\eta_{Nk,t}^{-1}(h) \subset Z_x(k,N)$, where $\eta_{Nk,t}$ is defined as in (6.6).*

Proof. Let $\theta \in (0, r_*]$ and let $h \in A_x^N\big(kt, \theta, X_{>C^2\ell_{c,t}^2}\big)$. Suppose that

$$\eta_{Nk,t}(h_1, \ldots, h_{Nk}) = h.$$

Then for any $1 \le i \le N$ we have

$$\begin{aligned} \operatorname{dist}(g_{ikt}h, g_t h_{ik} \cdots g_t h_1) &\underset{(6.6)}{=} \operatorname{dist}(g_{ikt}\tilde{h}_{Nk} \cdots \tilde{h}_1, g_{ikt}\tilde{h}_{ik} \cdots \tilde{h}_1) \\ &= \begin{cases} \operatorname{dist}(g_{ikt}\tilde{h}_{Nk} \cdots \tilde{h}_{ik+1} g_{-ikt}, e) & \text{if } i < N, \\ 0 & \text{if } i = N. \end{cases} \end{aligned}$$

Moreover, if $i < N$ one has

$$\begin{aligned} &\operatorname{dist}(g_{ikt}\tilde{h}_{Nk} \cdots \tilde{h}_{ik+1} g_{-ikt}, e) \\ &\quad \le \operatorname{dist}(g_{ikt}\tilde{h}_{ik+1} g_{-ikt}, e) + \cdots + \operatorname{dist}(g_{ikt}\tilde{h}_{Nk} g_{-ikt}, e) \\ &\quad \underset{(6.6)}{=} \operatorname{dist}(h_{ik+1}, e) + \operatorname{dist}(g_{-t}h_{ik+2}g_t, e) + \operatorname{dist}(g_{-2t}h_{ik+3}g_{2t}, e) \\ &\qquad\qquad + \cdots + \operatorname{dist}(g_{-((N-i)k-1)t}h_{Nk}g_{((N-i)k-1)t}, e) \\ &\quad \underset{(6.2)}{\le} 1 + \frac{1}{4} + \frac{1}{4^2} + \cdots \frac{1}{4^{(N-i)k-1}} < 2. \end{aligned}$$

Hence, in view of (7.1), for any $1 \le i \le N$, $g_{ikt}hx \in X_{>C^2\ell_{c,t}^2}$ implies that

$$g_t h_{ik} \cdots g_t h_1 x \in X_{>C\ell_{c,t}^2}.$$

This finishes the proof. □

Now let $\theta \in (0, r_*]$. Note that in view of (3.2) we have $\overline{V_\theta} \subset B^P(r_*/2) \subset B^P(1/2)$; moreover, $kt \ge kt_c \ge t_1$. Thus, by Lemma 7.2 and Lemma 6.3 we have

$$\begin{aligned} \nu\Big(A_x^N\big(kt, \theta, X_{>C^2\ell_{c,t}^2}\big)\Big) &\le \nu_{Nk,t}\Big(A_x^N\big(kt, \theta, X_{>C^2\ell_{c,t}^2}\big)\Big) \\ &\le \nu^{\otimes Nk}\big(Z_x(k,N)\big), \end{aligned} \tag{7.6}$$

where $\nu_{Nk,t}$ is defined as in (6.7). So it suffices to estimate $\nu^{\otimes Nk}\big(Z_x(k,N)\big)$. Define

$$s(k,N,x) := \int_{Z_x(k,N)} u(g_t h_{Nk} \cdots g_t h_1 x)\, d\nu^{\otimes Nk}(h_1, \ldots, h_{Nk}).$$

Since $\ell_{c,t} \ge e^{\alpha t}$, in view of (7.1) and (7.2) we have $u(g_t h_{k-1} \cdots g_t h_1 x) > \ell_{c,t}$ whenever $(h_1, \ldots, h_k) \in Z_x(k,1)$. Hence,

$$\begin{aligned} &s(k,1,x) \\ &\le \int_{B^P(1/2)^{k-1}} 1_{X_{>\ell_{c,t}}}(g_t h_{k-1} \cdots g_t h_1 x) u(g_t h_k \cdots g_t h_1 x)\, d\nu^{\otimes k-1}(h_1, \ldots, h_{k-1}) \\ &\le 2c \int_{B^P(1/2)^{k-1}} u(g_t h_{k-1} \cdots g_t h_1 x)\, d\nu^{\otimes k-1}(h_1, \ldots, h_{k-1}), \end{aligned} \tag{7.7}$$

where the second inequality follows from (7.3) applied with x replaced by $g_t h_{k-1} \cdots g_t h_1 x$, and from the fact that $\ell_{c,t} \ge \frac{d}{c}$.

Again recall that $\nu\big(B^P(1)\big) = 1$. By applying (6.5) we get

$$\begin{aligned}
&\int_{B^P(1/2)^{k-1}} u(g_t h_{k-1}\cdots g_t h_1 x)\, d\nu^{\otimes k-1}(h_1,\cdots,h_{k-1}) \\
&\le c\int_{B^P(1/2)^{k-2}} u(g_t h_{k-2}\cdots g_t h_1 x)\, d\nu^{\otimes k-2}(h_1,\cdots,h_{k-2}) + d\cdot\nu\Big(B^P(1/2)^{k-2}\Big) \\
&\le c\int_{B^P(1/2)^{k-2}} u(g_t h_{k-2}\cdots g_t h_1 x)\, d\nu^{\otimes k-2}(h_1,\cdots,h_{k-2}) + d.
\end{aligned}$$

Therefore, if we apply (6.5) repeatedly, similarly to (6.10) we get

$$\begin{aligned}
&\int_{B^P(1/2)^{k-1}} u(g_t h_{k-1}\cdots g_t h_1 x)\, d\nu^{\otimes k-1}(h_1,\cdots,h_{k-1}) \\
&\le c^{k-1}u(x) + \frac{d}{1-c} \le \frac{2}{1-c}\cdot\max\big(u(x),d\big).
\end{aligned} \tag{7.8}$$

So by combining (7.7) and (7.8) we have

$$s(k,1,x) \le \frac{4c}{1-c}\cdot\max\big(u(x),d\big) \text{ for all } x\in X. \tag{7.9}$$

Note that since $\ell_{c,t} \ge e^{\alpha t}$, in view of (7.1), (7.2) and (7.4)

$$(h_1,\dots,h_{ik}) \in Z_x(k,i) \Rightarrow u(g_t h_{(i-1)k}\cdots g_t h_1 x) \ge \ell_{c,t} \ge \frac{d}{c} \ge d. \tag{7.10}$$

Now for any $2 \le i \in \mathbb{N}$ we can write

$$\begin{aligned}
s(k,i,x) &= \int_{Z_x(k,i)} u(g_t h_{ik}\cdots g_t h_1 x) d\nu^{\otimes ik}(h_1,\dots,h_{ik}) \\
&= \int_{Z_x(k,i-1)}\int_{Z_{g_t h_{(i-1)k}\cdots g_t h_1 x}(k,1)} u(g_t h_{ik}\cdots g_t h_1 x) \\
&\qquad\qquad d\nu^{\otimes k}(h_{(i-1)k+1},\dots,h_{ik})\, d\nu^{\otimes(i-1)k}(h_1,\dots,h_{(i-1)k}) \\
&= \int_{Z_x(k,i-1)} s(k,1,g_t h_{(i-1)k}\cdots g_t h_1 x)\, d\nu^{\otimes(i-1)k}(h_1,\dots,h_{(i-1)k}) \\
&\underset{\substack{(7.9)\\(7.10)}}{\le} \int_{Z_x(k,i-1)} \frac{4c}{1-c}\cdot u(g_t h_{(i-1)k}\cdots g_t h_1 x)\, d\nu^{\otimes(i-1)k}(h_1,\dots,h_{(i-1)k}) \\
&= \frac{4c}{1-c}\cdot s(k,i-1,x)
\end{aligned}$$

