ACCESSIBILITY AND ERGODICITY FOR COLLAPSED ANOSOV FLOWS

SERGIO R. FENLEY AND RAFAEL POTRIE

Abstract. We consider a class of partially hyperbolic diffeomorphisms introduced in [BFP] which is open and closed and contains all known examples. If in addition the diffeomorphism is non-wandering, then we show it is accessible unless it contains a *su*-torus. This implies that these systems are ergodic when they preserve volume, confirming a conjecture by Hertz-Hertz-Ures [CHHU, Conjecture 2.11] for this class of systems.

Keywords: Partial hyperbolicity, ergodicity, accessibility, Anosov flows, foliations. 3-manifold topology, foliations.

1. Introduction

In this paper we study the ergodicity problem for 3 dimensional partially hyperbolic diffeomorphisms. It has been shown in [HHU] that ergodicity (in fact the K-property) is abundant, establishing a conjecture of Pugh and Shub in this setting [PS]. This lead to the belief that non-ergodic partially hyperbolic diffeomorphisms in dimension 3 can be described [HHU_2] (see also [CHHU]).

Many recent works including our own, have tried to show that in certain manifolds or isotopy classes ergodicity holds for *all* volume preserving partially hyperbolic diffeomorphisms (see [HHU_2, HU, GS, FP, HRHU]). Thanks to the reductions of [GPS, BuW, HHU] the problem boils down to the study of accessibility, and this has become a problem on its own (see [HP, H, HS, FP] for general results about accessibility).

Here, instead of fixing a manifold or isotopy class, we work on a specific class of partially hyperbolic diffeomorphisms that were introduced in [BFP] motivated by [BFFP, $BFFP_2$, FP_2]. They are called *collapsed Anosov flows*. Very roughly the dynamics of the partially hyperbolic diffeomorphism is semiconjugated to a self orbit equivalence of an Anosov flow, and the semiconjugacy sends flow lines to curves tangent to the center bundle. See section 2 for a precise definition. The class of partially hyperbolic diffeomorphisms which are collapsed Anosov flows is an open and closed subset of all partially hyperbolic diffeomorphisms, and in addition it includes all known examples in manifolds with non-solvable fundamental group. We note that this class is strictly larger than the ones previously known to satisfy [CHHU, Conjecture 2.11] as it contains the partially hyperbolic diffeomorphisms in the connected component of the examples constructed in [BGP, BGHP].

Our main result is the following:

Theorem A. *Let $f : M \to M$ be a collapsed Anosov flow of a closed 3-manifold M whose fundamental group is not virtually solvable. Assume that the non-wandering set of f is all of M. Then, f is accessible. In particular, if f is C^{1+} and volume preserving it is a K-system (and consequently ergodic and mixing).*

S.F. was partially supported by Simons Foundation grant numbers 280429 and 637554. R. P. was partially supported by CSIC 618.

Again, precise definitions are given in § 2. It is worth noting that in [BGP, BGHP] the argument to get ergodic examples is perturbative, so not even the concrete examples constructed there were known to be ergodic (but it was known that an open and dense subset of the examples in a volume preserving neighborhood were). The ergodicity for specific examples follows from the results of this paper.

We note that the assumption that M does not have solvable fundamental group is necessary since the time one map of a suspension of a toral automorphism is not accessible (and in fact contains a *su*-torus). We refer the reader to [H] for a complete treatment of accessibility in the class of 3-manifolds with virtually solvable fundamental group.

Theorem A is complementary to what was done in [FP] but some results have intersection (in [FP] it is proved that a more restrictive class, that of *discretized Anosov flows*, are always accessible without the non-wandering assumption). In fact, the proof of Theorem A uses [FP] at some point. However, a slightly weaker version of Theorem A can be proven independently of [FP] (see § 6.2).

In [FP] accessibility and ergodicity are established unconditionally in certain manifolds or isotopy classes of diffeomorphisms. On the other hand, in [FP$_2$] we showed that every partially hyperbolic diffeomorphism in a hyperbolic 3-manifold is a collapsed Anosov flow, so this paper gives a different proof[1] of some of the results of [FP]. Note that in [FP$_3$] we plan to show that every partially hyperbolic diffeomorphism in a Seifert 3-manifold is a collapsed Anosov flow, so this paper will imply that [CHHU, Conjecture 2.11] also holds for Seifert manifolds extending [HRHU, FP] where this was shown for certain isotopy classes.

The results of this article emphasize what [BFP] proposes: it is valuable to understand the dynamics of collapsed Anosov flows and separate it from the classification of general partially hyperbolic diffeomorphisms. The proof uses some fine properties of (topological) Anosov flows that had not been previously used in the study of partially hyperbolic dynamics (see § 2.1).

2. Background

In this paper M will always denote a closed 3-manifold. In this section we briefly recall some basic properties of Anosov flows, partially hyperbolic diffeomorphisms and collapsed Anosov flows. Except in § 2.1 we only present definitions and quote results from elsewhere. In § 2.1 we prove Proposition 2.5 which is an observation about self orbit equivalences of topological Anosov flows that we will use in the proof of Theorem A. We refer the reader to [BFP, FP] for more detailed information.

For simplicity and since Theorem A can be shown in finite lifts we will assume without explicit mention that all objects are orientable and foliations are transversally orientable; we comment on this assumption in § 2.4.

2.1. **Topological Anosov flows.** A flow $\phi_t : M \to M$ generated by a continuous vector field X is said to be a *topological Anosov flow* if it is expansive and preserves a topological foliation (see [BFP, §5] for other equivalent definitions). A topological Anosov flow ϕ_t preserves two transverse foliations $\mathcal{F}^{ws}$ and $\mathcal{F}^{wu}$ with the property that orbits in $\mathcal{F}^{ws}(x)$ approach the orbit of x in the future while orbits in $\mathcal{F}^{wu}(x)$ approach the orbit of x in the past. Leaves of $\mathcal{F}^{ws}$ and $\mathcal{F}^{wu}$ are planes or cylinders[2]

[1] The argument of [FP] achieves accessibility in hyperbolic 3-manifolds in a shorter way as it uses much less of the classification. In particular to use the proof we present here to deduce ergodicity in hyperbolic 3-manifolds one should use the full classification of partially hyperbolic diffeomorphisms in hyperbolic 3-manifolds given in [FP$_2$] which is way longer.

[2] One could have Möbius bands but our standing assumption that everything is orientable excludes this.

and all cylinder leaves contain a unique periodic orbit. We refer the reader to [Bar] for generalities on Anosov flows in dimension 3.

We denote by $\widetilde{\phi}_t : \widetilde{M} \to \widetilde{M}$ the lift of the flow to the universal cover. A fundamental property for us is [Fen$_1$, Ba] that the quotient $\mathcal{O}_\phi$ of $\widetilde{M}$ by the orbits of $\widetilde{\phi}_t$ is homeomorphic to $\mathbb{R}^2$. In other words, the flow $\widetilde{\phi}_t$ in $\widetilde{M}$ is *topologically* a product flow. However, geometrically this is rarely the case, and this will be a key fact (see Theorem 2.2 below for a precise statement) that we will exploit to show accesibility.

We call $\mathcal{O}_\phi$ the *orbit space* of ϕ. The lifts $\widetilde{\mathcal{F}^{ws}}$ and $\widetilde{\mathcal{F}^{wu}}$ are flow invariant and therefore induce transverse one dimensional foliations $\mathcal{G}^s$ and $\mathcal{G}^u$ in the orbit space $\mathcal{O}_\phi$. The fundamental group $\pi_1(M)$ preserves the collection of flow lines, hence induces an action by homeomorphisms on $\mathcal{O}_\phi$ which preserves the foliations $\mathcal{G}^s$ and $\mathcal{G}^u$. Given an orbit $o \in \mathcal{O}_\phi$ we call *half leaf* of $\mathcal{G}^s(o)$ (or $\mathcal{G}^u(o)$) to a connected component of $\mathcal{G}^s(o)\backslash\{o\}$ (or $\mathcal{G}^u(o)\backslash\{o\}$).

Given an orbit o of $\widetilde{\phi}_t$ we view it as both an element or point in $\mathcal{O}_\phi$ and as a subset of $\widetilde{M}$ consisting of all points in the orbit.

