GEOGRAPHY OF PINCHED FOUR-MANIFOLDS

RENATO G. BETTIOL, MARIO KUMMER, AND RICARDO A. E. MENDES

ABSTRACT. We prove several new restrictions on the Euler characteristic and signature of oriented 4-manifolds with (positively or negatively) pinched sectional curvature. In particular, we show that simply connected 4-manifolds with $\delta \leq \sec \leq 1$, where $\delta = \frac{1}{1+3\sqrt{3}} \cong 0.161$, are homeomorphic to S^4 or $\mathbb{C}P^2$.

1. INTRODUCTION

The so-called *Geography Problem* in 4-manifold topology is to determine which pairs $(\sigma, \chi) \in \mathbb{Z}^2$ can be realized as signature $\sigma = \sigma(M)$ and Euler characteristic $\chi = \chi(M)$ of a certain class of 4-manifolds M; e.g., complex surfaces [Per87], irreducible 4-manifolds [FS94], or those with a given fundamental group [KL09]. In this paper, we systematically investigate a geometric version of this problem, where the condition imposed on M is the existence of a Riemannian metric with pinched curvature. We say that a Riemannian manifold (M, g) is *positively* or *negatively* δ*-pinched*, $\delta \in (0, 1]$, if the sectional curvature of every tangent 2-plane Π satisfies

$$\delta \leq \sec(\Pi) \leq 1, \quad \text{or} \quad -1 \leq \sec(\Pi) \leq -\delta,$$

respectively. Collectively, (M, g) is called δ*-pinched* if it is either positively or negatively δ-pinched. It is well-known that δ-pinched 4-manifolds have $\chi(M) > 0$; our first main result provides an explicit upper bound for the ratio $|\sigma(M)|/\chi(M)$:

THEOREM A. *If (M^4, g) is a δ-pinched oriented 4-manifold with finite volume, then*

$$|\sigma(M)| \leq \lambda(\delta)\, \chi(M), \tag{1.1}$$

where $\lambda\colon (0, 1] \to \mathbb{R}$ is an explicit *continuous function, given in (1.5), that is strictly decreasing and satisfies $\lim_{\delta \searrow 0} \lambda(\delta) = +\infty$, $\lambda\Big(\frac{1}{1+3\sqrt{3}}\Big) < \frac{1}{2}$, $\lambda\big(\frac{1}{4}\big) = \frac{1}{3}$, and $\lambda(1) = 0$.*

In the above statement, and throughout this paper, all manifolds are assumed complete and without boundary. If (M^4, g) is negatively δ-pinched, then M need not be closed, in which case $\sigma(M)$ is to be understood as the proper homotopy invariant given by the L^2-signature $\sigma_{(2)}(M^4, \mathrm{g})$, see Section 2.6 and [CG85a, CG85b].

The fact that λ is an *explicit* function of δ is the crucial component of Theorem A, as the existence of *some* function $\lambda\colon (0, 1] \to \mathbb{R}$ satisfying (1.1) and $\lambda(1) = 0$ can be shown with routine arguments. Theorem A is a concoction of our main technical result (Theorem 6.2) and a thorough extension (Theorem A.1) of the seminal works of Ville [Vil85, Vil89], carried out in Appendix A. While Theorem 6.2 gives a continuously differentiable function $\lambda^*\colon (0, 1] \to \mathbb{R}$ satisfying (1.1) and all other conditions in Theorem A except $\frac{1}{4} \mapsto \frac{1}{3}$, Theorem A.1 yields a continuous function $\lambda^{\mathrm{V}}\colon [\delta_0^{\mathrm{V}}, 1] \to \mathbb{R}$, where $\delta_0^{\mathrm{V}} \cong 0.163$, satisfying (1.1) and $\frac{1}{4} \mapsto \frac{1}{3}$. Combining these,

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we have the function $\lambda(\delta)=\min\{\lambda^*(\delta),\lambda^{\mathrm{V}}(\delta)\}$ in Theorem A; see Section 1.4 for details. Theorem A also improves a similar result of Polombo [Pol78] for $0<\delta\leq\frac{1}{4}$.

Since δ-pinching becomes less stringent as $\delta\searrow 0$, it is natural that a function $\lambda\colon(0,1]\to\mathbb{R}$ satisfying (1.1) be decreasing and $\lambda(0_+)=+\infty$. On the other hand, $\lambda(1)=0$ indicates a certain sharpness regarding (1.1), since 1-pinched manifolds are either spherical or hyperbolic spaceforms, which, being locally conformally flat, have vanishing signature. Also $\lambda\big(\frac{1}{4}\big)=\frac{1}{3}$ is sharp, as the complex projective plane $M=\mathbb{C}P^2$ and compact quotients $M=\mathbb{C}H^2/\Gamma$ of the complex hyperbolic plane are all $\frac{1}{4}$-pinched and have $\sigma(M)=\frac{1}{3}\chi(M)$.

Let us further analyze the positively and negatively pinched cases separately.

1.1. **Positively pinched 4-manifolds.** By the celebrated 1/4-pinched Sphere Theorem and its several improvements [BS11, PT09], it is known that there exists $\varepsilon>0$ such that if an oriented 4-manifold (M^4,g) is δ-pinched with $\delta\geq\frac{1}{4}-\varepsilon$, then M is diffeomorphic to S^4 or $\mathbb{C}P^2$. Moreover, every positively δ-pinched oriented 4-manifold M is closed and has $b_1(M)=0$, hence $|\sigma(M)|<\chi(M)$, see (2.8).

Thus, in this case, Theorem A only provides new information in the range

$$\lambda^{-1}(1)<\delta<\tfrac{1}{4}-\varepsilon, \tag{1.2}$$

where $\lambda^{-1}(1)=\frac{39-5\sqrt{57}}{24}\cong 0.052$, according to (1.5). A particularly interesting value in the above range is $\frac{1}{1+3\sqrt{3}}\cong 0.161$, since positively $\frac{1}{1+3\sqrt{3}}$-pinched oriented 4-manifolds were recently shown by Diógenes and Ribeiro [DR19, Thm. 1] to have definite intersection form. Combined with Theorem A, that gives $|\sigma(M)|<\frac{1}{2}\chi(M)$, and the classical works of Donaldson [Don83] and Freedman [Fre82], this yields:

Theorem B. *If (M^4,g) is a positively δ-pinched simply-connected 4-manifold, with $\delta\geq\frac{1}{1+3\sqrt{3}}\cong 0.161$, then M is homeomorphic to S^4 or $\mathbb{C}P^2$.*

Theorem B improves on results of Ville [Vil89], Seaman [Sea89], and Ko [Ko05], where the same conclusion is obtained under stricter δ-pinching: $\delta\geq\frac{4}{19}\cong 0.211$, $\delta\geq 0.188$, and $\delta\geq 0.171$, respectively. Although it is widely expected that closed simply-connected 4-manifolds (M^4,g) with $\sec>0$ be diffeomorphic to S^4 or $\mathbb{C}P^2$ (see e.g. [Zil14]), this remains a difficult open problem. Perhaps the most compelling evidence for this conjecture is that it holds if (M^4,g) has an isometric circle action [HK89, GW14]. From this point of view, Theorem B provides new evidence *without any symmetry assumptions*, cf. also [BM22, Thm. C].

Our next main result gives upper bounds on the region in the $(|\sigma|,\chi)$-plane reachable by positively δ-pinched 4-manifolds, refining an observation of Berger [Ber62] that such 4-manifolds are either homology spheres or have $\chi(M)\leq\frac{1}{\delta^2}+\frac{8}{27}\left(\frac{1}{\delta}-1\right)^2$.

Theorem C. *If (M^4,g) is a positively δ-pinched oriented 4-manifold, then either M is diffeomorphic to S^4, or $\chi(M)\leq\frac{8}{9}(\frac{1}{\delta}-1)^2$ and $|\sigma(M)|\leq\frac{8}{27}(\frac{1}{\delta}-1)^2$.*

To the best of our knowledge, no restrictions on $\sigma(M)$, aside those inherited from $|\sigma(M)|\leq\chi(M)-2$, were previously known for positively δ-pinched manifolds.

Even though the upper bounds in Theorem C diverge to $+\infty$ as $\delta\searrow 0$, $\chi(M)$ and $|\sigma(M)|$ are known to be bounded above by a universal constant $C(4)$ for any closed 4-manifold (M^4,g) with $\sec>0$. This is a consequence of the celebrated total Betti number bound of Gromov [Gro81], see Remark 7.5 for details. Gromov conjectured that $C(4)=2^4$, but the best available estimates (due to Abresch [Abr87]) only give

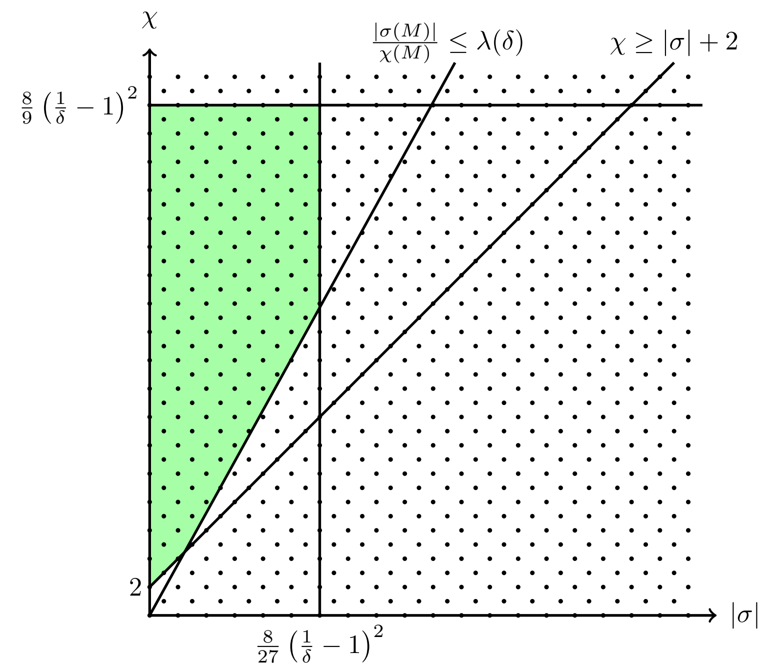

FIGURE 1. Admissible range (green) for $\big(|\sigma(M)|, \chi(M)\big)$ if M is positively δ-pinched by (2.8) and Theorems A and C; drawn with $\delta = \frac{9\sqrt{2}-2}{79} \cong 0.136$, so $\lambda(\delta) \cong 0.552$, $\chi(M) \leq 36$, and $\sigma(M) \leq 12$.

$C(4) \lesssim 2.731 \times 10^{232}$. The latter is a smaller upper bound on $\chi(M)$ than that in Theorem C if $\delta \lesssim 5.705 \times 10^{-117}$, but it is *hundreds* of orders of magnitude larger if, e.g., $\delta \gtrsim 0.086$, in which case Theorem C gives $\chi(M) \leq 10^2$. Still, according to the conjectured classification of closed simply-connected 4-manifolds with $\sec > 0$, one ought to have $\chi(M) \leq 3$ and $|\sigma(M)| \leq 1$, no matter how small is $\delta > 0$.

The admissible region in the $(|\sigma|, \chi)$-plane for positively δ-pinched 4-manifolds, as constrained by Theorems A and C, is illustrated in Figure 1. By Synge's Theorem, such a 4-manifold M has $\pi_1(M) \cong \{1\}$ if it is oriented, and $\pi_1(M) \cong \mathbb{Z}_2$ otherwise. Thus, resorting again to the works of Donaldson [Don83] and Freedman [Fre82] for the simply-connected case, and to Hambleton, Kreck, and Teichner [HKT94] for the non-simply-connected case, one derives the following from Theorems A and C.

COROLLARY D. *For all $\delta > 0$, there exists an* explicit *finite list of possible homeomorphism types for positively δ-pinched 4-manifolds.*

While Corollary D also follows from the above mentioned bound on $\chi(M)$ for each $\delta > 0$ by Berger [Ber62], together with $|\sigma(M)| + 2 \leq \chi(M)$, the explicit list of homeomorphism types obtained using Theorems A and C is substantially shorter.

1.2. **Negatively pinched 4-manifolds.** In light of constructions of Gromov and Thurston [GT87], classification results similar to the 1/4-pinched Sphere Theorem are impossible in the negatively pinched case. Indeed, for all $\varepsilon > 0$, there are closed negatively $(1-\varepsilon)$-pinched 4-manifolds that do not admit metrics of constant curvature. While the examples in [GT87] have zero signature, similar examples with *nonzero signature* were recently found by Ontaneda [Ont20, Cor. 4]. Thus, in stark contrast to the positively pinched case (1.2), the conclusion of Theorem A in the negatively pinched case is nontrivial in the full range $0 < \delta < 1$. As $\delta \nearrow 1$, it follows from Theorem A that negatively δ-pinched oriented 4-manifolds (M^4, g) with $\sigma(M) \neq 0$ must have $\chi(M) \nearrow +\infty$; namely, by (1.5), we have that

$$\chi(M) \geq \frac{24\delta^2 - 12\delta + 15}{8(1-\delta)^2}$$

if δ is sufficiently close to 1. In particular, solving for δ, one sees that closed oriented 4-manifolds M with $\sigma(M) \neq 0$ and fixed Euler characteristic $\chi(M) = \chi$ can only admit metrics that are negatively δ-pinched if $\delta \leq \delta_\chi < 1$, where δ_χ is explicit. A similar gap $\delta \leq \delta_\pi$ holds [Bel99, Cor. 1.3] weakening $\sigma(M) \neq 0$ to M not admitting hyperbolic metrics, and fixing $\pi_1(M) \cong \pi$ instead of $\chi(M)$, but δ_π is not explicit.

We also remark that although the lower bound (1.1) becomes arbitrarily weak as $\delta \searrow 0$, it follows from Gromov [Gro78, §1.7 (2)] that there exists $C > 0$ such that, for all $\delta > 0$, negatively δ-pinched closed 4-manifolds with $\sigma(M) \neq 0$ have $\chi(M) \geq C$. Combined with (1.1), one has the lower bound

$$\chi(M) \geq \max\left\{ \frac{|\sigma(M)|}{\lambda(\delta)}, C \right\},$$

which remains uniformly away from zero for all δ, and diverges to $+\infty$ as $\delta \nearrow 1$.

Another striking difference arises from negatively δ-pinched closed 4-manifolds M not necessarily having $b_1(M) = 0$. However, as these manifolds have $\chi(M) > 0$,

$$b_1(M) \leq 1 + \tfrac{1}{2} b_+(M) + \tfrac{1}{2} b_-(M). \tag{1.3}$$

A consequence of Theorem A is that this upper bound can be improved to

$$b_1(M) \leq 1 + \tfrac{\lambda-1}{2\lambda} \max\big\{b_+(M), b_-(M)\big\} + \tfrac{\lambda+1}{2\lambda} \min\big\{b_+(M), b_-(M)\big\}, \tag{1.4}$$

where $\lambda = \lambda(\delta) > 0$. Note that the above weighted average of $b_\pm(M)$ is strictly smaller than the simple average in (1.3) whenever $\sigma(M) \neq 0$.

Similarly to Theorem C, upper bounds (depending on volume) can be given on the admissible region in the $(|\sigma|, \chi)$-plane for negatively δ-pinched 4-manifolds:

Theorem E. *If (M^4, g) is a negatively δ-pinched oriented 4-manifold with finite volume, then*

$$\chi(M) \leq \frac{3}{4\pi^2} \operatorname{Vol}(M, \mathrm{g}), \quad \text{and} \quad |\sigma(M)| \leq \frac{2}{9\pi^2}(1-\delta)^2 \operatorname{Vol}(M, \mathrm{g}).$$

Equality in the upper bound for $\chi(M)$ is achieved if and only if (M^4, g) is hyperbolic.

The above upper bound on $\chi(M)$ is likely known among experts (e.g., it follows from [Vil87, Thm. 1]), but we are unaware of any such prior results for $\sigma(M)$. As in Theorem A, $\sigma(M)$ is to be understood as the L^2-signature if (M^4, g) is noncompact.

Using Bishop Volume Comparison in the inequalities in Theorem E, we see that negatively δ-pinched closed oriented 4-manifolds (M^4, g) with $\operatorname{diam}(M, \mathrm{g}) \leq D$ have

$$\chi(M) \leq 2(2 + \cosh D) \sinh^4 \tfrac{D}{2}, \quad \text{and} \quad |\sigma(M)| \leq \tfrac{16}{27}(1-\delta)^2(2 + \cosh D) \sinh^4 \tfrac{D}{2},$$

and, once again, equality in the upper bound for $\chi(M)$ holds if and only if (M^4, g) is hyperbolic. It follows from Gromov [Gro78, §1.7 (1)] that, for all $D > 0$ and $V > 0$, there exists $0 < \delta(D, V) < 1$ such that if a closed 4-manifold M is negatively $\delta(D, V)$-pinched and $\sigma(M) \neq 0$, then $\operatorname{diam}(M, \mathrm{g}) \geq D$ and $\operatorname{Vol}(M, \mathrm{g}) \geq V$. Note that Theorem E *quantifies* this result, yielding explicit estimates on $\delta(D, V)$, since it implies that closed δ-pinched 4-manifolds with $\sigma(M) \neq 0$ have $\mathrm{Vol}(M, \mathrm{g}) \geq \frac{9\pi^2}{2(1-\delta)^2}$ and $\mathrm{diam}(M, \mathrm{g}) \geq D$ where $D > 0$ satisfies $(2 + \cosh D)\sinh^4 \frac{D}{2} = \frac{27}{16(1-\delta)^2}$.

1.3. **Methods of proof.** Our results on pinched 4-manifolds are proven with a blend of Differential Geometry and Convex Algebraic Geometry. By the Chern–Gauss–Bonnet formula and Hirzebruch signature formula, the Euler characteristic and signature of (M^4, g) can be computed as integrals (2.9) of quadratic forms $\underline{\chi}(R)$ and $\underline{\sigma}(R)$ on its curvature operator R, respectively. (These formulas generalize to the case in which M is noncompact, see Section 2.6.) We use these to estimate $\chi(M)$ and $\sigma(M)$ combining pointwise bounds on such integrands, obtained through optimization methods; and global restrictions on the diameter and volume of such manifolds, obtained with standard Comparison Geometry techniques. Moreover, since $\underline{\chi}(R)$ and $\underline{\sigma}(R)$ only have degree 2 terms, any pointwise bounds on *positively* δ-pinched operators automatically hold for *negatively* δ-pinched operators.

The key link with Convex Algebraic Geometry is the Finsler–Thorpe Trick (Lemma 2.3), a distinctly 4-dimensional phenomenon (see [BKM21]) that characterizes the set of positively δ-pinched (algebraic) curvature operators $R\colon \wedge^2\mathbb{R}^4 \to \wedge^2\mathbb{R}^4$ as a *spectrahedral shadow*. More precisely, it is the intersection of the projections onto the space $\mathrm{Sym}^2_b(\wedge^2\mathbb{R}^4)$ of the *spectrahedra* of operators in $\mathrm{Sym}^2(\wedge^2\mathbb{R}^4)$ with all eigenvalues $\geq \delta$ and ≤ 1, respectively. Recall that $\mathrm{Sym}^2_b(\wedge^2\mathbb{R}^4) \subset \mathrm{Sym}^2(\wedge^2\mathbb{R}^4)$ is the subspace of operators satisfying the first Bianchi identity, and its orthogonal complement is spanned by the Hodge star operator $*$. Moreover, a spectrahedral shadow is (by definition) a linear projection of a spectrahedron, and the intersection of two spectrahedral shadows is also a spectrahedral shadow.

As explained in Section 4, it also follows from Finsler–Thorpe's Trick that projecting away the traceless Ricci part $R_{\mathcal{L}}$ of a pinched curvature operator R produces an Einstein curvature operator $R - R_{\mathcal{L}}$ which is *at least* as pinched as R. The set of such operators is a far simpler convex set: it is the orbit under the $\mathsf{SO}(4)$-action on $\mathrm{Sym}^2_b(\wedge^2\mathbb{R}^4)$ of a set affinely equivalent to a simplex $\Delta^5_\delta \subset \mathbb{R}^5$, which we call the *Einstein simplex*. Similarly, there is an *augmented Einstein simplex* $\Delta^6_\delta \subset \mathbb{R}^6$ that parametrizes $\mathsf{SO}(4)$-orbits of δ-pinched Einstein curvature operators R and the corresponding $t_1 \in \mathbb{R}$ such that $R + t_1 * \in \mathrm{Sym}^2(\wedge^2\mathbb{R}^4)$ has all eigenvalues $\geq \delta$.

Thus, finding extrema of $\mathsf{SO}(4)$-invariant quadratic forms on R that *do not* depend on $R_{\mathcal{L}}$ reduces to a quadratic optimization problem on the simplex Δ^5_δ. Even though general quadratic programming is NP-hard [Sah74, PV91], the cases at hand are manageable. For example, we are able to explicitly compute the maximum of $a|W_+|^2 + b|W_-|^2$, for any $a, b \in \mathbb{R}$, where $W_\pm$ are the self-dual and anti-self-dual Weyl parts of a δ-pinched curvature operator, and also characterize the equality case, see Proposition 7.1. This sharp pointwise estimate, which is likely to have other applications, is the main new input to prove Theorems C and E.

On the other hand, the 1-parameter family of $\mathsf{SO}(4)$-invariant quadratic forms $I_\lambda(R) = \underline{\chi}(R) - \frac{1}{\lambda}\underline{\sigma}(R)$ used to prove Theorem A, and the particular case $I_{\frac{1}{2}}(R)$ needed for Theorem B, *do depend* on the traceless Ricci part $R_{\mathcal{L}}$. To overcome this, we prove (Proposition 5.3) an upper bound on $|R_{\mathcal{L}}|^2$ for any $R \in \mathrm{Sym}^2_b(\wedge^2\mathbb{R}^4)$ in terms of its minimal (or maximal) sectional curvature $k \in \mathbb{R}$, scal, $W_\pm$, and a value $t \in \mathbb{R}$ such that $\pm(R - k\,\mathrm{Id}) + t\,*$ is positive-semidefinite (which exists by the Finsler–Thorpe Trick). Aside from its independent interest, Proposition 5.3 can be used to eliminate the dependence on $R_{\mathcal{L}}$ and find a quadratic form $Q_\lambda\colon \Delta^6_\delta \to \mathbb{R}$

on the augmented Einstein simplex Δ^6_δ whose values bound those of I_λ from below. Then, optimizing Q_λ on Δ^6_δ and requesting that its minimum be nonnegative determines an explicit description of a semialgebraic set in the (δ, λ)-plane. Applying Cylindrical Algebraic Decomposition to this set gives, for each δ, the desired explicit lower bound on λ that is sufficient to ensure $I_\lambda(R) \geq 0$ for all δ-pinched curvature operators, and hence $\sigma(M) \leq \lambda \cdot \chi(M)$ for all δ-pinched 4-manifolds (M^4, g). This yields Theorem 6.2, which is our main technical result leading to Theorem A.

