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Nonlinear Interaction of Three Impulsive Gravitational Waves II: The Wave Estimates

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Abstract

This is the second and last paper of a series aimed at solving the local Cauchy problem for polarized $\mathbb{U}(1)$ symmetric solutions to the Einstein vacuum equations featuring the nonlinear interaction of three small amplitude impulsive gravitational waves. Such solutions are characterized by their three singular "wave-fronts" across which the curvature tensor is allowed to admit a delta singularity. Under polarized $\mathbb{U}(1)$ symmetry, the Einstein vacuum equations reduce to the Einstein–scalar field system in $(2+1)$ dimensions. In this paper, we focus on the wave estimates for the scalar field in the reduced system. The scalar field terms are the most singular ones in the problem, with the scalar field only being Lipschitz initially. We use geometric commutators to prove energy estimates which reflect that the singularities are localized, and that the scalar field obeys additional fractional-derivative regularity, as well as regularity along appropriately defined "good directions". The main challenge is to carry out all these estimates using only the low-regularity properties of the metric. Finally, we prove an anisotropic Sobolev embedding lemma, which when combined with our energy estimates shows that the scalar field is everywhere Lipschitz, and that it obeys additional $C^{1,\theta}$ estimates away from the most singular region.

Contents

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1 Introduction

The impulsive gravitational waves. In this paper and [25], our main goal is to construct and give a precise description of a large class of local solutions to the Einstein vacuum equations

$$Ric({}^{(4)}g) = 0 \tag{1.1}$$

which feature the nonlinear, transversal interaction of **three impulsive gravitational waves**. An impulsive gravitational wave is a (weak) solution to the Einstein vacuum equations for which the Riemann curvature tensor has a delta singularity supported on a null hypersurface. Interaction of impulsive gravitational waves is then represented by solutions to (1.1) featuring the transversal intersection of such singular hypersurfaces.

In our work, we impose a polarized $\mathbb{U}(1)$ symmetry assumption. In other words, we consider a $(3+1)$-dimensional Lorentzian manifold $(I \times \mathbb{R}^2 \times \mathbb{S}^1, {}^{(4)}g)$, where $I \subseteq \mathbb{R}$ is an interval, and assume that the metric takes the following form

$${}^{(4)}g = e^{-2\phi} g + e^{2\phi}(dx^3)^2, \tag{1.2}$$

where $\phi : I \times \mathbb{R}^2 \to \mathbb{R}$ is a scalar function and g is a Lorentzian metric on $I \times \mathbb{R}^2$, i.e. they are independent of the $\mathbb{S}^1 = \mathbb{R}/(2\pi\mathbb{Z})$-direction, which we parameterize by the coordinate x^3. The Einstein vacuum equations then reduce to the $(2+1)$-

dimensional Einstein–scalar field system

$$\begin{cases} Ric(g) = 2\mathrm{d}\phi \otimes \mathrm{d}\phi, \\ \Box_g \phi = 0. \end{cases} \tag{1.3}$$

The following is an informal version of our main theorem (see [25, Theorem 5.2] for a precise statement):

Theorem 1.1 (Informal main theorem for impulsive gravitational waves) *Given a polarized* $\mathbb{U}(1)$*-symmetric initial data set corresponding to three (non-degenerate) small-amplitude impulsive gravitational waves propagating towards each other, there exists a weak solution to the Einstein vacuum equations corresponding to the given data up to and beyond the transversal interaction of these waves. In particular, in the solution, the metric is everywhere Lipschitz and is* $H^2_{loc} \cap C^{1,\theta}_{loc}$ *for some* $\theta \in (0, \frac{1}{4})$ *away from the three null hypersurfaces corresponding to impulsive gravitational waves.*

The δ-impulsive gravitational waves. We began the proof of Theorem 1.1 in part I of our series [25]. We introduced the notion of δ-impulsive gravitational waves, which are smooth approximations of the impulsive gravitational waves at a length scale $\delta > 0$. In our setup, these waves are of small, $O(\epsilon)$, amplitude, but being δ-impulsive means that their second derivatives could be of pointwise size $O(\epsilon\delta^{-1})$ in δ-neighborhoods around the null hypersurfaces on which the singularity propagates. They can be viewed as more realistic solutions to (1.1) which are "quantitatively impulsive" but without an actual singularity. For this reason, the study of δ-impulsive waves is a problem of independent interest that we will also address: we give below an informal version of our result on δ-impulsive waves (see [25, Theorem 5.6] for a precise statement).

Theorem 1.2 (Informal main theorem for δ-impulsive gravitational waves) *Given a polarized* $\mathbb{U}(1)$*-symmetric initial data set corresponding to three small-amplitude* δ*-impulsive gravitational waves propagating towards each other, there exists a smooth solution to the Einstein vacuum equations corresponding to the given data up to and beyond the transversal interaction of these waves.*

Moreover, for all sufficiently small $\delta > 0$*, the following holds:*

- *[Local existence]. The solution exists up to time* 1*, independently of* δ*.*
- *[Uniform estimates]. The solution satisfies* δ*-dependent estimates consistent with* δ*-approximations of actual impulsive waves.*
- *In particular, the metric is uniformly Lipschitz in* δ *everywhere, and obeys uniform-in-*δ $H^2 \cap C^{1,\theta}$ *(for* $\theta \in (0, \frac{1}{4})$*) estimates away from the* δ*-impulsive gravitational waves.*

As it turns out, the proof of our main Theorem 1.1 regarding actual impulsive waves reduces to the proof of Theorem 1.2 on δ-impulsive waves. We indeed proved on the one hand in [25] that given any non-degenerate.[1] initial data representing three small

[1] We recall that the non-degeneracy assumption in [25] is only used to solve the constraint equations. Moreover, given $O(\epsilon)$ data, the non-degeneracy assumption can be guaranteed by adding an $O(\epsilon^{\frac{6}{5}})$ smooth perturbation; see [25, Remark 4.7].

amplitude impulsive gravitational waves propagating towards each other, the initial data can be approximated by those for δ-impulsive gravitational waves for all small enough $\delta > 0$. On the other hand, we proved in [25] via a limiting argument that to any such one-parameter (indexed by δ) family of δ-impulsive gravitational waves solutions *corresponds an actual impulsive gravitational waves solution*, provided that the δ-impulsive waves satisfy specific quantitative estimates for all small $\delta > 0$.

Because of the above reduction, the remaining goal is to prove the quantitative wave estimates for the δ-impulsive waves as stated in Theorem 1.2. By the above, this step completes our resolution of the local Cauchy problem for three actual impulsive gravitational waves, i.e. it completes the proof of Theorem 1.1.

Wave estimates for the δ-impulsive waves. In view of the form of the metric (1.2) in polarized $\mathbb{U}(1)$ symmetry, the estimates for the original $(3+1)$-dimensional metric ${}^{(4)}g$ naturally separate into those for the reduced metric g and for the scalar field ϕ. From now on, we will work in the reduced picture: we will refer to g as the "metric" part, and ϕ as the "wave" part.

In the context of Theorems 1.1 and 1.2, the wave part is more singular. Indeed, for an impulsive gravitational wave, $\partial\phi$ has a jump discontinuity across a null hypersurface, while g is more regular. Correspondingly, for a δ-impulsive gravitational wave, $|\partial\phi|$ is of size $O(1)$, and $|\partial^2\phi|$ is of size $O(\delta^{-1})$ in a δ-neighborhood of a null hypersurface. Thus, in Theorem 1.2, when we prove that the $(3+1)$-dimensional metric ${}^{(4)}g$ is uniformly Lipschitz in δ everywhere and obeys uniform-in-δ $H^2 \cap C^{1,\theta}$ estimates (for $\theta \in (0, \frac{1}{4})$) away from the δ-impulsive gravitational waves, the main challenge is to prove these bounds for ϕ.

In part I of our series [25], we proved estimates for the metric g, as well as for some associated null hypersurfaces, assuming estimates for ϕ which are consistent with the spacetime having three interacting δ-impulsive waves.

In this paper, we carry out the remaining task, which is to obtain the estimates for ϕ assumed in [25], thus closing a bootstrap argument.

In fact, given the estimates in [25], and recalling from (1.3) that ϕ satisfies a linear wave equation, we can think of this as a statement concerning the linear wave equation with δ-impulsive wave data on a background with rough metric. (See Sect. 1.2.2 for further discussions.) The following is an informal version of the main result in this paper:

Theorem 1.3 (Informal version of the main result in this paper) *Suppose that*

- *the initial data for ϕ correspond to three small-amplitude δ-impulsive gravitational wave propagating towards each other, and*
- *there is a smooth Lorentzian metric g in $[0, T_B) \times \mathbb{R}^2$ such that the geometric estimates for the reduced $(2+1)$-dimensional metric and null hypersurfaces in [25] hold.*

Suppose ϕ is the solution to the linear wave equation $\Box_g\phi = 0$ with the prescribed data. Then, for all sufficiently small $\delta > 0$, the following holds in $[0, T_B) \times \mathbb{R}^2$:

- *The solution ϕ satisfies δ-dependent estimates consistent with δ-approximations of actual impulsive waves.*

- *ϕ is Lipschitz uniformly-in-δ everywhere, and obeys uniform-in-δ $H^2 \cap C^{1,\theta}$ estimates (for $\theta \in (0, \frac{1}{4})$) away from the δ-impulsive gravitational waves.*

The precise version of Theorem 1.3 can be found in[2] Theorem 4.2 and Theorem 4.3. In particular, we refer the reader

- to Sect. 4.1 for the precise assumptions on the initial data of the δ-impulsive gravitational waves,
- to Sect. 5 for the geometric estimates that we need, and
- to Sect. 4.3 for the precise wave estimates that we prove.

According to the results in [25], the estimates in Theorem 1.3 complete the proof of Theorem 1.2.

Comments on the wave estimates. The main issue at stake is that we want to propagate a bound for $\|\partial\phi\|_{L^\infty(\mathbb{R}^2)}$ everywhere and a bound for $\|\partial\phi\|_{C^\theta(\mathbb{R}^2)}$ (for $\theta \in (0, \frac{1}{4})$) away from the most singular region, while the initial data of ϕ are very rough from the point of view of isotropic L^2-based Sobolev spaces. Indeed, recall that for an impulsive gravitational wave, $\partial\phi$ initially has a jump discontinuity across a curve. Thus, for the δ-impulsive wave, in terms of isotropic L^2-based Sobolev spaces H^s, the data for ϕ only obey the following[3] δ-independent bound:

$$\|\partial\phi\|_{H^{\frac{1}{2}-}(\mathbb{R}^2)} \leq \epsilon. \tag{1.4}$$

This is far too weak to control the Lipschitz and Hölder norms (and is even below the threshold to close the estimates for local existence of the quasilinear problem).

It turns out that in order to close a bootstrap argument, to propagate uniform-in-δ Lipschitz bounds for ϕ, and to obtain improved Hölder regularity away from the wave fronts, we need to design energies that exploit the specific nature of the δ-impulsive waves. More precisely, we will use the following more subtle "improved regularity" in the problem:

1. *[Anisotropy]*. We prove that each of the three impulsive waves propagates along specific directions: this property can be proven by differentiating ϕ by vector fields tangential to the wave front.
2. *[Hierarchy of δ-dependent estimates: "short pulse bounds"]*. Related to the localization, the solution satisfies a hierarchy of δ-dependent bounds involving large and small quantities, in a manner that is similar to Christodoulou's short pulse estimates in [11].
3. *[Localization]*. The singular parts are initially localized, and we prove that they remain localized in δ-neighborhoods of 3 null hypersurfaces throughout the evolution.