Note that if for some $o \in \mathcal{O}_\phi$ there is some non trivial $\gamma \in \pi_1(M)$ such that $\gamma o = o$ then this means that o corresponds to a periodic orbit $\alpha = \pi(o)$ of ϕ_t. In fact this is an if and only if property. We say that γ acts *increasingly* (resp. *decreasingly*) on o if $\gamma x = \widetilde{\phi}_t(x)$ for some $t > 0$ (resp. $t < 0$) and some $x \in o$ (since γ does not have fixed points in $\widetilde{M}$ it is easy to see that this is independent on $x \in o$). Our orientability assumptions imply that if a deck transformation fixes some orbit, then it also fixes all the *half leaves* of $\mathcal{G}^s(o)$ and $\mathcal{G}^u(o)$, i.e. the connected components of $\mathcal{G}^s(o)\backslash\{o\}$, $\mathcal{G}^u(o)\backslash\{o\}$.

Let α_1, α_2 be periodic orbits of ϕ_t. We say that they are *freely homotopic* if the unoriented curves α_1, α_2 are freely homotopic. With our orientation conditions this is equivalent to saying that there is a non trivial deck transformation $\gamma \in \pi_1(M)$ and lifts o_1, o_2 of α_1, α_2 respectively such that $\gamma o_1 = o_1$ and $\gamma o_2 = o_2$ (note that we do not require γ to act increasingly on both, or any of them). There are many subtleties with defining freely homotopic orbits, having to do with orientation on the orbits, taking powers of orbits; but we will not enter into them here. We refer the reader to [Fen$_3$] for more details.

A key object for the main result here will be lozenges, see figure 1.

Definition 2.1. An open subset $\mathcal{L}$ of the orbit space $\mathcal{O}_\phi$ is said to be a *lozenge* if there are two orbits o_1, o_2 called the *corners* of the lozenge which verify that they have half leaves A_1^s of $\mathcal{G}^s(o_1)$ and A_1^u of $\mathcal{G}^u(o_1)$ which are disjoint from half leaves A_2^s of $\mathcal{G}^s(o_2)$ and A_2^u of $\mathcal{G}^u(o_2)$ and satisfy the following properties:

- A leaf of $\mathcal{G}^s$ intersects A_1^u if and only if it intersects A_2^u.
- A leaf of $\mathcal{G}^u$ intersects A_1^s if and only if intersects A_2^s.
- every point $p \in \mathcal{L}$ satisfies that $\mathcal{G}^s(p)$ intersects A_1^u and separates o_1 from o_2. In addition $\mathcal{G}^u(p)$ intersects A_1^s and separates o_1 from o_2.

The half leaves $A_1^s, A_2^s, A_1^u, A_2^u$ are called the *sides* of the lozenge.

A *chain of lozenges* is a sequence $\mathcal{L}_1, \dots, \mathcal{L}_k$ of lozenges such that the closures of $\mathcal{L}_i \cap \mathcal{L}_{i+1}$ in the orbit space intersect either at a side or at a corner. A chain is minimal if a given lozenge occurs at most once in the chain − no backtracking.

We will need the following result, the first claim of which is [Fen$_2$, Theorem 3.3] and the second is [Fen$_3$, Corollary E]. For the last claim see [Fen$_3$] or [BaF, Theorem C].

Theorem 2.2. *Let $\phi_t : M \to M$ be a topological Anosov flow. Then,*

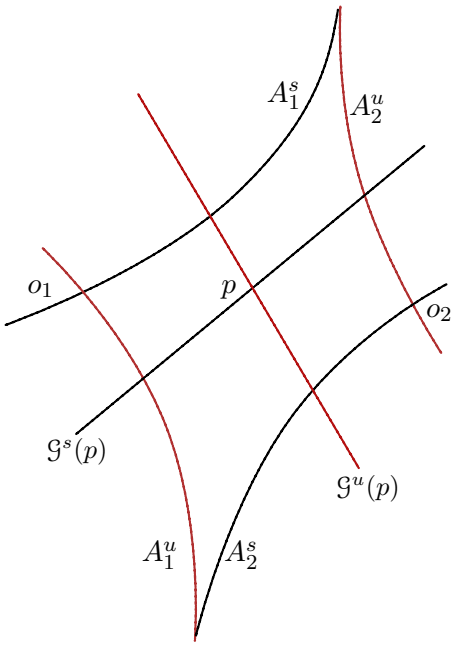

FIGURE 1. A lozenge. This figure is in the orbit space. So topologically the region of $\widetilde{M}$ which projects into the lozenge is this lozenge times the reals.

(i) *Let o_1, o_2 orbits in $\mathcal{O}_\phi$ which are fixed by a non trivial $\gamma \in \pi_1(M)$. Then o_1 and o_2 are connected by a finite chain of lozenges whose corners are fixed by γ.*
(ii) *If ϕ_t is not a suspension then there are freely homotopic periodic orbits of ϕ_t.*

Moreover, every $\mathbb{Z}^2$ subgroup of the fundamental group is associated to at most two bi-infinite minimal chains of lozenges. The subgroup fixes each such chain of lozenges. There are infinitely many non trivial elements in the $\mathbb{Z}^2$ subgroup which fix every lozenge in these chains.

To get the last statement one uses that the $\mathbb{Z}^2$ subgroup acts on the linearly ordered set of lozenges in the bi-infinite chain of lozenges, which is order isomorphic to $\mathbb{Z}$.

As a consequence one deduces that every topological Anosov flow $\phi_t : M \to M$ which is not a suspension contains a lozenge whose corners are lifts of periodic orbits of ϕ_t to $\widetilde{M}$. This is the crucial place where we will use that the fundamental group is not solvable since suspensions are very different from the rest of Anosov flows in this respect.

Corollary 2.3. *If $\phi_t : M \to M$ is a topological Anosov flow which is not a suspension, then there exists a lozenge $\mathcal{L}$ in $\mathcal{O}_\phi$ which is fixed by some non trivial $\gamma \in \pi_1(M)$.*

Note that a topological Anosov flow is a suspension if and only if the manifold has (virtually) solvable fundamental group [Bru].

An important property of lozenges that we will use is the following (see figures 2 and 3):

Proposition 2.4. *Let $\mathcal{L} \subset \mathcal{O}_\phi$ be a lozenge in $\mathcal{O}_\phi$ fixed by $\gamma \in \pi_1(M)$. Denote by o_1, o_2 the corners of $\mathcal{L}$ which correspond to periodic orbits of ϕ_t. Then, if γ acts increasingly on o_1 then it acts decreasingly on o_2 and vice versa.*

Proof. Note that if γ acts increasingly on o_1 then γ expands points in $\mathcal{G}^s(o_1)$ and contracts points in $\mathcal{G}^u(o_1)$. This is counterintuitive so we explain for the action on $\mathcal{G}^s(o_1)$. Let L be stable leaf of $\widetilde{\phi}_t$ which projects to $\mathcal{G}^s(o_1)$ in $\mathcal{O}_\phi$. Then γ acts as an isometry in L and sends a point x in o_1 to $\widetilde{\phi}_t(x)$ where $t > 0$. Let y be in L in a nearby orbit at distance d_0 from x. Then $\gamma(y)$ is at distance d_0 from $\widetilde{\phi}_t(x)$ since γ acts as an isometry on L. But the orbit through y is closer to $\widetilde{\phi}_t(x)$ than y is to x because orbits in a stable leaf converge together. So the orbit through $\gamma(y)$ is farther from o_1 than the orbit through y is, that is γ acts as an expansion on the set of orbits in L.

Let A_i^s be the half leaves of $\mathcal{G}^s(o_i)$ which are in the boundary of the lozenge, and A_i^u the half leaves of $\mathcal{G}^u(o_i)$ which are in the boundary of the lozenge. We proved that γ acts in an exanding manner in $\mathcal{G}^s(o_1)$. Let q in $\mathcal{G}^s(o_1)$. Then $\mathcal{G}^u(q)$ intersects A_2^s and separates o_1 from o_2 by the lozenge property. This implies that γ contracts points in $\mathcal{G}^s(o_2)$.