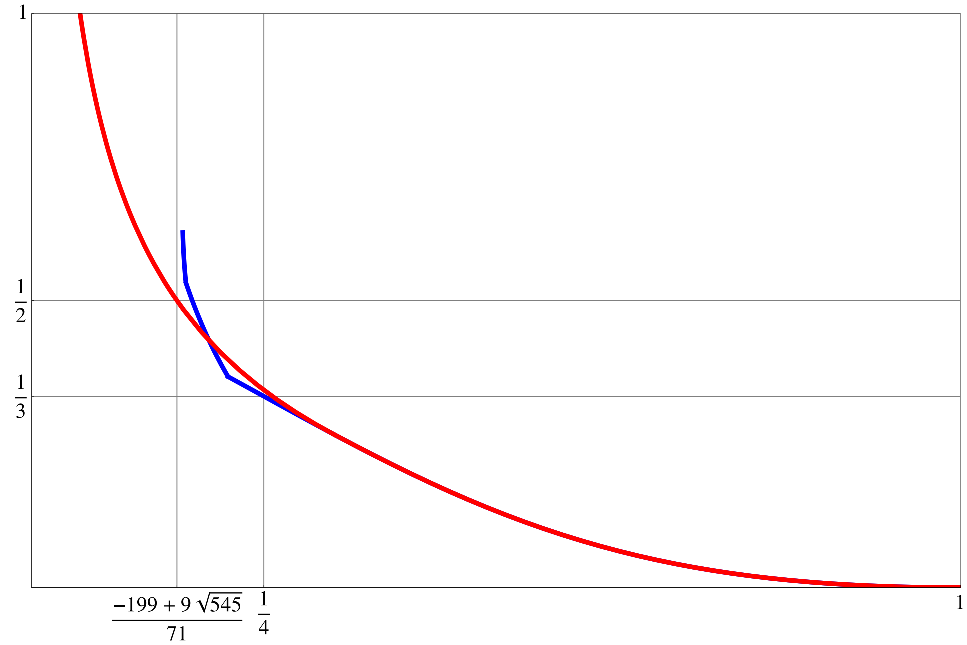

FIGURE 2. Graphs of functions λ^* from Theorem 6.2 (red) and λ^{V} from Theorem A.1 (blue); whose minimum is λ given by (1.5).

1.4. **Explicit function $\lambda(\delta)$.** As mentioned above, Theorem A is the combination of Theorems 6.2 and A.1, each of which proves (1.1) for a certain explicit function of δ. Comparing these functions and extracting $\lambda(\delta) = \min\{\lambda^*(\delta), \lambda^{\mathrm{V}}(\delta)\}$ gives:

$$\lambda(\delta) = \begin{cases} \dfrac{\sqrt{\frac{24}{\delta} + 8 - 8\delta + \delta^2} + \delta - 4}{6(3-\delta)}, & 0 < \delta < \delta_1, \\[2mm] \dfrac{4}{3\sqrt{15}} \dfrac{1-\delta}{\sqrt{\delta(\delta+2)}}, & \delta_1 \leq \delta < \delta_2, \\[2mm] \dfrac{26\delta^2 + 8\delta + 2 - 2\sqrt{3}\sqrt{55\delta^4 + 40\delta^3 + 6\delta^2 + 8\delta - 1}}{3(1-\delta)^2}, & \delta_2 \leq \delta \leq \delta_3, \\[2mm] \dfrac{8(1-\delta)^2}{24\delta^2 - 12\delta + 15}, & \delta_3 \leq \delta \leq 1, \end{cases} \tag{1.5}$$

where

(i) $\delta_1 \cong 0.069$ is the smallest real root of the polynomial $2\delta^3 - 40\delta^2 + 89\delta - 6$,

(ii) $\delta_2 \cong 0.191$ is the largest real root of the polynomial $2279\delta^6+6246\delta^5+4470\delta^4+2060\delta^3-450\delta^2-24\delta-1$,
(iii) $\delta_3 \cong 0.211$ is the largest real root of the polynomial $140\delta^4+40\delta^3-6\delta^2+88\delta-19$.

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2. Preliminaries on 4-manifolds

In this section, we discuss a multitude of basic topics regarding 4-manifolds needed in the remainder of the paper, also fixing notations and conventions.

2.1. **Four-dimensional curvature operators.** We now briefly recall some well-known facts about curvature operators of oriented 4-manifolds, for details, we refer the reader to [Bes08, Chap. 1.G-H].

The space $\mathrm{Sym}^2(\wedge^2\mathbb{R}^4)$ of symmetric endomorphisms of $\wedge^2\mathbb{R}^4$ decomposes as the orthogonal direct sum of four irreducible pairwise non-isomorphic $\mathsf{O}(4)$-submodules,

$$\mathrm{Sym}^2(\wedge^2\mathbb{R}^4) = \mathcal{U}\oplus\mathcal{L}\oplus\mathcal{W}\oplus\wedge^4\mathbb{R}^4, \tag{2.1}$$

and, accordingly, we write $R = R_{\mathcal{U}} + R_{\mathcal{L}} + R_{\mathcal{W}} + R_{\wedge^4}$ to indicate the components of an element $R\in\mathrm{Sym}^2(\wedge^2\mathbb{R}^4)$. The (real) dimensions of the spaces in (2.1) are 1, 9, 10, and 1, respectively. In particular, the summand $R_{\wedge^4}$ is a scalar multiple of the Hodge star operator $*\in\mathrm{Sym}^2(\wedge^2\mathbb{R}^4)$, and it vanishes if and only if R satisfies the first Bianchi identity. We denote by $\mathrm{Sym}^2_b(\wedge^2\mathbb{R}^4)$ the subspace of elements $R\in\mathrm{Sym}^2(\wedge^2\mathbb{R}^4)$ with $R_{\wedge^4}=0$. Such R are called *algebraic curvature operators*, while general elements of $\mathrm{Sym}^2(\wedge^2\mathbb{R}^4)$ are often called *modified curvature operators*.

The curvature operator of a Riemannian 4-manifold (M^4,g), at a point $p\in M$, is an element of $\mathrm{Sym}^2_b(\wedge^2\mathbb{R}^4)$ once T_pM is isometrically identified with $\mathbb{R}^4$; and, conversely, every element of $\mathrm{Sym}^2_b(\wedge^2\mathbb{R}^4)$ can be realized as the curvature operator of a Riemannian 4-manifold at a point. In terms of the Kulkarni–Nomizu product,

$$R_{\mathcal{U}} = \tfrac{\mathrm{scal}}{24}\,\mathrm{g}\owedge\mathrm{g}, \quad \text{and} \quad R_{\mathcal{L}} = \tfrac12\mathrm{g}\owedge\big(\mathrm{Ric}-\tfrac14\,\mathrm{scal}\big),$$

i.e., geometrically, $R_{\mathcal{U}}$ encodes the scalar curvature, $R_{\mathcal{L}}$ the traceless Ricci tensor, and $R_{\mathcal{W}}$ the Weyl tensor, so an algebraic curvature operators R is called *scalar flat* if $R_{\mathcal{U}}=0$, *Einstein* if $R_{\mathcal{L}}=0$, and *locally conformally flat* if $R_{\mathcal{W}}=0$.

2.2. **Further decompositions.** The Hodge star operator $*\in\mathrm{Sym}^2(\wedge^2\mathbb{R}^4)$ has eigenvalues ±1, and the corresponding eigenspaces $\wedge^2_+\mathbb{R}^4$ and $\wedge^2_-\mathbb{R}^4$ are called the self-dual and anti-self-dual subspaces, respectively. The orthogonal direct sum

$$\wedge^2\,\mathbb{R}^4 = \wedge^2_+\mathbb{R}^4\oplus\wedge^2_-\mathbb{R}^4 \tag{2.2}$$

is preserved by $\mathsf{SO}(4)$, whose action on (2.2) factors through the standard product action of its $\mathbb{Z}_2$-quotient $\mathsf{SO}(3)\times\mathsf{SO}(3)$ on $\mathbb{R}^3\oplus\mathbb{R}^3$. Restricting the $\mathsf{O}(4)$-representation (2.1) to $\mathsf{SO}(4)\subset\mathsf{O}(4)$, the subspace $\mathcal{W}$ further decomposes into two $\mathsf{SO}(4)$-irreducibles $\mathcal{W}=\mathcal{W}_+\oplus\mathcal{W}_-$, which are the 5-dimensional subspaces $\mathcal{W}_\pm\cong$

$\mathrm{Sym}^2_0(\wedge^2_\pm\mathbb{R}^4)$. In particular, if (M^4,g) is an *oriented* Riemannian 4-manifold, then the summand $R_{\mathcal{W}}$ of its curvature operator splits accordingly as $R_{\mathcal{W}_+}+R_{\mathcal{W}_-}$.

Restricting the $\mathsf{O}(4)$-representation (2.1) once more, to $\mathsf{U}(2)\subset\mathsf{SO}(4)$, the subspace $\wedge^2_+\mathbb{R}^4$ decomposes into two $\mathsf{U}(2)$-irreducibles, of dimensions 1 and 2, and thus $\mathcal{W}_+$ decomposes into three $\mathsf{U}(2)$-irreducibles, of dimensions 1, 2, and 2, while $\mathcal{L}$ decomposes into two $\mathsf{U}(2)$-irreducibles, of dimensions 3 and 6, see [Arm97, TV81].

2.3. **Canonical form.** Given $R\in\mathrm{Sym}^2_b(\wedge^2\mathbb{R}^4)$, the above $\mathsf{SO}(4)$-action on (2.2) can be used to diagonalize R on each subspace $\wedge^2_\pm\mathbb{R}^4$, obtaining orthonormal bases of $\wedge^2_+\mathbb{R}^4$ and $\wedge^2_-\mathbb{R}^4$ so that the matrix representing R has the block structure

$$R=\begin{pmatrix} u\,\mathrm{Id}+W_+ & C^{\mathrm{t}} \\ C & u\,\mathrm{Id}+W_- \end{pmatrix}, \tag{2.3}$$

where $u\in\mathbb{R}$ is a scalar, $W_\pm$ are traceless diagonal 3×3 matrices, with eigenvalues $w_1^\pm\leq w_2^\pm\leq w_3^\pm$, and C is a 3×3 matrix. To simplify notation, we often use vectors

$$\vec{w}_\pm := \big(w_1^\pm, w_2^\pm, w_3^\pm\big), \tag{2.4}$$

and write $W_\pm=\mathrm{diag}(w_1^\pm,w_2^\pm,w_3^\pm)$. For convenience, we assume henceforth that every algebraic curvature operator R is in the above canonical form (2.3).

The components in (2.3) correspond precisely to $R_{\mathcal{U}}$, $R_{\mathcal{L}}$, and $R_{\mathcal{W}_\pm}$. Note that

$$u=\tfrac{1}{12}\,\mathrm{scal}=\tfrac{1}{6}\,\mathrm{tr}\,R$$

and R is Einstein if and only if $C=0$, which, in turn, is equivalent to R and $*$ commuting. Moreover, R is called *half conformally flat* if one of the self-dual or anti-self-dual Weyl tensors $W_\pm$ vanishes. The involution of (2.2) given by reversing orientation interchanges $\wedge^2_\pm\mathbb{R}^4$, and hence interchanges $W_\pm$, transposes C, but leaves u invariant. This is the effect on the curvature operator of an oriented Riemannian 4-manifold if its orientation is reversed.

The algebraic curvature operator R is *Kähler* if $R\in\mathrm{Sym}^2_b(\mathfrak{u}(2))$, where $\mathfrak{u}(2)\subset\mathfrak{o}(4)\cong\wedge^2\mathbb{R}^4$ is the Lie algebra of $\mathsf{U}(2)\subset\mathsf{O}(4)$, or, equivalently, if $JR=RJ=R$, where $J\in\mathrm{Sym}^2(\wedge^2\mathbb{R}^4)\cong\mathrm{Sym}^2(\wedge^2\mathbb{C}^2)$ is the map $J(v\wedge w)=\sqrt{-1}\,v\wedge\sqrt{-1}\,w$. In terms of the canonical form (2.3), this means that $\vec{w}_+=(-u,-u,2u)$, or $\vec{w}_+=(2u,-u,-u)$, depending on the sign of u, and C^{t} has at most one nonzero row, so that $\ker R$ contains $\mathfrak{u}(2)^\perp\subset\wedge^2_+\mathbb{R}^4$.

Remark 2.1. There are other *canonical* ways to represent the matrix of a curvature operator $R\in\mathrm{Sym}^2_b(\wedge^2\mathbb{R}^4)$ aside from the above (2.3), see e.g. Ville [Vil85, Vil89] and Appendix A. Furthermore, for Einstein curvature operators, see [Ber61, ST69].

2.4. **Sectional curvature and pinching.** Consider the oriented Grassmannian

$$\mathrm{Gr}_2^+(\mathbb{R}^4)=\big\{\alpha\in\wedge^2\mathbb{R}^4 : |\alpha|^2=1,\ \alpha\wedge\alpha=0\big\}\subset\wedge^2\mathbb{R}^4, \tag{2.5}$$

which, using (2.2), can be realized as $S^2\times S^2\subset\mathbb{R}^3\oplus\mathbb{R}^3$, cf. (A.6). Given a modified curvature operator $R\in\mathrm{Sym}^2(\wedge^2\mathbb{R}^4)$, the *sectional curvature* of $\alpha\in\mathrm{Gr}_2^+(\mathbb{R}^4)$ is

$$\sec_R(\alpha):=\langle R(\alpha),\alpha\rangle.$$

Clearly, $\sec_R$ depends linearly on R. Moreover, given $k\in\mathbb{R}$, we write $\sec_R\geq k$ if $\sec_R(\alpha)\geq k$ for all $\alpha\in\mathrm{Gr}_2^+(\mathbb{R}^4)$, and analogously for $\sec_R\leq k$.

Definition 2.2. Given $0<\delta\leq1$, an algebraic curvature operator $R\in\mathrm{Sym}^2_b(\wedge^2\mathbb{R}^4)$ is called *positively δ-pinched* if $\delta\leq\sec_R\leq1$, *negatively δ-pinched* if $-R$ is positively δ-pinched, and *δ-pinched* if either R or $-R$ is positively δ-pinched.

The following characterization of sectional curvature bounds for algebraic curvature operators is a consequence of Finsler's Lemma in Optimization Theory, that became known as Thorpe's trick in the Geometric Analysis community, see [BKM21].

Lemma 2.3 (Finsler–Thorpe Trick)**.** *Let* $R \in \mathrm{Sym}^2_b(\wedge^2\mathbb{R}^4)$ *be an algebraic curvature operator. Then* $\sec_R \geq 0$ *if and only if there exists* $t \in \mathbb{R}$ *such that* $R + t\,* \succeq 0$.

In other words, the set of algebraic curvature operators with $\sec \geq 0$ is the image of the set of positive-semidefinite operators in $\mathrm{Sym}^2(\wedge^2\mathbb{R}^4)$ under the orthogonal projection onto $\mathrm{Sym}^2_b(\wedge^2\mathbb{R}^4)$. Since $\sec_R$ depends linearly on R, and $\sec_{\mathrm{Id}} = 1$, the following is an immediate consequence of Lemma 2.3.

Corollary 2.4. *An algebraic curvature operator* $R \in \mathrm{Sym}^2_b(\wedge^2\mathbb{R}^4)$ *is positively* δ*-pinched if and only if there exist* $t_1, t_2 \in \mathbb{R}$ *such that*

$$R - \delta\,\mathrm{Id} + t_1 * \succeq 0, \qquad \textit{and} \qquad \mathrm{Id} - R + t_2 * \succeq 0. \tag{2.6}$$

Example 2.5. Useful models of algebraic curvature operators are given by the curvature operators of symmetric spaces, which are trivially constant. Recall that each compact symmetric space has a noncompact dual, and their curvature operators are the opposite of one another. For instance, the round 4-sphere S^4, and its noncompact dual, the hyperbolic space H^4, have curvature operators

$$R_{S^4} = \mathrm{Id}, \quad \text{and} \quad R_{H^4} = -\,\mathrm{Id},$$

and hence have constant sectional curvature 1 and -1, respectively. Both are obviously Einstein and locally conformally flat. Moreover, R_{S^4} is clearly the unique algebraic curvature operator that is positively δ-pinched for all $0 < \delta \leq 1$.

The curvature operators of the complex projective plane $\mathbb{C}P^2$, and its noncompact dual, the complex hyperbolic plane $\mathbb{C}H^2$, written in the form (2.3), are:

$$R_{\mathbb{C}P^2} = \begin{pmatrix} \mathrm{diag}(0,0,6) & \\ & 2\,\mathrm{Id} \end{pmatrix} \quad \text{and} \quad R_{\mathbb{C}H^2} = \begin{pmatrix} \mathrm{diag}(-6,0,0) & \\ & -2\,\mathrm{Id} \end{pmatrix}, \tag{2.7}$$

and hence have sectional curvatures $1 \leq \sec_{R_{\mathbb{C}P^2}} \leq 4$ and $-4 \leq \sec_{R_{\mathbb{C}H^2}} \leq -1$, respectively. In particular, $R = \frac{1}{4}R_{\mathbb{C}P^2}$ is positively $\frac{1}{4}$-pinched, with (2.6) satisfied setting $t_1 = \frac{1}{4}$ and $t_2 = \frac{1}{2}$. Both curvature operators (2.7) are half conformally flat, Kähler, and Einstein. Reversing orientations, one obtains the curvature operators $R_{\overline{\mathbb{C}P^2}}$ and $R_{\overline{\mathbb{C}H^2}}$, having the same diagonal blocks as (2.7) but in the reverse order.

2.5. **Topology of closed 4-manifolds.** Let M be a closed oriented (smooth) 4-manifold, and denote by $b_k(M) = \operatorname{rank} H_k(M,\mathbb{Z})$ its Betti numbers. The *intersection form* of M is the unimodular symmetric bilinear form defined on the torsion-free part of the second cohomology $H^2(M,\mathbb{Z})$ as

$$Q_M \colon H^2(M,\mathbb{Z})/\text{torsion} \times H^2(M,\mathbb{Z})/\text{torsion} \longrightarrow \mathbb{Z}$$
$$Q_M(\alpha,\beta) = (\alpha \smile \beta)([M]),$$

where $[M] \in H_4(M,\mathbb{Z})$ denotes the fundamental class of M. Alternatively, using de Rham cohomology and representing $\alpha, \beta \in \Omega^2(M)$ as 2-forms on M, one has $Q_M(\alpha,\beta) = \int_M \alpha \wedge \beta$. The number of positive and negative eigenvalues of Q_M, counted with multiplicities, are denoted $b_+(M)$ and $b_-(M)$, respectively. By Hodge Theory, $b_\pm(M) = \dim\{\alpha \in \Omega^2(M) : \Delta\alpha = 0,\ *\alpha = \pm\alpha\}$.

The manifold M is called *definite* if Q_M is definite, i.e., if either $b_+(M) = 0$ or $b_-(M) = 0$, and called *indefinite* otherwise. Denoting by $\overline{M}$ the manifold M with its reverse orientation, $Q_{\overline{M}} = -Q_M$, so $b_\pm(\overline{M}) = b_\mp(M)$. Moreover, $Q_{M_1\# M_2} = Q_{M_1}\oplus Q_{M_2}$. For instance, $\mathbb{C}P^2$ is definite, since $b_+(\mathbb{C}P^2)=1$ and $b_-(\mathbb{C}P^2)=0$, and $b_+(\#^r\mathbb{C}P^2\#^s\overline{\mathbb{C}P^2}) = r$ and $b_-(\#^r\mathbb{C}P^2\#^s\overline{\mathbb{C}P^2}) = s$, while $S^2\times S^2$ is indefinite, as $b_+(S^2\times S^2) = b_-(S^2\times S^2) = 1$, and $b_+(\#^r(S^2\times S^2)) = b_-(\#^r(S^2\times S^2)) = r$. Recall that $\overline{S^2\times S^2} = S^2\times S^2$, since the antipodal map on S^2 is orientation-reversing.

Clearly, $b_2(M) = b_+(M) + b_-(M)$. Since M is oriented, by Poincaré duality, $b_0(M) = b_4(M) = 1$ and $b_1(M) = b_3(M)$. The *Euler characteristic* and *signature* of M are given by

$$\chi(M) = 2 - 2b_1(M) + b_+(M) + b_-(M)$$
$$\sigma(M) = b_+(M) - b_-(M).$$

In particular, it follows that, for all closed oriented 4-manifolds,

$$\chi(M) \equiv \sigma(M) \mod 2, \quad \text{and} \quad \chi(M) \geq |\sigma(M)| - 2b_1(M) + 2. \tag{2.8}$$

If M is simply-connected, then $H^2(M,\mathbb{Z})$ is free and $b_1(M) = b_3(M) = 0$. In this case, the celebrated works of Donaldson [Don83] and Freedman [Fre82], combined with the $\widehat{A}$-genus obstruction to $\mathrm{scal} > 0$ for spin manifolds yields the following:

Theorem 2.6 (Donaldson, Freedman, Lichnerowicz)**.** *Let M be a smooth, closed, oriented, simply-connected 4-manifold that admits a metric with* $\mathrm{scal} > 0$*.*

(i) *If M is non-spin, then M is homeomorphic to $\#^r\mathbb{C}P^2\#^s\overline{\mathbb{C}P^2}$,*
(ii) *If M is spin, then $\sigma(M) = 0$ and M is homeomorphic to $\#^r(S^2\times S^2)$,*

where $r = b_+(M)$ and $s = b_-(M)$; and if $r = 0$ or $s = 0$, then the corresponding (trivial) summand is S^4.

Proof. By Donaldson [Don83] and Freedman [Fre82], every smooth, closed, oriented, simply-connected 4-manifold M is homeomorphic to either:

- a connected sum of $\mathbb{C}P^2$'s and $\overline{\mathbb{C}P^2}$'s, if M is non-spin,
- a connected sum of $S^2\times S^2$'s and M_{E_8}'s, if M is spin.

