All-set-homogeneous spaces

Nina Lebedeva and Anton Petrunin

Abstract

A metric space is said to be all-set-homogeneous if any isometry between its subsets can be extended to an isometry of the whole space. We give a classification of a certain subclass of all-set-homogeneous length spaces.

1 Main result

The distance between two points x and y in a metric space M will be denoted by $|x-y|_M$. Recall that M is called *length* (or *geodesic*) space if any two points $x, y \in M$ can be connected by a path γ such that $|x-y|_M$ is arbitrarily close to the length of γ (or $|x-y|_M = \operatorname{length}\gamma$ respectively). Evidently, any geodesic space is a length space, but not the other way around.

A metric space M is said to be *all-set-homogeneous* if any isometry $A \to A'$ between its subsets can be extended to an isometry of the whole space $M \to M$.

Examples of geodesic all-set-homogeneous spaces include complete simply-connected Riemannian manifolds with constant curvature and the circle equipped with length metrics. These will be referred further as *classical spaces*; they are closely related to classical Euclidean/non-Euclidean geometry.

It is worth mentioning that an infinite-dimensional Hilbert space is *not* all-set-homogeneous; indeed, it is isometric to its proper subset. Also, for $n \geqslant 2$, the real projective space $\mathbb{R}\mathrm{P}^n$ with canonical metric is not all-set-homogeneous. Indeed, it contains two isometric but noncongruent triples of points with pairwise distance $\frac{\pi}{3}$ (we assume that a closed geodesic on $\mathbb{R}\mathrm{P}^n$ has length π); one triple lies on a closed geodesic and another does not.

Nonclassical examples include the universal metric trees of finite valence; these are discussed in the next section.

Given a metric space M and a positive integer n, consider all pseudometrics induced on n points $x_1, \dots, x_n \in M$. Any such metric is completely described by $N = \frac{n\cdot(n-1)}{2}$ distances $|x_i - x_j|_M$ for $i < j$, so it can be encoded by a point in $\mathbb{R}^N$. The set of all these points $F_n(M) \subset \mathbb{R}^N$ will be called n^{th} *fingerprint* of M.

Theorem. *Let M be a complete all-set-homogeneous length space. Suppose that all fingerprints of M are closed. Then M is classical.*

The following two results are closely related to our theorem.

- *Any complete all-set-homogeneous geodesic space with locally unique nonbifurcating geodesics is classical;* it was proved by Garrett Birkhoff [2].
- *Any locally compact three-point-homogeneous length space is classical.* This result was proved by Herbert Busemann [4]; it also follows from the more

general result of Jacques Tits [8] about two-point-homogeneous spaces. (A space is called n-point homogeneous if any isometry between its subsets with at most n points in each can be extended to an isometry of the whole space.)

For more related results, see the survey by Semeon Bogatyi [3] and the references therein.

Proof. If M is locally compact, then the statement follows from the Busemann–Tits result stated above. Therefore, we can assume that M is not locally compact.

In this case, there is an infinite sequence of points $x_1, x_2, \dots$ such that $\varepsilon < |x_i - x_j|_M < 1$ for some fixed $\varepsilon > 0$ and all $i \neq j$. Applying the Ramsey theorem, we get that for arbitrary positive integer n there is a sequence $x_1, x_2, \dots, x_n$ such that all the distances $|x_i - x_j|_M$ lie in an arbitrarily small subinterval of $(\varepsilon, 1)$. Since the fingerprints are closed, there is an arbitrarily long sequence $x_1, x_2, \dots, x_n$ such that $|x_i - x_j|_M = r$ for some fixed $r > 0$.

Choose a maximal (with respect to inclusion) set of points A with distance r between any pair. Since M is all-set-homogeneous, we get that A has to be infinite. In particular, there is a map $f\colon A \to A$ that is injective, but not surjective.

Note that f is distance-preserving. Since A is maximal, for any $y \notin A$ we have that $|y - x|_M \neq r$ for some $x \in A$. It follows that a distance-preserving map $M \to M$ that agrees with f cannot have points of $A \setminus f(A)$ in its image. In particular, no isometry $M \to M$ agrees with the map f — a contradiction. $\square$

2 Example

Recall that geodesic space T is called a *metric tree* if any pair of points $x, y \in T$ are connected by a unique geodesic $[xy]_T$, and the union of any two geodesics $[xy]_T$, and $[yz]_T$ contains $[xz]_T$. The *valence* of $x \in T$ is defined as the cardinality of connected components in $T \setminus \{x\}$.

It is known that for any cardinality $n \geqslant 2$, there is a space $\mathbb{T}_n$ that satisfies the following properties:

- ⋄ The space $\mathbb{T}_n$ is a complete metric tree with valence n at any point.
- ⋄ $\mathbb{T}_n$ is homogeneous; that is, its group of isometries acts transitively.

Moreover, this space is uniquely defined up to isometry and n-universal; the latter means that $\mathbb{T}_n$ includes an isometric copy of any metric tree of maximal valence at most n.

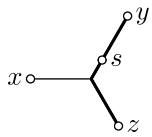
The space $\mathbb{T}_n$ is called a *universal metric tree of valence n*. An explicit construction of $\mathbb{T}_n$ is given by Anna Dyubina and Iosif Polterovich [5]. Their proof of the universality of $\mathbb{T}_n$ admits a straightforward modification that proves the following claim.

Claim. *If n is finite, then $\mathbb{T}_n$ is all-set-homogeneous.*

Note that the claim implies that the condition on fingerprints in our theorem is necessary. In fact, if $n \geqslant 3$, then the $(n+1)^{\text{th}}$ fingerprint of $\mathbb{T}_n$ is not closed — $\mathbb{T}_n$ does not contain $n+1$ points on distance 1 from each other, but it contains an arbitrarily large set with pairwise distances arbitrarily close to 1.