[2] Notice that Theorems 4.2 and 4.3 do not explicitly refer to the geometric estimates in [25]. Nonetheless, in the proof we will indeed first use the bootstrap assumptions and results in [25] to obtain the geometric estimates; see Sect. 5.

[3] Indeed, it is easy to check that a function with a jump discontinuity along a smooth curve in $\mathbb{R}^2$ is locally in $H^{\frac{1}{2}-}(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$.

In the energy estimates, it is important that we employ a combination of geometric and fractional derivatives so as to capture the above features. The main challenge for closing these energy estimates is that due to the quasilinear coupling, the metric is of very limited regularity, and we need to propagated the energy bounds for such rough metrics.

We will further discuss these estimates and sketch the main ideas of the proof in **Section** 1.1. After the discussion of the proof, we will discuss some related works in **Section** 1.2. Finally, we will outline the remainder of the paper in **Section** 1.3.

1.1 Ideas of the proof

This section will be organized as follows.

We begin with the geometric setup in **Section** 1.1.1. Then in **Section** 1.1.2, we briefly recall the estimates for the geometric quantities derived in [25].

Turning to the scalar wave, we first introduce in **Section** 1.1.3 the regular-singular composition of the scalar wave, which plays an important role in the analysis. Roughly speaking, this decomposes the scalar wave into a regular part and singular parts, where the latter are localized and propagating in specific directions.

We then address the proof of the wave estimates, which is the focus of this paper. Our main wave estimates are L^2-based. However, importantly, our L^2-based energies are designed so as to obtain the global Lipschitz estimates as well as the improved Hölder bounds away from the singular region (cf. Theorem 1.1).

- In **Section** 1.1.4, we discuss the L^2-based estimates up to the second derivative. These estimates already capture particular features of the δ-impulsive waves, including its anisotropy and localization.
- In **Section** 1.1.5, we motivate the various higher order L^2 norms that we use by two anisotropic Sobolev-type embedding results. This is related to the Lipschitz and improved Hölder estimates.
- Finally, in **Section** 1.1.6, we explain the ideas in the proof of the higher order L^2-based estimates. In particular, we will discuss how the proof of these estimates are intertwined with the control for the geometry that we discussed in Sect. 1.1.2.

1.1.1 The basic geometric setup

Elliptic gauge. We recall the basic geometric setup in [25]. First, we construct a solution in an elliptic gauge, i.e. the ((2+1)-dimensional reduced) Lorentzian manifold $(I \times \mathbb{R}^2), g)$ takes the form $I \times \mathbb{R}^2 = \underset{t \in I}{\cup} \Sigma_t$ and

$$g = -N^2 dt^2 + e^{2\gamma}\delta_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt), \tag{1.5}$$

where δ_{ij} is the Kronecker symbol, the constant-t hypersurfaces Σ_t are maximal, and (as a consequence) the metric components $\mathfrak{g} \in \{N, \gamma, \beta^i\}$ satisfy semilinear elliptic equations which are schematically of the form

$$\Delta \mathfrak{g} = (\partial\phi)^2 + (\partial_x \mathfrak{g})^2. \tag{1.6}$$

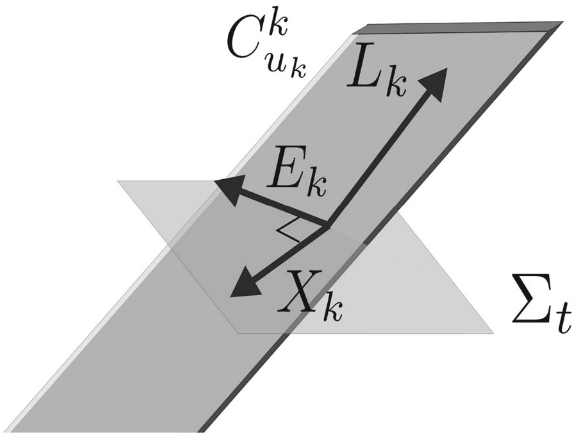

Fig. 1 The vector fields $\{L_k, E_k, X_k\}$: L_k is null, E_k, X_k are space-like and tangent to Σ_t

Eikonal functions and geometric vector fields. In addition to the metric itself, we constructed — dynamically defined — eikonal functions $\{u_k\}_{k=1,2,3}$, satisfying $g^{-1}(du_k, du_k) = 0$, which capture the direction of propagation of the δ-impulsive gravitational waves. Associated with each eikonal function u_k, we constructed a frame of vector fields $\{L_k, E_k, X_k\}$, where L_k and E_k are tangential to the constant-u_k (null) hypersurfaces $C^k_{u_k}$ and X_k is tangent to Σ_t and orthogonal to E_k as depicted in Fig. 1. These eikonal functions and geometric vector fields are important for capturing the propagation and interaction of the δ-impulsive waves, as we will further explain in Sects. 1.1.3 and 1.1.4 below.

1.1.2 Summary of the geometric estimates from part I

Continuing our discussion on geometry, we recall some of the estimates for the geometric quantities that we obtained in [25]. As we will see, one of the challenges in proving the wave estimates is to contend with the low regularity of the metric.

Different components of the metric components in the elliptic gauge (1.5) and different derivatives of the Ricci coefficients with respect to the $\{L_k, E_k, X_k\}$ frame obey different bounds. Especially for the highest order wave estimates, we will use the precise bounds for these geometric objects.

1. For the metric components in the elliptic gauge, denoted with the schematic notation $\mathfrak{g} \in \{N, \gamma, \beta_i\}$, we have the following regularity estimates for all $R > 0$:

$$\|\partial_i \mathfrak{g}\|_{W^{1,\infty}\cap W^{1+s',2}(\Sigma_t)} \lesssim \epsilon^2, \quad \|\partial_t \mathfrak{g}\|_{W^{1,\frac{2}{s'-s''}}(\Sigma_t\cap B(0,R))} \lesssim_R \epsilon^2, \tag{1.7}$$

   where $0 < s'' < s' < \frac{1}{2}$ are fixed but arbitrary parameters, to be explained later. Note that no estimates were obtained for $\partial_t^2 \mathfrak{g}$.
2. The Ricci coefficients $\chi_k := g(\nabla_{E_k} L_k, E_k)$ and $\eta_k := g(\nabla_{X_k} L_k, E_k)$ associated to the null frame $\{L_k, E_k, X_k\}$ are considerably less regular. Denote $\kappa_k \in \{\chi_k,\ \eta_k\}$, and introduce coordinates (t_k, u_k, θ_k) with u_k the eikonal function from Sect. 1.1.1,

$t_k = t$ and θ_k such that $L_k\theta_k = 0$. Then [25] gives

$$\sum_{\kappa_k \in \{\chi_k, \eta_k\}} \left(\|\kappa_k\|_{L^\infty(\Sigma_t)} + \|L_k\kappa_k\|_{L^\infty(\Sigma_t)} \right) \lesssim \epsilon^2. \tag{1.8}$$

Observe that $L_k\kappa_k$ is estimated at the same regularity as κ_k: this is because κ_k satisfies a transport equation in the L_k direction due to the Einstein equations (see [25, Lemma 2.22]).

The other E_k and X_k derivatives are less regular and only obey mixed L^2/L^∞ or L^2 bounds:

$$\|E_k\kappa_k\|_{L^\infty_{t_k} L^\infty_{u_k} L^2_{\theta_k}} + \|X_k\chi_k\|_{L^\infty_{t_k} L^\infty_{u_k} L^2_{\theta_k}} \lesssim \epsilon^2, \quad \|X_k\eta_k\|_{L^2(\Sigma_t \cap B(0,R))} \lesssim_R \epsilon^2. \tag{1.9}$$

Note that $X_k\chi_k$ obeys a similar bound as $E_k\kappa_k$, but to bound $X_k\eta_k$, we need L^2 in both u_k and θ_k.

To obtain higher order estimates, we are only allowed to commute with an extra L_k derivative:

$$\begin{aligned} &\|L_k X_k \chi_k\|_{L^2(\Sigma_t \cap B(0,R))}, \sum_{\kappa_k \in \{\chi_k, \eta_k\}} \|L_k^2 \kappa_k\|_{L^2(\Sigma_t \cap B(0,R))}, \\ &\sum_{\kappa_k \in \{\chi_k, \eta_k\}} \|L_k E_k \kappa_k\|_{L^2(\Sigma_t \cap B(0,R))} \lesssim_R \epsilon^2. \end{aligned} \tag{1.10}$$

Notice that as in (1.9) η_k obeys slightly weaker bounds than χ_k and moreover there is no estimate to control $L_k X_k \eta_k$.

In general, the derivatives of $\mathfrak{g}$ obey better bounds than χ_k, η_k (due to ellipticity of (1.6)). However, spatial ellipticity does not merge well with ∂_t derivatives: $\partial_t\mathfrak{g}$ only obeys weaker bounds, and $\partial_t^2\mathfrak{g}$ is not controlled in our argument at all. On the other hand, while χ_k, η_k obeys weaker bounds, they behave better with respect to L_k derivatives (which contains a ∂_t component); see (1.8), (1.10). (Additionally, one needs to control various non-trivial commutators when going back and forth between (1) the eikonal quantities constructed with L_k and E_k and (2) the metric coefficients in the elliptic gauge (1.5). We will not get into details here, except for remarking that they can be controlled using the geometric estimates in [25].)

1.1.3 Regular-singular decomposition and the singular zones

We will impose that the δ-impulsive waves are of small amplitude $\epsilon > 0$. The length scale δ at which each δ-impulsive wave is localized is required to satisfy $0 < \delta \ll \epsilon$.

We begin by decomposing ϕ into a regular and three singular parts. This is achieved by solving an auxiliary characteristic-Cauchy problem so that

$$\phi = \phi_{reg} + \sum_{k=1}^{3} \widetilde{\phi}_k, \text{ where } \Box_g\phi_{reg} = 0 \text{ and } \Box_g\widetilde{\phi}_k = 0 \text{ for } k = 1, 2, 3,$$

where each $\widetilde{\phi}_k$ corresponds to a δ-impulsive wave propagating along the constant-u_k null hypersurfaces $C^k_{u_k}$ and ϕ_{reg} is an error term which is more regular. The part ϕ_{reg} is regular everywhere in the sense that

$$\|\phi_{reg}\|_{H^{2+s'}(\mathbb{R}^2)} \lesssim \epsilon,$$

for some $s' \in (0, \frac{1}{2})$; see Sect. 13. The remainder of Sect. 1.1 will thus be devoted to the discussion of the singular parts $\widetilde{\phi}_k$.

Each $\widetilde{\phi}_k$ is initially regular away from the region $\{-\delta \leq u_k \leq 0\}$ and is in fact constructed to *vanish* for $u_k \leq -\delta$. In the region $\{-\delta \leq u_k \leq 0\}$, the first and second derivatives of $\widetilde{\phi}_k$ only obey initially the following schematic bounds

$$|\partial\widetilde{\phi}_k| \lesssim \epsilon, \ |\partial^2\widetilde{\phi}_k| \lesssim \epsilon\delta^{-1}. \tag{1.11}$$

(Notice that these are exactly the size estimates one obtains by smoothing out at a scale $u_k \approx \delta$ an initial function ϕ_{rough} of amplitude ϵ whose generic first derivatives $\partial\phi_{rough}$ have a jump continuity across the curve given by $\{u_k = 0\}$ and whose generic second (distributional) derivatives $\partial^2\phi_{rough}$ have a delta singularity supported on $\{u_k = 0\}$.) Because of (1.11), $\widetilde{\phi}_k$ is initially no better than $\|\widetilde{\phi}_k\|_{H^2(\Sigma_0)} \lesssim \epsilon\delta^{-\frac{1}{2}}$ and, in terms of L^2-based Sobolev spaces, it is only the $\|\partial\widetilde{\phi}_k\|_{H^s(\Sigma_0)}$ norms, for $s < \frac{1}{2}$, that obey the uniform-in-δ bounds $\|\partial\widetilde{\phi}_k\|_{H^s(\Sigma_0)} \lesssim_s \epsilon$.