In the above, M_{E_8} is a certain non-smooth 4-manifold with $\sigma(M_{E_8}) = 8$, see [DK90, Chap. 1] for details. However, if M is spin, then the existence of a metric with $\mathrm{scal} > 0$ implies that $\widehat{A}(M) = 0$, and hence $\sigma(M) = 0$, see e.g. [Bes08, §6.72]. Therefore, no copies of M_{E_8} may appear in this situation, concluding the proof. □

Remark 2.7. The *converse* to Theorem 2.6 also holds, in the sense that for all $r,s\in\mathbb{N}\cup\{0\}$, the 4-manifolds $\#^r\mathbb{C}P^2\#^s\overline{\mathbb{C}P^2}$ and $\#^r(S^2\times S^2)$ admit (smooth structures that support) metrics with $\mathrm{scal} > 0$; in fact $\mathrm{Ric} > 0$, see [SY93, Per97].

2.6. **Integral formulas.** According to the Chern–Gauss–Bonnet formula and the Hirzebruch signature formula, the Euler characteristic and signature of a closed oriented Riemannian 4-manifold (M^4,g) can be respectively expressed as

$$\chi(M) = \frac{1}{\pi^2}\int_M \underline{\chi}(R)\,\mathrm{vol_g}, \quad \text{and} \quad \sigma(M) = \frac{1}{\pi^2}\int_M \underline{\sigma}(R)\,\mathrm{vol_g}, \tag{2.9}$$

where $\underline{\chi}(R)$ and $\underline{\sigma}(R)$ are quadratic forms on its curvature operator R, given by

$$\underline{\chi}(R) = \tfrac{1}{8}\left(6u^2 + |W_+|^2 + |W_-|^2 - 2|C|^2\right), \tag{2.10}$$
$$\underline{\sigma}(R) = \tfrac{1}{12}\left(|W_+|^2 - |W_-|^2\right), \tag{2.11}$$

if R is written in the canonical form (2.3), see e.g. [Bes08, p. 161]. We denote by $|A|$ the Hilbert–Schmidt norm of $A \in \mathrm{Mat}_{n\times n}(\mathbb{R})$, i.e., $|A|^2 = \sum_{i,j} a_{ij}^2$. In particular,

$$|W_\pm|^2 = |\vec{w}_\pm|^2 = (w_1^\pm)^2 + (w_2^\pm)^2 + (w_3^\pm)^2,$$

is the usual Euclidean norm of the vectors $\vec{w}_\pm$ as in (2.4). Since all terms in (2.10) and (2.11) are of degree 2, it follows that $\underline{\sigma}(-R) = \underline{\sigma}(R)$ and $\underline{\chi}(-R) = \underline{\chi}(R)$. Moreover, reversing the orientation changes the sign of $\underline{\sigma}$ but leaves $\underline{\chi}$ invariant.

Remark 2.8. If R is Einstein, i.e., $C = 0$, then $2\underline{\chi}(R) - 3\underline{\sigma}(R) = \frac{3}{2}u^2 + \frac{1}{2}|W_-|^2 \geq 0$ by (2.10) and (2.11). Therefore, any closed oriented Einstein 4-manifold (M^4, g) satisfies the *Hitchin–Thorpe inequality* $|\sigma(M)| \leq \frac{2}{3}\chi(M)$, see e.g. [Bes08, Thm. 6.35].

If (M^4, g) is a noncompact negatively δ-pinched 4-manifold with *finite volume*, then the integrals (2.9) converge. Such manifolds have *finite topological type*; i.e., are diffeomorphic to the interior of a compact manifold with boundary, see e.g. [BGS85, Thm. 10.5]. In particular, $\chi(M)$ is well-defined. Moreover, (M^4, g) has *bounded geometry* in the sense of Cheeger and Gromov [CG85a, CG85b], since its universal cover has infinite injectivity radius. Thus, by [CG85a, Thm. 3.1 (3), Thm. 6.1], both equalities in (2.9) hold, where $\sigma(M)$ is to be understood as the L^2-signature $\sigma_{(2)}(M^4, \mathrm{g})$, which is a proper homotopy invariant.

We shall also need the following consequence of [CGY03, Thm A].

Lemma 2.9 (Chang–Gursky–Yang)**.** *If* (M^4, g) *is a closed oriented* 4*-manifold with* $\sec > 0$*, then either* M *is diffeomorphic to* S^4*, or*

$$\chi(M) \leq \frac{1}{4\pi^2} \int_M |W_+|^2 + |W_-|^2 \ \mathrm{vol}_{\mathrm{g}}.$$

Proof. To apply [CGY03, Thm A], we need to show that the Yamabe invariant $Y(M, \mathrm{g})$ is positive. By the solution of the Yamabe problem (see [Aub98, Thms 5.11, 5.30]), there exists a positive function $\varphi \in C^\infty(M)$ such that the metric $\mathrm{g}' = \varphi^2 \mathrm{g}$ has constant scalar curvature c, and such that the infimum in the definition of $Y(M, \mathrm{g})$ is achieved at g'. Thus $Y(M, \mathrm{g}) = c\,\mathrm{Vol}(M, \mathrm{g}')^{1/2}$. Let $p \in M$ be the global maximum of the conformal factor φ, so that $\Delta\varphi(p) \geq 0$. Since $6\Delta\varphi + \mathrm{scal}_{\mathrm{g}}\,\varphi = c\,\varphi^3$, we have that $c > 0$, so $Y(M, \mathrm{g}) > 0$. The conclusion now follows from [CGY03, Thm A], keeping in mind the different norm conventions, see [CGY03, Rmk 2]. □

2.7. **Sphere theorem and volume bound.** Finally, for the reader's convenience, we now explain how to use standard techniques in Comparison Geometry to prove:

Lemma 2.10. *Let* (M^4, g) *be a* 4*-manifold with* $\sec \geq \delta > 0$*. If* M *is not homeomorphic to* S^4*, then* $\mathrm{Vol}(M, \mathrm{g}) \leq \frac{4\pi^2}{3\delta^2}$*.*

Proof. Since M is not homeomorphic to S^4, the Grove–Shiohama Diameter Sphere Theorem [GS77] implies that $\mathrm{diam}(M) \leq \frac{1}{2}\,\mathrm{diam}\big(S^4\big(\frac{1}{\sqrt{\delta}}\big)\big) = \frac{\pi}{2\sqrt{\delta}}$. In particular, choosing any $p \in M$, we have $M = B\big(p, \frac{\pi}{2\sqrt{\delta}}\big)$. Since $\sec \geq \delta$, the Ricci curvature of (M^4, g) is at least that of the sphere $S^4\big(\frac{1}{\sqrt{\delta}}\big)$, and hence, by the Bishop Volume Comparison Theorem (see e.g. [Bes08, Chap. 0.H]), the volume of the ball $B\big(p, \frac{\pi}{2\sqrt{\delta}}\big)$ is at most the volume of a hemisphere in $S^4\big(\frac{1}{\sqrt{\delta}}\big)$. In conclusion:

$$\mathrm{Vol}(M, \mathrm{g}) = \mathrm{Vol}\big(B\big(p, \tfrac{\pi}{2\sqrt{\delta}}\big)\big) \leq \tfrac{1}{2}\,\mathrm{Vol}\big(S^4\big(\tfrac{1}{\sqrt{\delta}}\big)\big) = \tfrac{1}{2\delta^2}\,\mathrm{Vol}\big(S^4(1)\big) = \tfrac{4\pi^2}{3\delta^2}. \qquad \square$$

3. Preliminaries on Optimization

This short section discusses an elementary (yet quite useful) approach to optimize quadratic forms on compact convex sets, particularly polytopes and simplices.

Let $Q\colon \mathbb{R}^n \to \mathbb{R}$ be a polynomial function of degree 2. We say Q is *positive-* or *negative-(semi)definite* if its Hessian matrix is positive- or negative-(semi)definite, and *indefinite* otherwise. It is noteworthy that even though the quadratic program

$$\begin{aligned}&\text{maximize } Q(x),\\ &\text{subject to } A\,x + b \preceq 0,\end{aligned}$$

where $A \in \mathrm{Mat}_{n\times n}(\mathbb{R})$ and $b \in \mathbb{R}^n$, can be solved in polynomial time if Q is negative-semidefinite [KTH79], it becomes NP-hard if the Hessian of Q is allowed to have positive eigenvalues [Sah74, PV91].

A *face* of a closed convex set $K \subset \mathbb{R}^n$ is a convex subset $F \subset K$ such that, whenever $\frac{x+y}{2} \in F$ for some $x, y \in K$, then both $x, y \in F$. Given a closed convex set $K \subset \mathbb{R}^n$, let $\mathcal{F}(K) = \{F \subset K : F \text{ is a face of } K\}$. Then

$$K = \bigsqcup_{F \in \mathcal{F}(K)} \mathrm{relint}(F),$$

where $\mathrm{relint}(F)$ is the *relative interior* of F, i.e., the interior of F inside its *affine hull* $\mathrm{aff}(F)$, which is the smallest affine subspace of $\mathbb{R}^n$ containing F. The dimension of a face F is defined as the dimension of its supporting affine space $\mathrm{aff}(F)$.

Proposition 3.1. *Let $Q\colon \mathbb{R}^n \to \mathbb{R}$ be a polynomial function of degree 2, and let $K \subset \mathbb{R}^n$ be a compact convex set. There is a face F of K such that $Q' = Q|_{\mathrm{aff}(F)}$ is negative-definite and a point $x_0 \in \mathrm{relint}(F)$ such that $Q(x_0) = \max_{x\in K} Q(x)$. Furthermore, x_0 is the unique point in $\mathrm{aff}(F)$ such that $\nabla Q'(x_0) = 0$.*

Proof. Let $x_0 \in K$ be such that $Q(x_0) = \max_{x\in K} Q(x)$ and $F \in \mathcal{F}(K)$ the face whose relative interior contains x_0. We clearly have $Q'(x_0) = \max_{x\in F} Q'(x)$. If Q' is not negative-definite, then there is an affine line $L \subset \mathrm{aff}(F)$ through x_0, such that the restriction of Q' to L is a convex function. So there exists x_1 in the relative boundary of F, and hence in a face of smaller dimension, with $Q(x_1) \geq Q(x_0)$. This implies the first claim. Furthermore, since Q' is negative-definite, it has a unique critical point, which is the global maximum of Q'. Since x_0 is in the relative interior of F, it must be the global maximum of Q'. □

Note that if $F = \{x_0\}$ is a singleton, then $\mathrm{aff}(F) = \mathrm{relint}(F) = F$, the restriction of $Q|_{\mathrm{aff}(F)}$ is simultaneously negative- and positive-definite, and $\nabla Q|_{\mathrm{aff}(F)}(x_0) = 0$.

Corollary 3.2. *Let $Q\colon \mathbb{R}^n \to \mathbb{R}$ be a polynomial function of degree 2 whose Hessian matrix has d eigenvalues in $(-\infty, 0)$, and $K \subset \mathbb{R}^n$ be a compact convex set. Then*

$$\max_{x\in K} Q(x) = \max_{\substack{x\in \mathrm{relint}(F),\\ F\in\mathcal{F}(K),\, \dim F \leq d}} Q(x).$$

Proof. By Proposition 3.1 there exists a face F of K such that $Q|_{\mathrm{aff}(F)}$ is negative-definite and $x_0 \in \mathrm{relint}(F)$ with $Q(x_0) = \max_{x\in K} Q(x)$. Negative-definiteness of $Q|_{\mathrm{aff}(F)}$ implies that $\dim(F) \leq d$. □

3.1. **Optimizing quadratic polynomials on simplices.** Suppose $\Delta^k \subset \mathbb{R}^n$ is a k-simplex with vertices $V = \{v_1, \dots, v_{k+1}\}$, i.e., V is affinely independent and $\Delta^k = \mathrm{conv}(V)$, and $Q\colon \mathbb{R}^n \to \mathbb{R}$ is a polynomial function of degree 2. Recall that

$$\mathrm{conv}(S) := \left\{ \sum_{j=1}^{N} x_j s_j \ : \ s_j \in S,\, x_j \geq 0,\, \sum_{j=1}^{N} x_j = 1,\, N \in \mathbb{N} \right\}$$

denotes the *convex hull* of a set $S \subset \mathbb{R}^n$. Note that Proposition 3.1 gives a method to compute $\max\limits_{x\in\Delta^k} Q(x)$ that only involves Linear Algebra, and Corollary 3.2 renders this computation significantly easier if d is small.

By Proposition 3.1, the maximum of Q on Δ^k is attained in the relative interior of a face on whose affine hull Q is negative-definite. Since Δ^k is a simplex, any subset $S \subset V$ is affinely independent, its convex hull $\mathrm{conv}(S)$ is a face of Δ^k, and every face of Δ^k is of this form. Thus, in order to find $\max\limits_{x\in\Delta^k} Q(x)$, first compute

$$\mathcal{S} = \big\{ S \subset V \, : \, Q|_{\mathrm{aff}(S)} \text{ is negative-definite} \big\}. \tag{3.1}$$

Then, for each $S \in \mathcal{S}$, compute the unique point $x_S \in \mathrm{aff}(S)$ with $\nabla Q|_{\mathrm{aff}(S)}(x_S) = 0$ and consider the set

$$\mathcal{S}' = \big\{ S \in \mathcal{S} \, : \, x_S \in \mathrm{relint}(\mathrm{conv}(S)) \big\}. \tag{3.2}$$

Finally, Proposition 3.1 implies that

$$\max_{x\in\Delta^k} Q(x) = \max_{S\in\mathcal{S}'} Q(x_S). \tag{3.3}$$

4. Pinched curvature operators

In this section, we establish foundational results on convex sets of δ-pinched curvature operators that are needed throughout the rest of the paper.

4.1. **Projection onto Einstein curvature operators.** Using the same notation in the decomposition (2.1) of $\mathrm{Sym}^2(\wedge^2\mathbb{R}^4)$, define the sets

$$\begin{aligned} \Omega_\delta &:= \big\{ R \in \mathrm{Sym}^2_b(\wedge^2\mathbb{R}^4) : R \text{ is positively } \delta\text{-pinched} \big\}, \\ \mathrm{E}_\delta &:= \Omega_\delta \cap (\mathcal{U} \oplus \mathcal{W}), \end{aligned} \tag{4.1}$$

and consider the orthogonal projection onto Einstein (modified) curvature operators

$$\begin{aligned} \mathrm{pr}\colon\ \mathrm{Sym}^2(\wedge^4\mathbb{R}^4) &\longrightarrow \mathcal{U} \oplus \mathcal{W} \oplus \wedge^4\mathbb{R}^4 \\ \mathrm{pr}(R) &= R - R_{\mathcal{L}}. \end{aligned} \tag{4.2}$$

Lemma 4.1. *If $R \in \mathrm{Sym}^2(\wedge^2\mathbb{R}^4)$ satisfies $R \succeq 0$, then also $\mathrm{pr}(R) \succeq 0$.*

Proof. Consider the decomposition $\wedge^2\mathbb{R}^4 = \wedge^2_+\mathbb{R}^4 \oplus \wedge^2_-\mathbb{R}^4$ as in (2.2), and denote by $\pi_\pm\colon \wedge^2\mathbb{R}^4 \to \wedge^2_\pm\mathbb{R}^4$ the corresponding orthogonal projections. Since $R \succeq 0$, we have $\mathrm{pr}(R) = \pi_+ \circ R \circ \pi_+ + \pi_- \circ R \circ \pi_- \succeq 0$. ☐

Lemma 4.2. *Let $R \in \mathrm{Sym}^2_b(\wedge^2\mathbb{R}^4)$ be an algebraic curvature operator.*

(a) *If $\sec_R \geq 0$, then $\sec_{\mathrm{pr}(R)} \geq 0$.*
(b) *If $R \in \Omega_\delta$, then $\mathrm{pr}(R) \in \mathrm{E}_\delta$.*

Proof. By the Finsler–Thorpe Trick (Lemma 2.3), if $\sec_R \geq 0$, then there exists $S \in \mathrm{Sym}^2(\wedge^2\mathbb{R}^4)$, $S \succeq 0$, whose orthogonal projection on $\mathrm{Sym}^2_b(\wedge^2\mathbb{R}^4)$ is R. Clearly, $\mathrm{pr}(R)$ is the orthogonal projection on $\mathrm{Sym}^2_b(\wedge^2\mathbb{R}^4)$ of $\mathrm{pr}(S)$. Thus, in order to prove (a), it suffices to show that $\mathrm{pr}(S) \succeq 0$, which holds by Lemma 4.1. As $\sec_R$ depends linearly on R, and $\sec_{\mathrm{Id}} = 1$, applying (a) to $R - \delta\,\mathrm{Id}$ and $\mathrm{Id} - R$ yields (b). □

Proposition 4.3. $\mathrm{pr}(\Omega_\delta) = \mathrm{E}_\delta$.

Proof. Lemma 4.2 (b) gives one inclusion, the other is clear from Definition 2.2. □

Remark 4.4. The properties of the projection (4.2) onto Einstein curvature operators in Lemma 4.2 and Proposition 4.3 *do not* hold for the projection onto locally conformally flat curvature operators. Namely, there exists $R \in \mathrm{Sym}^2_b(\wedge^2\mathbb{R}^4)$ with $\sec_R \geq 0$ whose locally conformally flat part $R_{\mathcal{U}} + R_{\mathcal{L}}$ does not have $\sec \geq 0$.

4.2. **Einstein simplices.** Fix $0 < \delta < 1$, and consider the curvature operators

$$R(\vec{w}_+, \vec{w}_-, u) := \mathrm{diag}(u + w_i^+, u + w_i^-) \in \mathcal{U} \oplus \mathcal{W} \tag{4.3}$$

as in (2.3), with $C = 0$, and recall that $\vec{w}_\pm = \big(w_1^\pm, w_2^\pm, w_3^\pm\big)$ satisfy

$$w_1^\pm + w_2^\pm + w_3^\pm = 0, \qquad w_1^\pm \leq w_2^\pm \leq w_3^\pm. \tag{4.4}$$

By the Finsler-Thorpe Trick (Lemma 2.3), such a curvature operator $R(\vec{w}_+, \vec{w}_-, u)$ is positively δ-pinched if and only if there exist $t_1, t_2 \in \mathbb{R}$ such that (2.6) holds, i.e.,

$$\begin{aligned} \delta &\leq w_i^+ + u + t_1, & w_i^+ + u + t_2 &\leq 1, \\ \delta &\leq w_i^- + u - t_1, & w_i^- + u - t_2 &\leq 1, \end{aligned} \qquad i = 1, 2, 3. \tag{4.5}$$

We first get rid of some redundant inequalities:

Lemma 4.5. $R(\vec{w}_+, \vec{w}_-, u) \in \mathrm{E}_\delta$ *if and only if there exist* $t_1, t_2 \in \mathbb{R}$ *such that*

$$\begin{aligned} \delta &\leq w_1^+ + u + t_1, & w_3^+ + u + t_2 &\leq 1, \\ \delta &\leq w_1^- + u - t_1, & w_3^- + u - t_2 &\leq 1. \end{aligned} \tag{4.6}$$

Proof. The inequalities in (4.6) are a subset of those in (4.5), so (4.6) obviously holds if $R(\vec{w}_+, \vec{w}_-, u) \in \mathrm{E}_\delta$. For the converse, observe that $\delta \leq w_1^+ + u + t_1$ together with $w_1^+ \leq w_i^+$ imply that $\delta \leq w_i^+ + u + t_1$ for $i = 1, 2, 3$. The other inequalities in (4.5) missing from (4.6) can be obtained in the same way, using (4.4). □

The set of points $(\vec{w}_+, \vec{w}_-, u, t_1, t_2) \in \mathbb{R}^9$ that satisfy (4.4) and (4.6) is clearly an intersection of linear subspaces and affine half-spaces in $\mathbb{R}^9$. In order to eliminate the variables t_1 and t_2, we show it is actually a simplex and compute its vertices.

Proposition 4.6. *Let $\Delta^7_\delta \subset \mathbb{R}^7$ be the 7-simplex defined as convex hull of the rows $v_1, \dots, v_8$ of the matrix:*

$$\begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \\ v_7 \\ v_8 \end{array}\begin{array}{c} \begin{array}{ccccccc} w_1^+ & w_2^+ & w_1^- & w_2^- & u & t_1 & t_2 \end{array} \\ \begin{pmatrix} \frac{2}{3}(\delta-1) & \frac{2}{3}(\delta-1) & 0 & 0 & \frac{1}{3}(2\delta+1) & \frac{1}{3}(1-\delta) & \frac{2}{3}(\delta-1) \\ \frac{4}{3}(\delta-1) & \frac{2}{3}(1-\delta) & 0 & 0 & \frac{1}{3}(\delta+2) & \frac{2}{3}(1-\delta) & \frac{1}{3}(\delta-1) \\ 0 & 0 & \frac{2}{3}(\delta-1) & \frac{2}{3}(\delta-1) & \frac{1}{3}(2\delta+1) & \frac{1}{3}(\delta-1) & \frac{2}{3}(1-\delta) \\ 0 & 0 & \frac{4}{3}(\delta-1) & \frac{2}{3}(1-\delta) & \frac{1}{3}(\delta+2) & \frac{2}{3}(\delta-1) & \frac{1}{3}(1-\delta) \\ 0 & 0 & 0 & 0 & 1 & \delta-1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1-\delta & 0 \\ 0 & 0 & 0 & 0 & \delta & 0 & \delta-1 \\ 0 & 0 & 0 & 0 & \delta & 0 & 1-\delta \end{pmatrix} \end{array}$$

and consider its image under the linear map $\iota_7 \colon \mathbb{R}^7 \to \mathbb{R}^9$, given by

$$\iota_7\big(w_1^+, w_2^+, w_1^-, w_2^-, u, t_1, t_2\big) = \big(w_1^+, w_2^+, -w_1^+ - w_2^+, w_1^-, w_2^-, -w_1^- - w_2^-, u, t_1, t_2\big).$$

Then $(\vec{w}_+, \vec{w}_-, u, t_1, t_2) \in \mathbb{R}^9$ is in $\iota_7(\Delta^7_\delta)$ if and only if it satisfies (4.4) and (4.6).