Similarly γ contracts points in $\mathcal{G}^u(o_1)$ and expands points in $\mathcal{G}^u(o_2)$. □

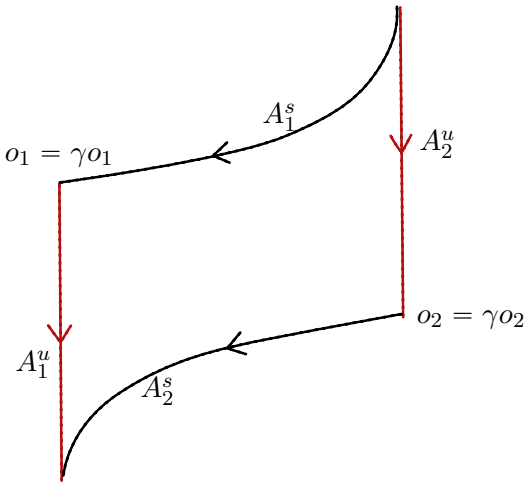

FIGURE 2. A periodic lozenge fixed by a non trivial deck transformation γ which also fixes the corners o_1, o_2 of the lozenge. The arrows indicate the action of γ on the stable and unstable leaves of o_i. Notice again that this figure depicts the situation in the orbit space which is two dimensional. The figure depicts the situation that $\gamma(x) = \widetilde{\Phi}_t(x)$ with $t < 0$ for x in o_1.

We now clarify what we mean by the flow not being a geometric product. Consider the setup as in Proposition 2.4. Let $\alpha_i = \pi(o_i)$ be the corresponding periodic orbits of ϕ_t in M ($\pi : \widetilde{M} \to M$ is the universal covering map). Proposition 2.4 implies that α_1, α_2 are freely homotopic to the inverses of each other. In particular this implies that a positive ray of o_1 is a bounded Hausdorff distance in $\widetilde{M}$ from a *negative* ray of o_2 and vice versa. Also a positive ray of o_1 is not a bounded distance from a positive ray of o_2. Hence the flow $\widetilde{\Phi}$ in $\widetilde{M}$ is very far from being a geometric product. The prototype flow for this behavior is the geodesic flow on a closed, orientable hyperbolic surface. Here every periodic orbit is freely homotopic to the inverse of another periodic orbit (the same geodesic traversed in the opposite direction).

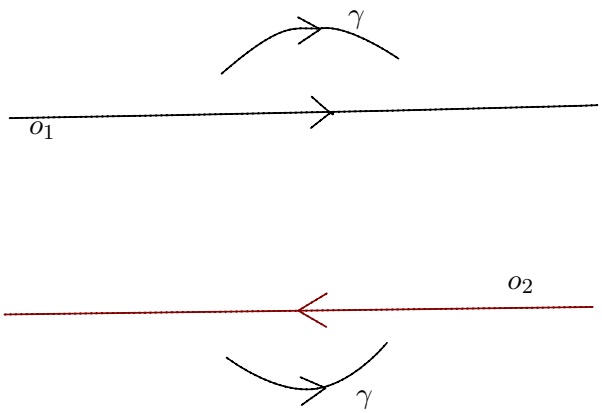

FIGURE 3. Lifts o_1, o_2 of periodic orbits α_1, α_2 which are freely homotopic to the inverses of each other. The arrows in the flow lines indicate the positive flow direction in each orbit. The action of γ in each orbit is also indicated. This figure is supposed to be 3-dimensional in $\widetilde{M}$ and geometrically correct in that positive rays of o_1 are a bounded distance from negative rays of o_2 and so on. Hence geometrically the flow is very far from being a product.

A *self orbit equivalence* of an Anosov flow ϕ_t is a homeomorphism $\beta : M \to M$ sending orbits of ϕ_t to orbits of ϕ_t and preserving their orientation. See [BaG, BFP] for more information on them. A self orbit equivalence is *trivial* if it leaves invariant every orbit.

We show the following proposition about self orbit equivalences that will be an important technical ingredient to prove our main result. The proof and statement will assume some familiarity with the theory of Anosov flows (we note that very similar arguments can be found in [BaG]; we include the proof since this is not stated explicitly).

The proposition refers to $\mathbb{R}$-covered Anosov flows. This means that the leaf space of the stable foliation of the flow lifted to $\widetilde{M}$ is homeomorphic to the reals. In case Φ is $\mathbb{R}$-covered and not conjugate to a suspension it has what is called *skewed* type. The skewed type has a well defined structure, in particular every orbit (periodic or not) is one corner of a bi-infinite chain of lozenges. If the orbit is periodic then there is a $\mathbb{Z}^2$ subgroup leaving this bi-infinite chain invariant. Any such flow has what is called a *one step up map*: it is self orbit equivalence of the flow Φ which is homotopic to the identity and has a lift to $\widetilde{M}$ which induces a shift by one in any of these bi-infinite chains. We refer the reader to [Fen$_1$] and [Bar] for details on $\mathbb{R}$-covered Anosov flows, including the one step up map.

Proposition 2.5. *Let $\beta : M \to M$ be a self orbit equivalence of a topological Anosov flow $\phi_t : M \to M$ on a manifold with non-virtually solvable fundamental group. Then, one of the following three possibilities happens:*

(i) *there is a lozenge $\mathcal{L}$ fixed by some non trivial deck transformation γ and there is a lift $\hat{\beta}$ of an iterate of β such that $\hat{\beta}(\mathcal{L}) = \mathcal{L}$*
(ii) *M is hyperbolic, ϕ_t is $\mathbb{R}$-covered and β is a non trivial power of an one step up map, or*
(iii) *M is Seifert and β acts in the base as a pseudo-Anosov.*

Proof. If a 3-manifold M admits an Anosov flow then it is irreducible and the universal cover is homeomorphic to $\mathbb{R}^3$. It follows that the manifold is either hyperbolic, Seifert fibered or it has a non-trivial JSJ decomposition (see e.g. [Bar]).

If M has a non-trivial JSJ decomposition, then there is some iterate of β that fixes all tori of the decomposition up to homotopy. Each torus in the torus decomposition is associated with a $\mathbb{Z}^2$ subgroup of $\pi_1(M)$ which leaves invariant a minimal chain of lozenges $\mathcal{C}$ (cf. Theorem 2.2). Let $\mathcal{L}$ be a lozenge in $\mathcal{C}$, which has to be fixed by infinitely many γ in the $\mathbb{Z}^2$ subgroup associated with the torus. Since an iterate of β leaves the torus invariant up to homotopy, this implies that some lift $\hat{\beta}_0$ of an iterate of β leaves invariant $\mathcal{C}$. If β_0 does not fix a lozenge in $\mathcal{C}$ then it acts as a translation on the set of lozenges in $\mathcal{C}$. A $\mathbb{Z}^2$ subgroup leaving $\mathcal{C}$ invariant has a non trivial element γ_1 acting as translation on $\mathcal{C}$, hence some iterate $\hat{\beta}_0^i\gamma_1^j$ with i not zero fixes $\mathcal{L}$, and this is a lift $\hat{\beta}$ of a non trivial iterate of β.

This shows that either the first possibility holds or the manifold is hyperbolic or Seifert fibered.

If M is hyperbolic then up to an iterate we can assume that β is homotopic to the identity. If β has a finite iterate which is trivial, then the finite iterate has a lift fixing all orbits in $\widetilde{M}$. Since M is hyperbolic then ϕ_t is not conjugate to a suspension Anosov flow. Then the first possibility of the proposition is satisfied automatically because of item (ii) of Theorem 2.2. If no power of β is trivial, then in [BaG] it is shown that ϕ_t must be $\mathbb{R}$-covered and β is a power of a one step up map.

If M is Seifert fibered, then the flow is topologically equivalent to a lift of a geodesic flow in a hyperbolic orbifold [Bru], in particular, every periodic orbit corresponds to a closed curve in the base and all periodic orbits lifted to the universal cover are the corner of some lozenge. If the action in the base of β is not pseudo-Anosov, then one of these lozenges will be periodic.

This finishes the proof of the proposition. □

2.2. **Partially hyperbolic diffeomorphisms.** A diffeomorphism $f : M^3 \to M^3$ is said to be partially hyperbolic if there exists a continuous Df-invariant splitting $TM = E^s \oplus E^c \oplus E^u$ into one dimensional bundles such that there exists $\ell > 0$ so that for every $x \in M$ and v^σ unit vectors in E^σ_x $(\sigma = s, c, u)$ we have:

$$\|Df^\ell v^s\| < \min\{1, \|Df^\ell v^c\|\} \leqslant \max\{1, \|Df^\ell v^c\|\} < \|Df^\ell v^u\|.$$

It is well known that E^s and E^u integrate uniquely into f-invariant foliations $\mathcal{W}^s$ and $\mathcal{W}^u$ (see e.g. [HP$_2$]). In general, the center foliation does not integrate into a foliation, though one dimensionality allows to get some structures that we will not use explicitely here [BI].