Thus, by repeatedly using the above computation, for any $N\in\mathbb{N}$ we conclude that

$$s(k,N,x) \le \left(\frac{4c}{1-c}\right)^{N-1} s(k,1,x) \underset{(7.9)}{\le} \left(\frac{4c}{1-c}\right)^{N}\max\big(u(x),d\big).$$

Note that $s(k,N,x) \ge \ell_{c,t}^2\cdot\nu^{\otimes Nk}\big(Z_x(k,N)\big)$. Hence (7.5) follows from the above inequality and (7.6). □

As a corollary, we get the following crucial covering result:

COROLLARY 7.3. *Let P be a subgroup of G with property* (ENDP). *Then for any $0 < c < 1$ there exists $t_c > 0$ such that for all $t \in \mathbb{N}t_c$ and $2 \le k \in \mathbb{N}$ satisfying $kt \ge \frac{\log(8\sqrt{p})}{\lambda_{\min}}$, all $\theta \in (0, r_*/2]$, all $N \in \mathbb{N}$, and for all $x \in X$, the set*

$$A_x^N\big(kt, \theta, X_{>C^3\ell_{c,t}^2}\big) = \left\{h \in \overline{V_\theta} : u(g_{ikt}hx) > C^3\ell_{c,t}^2 \ \forall\, i \in \{1, \dots, N\}\right\}$$

can be covered with at most

$$\frac{e^{\delta Nkt}}{\nu(V_\theta)}\left(\frac{4c}{1-c}\right)^N \frac{\max\big(u(x), d\big)}{\ell_{c,t}^2}$$

Bowen (Nkt, θ)-balls in P.

Proof. Let $0 < c < 1$, take t_c as in Corollary 6.2, and let $t \in \mathbb{N}t_c$ and $2 \le k \in \mathbb{N}$ be such that $kt \ge \frac{\log(8\sqrt{p})}{\lambda_{\min}}$. Also let $\theta \in (0, r_*/2]$, $N \in \mathbb{N}$, and $x \in X$. Take a covering of $\overline{V_\theta}$ with Bowen (Nkt, θ)-boxes in P. Now let R be one of the Bowen boxes in this cover which has non-empty intersection with $A_x^N\big(kt, \theta, X_{>C^3\ell_{c,t}^2}\big)$. Note that in view of (3.4), we have

$$\operatorname{diam}(R) \le \frac{\theta}{2} e^{-\lambda_{\min} Nkt} \le \frac{\theta}{2} e^{-\lambda_{\min} kt} \le \frac{\theta}{16\sqrt{p}}.$$

So, since $R \cap \overline{V_\theta} \neq \varnothing$, we must have

$$R \subset \partial_{\frac{\theta}{16\sqrt{p}}} \overline{V_\theta} \subset V_{2\theta}, \tag{7.11}$$

where in the last inclusion we use the 2-bi-Lipschitz property of exp.

Now let $h \in R \cap A_x^N\big(kt, \theta, X_{>C^3\ell_{c,t}^2}\big)$. Then

$$u(g_{ikt}hx) > C^3\ell_{c,t}^2 \text{ for all } 1 \le i \le N.$$

On the other hand, if we denote the center of R by h_0, then for all $1 \le i \le N$ we have for all $h' \in R$:

$$\begin{aligned} g_{ikt}h'x &= \big(g_{ikt}h'h_0^{-1}g_{-ikt}\big)\, g_{ikt}h_0x \\ &\in \Big(g_{-(N-i)kt}\overline{V_\theta}g_{(N-i)kt}\Big)\, g_{ikt}h_0x \underset{(3.4)}{\in} B\left(\frac{\theta}{2} \cdot e^{-\lambda_{\min}(N-i)kt}\right) g_{ikt}h_0x \\ &\subset B(\theta/2)g_{ikt}h_0x \subset B(1/2)g_{ikt}h_0x. \end{aligned}$$

This implies that

$$g_{ikt}Rx \subset B(1)g_{ikt}hx \text{ for all } 1 \le i \le N. \tag{7.12}$$

Now in view of (7.11) and (7.12) we can conclude that

$$R \subset A_x^N\big(kt, 2\theta, X_{>C^2\ell_{c,t}^2}\big). \tag{7.13}$$

Therefore, by (7.5) and (7.13) applied with θ replaced with 2θ, the set

$$A_x^N\big(kt, \theta, X_{>C^3\ell_{c,t}^2}\big)$$

can be covered with at most

$$\frac{\nu\Big(A_x^N\big(kt, 2\theta, X_{>C^2\ell_{c,t}^2}\big)\Big)}{\nu(g_{-Nkt}V_\theta g_{Nkt})} \le \frac{(\frac{4c}{1-c})^N \max\big(u(x), d\big)}{\nu(g_{-Nkt}V_\theta g_{Nkt})\cdot \ell_{c,t}^2}$$
$$= \frac{e^{\delta Nkt}}{\nu(V_\theta)}\left(\frac{4c}{1-c}\right)^N \frac{\max\big(u(x), d\big)}{\ell_{c,t}^2}$$

Bowen (Nkt, θ)-boxes in P. This finishes the proof. □

8. COMBINING THE ESTIMATES OF §5 AND §7

The goal of this section is to describe a method making it possible to put together properties (EEP) and (ENDP). In the next proposition neither (EEP) nor (ENDP) are assumed to hold. Instead we will assume certain covering estimates (similar to those we derived from (EEP) and (ENDP) respectively in the previous sections) and then combine them to derive an estimate on which our dimension bound is based. This formalizes the argument which first appeared in [15] and then was used in [20] to solve DDC in the case (1.2).

PROPOSITION 8.1. *Let P be a connected subgroup of H normalized by F. Let $S, Q \subset X$, t satisfying (3.6), $r > 0$, $\theta \in [r, r_*/2]$, and let $k_1, k_2, a_1, a_2 \ge 1$ be given. Suppose that for any $N \in \mathbb{N}$ the following two conditions hold:*

(a) *For all $x \in \partial_r(S\cap Q)$ the set $A_x^N(t, r, S\cap Q)$ can be covered with at most $k_1 e^{\delta Nt} a_1^N$ Bowen (Nt, θ)-boxes in P.*
(b) *For all $x \in \partial_\theta(S\cap Q)$ the set $A_x^N(t, \theta, Q^c)$ can be covered with at most $k_2 e^{\delta Nt} a_2^N$ Bowen (Nt, θ)-boxes in P.*

Then for all $x \in \partial_r(S\cap Q)$ the set $A_x^N(t, r, S)$ can be covered with at most $k_3 e^{\delta Nt} a_3^N$ Bowen (Nt, θ)-boxes in P, where

$$k_3 = (1+C_0)\frac{c_2}{c_1}\left(\frac{\theta}{r} + 8\sqrt{p}\right)^p k_1 k_2^2, \quad a_3 = a_1 + a_2 + \sqrt{k_3 a_2}. \tag{8.1}$$