Proof. Let P_δ be the set of $(\vec{w}_+, \vec{w}_-, u, t_1, t_2) \in \mathbb{R}^9$ that satisfy (4.4) and (4.6). Since these are points such that $R(\vec{w}_+, \vec{w}_-, u) \in \mathrm{E}_\delta$ by Lemma 4.5, the set P_δ is bounded. Indeed, each entry $R_{ijkl} = \langle R(e_i \wedge e_j), e_k \wedge e_l\rangle$ of the matrix representing a positively δ-pinched curvature operator $R \in \Omega_\delta$, where $\{e_i\}$ is an orthonormal basis, satisfies $|R_{ijkl}| \leq 1$ by Berger's classical estimates, see e.g. [Kar70]. Thus, if $(\vec{w}_+, \vec{w}_-, u, t_1, t_2) \in P_\delta$, then $|u|$ and $|\vec{w}_\pm|$ are bounded, and hence so are $|t_1|$ and $|t_2|$ by (4.6). Therefore, $P_\delta \subset \mathbb{R}^9$ is a polytope. Moreover, $P_\delta \subset \iota_7(\mathbb{R}^7)$ by (4.4), so $\dim P_\delta \leq 7$. On the other hand, $\iota_7\left(\frac{\delta-1}{3}, 0, \frac{\delta-1}{3}, 0, \frac{\delta+1}{2}, 0, 0\right)$ is in the relative interior of P_δ, and hence $\dim P_\delta = 7$. Since P_δ is defined by 8 inequalities, namely, 4 in (4.4) and 4 in (4.6), it is a 7-simplex. Its vertices are the points where 7 of the 8 inequalities are equalities. These are exactly the points $\iota_7(v_j)$, $1 \leq j \leq 8$, where v_j are the rows of the matrix above. For example, $\iota_7(v_1)$ saturates all inequalities in (4.4) and (4.6) except for $w_2^+ \leq w_3^+$. Thus, $P_\delta = \iota_7(\Delta^7_\delta)$, concluding the proof. □

Remarkably, the 7-simplex Δ^7_δ in Proposition 4.6 is such that its images under projections that eliminate one or both of the variables t_1 and t_2 are also simplices.

Proposition 4.7. *Let $\Delta^5_\delta \subset \mathbb{R}^5$ be the 5-simplex defined as convex hull of the rows $p_1, \dots, p_6$ of the matrix:*

$$\begin{array}{c} \\ p_1 \\ p_2 \\ p_3 \\ p_4 \\ p_5 \\ p_6 \end{array}\begin{array}{c} \begin{array}{ccccc} w_1^+ & w_2^+ & w_1^- & w_2^- & u \end{array} \\ \begin{pmatrix} \frac{2}{3}(\delta-1) & \frac{2}{3}(\delta-1) & 0 & 0 & \frac{1}{3}(2\delta+1) \\ \frac{4}{3}(\delta-1) & \frac{2}{3}(1-\delta) & 0 & 0 & \frac{1}{3}(\delta+2) \\ 0 & 0 & \frac{2}{3}(\delta-1) & \frac{2}{3}(\delta-1) & \frac{1}{3}(2\delta+1) \\ 0 & 0 & \frac{4}{3}(\delta-1) & \frac{2}{3}(1-\delta) & \frac{1}{3}(\delta+2) \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & \delta \end{pmatrix} \end{array}$$

and consider its image under the linear map $\iota_5 \colon \mathbb{R}^5 \to \mathbb{R}^7$, given by

$$\iota_5\big(w_1^+, w_2^+, w_1^-, w_2^-, u\big) = \big(w_1^+, w_2^+, -w_1^+ - w_2^+, w_1^-, w_2^-, -w_1^- - w_2^-, u\big).$$

Then $(\vec{w}_+, \vec{w}_-, u) \in \mathbb{R}^7$ is in $\iota_5(\Delta^5_\delta)$ if and only if $R(\vec{w}_+, \vec{w}_-, u) \in \mathrm{E}_\delta$.

Proof. By Lemma 4.5 and Proposition 4.6, we have that $R(\vec{w}_+,\vec{w}_-,u)\in \mathrm{E}_\delta$ if and only if $(\vec{w}_+,\vec{w}_-,u)\in\mathbb{R}^7$ is such that $(\vec{w}_+,\vec{w}_-,u,t_1,t_2)\in\iota_7(\Delta^7_\delta)$ for some $t_1,t_2\in\mathbb{R}$. In other words, if and only if $(\vec{w}_+,\vec{w}_-,u)\in\Pi\big(\iota_7(\Delta^7_\delta)\big)$ where $\Pi\colon\mathbb{R}^9\to\mathbb{R}^7$ is the projection that eliminates the last two coordinates. The conclusion follows, since

$$\begin{aligned}\Pi\big(\iota_7(\Delta^7_\delta)\big)&=\Pi\big(\iota_7(\mathrm{conv}(v_1,\dots,v_8))\big)\\&=\mathrm{conv}\big(\Pi(\iota_7(v_1)),\dots,\Pi(\iota_7(v_8))\big)\\&=\mathrm{conv}(\iota_5(p_1),\dots,\iota_5(p_6))\\&=\iota_5(\Delta^5_\delta),\end{aligned}$$

where the third equality holds because $\Pi(\iota_7(v_j))=p_j$ if $1\le j\le 4$, $\Pi(\iota_7(v_j))=p_5$ if $j=5,6$, and $\Pi(\iota_7(v_j))=p_6$ if $j=7,8$. □

Proposition 4.8. *Let $\Delta^6_\delta\subset\mathbb{R}^6$ be the 6-simplex defined as convex hull of the rows $q_1,\dots,q_7$ of the matrix:*

$$\begin{array}{c}\\ q_1\\ q_2\\ q_3\\ q_4\\ q_5\\ q_6\\ q_7\end{array}\begin{array}{c}\begin{array}{cccccc} w_1^+ & w_2^+ & w_1^- & w_2^- & u & t_1\end{array}\\ \begin{pmatrix}\frac23(\delta-1) & \frac23(\delta-1) & 0 & 0 & \frac13(2\delta+1) & \frac13(1-\delta)\\ \frac43(\delta-1) & \frac23(1-\delta) & 0 & 0 & \frac13(\delta+2) & \frac23(1-\delta)\\ 0 & 0 & \frac23(\delta-1) & \frac23(\delta-1) & \frac13(2\delta+1) & \frac13(\delta-1)\\ 0 & 0 & \frac43(\delta-1) & \frac23(1-\delta) & \frac13(\delta+2) & \frac23(\delta-1)\\ 0&0&0&0&1&\delta-1\\ 0&0&0&0&1&1-\delta\\ 0&0&0&0&\delta&0\end{pmatrix}\end{array}$$

and consider its image under the linear map $\iota_6\colon\mathbb{R}^6\to\mathbb{R}^8$, given by

$$\iota_6\big(w_1^+,w_2^+,w_1^-,w_2^-,u,t_1\big)=\big(w_1^+,w_2^+,-w_1^+-w_2^+,w_1^-,w_2^-,-w_1^--w_2^-,u,t_1\big).$$

Then $(\vec{w}_+,\vec{w}_-,u,t_1)\in\mathbb{R}^8$ is in $\iota_6(\Delta^6_\delta)$ if and only if $R(\vec{w}_+,\vec{w}_-,u-\delta)+t_1*\succeq 0$ and $R(\vec{w}_+,\vec{w}_-,u)\in\mathrm{E}_\delta$.

Proof. Lemma 4.5 and Proposition 4.6 imply that $R(\vec{w}_+,\vec{w}_-,u-\delta)+t_1*\succeq 0$ and $R(\vec{w}_+,\vec{w}_-,u)\in\mathrm{E}_\delta$ if and only if $(\vec{w}_+,\vec{w}_-,u,t_1)\in\iota_7(\Delta^7_\delta)$ for some $t_2\in\mathbb{R}$, i.e., $(\vec{w}_+,\vec{w}_-,u,t_1)\in\Pi'\big(\iota_7(\Delta^7_\delta)\big)$, where $\Pi'\colon\mathbb{R}^9\to\mathbb{R}^8$ is the projection that eliminates the last coordinate. Similarly to Proposition 4.7, we have that $\Pi'\big(\iota_7(\Delta^7_\delta)\big)=\iota_6(\Delta^6_\delta)$, as $\Pi'(\iota_7(v_j))=q_j$ if $1\le j\le 6$, and $\Pi'(\iota_7(v_j))=q_7$ if $j=7,8$. □

We shall refer to Δ^5_δ as the *Einstein simplex*, and to Δ^6_δ and Δ^7_δ as *augmented Einstein simplices*. The rationale for this nomenclature is that, by Proposition 4.7 and (2.3), the set of conjugacy classes of positively δ-pinched Einstein curvature operators is parametrized by Δ^5_δ. Indeed, $\varphi\colon\Delta^5_\delta\to\mathrm{E}_\delta$, where $\varphi=R\circ\iota_5$ and $R\colon\mathbb{R}^7\to\mathcal{U}\oplus\mathcal{W}$ is given by (4.3), is an affine map whose image is a section for the change of basis $\mathsf{SO}(4)$-action on $\mathrm{E}_\delta\subset\mathcal{U}\oplus\mathcal{W}$. Analogously, Δ^6_δ and Δ^7_δ parametrize this set together with the corresponding t_1 and t_2 for which (2.6) holds.

Remark 4.9. The vertices $p_1,\dots,p_6$ of the Einstein simplex Δ^5_δ correspond to geometrically meaningful curvature operators. Namely, using $\varphi=R\circ\iota_5$, we have:

$$\varphi(p_1)=\tfrac{1-\delta}{3}R_{\mathbb{C}P^2}+\tfrac{4\delta-1}{3}R_{S^4},\quad \varphi(p_3)=\tfrac{1-\delta}{3}R_{\overline{\mathbb{C}P^2}}+\tfrac{4\delta-1}{3}R_{S^4},\quad \varphi(p_5)=R_{S^4},$$
$$\varphi(p_2)=\tfrac{1-\delta}{3}R_{\mathbb{C}H^2}+\tfrac{4-\delta}{3}R_{S^4},\quad \varphi(p_4)=\tfrac{1-\delta}{3}R_{\overline{\mathbb{C}H^2}}+\tfrac{4-\delta}{3}R_{S^4},\quad \varphi(p_6)=\delta R_{S^4},$$

where $R_{S^4} = \mathrm{Id}$, while $R_{\mathbb{C}P^2}$ and $R_{\mathbb{C}H^2}$ are given in (2.7), and satisfy $1 \leq \sec \leq 4$ and $-4 \leq \sec \leq -1$ respectively. Recall that $\overline{\mathbb{C}P^2}$ and $\overline{\mathbb{C}H^2}$ are the manifolds $\mathbb{C}P^2$ and $\mathbb{C}H^2$ with the opposite orientation. Being positively δ-pinched is invariant under change of orientation (which interchanges $\vec{w}_+$ and $\vec{w}_-$ and fixes u), so the collection of vertices also has this symmetry. This is clear by comparing the first two columns above and recalling that S^4 has orientation-reversing isometries. Finally, note that $\varphi(p_j)$ depend affinely on δ, and, of course, become equal to R_{S^4} if $\delta = 1$.

5. Traceless Ricci bounds

The purpose of this section is to prove a new upper bound (Proposition 5.3) on the norm of the traceless Ricci part of 4-dimensional curvature operators with either a lower or upper sectional curvature bound. In addition to its role in the proof of Theorem A, we believe this result is of independent interest and may have other applications; e.g., it yields a simple proof of the algebraic Hopf question in dimension 4, see Corollary 5.4. We begin with two algebraic lemmas.

Lemma 5.1. *Let $0 < \lambda_1 < \cdots < \lambda_n$ and $0 < \mu_1 < \cdots < \mu_n$. For all permutations $\phi \in \mathfrak{S}_n$ we have*

$$\sum_{i=1}^{n} \lambda_i \mu_{\phi(i)} \leq \sum_{i=1}^{n} \lambda_i \mu_i.$$

Proof. Suppose, by contradiction, that the permutation $\phi \in \mathfrak{S}_n$ that maximizes $\sum_{i=1}^n \lambda_i \mu_{\phi(i)}$ is not the identity. Then, there are $1 \leq i < j \leq n$ with $\phi(i) > \phi(j)$. We have

$$(\lambda_i \mu_{\phi(j)} + \lambda_j \mu_{\phi(i)}) - (\lambda_i \mu_{\phi(i)} + \lambda_j \mu_{\phi(j)}) = (\lambda_j - \lambda_i)(\mu_{\phi(i)} - \mu_{\phi(j)}) > 0,$$

contradicting the maximality of $\sum_{i=1}^n \lambda_i \mu_{\phi(i)}$. □

Lemma 5.2. *Let $0 \leq \lambda_1 \leq \cdots \leq \lambda_n$, $0 \leq \mu_1 \leq \cdots \leq \mu_n$, and $C \in \mathrm{Mat}_{n\times n}(\mathbb{R})$ be such that*

$$\begin{pmatrix} \mathrm{diag}(\lambda_1, \ldots, \lambda_n) & C^{\mathrm{t}} \\ C & \mathrm{diag}(\mu_1, \ldots, \mu_n) \end{pmatrix} \succeq 0. \tag{5.1}$$

Then $|C|^2 \leq \sum_{i=1}^{n} \lambda_i \mu_i$.

Proof. By continuity, we shall assume $0 < \lambda_1 < \cdots < \lambda_n$ and $0 < \mu_1 < \cdots < \mu_n$. Using Schur complements, we see that (5.1) holds if and only if

$$\mathrm{diag}(\mu_1, \ldots, \mu_n) - C\,\mathrm{diag}(\lambda_1^{-1}, \ldots, \lambda_n^{-1})C^{\mathrm{t}} \succeq 0.$$

This is equivalent to $D = (d_{ij})_{1\leq i,j\leq n}$ lying in the unit ball with respect to the spectral norm, where $d_{ij} = \frac{c_{ij}}{\sqrt{\lambda_i \mu_j}}$. We thus want to bound

$$|C|^2 = \sum_{i,j=1}^{n} \lambda_i \mu_j \, d_{ij}^2 \tag{5.2}$$

from above. The extreme points of the unit ball in $\mathrm{Mat}_{n\times n}(\mathbb{R})$ with respect to the spectral norm are orthogonal matrices (see e.g. [GM77, Thm. 4(i)]), so we may assume D is orthogonal, as the right-hand side of (5.2) is a convex function in its entries. In that case, the matrix $D_2 = (d_{ij}^2)_{1\leq i,j\leq n}$ is *doubly stochastic*, i.e., each of its rows and columns sums to 1. By the Birkhoff–von Neumann Theorem (see e.g. [Bar02, Thm. II.5.2]), every doubly stochastic matrix lies in the convex hull of permutation matrices. Thus, for bounding (5.2) from above, we may further assume $D = D_2$ is a permutation matrix, so the conclusion follows from Lemma 5.1. $\square$

We are now ready for the main result of this section. Although it solely regards algebraic curvature operators in $\mathrm{Sym}^2_b(\wedge^2\mathbb{R}^4)$, we state it as a pointwise estimate on a Riemannian 4-manifold to render it more easily applicable elsewhere.

Proposition 5.3. *Let (M^4, g) be a 4-manifold, $p \in M$, and $k \in \mathbb{R}$. Let R be the curvature operator at $p \in M$, and u, $\vec{w}_\pm$, and C as in (2.3) and (2.4).*

(i) *If all 2-planes in T_pM have either $\sec \geq k$ or $\sec \leq k$, then*
$$|C|^2 \leq 3(u-k)^2 + \langle \vec{w}_+, \vec{w}_- \rangle,$$
(ii) *If $t \in \mathbb{R}$ is such that $\pm(R - k\,\mathrm{Id}) + t* \succeq 0$ on $\wedge^2 T_pM$, then*
$$|C|^2 \leq 3(u-k)^2 - 3t^2 + \langle \vec{w}_+, \vec{w}_- \rangle.$$

Proof. As elsewhere in the paper, we identify $T_pM \cong \mathbb{R}^4$ and assume the curvature operator R is in the canonical form (2.3). By Finsler–Thorpe's Trick (Lemma 2.3), if either $\sec \geq k$ or $\sec \leq k$, then there exists $t \in \mathbb{R}$ such that $R - k\,\mathrm{Id} + t* \succeq 0$ or $-(R - k\,\mathrm{Id}) + t* \succeq 0$, respectively; so it suffices to prove (ii).

Condition (5.1) is verified since $\pm(R - k\,\mathrm{Id}) + t* \succeq 0$, so we may apply Lemma 5.2 with $n = 3$, $\lambda_i = u - k + w_i^+ + t$, and $\mu_i = u - k + w_i^- - t$, concluding that
$$\begin{aligned} |C|^2 &\leq \sum_{i=1}^{3} (u - k + w_i^+ + t)(u - k + w_i^- - t) \\ &= \sum_{i=1}^{3} (u - k + t)(u - k - t) + w_i^+ w_i^- \\ &= 3(u-k)^2 - 3t^2 + \langle \vec{w}_+, \vec{w}_- \rangle, \end{aligned}$$
where the first equality uses (4.4). $\square$

While the general case of the Hopf question asking whether closed $2d$-dimensional manifolds (M^{2d}, g) with $\sec \geq 0$ or $\sec \leq 0$ have $(-1)^d \chi(M) \geq 0$ remains an important open problem, its *algebraic* variant asking whether the Chern–Gauss–Bonnet integrand $\underline{\chi}(R)$ computed at an algebraic curvature operator $R \in \mathrm{Sym}^2(\wedge^2\mathbb{R}^{2d})$ with $\sec_{\pm R} \geq 0$ satisfies $(-1)^d \underline{\chi}(R) \geq 0$ was answered affirmatively if $2d = 4$ by Milnor [Che55, BG64], and negatively if $2d \geq 6$ by Geroch [Ger76, Kle76]. The former result of Milnor can be easily recovered with Proposition 5.3, which also allows to characterize the equality case, as follows.

Corollary 5.4 (Algebraic Hopf question in dimension 4)**.** *If $R \in \mathrm{Sym}^2(\wedge^2\mathbb{R}^4)$ has $\sec_{\pm R} \geq 0$, then $\underline{\chi}(R) \geq 0$. Moreover, $\underline{\chi}(R) = 0$ if and only if $\pm R \succeq 0$, $W_+ = W_-$, and $|C|^2 = 3u^2 + |W_\pm|^2$; in particular, if R is Einstein, then $R = 0$.*

Proof. As before, let $t \in \mathbb{R}$ be such that $\pm R + t* \succeq 0$, see Finsler–Thorpe's Trick (Lemma 2.3). By Proposition 5.3 (ii) and (2.10), we obtain
$$\begin{aligned} 8\underline{\chi}(R) &= 6u^2 + |W_+|^2 + |W_-|^2 - 2|C|^2 \\ &\geq 6t^2 + |\vec{w}_+|^2 + |\vec{w}_-|^2 - 2\langle \vec{w}_+, \vec{w}_- \rangle \\ &= 6t^2 + \left|\vec{w}_+ - \vec{w}_-\right|^2 \geq 0. \end{aligned}$$

Moreover, $\underline{\chi}(R) = 0$ if and only if equality holds in all above inequalities. □

6. Lower bounds

In this section, we establish a (pointwise) lower bound for the quadratic form I_λ in the curvature operator of an oriented 4-manifold M that integrates to

$$\chi(M) - \tfrac{1}{\lambda}\sigma(M), \quad \lambda > 0, \tag{6.1}$$

see (6.10) for details. Given $0 < \delta \leq 1$, this lower bound gives sufficient conditions on λ for the integrand I_λ to be nonnegative on δ-pinched curvature operators, hence for (6.1) to be nonnegative if (M^4, g) is δ-pinched, see Theorem 6.2. Combined with Theorem A.1, this yields Theorem A in the Introduction.

First, we focus on the particular case $\lambda = \frac{1}{2}$, to demonstrate the optimization arguments used in the general case more concretely, and simplify the exposition for readers mainly interested in a self-contained proof of Theorem B, given below.

Theorem 6.1. *If (M^4, g) is a δ-pinched oriented 4-manifold, with finite volume and $\delta > \frac{-199+9\sqrt{545}}{71} \cong 0.156$, then $\chi(M) - 2|\sigma(M)| > 0$.*

Proof of Theorem B. Since (M^4, g) is positively δ-pinched, $\delta \geq \frac{1}{1+3\sqrt{3}}$, its intersection form is definite [DR19, Thm. 1]. Up to reversing orientation, we assume it is positive-definite, i.e., $b_-(M) = 0$, so $\chi(M) = 2 + b_+(M)$ and $\sigma(M) = b_+(M) \geq 0$. By Theorem 6.1, we have $0 < \chi(M) - 2|\sigma(M)| = 2 - b_+(M)$, so $b_+(M) = 0$ or 1. Therefore, by Theorem 2.6, we conclude M is homeomorphic to S^4 or $\mathbb{C}P^2$. □

Proof of Theorem 6.1. Given a δ-pinched oriented 4-manifold (M^4, g) with finite volume, up to reversing its orientation, we shall assume that $\sigma(M) \geq 0$. Moreover, at each point $p \in M$, its curvature operator $R_p \in \mathrm{Sym}^2_b(\wedge^2\mathbb{R}^4)$ satisfies $\pm R \in \Omega_\delta$, see (4.1). Writing R in the canonical form (2.3), we have from (2.9), (2.10), and (2.11), that

$$I_{\frac{1}{2}}(R) := \underline{\chi}(R) - 2\underline{\sigma}(R) = \tfrac{3}{4}u^2 - \tfrac{1}{24}|W_+|^2 + \tfrac{7}{24}|W_-|^2 - \tfrac{1}{4}|C|^2 \tag{6.2}$$

satisfies $\int_M I_{\frac{1}{2}}(R)\, \mathrm{vol}_\mathrm{g} = \chi(M) - 2\sigma(M)$, and $I_{\frac{1}{2}}(-R) = I_{\frac{1}{2}}(R)$. Thus, it suffices to prove that

$$\min_{R \in \Omega_\delta} I_{\frac{1}{2}}(R) > 0, \quad \text{if} \quad \delta > \tfrac{-199+9\sqrt{545}}{71}. \tag{6.3}$$

Suppose $R \in \Omega_\delta$, and let $t_1, t_2 \in \mathbb{R}$ be as in (2.6), see Lemma 2.3. From Proposition 5.3 (ii) with $k = \delta$ and $t = t_1$, we have:

$$|C|^2 \leq 3(u-\delta)^2 - 3t_1^2 + \langle \vec{w}_+, \vec{w}_- \rangle. \tag{6.4}$$

Therefore, we may bound (6.2) from below using (6.4) as follows:

$$I_{\frac{1}{2}}(R) \geq \tfrac{3}{4}u^2 - \tfrac{1}{24}|W_+|^2 + \tfrac{7}{24}|W_-|^2 - \tfrac{3}{4}(u-\delta)^2 + \tfrac{3}{4}t_1^2 - \tfrac{1}{4}\langle \vec{w}_+, \vec{w}_- \rangle.$$

Moreover, using (4.4), this lower bound can be written as the quadratic polynomial

$$\begin{aligned} Q_{\frac{1}{2}}(w_1^+, w_2^+, w_1^-, w_2^-, u, t_1) := & -\tfrac{1}{12}\big((w_1^+)^2 + (w_2^+)^2 + w_1^+w_2^+\big) \\ & + \tfrac{7}{12}\big((w_1^-)^2 + (w_2^-)^2 + w_1^-w_2^-\big) \\ & - \tfrac{1}{2}\big(w_1^+w_1^- + w_2^+w_2^-\big) - \tfrac{1}{4}\big(w_1^+w_2^- + w_2^+w_1^-\big) \\ & + \tfrac{3}{4}t_1^2 + \tfrac{3}{2}\delta u - \tfrac{3}{4}\delta^2, \end{aligned}$$

which depends solely on t_1 and the 5 variables $w_1^+, w_2^+, w_1^-, w_2^-, u$ that determine $\mathrm{pr}(R) = R(\iota_5(w_1^+, w_2^+, w_1^-, w_2^-, u)) \in \mathrm{E}_\delta$. Therefore, by Propositions 4.3 and 4.8,

$$\min_{R\in\Omega_\delta} I_{\frac{1}{2}}(R) \;\geq\; \min_{x\in\Delta^6_\delta} Q_{\frac{1}{2}}(x), \tag{6.5}$$

where $\Delta^6_\delta = \mathrm{conv}(q_1, \dots, q_7)$ is the augmented Einstein simplex in Proposition 4.8.