Definition 2.6. We say that f is *accessible* if for every $x, y \in M$ there exists a (piecewise smooth) curve tangent to $E^s \cup E^u$ joining x to y.

There is a strong link between accessibility and ergodicity suggested by the Pugh-Shub conjectures [PS]. In our setting the following result is true (see [BuW] for stronger results in higher dimensions that also imply the following):

Theorem 2.7 (Burns-Wilkinson)**.** *If $f : M \to M$ is a volume preserving partially hyperbolic diffeomorphism in a closed 3-manifold M which is accessible and of class C^{1+}. Then, f is a K-system, in particular it is ergodic and mixing.*

Recall that a measure preserving system is a *K-system* if every non-trivial finite partition has positive metric entropy, this implies the system is ergodic (i.e. invariant sets have zero or full measure). See [BuW, CHHU] for more information.

Lack of accessibility in dimension 3 allows to produce a strong structure that is what we will analyse here (see [HHU, HHU$_2$, CHHU]):

Theorem 2.8 (Hertz-Hertz-Ures)**.** *Let $f : M \to M$ be a partially hyperbolic diffeomorphism on a closed 3-manifold with non-solvable fundamental group whose non-wandering set is all of M and such that f is not accessible. Then, there exists a lamination Λ^{su} tangent to $E^s \oplus E^u$ such that:*

- *Λ does not have closed leaves,*
- *the closure of the complementary regions of Λ^{su} (if existing) are I-bundles where E^c is uniquely integrable and such that center curves form the I-bundle structure.*

See [FP, §2.3] for more discussion on this result. The case where the fundamental group is solvable is related with the existence of *su*-tori, see [CHHU].

We will use the following easy property of partially hyperbolic diffeomorphisms that follows directly from uniform transversality of the bundles after iteration (see e.g. [HP$_2$, Proposition 4.2]):

Proposition 2.9. *There are no closed curves tangent to E^s or E^u.*

A generalization will be given in Proposition 6.3 below.

2.3. **Collapsed Anosov flows.** We introduce here the notion of collapsed Anosov flows from [BFP]. As mentioned earlier, it corresponds to an open and closed class of partially hyperbolic diffeomorphisms (see [BFP, Theorem C]) that contains all known examples of partially hyperbolic diffeomorphisms in manifolds with non virtually solvable fundamental group (see [BFP, Theorem A]).

Definition 2.10. A partially hyperbolic diffeomorphism $f : M \to M$ is a *collapsed Anosov flow* if there exists a (topological) Anosov flow $\phi_t : M \to M$, a self orbit equivalence $\beta : M \to M$ and a continuous map $h : M \to M$ homotopic to the identity which is C^1 along orbits of the flow and such that $\partial_t h(\phi_t(x))|_{t=0} \in E^c(h(x))\backslash\{0\}$ and such that $f \circ h = h \circ \beta$.

In [BFP, Theorem A] we actually show that all known examples verify a slightly stronger assumption that we call *strong collapsed Anosov flow* which under our orientability assumptions is also open and closed among partially hyperbolic diffeomorphisms. In §6.2 we introduce this notion since the proof of Theorem A admits a shortcut if one restricts to this class. (Note that it is an open question if being collapsed Anosov flow implies being strong collapsed Anosov flow.)

2.4. **Orientability assumptions.** Here we comment on the assumption we have made that all bundles are orientable and that Df preserves their orientation. This is no loss of generality as we shall now explain.

Firstly, let us remark that if f is a partially hyperbolic diffeomorphism of a manifold M and g is a lift of an iterate of f to a finite cover $\hat{M}$ of M, then, accessibility of g implies accessibility of f. To see this, notice that the strong stable and strong unstable manifolds of an iterate of f are the same as those of f, so taking iterates does not change the fact that the whole manifold is an accessibility class. Further, an accessibility class in $\hat{M}$ projects to an accessibility class in M, thus, accessibility in $\hat{M}$ for g implies accessibility of f as desired.

Now we need to justify why taking a lift of an iterate of a collapsed Anosov flow to a finite cover is still a collapsed Anosov flow which will end the justification that our assumptions are in fact no loss of generality.

First, notice that taking an iterate of a collapsed Anosov flow is still a collapsed Anosov flow: one just needs to keep the same flow ϕ_t and map h in Definition 2.10 and pick an iterate of β. Consider now a collapsed Anosov flow $f : M \to M$ and let $\pi : \hat{M} \to M$ be a finite lift corresponding to the orientation of the bundles (i.e. $\pi : \hat{M} \to M$ is a lift so that the bundles E^s, E^c, E^u are orientable in the cover). In

addition we assume that f lifts to $\hat{f} : \hat{M} \to \hat{M}$ so that $\hat{f}$ preserves all orientations of the bundles − this can be achieved by further lifts and iterates. We claim that $\hat{f}$ is also a collapsed Anosov flow. The flow ϕ_t lifts to $\hat{\phi}_t$, which is a topological Anosov flow in $\hat{M}$. In addition since h is homotopic to the identity the actions on homotopy induced by f and β on $\pi_1(M)$ are the same. Therefore β also lifts to $\hat{\beta}$ in $\hat{M}$. Clearly $\hat{\beta}$ is a self orbit equivalence of $\hat{\phi}_t$. Finally the homotopy from the identity to h, lifts h to $\hat{h}$ in $\hat{M}$. By construction $\hat{f} \circ \hat{h} = \hat{h} \circ \hat{\beta}$. In other words $\hat{f}$ is also a collapsed Anosov flow. This concludes the proof of our claim.

3. Idea of the proof of Theorem A

Here we explain some of the main ideas to prove Theorem A.

Since f is a collapsed Anosov flow associated with an Anosov flow ϕ_t, there is a self orbit equivalence β of the flow ϕ_t, and a map h homotopic to the identity so that $f \circ h = h \circ \beta$.

Our assumptions give that there are periodic orbits α_1, α_2 of ϕ_t, which are freely homotopic to the inverses of each other and which lift to orbits $\widetilde{\alpha}_1, \widetilde{\alpha}_2$ that are corners of a lozenge $\mathcal{L}$ invariant by deck transformation γ associated to α_1 (c.f. Corollary 2.3).

By Proposition 2.5 (and because the other cases where dealt in [FP]) we can choose α_1, α_2 so that the lozenge $\mathcal{L}$ is also invariant by a lift of a power of the self orbit equivalence β. Let $o_i = \widetilde{\alpha}_i$.

Roughly, the idea of the proof of Theorem A is that the orbits α_1, α_2 above and the fact that they are (essentially) transverse to the lamination Λ^{su} force a closed curve in a leaf of Λ^{su} which is invariant by a power of f. It is easy to show that this leads to a contradiction (see Proposition 6.3).

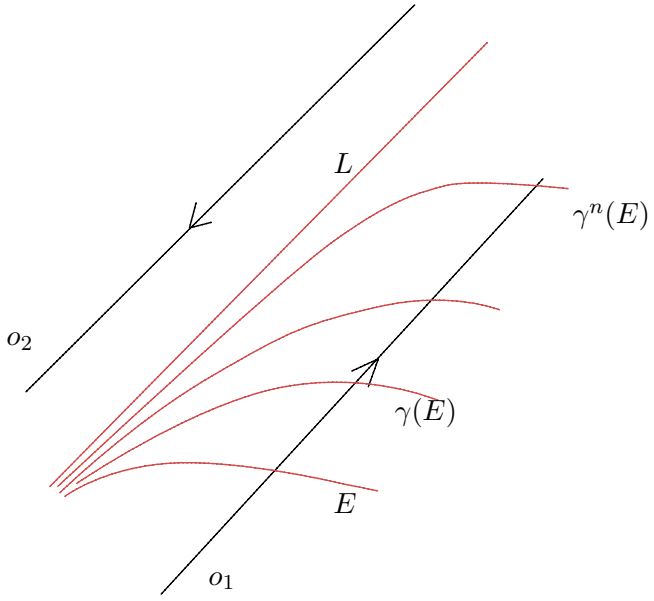

Figure 4. E is a leaf of $\widetilde{\Lambda}^{su}$ intersecting o_1. It cannot intersect o_2 and neither its iterates by γ. Hence $\gamma^n(E)$ $(n \to \infty)$ converges to a unique leaf L of $\widetilde{\Lambda}^{su}$ which separates o_1 from o_2 (the sequence $\gamma^n(E)$ may also converge to other leaves as well, but L is uniquely determined). This figure is supposed to be 3-dimensional, we draw a 2 dimensional slice of it.