An important use of the dynamically constructed eikonal functions that we mentioned earlier is they can track the location of singularities. For each $k = 1, 2, 3$, define the corresponding singular zone by

$$S^k_\delta := \{-\delta \leq u_k \leq \delta\} \tag{1.12}$$

(slightly larger than the initial singular zone $\{-\delta \leq u_k \leq 0\}$), measured with respect to the eikonal functions. We will show that throughout the evolution, the most singular part of $\widetilde{\phi}_k$ is localized in S^k_δ. As a first guide to the estimates, the reader can keep in mind that we will prove the following bounds inside and outside S^k_δ:

- *[Interior of the singular zone S^k_δ].* Within this singular region S^k_δ (see (1.12)), our bounds can be no better than the initial estimates (1.11). We will in fact prove estimates consistent with the δ-weights in (1.11). Namely, we prove the L^2-based bound

$$\|\widetilde{\phi}_k\|_{H^2(S^k_\delta)} \lesssim \epsilon \cdot \delta^{-\frac{1}{2}}, \tag{1.13}$$

  as well as the Lipschitz bound for $\widetilde{\phi}_k$

$$\|\partial\widetilde{\phi}_k\|_{L^\infty(S^k_\delta)} \lesssim \epsilon. \tag{1.14}$$

- *[Exterior of the singular zone S^k_δ].* We prove that the following estimate holds

$$\|\widetilde{\phi}_k\|_{H^2(\mathbb{R}^2\setminus S^k_\delta)} \lesssim \epsilon. \tag{1.15}$$

Note that this is better than the bounds (1.11) in the singular zone for the initial data.

Moreover, in terms of L^∞ based norms, we will show an improved Hölder estimate (compare with the Lipschitz estimate (1.14) above) for $\widetilde{\phi}_k$ outside of the singular zone S^k_δ, i.e. for some $\theta \in (0, \frac{1}{4})$

$$\|\widetilde{\phi}_k\|_{C^{1,\theta}(\mathbb{R}^2 \setminus S^k_\delta)} \lesssim \epsilon. \tag{1.16}$$

We will further explain the proof of the estimates (1.13)–(1.16). In order to derive these bounds, we will need to prove that improved regularity is exhibited for derivatives with respect to $\{L_k, E_k\}$, the vector fields tangential to constant-u_k hypersurfaces, as well as to derive higher order estimates.

1.1.4 The H^2 energy estimates: anisotropic estimates, short-pulse bounds and slice-picking

We first discuss our L^2 based energy estimates for $\widetilde{\phi}_k$ up to the second derivative. (The L^∞-based estimates will be discussed in Sect. 1.1.5 and the higher order L^2-based estimates will be explained in Sect. 1.1.6.) One of the main challenges of this problem is that the H^2 norm of ϕ is no better than $\delta^{-\frac{1}{2}}$ (recall (1.13)). Already at the H^2 level, we capture the following features of the solutions in our energy estimates (these will again play a role in the Lipschitz (as in (1.14)) and improved Hölder bounds (as in (1.16)); see Sect. 1.1.5):

1. [*Anisotropy*]. Derivatives in the geometric directions L_k and E_k are "good" derivatives for $\widetilde{\phi}_k$ that are better behaved than others. This phenomenon will allow us to prove anisotropic H^2-estimates where one general derivative is replaced by a "good derivative".
2. [*Short pulse bounds*]. As we mentioned above, the singularity leading to a large H^2 norm is only localized in a "small" region of length $\sim \delta$. At the same time, in the singular region, some (integrated) bounds can be proven to be δ-*small* using the small δ length as a source of smallness.
3. [*Localization*]. We prove that the singularity for $\widetilde{\phi}_k$ is localized in a small region S^k_δ around a null hypersurface. Indeed, we show that $\widetilde{\phi}_k$ obeys uniform-in-δ H^2 bounds *away* from S^k_δ as in (1.15). To show such bounds, we rely on a novel *slice-picking argument* exploiting the anisotropic bounds and the short pulse bounds. A δ-independent H^2 bound can then be propagated towards the future of this good hypersurface.

In the steps below, we explain in more detail these features of our (up to H^2 level) energy estimates.

Step 1: Anisotropic energy estimates captured by the good geometric derivatives. At the lowest order, our regularity assumption allows us to easily prove a δ-independent bound

$$\|\partial\widetilde{\phi}_k\|_{L^2(\Sigma_t)} \lesssim \epsilon. \tag{1.17}$$

In fact, we can put in an extra fractional $s' \in (0, \frac{1}{2})$ derivative (cf. (1.4)) and prove

$$\|\partial \langle D_x \rangle^{s'} \widetilde{\phi}_k\|_{L^2(\Sigma_t)} \lesssim \epsilon. \tag{1.18}$$

However, as mentioned in (1.13), at the second derivative level, we prove an estimate no better than the following:

$$\|\partial^2 \widetilde{\phi}_k\|_{L^2(\Sigma_t \cap S_\delta^k)} \sim \|\partial^2 \widetilde{\phi}_k\|_{L^2(\Sigma_t)} \lesssim \epsilon \cdot \delta^{-\frac{1}{2}}. \tag{1.19}$$

As we indicated above, despite (1.19), not all derivatives are equally bad. Since $\widetilde{\phi}_k$ is essentially propagating along constant-u_k hypersurfaces C_{u_k}, we have better regularity properties for derivatives in the directions tangential to C_{u_k} i.e. directions spanned by $\{L_k, E_k\}$ (see Sect. 1.1.1 and Fig. 1). Indeed, we prove that $\partial L_k \widetilde{\phi}_k$ and $\partial E_k \widetilde{\phi}_k$ are more regular and on constant-t hypersurfaces Σ_t:

$$\sum_{Y_k \in \{L_k, E_k\}} \|\partial Y_k \widetilde{\phi}_k\|_{L^2(\Sigma_t)} \lesssim \epsilon. \tag{1.20}$$

Step 2: The short pulse bounds in the singular region. The next feature of $\widetilde{\phi}_k$ to be emphasized is that the large H^2 norm (recall (1.19)) is only localized in a small region S_δ^k (recall (1.12)) of length scale $\sim \delta$. The first observation towards proving the localization is the following: while S_δ^k is a singular region for $\widetilde{\phi}_k$ in the sense that (1.19) cannot be improved, some small-in-δ bounds hold for the lower derivatives in S_δ^k.

To see this, first observe that since the initial data for $\widetilde{\phi}_k$ is chosen so that $\widetilde{\phi}_k = 0$ for $u_k \leq -\delta$, finite speed of propagation implies that $\widetilde{\phi}_k = 0$ on the null hypersurface $\{u_k = -\delta\}$ and in fact on the whole half-space $\{u_k \leq -\delta\}$. Using this vanishing and the smallness of the δ length scale, we can propagate a hierarchy of δ-dependent estimates for $\widetilde{\phi}_k$ and its derivatives in the singular region S_δ^k. (This is reminiscent of the short pulse estimates of Christodoulou, originally introduced to tackle the problem of the formation of trapped surfaces for the Einstein vacuum equations [11].) In particular, we prove the *smallness* estimate for the H^1 norm of $\widetilde{\phi}_k$:

$$\|\partial \widetilde{\phi}_k\|_{L^2(\Sigma_t \cap S_\delta^k)} \lesssim \epsilon \cdot \delta^{\frac{1}{2}}. \tag{1.21}$$

This is consistent with the initial data bound (1.11) (and the Lipschitz estimate (1.14) that we hope to prove): $\partial \widetilde{\phi}_k$ is bounded by ϵ pointwise, and the smallness arises from the smallness of the δ-length scale. Moreover, in this region, $\partial L_k \widetilde{\phi}_k$ and $\partial E_k \widetilde{\phi}_k$ also obey similar smallness bounds, which are better than (1.20):

$$\sum_{Y_k \in \{L_k, E_k\}} \|\partial Y_k \widetilde{\phi}_k\|_{L^2(\Sigma_t \cap S_\delta^k)} \lesssim \epsilon \cdot \delta^{\frac{1}{2}}. \tag{1.22}$$

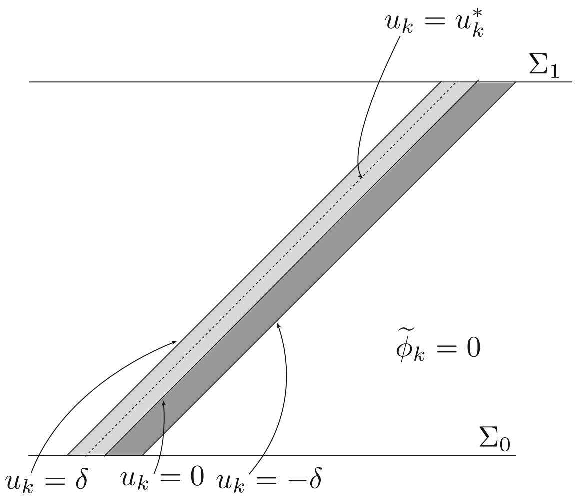

Fig. 2 The regions for the slice-picking argument

Step 3: Localization using a slice-picking argument. The short pulse bounds (1.21)–(1.22) allow us to use a slice-picking argument to prove that $\widetilde{\phi}_k$ obeys H^2 bounds *with no* δ*-weights* when $u_k \geq \delta$, i.e. beyond the singular region S_δ^k.

Consider Fig. 2. For $u_k \leq -\delta$, we have $\phi_k \equiv 0$. The initial $\|\partial^2\widetilde{\phi}_k\|_{L^2(\{-\delta\leq u_k\leq 0\})}$ norm is large — of size $O(\epsilon\delta^{-\frac{1}{2}})$— when $-\delta \leq u_k \leq 0$ (the darker shaded region), while the initial $\|\partial^2\widetilde{\phi}_k\|_{L^2(\{u_k\geq 0\})}$ norm is of size $O(\epsilon)$ away from the darker shaded region (including in the lightly shaded region, which is also of length scale δ). In both the darker shaded region and and lightly shaded region, we can prove the estimates (1.19), (1.21) and (1.22).