In order to compute the minimum value of $Q_{\frac{1}{2}}\colon \mathbb{R}^6 \to \mathbb{R}$ on Δ^6_δ, we apply the optimization method discussed in Section 3.1 to maximize $-Q_{\frac{1}{2}}$ on Δ^6_δ. The first step is to compute the collection $\mathcal{S}$ of subsets $S \subset \{q_1, \dots, q_7\}$ on whose affine hull $\mathrm{aff}(S)$ the restriction of $Q_{\frac{1}{2}}$ is positive-definite, see (3.1). Note that the restriction of $Q_{\frac{1}{2}}\colon \mathbb{R}^6 \to \mathbb{R}$ to any such affine subspace $\mathrm{aff}(S) \subset \mathbb{R}^6$ is positive-definite for *some* $0 < \delta < 1$ if and only if it is positive-definite for *all* $0 < \delta < 1$. Indeed, from Proposition 4.8, each coordinate of q_j is a scalar multiple of $(1-\delta)$, except for the u-coordinate, on which $Q_{\frac{1}{2}}$ has no degree 2 term. Therefore, the eigenvalues of $\mathrm{Hess}\big(Q_{\frac{1}{2}}|_{\mathrm{aff}(S)}\big)$ are scalar multiples of $(1-\delta)^2$.

A direct computation shows that $\mathrm{Hess}\, Q_{\frac{1}{2}}$ has eigenvalues

$$\tfrac{2}{3},\ 2,\ \tfrac{3}{2},\ -\tfrac{1}{2},\ -\tfrac{1}{6},\ 0.$$

As there are $d = 3$ positive eigenvalues, by Corollary 3.2, it suffices to consider subsets S consisting of at most 4 vertices, i.e., such that the face $\mathrm{conv}(S) \subset \Delta^6_\delta$ has dimension ≤ 3.

All 0-dimensional faces of Δ^6_δ, i.e., singletons $S = \{q_j\}$, $1 \leq j \leq 7$, trivially belong to $\mathcal{S}$. Regarding 1-dimensional faces, it is straightforward to verify that 18 of the $21 = \binom{7}{2}$ subsets $S = \{q_{j_1}, q_{j_2}\}$ of 2 vertices belong to $\mathcal{S}$; namely, all except for $\{q_1, q_2\}$, $\{q_1, q_7\}$ and $\{q_2, q_6\}$. For instance, the Hessian 1×1-matrix of the restriction of $Q_{\frac{1}{2}}$ to $\mathrm{aff}(q_1, q_2)$ is $-\frac{1}{18}(1-\delta)^2$, while to $\mathrm{aff}(q_1, q_3)$ it is $\frac{10}{3}(1-\delta)^2$. Similarly, concerning 2-dimensional faces, 18 of the $35 = \binom{7}{3}$ subsets of 3 vertices $S = \{q_{j_1}, q_{j_2}, q_{j_3}\}$ belong to $\mathcal{S}$; namely,

$$\begin{aligned}
&\{q_1,q_3,q_4\},\ \{q_1,q_3,q_5\},\ \{q_1,q_3,q_6\},\ \{q_1,q_4,q_5\},\ \{q_1,q_4,q_6\},\ \{q_2,q_3,q_4\},\\
&\{q_2,q_3,q_5\},\ \{q_2,q_3,q_7\},\ \{q_2,q_4,q_5\},\ \{q_3,q_4,q_5\},\ \{q_3,q_4,q_6\},\ \{q_3,q_4,q_7\},\\
&\{q_3,q_5,q_6\},\ \{q_3,q_5,q_7\},\ \{q_3,q_6,q_7\},\ \{q_4,q_5,q_6\},\ \{q_4,q_5,q_7\},\ \{q_4,q_6,q_7\}.
\end{aligned} \tag{6.6}$$

For example, the Hessian 2×2-matrix of the restrictions of $Q_{\frac{1}{2}}$ to $\mathrm{aff}(q_1, q_2, q_3)$ and $\mathrm{aff}(q_1, q_3, q_4)$ have eigenvalues $\frac{59\pm\sqrt{3737}}{36}(1-\delta)^2$ and $\frac{123\pm\sqrt{8473}}{36}(1-\delta)^2$, respectively. Finally, in regard to 3-dimensional faces, 6 of the $35 = \binom{7}{4}$ subsets of 4 vertices $S = \{q_{j_1}, q_{j_2}, q_{j_3}, q_{j_4}\}$ are in $\mathcal{S}$; namely

$$\begin{aligned}
&\{q_1,q_3,q_4,q_5\},\quad \{q_1,q_3,q_4,q_6\},\quad \{q_2,q_3,q_4,q_5\},\\
&\{q_3,q_4,q_5,q_6\},\quad \{q_3,q_4,q_5,q_7\},\quad \{q_3,q_4,q_6,q_7\}.
\end{aligned} \tag{6.7}$$

For instance, the Hessian 3×3-matrix of the restriction of $Q_{\frac{1}{2}}$ to $\mathrm{aff}(q_1, q_2, q_3, q_4)$ and $\mathrm{aff}(q_1, q_3, q_4, q_5)$ have eigenvalues

$$\tfrac{\alpha_1}{18}(1-\delta)^2,\ \tfrac{\alpha_2}{18}(1-\delta)^2,\ \tfrac{\alpha_3}{18}(1-\delta)^2,\quad \text{and}\quad \tfrac{\beta_1}{18}(1-\delta)^2,\ \tfrac{\beta_2}{18}(1-\delta)^2,\ \tfrac{\beta_3}{18}(1-\delta)^2,$$

respectively, where $\alpha_1 \cong 109.22$, $\alpha_2 \cong 20.12$, and $\alpha_3 \cong -7.34$ are the roots of the polynomial $x^3 - 122x^2 + 1248x + 16128$, and $\beta_1 \cong 134.65$, $\beta_2 \cong 19.27$, and $\beta_3 \cong 13.06$ are the roots of the polynomial $x^3 - 167x^2 + 4608x - 33936$.

The second step in the optimization procedure is to compute the unique critical point $x_S \in \mathrm{aff}(S)$ of $Q_{\frac{1}{2}}|_{\mathrm{aff}(S)}$ for each of the above 49 subsets $S \in \mathcal{S}$, and build the subcollection $\mathcal{S}' \subset \mathcal{S}$ consisting of the $S \in \mathcal{S}$ such that x_S is in the relative interior of the face $\mathrm{conv}(S)$, see (3.2). Differently from the above, this step *depends* on the value of $0 < \delta < 1$, and several $S \in \mathcal{S}$ only join the collection $\mathcal{S}'$ for $\delta > 0$ sufficiently small. Explicitly parametrizing each face $\mathrm{conv}(S)$ for $S \in \mathcal{S}$ with a standard simplex, and solving the corresponding inequalities in δ to determine if $x_S \in \mathrm{relint}(\mathrm{conv}(S))$, we compute the conditions for which $S \in \mathcal{S}'$ and the corresponding list of values $Q_{\frac{1}{2}}(x_S)$ where the minimum of $Q_{\frac{1}{2}}|_{\mathrm{aff}(S)}$ is achieved.

All singletons $S = \{q_j\}$ trivially belong to $\mathcal{S}'$ for all $0 < \delta < 1$, and have $x_S = q_j$. The value $Q_{\frac{1}{2}}(x_S) = Q_{\frac{1}{2}}(q_j)$ for each of these points is listed in Table 6.1.

S	$Q_{\frac{1}{2}}(x_S)$	S	$Q_{\frac{1}{2}}(x_S)$
$\{q_1\}$	$\frac{2}{9}\delta^2 + \frac{5}{9}\delta - \frac{1}{36}$	$\{q_5\}$	$\frac{3}{4}$
$\{q_2\}$	$-\frac{1}{36}\delta^2 + \frac{5}{9}\delta + \frac{2}{9}$	$\{q_6\}$	$\frac{3}{4}$
$\{q_3\}$	$\frac{10}{9}\delta^2 - \frac{11}{9}\delta + \frac{31}{36}$	$\{q_7\}$	$\frac{3\delta^2}{4}$
$\{q_4\}$	$\frac{31}{36}\delta^2 - \frac{11}{9}\delta + \frac{10}{9}$		

TABLE 6.1. Values of $Q_{\frac{1}{2}}$ on the 0-dimensional faces of Δ^6_δ.

In 12 of the 18 subsets $S \in \mathcal{S}$ with 2 vertices, the critical point $x_S \in \mathrm{aff}(S)$ of $Q_{\frac{1}{2}}|_{\mathrm{aff}(S)}$ lies in the relative interior of the 1-dimensional face $\mathrm{conv}(S)$ for some value of $0 < \delta < 1$, as listed in Table 6.2. For instance, the critical point of $Q_{\frac{1}{2}}|_{\mathrm{aff}(q_1,q_3)}$ is $x_{\{q_1,q_3\}} = \frac{23}{30}\, q_1 + \frac{7}{30}\, q_3$, which clearly lies in $\mathrm{relint}(\mathrm{conv}(q_1, q_3))$ for all $0 < \delta < 1$. Meanwhile, the critical point of $Q_{\frac{1}{2}}|_{\mathrm{aff}(q_1,q_4)}$ is $x_{\{q_1,q_4\}} = \frac{52-43\delta}{63(1-\delta)}\, q_1 + \frac{11-20\delta}{63(1-\delta)}\, q_4$, which lies in $\mathrm{relint}(\mathrm{conv}(q_1, q_4))$ if and only if $0 < \delta < \frac{11}{20}$.

S	$Q_{\frac{1}{2}}(x_S)$	δ_S	S	$Q_{\frac{1}{2}}(x_S)$	δ_S
$\{q_1, q_3\}$	$\frac{71}{540}\delta^2 + \frac{199}{270}\delta - \frac{16}{135}$	1	$\{q_3, q_4\}$	$\frac{70}{93}\delta^2 - \frac{77}{93}\delta + \frac{70}{93}$	$\frac{11}{20}$
$\{q_1, q_4\}$	$\frac{26}{567}\delta^2 + \frac{425}{567}\delta - \frac{46}{567}$	$\frac{11}{20}$	$\{q_3, q_5\}$	$\frac{21}{40}$	$\frac{11}{20}$
$\{q_1, q_5\}$	$-\frac{9}{44}\delta^2 + \frac{9}{11}\delta - \frac{3}{44}$	$\frac{4}{13}$	$\{q_3, q_6\}$	$-\frac{9}{76}\delta^2 + \frac{9}{19}\delta + \frac{21}{76}$	$\frac{20}{29}$
$\{q_2, q_3\}$	$-\frac{46}{567}\delta^2 + \frac{425}{567}\delta + \frac{26}{567}$	$\frac{43}{52}$	$\{q_4, q_5\}$	$\frac{21}{31}$	$\frac{22}{31}$
$\{q_2, q_4\}$	$-\frac{35}{108}\delta^2 + \frac{31}{27}\delta - \frac{2}{27}$	1	$\{q_4, q_6\}$	$-\frac{36}{103}\delta^2 + \frac{90}{103}\delta + \frac{21}{103}$	$\frac{58}{67}$
$\{q_2, q_5\}$	$-\frac{36}{71}\delta^2 + \frac{90}{71}\delta - \frac{3}{71}$	$\frac{26}{35}$	$\{q_5, q_6\}$	$-\frac{3}{4}\delta^2 + \frac{3}{2}\delta$	1

TABLE 6.2. Minimum of $Q_{\frac{1}{2}}|_{\mathrm{aff}(S)}$, attained at $x_S \in \mathrm{aff}(S)$, which is in the relative interior of $\mathrm{conv}(S)$ if and only if $0 < \delta < \delta_S$, for each $S \in \mathcal{S}$ such that $\mathrm{conv}(S)$ is a 1-dimensional face of Δ^6_δ. If $x_S \notin \mathrm{relint}(\mathrm{conv}(S))$ for all $0 < \delta < 1$, then the corresponding entry $S \in \mathcal{S}$ is suppressed.

Among the 18 subsets $S \in \mathcal{S}$ with 3 vertices, listed in (6.6), only 9 are such that the critical point x_S of $Q_{\frac{1}{2}}|_{\mathrm{aff}(S)}$ lies in the relative interior of the 2-dimensional face conv(S) for some value of $0<\delta<1$, as listed in Table 6.3. For example, the critical point of the restriction of $Q_{\frac{1}{2}}$ to aff(q_1,q_3,q_4) is

$$x_{\{q_1,q_3,q_4\}}=\tfrac{3(212-191\delta)}{832(1-\delta)}\,q_1+\tfrac{19\delta+188}{832(1-\delta)}\,q_3+\tfrac{4-139\delta}{416(1-\delta)}\,q_4,$$

which lies in the relative interior of conv(q_1,q_3,q_4) if and only if $0<\delta<\frac{4}{139}$.

S	$Q_{\frac{1}{2}}(x_S)$	δ_S
$\{q_1,q_3,q_4\}$	$\frac{227}{4992}\delta^2+\frac{463}{624}\delta-\frac{37}{312}$	$\frac{4}{139}$
$\{q_1,q_3,q_5\}$	$-\frac{45}{202}\delta^2+\frac{153}{202}\delta-\frac{12}{101}$	$\frac{4}{139}$
$\{q_1,q_4,q_5\}$	$\frac{1075503}{1763584}\delta^2+\frac{439605}{881792}\delta-\frac{162729}{1763584}$	$\frac{43}{439}$
$\{q_2,q_3,q_4\}$	$\frac{15719}{14884}\delta^2+\frac{1601}{3721}\delta-\frac{121}{3721}$	$\frac{4}{31}<\delta<\frac{23}{41}$
$\{q_2,q_3,q_5\}$	$-\frac{1137329}{2883204}\delta^2+\frac{6550079}{5766408}\delta-\frac{2782793}{46131264}$	$\frac{349}{844}$
$\{q_2,q_4,q_5\}$	$\frac{410727}{1201216}\delta^2+\frac{202671}{300304}\delta+\frac{10935}{300304}$	$\frac{10}{37}$
$\{q_3,q_4,q_6\}$	$\frac{44103}{153664}\delta^2+\frac{4329}{10976}\delta+\frac{423}{3136}$	$\frac{7}{13}$
$\{q_3,q_5,q_6\}$	$-\frac{2321}{7056}\delta^2+\frac{353}{504}\delta+\frac{31}{144}$	$\frac{7}{13}$
$\{q_4,q_5,q_6\}$	$\frac{2091}{12544}\delta^2+\frac{57}{224}\delta+\frac{3}{16}$	$\frac{28}{43}$

TABLE 6.3. Minimum of $Q_{\frac{1}{2}}|_{\mathrm{aff}(S)}$, attained at $x_S\in\mathrm{aff}(S)$, which is in the relative interior of conv(S) if and only if $0<\delta<\delta_S$, for each $S\in\mathcal{S}$ such that conv(S) is a 2-dimensional face of Δ^6_δ; except for $S=\{q_2,q_3,q_4\}$, for which $x_S\in\mathrm{relint}(\mathrm{conv}(S))$ if and only if $\frac{4}{31}<\delta<\frac{23}{41}$. If $x_S\notin\mathrm{relint}(\mathrm{conv}(S))$ for all $0<\delta<1$, then the corresponding entry $S\in\mathcal{S}$ is suppressed.

Lastly, of the 6 subsets $S\in\mathcal{S}$ with 4 vertices, see (6.7), only $S=\{q_2,q_3,q_4,q_5\}$ is such that the critical point x_S of $Q_{\frac{1}{2}}|_{\mathrm{aff}(S)}$ lies in the relative interior of the 3-dimensional face conv(S) for some $0<\delta<1$. Namely, we have that

$$x_{\{q_2,q_3,q_4,q_5\}}=\tfrac{63(19-20\delta)}{1846(1-\delta)}\,q_2+\tfrac{9(57\delta-8)}{923(1-\delta)}\,q_3+\tfrac{27(19-20\delta)}{1846(1-\delta)}\,q_4+\tfrac{4(35-134\delta)}{923(1-\delta)}\,q_5$$

is in the relative interior of conv(q_2,q_3,q_4,q_5) if and only if $\frac{8}{57}<\delta<\frac{35}{134}$, and

$$Q_{\frac{1}{2}}\big(x_{\{q_2,q_3,q_4,q_5\}}\big)=\tfrac{634359}{851929}\delta^2+\tfrac{372771}{851929}\delta-\tfrac{309393}{13630864}. \tag{6.8}$$

Altogether, it follows that the minimum of $Q_{\frac{1}{2}}\colon\Delta^6_\delta\to\mathbb{R}$ is equal to the smallest $Q_{\frac{1}{2}}(x_S)$ among the $S\in\mathcal{S}$ that are in the subcollection $\mathcal{S}'$ for the given value of δ, as listed in Tables 6.1 to 6.3 and (6.8). By direct inspection, setting $\delta=\frac{-199+9\sqrt{545}}{71}$, all $Q_{\frac{1}{2}}(x_S)$ for which $S\in\mathcal{S}'$ are strictly positive, except for $Q_{\frac{1}{2}}\big(x_{\{q_1,q_3\}}\big)=0$. Furthermore, subsets $S\in\mathcal{S}$ only enter or leave the subcollection $\mathcal{S}'$ at one of *finitely many* possible values of δ, so provided $\varepsilon>0$ is sufficiently small,

$$\min_{x\in\Delta^6_\delta}Q_{\frac{1}{2}}(x)=Q_{\frac{1}{2}}\big(x_{\{q_1,q_3\}}\big)=\tfrac{71}{540}\delta^2+\tfrac{199}{270}\delta-\tfrac{16}{135},\ \text{for all}\ \left|\delta-\tfrac{-199+9\sqrt{545}}{71}\right|<\varepsilon,$$

and, linearizing this quadratic polynomial at $\delta = \frac{-199+9\sqrt{545}}{71}$, one easily sees that $\min_{x\in\Delta^6_\delta} Q_{\frac{1}{2}}(x) > 0$ for $\frac{-199+9\sqrt{545}}{71} < \delta < \frac{-199+9\sqrt{545}}{71} + \varepsilon$. The above, combined with the fact that $\Delta^6_\delta \subset \Delta^6_{\delta'}$ if $\delta' < \delta$ and hence $\min_{x\in\Delta^6_\delta} Q_{\frac{1}{2}}(x)$ is a monotonically increasing function of $0 < \delta < 1$, implies that

$$\min_{x\in\Delta^6_\delta} Q_{\frac{1}{2}}(x) > 0, \qquad \text{for all} \qquad \delta > \tfrac{-199+9\sqrt{545}}{71}.$$

The above inequality and (6.5) imply (6.3), concluding the proof. □

We now proceed to the case of general $\lambda > 0$, leading to Theorem A in the Introduction. The method of proof follows the same outline of Theorem 6.1.

Theorem 6.2. *If (M^4, g) is a δ-pinched oriented 4-manifold with finite volume, then*

$$|\sigma(M)| \leq \lambda^*(\delta)\, \chi(M),$$

where $\lambda^*(\delta)$ is the continuously differentiable function given by

$$\lambda^*(\delta) = \begin{cases} \dfrac{\sqrt{\frac{24}{\delta} + 8 - 8\delta + \delta^2} + \delta - 4}{6(3-\delta)}, & \text{if } 0 < \delta < \delta_1^*, \\ \dfrac{4}{3\sqrt{15}} \dfrac{1-\delta}{\sqrt{\delta(\delta+2)}}, & \text{if } \delta_1^* \leq \delta < \delta_2^*, \\ \dfrac{8(1-\delta)^2}{24\delta^2 - 12\delta + 15}, & \text{if } \delta_2^* \leq \delta \leq 1, \end{cases} \tag{6.9}$$

and

(i) $\delta_1^* \cong 0.069$ *is the smallest real root of the polynomial* $2\delta^3 - 40\delta^2 + 89\delta - 6$*,*
(ii) $\delta_2^* = 4 - \frac{3\sqrt{6}}{2} \cong 0.326$*.*

Remark 6.3. The above semialgebraic function λ^* is C^1, but not C^2.

Proof. Just as in the proof of Theorem 6.1, up to reversing orientation, we may assume $\sigma(M) \geq 0$; and, at every point $p \in M$, we have that $\pm R \in \Omega_\delta$. Writing R in the canonical form (2.3), we have from (2.9), (2.10), and (2.11), that for all $\lambda > 0$,

$$\begin{aligned} I_\lambda(R) &:= \underline{\chi}(R) - \tfrac{1}{\lambda}\underline{\sigma}(R) \\ &= \tfrac{3}{4}u^2 + \left(\tfrac{1}{8} - \tfrac{1}{12\lambda}\right)|W_+|^2 + \left(\tfrac{1}{8} + \tfrac{1}{12\lambda}\right)|W_-|^2 - \tfrac{1}{4}|C|^2 \end{aligned} \tag{6.10}$$

satisfies $\int_M I_\lambda(R)\, \mathrm{vol_g} = \chi(M) - \frac{1}{\lambda}\sigma(M)$, and $I_\lambda(-R) = I_\lambda(R)$. Thus, it suffices to prove that for all $0 < \delta < 1$,

$$\min_{R\in\Omega_\delta} I_\lambda(R) \geq 0, \quad \text{if} \quad \lambda = \lambda^*(\delta). \tag{6.11}$$

Note that the conclusion holds in the trivial case $\delta = 1$ and $\lambda^*(1) = 0$, as $\sigma(M) = 0$ if (M^4, g) is 1-pinched, i.e., has constant curvature; so we shall assume $0 < \delta < 1$.