Let us explain how the curve is found: to simplify the explanation let us assume in this section that h is injective, hence a homeomorphism. Notice that h sends flow lines of ϕ_t to C^1 curves tangent to E^c and hence transverse to the lamination Λ^{su}.

Lift the homotopy of h to the identity to produce $\widetilde{h}$.

We consider the action of γ on $\widetilde{h}(\widetilde{\alpha}_1)$ and $\widetilde{h}(\widetilde{\alpha}_2)$. By Proposition 2.4, γ acts increasingly on $\widetilde{h}(\widetilde{\alpha}_1)$ and decreasingly on $\widetilde{h}(\widetilde{\alpha}_2)$. This is the crucial fact. In § 5 we show that this implies that for any leaf E of $\widetilde{\Lambda}^{su}$ then E cannot intersect both $\widetilde{h}(\widetilde{\alpha}_1)$ and $\widetilde{h}(\widetilde{\alpha}_2)$ because E separates $\widetilde{M}$. Hence, starting with E intersecting $o_1 = \widetilde{\alpha}_1$, we can produce (see figure 4) a unique leaf L of $\widetilde{\Lambda}^{su}$ which separates in a specific way $\widetilde{h}(\widetilde{\alpha}_1)$ from $\widetilde{h}(\widetilde{\alpha}_2)$ (see Proposition 5.1). This leaf L is fixed by γ.

In § 4 we do a careful analysis of how a surface like L can intersect the image under $\widetilde{h}$ of the stable and unstable leaves of $\widetilde{\alpha}_i$ and Proposition 4.6 produces a curve in the intersection which is invariant by γ. By the choice above we can prove this curve is also invariant by a power of a lift of an iterate of f. This is where the semiconjugacy betweeen f and β by h is used. This projects to a closed curve in a leaf of Λ^{su} which is invariant by a power of f that gives a contradiction as explained above.

In the rest of the article we carefully carry out this strategy.

4. Surfaces transverse to collapsed Anosov flows

In this section we somewhat extend [Fen$_4$, Proposition 4.3]. The context is slightly different since we need to take care of a different setup, but some of the ideas are very similar. Our result in this direction is Proposition 4.5 below, stating that in the orbit space of an Anosov flow, the boundary of the orbits that intersect a surface uniformly transverse to the flow is made of entire weak stable and weak unstable leaves.

The goal is to understand the intersection between a surface transverse to the center bundle of a (collapsed) Anosov flow and how it behaves in its boundary in the "collapsed orbit space". The main result of this section is Proposition 4.6 which produces some closed curves in the intersection of certain surfaces transverse to the center bundle and the image under h of the weak stable or weak unstable foliation of the Anosov flow. This will be used later for certain su-surfaces to find some contradiction assuming non accessibility.

4.1. **Setup.** Let $\phi_t : M \to M$ be a (topological) Anosov flow and let Y be a non-singular vector field in M such that there is a map $h : M \to M$ continuous and homotopic to the identity, with h being C^1 along orbits of the flow, and in addition such that $Y(h(x))$ belongs to $\mathbb{R}_+ \partial_t h(\phi_t(x))|_{t=0}$. By non singular we mean that Y is continuous and never zero.

We consider in $\widetilde{M}$, the universal cover of M, the orbit space $\mathcal{O}_\phi$ of the flow $\widetilde{\phi}_t$ lifted to $\widetilde{M}$. We can take $\widetilde{h}$ a lift of h commuting with deck transformations and at bounded distance from the identity (i.e. take a lift of a homotopy to the identity to construct $\widetilde{h}$) and we denote by $\widetilde{Y}$ the lifted vector field.

For $x \in \widetilde{M}$ we denote o_x to the orbit of $\widetilde{\phi}_t$ containing it.

Definition 4.1. Given an orbit o of $\widetilde{\phi}_t$ we define c_o to be the image by $\widetilde{h}$ of o which is a curve tangent to $\widetilde{Y}$.

We consider the sets $\widehat{\mathcal{W}^{cs}}(c_o)$ and $\widehat{\mathcal{W}^{cu}}(c_o)$ to be the images by $\widetilde{h}$ of the weak stable and weak unstable foliations of o. Note that these are a priori only topological objects and not necessarily C^1 immersed surfaces. In addition a priori there may

be topological crossings, so the collection of these may not form a (topological) branching foliation.

4.2. **Transverse surfaces.** We consider S a properly embedded surface in $\widetilde{M}$ which is uniformly transverse to $\widetilde{Y}$. We will assume that for each $o \in \mathcal{O}_\phi$ the curve c_o intersects S in at most one point[3].

It will be important to understand better which curves c_o the surface S intersects. Since Y is only continuous, distinct curves c_o may intersect. Hence it is easier and more convenient to understand this set of intersections with S from the point of view of the orbit space of $\widetilde{\phi}_t$ as follows. We define:

$$S_{\mathcal{O}} = \{o \in \mathcal{O}_\phi \ : \ c_o \cap S \neq \varnothing\}.$$

Lemma 4.2. *The set $S_{\mathcal{O}}$ is open in $\mathcal{O}_\phi$.*

Proof. This is just transversality of S and $\widetilde{Y}$ and that $\widetilde{h}$ maps orbits of $\widetilde{\Phi}$ to curves tangent to $\widetilde{Y}$. □

We will define a function that indicates how the surface approaches c_o, particularly when o is a boundary point of $S_{\mathcal{O}}$. This function depends on some choices, but its asymptotic behavior for points in the boundary is well defined. Fix $x \in o$ and a transversal D to o at x. We can define $\tau^S : D \to \mathbb{R} \cup \{\infty\}$ as

$$\tau^S(y) = \infty \quad \text{if} \quad \widetilde{h}(o_y) \cap S = \varnothing, \text{ and}$$

$$\tau^S(y) = t_y \in \mathbb{R} \quad \text{if} \quad \widetilde{h}(\widetilde{\phi_{t_y}}(y)) \in S.$$

The funtion τ^S also depends on the choice of D which is left implicit. Suppose that the orbit o of $\widetilde{\Phi}$ intersects D. Then clearly c_o intersects S if and only if $\tau^S(o \cap D)$ is a real number, that is, it is finite. But in fact one deduces that τ^S is uniformly bounded in a neighborhood of $o \cap D$ when c_o cuts S. Intuitively if $\tau^S(y) > 0$ then S is "above" (or flow forward of) $\widetilde{h}(y)$ and if $\tau^S(y) < 0$, then S is "below" $\widetilde{h}(y)$. By uniform transversality, it is easy to see that:

Lemma 4.3. *The function τ^S is continuous in every point on which τ^S is finite.*

4.3. **Boundaries.** We want to understand how the function τ^S goes to infinity when one approaches from $S_{\mathcal{O}}$ the points of $\mathcal{O}_\phi$ which are in the boundary of $S_{\mathcal{O}}$.

Lemma 4.4. *If $o \notin S_{\mathcal{O}}$ then the function τ^S is uniformly bounded above in $\widetilde{\mathcal{F}^{ws}}(o) \cap D$ and uniformly bounded below in $\widetilde{\mathcal{F}^{wu}}(o) \cap D$. More precisely, for every $x \in o$ and transverse disk D we have that there exists $K > 0$ such that for every $z \in D$ such that $o_z \notin S_{\mathcal{O}}$ then:*

- *If $y \in \widetilde{\mathcal{F}^{ws}}(o_z) \cap D$ we have that $\tau^S(y) \in (-\infty, K) \cup \{\infty\}$.*
- *If $y \in \widetilde{\mathcal{F}^{wu}}(o_z) \cap D$ then we have that $\tau^S(y) \in (-K, +\infty) \cup \{\infty\}$.*

Proof. By uniform transversality between S and the vector field Y, we know that if points $x, y \in \widetilde{M}$ verify that $d(x, y) < \varepsilon$ and $x \in S$ then every curve tangent to Y through y will intersect S.