Squaring, integrating (1.22) over t and using Fubini's theorem to switch the t and u_k integrals, we have

$$\sum_{Y_k\in\{L_k,E_k\}}\int_0^\delta \|\partial Y_k\widetilde{\phi}_k\|^2_{L^2(C^k_{u_k})}\,du_k$$
$$\lesssim \sum_{Y_k\in\{L_k,E_k\}}\int_0^T \|\partial Y_k\widetilde{\phi}_k\|^2_{L^2(\Sigma_t\cap S^k_\delta)}\,dt \lesssim (\epsilon\delta^{\frac{1}{2}})^2 \lesssim \epsilon^2\delta,$$

where $C^k_{u_k}$ is a constant u_k-null hypersurface. The mean value theorem implies that **there exists** $u_k^* \in [0,\delta]$ (the dotted line in the lightly shaded region after the short pulse) such that the integral over the $u_k = u_k^*$ null hypersurface $C^k_{u_k^*}$ satisfies

$$\sum_{Y_k\in\{L_k,E_k\}} \|\partial Y_k\widetilde{\phi}_k\|^2_{L^2(C^k_{u_k^*})} \lesssim \epsilon^2. \tag{1.23}$$

Using standard energy estimates (assuming sufficient bounds for the metric), in order to estimate $\|\partial^2\widetilde{\phi}_k\|_{L^2(\{u_k\geq u_k^*\})}$ after $C^k_{u_k^*}$, it suffices to bound (a) the data on Σ_0

in the region $\{u_k \geq u_k^*\}$ and (b) the flux

$$\sum_{Y_k \in \{L_k, E_k\}} (\|Y_k \widetilde{\phi}_k\|^2_{L^2(C^k_{u_k^*})} + \|\partial Y_k \widetilde{\phi}_k\|^2_{L^2(C^k_{u_k^*})}),$$

i.e. on $C^k_{u_k^*}$ we only need bounds where at least one derivative is tangential to $C^k_{u_k^*}$. Since (a) we have improved data bound on $\Sigma_0 \cap \{u_k \geq 0\}$ and (b) u_k^* is picked so that we have a δ-independent bound (1.23) of this flux, we obtain H^2 estimates in the region $\{u_k \geq \delta\} \subseteq \{u_k \geq u_k^*\}$ *with no δ weights*, i.e.

$$\|\partial^2 \widetilde{\phi}_k\|_{L^2(\Sigma_t \setminus S^k_\delta)} \lesssim \epsilon. \tag{1.24}$$

In other words, the worst bound (1.19) is indeed only saturated in S^k_δ.

1.1.5 Anisotropic embedding results and the Lipschitz and Hölder estimates

Recall that we aim at proving the Lipschitz bound (1.14) and the Hölder bound (1.16). This necessitates L^2 estimates beyond those discussed in Sect. 1.1.4. Below, we will explain the embedding results adapted to our setting, and the precise higher order L^2 estimates that we will need.

Our embedding results will be used to control $\partial \widetilde{\phi}_k$, where ∂ denotes a derivative in the (original) coordinates of the elliptic gauge. In order to take advantage of the good derivatives, we will also introduce another coordinate system on each Σ_t as follows. Given $k \in \{1, 2, 3\}$, pick $k' \in \{1, 2, 3\}$ with $k \neq k'$. Then $(u_k, u_{k'})$ forms a coordinate system on $\mathbb{R}^2$ for any fixed t. Denote by $(\not\partial_{u_k}, \not\partial_{u_{k'}})$ the corresponding coordinate derivatives.

The reader should already think that $\not\partial_{u_{k'}}$ is the "good" derivative for $\widetilde{\phi}_k$, i.e. it is parallel to E_k, while $\not\partial_{u_k}$ is a "bad" derivative, and that $\not\partial$ denotes a general derivative in the $(u_k, u_{k'})$ coordinates.

Almost Lipschitz bounds. By comparing with the initial data estimates, one sees that the bounds (1.19), (1.22) and (1.24) in Sect. 1.1.4 are already the best H^2 estimates that can be proven. Heuristically, the bounds of Sect. 1.1.4 are almost sufficient to obtain the desired Lipschitz estimate (1.14) for ϕ except that when trying to use Sobolev embedding, one encounters a logarithmic divergence in the summation over frequency scales in a Littlewood–Paley decomposition. However, for any fixed $p \in [1, \infty)$, the H^2 bounds (1.19), (1.22) and (1.24) are still sufficient to give an L^p bound:

- *[L^p bounds away from the singular zone]*. Away from the singular zone S^k_δ, the standard Sobolev embedding $H^1(B(0, R)) \to L^p(B(0, R))$ give

$$\|\partial \widetilde{\phi}_k\|_{L^p(\Sigma_t \setminus S^k_\delta)} \lesssim \|\partial \widetilde{\phi}_k\|_{L^2(\Sigma_t \setminus S^k_\delta)} + \|\not\partial \partial \widetilde{\phi}_k\|_{L^2(\Sigma_t \setminus S^k_\delta)}. \tag{1.25}$$

  By (1.17) and (1.24) (after justifying that $\not\partial \partial \widetilde{\phi}_k$ and $\partial^2 \widetilde{\phi}_k$ are comparable), the right-hand side of (1.25) is bounded by ϵ, independently of δ.

- *[L^p bounds inside the singular zone]*. To treat the singular region, note that one can prove a refined version of Sobolev embedding that takes into account the directions of the derivatives and makes use of the localization of the singular region. Introducing a cutoff function ρ_k localizing $\partial\widetilde{\phi}_k$ near S^k_δ, we have

$$\begin{aligned} \|\rho_k\partial\widetilde{\phi}_k\|_{L^p(\Sigma_t)} &\lesssim \|\partial\!\!\!/(\rho_k\partial\widetilde{\phi}_k)\|^{\frac12}_{L^2(\Sigma_t)}\|\partial\!\!\!/_{u_{k'}}(\rho_k\partial\widetilde{\phi}_k)\|^{\frac12}_{L^2(\Sigma_t)} + \|\rho_k\partial\widetilde{\phi}_k\|_{L^2(\Sigma_t)} \\ &\lesssim \delta^{\frac12}\|\partial\!\!\!/(\rho_k\partial\widetilde{\phi}_k)\|_{L^2(\Sigma_t)} + \delta^{-\frac12}\|\partial\!\!\!/_{u_{k'}}(\rho_k\partial\widetilde{\phi}_k)\|_{L^2(\Sigma_t)} + \|\rho_k\partial\widetilde{\phi}_k\|_{L^2(\Sigma_t)}, \end{aligned} \tag{1.26}$$

  where the second line follows from the first using the Cauchy–Schwarz inequality. Now even though in our setting $\|\widetilde{\phi}_k\|_{H^2(\Sigma_t\cap S^k_\delta)} \sim \epsilon\delta^{-\frac12}$, we have *smallness* in the good derivatives estimate (1.22). Thus, modulo controlling the coordinate change and the vector field E_k, (1.19) and (1.22) respectively imply that $\delta^{\frac12}\|\partial\!\!\!/(\rho_k\partial\widetilde{\phi}_k)\|_{L^2(\Sigma_t)} \lesssim \epsilon$ and $\delta^{-\frac12}\|\partial\!\!\!/_{u_{k'}}(\rho_k\partial\widetilde{\phi}_k)\|_{L^2(\Sigma_t)} \lesssim \epsilon$. Using also (1.17) to control $\|\rho_k\partial\widetilde{\phi}_k\|_{L^2(\Sigma_t)}$, this shows that $\|\rho_k\partial\widetilde{\phi}_k\|_{L^p(\Sigma_t)} \lesssim \epsilon$.

Anisotropic Sobolev embedding adapted to the problem. In order to improve (1.25), (1.26), we prove two anisotropic embedding results, designed particularly for our setting for which we can exploit the anisotropy and localization of our L^2 estimates. In the following, we will only give the embedding estimates when applied to $\partial\widetilde{\phi}_k$ (or a cutoff version of $\partial\widetilde{\phi}_k$). These estimates are key ingredients in our proof of (1.14) and (1.16), since they provide the summability over all frequencies that we were lacking in the above paragraph.

- Our first embedding result (cf. Theorem 15.5) is a Hölder estimate on a half space[4]. For $s'' \in (0, \frac12)$,

$$\begin{aligned} \|\partial\widetilde{\phi}_k\|_{C^{0,\frac{s}{2}}(\Sigma_t\setminus S^k_\delta)} \lesssim_s\ & \|\partial\widetilde{\phi}_k\|_{L^2(\Sigma_t\setminus S^k_\delta)} + \|\partial\!\!\!/\partial\widetilde{\phi}_k\|_{L^2(\Sigma_t\setminus S^k_\delta)} \\ & + \|\partial\!\!\!/_{u_{k'}}\langle D_{u_k,u_{k'}}\rangle^{s''}\partial\widetilde{\phi}_k\|_{L^2(\Sigma_t)}, \end{aligned} \tag{1.27}$$

  where $\langle D_{u_k,u_{k'}}\rangle^{s''}$ is the fractional derivative operator in the $(u_k, u_{k'})$ coordinates. The estimate (1.27) could be compared with (1.25), where the extra term $\|\partial\!\!\!/_{u_{k'}}\langle D_{u_k,u_{k'}}\rangle^{s''}\partial\widetilde{\phi}_k\|_{L^2(\Sigma_t)}$ on the right-hand side not only allows us to sum over all frequencies in a Littlewood–Paley decomposition, but also lets us obtain extra Hölder regularity (as long as we are away from S^k_δ).
- Our second embedding result (cf. Theorem 15.3) is an L^∞ estimate, involving δ weights on the right-hand side:

$$\begin{aligned} \|\rho_k\partial\widetilde{\phi}_k\|_{L^\infty(\Sigma_t)} \lesssim\ & \delta^{-\frac12}\|\rho_k\partial\widetilde{\phi}_k\|_{L^2(\Sigma_t)} + \delta^{\frac12}\|\partial\!\!\!/_{u_{k'}}\partial\!\!\!/(\rho_k\partial\widetilde{\phi}_k)\|_{L^2(\Sigma_t)} \\ & + \delta^{\frac12}\|\partial\!\!\!/(\rho_k\partial\widetilde{\phi}_k)\|_{L^2(\Sigma_t)} + \delta^{-\frac12}\|\partial\!\!\!/_{u_{k'}}(\rho_k\partial\widetilde{\phi}_k)\|_{L^2(\Sigma_t)}, \end{aligned} \tag{1.28}$$

[4] We overlook here the ambiguity in whether the L^2, C^θ, norms are taken with respect to the (x^1, x^2) coordinates or the $(u_k, u_{k'})$ coordinates, since we showed in [25] that $(x^1, x^2) \mapsto (u_k, u_{k'})$ is a C^1 diffeomorphism. A similar comment applies to (1.28) below.

where ρ_k is a cutoff as in (1.26).

Notice that (1.27) and (1.28) in particular gives the global Lipschitz estimate (cf. (1.14)):

$$\|\partial\widetilde{\phi}_k\|_{L^\infty(\Sigma_t)} \lesssim \text{RHSs of (1.27) and (1.28)}. \tag{1.29}$$

The reader may want to compare (1.28) with (1.26). The two new δ-weighted terms $\delta^{-\frac{1}{2}}\|\rho_k\partial\widetilde{\phi}_k\|_{L^2(\Sigma_t)}$ and $\delta^{\frac{1}{2}}\|\not\partial_{u_{k'}}\not\partial(\rho_k\partial\widetilde{\phi}_k)\|_{L^2(\Sigma_t)}$ allow us to sum over all frequencies. In fact, this even allows us to control a Besov norm $\|\rho_k\partial\widetilde{\phi}_k\|_{B^0_{\infty,1}(\mathbb{R}^2)}$, which is crucial for closing an endpoint elliptic estimate in part I; see [25, Section 1.1.4]. Notice also that $\delta^{-\frac{1}{2}}\|\rho_k\partial\widetilde{\phi}_k\|_{L^2(\Sigma_t)} \lesssim \epsilon$ by (1.21).

