



METRICAL PROPERTIES OF FUNCTIONS OF CONSECUTIVE DIGITS IN LÜROTH SERIES

Dmitry Kleinbock

Department of Mathematics, Brandeis University, Waltham, Massachusetts
 kleinboc@brandeis.edu

Yuqing Zhang¹

Department of Basic Courses, Wuhan Donghu University, Hubei, China
 yuqingnanjing@gmail.com

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Abstract

Recently, the growth of the products of Lüroth quotients $d_i(x)$ in the Lüroth expansion of a real number x was studied in connection with improvements to Dirichlet's theorem. In this paper, for a non-decreasing positive measurable function $F(x_1, \dots, x_m)$ and a function $\phi : \mathbb{N} \rightarrow \mathbb{R}_{>0}$, we consider the following set:

$$\mathcal{E}_F(\phi) = \{x \in [0, 1] : F(d_n(x), \dots, d_{n+m-1}(x)) \geq \phi(n) \text{ for infinitely many } n \in \mathbb{N}\},$$

and obtain its Lebesgue measure $\lambda(\mathcal{E}_F(\phi))$. As an application of our result, we consider the case when $F(x_1, \dots, x_m) = x_1 + \dots + x_m$.

1. Introduction

Every $x \in (0, 1]$ can be uniquely expressed in the form of a Lüroth series

$$x = \frac{1}{d_1(x)} + \frac{1}{d_1(x)(d_1(x) - 1)d_2(x)} + \frac{1}{d_1(x)(d_1(x) - 1)d_2(x)(d_2(x) - 1)d_3(x)} + \dots,$$

where $d_n(x) \in \mathbb{N}$, $d_n(x) \geq 2$, $n = 1, 2, 3, \dots$. Lüroth series can be studied through the Lüroth map $\mathcal{L}$, whose definition will be addressed in Section 2.

Let λ denote the Lebesgue measure on $\mathbb{R}$. H. Jager and C. de Vroedt [3] showed that $\mathcal{L}$ is measure-preserving and ergodic. Also, they noted that the Lüroth quotients $d_1(x), d_2(x), \dots$ can be viewed as random variables and are independent.

Throughout this paper, let $\Psi : \mathbb{N} \rightarrow \mathbb{R}_{>0}$ be a positive function. We consider the Lebesgue measure of the following set:

$$\mathcal{E}(\Psi) := \{x \in (0, 1] : d_n(x) \geq \Psi(n) \text{ for infinitely many } n \in \mathbb{N}\}.$$

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¹Corresponding author

The following theorem is the Lüroth series analogue of the Borel-Bernstein Theorem for continued fractions.

Theorem 1 ([3]). *The Lebesgue measure of $\mathcal{E}(\Psi)$ is given by*

$$\lambda(\mathcal{E}(\Psi)) = \begin{cases} 0 & \text{if } \sum_{n=1}^{\infty} \Psi(n)^{-1} < \infty \\ 1 & \text{if } \sum_{n=1}^{\infty} \Psi(n)^{-1} = \infty. \end{cases}$$

Brown-Sarre, Robert, and Hussain [2] considered a weighted product of consecutive Lüroth digits: given $m \in \mathbb{N}$ and $\mathbf{t} = (t_0, \dots, t_{m-1}) \in \mathbb{R}_{>0}^m$, they defined

$$\mathcal{E}_{\mathbf{t}}(\Psi) := \left\{ x \in (0, 1] : \prod_{i=0}^{m-1} d_{n+i}^{t_i}(x) \geq \Psi(n) \text{ for infinitely many } n \in \mathbb{N} \right\}$$

and proved the following dichotomy statement for the Lebesgue measure of $\mathcal{E}_{\mathbf{t}}(\Psi)$.