We prove the first property as the second is entirely analogous. Since h is continuous there is $\delta > 0$ so that if $d(x, y) < \delta$ then $d(\widetilde{h}(x), \widetilde{h}(y)) < \varepsilon$. By the contraction property of Anosov flows we know that there exists $t_0 > 0$ such that for every $z \in D$ we have that if $y \in \widetilde{\mathcal{F}^{ws}}(o_z) \cap D$ and $t > t_0$ then $d(\widetilde{\phi}_t(y), \widetilde{\phi}_{\mathbb{R}}(z)) < \delta$. It follows that

[3]This is the case for instance, when S separates $\widetilde{M}$ in two connected components, e.g. if S is the lift of a leaf of a Reebless foliation.

if $o_z \notin S_{\mathcal{O}}$ but $o_y \in S_{\mathcal{O}}$, then $\tau^S(y) < t_0$. Since the t_0 above can be chosen uniform over D, this implies the result. □

Recall that we denote by $\mathcal{G}^s$ and $\mathcal{G}^u$ to the foliations in $\mathcal{O}_\phi$ induced by $\widetilde{\mathcal{F}^{ws}}$ and $\widetilde{\mathcal{F}^{wu}}$. As a consequence, we get:

Proposition 4.5. *If $o \in \partial S_{\mathcal{O}}$ then either $\mathcal{G}^s(o)$ or $\mathcal{G}^u(o)$ is contained in $\partial S_{\mathcal{O}}$. The first case happens if there is V a neighborhood of $o \cap D$ in D such that τ^S is bounded below in V and the second case happens if τ^S is bounded above in a similar neighborhood.*

Proof. As in the previous setup, pick $x \in o$ and a disk D transverse to $\widetilde{\phi}_t$ through x. Consider $x_n \in D$ so that $x_n \in D$ verify that $o_{x_n} \in S_{\mathcal{O}}$, and x_n converges to x. In particular $\tau(x_n)$ is a real number for all n. First notice that $|\tau^S(x_n)|$ converges to ∞. Otherwise up to subsequence it converges to a real number and then by continuity S intersects $\widetilde{h}(o_x)$, contradiction.

Up to subsequence and without loss of generality we can assume that $\tau^S(x_n) \to +\infty$. Since $\tau(x_n)$ is a real number, Lemma 4.4 implies that for n sufficiently big and for every $y \in \widetilde{\mathcal{F}^{ws}}(o_{x_n}) \cap D$ we have that $o_y \in S_{\mathcal{O}}$. Therefore we get that given $z \in \widetilde{\mathcal{F}^{ws}}(o) \cap D$ there is a sequence $z_n \in D$ such that $z_n \to z$ and $o_{z_n} \in S_{\mathcal{O}}$. Moreover, $o_z \notin S_{\mathcal{O}}$ since $\tau^S(z_n) \to +\infty$: this is because if $o_z \in S_{\mathcal{O}}$ then $\tau^S(z)$ is real and uniformly bounded in a neighborhood of z in D. This shows that the local weak stable of the manifold of o is contained in $\partial S_{\mathcal{O}}$.

Now we iterate this analysis. For any v in $\mathcal{G}^s(o)$ we find $D_1 = D, D_2, ..., D_n$ small disks transverse to $\widetilde{\phi}_t$, consecutive ones intersecting a common orbit of $\widetilde{\phi}_t$, and so that the segment in $\mathcal{G}^s(o)$ between o and v is contained in the union of the orbits of $\widetilde{\phi}_t$ through the D_i. In the last paragraph we showed that for any $z \in \widetilde{\mathcal{F}^{ws}}(o) \cap D_1$ we find

$$z_n \in D \text{ with } z_n \to z, \quad o_{z_n} \in S_{\mathcal{O}}, \quad \text{and } \tau^s(z_n) \to +\infty.$$

Choose z so that o_z also intersects D_2 and now apply the result to D_2. We get that for every $y \in \widetilde{\mathcal{F}^{ws}}(o) \cap D_2$ then $o_y \in \partial S_{\mathcal{O}}$. Induction proves that $\mathcal{G}^s(o) \subset \partial S_{\mathcal{O}}$.

In addition $S_{\mathcal{O}}$ is connected and hence in this case intersects only one complementary component of $\mathcal{G}^s(o)$ in $\mathcal{O}_\phi$.

The remaining option in the analysis above is that $\tau^S(x_n)$ converges to $-\infty$ as x_n converges to x. Then the same type of arguments show that $\mathcal{G}^u(o) \subset \partial S_{\mathcal{O}}$. In addition for any $z \in \widetilde{\mathcal{F}^{wu}}(o)$ and for z_n in D with $o_{z_n} \in S_{\mathcal{O}}$ converging to z then $\tau^S(z_n)$ converges to $-\infty$.

By the complementary condition these two possibilities are incompatible. This proves the proposition. □

4.4. **Additional invariance.** We now assume in addition that there is a nontrivial $\gamma \in \pi_1(M)$ such that:

(i) γ fixes a lozenge $\mathcal{L}$ of $\widetilde{\phi}_t$ whose corners we denote by e_1 and e_2 and such that γ acts increasingly in the orientation of e_1 and decreasingly in the orientation of e_2,
(ii) $\gamma(S) = S$,
(iii) S does not intersect $c_1 := c_{e_1} = \widetilde{h}(e_1)$ nor $c_2 := c_{e_2} = \widetilde{h}(e_2)$,
(iv) S intersects some orbit of the lozenge, that is, there is $o \in \mathcal{L}$ such that $\widetilde{h}(o) \cap S \neq \varnothing$.

Our purpose is to show:

Proposition 4.6. *Under the above assumptions it follows that there is a γ-invariant curve in $\widehat{\mathcal{W}^{cs}}(c_1) \cap S$ or in $\widehat{\mathcal{W}^{cu}}(c_1) \cap S$. Moreover, if a homeomorphism preserves S, $\widehat{\mathcal{W}^{cs}}(c_1)$ and $\widehat{\mathcal{W}^{cu}}(c_1)$ then it also preserves this γ-invariant curve.*

The first thing we need to prove is that:

Lemma 4.7. *The set $S_{\mathcal{O}}$ contains $\mathcal{L}$.*

Proof. The set $S_{\mathcal{O}}$ is an open connected set which intersects $\mathcal{L}$ and it is γ-invariant, because $S, \mathcal{L}$ and the sides are γ invariant.

Assume that there is a point $o \in \mathcal{L} \cap \partial S_{\mathcal{O}}$. Using Proposition 4.5 assume without loss of generality that $\mathcal{G}^u(o) \subset \partial S_{\mathcal{O}}$. Since o is in the interior of the lozenge it follows that $\mathcal{G}^u(o) \cap \mathcal{G}^s(e_i)$ for $i = 1, 2$. Since $S_{\mathcal{O}}$ is γ-invariant and γ acts as an expansion on $\mathcal{G}^s(e_1)$ (resp γ acts as a contraction on $\mathcal{G}^s(e_2)$) we get that $S_{\mathcal{O}}$ must accumulate in both e_1 and e_2 from inside $\mathcal{L}$.

However, since $S_{\mathcal{O}}$ is connected this is forbiden by the fact that $\mathcal{G}^u(o) \subset \partial S_{\mathcal{O}}$ and so $S_{\mathcal{O}}$ cannot accumulate in both. This contradiction proves the lemma. □

We can now prove

Proof of Proposition 4.6. Since $\mathcal{L} \subset S_{\mathcal{O}}$ and $e_1, e_2 \notin S_{\mathcal{O}}$ by assumption we deduce that $e_1, e_2 \in \partial S_{\mathcal{O}}$.

We can then apply Proposition 4.5 and without loss of generality we assume that $\mathcal{G}^s(e_1) \subset \partial S_{\mathcal{O}}$. We will find a curve in $\widehat{\mathcal{W}^{cu}}(c_1) \cap S$ which is invariant under γ.

First, we note that since e_1 is in the boundary of $S_{\mathcal{O}}$ and $\mathcal{G}^s(e_1) \subset \partial S_{\mathcal{O}}$ it follows that $\mathcal{G}^u(e_1)$ is not contained in $\partial S_{\mathcal{O}}$. Since $\mathcal{L}$ is contained in $S_{\mathcal{O}}$ which is γ invariant, then it follows that there is a half leaf A of $\mathcal{G}^u(e_1)$ (i.e. a connected component of $\mathcal{G}^u(e_1) \backslash \{e_1\}$) such that every $o \in A$ verifies that $\widetilde{h}(o)$ intersects S. In other words $A \subset S_{\mathcal{O}}$.