By (1.27) and (1.29), proving that ϕ is Lipschitz uniformly-in-δ with additional Hölder regularity away from the δ-impulsive waves reduces to showing RHSs of (1.27) and (1.28) $\lesssim \epsilon$, and will thus require the following main higher order estimates

$$\|\not\partial_{u_{k'}}\langle D_{u_k,u_{k'}}\rangle^{s''}\partial\widetilde{\phi}_k\|_{L^2(\Sigma_t)} \lesssim \epsilon, \quad \delta^{\frac{1}{2}}\|\not\partial_{u_{k'}}\not\partial(\rho_k\partial\widetilde{\phi}_k)\|_{L^2(\Sigma_t)} \lesssim \epsilon. \tag{1.30}$$

Recalling that $\not\partial_{u_{k'}}$ can be thought of as a good derivative, we see that the first bound is an estimate combining fractional and good geometric derivatives while the second bound is a higher order δ-weighted estimates involving a good geometric derivative.

The most difficult part of the paper is then to obtain the bounds in (1.30) under the very limited regularity of the metric. We will explain these L^2 estimates in the next subsection.

1.1.6 The higher order energy estimates

The main higher order energy estimates. We now explain the higher order energy estimates we prove to obtain (1.30). Corresponding to the first term in (1.30), we prove

$$\|\partial E_k\langle D_x\rangle^{s''}\widetilde{\phi}_k\|_{L^2(\Sigma_t)} \lesssim \epsilon, \quad \|\partial L_k\langle D_x\rangle^{s''}\widetilde{\phi}_k\|_{L^2(\Sigma_t)} \lesssim \epsilon. \tag{1.31}$$

Corresponding to the second term in (1.30), we prove

$$\|\partial E_k\partial_i\widetilde{\phi}_k\|_{L^2(\Sigma_t)} \lesssim \epsilon\delta^{-\frac{1}{2}}, \quad \|\partial L_k L_k\widetilde{\phi}_k\|_{L^2(\Sigma_t)} \lesssim \epsilon\delta^{-\frac{1}{2}}. \tag{1.32}$$

One can think that the E_k's in the first terms in (1.27), (1.28) above are the good derivatives $\not\partial_{u_{k'}}$, since $\not\partial_{u_{k'}}$ is parallel to E_k. Once (1.31) and (1.32) are obtained, the bounds from [25] allow us to control all necessary commutator terms (even though some of them are top order), convert (1.31)–(1.32) into estimates in the $(u_k, u_{k'})$ coordinate system, and to apply them for (1.30); see Sect. 15.

(1.31) and (1.32) are satisfied initially. Notice that (1.31) and (1.32) are consistent with the initial regularity of the wave. In particular, (1.31) is a statement that the fractional regularity energy estimate (1.18) still holds after a suitable commutation with the good derivatives E_k and L_k. The main challenge, however, is to *propagate* such regularity *with only very limited regularity of the metric*.

The estimates involving L_k in (1.31) and (1.32). Furthermore, notice that only the respective first bounds in (1.31) and (1.32) are used for the anisotropic Sobolev embedding. However, in order to handle some commutators that arise, it is important to simultaneously prove the second bounds in (1.31) and (1.32).

Ideas of proof of (1.31). We now explain the proof of (1.31).

- To prove (1.31), we bound the commutator terms $[\Box_g, E_k\langle D_x\rangle^{s''}]\widetilde{\phi}_k$ and $[\Box_g, L_k\langle D_x\rangle^{s''}]\widetilde{\phi}_k$. It is important to both (1) use fractional derivatives with respect to the elliptic gauge (as opposed to geometric) coordinates and (2) commute with $\langle D_x\rangle^{s''}$ first before commuting with the geometric vector fields. This way we exploit the better regularity of the metric components in the elliptic gauge. (Indeed, we will not be able to control either $[\Box_g, E_k\langle D_{u_k,u_{k'}}\rangle^{s''}]\widetilde{\phi}_k$ or $[\Box_g, \langle D_x\rangle^{s''}E_k]\widetilde{\phi}_k$.)
- The commutator term $[\Box_g, E_k\langle D_x\rangle^{s''}]\widetilde{\phi}_k$ schematically gives rise to error terms of the form

$$(\langle D_x\rangle^{s''}\partial\partial_i\mathfrak{g})(\partial\widetilde{\phi}_k),\quad (\partial\partial_i\mathfrak{g})(\partial\langle D_x\rangle^{s''}\widetilde{\phi}_k),\quad (\partial\mathfrak{g})(\partial^2\langle D_x\rangle^{s''}\widetilde{\phi}_k). \tag{1.33}$$

  The terms $(\langle D_x\rangle^{s''}\partial\partial_i\mathfrak{g})(\partial\widetilde{\phi}_k)$ can be controlled using the metric bound (1.7) together with (1.14). To control the terms $(\partial\partial_i\mathfrak{g})(\partial\langle D_x\rangle^{s''}\widetilde{\phi}_k)$, we use (1.18) and combine it with (1.7) (recall that $0 < s'' < s' < \frac{1}{2}$). There is a slight subtlety here: the reason that we need to introduce two different exponents $0 < s'' < s' < \frac{1}{2}$ and estimate $\partial\langle D_x\rangle^{s'}\widetilde{\phi}_k$, $\partial E_k\langle D_x\rangle^{s''}\widetilde{\phi}_k$, $\partial L_k\langle D_x\rangle^{s''}\widetilde{\phi}_k$ with the slightly different order of derivatives is because for the term $(\partial_t\partial_i\mathfrak{g})(\partial\langle D_x\rangle^{s''}\widetilde{\phi}_k)$, we do not have L^∞ bounds for $\partial_t\partial_i\mathfrak{g}$ (see (1.7)).
- The third type of error terms in (1.33), i.e. the terms $(\partial\mathfrak{g})(\partial^2\langle D_x\rangle^{s''}\widetilde{\phi}_k)$, are more subtle because we do not control general derivatives $\partial^2\langle D_x\rangle^{s''}\widetilde{\phi}_k$. To close our argument, we need show that the only such term arising in the commutator is schematically of the form $\partial E_k\langle D_x\rangle^{s''}\widetilde{\phi}_k$. To achieve this, we need to give a sharp expression for the commutator with fractional derivatives to isolate the main $\partial E_k\langle D_x\rangle^{s''}\widetilde{\phi}_k$ term. This in turn requires a refinement of the usual Kato–Ponce type commutator estimates; see already Proposition 12.9.
- When showing that the top-order derivative $\partial^2\langle D_x\rangle^{s''}\widetilde{\phi}_k$ from the above bullet point is morally $\partial E_k\langle D_x\rangle^{s''}\widetilde{\phi}_k$, the term we obtain is $E_k\langle D_x\rangle^{s''-2}\partial^3_{i\nu\beta}\widetilde{\phi}_k$. Since $\langle D_x\rangle^{-2}\partial^2_{ij}$ is a bounded operator on L^2-based Sobolev spaces, the term can be thought of as like $\partial E_k\langle D_x\rangle^{s''}\widetilde{\phi}_k$ if at least one of ν, β is a spatial index. However, the term becomes much more challenging when $(\nu,\beta) = (t,t)$ so that we need to use the wave equation to convert the times indices into spatial ones, and in the process we are required to handle a large number of commutator terms.
- Since we consider the nonlocal operator $\langle D_x\rangle$, the terms involved are no longer compacted supported. An additional challenge is that the metric components diverge logarithmically near spatial infinity (a difficulty well-known in the $(2+1)$-dimensional case); and moreover the components L^i_k, E^i_k of the commutators $L_k = L^i_k\partial_i$ and $E_k = E^i_k\partial_i$ also grow near spatial infinity. We therefore use

weighted estimates[5], including when understanding terms like $E_k\langle D_x\rangle^{s''-2}\partial^3_{i\nu\beta}\widetilde{\phi}_k$ described in the above point.

- Finally, the considerations above by themselves cannot control $[\Box_g, L_k\langle D_x\rangle^{s''}]\widetilde{\phi}_k$. This is because L_k (in the ∂_t, ∂_1, ∂_2 basis from the elliptic gauge (1.5)) has a ∂_t component and thus the result is a term schematically like

$$(\langle D_x\rangle^{s''}\partial_t^2\mathfrak{g})(\partial\widetilde{\phi}_k),$$

in addition to terms similar to those we encountered in $[\Box_g, E_k\langle D_x\rangle^{s''}]\widetilde{\phi}_k$. Recall that (see Sect. 1.1.2) we do not have <u>any</u> bounds for $\partial_t^2\mathfrak{g}$. To resolve this issue, we note that such a term can be traced back to a total ∂_t-derivative, i.e. we can write

$$\Box_g L_k\langle D_x\rangle^{s''}\widetilde{\phi}_k = F + \partial_t C. \tag{1.34}$$

While $\partial_t C$ cannot be controlled, F, C and $\partial_i C$ can be controlled in $L^2(\Sigma_t)$ using the same methods as for $[\Box_g, E_k\langle D_x\rangle^{s''}]\widetilde{\phi}_k$. Now the key observation is that in the energy estimate, we schematically have a bulk integral of the form

$$\int(\partial_t L_k\langle D_x\rangle^{s''}\widetilde{\phi}_k)(\partial_t C). \tag{1.35}$$

To address (1.35), we *integrate by parts* in t. For the $\partial_t^2 L_k\langle D_x\rangle\widetilde{\phi}_k$ term, we can use the wave equation (1.34) so that up to lower order terms, we obtain three terms to be controlled

$$\int(\partial^2_{\nu i}L_k\langle D_x\rangle^{s''}\widetilde{\phi}_k)C + \int FC + \int C^2. \tag{1.36}$$

For the first term, we integrate by parts again in the spatial ∂_i derivative. We can thus bound these terms using the estimates we have for F, C and $\partial_i C$.

Ideas of proof of (1.32). Finally, we explain the proof of (1.32).

- Similar to the proof of (1.31), the exact choice of commutators matters. We will use $E_k\partial_i$ and L_k^2 as commutators, so that we need to bound $[\Box_g, E_k\partial_i]\widetilde{\phi}_k$ and $[\Box_g, L_k^2]\widetilde{\phi}_k$. By contrast, we could for instance neither control $[\Box_g, E_kE_k]\widetilde{\phi}_k$, $[\Box_g, \partial_i E_k]\widetilde{\phi}_k$ (since we lack general second derivative control of χ_k and η_k) nor $[\Box_g, L_k\partial_i]\widetilde{\phi}_k$ (since we lack L^∞ estimates for $\partial_t\partial_i\mathfrak{g}$).
- Terms that arise in $[\Box_g, E_k\partial_i]\widetilde{\phi}_k$ are schematically

$$\partial\mathfrak{g}E_i\partial^2\widetilde{\phi}_k,\ \partial\mathfrak{g}L_k^2\partial\widetilde{\phi}_k,\ \partial^2\mathfrak{g}\partial^2\widetilde{\phi}_k,\ \partial^3\mathfrak{g}\partial\widetilde{\phi}_k.$$

[5] We would like to thank an anonymous referee for suggesting us to handle this instead with a commutator of the form $\varpi\langle D_x\rangle^{s''}$, where ϖ is compactly supported. While we have not implemented this, we do believe that this would lead to some simplifications of our arguments.

As in many of the previous estimates, it is important that these terms have some structure. First, in $\partial^2\mathfrak{g}$ and $\partial^3\mathfrak{g}$ there are at most one ∂_t derivative (recall that we do not control $\partial_t^2\mathfrak{g}$). Second, because we do not control $\partial_i\partial_t\mathfrak{g}$ in L^∞ (see (1.7)), we would not be able to bound $\partial_i\partial_t\mathfrak{g}\,\partial^2\widetilde{\phi}_k$ in general. Fortunately, the commutator has a useful structure in that only $\partial_i\partial_t\mathfrak{g}\,\partial L_k\widetilde{\phi}_k$ or $\partial_i\partial_t\mathfrak{g}\,\partial E_k\widetilde{\phi}_k$ arise.