Theorem 2 ([2]). *Let $m \in \mathbb{N}$, $\Psi : \mathbb{N} \rightarrow \mathbb{R}_{>0}$ and $\mathbf{t} = (t_0, \dots, t_{m-1}) \in \mathbb{R}_{>0}^m$. If*

$$\liminf_{n \rightarrow \infty} \Psi(n) > 1, \quad (1)$$

then

$$\lambda(\mathcal{E}_{\mathbf{t}}(\Psi)) = \begin{cases} 0 & \text{if } \sum_{n=1}^{\infty} \frac{(\log \Psi(n))^{\ell(\mathbf{t})-1}}{\Psi(n)^{1/T}} < \infty \\ 1 & \text{if } \sum_{n=1}^{\infty} \frac{(\log \Psi(n))^{\ell(\mathbf{t})-1}}{\Psi(n)^{1/T}} = \infty \end{cases}$$

where

$$T := \max\{t_0, t_1, \dots, t_{m-1}\}, \quad \ell(\mathbf{t}) := \#\{j \in \{0, \dots, m-1\} : t_j = T\}. \quad (2)$$

A similar result for weighted products of multiple partial quotients in continued fractions was obtained earlier in [1].

In this paper, for a fixed $m \in \mathbb{N}$, we consider positive non-decreasing measurable m -variable function $F(x_1, x_2, \dots, x_m)$ defined on $[2, \infty) \times [2, \infty) \times \dots \times [2, \infty)$. The function F is said to be non-decreasing if for all fixed a_i, b_i ($a_i \geq 2, b_i \geq 2, 1 \leq i \leq m$), the following condition holds:

$$a_i \leq b_i \text{ for all } i, 1 \leq i \leq m \Rightarrow F(a_1, a_2, \dots, a_m) \leq F(b_1, b_2, \dots, b_m).$$

We define

$$\mathcal{E}_F(\Psi) = \{x \in [0, 1] : F(d_n, \dots, d_{n+m-1}) \geq \Psi(n) \text{ for infinitely many } n \in \mathbb{N}\}.$$

It is clear that $\mathcal{E}_F(\Psi)$ generalizes $\mathcal{E}_{\mathbf{t}}(\Psi)$ in Theorem 2, with

$$F(x_1, x_2, \dots, x_m) = \prod_{i=1}^m x_i^{t_i}. \quad (3)$$

Our first theorem gives a general zero – one law for the sets $\mathcal{E}_F(\Psi)$. Let $\mathbb{D} = \mathbb{N}_{\geq 2}$, and define

$$H(x) = \sum_{\substack{F(x_1, x_2, \dots, x_m) \geq x, \\ x_1, \dots, x_m \in \mathbb{D}}} \prod_{i=1}^m \frac{1}{x_i(x_i - 1)}. \quad (4)$$

Theorem 3. *Let $\Psi : \mathbb{N} \rightarrow \mathbb{R}_{>0}$. Suppose $F(x_1, \dots, x_m) > 0$ for $x_1, \dots, x_m \in \mathbb{D}$, and let $H(x)$ be defined as in (4). Then*

$$\lambda(\mathcal{E}_F(\Psi)) = \begin{cases} 0 & \text{if } \sum_{n=1}^{\infty} H(\Psi(n)) < \infty \\ 1 & \text{if } \sum_{n=1}^{\infty} H(\Psi(n)) = \infty. \end{cases}$$

Proof of Theorem 2 assuming Theorem 3. Let $t := \min\{t_0, t_1, \dots, t_{m-1}\}$. According to [2, Lemma 3.4], if $m \in \mathbb{N}$, $\mathbf{t} \in \mathbb{R}_{>0}^m$, and $g \geq 2^{mt}$, then

$$\sum_{\substack{d_1^{t_0} \dots d_m^{t_{m-1}} \geq g \\ d_1, \dots, d_m \in \mathbb{D}}} \prod_{j=1}^m \frac{1}{d_j(d_j - 1)} \asymp_{m, \mathbf{t}} \frac{\log^{\ell(\mathbf{t})-1} g}{g^{1/T}},$$

where T and $\ell(\mathbf{t})$ are as in (2). Thus, if we assume that $\Psi(n) > 2^{mt}$ for all large enough $n \in \mathbb{N}$ and take F as in (3), it follows that

$$H(\Psi(n)) \asymp_{m, \mathbf{t}} \frac{(\log \Psi(n))^{\ell(\mathbf{t})-1}}{\Psi(n)^{1/T}},$$

which makes the convergence/divergence conditions of Theorems 2 and 3 coincide.