Given $R \in \Omega_\delta$, let $t_1, t_2 \in \mathbb{R}$ be as in (2.6), see Lemma 2.3. Using Lemma 5.2 and arguing exactly as in (6.4), we may bound (6.10) from below:

$$I_\lambda(R) \geq \tfrac{3}{4}u^2 + \left(\tfrac{1}{8} - \tfrac{1}{12\lambda}\right)|W_+|^2 + \left(\tfrac{1}{8} + \tfrac{1}{12\lambda}\right)|W_-|^2 - \tfrac{3}{4}(u-\delta)^2 + \tfrac{3}{4}t_1^2 - \tfrac{1}{4}\langle \vec{w}_+, \vec{w}_-\rangle,$$

and, using (4.4), this lower bound can be written as the quadratic polynomial

$$\begin{aligned} Q_\lambda(w_1^+, w_2^+, w_1^-, w_2^-, u, t_1) := &\left(\tfrac{1}{4} - \tfrac{1}{6\lambda}\right)\left((w_1^+)^2 + (w_2^+)^2 + w_1^+ w_2^+\right) \\ &+ \left(\tfrac{1}{4} + \tfrac{1}{6\lambda}\right)\left((w_1^-)^2 + (w_2^-)^2 + w_1^- w_2^-\right) \\ &- \tfrac{1}{2}\left(w_1^+ w_1^- + w_2^+ w_2^-\right) - \tfrac{1}{4}\left(w_1^+ w_2^- + w_2^+ w_1^-\right) \\ &+ \tfrac{3}{4} t_1^2 + \tfrac{3}{2}\delta u - \tfrac{3}{4}\delta^2 \end{aligned}$$

in t_1, and $w_1^\pm, w_2^\pm, u$, which determine $\mathrm{pr}(R) = R(\iota_5(w_1^+, w_2^+, w_1^-, w_2^-, u)) \in \mathrm{E}_\delta$. Thus, analogously to (6.5), Propositions 4.3 and 4.8 imply that for all $\lambda > 0$,

$$\min_{R \in \Omega_\delta} I_\lambda(R) \;\geq\; \min_{x \in \Delta_\delta^6} Q_\lambda(x), \tag{6.12}$$

where $\Delta_\delta^6 = \mathrm{conv}(q_1, \dots, q_7)$ is the augmented Einstein simplex in Proposition 4.8.

Once again, we apply the optimization method in Section 3.1 to maximize $-Q_\lambda$ on Δ_δ^6. The first step is to determine the collection $\mathcal{S}_\lambda$ of subsets $S \subset \{q_1, \dots, q_7\}$ on whose affine hull $\mathrm{aff}(S)$ the restriction of Q_λ is positive-definite, see (3.1). As observed in the proof of Theorem 6.1, since the coordinates of q_j are scalar multiples of $(1-\delta)$, except for the u-coordinate, on which Q_λ has no degree 2 term, the eigenvalues of $\mathrm{Hess}\big(Q_\lambda|_{\mathrm{aff}(S)}\big)$ are scalar multiples of $(1-\delta)^2$ for any subset $S \subset \{q_1, \dots, q_7\}$. Thus, the restriction of $Q_\lambda \colon \mathbb{R}^6 \to \mathbb{R}$ to $\mathrm{aff}(S) \subset \mathbb{R}^6$ is either positive-definite for every $0 < \delta < 1$, or for no $0 < \delta < 1$ at all. However, for a fixed S, the restriction $Q_\lambda|_{\mathrm{aff}(S)}$ may be positive-definite for some values of $\lambda > 0$, and indefinite or negative-definite for other values of $\lambda > 0$. Thus, even though the collection $\mathcal{S}_\lambda$ is independent of δ, it *does* depend on λ.

A simple computation shows that $\mathrm{Hess}\, Q_\lambda$ has the following eigenvalues:

$$\tfrac{3\lambda + \sqrt{9\lambda^2+4}}{12\lambda},\ \tfrac{3\lambda + \sqrt{9\lambda^2+4}}{4\lambda},\ \tfrac{3}{2},\ \tfrac{3\lambda - \sqrt{9\lambda^2+4}}{4\lambda},\ \tfrac{3\lambda - \sqrt{9\lambda^2+4}}{12\lambda},\ 0,$$

so, for all $\lambda > 0$, there are exactly $d = 3$ positive eigenvalues. Thus, by Corollary 3.2, it suffices to inspect faces of Δ_δ^6 that have dimension ≤ 3.

All singletons $S = \{q_j\}$, $1 \leq j \leq 7$, i.e., 0-dimensions faces, trivially belong to $\mathcal{S}_\lambda$, for all $\lambda > 0$. Regarding 1-dimensional faces, all of the $21 = \binom{7}{2}$ subsets of 2 vertices belong to $\mathcal{S}_\lambda$ for large enough $\lambda > 0$. For instance, the Hessian 1×1-matrix of the restriction of Q_λ to $\mathrm{aff}(q_1, q_2)$ is $\frac{15\lambda - 8}{18\lambda}(1-\delta)^2$, which is positive if and only if $\lambda > \frac{8}{15}$. In general, $S = \{q_{j_1}, q_{j_2}\} \in \mathcal{S}_\lambda$ if and only if $\lambda > \lambda_{j_1,j_2}$, where

$$\lambda_{1,2} = \lambda_{1,7} = \lambda_{2,6} = \tfrac{8}{15},\ \lambda_{1,6} = \lambda_{2,7} = \tfrac{1}{3},\ \lambda_{1,5} = \tfrac{2}{15},\ \lambda_{2,5} = \tfrac{8}{87},$$

and $\lambda_{j_1,j_2} = 0$ for all other $1 \leq j_1 < j_2 \leq 7$. Regarding 2-dimensional faces, 32 of the $35 = \binom{7}{3}$ subsets of 3 vertices belong to $\mathcal{S}_\lambda$ for large enough $\lambda > 0$. Namely, $\{q_1, q_3, q_7\}$, $\{q_2, q_4, q_7\}$, and $\{q_5, q_6, q_7\}$ do not belong to $\mathcal{S}_\lambda$ for any $\lambda > 0$, and the remaining $S = \{q_{j_1}, q_{j_2}, q_{j_3}\}$ belong to $\mathcal{S}_\lambda$ if and only if $\lambda > \lambda_{j_1,j_2,j_3}$, where

$$\begin{aligned} &\lambda_{1,2,3} = \tfrac{9+\sqrt{105}}{36}, &&\lambda_{1,3,4} = \tfrac{-9+\sqrt{105}}{36}, &&\lambda_{1,2,4} = \tfrac{6+\sqrt{42}}{18}, &&\lambda_{2,3,4} = \tfrac{-6+\sqrt{42}}{18}, \\ &\lambda_{1,3,5} = \tfrac{-1+\sqrt{5}}{9}, &&\lambda_{1,3,6} = \tfrac{1+\sqrt{5}}{9}, &&\lambda_{1,4,5} = \tfrac{-15+\sqrt{609}}{72}, &&\lambda_{2,3,6} = \tfrac{15+\sqrt{609}}{72}, \\ &\lambda_{1,4,6} = \tfrac{21+\sqrt{1113}}{126}, &&\lambda_{2,3,5} = \tfrac{-21+\sqrt{1113}}{126}, &&\lambda_{1,4,7} = \tfrac{3+\sqrt{105}}{18}, &&\lambda_{2,3,7} = \tfrac{-3+\sqrt{105}}{18}, \\ &\lambda_{2,4,5} = \tfrac{-2+2\sqrt{2}}{9}, &&\lambda_{2,4,6} = \tfrac{2+2\sqrt{2}}{9}, \end{aligned}$$

$$\lambda_{1,2,5} = \lambda_{1,2,6} = \lambda_{1,2,7} = \lambda_{1,5,6} = \lambda_{1,5,7} = \lambda_{1,6,7} = \lambda_{2,5,6} = \lambda_{2,5,7} = \lambda_{2,6,7} = \tfrac{2}{3},$$

and $\lambda_{j_1,j_2,j_3} = 0$ for all other $1 \leq j_1 < j_2 < j_3 \leq 7$. For instance, the eigenvalues of the Hessian 2×2-matrix of $Q_\lambda|_{\mathrm{aff}(q_1,q_2,q_3)}$ are $\frac{75\lambda-8\pm\sqrt{2169\lambda^2+528\lambda+128}}{36\lambda}(1-\delta)^2$, which are positive if and only if $\lambda > \lambda_{1,2,3} = \frac{9+\sqrt{105}}{36}$. Finally, regarding 3-dimensional faces, 20 of the $35 = \binom{7}{4}$ subsets of 4 vertices belong to $\mathcal{S}_\lambda$ for large enough $\lambda > 0$. Namely, $S = \{q_{j_1}, q_{j_2}, q_{j_3}, q_{j_4}\}$ belongs to $\mathcal{S}_\lambda$ if and only if $\lambda > \lambda_{j_1,j_2,j_3,j_4}$, where

$$\begin{aligned}
&\lambda_{1,2,3,6} = \lambda_{1,2,5,6} = \lambda_{1,2,5,7} = \lambda_{1,2,6,7} = \tfrac{2}{3},\ \lambda_{1,3,4,5} = \tfrac{-1+\sqrt{5}}{9} \\
&\lambda_{1,4,5,6} = \lambda_{1,4,5,7} = \lambda_{1,4,6,7} = \lambda_{2,3,5,6} = \lambda_{2,3,5,7} = \lambda_{2,3,6,7} = \tfrac{4}{3\sqrt{3}}, \\
&\lambda_{1,2,3,5} \cong 0.818 \text{ is the largest real root of } 243\lambda^3 - 108\lambda^2 - 84\lambda + 8, \\
&\lambda_{1,2,3,7} \cong 0.701 \text{ is the largest real root of } 243\lambda^3 - 54\lambda^2 - 93\lambda + 8, \\
&\lambda_{1,2,4,6} \cong 0.795 \text{ is the largest real root of } 243\lambda^3 - 270\lambda^2 + 51\lambda + 8, \\
&\lambda_{1,2,4,6} \cong 0.461 \text{ is the largest real root of } 243\lambda^3 + 108\lambda^2 - 84\lambda - 8, \\
&\lambda_{2,3,4,5} \cong 0.0996 \text{ is the largest real root of } 243\lambda^3 + 270\lambda^2 + 51\lambda - 8, \\
&\lambda_{2,3,4,6} \cong 0.562 \text{ is the largest real root of } 243\lambda^3 + 54\lambda^2 - 93\lambda - 8, \\
&\lambda_{3,4,5,6} = \lambda_{3,4,5,7} = \lambda_{3,4,6,7} = 0,
\end{aligned}$$

and the remaining 15 subsets do not belong to $\mathcal{S}_\lambda$ for any $\lambda > 0$.

The second step is to compute the critical point $x_S \in \mathrm{aff}(S)$ of $Q_\lambda|_{\mathrm{aff}(S)}$ for each of the above subsets $S \in \mathcal{S}_\lambda$, and determine the values of $0 < \delta < 1$ and $\lambda > 0$ such that x_S is in the relative interior of the face $\mathrm{conv}(S)$. These subsets S define a subcollection $\mathcal{S}'_\lambda$ of $\mathcal{S}_\lambda$, which depend on both δ and λ, such that (cf. (3.3))

$$\min_{\Delta^6_\delta} Q_\lambda = \min_{S\in\mathcal{S}'_\lambda} Q_\lambda(x_S). \tag{6.13}$$

All singletons $S = \{q_j\}$ belong to $\mathcal{S}'_\lambda$ for all $0 < \delta < 1$ and $\lambda > 0$, and $x_S = q_j$. The value $Q_\lambda(x_S) = Q_\lambda(q_j)$ for each of these points is listed below in Table 6.4.

S	$Q_\lambda(x_S)$	S	$Q_\lambda(x_S)$
$\{q_1\}$	$\frac{2(3\lambda-1)}{9\lambda}\delta^2 - \frac{3\lambda-4}{9\lambda}\delta + \frac{15\lambda-8}{36\lambda}$	$\{q_5\}$	$\frac{3}{4}$
$\{q_2\}$	$\frac{15\lambda-8}{36\lambda}\delta^2 - \frac{3\lambda-4}{9\lambda}\delta + \frac{24\lambda-8}{36\lambda}$	$\{q_6\}$	$\frac{3}{4}$
$\{q_3\}$	$\frac{2(3\lambda+1)}{9\lambda}\delta^2 - \frac{3\lambda+4}{9\lambda}\delta + \frac{15\lambda+8}{36\lambda}$	$\{q_7\}$	$\frac{3\delta^2}{4}$
$\{q_4\}$	$\frac{15\lambda+8}{36\lambda}\delta^2 - \frac{3\lambda+4}{9\lambda}\delta + \frac{24\lambda+8}{36\lambda}$		

TABLE 6.4. Values of Q_λ on the 0-dimensional faces of Δ^6_δ.

There are 15 of the 21 subsets $S \in \mathcal{S}_\lambda$ with 2 vertices for which the critical point $x_S \in \mathrm{aff}(S)$ of $Q_\lambda|_{\mathrm{aff}(S)}$ lies in the relative interior of $\mathrm{conv}(S)$ for some value of $0 < \delta < 1$ and $\lambda > 0$, as listed in Table 6.5. Similarly, there are 15 of the 32 subsets $S \in \mathcal{S}_\lambda$ with 3 vertices, and 4 of the 20 subsets $S \in \mathcal{S}_\lambda$ with 4 vertices for which that happens. For instance, $S = \{q_1, q_2, q_5\} \in \mathcal{S}'_\lambda$ if and only if $\delta < \frac{1}{3}$ and $\lambda > \frac{2(1-\delta)}{3(1-3\delta)}$, in which case $Q_\lambda|_{\mathrm{aff}(q_1,q_2,q_5)}$ has minimum

$$Q_\lambda\big(x_{\{q_1,q_2,q_5\}}\big) = \tfrac{9\lambda(8-21\lambda)}{8(36\lambda^2-27\lambda+2)}\delta^2 + \tfrac{45\lambda}{48\lambda-4}\delta + \tfrac{6-9\lambda}{8-96\lambda},$$

S	$Q_\lambda(x_S)$	Conditions for $S \in \mathcal{S}'_\lambda$
$\{q_1,q_2\}$	$\frac{18(\lambda-1)\lambda+4}{3\lambda(15\lambda-8)}\delta^2+\frac{(9(\lambda-2)\lambda+8)}{3(8-15\lambda)\lambda}\delta+\frac{18(\lambda-1)\lambda+4}{3\lambda(15\lambda-8)}$	$\delta<\frac{1}{4},\quad \lambda>\frac{4(1-\delta)}{3-12\delta}$
$\{q_1,q_3\}$	$\left(\frac{1}{4}-\frac{4}{135\lambda^2}\right)\delta^2+\left(\frac{8}{135\lambda^2}+\frac{1}{2}\right)\delta-\frac{4}{135\lambda^2}$	$\lambda>\frac{4}{15}$
$\{q_1,q_4\}$	$\frac{18\lambda(3\lambda+1)-16}{567\lambda^2}\delta^2+\left(\frac{32}{567\lambda^2}+\frac{11}{21}\right)\delta+\frac{18\lambda(3\lambda-1)-16}{567\lambda^2}$	$\delta<\frac{3}{4},\quad \lambda>\frac{8(1-\delta)}{27-36\delta}$
$\{q_1,q_5\}$	$\frac{9\lambda}{8-60\lambda}\delta^2+\frac{9\lambda}{15\lambda-2}\delta+\frac{6-9\lambda}{8-60\lambda}$	$\delta<\frac{4}{7},\quad \lambda>\frac{4(1-\delta)}{12-21\delta}$
$\{q_1,q_6\}$	$\frac{6-9\lambda}{8-24\lambda}$	$\delta<\frac{1}{4},\quad \lambda>\frac{4(1-\delta)}{3-12\delta}$
$\{q_2,q_3\}$	$\frac{18\lambda(3\lambda-1)-16}{567\lambda^2}\delta^2+\left(\frac{32}{567\lambda^2}+\frac{11}{21}\right)\delta+\frac{18\lambda(3\lambda+1)-16}{567\lambda^2}$	$\delta\le\frac{3}{4},\quad \lambda>\frac{8(1-\delta)}{36-27\delta}$ $\delta>\frac{3}{4},\quad \frac{8(1-\delta)}{36-27\delta}<\lambda<\frac{8(1-\delta)}{36\delta-27}$
$\{q_2,q_4\}$	$-\left(\frac{1}{54\lambda^2}+\frac{1}{4}\right)\delta^2+\left(\frac{1}{27\lambda^2}+1\right)\delta-\frac{1}{54\lambda^2}$	$\lambda>\frac{1}{6}$
$\{q_2,q_5\}$	$\frac{36\lambda}{8-87\lambda}\delta^2+\frac{90\lambda}{87\lambda-8}\delta+\frac{6-9\lambda}{8-87\lambda}$	$\delta<\frac{14}{17},\ \lambda>\frac{8(1-\delta)}{42-51\delta}$
$\{q_2,q_6\}$	$\frac{6-9\lambda}{8-15\lambda}$	$\delta<\frac{2}{5},\quad \lambda>\frac{8(1-\delta)}{6-15\delta}$
$\{q_3,q_4\}$	$\frac{18\lambda(\lambda+1)+4}{3\lambda(15\lambda+8)}\delta^2+\frac{1}{15}\left(\frac{9}{15\lambda+8}-\frac{5}{\lambda}-3\right)\delta+\frac{18\lambda(\lambda+1)+4}{3\lambda(15\lambda+8)}$	$\delta\le\frac{1}{4}$ $\delta>\frac{1}{4},\ \lambda<\frac{4(1-\delta)}{12\delta-3}$
$\{q_3,q_5\}$	$\frac{6+9\lambda}{8+24\lambda}$	$\delta\le\frac{1}{4}$ $\delta>\frac{1}{4},\ \lambda<\frac{4(1-\delta)}{12\delta-3}$
$\{q_3,q_6\}$	$-\frac{9\lambda}{60\lambda+8}\delta^2+\frac{9\lambda}{15\lambda+2}\delta+\frac{9\lambda+6}{60\lambda+8}$	$\delta\le\frac{4}{7}$ $\delta>\frac{4}{7},\ \lambda<\frac{4(1-\delta)}{21\delta-12}$
$\{q_4,q_5\}$	$\frac{6+9\lambda}{8+15\lambda}$	$\delta\le\frac{2}{5}$ $\delta>\frac{2}{5},\ \lambda<\frac{8(1-\delta)}{15\delta-6}$
$\{q_4,q_6\}$	$-\frac{36\lambda}{87\lambda+8}\delta^2+\frac{90\lambda}{87\lambda+8}\delta+\frac{9\lambda+6}{87\lambda+8}$	$\delta\le\frac{14}{17}$ $\delta>\frac{14}{17},\ \lambda<\frac{8(1-\delta)}{51\delta-42}$
$\{q_5,q_6\}$	$-\frac{3}{4}\delta^2+\frac{3}{2}\delta$	$0<\delta<1,\ \lambda>0$

TABLE 6.5. Minimum of $Q_\lambda|_{\mathrm{aff}(S)}$, attained at $x_S\in\mathrm{aff}(S)$, with necessary and sufficient conditions on δ and λ for $Q_\lambda|_{\mathrm{aff}(S)}$ to be positive-definite and $x_S\in\mathrm{relint}(\mathrm{conv}(S))$, i.e., for $S\in\mathcal{S}'_\lambda$.

while $S=\{q_2,q_3,q_5,q_6\}$ belongs to $\mathcal{S}'_\lambda$ if and only if $\delta<\frac{1}{6}$ and $\lambda>\frac{\delta+\sqrt{289\delta^2-336\delta+48}}{9(1-6\delta)}$, in which case $Q_\lambda|_{\mathrm{aff}(q_2,q_3,q_5,q_6)}$ has minimum

$$Q_\lambda\big(x_{\{q_2,q_3,q_5,q_6\}}\big)=\frac{3\big(3321\lambda^4+270\lambda^3-1287\lambda^2-144\lambda+64\big)}{4(16-27\lambda^2)^2}\delta^2+\frac{3\big(36\lambda^2-3\lambda+8\big)}{64-108\lambda^2}\delta+\frac{3}{16}.$$

The remaining values $Q_\lambda(x_S)$ are omitted to simplify the exposition, but the reader may find them in the particular case $\lambda=\frac{1}{2}$ in Table 6.3 and (6.8).