- There are some further subtleties in the bounds for $[\Box_g, L_k^2]\widetilde{\phi}_k$.
  - $[\Box_g, L_k^2]\widetilde{\phi}_k$ contains terms with second derivatives of χ_k and η_k (which we do not in general control). Importantly, exactly because we are commuting with L_k twice, one of the two derivatives on χ_k and η_k must be L_k so that we can use (1.10).
  - Another dangerous term that arises is $(L_kL_kX_k\log N)(L_k\widetilde{\phi}_k)$, since schematically it is of the form $\partial_t^2\partial_i\mathfrak{g}$ (recall we do not have any control over two time derivatives of $\mathfrak{g}$!). This can be treated with an integration by parts argument similar to (1.34)–(1.36) in the proof of (1.31).

1.2 Comments and related works

We refer the reader to the introduction of [25] for discussions on impulsive gravitational waves and other related works in general relativity. Instead, we restrict ourselves to discussing previous works on wave estimates (for linear and nonlinear wave equations) related to those in this paper and how our work connects to this existing literature.

1.2.1 Geometric and harmonic analysis techniques for quasilinear wave equations

As we saw from Sect. 1.1, our result in this paper is based on a combination of techniques from geometric analysis and harmonic analysis. Related techniques are used in many low-regularity problems for quasilinear wave equations. We refer the readers to [2, 12, 15–17, 30, 31, 37] for a sample of results.

In the specific context of low-regularity solutions to quasilinear hyperbolic equations featuring one or more singularities propagating along null hypersurfaces, geometric methods using well-chosen coordinate systems and commuting vector fields are often employed; see [3, 14, 21–24]. In the present paper, we extend the methods in these works but further combine them with techniques from harmonic analysis to handle the interaction of three (δ-)impulsive waves.

1.2.2 Linear wave equations with rough coefficients

While our main goal in this paper is to prove wave estimates so as to complete the program in [25], when taken on its own, the present paper concerns proving estimates for a linear scalar wave equation with rough coefficients. Indeed, as seen in Theorem 1.3, the main result in this paper takes the following form: assuming certain bounds on the metric and suitable commuting vector fields, then one can propagate δ-impulsive waves type estimates under the flow of the linear wave equation. Such a formulation does not explicitly refer to general relativity. In this context, let us also remark that the techniques we introduce can also be easily adapted to deal with linear wave equations of the form

$$g^{\nu\beta}\partial^2_{\nu\beta}\phi + B^\nu\partial_\nu\phi + V\phi = 0 \tag{1.37}$$

with suitable regularity assumptions on $g^{\nu\beta}$, B^ν and V.

Though not directly related to this paper, we mention a small sample of works concerning estimates for linear wave equations with rough coefficients; see [13, 34, 35].

1.2.3 Interactions of singularities for semilinear wave equations

Our main result Theorem 1.1 can be viewed as a result on the interaction of singularities. In the setup of (1.3), the nonlinear interaction is hidden in the coupling between the scalar wave and the metric. In the literature, interaction of singularity results are often studied for the following type of simpler semilinear models:

$$\Box\phi = F(\phi), \tag{1.38}$$

where $\Box$ is the standard wave operator on $\mathbb{R}^{2+1}$ and $F : \mathbb{R} \to \mathbb{R}$ is a smooth function. See for instance [4–9, 18, 19, 26–28, 36, 38].

We remark that even though our methods are specifically designed to handle the rough metric, they can be easily applied to the model problem (1.38). Indeed, given initial data which represent three small-amplitude impulsive waves, we can smooth them out to δ-impulsive waves and introduce the decomposition $\phi = \phi_{reg} + \sum_{k=1}^{3}\widetilde{\phi}_k$, where

$$\Box\widetilde{\phi}_k = 0, \quad \Box\phi_{reg} = F(\phi).$$

(Notice that this is slightly different from Sect. 1.1.3.) It is then not difficult to see that one can propagate all the L^2 estimates that we prove in this paper. (In fact, the proof would be by far easier than that in this paper.) In particular, after taking the $\delta \to 0$ limit, this shows that the solution remains Lipschitz everywhere and has additional H^2 and Hölder regularity away from propagating singularities.

Let us note that it is also interesting to study interactions of singularities for semilinear wave equations where the nonlinearity depends also on the derivative of the solution [10, 29] (e.g., nonlinearities satisfying the classical null condition). However, the techniques introduced in this paper do not immediately apply to these models.

1.3 Outline of the paper

The remainder of the paper is structured as follows.

- In **Section** 2, we introduce the geometric setup, the equations in various coordinate systems and the main notations that will be used throughout the paper.

- In **Section** 3, we introduce the function spaces and norms that we will use in the paper.
- In **Section** 4, we give a precise version of our main results, whose rough versions were already presented as Theorem 1.1 and Theorem 1.2.
- In **Section** 5, we recall the main results of Part I [25], including the estimates for the metric components and for the null hypersurfaces.
- Most of the remainder of the paper is devoted to the proof of the energy estimates. We begin with some preliminaries towards the energy estimates.
  - In **Section** 6, we prove a technical integration by parts lemma that will be important in the proof of the energy estimates.
  - In **Section** 7, we give the proof of basic energy estimates with an arbitrary source term.
  - In **Section** 8, we compute and estimate the commutators between various vector fields and the wave operator in preparation for the proof of higher order energy estimates.
- Using the above preliminaries, we first prove energy estimates for $\widetilde{\phi}_k$ up to second derivatives (see Sect. 1.1.4):
  - In **Section** 9, we prove our basic energy estimates up to second derivatives.
  - In **Section** 10, we obtain improved energy estimates up to second derivatives (see (1.21), (1.22), (1.24)).
- We then prove higher order energy estimates for $\widetilde{\phi}_k$ (see (1.31) and (1.32)):
  - In **Section** 11, we prove energy estimates involving up to three derivatives of $\widetilde{\phi}_k$ (and ϕ_{reg}).
  - In **Section** 12, we prove fractional energy estimates for $\widetilde{\phi}_k$ and its good derivatives.
- In **Section** 13, we prove energy estimates for ϕ_{reg}, the regular part of the solution.
- In **Section** 14, we combine the results of all previous sections to conclude the proof of our energy estimates.
- In **Section** 15, we prove an anisotropic Sobolev embedding result. Using our energy estimates from Sect. 14, we apply the embedding result to obtain Lipschitz and improved Hölder bounds.

2 Summary of the geometric setup

In this section, we recall the geometric setup introduced in [25], as well as some useful computations.

In Sect. 2.1, we introduce the symmetry assumption and the elliptic gauge in the symmetry-reduced spacetime.

In Sect. 2.2, we introduce the eikonal functions u_k and the geometric vector field (L_k, E_k, X_k) for $k = 1, 2, 3$ (see Sect. 1.1.1). In Sect. 2.3, we compute the covariant derivatives and commutators with respect to these geometric vector fields.

In connection with eikonal functions, we introduce in Sect. 2.4 various different systems of coordinates. Some computations regarding the change of coordinates between these coordinate systems are given in Sect. 2.5.

2.1 Elliptic gauge and conformally flat spatial coordinates

Definition 2.1 *($\mathbb{U}(1)$ symmetry)* We say that a (3+1) Lorentzian manifold $(\mathcal{M} = \mathbb{R}^2 \times \mathbb{S}^1 \times I, {}^{(4)}g)$, where $I \subseteq \mathbb{R}$ is an open interval, has **polarized** $\mathbb{U}(1)$ **symmetry** if the metric ${}^{(4)}g$ can be expressed as:

$$ {}^{(4)}g = e^{-2\phi} g + e^{2\phi}(dx^3)^2, \tag{2.1}$$

where ϕ is a scalar function on $I \times \mathbb{R}^2$ and g is a $(2+1)$ Lorentzian metric[6] on $I \times \mathbb{R}^2$.

Definition 2.2 *(The foliation Σ_t)* Given a space-time as in Definition 2.1, we foliate the $2+1$ space-time $(I \times \mathbb{R}^2, g)$ with slices $\{\Sigma_t\}_{t\in I}$ where Σ_t are spacelike. We will later make a particular choice of t; see Definition 2.4. The metric can then be written as

$$ g = -N^2 dt^2 + \bar{g}_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt). \tag{2.2}$$

In the above, and the remainder of the paper, we use the convention that **lower case Latin indices refer the the spatial coordinates** (x^1, x^2), and **lower case Greek indices to refer to spacetime coordinates** $(x^0, x^1, x^2) := (t, x^1, x^2)$. **Repeated indices are always summed over**: repeated lower case Latin indices are summed over $i, j, \cdots = 1, 2$ and repeated lower case Greek indices are summed over $\mu, \nu, \cdots = 0, 1, 2$.

Definition 2.3 Given $(I \times \mathbb{R}^2, g)$ and $\{\Sigma_t\}_{t\in I}$ as in Definition 2.2.

1. (Space-time connection) Denote by ∇ the Levi–Civita connection for g.
2. (Induced metric) Denote by $\bar{g}$ the induced metric on the two-dimensional slice Σ_t.
3. (Spatial connection) Denote by $\bar{\nabla}$ the orthogonal projection of ∇ onto $T\Sigma_t$ and $T^*\Sigma_t$ (and their tensor products).[7]
4. (Normal to Σ_t) Denote by $\vec{n}$ the future-directed unit normal to Σ_t; $\vec{n}$ admits the following expression

$$ \vec{n} = \frac{\partial_t - \beta^i \partial_i}{N}. \tag{2.3}$$

and satisfies $g(\vec{n}, \vec{n}) = -1$. Note that we have the following commutation formula

$$ \left[\vec{n}, \partial_q\right] = \partial_q \log(N) \cdot \vec{n} + \frac{\partial_q \beta^i}{N} \cdot \partial_i. \tag{2.4}$$

[6] Note that since ϕ and g are defined on $\mathbb{R}^2 \times \mathbb{R}$, they do not depend on x^3, the coordinate on $\mathbb{S}^1$.

[7] We remark that for $Y, Z \in \Gamma(T\Sigma_t)$, $\bar{\nabla}_Y Z$ coincides with the derivative with respect to the Levi–Civita connection for $\bar{g}$.

Define also e_0 to be the vector field

$$e_0 = \partial_t - \beta^i \partial_i = N \cdot \vec{n}. \tag{2.5}$$

5. (Second fundamental form) Define K to be the second fundamental form on Σ_t:

$$K(Y, Z) = g(\nabla_Y \vec{n}, Z), \tag{2.6}$$

for every $Y,\ Z \in T\Sigma_t$.

Definition 2.4 *(Gauge conditions)* We define our gauge conditions (assuming already (2.1)) by the following:

1. For every $t \in I$, Σ_t is required to be maximal, i.e.

$$(\bar{g}^{-1})^{ij} K_{ij} = 0. \tag{2.7}$$

Note that (2.7) defines the coordinate t.
2. We choose the coordinate system on Σ_t so that $\bar{g}_{ij}$ is conformally flat: this gauge condition is written as

$$\bar{g}_{ij} = e^{2\gamma} \delta_{ij}, \tag{2.8}$$

where from now on δ denotes the Kronecker delta.