It remains to consider the case when there are infinitely many $n \in \mathbb{N}$ such that $1 < \Psi(n) \leq 2^{mt}$. Then, in view of (1), we can choose a real number y and an increasing sequence of natural numbers $(n_j)_{j \geq 1}$ such that

$$1 < y < \Psi(n_j) \leq 2^{mt} \quad \text{for all } j \in \mathbb{N}. \quad (5)$$

Then

$$\log^{\ell(\mathbf{t})-1} y < \log^{\ell(\mathbf{t})-1} \Psi(n_j) \quad \text{and} \quad 2^{-mt/T} \leq \Psi(n_j)^{-1/T} \quad \text{for all } j \in \mathbb{N},$$

and therefore,

$$\sum_{j=1}^{\infty} \frac{\log^{\ell(\mathbf{t})-1} \Psi(n_j)}{\Psi(n_j)^{1/T}} = \infty.$$

Meanwhile, for any $x \in (0, 1]$ and any $n \in \mathbb{N}$, we have $\prod_{i=0}^{m-1} d_{n+i}^{t_i}(x) \geq 2^{mt}$, and thus from (5) we have $\mathcal{E}_{\mathbf{t}}(\Psi) = (0, 1]$. Hence Theorem 2 follows from Theorem 3. $\square$

For computations it is convenient to replace the summation in (4) with integration. For that, one needs to impose some regularity conditions on F . Namely, suppose that

$$\sup_{x_i \geq 2, 1 \leq i \leq m} \frac{F(x_1, x_2, \dots, x_m)}{F(\lfloor x_1 \rfloor, \lfloor x_2 \rfloor, \dots, \lfloor x_m \rfloor)} = M < \infty. \quad (6)$$

Set

$$G(x) = \int \cdots \int_{\substack{F(x_1, \dots, x_m) \geq x, \\ x_1 \geq 2, \dots, x_m \geq 2}} \frac{1}{x_1^2 \cdots x_m^2} dx_1 \cdots dx_m. \quad (7)$$

Theorem 4. *Let $\Psi : \mathbb{N} \rightarrow \mathbb{R}_{>0}$. Suppose $F(x_1, \dots, x_m) > 0$ for $x_i \geq 2$, $1 \leq i \leq m$, and that F is non-decreasing and satisfies (6). Let $G(x)$ be defined as in (7). Then*

$$\lambda(\mathcal{E}_F(\Psi)) = \begin{cases} 0 & \text{if } \sum_{n=1}^{\infty} G(\Psi(n)) < \infty \\ 1 & \text{if } \sum_{n=1}^{\infty} G(M\Psi(n)) = \infty, \end{cases}$$

where M is as in (6).

As an example, this theorem can be applied to the function

$$F(x_1, \dots, x_m) = x_1 + \dots + x_m. \quad (8)$$

in the following way: if $\Psi : \mathbb{N} \rightarrow \mathbb{R}_{\geq m}$ and $F(x)$ is defined as in (8), then

$$\lambda(\mathcal{E}_F(\Psi)) = \begin{cases} 0 & \text{if } \sum_{n=1}^{\infty} \frac{1}{\Psi(n)} < \infty \\ 1 & \text{if } \sum_{n=1}^{\infty} \frac{1}{\Psi(n)} = \infty \end{cases} \quad (9)$$

Indeed, one can show that for any $m \in \mathbb{N}$ and $a \geq m$,

$$\int \cdots \int_{\substack{x_1 + x_2 + \dots + x_m \geq a, \\ x_1 \geq 1, \dots, x_m \geq 1}} \frac{1}{x_1^2 \cdots x_m^2} dx_1 \cdots dx_m \asymp_m \frac{1}{a}$$

(the upper bound is derived in [6, (3.2)], and the lower bound can be established by a similar method).