We know that both A and S are γ-invariant. Since for each orbit o we have that the intersection point $S \cap \widetilde{h}(o)$ is unique, we deduce that $S \cap \widehat{\mathcal{W}^{cu}}(c_1)$ is connected. This implies that it is a γ-invariant curve as desired and that if a homeomorphism preserves S and $\widehat{\mathcal{W}^{cu}}(c_1)$ then it must preserve this curve. □

Remark 4.8. The curve may auto-intersect both in $\widetilde{M}$ as well as in its projection in M. But the point is that its projection to M is the image by a continuous map of a circle. This is because the curve is preserved by the non trivial element $\gamma \in \pi_1(M)$.

5. Transverse laminations

We continue in the setup of § 4.1.

Let Λ be a lamination transverse to Y which verifies that:

- Λ does not have compact leaves,
- the closure of the complementary regions of Λ (if existing) are I-bundles where Y is uniquely integrable and such that flowlines of Y form the I-bundle structure.

The standing assumption will be that the Anosov flow ϕ_t is not a suspension (see Corollary 2.3). Under this assumption we can show that the assumptions in § 4.4 are verified for any surface L in the lift of the lamination Λ to the universal cover, as follows. First notice that since Λ is transverse to Y and Λ is closed, then Λ is uniformly transverse to Y. In addition, the second condition above implies that Λ can be completed to a foliation $\mathcal{F}$ so that it does not have compact leaves and $\mathcal{F}$ is transverse to Y (this is well known, but see explicit proofs and explanations in [FP, Lemma 3.9]). Since $\mathcal{F}$ does not have compact leaves it is Reebless, therefore any

curve in $\widetilde{M}$ transverse to $\widetilde{\mathcal{F}}$ intersects a leaf of $\widetilde{\mathcal{F}}$ at most once. In particular any curve tangent to $\widetilde{Y}$ intersects a leaf of $\widetilde{\Lambda}$ at most once as required in § 4.2.

Proposition 5.1. *For every lozenge $\mathcal{L}$ fixed by a non trivial deck transformation $\gamma \in \pi_1(M)$ there is a γ-invariant leaf L of $\widetilde{\Lambda}$ so that L intersects the image by $\widetilde{h}$ of some orbit in the interior of the lozenge but does not intersect the image of the corner orbits of $\mathcal{L}$.*

Proof. Let o_1, o_2 be the corners of the lozenge $\mathcal{L}$. Take a leaf $E \in \widetilde{\Lambda}$ intersecting $c_1 = \widetilde{h}(o_1)$. There is always such a leaf because c_1 is a properly embedded curve tangent to Y and the closures of complementary regions of Λ are I-bundles where Y is uniquely integrable.

Claim 5.2. *The leaf E cannot intersect $c_2 = \widetilde{h}(o_2)$.*

Proof. Assume that γ acts increasingly on o_1. Proposition 2.4 implies that γ acts decreasingly on o_2. Since $\widetilde{h}$ commutes with deck transformations we get that the same happens in c_1 and c_2. Now we again use that Λ can be completed to a Reebless foliation. This implies that any leaf of $\widetilde{\Lambda}$ separates $\widetilde{M}$. Since γ acts increasingly in o_1 and hence also in c_1, it follows that $\gamma(E)$ is on the positive side of E with respect to Y. If E intersects c_2 then since γ acts decreasingly on o_2, the same argument shows that $\gamma(E)$ is contained in the negative side of E with respect to Y. This is a contradiction and proves the claim. □

Therefore, we can consider V to be the region in $\widetilde{M}$ between E and $\gamma(E)$. We claim that c_2 cannot intersect V: c_2 is γ-invariant, and if it intersects V then it must intersect $\gamma^{-1}(V)$ and thus intersect E a contradiction.

Therefore, the open region

$$R = \bigcup_n \gamma^n(V \cup E)$$

contains c_1 and is disjoint from c_2. The boundary of R is accumulated by translates of E under $\gamma^n, n \to \infty$, therefore is saturated by leaves of $\widetilde{\Lambda}$. There must be a single leaf $L \in \widetilde{\Lambda}$ in the boundary of R that separates c_1 from c_2. By construction this leaf does not intersect c_1 nor c_2 and is invariant by γ.

We need to show that L intersects the image of some orbit in the lozenge. But this is true because otherwise one could connect c_1 and c_2 by a path with endpoints one in c_1 one in c_2 and the interior a path which is the projection of a path in the lozenge $\mathcal{L}$. Hence c_1, c_2 would be in the same connected component of the complement of L. This completes the proof. □

As a consequence of Proposition 5.1 and Proposition 4.6 we deduce:

Corollary 5.3. *A lamination with the properties stated in the beginning of this section verifies that there is a leaf $L \in \widetilde{\Lambda}$ invariant under a non trivial $\gamma \in \pi_1(M)$ and a γ-invariant orbit $o \in \mathcal{O}_\phi$ such that $\widehat{\mathcal{W}^{cs}}(c_o) \cap L$ contains a curve which is γ-invariant.*

6. Accessibility and ergodicity of partially hyperbolic diffeomorphisms

6.1. **Setup.** We let $f : M \to M$ be a collapsed Anosov flow with respect to an Anosov flow $\phi_t : M \to M$ which is not a suspension. There is a self orbit equivalence $\beta : M \to M$ and $h : M \to M$ a map homotopic to the identity, so that h sends orbits of ϕ_t injectively onto curves tangent to E^c. In addition $f \circ h = h \circ \beta$.

In addition Remark [BFP, Remark 2.6] shows that for a collapsed Anosov flow, the center bundle E^c is orientable, and it shows that $\partial_t h((\phi_t(x))|_{t=0}$ induces an orientation on the center bundle E^c. Therefore we choose Y to be the vector field of norm one such that

$$Y(h(x)) \quad \in \quad \mathbb{R}_+ \partial_t h(\phi_t(x))|_{t=0}.$$

The vector field Y is contained in the E^c bundle. Note that this is the context of §4.1.

To prove Theorem A we will assume that the non-wandering set of f is all of M. This allows us to apply Theorem 2.8. We will assume by contradiction that f is not accessible so that Theorem 2.8 implies that there is a lamination Λ^{su} tangent to $E^s \oplus E^u$ which satisfies the conditions of § 5.

As explained before, the proof of Theorem A is not affected if we take finite lifts and iterates, so we will assume for simplicity that all bundles of f are orientable and their orientation is preserved by Df. We will reach a contradiction that will prove that f is accessible, then the ergodicity part of Theorem A follows immediately from Theorem 2.7.

6.2. **Strong collapsed Anosov flows.** We first give a direct proof under the extra assumption that f is a *strong collapsed Anosov flow* (see [BFP] for discussions). This implies in particular that the map h maps weak stable and weak unstable leaves of ϕ_t into surfaces tangent to $E^{cs} = E^s \oplus E^c$ and $E^{cu} = E^c \oplus E^u$ respectively. The reason we first show this case is that here we will not need to use the dynamics at all, and will get a direct contradiction to Proposition 2.9.

The contradiction follows from applying Corollary 5.3 which in this case produces a closed curve tangent to E^s which is γ invariant, contradicting Proposition 2.9. This is because in this case $\widehat{\mathcal{W}^{cs}}$ is tangent to E^{cs} and Λ^{su} is tangent to $E^s \oplus E^u$, hence the intersection is tangent to E^s.

Remark 6.1. As explained above, in principle, collapsed Anosov flows may always have this stronger property (that is, we currently know no example which is a collapsed Anosov flow, but not a strong collapsed Anosov flow).

6.3. **Collapsed Anosov flows.** The proof of Theorem A is harder since it will need to appeal to [FP] for some cases. However, for the other cases it is just slightly harder: instead of using Proposition 2.9 we will use some properties of self-orbit equivalences (cf. Proposition 2.5) to show that the curve given by Corollary 5.3 is also $\hat{f}$-invariant for some lift $\hat{f}$ of some iterate of f which will be enough to get a contradiction.

We now come back to the setting of § 6.1 and do not make the assumption on h we did in the previous subsection (Subsection on strong collapsed Anosov flows).