Proof. For any $h \in A_x^N(t, r, S)$, let us define:

$$J_h := \big\{ j \in \{1, \ldots, N\} : g_{jt} h x \in Q^c \big\},$$

and for any $J \subset \{1, \ldots, N\}$, set:

$$Z(J) := \big\{ h \in A_x^N(t, r, S) : J_h = J \big\}.$$

Note that

$$A_x^N(t, r, S) = \bigcup_{J\subset\{1,\ldots,N\}} Z(J) \tag{8.2}$$

Let J be a subset of $\{1, \ldots, N\}$. We can decompose J and $I := \{1, \ldots, N\} \smallsetminus J$ into sub-intervals of maximal size $J_1, \ldots, J_q$ and $I_1, \ldots, I_{q'}$ so that

$$J = \bigsqcup_{j=1}^{q} J_j \text{ and } I = \bigsqcup_{i=1}^{q'} I_i.$$

Hence, we get a partition of the set $\{1,\ldots,N\}$ as follows:

$$\{1,\ldots,N\} = \bigsqcup_{j=1}^{q} J_j \sqcup \bigsqcup_{i=1}^{q'} I_i.$$

Now we inductively prove the following

CLAIM 8.2. For any integer $L \leq N$, if

$$\{1,\ldots,L\} = \bigsqcup_{j=1}^{\ell} J_j \sqcup \bigsqcup_{i=1}^{\ell'} I_i, \tag{8.3}$$

then the set $Z(J)$ can be covered with at most

$$k_2^{d'_{J,L}+1}\left((1+C_0)\frac{c_2}{c_1}\left(\frac{\theta}{r}+8\sqrt{p}\right)^p k_1\right)^{d_{J,L}+1} e^{\delta Lt} a_1^{\sum_{i=1}^{\ell'}|I_i|-d_{J,L}} a_2^{\sum_{j=1}^{\ell}|J_j|} \tag{8.4}$$

Bowen (Lt,θ)-boxes in P, where $d_{J,L}$, $d'_{J,L}$ are defined as follows:

$$d_{J,L} := \#\big\{i \in \{1,\ldots,L\} : \ i < L, i \in J \ and \ i+1 \in I\big\},$$
$$d'_{J,L} := \#\big\{i \in \{1,\ldots,L\} : \ i < L, i \in I \ and \ i+1 \in J\big\}.$$

Proof of Claim 8.2. We argue by induction on $\ell+\ell'$. When $\ell+\ell'=1$, we have $d_{J,L} = d'_{J,L} = 0$, and there are two cases: either $\ell = 1$ and $\{1,\ldots,L\} = J_1$, or $\ell' = 1$ and $\{1,\ldots,L\} = I_1$. In the first case

$$Z(J) \subset A_x^L\left(t,r,Q^c\right) \subset A_x^L\left(t,\theta,Q^c\right),$$

Therefore, condition (b) applied with $N = L$ implies that this set can be covered with at most

$$k_2 e^{\delta Lt} a_2^L < k_1 k_2 (1+C_0)\frac{c_2}{c_1}\left(\frac{\theta}{r}+8\sqrt{p}\right)^p e^{\delta Lt} a_2^L$$

Bowen (Lt,θ)-boxes in P. This finishes the proof of the first case.

In the second case, note that

$$Z(J) \subset A_x^L(t,r,S\cap Q).$$

Moreover, by condition (a) applied with $N = L$, $A_x^N(t,r,S\cap Q)$ can be covered by at most

$$k_1 e^{\delta Lt} a_1^L < k_1 k_2 (1+C_0)\frac{c_2}{c_1}\left(\frac{\theta}{r}+8\sqrt{p}\right)^p e^{\delta Lt} a_1^L$$

Bowen (Lt,θ)-balls in P. This ends the proof of the base of the induction.

In the inductive step, let $L' > L$ be the next integer for which an equation similar to (8.3) is satisfied. We have two cases. Either

$$\{1,\ldots,L'\} = \{1,\ldots,L\} \sqcup I_{\ell'+1} \tag{8.5}$$

or

$$\{1,\ldots,L'\} = \{1,\ldots,L\} \sqcup J_{\ell+1}. \tag{8.6}$$

We start with the case (8.5). Note that in this case we have

$$d_{J,L'} = d_{J,L}+1 \text{ and } d'_{J,L'} = d'_{J,L}. \tag{8.7}$$

By the induction hypothesis, an upper bound for the number of Bowen (Lt,θ)-boxes needed to cover $Z(J)$ is given by (8.4). Then observe that:

- In view of (3.6) and Lemma 3.2,
$$e^{\delta t}(1+C_0e^{-\lambda_{\min}kt}) \le e^{\delta t}(1+C_0) \tag{8.8}$$
is an upper bound for the number of Bowen $\big((L+1)t,\theta\big)$-boxes needed to cover an arbitrary Bowen (Lt,θ)-box;
- In view of Lemma 3.1,
$$\frac{c_2}{c_1}\left(\frac{\theta}{r}+8\sqrt{p}\right)^p \tag{8.9}$$
is an upper bound for the number of Bowen $\big((L+1)t,r\big)$-boxes needed to cover an arbitrary Bowen (Lt,θ)-box.

Now let B_r be a Bowen $\big((L+1)t,r\big)$-box that has non-empty intersection with $Z(J)$, and let $h\in B_r\cap Z(J)$. Since $h\in Z(J)$, it follows that $g_{(L+1)t}hx\in S\cap Q$. Therefore, if we denote the center of B_r by h_0, we have
$$g_{(L+1)t}h_0x\in \overline{V_r}(S\cap Q)\subset \partial_r(S\cap Q). \tag{8.10}$$
Moreover, for any $h\in B_r$ and any positive integer $1\le i\le L'-(L+1)$ we have
$$g_{(L+1+i)t}hx = g_{it}(g_{(L+1)t}hh_0^{-1}g_{-(L+1)t})(g_{(L+1)t}h_0x).$$
Since the map $h\to g_{(L+1)t}hh_0^{-1}g_{-(L+1)t}$ sends B_r into $\overline{V_r}$, the preceding equality implies that
$$\begin{aligned}&\big\{h'\in B_r : g_{(L+1+i)t}h'x\in S\cap Q\ \forall\, i\in\{1,\dots,L'-(L+1)\}\big\}\\ &\quad\subset g_{-(L+1)t}A^{L'-(L+1)}_{g_{(L+1)t}h_0x}(t,r,S\cap Q)\,g_{(L+1)t}h_0.\end{aligned}$$
So, in view of the above inclusion and (8.10), we can go through the same procedure and apply condition (a) with N replaced with $|I_{\ell'+1}|-1=L'-(L+1)$ and x replaced with $g_{(L+1)t}h_0x$, and conclude that $B_r\cap Z(J)$ can be covered with at most
$$k_1e^{\delta(|I_{\ell'+1}|-1)t}a_1^{|I_{\ell'+1}|-1} \tag{8.11}$$
Bowen $(L't,\theta)$-boxes in P. Multiplying the bounds (8.4), (8.8), (8.9) and (8.11), we conclude that $Z(J)$ can be covered with at most
$$\begin{aligned}&\frac{c_2}{c_1}\left(\frac{\theta}{r}+8\sqrt{p}\right)^p e^{\delta(|I_{\ell'+1}|-1)t}a_1^{|I_{\ell'+1}|-1}(1+C_0)\\ &\qquad\cdot k_2^{d'_{J,L}+1}\left((1+C_0)\frac{c_2}{c_1}\left(\frac{2\theta}{r}\right)^pk_1\right)^{d_{J,L}+1}e^{\delta(L+1)t}a_1^{\sum_{i=1}^{\ell'}|I_i|-d_{J,L}}a_2^{\sum_{j=1}^{\ell}|J_j|}\\ &\underset{(8.7)}{=} k_2^{d'_{J,L'}+1}\left((1+C_0)\frac{c_2}{c_1}\left(\frac{\theta}{r}+8\sqrt{p}\right)^pk_1\right)^{d_{J,L'}+1}e^{\delta L't}a_1^{\sum_{i=1}^{\ell'}|I_i|-d_{J,L'}}a_2^{\sum_{j=1}^{\ell}|J_j|}\end{aligned}$$
Bowen $(L't,\theta)$-boxes in P. This ends the proof of the claim in this case.