Altogether, there are 41 subsets S that belong to the collection $\mathcal{S}'_\lambda$ for some $(\delta,\lambda)\in H$, where $H=(0,1)\times(0,+\infty)$ is a vertical strip in $\mathbb{R}^2$. The corresponding sentences "if $S\in\mathcal{S}'_\lambda$, then $Q_\lambda(x_S)\geq 0$" give a description of the semialgebraic set

$$\mathfrak{X}:=\left\{(\delta,\lambda)\in H:\min_{S\in\mathcal{S}'_\lambda}Q_\lambda(x_S)\geq 0\right\}$$

involving (finitely many) polynomial inequalities in (δ,λ), connected by "and" and "or". Using Cylindrical Algebraic Decomposition, see e.g. [BPR06, Sec 5.1], any semialgebraic set in $\mathbb{R}^2$ can be written as a finite disjoint union of 2-*cells*, i.e., points, vertical open intervals, graphs of the form $\big\{(\delta,\lambda)\in\mathbb{R}^2 : a<\delta<b,\ \lambda=\varphi(\delta)\big\}$, and bands of the form $\big\{(\delta,\lambda)\in\mathbb{R}^2 : a<\delta<b,\ \varphi(\delta)<\lambda<\psi(\delta)\big\}$, where $a,b\in\mathbb{R}$, and $\varphi,\psi\colon (a,b)\to[-\infty,+\infty]$ are continuous semialgebraic functions. (The latter are similar to what Calculus textbooks often call *regions of type I* in integration of functions of two variables.) Applied to the semialgebraic set $\mathfrak{X}$ in the (δ,λ)-plane, cylindrical algebraic decomposition yields:

$$\mathfrak{X}=\left\{\delta\in(0,\delta_1^*],\ \lambda\geq\frac{\sqrt{\frac{24}{\delta}+8-8\delta+\delta^2}+\delta-4}{6(3-\delta)}\right\}\cup\left\{\delta\in[\delta_1^*,\delta_2^*],\ \lambda\geq\frac{4}{3\sqrt{15}}\frac{1-\delta}{\sqrt{\delta(\delta+2)}}\right\}$$
$$\cup\left\{\delta\in[\delta_2^*,1),\ \lambda\geq\frac{8(1-\delta)^2}{24\delta^2-12\delta+15}\right\},$$

i.e., $\mathfrak{X}=\big\{\delta\in(0,1),\ \lambda\geq\lambda^*(\delta)\big\}$, where $\lambda^*\colon(0,1)\to\mathbb{R}$ is the piecewise continuous function defined in (6.9). From (6.12) and (6.13), we have that $(\delta,\lambda)\in\mathfrak{X}$ implies $\min\limits_{R\in\Omega_\delta} I_\lambda(R)\geq 0$, so (6.11) holds, concluding the proof. □

Proof of Theorem A. Consider the functions $\lambda^*\colon(0,1]\to\mathbb{R}$ and $\lambda^{\mathrm{V}}\colon\big[\delta_0^{\mathrm{V}},1\big]\to\mathbb{R}$, defined in (6.9) and Theorem A.1, respectively. Define $\lambda\colon(0,1]\to\mathbb{R}$ as follows

$$\lambda(\delta)=\begin{cases}\lambda^*(\delta), & \text{if } \delta\in\big(0,\delta_0^{\mathrm{V}}\big],\\ \min\big\{\lambda^*(\delta),\lambda^{\mathrm{V}}(\delta)\big\}, & \text{if } \delta\in\big[\delta_0^{\mathrm{V}},1\big].\end{cases}$$

Routine computations show that λ agrees with (1.5), and satisfies $\lim\limits_{\delta\searrow 0}\lambda(\delta)=+\infty$, $\lambda\Big(\frac{1}{1+3\sqrt{3}}\Big)<\frac12$, $\lambda\big(\frac14\big)=\frac13$, and $\lambda(1)=0$. From Theorems 6.2 and A.1, a δ-pinched oriented 4-manifold (M^4,g) with finite volume satisfies $|\sigma(M)|\leq\lambda(\delta)\,\chi(M)$. □

Remark 6.4. Polombo [Pol78, Thm II.13] proved a similar *explicit* inequality for δ-pinched 4-manifolds, with $0<\delta\leq\frac14$; namely $|\sigma(M)|\leq\frac{2}{27}\left(\frac{2}{\delta^2}-\frac{7}{\delta}+5\right)\chi(M)$. It is straightforward to check that $\lambda(\delta)<\frac{2}{27}\left(\frac{2}{\delta^2}-\frac{7}{\delta}+5\right)$ for all $0<\delta\leq\frac14$.

7. Upper bounds

In this section, we discuss further applications of the optimization methods from Section 3, proving *upper bounds* for $\chi(M)$ and $\sigma(M)$ if M is a δ-pinched oriented 4-manifold with finite volume.

7.1. **Weyl tensor.** A key step towards the above goal is to establish an upper bound on $|W_\pm|^2$ for a δ-pinched curvature operator R. By Proposition 4.3, no generality is lost if we assume that R is Einstein. This pointwise problem has received great attention in the literature, see e.g. [Yan00, Lemma 4.1], [GL99, Lemma 1], and [CT18, Lemma 3.1]. The following result provides a useful *sharp* upper bound for any linear combinations of $|W_\pm|^2$ when R is a δ-pinched curvature operator.

Proposition 7.1. *For all $-1\leq\eta\leq1$ and $0<\delta\leq1$, if R is δ-pinched, then*

$$|W_+|^2+\eta\,|W_-|^2\leq\tfrac83(1-\delta)^2. \tag{7.1}$$

For $\eta\neq1$, equality in (7.1) holds if and only if $\mathrm{pr}(\pm R)=t\,\iota_5(p_1)+(1-t)\,\iota_5(p_2)$, $t\in[0,1]$, and, for $\eta=1$, if and only if $\mathrm{pr}(\pm R)$ is in the convex hull of $\iota_5(p_j)$, $1\leq j\leq4$, where $\pm R\in\Omega_\delta$, using the notation in (4.1), (4.2), and Proposition 4.7.

Proof. For all $-1 \leq \eta \leq 1$, given R in the canonical form (2.3), by (4.4),

$$|W_+|^2 + \eta\, |W_-|^2 = Q_\eta(w_1^+, w_2^+, w_1^-, w_2^-, u),$$

where $Q_\eta \colon \mathbb{R}^5 \to \mathbb{R}$ is the quadratic polynomial

$$\begin{aligned} Q_\eta(w_1^+, w_2^+, w_1^-, w_2^-, u) := 2\big((w_1^+)^2 + (w_2^+)^2 + w_1^+ w_2^+\big) \\ + 2\eta\big((w_1^-)^2 + (w_2^-)^2 + w_1^- w_2^-\big). \end{aligned}$$

Given $R \in \Omega_\delta$, let $\mathrm{pr}(R) = R(\vec{w}_+, \vec{w}_-, u) \in \mathrm{E}_\delta$, see Lemma 4.2, (4.2), and (4.3). By Proposition 4.7, we have that $(w_1^+, w_2^+, w_1^-, w_2^-, u) \in \Delta^5_\delta = \mathrm{conv}(p_1, \dots, p_6)$. By a straightforward computation, the eigenvalues of $\operatorname{Hess} Q_\eta$ are

$$1,\ 3,\ \eta,\ 3\eta,\ 0,$$

so the number of negative eigenvalues is either 0 or 2, according to whether $\eta \geq 0$ or $\eta < 0$, respectively. By Corollary 3.2, this means $\max_{x\in\Delta^5_\delta} Q_\eta(x)$ is achieved at some vertex p_j if $\eta \geq 0$, while we must inspect faces of dimension ≤ 2, i.e., convex combinations of up to 3 vertices p_j's, if $\eta < 0$. The values assumed by Q_η at p_j are:

$$\begin{aligned} Q_\eta(p_1) = Q_\eta(p_2) &= \tfrac{8}{3}(1-\delta)^2, \\ Q_\eta(p_3) = Q_\eta(p_4) &= \tfrac{8\eta}{3}(1-\delta)^2, \\ Q_\eta(p_5) = Q_\eta(p_6) &= 0, \end{aligned} \tag{7.2}$$

so the inequality $Q_\eta(x) \leq \frac{8}{3}(1-\delta)^2$ clearly holds for all $x \in \Delta^5_\delta$ and $0 \leq \eta \leq 1$, with equality achieved if and only if $x \in \mathrm{conv}(p_j)$, where $1 \leq j \leq 2$ if $0 \leq \eta < 1$, and $1 \leq j \leq 4$ if $\eta = 1$. Thus, let us now assume $-1 \leq \eta < 0$, and follow the optimization procedure in Section 3.1.

The first step is to determine the collection $\mathcal{S}$ of subsets $S \subset \{p_1, \dots, p_6\}$ such that the restriction of Q_η to the affine hull $\mathrm{aff}(S)$ is negative-definite, see (3.1). All singletons $S = \{p_j\}$, $1 \leq j \leq 6$ trivially belong to $\mathcal{S}$. Regarding 1-dimensional faces, we find that $Q_\eta|_{\mathrm{aff}(S)}$ is negative-definite if and only if $S = \{p_{j_1}, p_{j_2}\}$ is one of the following:

$$\{p_3,p_4\},\ \{p_3,p_5\},\ \{p_3,p_6\},\ \{p_4,p_5\},\ \{p_4,p_6\}. \tag{7.3}$$

In all cases, the single entry of $\operatorname{Hess}\big(Q_\eta|_{\mathrm{aff}(S)}\big)$ is $\frac{8\eta}{3}(1-\delta)^2$. Regarding 2-dimensional faces, $Q_\eta|_{\mathrm{aff}(S)}$ is negative-definite if and only if $S = \{p_3, p_4, p_5\}$ or $S = \{p_3, p_4, p_6\}$. In both cases, its eigenvalues of $\operatorname{Hess}\big(Q_\eta|_{\mathrm{aff}(S)}\big)$ are $4\eta(1-\delta)^2$ and $\frac{4\eta}{3}(1-\delta)^2$.

The second step is to extract the subcollection $\mathcal{S}' \subset \mathcal{S}$ such that the unique critical point $x_S \in \mathrm{aff}(S)$ of $Q_\eta|_{\mathrm{aff}(S)}$ is in the relative interior of the face $\mathrm{conv}(S)$, see (3.2). Every singleton $S = \{p_j\}$ is trivially in $\mathcal{S}'$ and has $x_S = p_j$; recall that the values of $Q_\eta(p_j)$ are given in (7.2). Among the subsets (7.3), only $\{p_3, p_4\} \in \mathcal{S}'$, since $x_{\{p_3,p_4\}} = \frac{1}{2}p_3 + \frac{1}{2}p_4 \in \mathrm{relint}(\mathrm{conv}(p_3, p_4))$, and Q_η assumes the value

$$Q_\eta\big(x_{\{p_3,p_4\}}\big) = 2\eta(1-\delta)^2. \tag{7.4}$$

In all other $S = \{p_{j_1}, p_{j_2}\}$, $j_1 < j_2$, one has $x_{\{p_{j_1},p_{j_2}\}} = p_{j_2} \notin \mathrm{relint}(\mathrm{conv}(p_{j_1}, p_{j_2}))$. Similarly, neither $\{p_3, p_4, p_5\}$ nor $\{p_3, p_4, p_6\}$ belong to $\mathcal{S}'$, since a direct computation shows that $x_{\{p_3,p_4,p_5\}} = p_5$ and $x_{\{p_3,p_4,p_6\}} = p_6$. Thus, by Corollary 3.2, see also (3.3), $\max_{x\in\Delta^5_\delta} Q_\eta(x) = Q_\eta(p_1) = Q_\eta(p_2) = \frac{8}{3}(1-\delta)^2$ is the largest value among (7.2) and (7.4). This concludes the proof of (7.1) and its equality case. □

Remark 7.2. Equality in (7.1) is achieved by δ-pinched curvature operators whose projection onto the set of Einstein δ-pinched curvature operators can be written as certain linear combinations of R_{S^4}, $R_{\mathbb{C}P^2}$, and $R_{\mathbb{C}H^2}$, see Remark 4.9.

Remark 7.3. Proposition 7.1 is reminiscent of a bound obtained by Yang [Yan00, Lemma 4.1(a)] for Einstein 4-manifolds. Namely, replacing $\delta \leq \sec \leq 1$ with $\delta \leq \sec \leq \Delta$, inequality (7.1) with $\eta = 1$ yields $|W|^2 \leq \frac{8}{3}(\Delta-\delta)^2$. Furthermore, if $\mathrm{Ric}_R = \mathrm{g}$, then $\Delta \leq 1-2\delta$, hence $|W|^2 \leq \frac{8}{3}(1-3\delta)^2$, cf. [Yan00, Equation (4.6)].

7.2. **Euler characteristic and signature.** We now use the bounds in Proposition 7.1 and Comparison Geometry to prove Theorems C and E, and Corollary D.

Proof of Theorem C. Suppose (M^4,g) is positively δ-pinched, so that $R \in \Omega_\delta$ at all points, and not diffeomorphic to S^4. We may then apply Lemmas 2.9 and 2.10, and Proposition 7.1 with $\eta = 1$, obtaining:

$$\chi(M) \leq \frac{1}{4\pi^2}\int_M |W_+|^2 + |W_-|^2 \ \mathrm{vol_g} \leq \frac{1}{4\pi^2}\,\frac{8}{3}(1-\delta)^2\,\frac{4\pi^2}{3\delta^2} = \frac{8}{9}\left(\frac{1}{\delta}-1\right)^2.$$

Up to reversing orientation, we assume without loss of generality that $\sigma(M) \geq 0$. Using (2.11) instead of Lemma 2.9, and Proposition 7.1 with $\eta = -1$, we have:

$$\sigma(M) = \frac{1}{12\pi^2}\int_M |W_+|^2 - |W_-|^2 \ \mathrm{vol_g} \leq \frac{1}{12\pi^2}\,\frac{8}{3}(1-\delta)^2\,\frac{4\pi^2}{3\delta^2} = \frac{8}{27}\left(\frac{1}{\delta}-1\right)^2. \quad \square$$

Remark 7.4. Given the generality afforded by $-1 \leq \eta \leq 1$ in Proposition 7.1, it is tempting to re-examine the proof of Theorem C with upper bounds of the form

$$\begin{aligned} a\,\chi(M) + b\,|\sigma(M)| &\leq \frac{1}{4\pi^2}\left(a+\frac{b}{3}\right)\int_M |W_+|^2 + \eta\,|W_-|^2 \ \mathrm{vol_g} \\ &\leq \frac{8}{9}\left(a+\frac{b}{3}\right)\left(\frac{1}{\delta}-1\right)^2, \end{aligned} \tag{7.5}$$

where $a, b \geq 0$ are not both zero, and $\eta = \frac{3a-b}{3a+b}$. However, all such bounds (7.5) are directly implied by the extreme cases $(a,b) = (1,0)$ and $(a,b) = (0,1)$, which form the statement of Theorem C. Indeed, the intersection of all affine half-spaces (7.5) in the $(|\sigma|,\chi)$-plane is precisely the rectangle $\chi \leq \frac{8}{9}\left(\frac{1}{\delta}-1\right)^2$ and $|\sigma| \leq \frac{8}{27}\left(\frac{1}{\delta}-1\right)^2$.

Remark 7.5. By a celebrated result of Gromov [Gro81], closed n-manifolds with $\sec \geq 0$ have bounded total Betti number $\sum_k b_k(M) \leq C(n)$. Thus, if (M^4,g) is a closed oriented 4-manifold with $\sec > 0$, then $\chi(M) \leq C(4)$, as $b_0(M) = b_4(M) = 1$ and $b_1(M) = b_3(M) = 0$, hence $\chi(M) = 2 + b_2(M) = \sum_k b_k(M)$. In particular, this also gives an upper bound $|\sigma(M)| \leq \chi(M) - 2 \leq C(4) - 2$ by (2.8).

Although Gromov conjectured that $C(n) = 2^n$, which would be sharp since the torus T^n has $\sum_k b_k(T^n) = 2^n$, the best known estimates for $C(n)$ grow exponentially in n^3, see Abresch [Abr87]. Using [Abr87, p. 477], we have that the Poincaré polynomial $P_t(M) = 1 + b_2(M)t^2 + t^4$ of (M^4,g) satisfies $P_{t(4)^{-1}}(M) \leq e^{466}$, where $t(4) = 5^{16}8^4e^{8/15}$. Thus, $b_2(M) \leq t(4)^2(e^{466}-1) + t(4)^{-2} \lesssim 2.731 \times 10^{232}$, so also

$$\chi(M) \lesssim 2.731 \times 10^{232}. \tag{7.6}$$

Therefore, the upper bound $\chi(M) \leq \frac{8}{9}\left(\frac{1}{\delta}-1\right)^2$ in Theorem C is smaller than (7.6) only if $\delta \gtrsim 5.705 \times 10^{-117}$. Nevertheless, it is *hundreds* of orders of magnitude smaller than (7.6) for larger δ; e.g., it gives $\chi(M) \leq 10^2$ if $\delta \gtrsim 0.086$.

Proof of Corollary D. Given $\delta > 0$, combining Theorem C and Theorem 2.6, it follows that an *orientable* positively δ-pinched 4-manifold (M^4, g) is homeomorphic to

(i) $\#^r \mathbb{C}P^2 \#^s \overline{\mathbb{C}P^2}$, $r+s+2 \leq \frac{8}{9}(\frac{1}{\delta}-1)^2$, $|r-s| \leq \frac{8}{27}(\frac{1}{\delta}-1)^2$, if M is non-spin;
(ii) $\#^r(S^2 \times S^2)$, $2r+2 \leq \frac{8}{9}(\frac{1}{\delta}-1)^2$ if M is spin.

Instead, a *non-orientable* positively δ-pinched 4-manifold (M^4, g) has $\pi_1(M) \cong \mathbb{Z}_2$, by Synge's Theorem. Applying Theorem C to its double-cover $(\widetilde{M}, \widetilde{\mathrm{g}})$, endowed with the pullback metric, we have $\chi(M) = \frac{1}{2}\chi(\widetilde{M}) \leq \frac{4}{9}(\frac{1}{\delta}-1)^2$. According to [HKT94, Thm. 1], for each given value of $\chi(M)$, the homeomorphism type of such M is completely determined by topological invariants that can only take finitely many different values. Moreover, an explicit list of closed non-orientable 4-manifolds with $\pi_1(M) \cong \mathbb{Z}_2$ realizing these homeomorphism types is given in [HKT94, Thm. 3]. □

Proof of Theorem E. The statement about $\sigma(M)$ follows exactly as in the proof of Theorem C. Without loss of generality, assume $\sigma(M) \geq 0$. Since $-R \in \Omega_\delta$ at all points of (M^4, g) and $\underline{\sigma}(-R) = \underline{\sigma}(R)$, from (2.11) and Proposition 7.1 with $\eta = -1$,

$$\sigma(M) = \frac{1}{12\pi^2}\int_M |W_+|^2 - |W_-|^2 \ \mathrm{vol_g} \leq \frac{2}{9\pi^2}(1-\delta)^2 \,\mathrm{Vol}(M, \mathrm{g}).$$

Regarding $\chi(M)$, due to the absence of a negatively pinched counterpart to Lemma 2.9, we use the optimization methods of Section 3.1 directly on (2.10). In order to have a quadratic form defined on a simplex, given R in the canonical form (2.3), we discard the nonpositive term $-\frac{1}{4}|C|^2$ in (2.10), and consider the quantity $\frac{3}{4}u^2 + \frac{1}{8}|W_+|^2 + \frac{1}{8}|W_-|^2 = Q(w_1^+, w_2^+, w_1^-, w_2^-, u)$, where $Q\colon \mathbb{R}^5 \to \mathbb{R}$ is given by

$$\begin{aligned} Q(w_1^+, w_2^+, w_1^-, w_2^-, u) := \tfrac{1}{4}\big((w_1^+)^2 + (w_2^+)^2 + w_1^+ w_2^+\big) \\ + \tfrac{1}{4}\big((w_1^-)^2 + (w_2^-)^2 + w_1^- w_2^-\big) + \tfrac{3}{4}u^2. \end{aligned}$$

For $R \in \Omega_\delta$, let $\mathrm{pr}(R) = R(\vec{w}_+, \vec{w}_-, u) \in \mathrm{E}_\delta$, see Lemma 4.2 and (4.3). By Proposition 4.7, we have that $(w_1^+, w_2^+, w_1^-, w_2^-, u) \in \Delta^5_\delta = \mathrm{conv}(p_1, \dots, p_6)$. Since Q is evidently positive-definite, Corollary 3.2 implies that its maximum is achieved at a vertex p_j, and is hence the largest among the following values:

$$\begin{aligned} &Q(p_1) = Q(p_3) = \tfrac{2}{3}\delta^2 - \tfrac{1}{3}\delta + \tfrac{5}{12}, \qquad && Q(p_2) = Q(p_4) = \tfrac{5}{12}\delta^2 - \tfrac{1}{3}\delta + \tfrac{2}{3}, \\ &Q(p_5) = \tfrac{3}{4}, && Q(p_6) = \tfrac{3}{4}\delta^2. \end{aligned} \tag{7.7}$$

Therefore, we have

$$\max_{R \in \Omega_\delta} \underline{\chi}(R) \leq \max_{x \in \Delta^5_\delta} Q(x) = \max_{1 \leq j \leq 6} Q(p_j) = Q(p_5) = \tfrac{3}{4}. \tag{7.8}$$

Thus, as $-R \in \Omega_\delta$ at all points of (M^4, g) and $\underline{\chi}(-R) = \underline{\chi}(R)$, from (7.8) and (2.10),

$$\chi(M) = \frac{1}{\pi^2}\int_M \underline{\chi}(R) \ \mathrm{vol_g} \leq \frac{3}{4\pi^2}\,\mathrm{Vol}(M, \mathrm{g}). \tag{7.9}$$

Clearly, equality in (7.9) holds if and only if $\underline{\chi}(R) = \frac{3}{4}$ at all points of (M^4, g), which, by (7.7) and (7.8) is equivalent to $-R = \iota_5(p_5) = R_{S^4}$, i.e., $\sec_M \equiv -1$. □

Remark 7.6. As stated in the Introduction, (7.9) was also observed by Ville [Vil87].

Appendix A. Revisiting Ville's estimates

The seminal works of Ville [Vil85, Vil89] on the geography of pinched 4-manifolds has been partially extended by several authors, see e.g. [Ko05, DRR]. In this Appendix, we give a uniform and general treatment of Ville's estimates, that pushes the method to its natural limit, yielding the following result.

Theorem A.1. *If (M^4, g) is a δ-pinched oriented 4-manifold, with finite volume and $\delta \geq \delta_0^{\mathrm{V}}$, then*

$$|\sigma(M)| \leq \lambda^{\mathrm{V}}(\delta)\, \chi(M),$$

where $\lambda^{\mathrm{V}} \colon [\delta_0^{\mathrm{V}}, 1] \to \mathbb{R}$ is given by

$$\lambda^{\mathrm{V}}(\delta) = \begin{cases} \dfrac{7\delta^2 + 10\delta + 1 - \sqrt{3}\sqrt{11\delta^4 + 68\delta^3 + 6\delta^2 + 28\delta - 5}}{6(1-\delta)^2}, & \text{if } \delta \in \left[\delta_0^{\mathrm{V}}, \delta_1^{\mathrm{V}}\right], \\ \dfrac{2}{3}\dfrac{13\delta^2 + 4\delta + 1 - \sqrt{3}\sqrt{55\delta^4 + 40\delta^3 + 6\delta^2 + 8\delta - 1}}{(1-\delta)^2}, & \text{if } \delta \in \left[\delta_1^{\mathrm{V}}, \delta_2^{\mathrm{V}}\right], \\ \dfrac{8(1-\delta)^2}{24\delta^2 - 12\delta + 15}, & \text{if } \delta \in \left[\delta_2^{\mathrm{V}}, 1\right], \end{cases}$$

and

(i) $\delta_0^{\mathrm{V}} \cong 0.163$ *is the smallest real root of the polynomial* $\delta^4 - 18\delta^3 + 2\delta^2 - 6\delta + 1$*,*

(ii) $\delta_1^{\mathrm{V}} \cong 0.166$ *is the only real root of the polynomial* $31\delta^3 + \delta^2 + 5\delta - 1$*,*

(iii) $\delta_2^{\mathrm{V}} \cong 0.211$ *is the largest real root of the polynomial* $140\delta^4 + 40\delta^3 - 6\delta^2 + 88\delta - 19$*.*

Remark A.2. The original instances of Theorem A.1 that appear in the works of Ville [Vil85, Vil89] are that negatively $\frac{1}{4}$-pinched oriented 4-manifolds with finite volume satisfy $|\sigma(M)| \leq \frac{1}{3}\,\chi(M)$, and positively $\frac{4}{19}$-pinched oriented 4-manifolds satisfy $|\sigma(M)| < \frac{1}{2}\,\chi(M)$. These statements derive, respectively, from $\lambda^{\mathrm{V}}\big(\frac{1}{4}\big) = \frac{1}{3}$ and $\lambda^{\mathrm{V}}\big(\frac{4}{19}\big) = \frac{2(97-7\sqrt{141})}{75} < \frac{1}{2}$.