Lemma 6.2. *If f is not accessible then there is some iterate of β which has a lift $\hat{\beta}$ to $\widetilde{M}$ which fixes some lozenge $\mathcal{L}$ in $\mathcal{O}_\phi$, and so that $\mathcal{L}$ is invariant by some non trivial $\gamma \in \pi_1(M)$..*

Proof. We apply Proposition 2.5. Unless the conclusion of the lemma is verified, we get that either M is hyperbolic or M is Seifert and β acts as pseudo-Anosov in the base. Both cases where treated in [FP] (see [FP, Theorem B] for the hyperbolic manifold case and [FP, Theorem D] for the Seifert case with base pseudo-Anosov). Specifically, in both cases, we proved in [FP] that f is accessible. In the other cases we obtain a lozenge as in Proposition 2.5. □

As a consequence we deduce from Corollary 5.3 that there is a leaf $L \in \widetilde{\Lambda^{su}}$ which is invariant under the deck tansformation γ as in the previous lemma, as well as L

is invariant under some lift $\hat{f} : \widetilde{M} \to \widetilde{M}$ of an iterate of f. The proof of Theorem A is therefore completed with the following extension of Proposition 2.9:

Proposition 6.3. *There are no closed curves (not necessarily injectively embedded) invariant under f in an f invariant surface tangent to $E^s \oplus E^u$.*

Proof. If an f-invariant curve is not tangent to E^s, then iterates of it have segments closer and closer to segments in unstable leaves. Again since the curve is invariant one obtains that the curve contains a segment which is contained in an unstable leaf. But then invariance implies that the entire curve is contained in an unstable leaf. This is impossible by Proposition 2.9. □

Remark 6.4. Another proof of the previous proposition follows from the fact that one can construct a *Lyapunov function* for the action of f in the image of η to show that f is an expansive homeomorphism in the image of η (see [Pot, §2]) . Since the circle does not admit expansive homeomorphisms, that gives a contradiction.

Acknowledgements: The authors would like to thank Santiago Martinchich for discussions and his comments on the paper.

References

[Ba] T. Barbot, Caractérisation des flots d'Anosov en dimension 3 par leurs feuilletages faibles. *Ergodic Theory Dynam. Systems* **15** (1995), no. 2, 247–270. 3

[BaF] T. Barbot, S. Fenley, Pseudo Anosov flows in toroidal manifolds, *Geom. Top.* **17** (2013), no. 4, 1877–1954. 3

[Bar] T. Barthelme, Anosov flows in dimension 3, lecture notes available at the authors' webpage. 3, 6

[BFFP] T. Barthelmé, S. Fenley, S. Frankel, R. Potrie, Partially hyperbolic diffeomorphisms homotopic to identity in dimension 3, Part I: The dynamically coherent case, arxiv:1908.06227v3. 1

[BFFP$_2$] T. Barthelmé, S. Fenley, S. Frankel, R. Potrie, Partially hyperbolic diffeomorphisms homotopic to identity in dimension 3, Part II: Branching foliations. arXiv:2008.04871 1

[BFP] T. Barthelmé, S. Fenley, R. Potrie, Collapsed Anosov flows and self orbit equivalences, arXiv:2008.06547 1, 2, 6, 8, 15

[BaG] T. Barthelmé, A. Gogolev, A note on self orbit equivalences of Anosov flows and bundles with fiberwise Anosov flows. *Math. Res. Lett.* **26** (2019), no. 3, 711–728. 6, 7

[BGP] C. Bonatti, A. Gogolev, R. Potrie, Anomalous partially hyperbolic diffeomorphisms II: stably ergodic examples. *Invent. Math.* **206** (2016), no. 3, 801–836. 1, 2

[BGHP] C. Bonatti, A. Gogolev, A. Hammerlindl, R. Potrie, Anomalous partially hyperbolic diffeomorphisms III: abundance and incoherence, *Geom. Topol.* **24** (2020), no. 4, 1751–1790. 1, 2

[Bru] M. Brunella, Expansive flows on Seifert manifolds and on torus bundles. *Bol. Soc. Brasil. Mat. (N.S.)* **24** (1993), no. 1, 89–104. 4, 7

[BI] D. Burago, S. Ivanov, Partially hyperbolic diffeomorphisms of 3-manifolds with abelian fundamental groups. *J. Mod. Dyn.* **2** (2008), no. 4, 541–580. 7

[BuW] K. Burns, A. Wilkinson, On the ergodicity of partially hyperbolic systems. *Ann. of Math.* (2) **171** (2010), no. 1, 451–489. 1, 7

[CHHU] P. Carrasco, F. Rodriguez Hertz, J. Rodriguez Hertz, R. Ures, Partially hyperbolic dynamics in dimension 3, *Ergodic Theory Dynam. Systems* **38** (2018), no. 8, 2801–2837. 1, 2, 7, 8

[Fen$_1$] S. Fenley, Anosov flows in 3-manifolds. *Ann. of Math. (2)* **139** (1994), no. 1, 79–115. 3, 6

[Fen$_2$] S. Fenley, Quasigeodesic Anosov flows and homotopic properties of closed orbits, *Jour. Diff. Geo.* **41** (1995), 479–514. 3

[Fen$_3$] S. Fenley, The structure of branching in Anosov flows of 3-manifolds, *Comm. Math. Helv.* **73** (1998) 259–297. 3

[Fen$_4$] S. Fenley, Surfaces transverse to pseudo-Anosov flows and virtual fibers in 3-manifolds. *Topology* **38** (1999), no. 4, 823–859. 10

[FP] S. Fenley, R. Potrie, Ergodicity of partially hyperbolic diffeomorphisms in hyperbolic 3-manifolds, arxiv:1809.02284v3. 1, 2, 8, 9, 13, 15

[FP$_2$] S. Fenley, R. Potrie, Partial hyperbolicity and pseudo-Anosov dynamics, arXiv:2102.02156. 1, 2

[FP$_3$] S. Fenley, R. Potrie, Partially hyperbolic diffeomorphisms in Seifert manifolds are collapsed Anosov flows, in preparation. 2

[GS] S. Gan, Y. Shi, Rigidity of center Lyapunov exponents and *su*-integrability, *Comment. Math. Helv.* **95** (2020), no. 3, 569–592. 1

[GPS] M. Grayson, C. Pugh, M. Shub, Stably ergodic diffeomorphisms. *Ann. of Math.* (2) **140** (1994), no. 2, 295–329. 1

[H] A. Hammerlindl, Ergodic components of partially hyperbolic systems. *Comment. Math. Helv.* **92** (2017), no. 1, 131–184. 1, 2

[HRHU] A. Hammerlindl, J. Rodriguez Hertz, R. Ures, Ergodicity and partial hyperbolicity on Seifert manifolds, *J. Mod. Dyn.* **16** (2020), 331–348. 1, 2

[HP] A. Hammerlindl, R. Potrie, Pointwise partial hyperbolicity in three-dimensional nilmanifolds. *J. Lond. Math. Soc. (2)* **89** (2014), no. 3, 853–875. 1

[HP$_2$] A. Hammerlindl, R. Potrie, Partial hyperbolicity and classification: a survey, *Ergodic Theory and Dyn. Sys.* **38** (2) (2018) 401–443. 7, 8

[HU] A. Hammerlindl, R. Ures, Ergodicity and partial hyperbolicity on the 3-torus. *Commun. Contemp. Math.* **16** (2014), no. 4, 1350038. 1

[HS] A. Hammerlindl, Y. Shi, Accessibility of derived-from-Anosov systems, to appear in *Transactions of the AMS*. 1

[HHU] F. Rodriguez Hertz, J. Rodriguez Hertz, R. Ures, Accessibility and stable ergodicity for partially hyperbolic diffeomorphisms with 1D-center bundle. *Invent. Math.* **172** (2008), no. 2, 353–381. 1, 7

[HHU$_2$] F. Rodriguez Hertz, J. Rodriguez Hertz, R. Ures, Partial hyperbolicity and ergodicity in dimension three. *J. Mod. Dyn.* **2** (2008), no. 2, 187-208. 1, 7

[Pot] R. Potrie, On the work of Jorge Lewowicz on expansive systems, *Publicaciones Matemáticas del Uruguay* **14** (2014) 6–22. 16

[PS] C. Pugh, M. Shub, Stable ergodicity and partial hyperbolicity. *International Conference on Dynamical Systems (Montevideo, 1995)*, 182?187, Pitman Res. Notes Math. Ser., 362, Longman, Harlow, 1996. 1, 7

Florida State University, Tallahassee, FL 32306
Email address: fenley@math.fsu.edu

Centro de Matemática, Universidad de la República, Uruguay
Email address: rpotrie@cmat.edu.uy
URL: http://www.cmat.edu.uy/~rpotrie/