Next assume (8.6). Note that in this case
$$d_{J,L'}=d_{J,L}\ \text{and}\ d'_{J,L'}=d'_{J,L}+1. \tag{8.12}$$

Take a covering of $Z(J)$ with Bowen (Lt,θ)-boxes in P. Suppose B' is one of the Bowen (Lt,θ)-boxes in the cover such that $B'\cap Z(J)\neq\varnothing$, and let h_1 be the center of B'. It is easy to see that $B'\cap Z(J)\neq\varnothing$ implies:

$$g_{Lt}h_1x\in\overline{V_\theta}(S\cap Q)\subset\partial_\theta(S\cap Q). \tag{8.13}$$

On the other hand, for any $s\in B'$ and any positive integer $1\le i\le L'-L$ we have

$$g_{(L+i)t}h_1x=g_{it}(g_{Lt}hh_1^{-1}g_{-Lt})(g_{Lt}h_1x).$$

Hence, since the map $h\to g_{Lt}hh_1^{-1}g_{-Lt}$ maps B' into $\overline{V_\theta}$, the above equality implies

$$\left\{h\in B':g_{(L+i)t}hx\in Q^c\ \forall\, i\in\{1,\cdots,L'-L\}\right\}\subset g_{-Lt}A^{L'-L}_{g_{Lt}h_1x}\left(t,\theta,Q^c\right)g_{Lt}h_1$$

So in view of the above inclusion and (8.13), we can apply condition (b) with $g_{Lt}h_1x$ in place of x, and $|J_{\ell+1}|=L'-L$ in place of N. This way, we get that the set $B'\cap Z(J)$ can be covered with at most $k_2a_2^{|J_{\ell+1}|}e^{\delta|J_{\ell+1}|t}$ Bowen $(L't,\theta)$-boxes in P. From this, combined with the induction hypothesis, we conclude that $Z(J)$ can be covered with at most

$$\begin{aligned}&k_2a_2^{|J_{\ell+1}|}e^{\delta|J_{\ell+1}|t}\cdot k_2^{d'_{J,L}+1}\left((1+C_0)\frac{c_2}{c_1}\left(\frac{\theta}{r}+8\sqrt{p}\right)^p k_1\right)^{d_{J,L}+1}e^{\delta Lt}a_1^{\sum_{i=1}^{\ell'}|I_i|-d_{J,L}}a_2^{\sum_{j=1}^{\ell}|J_j|}\\ &\underset{(8.12)}{=}k_2^{d'_{J,L'}+1}\left((1+C_0)\frac{c_2}{c_1}\left(\frac{\theta}{r}+8\sqrt{p}\right)^p k_1\right)^{d_{J,L'}+1}e^{\delta L't}a_1^{\sum_{i=1}^{\ell'}|I_i|-d_{J,L}}a_2^{\sum_{j=1}^{\ell+1}|J_j|}\end{aligned}$$

Bowen $(L't,\theta)$-boxes in P, finishing the proof of the claim. □

Now by letting $L=N$, we conclude that $Z(J)$ can be covered with at most

$$k_2^{d'_{J,N}+1}\left((1+C_0)\frac{c_2}{c_1}\left(\frac{\theta}{r}+8\sqrt{p}\right)^p k_1\right)^{d_{J,N}+1}e^{\delta Nt}a_1^{|I|-d_{J,N}}a_2^{|J|} \tag{8.14}$$

Bowen (Nt,θ)-boxes in P.

Clearly

$$d'_{J,N}\le d_{J,N}+1. \tag{8.15}$$

Also, note that since $d_{J,N}\le\max(|I|,|J|)$, the exponents $|I|-d_{J,N},|J|-d_{J,N}$ in (8.14) are non-negative integers. So, in view of (8.2) and (8.14), the set $A_x^N(t,r,S)$ can be covered with at most

$$\begin{aligned}&\sum_{J\subset\{1,\dots,N\}}k_2^{d'_{J,N}+1}\left((1+C_0)\frac{c_2}{c_1}\left(\frac{\theta}{r}+8\sqrt{p}\right)^p k_1\right)^{d_{J,N}+1}e^{\delta Nt}a_1^{|I|-d_{J,N}}a_2^{|J|}\\ &\quad\underset{(8.15)}{\le}e^{\delta Nt}\sum_{J\subset\{1,\dots,N\}}k_2^{d_{J,N}+2}\left((1+C_0)\frac{c_2}{c_1}\left(\frac{\theta}{r}+8\sqrt{p}\right)^p k_1\right)^{d_{J,N}+1}a_1^{|I|-d_{J,N}}a_2^{|J|}\\ &\quad\le k_3e^{\delta Nt}\sum_{J\subset\{1,\dots,N\}}a_1^{|I|-d_{J,N}}a_2^{|J|}k_3^{d_{J,N}}\\ &\quad=k_3e^{\delta Nt}\sum_{J\subset\{1,\dots,N\}}a_1^{N-|J|-d_{J,N}}a_2^{|J|-d_{J,N}}(k_3a_2)^{d_{J,N}}\end{aligned}$$

Bowen (Nt,θ)-boxes in P, where $k_3:=(1+C_0)\frac{c_2}{c_1}\left(\frac{\theta}{r}+8\sqrt{p}\right)^p k_1k_2^2$.

To simplify the last expression we will use the following:

LEMMA 8.3. [20, Lemma 5.4] *For any $n_1, n_2, n_3 > 0$ it holds that*

$$\sum_{J\subset\{1,\dots,N\}} n_1^{N-|J|-d_{J,N}} n_2^{|J|-d_{J,N}} n_3^{2d_{J,N}} \le (n_1+n_2+n_3)^N.$$

Applying the above lemma with $n_1 = a_1$, $n_2 = a_2$ and $n_3 = \sqrt{k_3 a_3}$, we conclude that $A_x^N(t,r,S)$ can be covered with at most

$$k_3 e^{\delta N t}\left(a_1 + a_2 + \sqrt{k_3 a_3}\right)^N$$

Bowen (Nt,θ)-boxes in P. The proof of Proposition 8.1 is now complete. □

9. PROOF OF THEOREM 2.1

Given $P \subset G$ satisfying (ENDP), $0 < c < 1$, and $t > 0$, let us fix a height function u on X as in Corollary 6.2 and define the compact subset $Q_{c,t}$ of X as follows:

$$Q_{c,t} := X_{\le C^3 \ell_{c,t}^2}, \tag{9.1}$$

where $\ell_{c,t}$ is as in (7.4) and C is as in (7.1).

LEMMA 9.1. *Let P be a subgroup of G that has properties* (EEP) *and* (ENDP). *Then there exist constants*

$$a', b', C_1, C_2, \lambda > 0,$$

and, given $0 < c < 1$ there exists $t_c > 0$ such that the following holds: for all $t \in \mathbb{N}t_c$, $0 < r < 1$, $2 \le k \in \mathbb{N}$ satisfying

$$e^{\frac{a'-kt}{b'}} \le r < \frac{1}{4}\min\left(r_0\left(\partial_1 Q_{c,t}\right), r_*\right), \tag{9.2}$$

any open subset O of X, any $N \in \mathbb{N}$ and $\theta \in \left[r, \frac{r_*}{2}\right]$, and for all $x \in \partial_r\left(Q_{c,t} \cap O^c\right)$, the set $A_x^N(kt, r, O^c)$ can be covered with at most

$$\frac{C_1}{\theta^{2p}} e^{\delta N k t}\left(1 - \mu\left(\sigma_{4\theta} O\right) + \frac{C_2}{r^p} e^{-\lambda k t} + \frac{8C_1}{\theta^p}\frac{\sqrt{c}}{1-c}\right)^N$$

Bowen (Nkt,θ)-boxes in P.