Proof of Theorem A.1. Given a δ-pinched oriented 4-manifold (M^4, g) with finite volume, up to reversing its orientation, we shall assume $\sigma(M) \geq 0$. Moreover, at each $p \in M$, its curvature operator $R_p \in \mathrm{Sym}^2_b(\wedge^2\mathbb{R}^4)$ satisfies $\pm R \in \Omega_\delta$, see (4.1).

For each $\lambda > 0$, consider the quadratic form $\underline{\Delta}_\lambda \colon \mathrm{Sym}^2_b(\wedge^2\mathbb{R}^4) \to \mathbb{R}$ defined as

$$\underline{\Delta}_\lambda(R) := 8\left(\underline{\chi}(R) - \tfrac{1}{\lambda}\,\underline{\sigma}(R)\right),$$

where $\underline{\chi}$ and $\underline{\sigma}$ are given by (2.10) and (2.11). That is, $\underline{\Delta}_\lambda(R) = 8I_\lambda(R)$, where I_λ is defined in (6.10), so writing R in the canonical form (2.3), we have:

$$\underline{\Delta}_\lambda(R) = 6u^2 + \left(1 - \tfrac{2}{3\lambda}\right)|W_+|^2 + \left(1 + \tfrac{2}{3\lambda}\right)|W_-|^2 - 2|C|^2. \tag{A.1}$$

Clearly, $\underline{\Delta}_\lambda(-R) = \underline{\Delta}_\lambda(R)$. Therefore, it follows from (2.9) that if

$$\min_{R \in \Omega_\delta} \underline{\Delta}_\lambda(R) \geq 0, \tag{A.2}$$

then $\sigma(M) \leq \lambda \cdot \chi(M)$. Thus, it suffices to prove (A.2) holds if $\lambda = \lambda^{\mathrm{V}}(\delta)$, cf. (6.11).

Fix $R \in \Omega_\delta$. Using the same notation as [Vil85, Vil89], let $H_i \in \wedge^2_+\mathbb{R}^4$, $i = 1, 2, 3$, be an orthonormal basis that diagonalizes W_+, and set

$$w_i^+ := \langle W_+ H_i, H_i\rangle, \quad i = 1, 2, 3, \tag{A.3}$$

where $w_1^+ \leq w_2^+ \leq w_3^+$. Consider the traceless Ricci component of R as a linear map $C\colon \wedge^2_+\mathbb{R}^4 \to \wedge^2_-\mathbb{R}^4$, which is denoted Z_1 in [Vil85, Vil89]. Let $K_i \in \wedge^2_-\mathbb{R}^4$ be unit vectors such that[1] $CH_i = c_i K_i$, where $c_i \in \mathbb{R}$, $i = 1,2,3$, and set

$$\begin{aligned} \widehat{w}_i^- &:= \langle W_- K_i, K_i\rangle, \quad i=1,2,3, \\ \alpha &:= \max_{1\leq i\leq 3} |\widehat{w}_i^-|. \end{aligned} \tag{A.4}$$

We stress that while w_i^+ are the eigenvalues of W_+ as in (2.4), the numbers $\widehat{w}_i^-$ defined in (A.4) in general *do not agree* with the eigenvalues w_i^- of W_-. Still, arguing as in [Vil85, Lemma 4] and [Vil89, Lemme 1.4], it follows from (A.4) that

$$|W_-|^2 \geq \tfrac{3}{2}\alpha^2. \tag{A.5}$$

Since the oriented Grassmannian (2.5) can be written as

$$\mathrm{Gr}_2^+(\mathbb{R}^4) = \left\{ \frac{H+K}{\sqrt{2}} \in \wedge^2\mathbb{R}^4 : H \in \wedge^2_+\mathbb{R}^4,\, K\in\wedge^2_-\mathbb{R}^4,\, \|H\| = \|K\| = 1\right\}, \tag{A.6}$$

it follows from $R \in \Omega_\delta$ and (A.3) that the quantities

$$v_i := u + \frac{w_i^+}{2} = \frac{\langle R\sigma_i,\sigma_i\rangle + \langle R*\sigma_i, *\sigma_i\rangle}{2}, \quad i=1,2,3, \tag{A.7}$$

where $\sigma_i := \frac{1}{\sqrt{2}}(H_i + K_0) \in \mathrm{Gr}_2^+(\mathbb{R}^4)$ and $K_0 \in \wedge^2_-\mathbb{R}^4$ is a unit vector chosen so that $\langle W_- K_0, K_0\rangle = 0$, satisfy

$$\delta \leq v_i \leq 1, \quad \text{and} \quad \sum_{i=1}^3 v_i = 3u. \tag{A.8}$$

Similarly, using that $R\in\Omega_\delta$ and $\frac{1}{\sqrt{2}}(H_i \pm K_i) \in \mathrm{Gr}_2^+(\mathbb{R}^4)$, we have

$$\delta \leq v_i + \frac{\widehat{w}_i^-}{2} \leq 1, \quad \text{and} \quad |c_i| \leq m\left(v_i + \frac{\widehat{w}_i^-}{2}\right), \quad i=1,2,3, \tag{A.9}$$

where m is the piecewise affine function

$$\begin{aligned} m\colon [\delta,1] &\longrightarrow \left[0, \tfrac{1-\delta}{2}\right] \\ m(x) &:= \min\{1-x, x-\delta\}. \end{aligned} \tag{A.10}$$

Thus, from (A.4) and (A.9),

$$\begin{aligned} |C|^2 = \sum_{i=1}^3 c_i^2 &\leq \sum_{i=1}^3 m\Big(v_i + \tfrac{\widehat{w}_i^-}{2}\Big)^2 \leq \sum_{i=1}^3 \Big(m(v_i) + \tfrac{|\widehat{w}_i^-|}{2}\Big)^2 \\ &\leq \sum_{i=1}^3 \big(m(v_i)^2 + \alpha\, m(v_i) + \tfrac14\alpha^2\big) = \sum_{i=1}^3 m(v_i)^2 + \alpha\sum_{i=1}^3 m(v_i) + \tfrac34\alpha^2. \end{aligned} \tag{A.11}$$

Combining (A.5), (A.7), (A.8), and (A.11), we arrive at Ville's main estimate:

$$\begin{aligned} \tfrac12 \underline{\Delta}_\lambda(R) &= 3u^2 + \big(\tfrac12 - \tfrac{1}{3\lambda}\big)|W_+|^2 + \big(\tfrac12 + \tfrac{1}{3\lambda}\big)|W_-|^2 - |C|^2 \\ &= \big(\tfrac{4}{9\lambda} - \tfrac13\big)\left(\sum_{i=1}^3 v_i\right)^2 + \big(2 - \tfrac{4}{3\lambda}\big)\sum_{i=1}^3 v_i^2 + \big(\tfrac12 + \tfrac{1}{3\lambda}\big)|W_-|^2 - |C|^2 \\ &\geq \big(\tfrac{4}{9\lambda} - \tfrac13\big)\left(\sum_{i=1}^3 v_i\right)^2 + \big(2 - \tfrac{4}{3\lambda}\big)\sum_{i=1}^3 v_i^2 + \tfrac{1}{2\lambda}\alpha^2 - \alpha\sum_{i=1}^3 m(v_i) - \sum_{i=1}^3 m(v_i)^2 \end{aligned}$$

[1] In other words, $K_i = \pm\frac{CH_i}{\|CH_i\|}$ if $CH_i \neq 0$, but K_i can be chosen arbitrarily if $CH_i = 0$.

$$= \left(\tfrac{4}{9\lambda} - \tfrac{1}{3}\right)\left(\sum_{i=1}^{3} v_i\right)^2 + \left(2 - \tfrac{4}{3\lambda}\right)\sum_{i=1}^{3} v_i^2 - \tfrac{\lambda}{2}\left(\sum_{i=1}^{3} m(v_i)\right)^2 - \sum_{i=1}^{3} m(v_i)^2$$
$$+ \left(\tfrac{1}{\sqrt{2\lambda}}\,\alpha - \sqrt{\tfrac{\lambda}{2}}\sum_{i=1}^{3} m(v_i)\right)^2$$
$$\geq F_\lambda(v_1, v_2, v_3),$$

where $F_\lambda \colon V_\delta \to \mathbb{R}$ is the piecewise quadratic function

$$F_\lambda(v_1, v_2, v_3) := \left(\tfrac{4}{9\lambda} - \tfrac{1}{3}\right)\left(\sum_{i=1}^{3} v_i\right)^2 + \left(2 - \tfrac{4}{3\lambda}\right)\sum_{i=1}^{3} v_i^2 - \tfrac{\lambda}{2}\left(\sum_{i=1}^{3} m(v_i)\right)^2 - \sum_{i=1}^{3} m(v_i)^2$$

on the polyhedron $V_\delta := \big\{(v_1, v_2, v_3) \in \mathbb{R}^3 : \delta \leq v_1 \leq v_2 \leq v_3 \leq 1\big\}$. More precisely, from (A.10), the restriction $F_\lambda^i := (F_\lambda)|_{V_\delta^i}$ of the above to each of the subpolyhedra

$$\begin{aligned} V_\delta^1 &:= \big\{(v_1, v_2, v_3) \in V_\delta : \tfrac{\delta+1}{2} \leq v_1 \leq v_2 \leq v_3 \leq 1\big\}, \\ V_\delta^2 &:= \big\{(v_1, v_2, v_3) \in V_\delta : \delta \leq v_1 \leq \tfrac{\delta+1}{2} \leq v_2 \leq v_3 \leq 1\big\}, \\ V_\delta^3 &:= \big\{(v_1, v_2, v_3) \in V_\delta : \delta \leq v_1 \leq v_2 \leq \tfrac{\delta+1}{2} \leq v_3 \leq 1\big\}, \\ V_\delta^4 &:= \big\{(v_1, v_2, v_3) \in V_\delta : \delta \leq v_1 \leq v_2 \leq v_3 \leq \tfrac{\delta+1}{2}\big\}, \end{aligned} \tag{A.12}$$

is a quadratic form $F_\lambda^i \colon V_\delta^i \to \mathbb{R}$; and, clearly, $V_\delta = \bigcup_{i=1}^4 V_\delta^i$. Therefore, in order to show that $\lambda = \lambda^{\mathrm{V}}(\delta)$ implies (A.2), it suffices to show that it implies

$$\min_{V_\delta^i} F_\lambda^i \geq 0, \quad 1 \leq i \leq 4; \tag{A.13}$$

which we shall now prove using Corollary 3.2 with $Q = -F_\lambda^i$ and $K = V_\delta^i$.

According to (A.12), we have that the vertices q_j^i of the polyhedron V_δ^i, where $1 \leq j \leq 4$ if $i \in \{1, 4\}$, and $1 \leq j \leq 6$ if $i \in \{2, 3\}$, are given as follows:

$$\begin{aligned} &q_1^1 = \left(\tfrac{\delta+1}{2}, \tfrac{\delta+1}{2}, \tfrac{\delta+1}{2}\right), \ q_2^1 = \left(\tfrac{\delta+1}{2}, \tfrac{\delta+1}{2}, 1\right), \ q_3^1 = \left(\tfrac{\delta+1}{2}, 1, 1\right), \ q_4^1 = (1, 1, 1), \\ &q_1^2 = \left(\delta, \tfrac{\delta+1}{2}, \tfrac{\delta+1}{2}\right), \ q_2^2 = \left(\delta, \tfrac{\delta+1}{2}, 1\right), \ q_3^2 = (\delta, 1, 1), \ q_4^2 = q_1^1, \ q_5^2 = q_2^1, \ q_6^2 = q_3^1, \\ &q_1^3 = \left(\delta, \delta, \tfrac{\delta+1}{2}\right), \ q_2^3 = (\delta, \delta, 1), \ q_3^3 = q_1^2, \ q_4^3 = q_2^2, \ q_5^3 = q_1^1, \ q_6^3 = q_2^1, \\ &q_1^4 = (\delta, \delta, \delta), \ q_2^4 = q_1^3, \ q_3^4 = q_1^2, \ q_4^4 = q_1^1. \end{aligned}$$

Note that V_δ^1 and V_δ^4 are 3-simplices (i.e., tetrahedra), but V_δ^2 and V_δ^3 are convex hulls of 6 distinct points in $\mathbb{R}^3$. However, it is easy to see that V_δ^2 and V_δ^3 are *prisms*, i.e., isometric to a product of a 2-simplex (the convex hull of 3 of its vertices) and a 1-simplex (the convex hull of 2 of its vertices). Thus, it is straightforward to check which subsets of q_j^2 and q_j^3 determine faces of V_δ^2 and V_δ^3, respectively.

Routine computations show that the Hessian of each F_λ^i is negative-definite if $\lambda < \frac{4}{3}$, and has exactly $d = 2$ positive eigenvalues if $\lambda > \frac{4}{3}$; so it suffices to inspect faces of $V_\delta^i = \operatorname{conv}(q_j^i)$ that have dimension ≤ 2, i.e., convex hulls of no more than 3 different q_j^i's. By direct inspection, we find that, for all $0 < \delta < 1$, $\lambda > 0$, and $1 \leq i \leq 4$, the Hessian of the restrictions of F_λ^i to all 1- and 2-dimensional faces of V_δ^i is *not* positive-definite. Thus, by Corollary 3.2, we have that $\min_{V_\delta^i} F_\lambda^i = \min F_\lambda^i(q_j^i)$ is the smallest among the values assumed by F_λ^i on the vertices q_j^i of V_δ^i, which are given in Table A.1.

Similarly to the last step in the proof of Theorem 6.2, applying Algebraic Cylindrical Decomposition to the semialgebraic set of $\mathbb{R}^2$ given by

$$\mathfrak{X}^{\mathrm{V}} := \left\{(\delta, \lambda) \in H : F_\lambda^i(q_j^i) \geq 0, \text{ for all } i, j\right\},$$

q^i_j	$F^i_\lambda(q^i_j)$
q^1_1	$-\frac{9\lambda}{8}\delta^2+\left(\frac{9\lambda}{4}+3\right)\delta-\frac{9\lambda}{8}$
q^1_2	$\left(\frac{1}{6}-\frac{\lambda}{2}-\frac{2}{9\lambda}\right)\delta^2+\left(\frac{5}{3}+\lambda+\frac{4}{9\lambda}\right)\delta+\frac{7}{6}-\frac{\lambda}{2}-\frac{2}{9\lambda}$
q^1_3	$\left(\frac{1}{6}-\frac{\lambda}{8}-\frac{2}{9\lambda}\right)\delta^2+\left(\frac{2}{3}+\frac{\lambda}{4}+\frac{4}{9\lambda}\right)\delta+\frac{13}{6}-\frac{\lambda}{8}-\frac{2}{9\lambda}$
q^1_4	3
q^2_1	$\left(\frac{7}{6}-\frac{\lambda}{2}-\frac{2}{9\lambda}\right)\delta^2+\left(\frac{5}{3}+\lambda+\frac{4}{9\lambda}\right)\delta+\frac{1}{6}-\frac{\lambda}{2}-\frac{2}{9\lambda}$
q^2_2	$\left(\frac{3}{2}-\frac{\lambda}{8}-\frac{2}{3\lambda}\right)\delta^2+\left(\frac{\lambda}{4}+\frac{4}{3\lambda}\right)\delta+\frac{3}{2}-\frac{\lambda}{8}-\frac{2}{3\lambda}$
q^2_3	$\left(\frac{5}{3}-\frac{8}{9\lambda}\right)\delta^2+\frac{4}{9}\left(\frac{4}{\lambda}-3\right)\delta+\frac{8}{3}-\frac{8}{9\lambda}$
q^3_1	$\left(\frac{13}{6}-\frac{\lambda}{8}-\frac{2}{9\lambda}\right)\delta^2+\left(\frac{2}{3}+\frac{\lambda}{4}+\frac{4}{9\lambda}\right)\delta+\frac{1}{6}-\frac{\lambda}{8}-\frac{2}{9\lambda}$
q^3_2	$\left(\frac{8}{3}-\frac{8}{9\lambda}\right)\delta^2+\frac{4}{9}\left(\frac{4}{\lambda}-3\right)\delta+\frac{5}{3}-\frac{8}{9\lambda}$
q^4_1	$3\delta^2$
q^4_2	$\left(\frac{13}{6}-\frac{\lambda}{8}-\frac{2}{9\lambda}\right)\delta^2+\left(\frac{2}{3}+\frac{\lambda}{4}+\frac{4}{9\lambda}\right)\delta+\frac{1}{6}-\frac{\lambda}{8}-\frac{2}{9\lambda}$

TABLE A.1. Values of F^i_λ on the vertices q^i_j of V^i_δ, $1\leq i\leq 4$. The suppressed entries are equal to some other entry in the table, namely $F^4(q^4_2)=F^3(q^3_1)$, $F^3(q^3_4)=F^2(q^2_2)$, $F^2(q^2_6)=F^1(q^1_3)$, $F^4(q^4_4)=F^3(q^3_5)=F^2(q^2_4)=F^1(q^1_1)$, $F^3(q^3_6)=F^2(q^2_5)=F^1(q^1_2)$, and $F^4(q^4_3)=F^3(q^3_3)=F^2(q^2_1)$.

where $H=(0,1)\times(0,+\infty)$, we obtain that $\mathfrak{X}^{\rm V}=\left\{\delta\in\left[\delta_0^{\rm V},1\right):\lambda^{\rm V}(\delta)\leq\lambda\leq\overline{\lambda}(\delta)\right\}$, where $\lambda^{\rm V}\colon\left[\delta_0^{\rm V},1\right]\to\mathbb{R}$ is the piecewise continuous function in the statement, and $\overline{\lambda}(\delta):=\frac{8\delta}{3(1-\delta)^2}$. Note that $\lambda^{\rm V}(\delta_0^{\rm V})=\overline{\lambda}(\delta_0^{\rm V})$, and at least one among $F^1_\lambda(q^1_1)$ and $F^2_\lambda(q^2_1)$ is negative if $\delta<\delta_0^{\rm V}$, so (A.13) does not hold. However, if $\delta\in\left[\delta_0^{\rm V},1\right)$, then $(\delta,\lambda^{\rm V}(\delta))\in\mathfrak{X}^{\rm V}$ implies that (A.13) and hence (A.2) hold. The desired conclusion also holds at the right endpoint $\delta=1$, where $\lambda^{\rm V}(1)=0$, since constant curvature manifolds are locally conformally flat, and thus have zero signature. □

Remark A.3. There are two noteworthy differences between the semialgebraic sets $\mathfrak{X}$ and $\mathfrak{X}^{\rm V}$ of (δ,λ) for which the crucial lower bounds in the proofs of Theorems 6.2 and A.1, respectively, yield nonnegative quantities (as desired). First, the projection of $\mathfrak{X}$ onto $0<\delta<1$ is surjective, which does not hold for $\mathfrak{X}^{\rm V}$. Second, for any given δ in this projection, the interval of $\lambda>0$ for which $(\delta,\lambda)\in\mathfrak{X}$ is not bounded from above, while it is for $\mathfrak{X}^{\rm V}$. Clearly, both stem from the presence of the upper bound $\overline{\lambda}(\delta)$ in the cylindrical decomposition of $\mathfrak{X}^{\rm V}$, which on $\mathfrak{X}$ it is simply $+\infty$.

Remark A.4. It was uncovered in our communications with Ville that there is a small mistake in [Vil85, p. 333-334] and [Vil89, p. 152], where the coefficient $-\frac{1}{7}$, respectively $-\frac{2}{5}$, of the term $\left(\sum_{i=1}^3 m(v_i)\right)^2$, should be replaced by $-\frac{1}{6}$, respectively $-\frac{1}{4}$; i.e., this coefficient should be equal to $-\frac{\lambda}{2}$, as in the definition of F_λ. This arises from erroneously assuming that the sum of the numbers $\widehat{w}_i^-$ in (A.4) vanishes, which would imply that the sum of their squares is bounded above by $2\alpha^2$. This need not be the case, and the sum of $(\widehat{w}_i^-)^2$ is only bounded above by $3\alpha^2$. Indeed, denoting by $K \in \mathrm{Mat}_{3\times 3}(\mathbb{R})$ the matrix whose columns are the coordinates of K_i with respect to a fixed orthonormal basis of $\wedge^2_- \mathbb{R}^4$, we have that

$$0 = \sum_{i=1}^{3} w_i^- = \operatorname{tr} W_- = \operatorname{tr}\big(K W_- K^{\mathrm{t}}\big) = \operatorname{tr}\big(W_-(K^{\mathrm{t}} K)\big),$$

since $K^{\mathrm{t}} K = \mathrm{Id}$; however

$$\sum_{i=1}^{3} \widehat{w}_i^- = \operatorname{tr}\big(K^{\mathrm{t}} W_- K\big) = \operatorname{tr}\big(W_-(K K^{\mathrm{t}})\big)$$

may not vanish, as KK^{t} may not be equal to Id. Fortunately, the rest of the proofs in [Vil85, Vil89] can be modified accordingly, e.g., following the above proof of Theorem A.1, without any impact on the main result.

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CUNY Lehman College
Department of Mathematics
250 Bedford Park Blvd W
Bronx, NY, 10468, USA

CUNY Graduate Center
Department of Mathematics
365 Fifth Avenue
New York, NY, 10016, USA

Email address: r.bettiol@lehman.cuny.edu

Technische Universität Dresden
Fakultät Mathematik
Institut für Geometrie
Zellescher Weg 12-14
01062 Dresden, Germany
Email address: mario.kummer@tu-dresden.de

University of Oklahoma
Department of Mathematics
601 Elm Ave
Norman, OK, 73019-3103, USA
Email address: ricardo.mendes@ou.edu