We collect some simple computations:

Lemma 2.5 *The following holds given g of the form (2.2) satisfying Definition 2.4:*

1. The inverse metric g^{-1} is given by

$$g^{-1} = \frac{1}{N^2} \begin{pmatrix} -1 & \beta^1 & \beta^2 \\ \beta^1 & N^2 e^{-2\gamma} - \beta^1\beta^1 & -\beta^1\beta^2 \\ \beta^2 & -\beta^1\beta^2 & N^2 e^{-2\gamma} - \beta^2\beta^2 \end{pmatrix}. \tag{2.9}$$

2. The space-time volume form associated to g is given by

$$dvol = Ne^{2\gamma} dx^1 dx^2 dt. \tag{2.10}$$

The volume form on the spacelike hypersurface Σ_t induced by g is given by

$$dvol_{\Sigma_t} = e^{2\gamma} dx^1 dx^2. \tag{2.11}$$

3. The wave operator (i.e. the Laplace–Beltrami operator associated to g) is given by

$$\Box_g f = \frac{-e_0^2 f}{N^2} + e^{-2\gamma} \delta^{ij} \partial^2_{ij} f + \frac{e_0 N}{N^3} e_0 f + \frac{e^{-2\gamma}}{N} \delta^{ij} \partial_i N \partial_j f$$

$$= -\vec{n}^2 f + e^{-2\gamma}\delta^{ij}\partial^2_{ij} f + \frac{e^{-2\gamma}}{N}\delta^{ij}\partial_i N \partial_j f, \tag{2.12}$$

where e_0 and $\vec{n}$ are as in Definition 2.3.

4. *The condition (2.7) can be rephrased as*

$$\partial_q \beta^q = 2e_0(\gamma), \tag{2.13}$$

5. *The second fundamental form is given by*[8]

$$K_{ij} = \frac{e^{2\gamma}}{2N} \cdot \left(\partial_q \beta^q \cdot \delta_{ij} - \partial_i \beta^q \cdot \delta_{qj} - \partial_j \beta^q \cdot \delta_{iq}\right) := -\frac{e^{2\gamma}}{2N}(\mathfrak{L}\beta)_{ij}, \tag{2.14}$$

where $\mathfrak{L}$ is the conformal Killing operator $(\mathfrak{L}\beta)_{ij} := -\partial_q \beta^q \cdot \delta_{ij} + \partial_i \beta^q \cdot \delta_{qj} + \partial_j \beta^q \cdot \delta_{iq}$.

Finally, we compute the connection coefficients with respect to $\{e_0, \partial_1, \partial_2\}$:

Lemma 2.6 *Given g of the form (2.2) satisfying Definition 2.4,*

$$g(\nabla_{e_0} e_0, e_0) = -N \cdot e_0 N, \tag{2.15}$$

$$g(\nabla_{e_0} e_0, \partial_i) = -g(\nabla_{\partial_i} e_0, e_0) = g(\nabla_{e_0} \partial_i, e_0) = N \cdot \partial_i N, \tag{2.16}$$

$$\begin{aligned} g(\nabla_{\partial_j} e_0, \partial_i) &= g(\nabla_{e_0} \partial_j, \partial_i) - e^{2\gamma} \cdot \partial_j \beta^l \delta_{il} = -g(\nabla_{\partial_j} \partial_i, e_0) \\ &= \frac{e^{2\gamma}}{2} \cdot \left(2e_0\gamma \cdot \delta_{ij} - \partial_i \beta^q \cdot \delta_{jq} - \partial_j \beta^q \cdot \delta_{iq}\right) \\ &= \frac{e^{2\gamma}}{2} \cdot \left(\partial_q \beta^q \cdot \delta_{ij} - \partial_i \beta^q \cdot \delta_{jq} - \partial_j \beta^q \cdot \delta_{iq}\right). \end{aligned} \tag{2.17}$$

Moreover,

$$\begin{aligned} \nabla_{\partial_i} \partial_j &= \frac{e^{2\gamma}}{2N} \cdot \left(\partial_q \beta^q \cdot \delta_{ij} - \partial_i \beta^q \cdot \delta_{jq} - \partial_j \beta^q \cdot \delta_{iq}\right)\vec{n} \\ &\quad + \left(\delta_i^q \partial_j \gamma + \delta_j^q \partial_i \gamma - \delta^{ij}\delta^{ql}\partial_l \gamma\right)\partial_q, \end{aligned} \tag{2.18}$$

$$\nabla_{e_0} e_0 = \frac{e_0 N}{N} \cdot e_0 + e^{-2\gamma}\delta^{ij} N \partial_i N \partial_j, \tag{2.19}$$

$$\begin{aligned} \nabla_{e_0} \partial_i &= \nabla_{\partial_i} e_0 + \partial_i \beta^j \partial_j = \frac{\partial_i N}{N} \cdot e_0 \\ &\quad + \frac{1}{2} \cdot \left(\partial_q \beta^q \cdot \delta_i^j + \partial_i \beta^j - \delta_{iq}\delta^{jl}\partial_l \beta^q\right)\partial_j, \end{aligned} \tag{2.20}$$

[8] This follows from

$$K_{ij} = \frac{2e_0(e^{2\gamma})}{N}\delta_{ij} - \frac{2e^{2\gamma}}{N}(\delta_{jl}\partial_i \beta^l + \delta_{il}\partial_j \beta^l)$$

together with (2.13).

2.2 Eikonal functions and null frames

We will define three eikonal functions together with null hypersurfaces and null frames. Each of these will later be chosen to be adapted to one propagating wave.

Definition 2.7 *(Eikonal functions)* Given a space-time $(I \times \mathbb{R}^2, g)$ of the form (2.2) satisfying Definition 2.4, define three eikonal functions u_k, $k = 1, 2, 3$ corresponding to the three impulsive waves, as the unique solutions to

$$(g^{-1})^{\nu\beta}\partial_\nu u_k \partial_\beta u_k = 0, \tag{2.21}$$

$$(u_k)_{|\Sigma_0} = a_k + c_{kj}x^j, \tag{2.22}$$

which satisfies $e_0 u_k > 0$. Here, $a_k, c_{kj} \in \mathbb{R}$ are constants obeying the following three conditions

$$\sqrt{c_{k1}^2 + c_{k2}^2} = 1, \tag{2.23}$$

$$\begin{aligned} |c_{k1} \cdot c_{k'1} + c_{k2} \cdot c_{k'2}| \geq \kappa_0, \\ |-c_{k2} \cdot c_{k'1} + c_{k1} \cdot c_{k'2}| = 1 - |c_{k1} \cdot c_{k'1} + c_{k2} \cdot c_{k'2}| \geq \kappa_0, \end{aligned} \tag{2.24}$$

for some fixed constant $\kappa_0 \in (0, \frac{\pi}{2})$ and for every $k \neq k' \in \{1, 2, 3\}$.

Definition 2.8 *(Sets associated with the eikonal functions)* Let u_k $(k = 1, 2, 3)$ satisfying (2.21) and (2.22) in $(I \times \mathbb{R}^2, g)$ be given.

1. For all $w \in \mathbb{R}$, define

$$C_w^k = \{(t, x) : u_k(t, x) = w\}, \quad C_{\leq w}^k := \bigcup_{u_k \leq w} C_{u_k}^k, \quad C_{\geq w}^k := \bigcup_{u_k \geq w} C_{u_k}^k. \tag{2.25}$$

2. For all $w_1, w_2 \in \mathbb{R}$, $w_2 \in \mathbb{R}$, define

$$S^k(w_1, w_2) := \bigcup_{w_1 \leq u_k \leq w_2} C_{u_k}^k. \tag{2.26}$$

3. Define (what we will later understand as) “the singular zone” for $\widetilde{\phi}_k$: for any $\delta_0 > 0$

$$S_{\delta_0}^k := S^k(-\delta_0, \delta_0) = \bigcup_{-\delta_0 \leq u_k \leq \delta_0} C_{u_k}^k. \tag{2.27}$$

Definition 2.9 *(Definition of the null frame)*

1. Define the null vector L_k^{geo} associated to the eikonal function u_k by

$$L_k^{geo} = -(g^{-1})^{\nu\beta}\partial_\beta u_k \cdot \partial_\nu. \tag{2.28}$$

2. Define L_k to be the vector field parallel to L_k^{geo} which satisfies $L_k t = N^{-1}$, i.e.

$$L_k = \mu_k \cdot L_k^{geo}, \quad \mu_k = (N \cdot L_k^{geo} t)^{-1}. \tag{2.29}$$

3. Define the vector field X_k to be the unique vector tangential to Σ_t which is everywhere orthogonal (with respect to $\bar{g}$) to $C_{u_k}^k \cap \Sigma_t$ and such that $g(X_k, L_k) = -1$.
4. Define E_k to be the unique vector field which is tangent to $C_{u_k}^k \cap \Sigma_t$, satisfies $g(E_k, E_k) = 1$ and such that (X_k, E_k) has the same orientation as (∂_1, ∂_2).

Lemma 2.10 *[25, Lemma 2.11]*

1. L_k^{geo} is null and geodesic, i.e.

$$g(L_k^{geo}, L_k^{geo}) = 0, \quad \nabla_{L_k^{geo}} L_k^{geo} = 0. \tag{2.30}$$

2. The following holds:

$$L_k u_k = E_k u_k = 0, \quad E_k t = X_k t = 0, \quad L_k t = N^{-1}, \quad X_k u_k = \mu_k^{-1}. \tag{2.31}$$

3. The normal $\vec{n}$ can be expressed in terms of X_k and L_k as:

$$\vec{n} = L_k + X_k. \tag{2.32}$$

4. The triplet (X_k, E_k, L_k) forms a null frame, i.e. it satisfies

$$\begin{aligned} &g(L_k, X_k) = -1, \quad g(E_k, L_k) = g(E_k, X_k) = g(L_k, L_k) = 0, \\ &g(E_k, E_k) = g(X_k, X_k) = 1. \end{aligned} \tag{2.33}$$

5. g^{-1} can be given in terms of the (X_k, E_k, L_k) frame by

$$g^{-1} = -L_k \otimes L_k - L_k \otimes X_k - X_k \otimes L_k + E_k \otimes E_k. \tag{2.34}$$

2.3 Ricci coefficients with respect to the $\{X_k, E_k, L_k\}$ frame

We now define some Ricci coefficients in terms of the frame $\{X_k, E_k, L_k\}$:

$$\chi_k = g(\nabla_{E_k} L_k, E_k) = -g(\nabla_{E_k} E_k, L_k), \tag{2.35}$$
$$\eta_k = g(\nabla_{X_k} L_k, E_k) = -g(\nabla_{X_k} E_k, L_k). \tag{2.36}$$

All the other Ricci coefficients can, in fact, be determined from χ_k, η_k and contractions of K.