Proof. Let $0 < c < 1$, take t_c as in Corollary 7.3, and let $0 < r < 1$, $2 \le k \in \mathbb{N}$, $t \in \mathbb{N}t_c$ be such that (9.2) is satisfied, where a', b' are as in Theorem 5.3. Also let $\theta \in \left[r, \frac{r_*}{2}\right]$, and take an open subset O of X. Note that the second inequality in (9.2), together with the fact that $r_0\left(\partial_1(O^c \cap Q_{c,t})\right) \ge r_0(\partial_1 Q_{c,t})$, implies condition (5.10) with O replaced by $O \cup Q_{c,t}^c$. Moreover, condition (5.3) with t replaced by kt follows from the first inequality in (9.2) . Hence, by applying Theorem 5.3 with O replaced with $O \cup Q_{c,t}^c$ and t replaced with kt, we get that for all $x \in \partial_r(O^c \cap Q_{c,t})$ and for all $N \in \mathbb{N}$, the set $A_x^N\left(kt, r, O^c \cap Q_{c,t}\right)$ can be covered with at most $k_1 e^{\delta N k t} a_1^N$ Bowen (Nkt,θ)-boxes in P, where

$$k_1 = \frac{c_2}{c_1}\left(\frac{2r}{\theta}\right)^p, \quad a_1 = 1 - \mu\left(\sigma_{4\theta} O\right) + \frac{C_2}{r^p} e^{-\lambda k t} \tag{9.3}$$

and C_2, λ are as in Theorem 5.3.

Moreover, in view of (9.1) and (7.4), for any $x \in \partial_\theta(O^c \cap Q_{c,t}) \subset \partial_2 Q_{c,t}$ we have

$$\frac{\max(u(x), d)}{\ell_{c,t}^2} \le \frac{\max(C^4 \ell_{c,t}^2, d)}{\ell_{c,t}^2} \underset{\ell_{c,t}^2 \ge \ell_{c,t} > d}{=} C^4.$$

Also, note that

$$kt \underset{(9.2)}{\ge} a' + b' \log\frac{1}{r} > b' \log\frac{1}{r} > b' \ge \frac{\log(8\sqrt{p})}{\lambda_{\min}}. \tag{9.4}$$

Thus, by applying Corollary 7.3 we get that for all $x \in \partial_\theta(O^c \cap Q_{c,t})$ and for all $N \in \mathbb{N}$, the set $A_x^N\left(kt, \theta, Q_{c,t}^c\right)$ can be covered with at most $k_2 e^{\delta N k t} a_2^N$ Bowen (Nkt, θ)-boxes in P, where

$$k_2 = \frac{C^4}{\nu(V_\theta)}, \quad a_2 = \frac{4c}{1-c}. \tag{9.5}$$

Now we put together the estimates we found to get an estimate for the number of Bowen (Nkt, θ)-boxes needed to cover the set $A_x^N(kt, r, O^c)$. Observe that in view of (9.4), we have $kt \ge \frac{\log(8\sqrt{p})}{\lambda_{\min}}$. So, we can apply Proposition 8.1 with $S = O^c$, $Q = Q_{c,t}$ and kt in place of t, and conclude that the set $A_x^N(kt, r, O^c)$ can be covered with at most $k_3 e^{\delta N k t} a_3^N$ Bowen (Nkt, θ)-boxes in P, where k_3, a_3 are as in (8.1), k_1, a_1 are as in (9.3), and k_2, a_2 are as in (9.5).

Finally, we need to estimate $k_3 e^{\delta N k t} a_3^N$ from above. We have

$$\begin{aligned} k_3 &\underset{(8.1)}{=} (1+C_0)\frac{c_2}{c_1}\left(\frac{\theta}{r} + 8\sqrt{p}\right)^p k_1 k_2^2 \\ &\underset{\substack{(9.3)\\(9.5)}}{=} (1+C_0)\left(\frac{c_2}{c_1}\right)^2 \left(\frac{\theta}{r} + 8\sqrt{p}\right)^p \left(\frac{2r}{\theta}\right)^p \left(\frac{C^4}{\nu(V_\theta)}\right)^2 \\ &\underset{(\theta \ge r)}{\le} (1+C_0)\left(\frac{c_2}{c_1}\right)^2 \left(2 + 16\sqrt{p}\right)^p \left(\frac{C^4}{\nu(V_\theta)}\right)^2 \\ &\underset{(3.3)}{\le} (1+C_0)\left(\frac{c_2}{c_1}\right)^2 \left(2 + 16\sqrt{p}\right)^p \left(\frac{(4\sqrt{p})^p}{c_1 \theta^p} C^4\right)^2 = \frac{C_1^2}{\theta^{2p}}, \end{aligned} \tag{9.6}$$

where $C_1 := \sqrt{1+C_0}\frac{c_2}{c_1}\left(2 + 16\sqrt{p}\right)^{p/2} \frac{(4\sqrt{p})^p}{c_1} C^4 \ge 1$. Furthermore, we have

$$\begin{aligned} a_3 &\underset{(8.1)}{=} a_1 + a_2 + \sqrt{k_3 a_2} \\ &\underset{(9.6)}{\le} a_1 + a_2 + \sqrt{\frac{C_1^2}{\theta^{2p}} \cdot a_2} \\ &\underset{\substack{(9.3)\\(9.5)}}{=} 1 - \mu\left(\sigma_{4\theta} O\right) + \frac{C_2}{r^p} e^{-\lambda k t} + \frac{4c}{1-c} + \sqrt{\frac{C_1^2}{\theta^{2p}} \cdot \frac{4c}{1-c}} \\ &\le 1 - \mu\left(\sigma_{4\theta} O\right) + \frac{C_2}{r^p} e^{-\lambda k t} + \frac{8C_1}{\theta^p} \cdot \frac{\sqrt{c}}{1-c}. \end{aligned} \tag{9.7}$$

Therefore, by combining (9.6) and (9.7) we obtain

$$k_3 e^{\delta Nkt} a_3^N \le \frac{C_1}{\theta^{2p}} e^{\delta Nkt}\left(1-\mu(\sigma_{4\theta}O)+\frac{C_2}{r^p}e^{-\lambda kt}+\frac{8C_1}{\theta^p}\frac{\sqrt{c}}{1-c}\right)^N.$$

This ends the proof of the lemma. □

Proof of Theorem 2.1. Let $0<c<1$. Take $t=t_c$ as in Lemma 9.1, and let $Q=Q_{c,t_c}$ be as in (9.1). Also let O be an open subset of X.

Proof of (1): Take $2\le k\in\mathbb{N}$ and $x\in X$. Our goal is to find an upper bound for the Hausdorff dimension of the set $S(k,t,x)$ defined in (2.5). In view of (2.5) and the countable stability of Hausdorff dimension it suffices to estimate the dimension of

$$\left\{h\in\overline{V_{r_*/2}} : g_{Nkt}hx\notin Q\ \forall N\in\mathbb{N}\right\},$$

which, due to (9.1), coincides with $\bigcap_{N\in\mathbb{N}} A_x^N\big(kt,\frac{r_*}{2},X_{>C^3\ell^2_{c,t}}\big)$.