We would like to remark though that (9) can be easily shown without using Theorem 4. Namely, if $\sum_{n \in \mathbb{N}} \frac{1}{\Psi(n)}$ converges, then the same is true for $\sum_{n \in \mathbb{N}} \frac{1}{\min(\Psi(n), \dots, \Psi(n-m+1))}$. The set

$$\{x \in (0, 1] : d_n + \dots + d_{n+m-1} > \Psi(n)\} \text{ for infinitely many } n\}$$

is contained in

$$\{x \in (0, 1] : d_n > \min(\Psi(n), \dots, \Psi(n-m+1)) \text{ for infinitely many } n\},$$

which has measure 0 from Theorem 1. On the other hand, if $\sum_{n=1}^{\infty} \frac{1}{\Psi(n)} = \infty$, then by the divergence case of Theorem 1, for Lebesgue-a.e. x , one has $d_n(x) \geq \Psi(n)$ for infinitely many $n \in \mathbb{N}$, and hence the same is true for $d_n(x) + \dots + d_{n+m-1}(x)$.

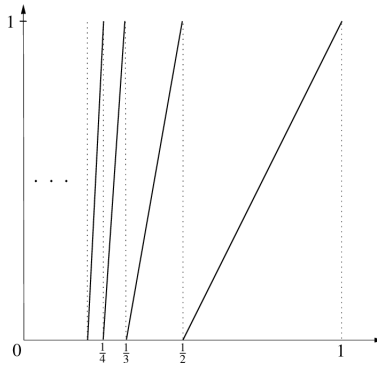


Figure 1: The Lüroth map

2. Lüroth Series

Let $d_1 : (0, 1] \rightarrow \mathbb{D} = \mathbb{N}_{\geq 2}$ be defined as $d_1(x) = \left\lfloor \frac{1}{x} \right\rfloor + 1$. Then d_1 assigns to each $x \in (0, 1]$ an integer greater than 1. The Lüroth map $\mathcal{L}(x) : [0, 1] \rightarrow [0, 1]$ is defined by

$$\mathcal{L}(x) = \begin{cases} d_1(x)(d_1(x) - 1)x - (d_1(x) - 1) & \text{if } x \in (0, 1] \\ 0 & \text{if } x = 0 \end{cases}$$

Figure 1 shows the graph of the Lüroth map. It is constructed similarly to the Gauss map used to define continued fractions. However the Gauss map is piecewise fractional-linear, while the Lüroth map has linear branches. This makes the analysis of Lüroth series easier than that of continued fractions.

For $x \in (0, 1]$ and $n \in \mathbb{D}$, we set $d_n(x) = d_1(\mathcal{L}^{n-1}(x))$, where the exponent $n - 1$ denotes the $(n - 1)$ -th iteration. We call $\mathcal{L}$ a shift operator. If x has Lüroth digits $(c_1, c_2, \dots)$, then $\mathcal{L}(x)$ has Lüroth digits $(c_2, c_3, \dots)$. For $\mathbf{c} = (c_1, \dots, c_n) \in \mathbb{D}^n$, we define the cylinder of level n based at $\mathbf{c}$ as

$$I_n(\mathbf{c}) := \{x \in (0, 1] : d_1(x) = c_1, \dots, d_n(x) = c_n\}.$$

We set $I_n(\mathbf{c})$ to be the set of all real numbers in $(0, 1]$ whose Lüroth expansion begin with $(c_1, \dots, c_n)$.

For each $\mathbf{c} = (c_1, \dots, c_n) \in \mathbb{D}^n$, set

$$\langle c_1, \dots, c_n \rangle := \frac{1}{c_1} + \frac{1}{c_1(c_1 - 1)c_2} + \dots + \frac{1}{c_1(c_1 - 1)c_2(c_2 - 1) \dots c_{n-1}(c_{n-1} - 1)c_n}.$$

Proposition 1 ([2]). For every $n \in \mathbb{N}$ and $\mathbf{c} = (c_1, \dots, c_n) \in \mathbb{D}^n$, we have

$$I_n(\mathbf{c}) = (\langle c_1, \dots, c_n \rangle, \langle c_1, \dots, c_{n-1} \rangle),$$

and therefore

$$\lambda(I_n(\mathbf{c})) = \prod_{j=1}^n \frac{1}{c_j(c_j - 1)}. \quad (10)$$

3. Proof of Theorem 3

In this section, our arguments largely follow those of Huang, Wu, and Xu [4]. Fix an $m \in \mathbb{N}$. Let $n \in \mathbb{N}$. Suppose that F is a positive non-decreasing m -variable measurable function. Define