Proof. Let $A, A' \subset \mathbb{T}_n$ and $x \mapsto x'$ be an isometry $A \to A'$. Applying the Zorn lemma, we can assume that A is maximal; that is, the domain A cannot be extended by a single point. It remains to show that $A = \mathbb{T}_n$ and $A' = \mathbb{T}_n$.

Note that A is closed.



Further, suppose $x, y, z \in A$ and $s \in [yz]_{\mathbb{T}_n}$. Since $\mathbb{T}_n$ is a metric tree, the distance $|x-s|_{\mathbb{T}_n}$ is completely determined by four values $|x-y|_{\mathbb{T}_n}$, $|x-z|_{\mathbb{T}_n}$, $|s-y|_{\mathbb{T}_n}$, $|s-z|_{\mathbb{T}_n}$.

Denote by s' the point on the geodesic $[y'z']_{\mathbb{T}_n}$ such that $|y'-s'|_{\mathbb{T}_n} = |y-s|_{\mathbb{T}_n}$ and therefore $|z'-s'|_{\mathbb{T}_n} = |z-s|_{\mathbb{T}_n}$. Since the map preserves distances $|x-y|_{\mathbb{T}_n}$ and $|x-z|_{\mathbb{T}_n}$, we get $|s'-x'|_{\mathbb{T}_n} = |s-x|_{\mathbb{T}_n}$; that is, the extension of the map by $s \mapsto s'$ is still distance-preserving.

Since A is maximal, $s \in A$. In other words, A is a convex subset of $\mathbb{T}_n$; in particular, A is a metric tree with maximal valence at most n.

Arguing by contradiction, suppose $A \neq \mathbb{T}_n$, choose $a \in A$ and $b \notin A$. Let $c \in A$ be the last point on the geodesic $[ab]_{\mathbb{T}_n}$. Note that the valence of c in A is smaller than n.

Since n is finite, at least one of the connected components in $\mathbb{T}_n \backslash \{c'\}$ does not intersect A'. Choose a point b' in this component such that $|c'-b'|_{\mathbb{T}_n} = |c-b|_{\mathbb{T}_n}$. Observe that the map can be extended by $b \mapsto b'$ — a contradiction. It follows that $A = \mathbb{T}_n$.

It remains to show that $A' = \mathbb{T}_n$. Note that A' is a closed convex set in $\mathbb{T}_n$ that is isometric to $\mathbb{T}_n$. In particular valence of any point in A' is n.

Assume A' is a proper subset of $\mathbb{T}_n$. Choose $a' \in A'$ and $b' \notin A'$. Let $c' \in A'$ be the last point on the geodesic $[a'b']_{\mathbb{T}_n}$. Observe that the valence of c' in A' is smaller than n — a contradiction. $\square$

3 Remarks

Let us list examples for related classification problems. We would like to see any other example or a proof of the corresponding classification.

First of all, we do not see other examples of complete all-set-homogeneous length spaces except those listed in the theorem and the claim.

Without length-metric assumption, we have a vast amount of examples. It includes finite discrete spaces, Cantor sets with natural ultrametrics; also note that snowflaking $(X, |\ -\ |^\theta)$ of any all-set-homogeneous spaces $(X, |\ -\ |)$ is all-set-homogeneous.

The definition of all-set-homogeneous spaces can be restricted to *small* subsets A and A'; for example, *finite* or *compact*. In these cases, we say that the space is *finite-set-homogeneous* or *compact-set-homogeneous* respectively.

Examples of complete separable compact-set-homogeneous length spaces include the spaces listed in the theorem, plus the Urysohn spaces $\mathbb{U}$ and $\mathbb{U}_d$ (the space $\mathbb{U}_d$ is isometric to a sphere of radius $\frac{d}{2}$ in $\mathbb{U}$). Without the separability condition, we get in addition the metric trees from the claim.

The finite-set-homogeneous spaces include, in addition, infinite-dimensional analogs of the classical spaces; in particular the Hilbert space.

Let us also mention that finite-set homogeneity is closely related to the *metric version of Fraïssé limit* introduced by Itay Ben Yaacov [1].

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References

[1] I. Ben Yaacov. "Fraïssé limits of metric structures". *J. Symb. Log.* 80.1 (2015), 100–115.

[2] G. Birkhoff. "Metric foundations of geometry. I". *Trans. Amer. Math. Soc.* 55 (1944), 465–492.

[3] S. A. Bogatyi. "Metrically homogeneous spaces". *Russian Math. Surveys* 57.2 (2002), 221–240.

[4] H. Busemann. *Metric methods in Finsler spaces and in the foundations of geometry.* Annals of Mathematics Studies, No. 8. 1942.

[5] A. Dyubina and I. Polterovich. "Explicit constructions of universal $\mathbb{R}$-trees and asymptotic geometry of hyperbolic spaces". *Bull. London Math. Soc.* 33.6 (2001), 727–734.

[6] J. Hanson. *All-set-homogeneous spaces.* MathOverflow. eprint: `https://mathoverflow.net/q/430738`.

[7] J. O'Rourke. *Which metric spaces have this superposition property?* MathOverflow. eprint: `https://mathoverflow.net/q/118008`.

[8] J. Tits. "Sur certaines classes d'espaces homogènes de groupes de Lie". *Acad. Roy. Belg. Cl. Sci. Mém. Coll. in 8°* 29.3 (1955), 268.