From Corollary 7.3 applied with $\theta=\frac{r_*}{2}$, combined with Lemma 3.3 applied with t replaced by Nkt and $r=\frac{r_*}{2}$, we get that for any $N\in\mathbb{N}$ the set

$$A_x^N\big(kt,\frac{r_*}{2},X_{>C^3\ell^2_{c,t}}\big)$$

can be covered with at most

$$\frac{e^{p\lambda_{\max}Nkt}}{\nu(V_{r_*/2})}\left(\frac{4c}{1-c}\right)^N\frac{\max\big(u(x),d\big)}{\ell^2_{c,t}}$$

balls of radius $\frac{r_*}{2}e^{-\lambda_{\max}Nkt}$ in P. Hence,

$$\begin{aligned}\dim\bigcap_{N\in\mathbb{N}} A_x^N\left(kt,\frac{r_*}{2},X_{>C^3\ell^2_{c,t}}\right) &\le \lim_{N\to\infty}\frac{\log\left(\frac{e^{p\lambda_{\max}Nkt}}{\nu(V_{r_*/2})}\left(\frac{4c}{1-c}\right)^N\frac{\max\left(u(x),d\right)}{\ell^2_{c,t}}\right)}{-\log\frac{r_*}{2}e^{-\lambda_{\max}Nkt}}\\ &=\frac{\log\frac{4c}{1-c}e^{p\lambda_{\max}kt}}{\lambda_{\max}kt}=p-\frac{1}{\lambda_{\max}kt}\log\frac{1-c}{4c}.\end{aligned}$$

Proof of (2): Let $2\le k\in\mathbb{N}$, and $x\in X$. Our goal is to find an upper bound for the Hausdorff dimension of the set

$$\{h\in P\smallsetminus S(k,t,x) : hx\in\widetilde{E}(F^+,O)\}$$

Recall that

$$S(k,t,x)^c=\big\{h\in P : g_{Nkt}hx\in Q \text{ for some } N\in\mathbb{N}\big\}.$$

Therefore

$$\begin{aligned}\{h\in P\smallsetminus S(k,t,x) : hx\in\widetilde{E}(F^+,O)\} &= \left\{h\in P : hx\in\widetilde{E}(F^+,O)\bigcap\left(\bigcup_{N\in\mathbb{N}} g_{-Nkt}Q\right)\right\}\\ &\subset\left\{h\in P : hx\in\bigcup_{N\in\mathbb{N}} g_{-Nkt}\big(Q\cap\widetilde{E}(F^+,O)\big)\right\}\\ &=\bigcup_{N\in\mathbb{N}}\big\{h\in P : hx\in g_{-Nkt}\big(Q\cap\widetilde{E}(F^+,O)\big)\big\}.\end{aligned}$$

Hence, since Hausdorff dimension is countably stable, to complete the proof of this part, it suffices to show that for any $N \in \mathbb{N}$ we have

$$\dim\left\{h \in P : hx \in g_{-Nkt}\big(Q \cap \widetilde{E}(F^+, O)\big)\right\} \leq p - \frac{\mu\big(\sigma_{4\theta} O\big) - \frac{C_2}{r^p} e^{-\lambda kt} - \frac{8C_1}{\theta^p} \frac{\sqrt{c}}{1-c}}{\lambda_{\max} kt}, \tag{9.8}$$

where C_1, C_2, λ as in Lemma 9.1.

Now let $N \in \mathbb{N}$ and suppose $r > 0$ is such that (2.7) is satisfied, where a', b' are as in Lemma 9.1. Note that, since P is normalized by F^+, we have $P = g_{-Nkt} P g_{Nkt}$. Moreover, V_r is a tessellation domain. Hence, by countable stability of Hausdorff dimension, in order to prove (9.8), it suffices to show that the Hausdorff dimension of the set

$$E'_{N,x,r} := \left\{h \in g_{-Nkt} \overline{V_r}\, g_{Nkt} : hx \in g_{-Nkt}\big(Q \cap \widetilde{E}(F^+, O)\big)\right\}$$

is not greater than the right-hand side of (9.8). For any $h \in E'_{N,x,r}$ we have

$$g_{ikt} g_{Nkt} hx = g_{ikt}(g_{Nkt} h g_{-Nkt}) g_{Nkt} x \in O^c \quad \forall\, i \in \mathbb{N},$$

and at the same time $g_{Nkt} h g_{-Nkt} \in \overline{V_r}$. Hence,

$$E'_{N,x,r} \subset g_{-Nkt} \left(\bigcap_{i \in \mathbb{N}} A^i_{g_{Nkt}x}\left(kt, r, O^c\right)\right) g_{Nkt}. \tag{9.9}$$

Also, it is easy to see that if $E'_{N,x,r}$ is non-empty, then

$$g_{Nkt} x \in \overline{V_r}\left(Q \cap O^c\right) \subset \partial_r\left(Q \cap O^c\right).$$

So by applying Lemma 3.3 with r replaced by θ and t replaced with ikt, and Lemma reffirst1 with t replaced by kt, we get that for any $i \in \mathbb{N}$ and any $\theta \in \left[r, \frac{r_*}{2}\right]$, the set $A^i_{g_{Nkt}x}(kt, r, O^c)$ can be covered with at most

$$\frac{C_1}{\theta^{2p}} e^{p\lambda_{\max} ikt} \left(1 - \mu\big(\sigma_{4\theta} O\big) + \frac{C_2}{r^p} e^{-\lambda ikt} + \frac{8C_1}{\theta^p} \frac{\sqrt{c}}{1-c}\right)^i$$

balls of radius $\theta e^{-\lambda_{\max} ikt}$ in P. Also, note that the Hausdorff dimension is preserved by conjugation. So, we have for any $\theta \in \left[r, \frac{r_*}{2}\right]$:

$$\begin{aligned}
\dim E'_{N,x,r} &\underset{(9.9)}{\leq} \dim\left(g_{-Nkt}\left(\bigcap_{i \in \mathbb{N}} A^i_{g_{Nkt}x}\left(kt, r, O^c\right)\right) g_{Nkt}\right) \\
&= \dim \bigcap_{i \in \mathbb{N}} A^i_{g_{Nkt}x}\left(kt, r, O^c\right) \\
&\leq \lim_{i \to \infty} \frac{\log\left(\frac{C_1}{\theta^{2p}} \cdot e^{p\lambda_{\max} ikt} \cdot \left(1 - \mu\big(\sigma_{4\theta} O\big) + \frac{C_2}{r^p} e^{-\lambda ikt} + \frac{8C_1}{\theta^p} \frac{\sqrt{c}}{1-c}\right)^i\right)}{-\log \theta e^{-\lambda_{\max} ikt}} \\
&= p - \frac{-\log\left(1 - \mu\big(\sigma_{4\theta} O\big) + \frac{C_2}{r^p} e^{-\lambda kt} + \frac{8C_1}{\theta^p} \frac{\sqrt{c}}{1-c}\right)}{\lambda_{\max} kt} \\
&\leq p - \frac{\mu\big(\sigma_{4\theta} O\big) - \frac{C_2}{r^p} e^{-\lambda kt} - \frac{8C_1}{\theta^p} \frac{\sqrt{c}}{1-c}}{\lambda_{\max} kt}.
\end{aligned}$$

This finishes the proof. □

10. CONCLUDING REMARKS

10.1. **Effective estimates.** It is a natural problem to effectivize the estimates showing up in the Dimension Drop Conjecture. Previous work of the authors on the subject [19, 20] contained explicit estimates, although with no claims of optimality. Namely, this has been done under the assumption that the complement of O is compact (in particular, when X is compact), and also in the special case (1.2).