$$A_n = \{x \in (0, 1] : F(d_1, \dots, d_m) \geq \Psi(n)\}.$$

It is clear that $x \in \mathcal{E}_F(\Psi)$ if and only if $\mathcal{L}^{n-1}(x) \in A_n$ for infinitely many $n \in \mathbb{N}$. Clearly A_n can be written as a union of a collection of m -th order cylinders:

$$A_n = \bigcup_{\substack{F(d_1, \dots, d_m) \geq \Psi(n) \\ d_1, \dots, d_m \in \mathbb{D}}} I_m(d_1, \dots, d_m).$$

Therefore,

$$\lambda(A_n) = \sum_{\substack{F(d_1, \dots, d_m) \geq \Psi(n) \\ d_1, \dots, d_m \in \mathbb{D}}} \lambda(I_m(d_1, \dots, d_m)).$$

From (10), we have

$$\lambda(A_n) = \sum_{\substack{F(d_1, \dots, d_m) \geq \Psi(n) \\ d_1, \dots, d_m \in \mathbb{D}}} \prod_{k=1}^m \frac{1}{d_k(d_k - 1)}. \quad (11)$$

The following theorem is Lemma 3.2 of [2].

Theorem 5. Fix $m \in \mathbb{N}$. Let $(A_n)_{n \geq 1}$ be a sequence of at most countable unions of cylinders of level m . We have

$$\lambda\left(\limsup_{n \rightarrow \infty} \mathcal{L}^{-n}[A_n]\right) = \begin{cases} 0 & \text{if } \sum_{n=1}^{\infty} \lambda(A_n) < \infty \\ 1 & \text{if } \sum_{n=1}^{\infty} \lambda(A_n) = \infty \end{cases}$$

Proof of Theorem 3. By (11), the series

$$\sum_{n=1}^{\infty} H((\Psi(n))) \quad (12)$$

converges (respectively, diverges) if and only if so does the series $\sum_{n=1}^{\infty} \lambda(A_n)$. By Theorem 5, this implies that the series (12) converges (respectively, diverges) if and only if the set

$$\{x \in (0, 1] : \mathcal{L}^n(x) \in A_n \text{ for infinitely many } n \in \mathbb{N}\}$$

has zero (respectively, full) measure. The latter, since $\mathcal{L}$ is measure-preserving, amounts to saying that $\lambda(\mathcal{E}_F\Psi)$ is equal to zero (respectively, one). $\square$

In order to prove Theorem 4, we give the upper and lower bounds for $\lambda(A_n)$.

Theorem 6. *For $n \in \mathbb{N}$, we have $\lambda(A_n) \leq 3^m G(\Psi(n))$.*

Proof. It is clear that

$$\frac{1}{d_k(d_k - 1)} \leq \frac{3}{d_k(d_k + 1)}, \quad d_k \geq 2.$$

From (11), we have

$$\begin{aligned} \lambda(A_n) &\leq 3^m \sum_{\substack{F(d_1, \dots, d_m) \geq \Psi(n) \\ d_1, \dots, d_m \in \mathbb{D}}} \frac{1}{d_k(d_k + 1)} = 3^m \sum_{\substack{F(d_1, \dots, d_m) \geq \Psi(n) \\ d_1, \dots, d_m \in \mathbb{D}}} \prod_{k=1}^m \int_{d_k}^{d_k+1} \frac{1}{x_k^2} dx_k \\ &= 3^m \sum_{\substack{F(d_1, \dots, d_m) \geq \Psi(n) \\ d_1, \dots, d_m \in \mathbb{D}}} \int_{d_1}^{d_1+1} \cdots \int_{d_m}^{d_m+1} \frac{1}{x_1^2 \cdots x_m^2} dx_1 \cdots dx_m. \end{aligned}$$