In the more general set-up of this paper it is also possible to make the estimates effective. This however would require an additional ingredient: finding a lower bound for the injectivity radii of compact sets $\{x : u_t(x) \le M\}$ arising from condition (ENDP). Such lower bounds can be obtained immediately whenever the following condition is satisfied: Let P be a subgroup of G that has property (ENDP), and let $\{u_t\}_{t\ge t_0}$ be the family of height functions as in Definition 1.5; then there exist positive constants m_0, m such that

$$r_0(x)^{-1} \ge m_0 u_t(x)^{-m} \quad \text{for every } x \in X,\ t \ge t_0. \tag{10.1}$$

This condition can be verified in many special cases. For example, in [28, 3] certain height functions are constructed on homogeneous spaces of semisimple Lie groups without compact factors, and for these height functions (10.1) is verified in [28, Proposition 26] and [3, Lemma 6.3] respectively. By using the same method one can easily show that (10.1) holds for height functions u_t as in Theorem 1.6, and also for the family of height functions constructed in [20] in the case (1.2). A variation of our argument shows that in the presence of (10.1) one has

$$\inf_{x\in X} \operatorname{codim}\big(\{h \in P : hx \in \widetilde{E}(F,O)\}\big) \gg \frac{\mu(O)}{\log \frac{1}{\min(\theta_O,\mu(O),r_1)}},$$

where θ_O is as in (2.9), and $0 < r_1 < \frac{1}{2}$ is a uniform constant independent of O.

10.2. **Removing the Ad-diagonalizability condition.** We expect that by a slight modification of the proof of Theorem 1.1 one can show that this theorem holds when F is an arbitrary one-parameter unbounded subsemigroup of a connected semisimple Lie group G; namely, the condition that F is Ad-diagonalizable is not necessary. Indeed, recall the Jordan decomposition of $F = \{g_t\}$: one can write $g_t = k_t a_t u_t$, where $K_F = \{k_t\}$ is bounded, $A_F = \{a_t\}$ is Ad-diagonalizable, and $U_F = \{u_t\}$ is Ad-unipotent. These subgroups are uniquely determined and commute with each other. If A_F is trivial (in other words, if F is Ad-quasiunipotent) and U_F is not, then Ratner's Measure Classification Theorem and the work of Dani and Margulis (see [30, Lemma 21.2] and [7, Proposition 2.1]) imply that whenever O is non-empty, the set $\widetilde{E}(F,O)$ is contained in a countable union of proper submanifolds of X; hence dimension drop takes place in a stronger form. On the other hand, if A_F is non-trivial, one can modify our argument following the lines of [13, §4], where an analog of (ENDP) was considered with $(I_{f,t}\psi)(x)$ as in (1.4) replaced by a family of operators

$$\psi(\cdot) \mapsto \int_P f(h)\psi(a_t g u_t g^{-1} h\cdot)\, d\nu(h),$$

and with g running through the centralizer of A_F in G.

10.3. **Jointly Dirichlet-improvable systems of linear forms: a dimension bound.** Fix $m, n \in \mathbb{N}$ and, given $c \le 1$, say that $Y \in M_{m,n}$ is *c-Dirichlet improvable* if for all sufficiently large N

$$\begin{gathered}\text{there exist } \mathbf{p} \in \mathbb{Z}^m \text{ and } \mathbf{q} \in \mathbb{Z}^n \smallsetminus \{0\} \text{ such that} \\ \|Y\mathbf{q} - \mathbf{p}\| < cN^{-n/m} \text{ and } 0 < \|\mathbf{q}\| < N.\end{gathered} \tag{10.2}$$

(In this subsection $\|\cdot\|$ stands for the supremum norm on $\mathbb{R}^m$, $\mathbb{R}^n$ and $\mathbb{R}^{m+n}$.) We let $\mathbf{DI}_{m,n}(c)$ be the set of c-Dirichlet improvable $Y \in M_{m,n}$. Dirichlet's theorem (see, e.g., [27]) implies that $\mathbf{DI}_{m,n}(1) = M_{m,n}$. Davenport and Schmidt [8] proved that the Lebesgue measure of $\mathbf{DI}_{m,n}(c)$ is zero for any $c < 1$, and also that $\bigcup_{c<1} \mathbf{DI}_{m,n}(c)$ contains the set of badly approximable $m \times n$ matrices, which is known [26] to have full Hausdorff dimension; in other words, $\dim \mathbf{DI}(c) \to mn$ as $c \to 1$.

Recently in [20] a solution of DDC for the case (1.2), that is for the space X of unimodular lattices in $\mathbb{R}^{m+n}$, was used to derive a dimension drop result for the family $\{\mathbf{DI}_{m,n}(c)\}$: namely, that $\dim \mathbf{DI}_{m,n}(c) < mn$ whenever $c < 1$. Moreover, as explained in [21, Remark 6], a combination of the methods from [20] with measure estimates obtained in [21] can produce an effective estimate for the codimension of $\mathbf{DI}_{m,n}(c)$. The reduction to dynamics goes back to Davenport, Schmidt and Dani [6]. It proceeds by assigning an element $h_Y := \begin{bmatrix} I_m & Y \\ 0 & I_n \end{bmatrix}$ of $G = \mathrm{SL}_{m+n}(\mathbb{R})$ to Y. Arguing as in [22, Proposition 2.1] or [20, Proof of Theorem 1.5], one can see that $Y \in \mathbf{DI}_{m,n}(c)$ if and only if $h_Y \mathbb{Z}^{m+n} \in \widetilde{E}(F, O)$, where

$$O = \left\{\Lambda \in X : \|\mathbf{v}\| \ge c^{\frac{m}{m+n}} \text{ for all } \mathbf{v} \in \Lambda \smallsetminus \{0\}\right\} \tag{10.3}$$

(a subset of X with non-empty interior), and X, F are as in (1.2).

Our new Diophantine application is motivated by [4, §2.7], where Beresnevich and Velani introduced the notion of *jointly singular* k-tuples of matrices. Namely, say that $(Y_1, \dots, Y_k) \in M_{n,m}^k$ is *c-Dirichlet improvable* if for all sufficiently large N

$$\begin{gathered}\text{there exist } \mathbf{p} \in \mathbb{Z}^m, \mathbf{q} \in \mathbb{Z}^n \smallsetminus \{0\} \text{ and } i \in \{1, \dots, k\} \text{ such that} \\ |Y_i\mathbf{q} - \mathbf{p}\| < cN^{-n/m} \text{ and } 0 < \|\mathbf{q}\| < N.\end{gathered} \tag{10.4}$$

Denote the set of c-Dirichlet improvable $(Y_1, \dots, Y_k)$ by $\mathbf{DI}_{m,n}^{(k)}(c)$. Applying Dirichlet's theorem for each k, it is easy to see that $\mathbf{DI}_{m,n}^{(k)}(1) = M_{n,m}^k$. When $c < 1$ one wants for each large N to improve the conclusion of Dirichlet's theorem for at least one of the matrices, and for different N it does not have to be the same matrix. If any one matrix within the tuple is c-Dirichlet improvable, then the entire k-tuple also possesses this property. However, it should be noted that in general, the set $\mathbf{DI}^{(k)}_{m,n}(c)$ could be much larger than the set

$$\{(Y_1,\dots,Y_k)\in M^k_{n,m} : Y_i\in \mathbf{DI}_{m,n}(c) \text{ for some } i=1,\dots,k\}.$$

This raises a problem of showing some sort of dimension drop, which is achieved by reducing the problem to a flow on the product of k copies of X as in (1.2). Indeed, it is not hard to see that the validity of (10.4) for all sufficiently large N is equivalent to the statement that for all sufficiently large t

$$\exists\, \mathbf{v}\in \mathbb{Z}^{m+n}\smallsetminus\{0\} \text{ and } i\in\{1,\dots,k\} \text{ with } \|g_t h_{Y_i}\mathbf{v}\| < c^{\frac{m}{m+n}}. \tag{10.5}$$

In its turn, (10.5) is equivalent to

$$(g_t h_{Y_1}\mathbb{Z}^{m+n},\dots,g_t h_{Y_k}\mathbb{Z}^{m+n}) \notin O\times\cdots\times O,$$

where O is as in (10.3). We conclude that $(Y_1,\dots,Y_k)\in \mathbf{DI}^{(k)}_{m,n}(c)$ if and only if $(h_{Y_1}\mathbb{Z}^{m+n},\dots,h_{Y_k}\mathbb{Z}^{m+n})\in \widetilde{E}(F^{(k)},O\times\cdots\times O)$, where

$$F^{(k)} := \{(g_t,\dots,g_t) : t\ge 0\}\subset \prod_{i=1}^k G$$

is acting on $X^{(k)} := \prod_{i=1}^k X$.

Since $F^{(k)}$ is a diagonalizable subsemigroup of $\prod_{i=1}^k G$ whose expanding horospherical subgroup is precisely

$$H^{(k)} := \prod_{i=1}^k \{h_Y : Y\in M_{m,n}\},$$

it follows from Theorem 1.6 that $H^{(k)}$ has property (ENDP) with respect to $(X^{(k)},F^{(k)})$. Moreover, since the action of F on X is exponentially mixing, by using Fubini's Theorem it is straightforward to check that the action of $F^{(k)}$ on $X^{(k)}$ is exponentially mixing as well; hence, by Theorem 1.4, $H^{(k)}$ has property (ENDP) with respect to $(X^{(k)},F^{(k)})$. Therefore, we can apply Theorem 1.1 with $P=H^{(k)}$ and arrive at

THEOREM 10.1. *The Hausdorff dimension of* $\mathbf{DI}^{(k)}_{m,n}(c)$ *is strictly less than* kmn *for any* $c<1$ *and* $k\in\mathbb{N}$.

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REFERENCES

[1] H. Al-Saqban, P. Apisa, A. Erchenko, O. Khalil, S. Mirzadeh and C. Uyanik, Exceptional directions for the Teichmüller geodesic flow and Hausdorff dimension, *J. Eur. Math. Soc.*, **23** (2021), 1423–1476.

[2] J. An, L. Guan, A. Marnat and R. Shi, Divergent trajectories on products of homogeneous spaces, *Adv. Math.*, **390** (2021), Paper No. 107910.

[3] Y. Benoist and J.-F. Quint, Mesures stationnaires et fermés invariants des espaces homogénes, *Ann. of Math.*, **174** (2011), 1111–1162.

[4] V. Beresnevich and S. Velani, Number theory meets wireless communications: An introduction for dummies like us, in *Number Theory Meets Wireless Communications*, 1–67, Math. Eng., Springer, 2020.

[5] Y. Cheung, Hausdorff dimension of the set of singular pairs, *Ann. of Math.*, **173** (2011), 127–167.

[6] S. G. Dani, Divergent trajectories of flows on homogeneous spaces and Diophantine approximation, *J. Reine Angew. Math.*, **359** (1985), 55–89.

[7] S. G. Dani and G. A. Margulis, Limit distributions of orbits of unipotent flows and values of quadratic forms, in *I. M. Gelfand Seminar*, 91–137, Adv. Soviet Math., **16**, Part 1, Amer. Math. Soc., Providence, RI, 1993.

[8] H. Davenport and W. M. Schmidt, Dirichlet's theorem on diophantine approximation, in *Symposia Mathematica, Vol. IV (INDAM, Rome, 1968/69)*, 113–132, Academic Press, 1970.

[9] M. Einsiedler, S. Kadyrov and A. Pohl, Escape of mass and entropy for diagonal flows in real rank one situations, *Israel J. Math.*, **210** (2015), 245–295.

[10] A. Eskin and G. Margulis, Recurrence properties of random walks on finite volume homogeneous manifolds, in *Random Walks and Geometry*, 431–444, de Gruiter, Berlin, 2004.

[11] A. Eskin, G. Margulis and S. Mozes, Upper bounds and asymptotics in a quantitative version of the Oppenheim conjecture, *Ann. Math.*, **147** (1998), 93–141.

[12] A. Eskin and S. Mozes, Margulis functions and their applications, in *Dynamics, Geometry, Number Theory: The Impact of Margulis on Modern Mathematics*, 342–361, University of Chicago Press, 2022.

[13] L. Guan and R. Shi, Hausdorff dimension of divergent trajectories on homogeneous spaces, *Compos. Math.*, **156** (2020), 340–359.

[14] S. Kadyrov, Exceptional sets in homogeneous spaces and Hausdorff dimension, *Dyn. Syst.*, **30** (2015), 149–157.

[15] S. Kadyrov, D. Kleinbock, E. Lindenstrauss and G. A. Margulis, Singular systems of linear forms and non-escape of mass in the space of lattices, *J. Anal. Math.*, **133** (2017), 253–277.

[16] D. Kelmer and P. Sarnak, Strong spectral gaps for compact quotients of products of $\mathrm{PSL}(2,\mathbb{R})$, *J. Eur. Math. Soc.*, **11** (2009), 283–313.

[17] D. Y. Kleinbock and G. A. Margulis, Bounded orbits of nonquasiunipotent flows on homogeneous spaces, in *Sinaí's Moscow Seminar on Dynamical Systems*, 141–172, Amer. Math. Soc. Trans. Ser. 2, vol. 171, Amer. Math. Soc., Providence, RI, 1996.

[18] D. Y. Kleinbock and G. A. Margulis, On effective equidistribution of expanding translates of certain orbits in the space of lattices, in *Number Theory, Analysis and Geometry*, 385–396, Springer, New York, 2012.

[19] D. Kleinbock and S. Mirzadeh, Dimension estimates for the set of points with non-dense orbit in homogeneous spaces, *Math. Z.*, **295** (2020), 1355–1383.

[20] D. Kleinbock and S. Mirzadeh, On the dimension drop conjecture for diagonal flows on the space of lattices, *Adv. Math.*, **425** (2023), Paper No. 109058.

[21] D. Kleinbock, A. Strömbergsson and S. Yu, A measure estimate in geometry of numbers and improvements to Dirichlet's theorem, *Proc. London Math. Soc.*, **125** (2022), 778–824.

[22] D. Kleinbock and B. Weiss, Dirichlet's theorem on Diophantine approximation and homogeneous flows, *J. Mod. Dyn.*, **2** (2008), 43–62.

[23] D. Kleinbock and B. Weiss, Modified Schmidt games and a conjecture of Margulis, *J. Mod. Dyn.*, **7** (2013), 429–460.

[24] G. A. Margulis, *On Some Aspects of the Theory of Anosov Systems*, Springer Monographs in Mathematics, Springer-Verlag, Berlin, 2004.

[25] F. Rodriguez Hertz and Z. Wang, On ε-escaping trajectories in homogeneous spaces, *Discrete Contin. Dyn. Syst.*, **41** (2021), 329–357.

[26] W. M. Schmidt, Badly approximable systems of linear forms, *J. Number Theory*, **1** (1969), 139–154.

[27] W. M. Schmidt, Diophantine Approximation, Lecture Notes in Mathematics, vol. 785, Springer-Verlag, Berlin, 1980.

[28] A. Sanchez and J. Seong, An avoidance principle and Margulis functions for expanding translates of unipotent orbits, *J. Mod. Dyn.*, **20** (2024), 409–439.

[29] R. Shi, Pointwise equidistribution for one parameter diagonalizable group action on homogeneous space, *Trans. Amer. Math. Soc.*, **373** (2020), 4189–4221.

[30] A. N. Starkov, *Dynamical Systems on Homogeneous Spaces*, Transl. Math. Monogr., vol. 190, Amer. Math. Soc., Providence, RI, 2000.

[31] H. Wegmann, Die Hausdorff-Dimension von kartesischen Produktmengen in metrischen Räumen, *J. Reine Angew. Math.*, **234** (1969), 163–171.

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