The domain of the above integration is

$$Q_1 = \{[d_1, d_1+1) \times \cdots \times [d_m, d_m+1) : F(d_1, \dots, d_m) \geq \Psi(n), d_1, \dots, d_m \in \mathbb{D}\}, \quad (13)$$

which is a subset of

$$\{(x_1, \dots, x_m) : x_1 \geq 2, \dots, x_m \geq 2, F(x_1, \dots, x_m) \geq \Psi(n)\}$$

since F is non-decreasing. Therefore,

$$\lambda(A_n) \leq 3^m \int \cdots \int_{\substack{F(x_1, \dots, x_m) \geq \Psi(n), \\ x_1 \geq 2, \dots, x_m \geq 2}} \frac{1}{x_1^2 \cdots x_m^2} dx_1 \cdots dx_m = 3^m G(\Psi(n)).$$

$\square$

Theorem 7. *For $n \in \mathbb{N}$, we have $\lambda(A_n) \geq G(M\Psi(n))$, where M is defined in (6).*

Proof. From (11) we have

$$\begin{aligned} \lambda(A_n) &= \sum_{\substack{F(d_1, \dots, d_m) \geq \Psi(n) \\ d_1, \dots, d_m \in \mathbb{D}}} \prod_{k=1}^m \frac{1}{d_k(d_k - 1)} \geq \sum_{\substack{F(d_1, \dots, d_m) \geq \Psi(n) \\ d_1, \dots, d_m \in \mathbb{D}}} \prod_{k=1}^m \int_{d_k}^{d_k+1} \frac{1}{x_k^2} dx_k \\ &= \int \cdots \int_{Q_1} \frac{1}{x_1^2 \cdots x_m^2} dx_1 \cdots dx_m, \end{aligned} \quad (14)$$

where Q_1 is defined in (13). Let

$$Q_2 = \{(x_1, \dots, x_m) : x_1 \geq 2, \dots, x_m \geq 2, F(x_1, \dots, x_m) \geq M\Psi(n)\}.$$

Then $Q_2 \subset Q_1$. To see this, note that if $(x_1, \dots, x_m) \in Q_2$, then $F(x_1, \dots, x_m) \geq M\Psi(n)$. From (6), we have

$$F(\lfloor x_1 \rfloor, \dots, \lfloor x_m \rfloor) \geq \Psi(n).$$

Therefore,

$$(x_1, \dots, x_m) \in [\lfloor x_1 \rfloor, \lfloor x_1 \rfloor + 1) \times \dots \times [\lfloor x_m \rfloor, \lfloor x_m \rfloor + 1) \subset Q_1.$$

From (14), we have

$$\mathcal{L}(A_n) \geq \int \dots \int_{Q_2} \frac{1}{x_1^2 \dots x_m^2} dx_1 \dots dx_m.$$

From (7), we have $\lambda(A_n) \geq G(M\Psi(n))$. □

Proof of Theorem 4. Set

$$\mathcal{R}_n = \{(d_1, \dots, d_m) \in \mathbb{D}^m : F(d_1, \dots, d_m) \geq \Psi(n)\},$$

and let $A_n = \bigcup_{\mathbf{d} \in \mathcal{R}_n} I_m(\mathbf{d})$.

If $\sum_{n=1}^{\infty} G(\Psi(n)) < \infty$, then by Theorem 6, we have

$$\sum_{n=1}^{\infty} \lambda(A_n) < \infty.$$

By Theorem 5, $\lambda(\{x \in (0, 1] : \mathcal{L}^n(x) \in A_n \text{ for infinitely many } n \in \mathbb{N}\}) = 0$. Therefore,

$$\lambda(\mathcal{E}_F(\Psi)) = \lambda(\{x \in (0, 1] : \mathcal{L}^{n-1}(x) \in A_n \text{ for infinitely many } n \in \mathbb{N}\}).$$

Since $\mathcal{L}$ is measure-preserving, we have $\lambda(\mathcal{E}_F(\Psi)) = 0$.

On the other hand, if $\sum_{n=1}^{\infty} G(M\Psi(n)) = \infty$, then by Theorem 7, we have

$$\sum_{n=1}^{\infty} \lambda(A_n) = \infty.$$

By Theorem 5, $\lambda(\{x \in (0, 1] : \mathcal{L}^n(x) \in A_n \text{ for infinitely many } n \in \mathbb{N}\}) = 1$. From the fact that $\mathcal{L}$ is measure-preserving, we have $\lambda(\mathcal{E}_F(\Psi)) = 1$. □

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