Lemma 2.11 *[25, Lemma 2.19] The following identities hold:*

$$\nabla_{E_k} L_k = \chi_k \cdot E_k - K(E_k, X_k) L_k, \tag{2.37}$$
$$\nabla_{L_k} E_k = (E_k \log(N) - K(E_k, X_k)) \cdot L_k, \tag{2.38}$$

$$
\begin{aligned}
[E_k, L_k] &= \chi_k \cdot E_k - E_k \log(N) \cdot L_k. && (2.39)\\
\nabla_{E_k} X_k &= K(E_k, X_k) X_k \\
&\quad + (K(E_k, E_k) - \chi_k) \cdot E_k + K(E_k, X_k) L_k, && (2.40)\\
\nabla_{X_k} E_k &= \eta_k X_k + K(E_k, X_k) L_k, && (2.41)\\
[E_k, X_k] &= (K(E_k, X_k) - \eta_k) \cdot X_k \\
&\quad + (K(E_k, E_k) - \chi_k) \cdot E_k, && (2.42)\\
\nabla_{L_k} X_k &= (-K(E_k, X_k) + E_k \log N) \cdot E_k \\
&\quad -(K(X_k, X_k) - X_k \log(N)) \cdot X_k - (K(X_k, X_k) - X_k \log(N)) \cdot L_k, && (2.43)\\
\nabla_{X_k} L_k &= \eta_k \cdot E_k - K(X_k, X_k) \cdot L_k, && (2.44)\\
[L_k, X_k] &= -(K(E_k, X_k) - E_k \log N + \eta_k) \cdot E_k \\
&\quad - (K(X_k, X_k) - X_k \log(N)) \cdot X_k + X_k \log(N) \cdot L_k, && (2.45)\\
\nabla_{E_k} E_k &= \chi_k \cdot X_k + K(E_k, E_k) \cdot L_k, && (2.46)\\
\nabla_{X_k} X_k &= K(X_k, X_k) \cdot X_k + (K(E_k, X_k) - \eta_k) \cdot E_k \\
&\quad + K(X_k, X_k) \cdot L_k, && (2.47)\\
\nabla_{L_k} L_k &= (K(X_k, X_k) - X_k \log(N)) \cdot L_k. && (2.48)
\end{aligned}
$$

2.4 Geometric coordinate systems (u_k, θ_k, t_k) and $(u_k, u_{k'})$

2.4.1 Spacetime coordinate system (u_k, θ_k, t_k)

We now introduce the coordinate θ_k such that (u_k, θ_k, t_k) is a regular coordinate system on $I \times \mathbb{R}^2$.

Definition 2.12 1. Given u_k satisfying (2.21)–(2.22), and fixing some constants b_k, define θ_k by

$$L_k \theta_k = 0, \tag{2.49}$$

$$(\theta_k)_{|\Sigma_0} = b_k + c^{\perp}_{kj} x^j, \tag{2.50}$$

where $c^{\perp}_{k1} = -c_{k2}$ and $c^{\perp}_{k2} = c_{k1}$, and c_{ki} are the constants in (2.22).

2. Let $t_k = t$.
3. Denote by $(\partial_{u_k}, \partial_{\theta_k}, \partial_{t_k})$ the coordinate vector fields in the (u_k, θ_k, t_k) coordinate system. (Note that we continue to use ∂_t to denote the coordinate derivative in the (x^1, x^2, t) coordinate system of Sect. 2.1.)

Lemma 2.13 *[25, Lemma 2.13] Defining $\Theta_k = (E_k \theta_k)^{-1}$ and $\Xi_k = X_k \theta_k$, we have*

$$L_k = \frac{1}{N} \cdot \partial_{t_k}, \quad E_k = \Theta_k^{-1} \cdot \partial_{\theta_k}, \quad X_k = \mu_k^{-1} \cdot \partial_{u_k} + \Xi_k \cdot \partial_{\theta_k}. \tag{2.51}$$

Lemma 2.14 *[25, (2.46)] The metric g in the (t_k, u_k, θ_k) coordinate system is given by*

$$g = \Theta_k^2\, d\theta_k^2 - 2\mu_k N\, dt_k\, du_k - 2\mu_k \Xi_k \Theta_k^2\, du_k\, d\theta_k + \mu_k^2(1 + \Xi_k^2\Theta_k^2)\, du_k^2. \tag{2.52}$$

2.4.2 Spatial coordinate system $(u_k, u_{k'})$ on Σ_t

Fix $k,\ k' \in \{1, 2, 3\}$ with $k \neq k'$. Introduce the spatial coordinate system $(u_k, u_{k'})$. So as to distinguish it from other coordinate derivatives, we define the coordinate vector fields on Σ_t in the $(u_k, u_{k'})$ coordinate system by $(\partial\!\!\!/_{u_k}, \partial\!\!\!/_{u_{k'}})$.

We now express $(\partial\!\!\!/_{u_k}, \partial\!\!\!/_{u_{k'}})$ in terms of (X_k, E_k) in the following lemma:

Lemma 2.15 *[25, Lemma 2.18] The vector fields X_k and E_k can be expressed in the $(u_k, u_{k'})$ coordinate system as follows:*

$$X_k = \mu_k^{-1} \cdot \partial\!\!\!/_{u_k} + \mu_{k'}^{-1} \cdot g(X_k, X_{k'}) \cdot \partial\!\!\!/_{u_{k'}}, \tag{2.53}$$

$$E_k = \mu_{k'}^{-1} \cdot g(E_k, X_{k'}) \cdot \partial\!\!\!/_{u_{k'}}. \tag{2.54}$$

The above transformation can be inverted to give

$$\partial\!\!\!/_{u_{k'}} = \mu_{k'} \cdot g(E_k, X_{k'})^{-1} E_k, \tag{2.55}$$

$$\partial\!\!\!/_{u_k} = \mu_k X_k - \frac{\mu_k \cdot g(X_k, X_{k'})}{g(E_k, X_{k'})} \cdot E_k. \tag{2.56}$$

2.5 Transformations between different vector field bases and different coordinate systems

2.5.1 Relations on Σ_t between (X_k, E_k) and the elliptic coordinate vector fields (∂_1, ∂_2)

Recall that we fixed the orientation of (X_k, E_k) to be the same as (∂_1, ∂_2).

Lemma 2.16 *[25, Lemma 2.16] We have the following identity between E_k^i and X_k^i:*

$$E_k^1 = -X_k^2, \quad E_k^2 = X_k^1. \tag{2.57}$$

Moreover, the coordinate vector fields (∂_1, ∂_2) can be expressed in terms of (E_k, X_k) as:

$$\partial_1 = e^{2\gamma} \cdot \left(-X_k^2 \cdot E_k + E_k^2 \cdot X_k\right), \tag{2.58}$$

$$\partial_2 = e^{2\gamma} \cdot \left(X_k^1 \cdot E_k - E_k^1 \cdot X_k\right). \tag{2.59}$$

2.5.2 Elliptic coordinate derivatives of (u_k, θ_k, t_k)

Recall that $t_k = t$ so by definition of the elliptic gauge, $\partial_i t_k = 0$ and $\partial_t t_k = 1$. Now we are going to compute the non-trivial coefficients of the Jacobian between the elliptic coordinate system (x^1, x^2, t) and the geometric coordinate system (u_k, θ_k, t_k).

Lemma 2.17 *[25, Lemma 2.17] We have the following identities:*

$$\partial_i u_k = e^{2\gamma} \cdot \mu_k^{-1} \cdot \delta_{ij} X_k^j, \tag{2.60}$$

$$\partial_t u_k = \beta^q \partial_q u_k + N \cdot \mu_k^{-1}. \tag{2.61}$$

$$\partial_i \theta_k = e^{2\gamma} \delta_{ij} \cdot \left(\Theta_k^{-1} \cdot E_k^j + \Xi_k \cdot X_k^j \right), \tag{2.62}$$

$$\partial_t \theta_k = \beta^i \partial_i \theta_k + N \cdot \Xi_k = e^{2\gamma} \cdot \beta_j \cdot \left(\Theta_k^{-1} \cdot E_k^j + \Xi_k \cdot X_k^j \right) + N \cdot \Xi_k. \tag{2.63}$$

Moreover, for all vector field Y in the tangent space of Σ_t we have

$$Y u_k = \mu_k^{-1} \cdot g(Y, X_k). \tag{2.64}$$

3 Function spaces and norms

This section is devoted to the definition of all the function spaces and norms that are used throughout the remainder of the paper.

3.1 Pointwise norms

Definition 3.1 Define the following pointwise norms in the coordinate system (t, x^1, x^2) associated to the elliptic gauge (see Sect. 2.1):

1. Given a scalar function f, define

$$|\partial_x f|^2 := \sum_{i=1}^{2} (\partial_i f)^2, \quad |\partial f|^2 := \sum_{\beta=0}^{2} (\partial_\beta f)^2.$$

2. Given a higher order tensor field, define its norm and the norms of its derivatives componentwise, e.g.

$$|\beta|^2 := \sum_{i=1}^{2} |\beta^i|^2, \quad |\partial_x \beta|^2 := \sum_{i,j=1}^{2} |\partial_i \beta^j|^2, \quad |K|^2 := \sum_{i,j=1}^{2} |K_{ij}|^2,$$
$$|\partial K|^2 := \sum_{\beta=0}^{2} \sum_{i,j=1}^{2} |\partial_\beta K_{ij}|^2 \quad etc.$$

3. Higher derivatives are defined analogously, e.g.

$$|\partial^2 f|^2 := \sum_{\beta,\sigma=0}^{2} (\partial^2_{\beta\sigma} f)^2, \quad |\partial\partial_x K|^2 := \sum_{\substack{\beta=0,1,2\\ i,j,l=1,2}} |\partial_\beta \partial_i K_{jl}|^2, \quad etc.$$

3.2 Lebesgue and Sobolev spaces on Σ_t

Unless otherwise stated, all Lebesgue spaces are defined with respect to the measure $dx^1\, dx^2$ (which is in general different from the volume form induced by $\bar{g}$).

Before we define the norms, we define the following weight function.

Definition 3.2 *(Japanese brackets)* Define $\langle x\rangle := \sqrt{1+|x|^2}$ for $x \in \mathbb{R}^2$ and $\langle s\rangle := \sqrt{1+s^2}$ for $s \in \mathbb{R}$.

Definition 3.3 *(C^k and Hölder norms)* For $k \in \mathbb{N}\cup\{0\}$ and $s \in (0,1)$, define $C^k(\Sigma_t)$ to be the space of continuously spatially k-differentiable functions with respect to elliptic gauge coordinate vector fields ∂_x with norm

$$\|f\|_{C^k(\Sigma_t)} := \sum_{|\beta|\le k} \sup_{\Sigma_t} |\partial_x^\beta f|,$$

and define $C^{k,s}(\Sigma_t) \subseteq C^k(\Sigma_t)$ with spatial Hölder norm defined with respect to the elliptic gauge coordinates as

$$\|f\|_{C^{k,s}(\Sigma_t)} := \|f\|_{C^k(\Sigma_t)} + \sup_{\substack{x,y\in\Sigma_t\\ x\neq y}} \sum_{|\beta|=k} \frac{|\partial_x^\beta f(x) - \partial_x^\beta f(y)|}{|x-y|^s}.$$

In the later parts of the paper, we will need to consider Hölder spaces in both the (x^1, x^2) coordinates and the $(u_k, u_{k'})$ coordinates. When we need to emphasize the distinction, we will use the notation $C^{0,\sigma}_{x^1,x^2}(\Sigma_t) = C^{0,\sigma}(\Sigma_t)$ and $C^{0,\sigma}_{u_k,u_{k'}}(\Sigma_t)$.

Definition 3.4 *(Standard Lebesgue and Sobolev norms)*

1. For $k \in \mathbb{N}\cup\{0\}$ and $p \in [1,+\infty)$, define the (unweighted) Sobolev norms

$$\|f\|_{W^{k,p}(\Sigma_t)} := \sum_{|\beta|\le k} \left(\int_{\Sigma_t} |\partial_x^\beta f|^p(t,x^1,x^2)\, dx^1 dx^2 \right)^{\frac{1}{p}}.$$

For $k \in \mathbb{N}\cup\{0\}$, define

$$\|f\|_{W^{k,\infty}(\Sigma_t)} := \sum_{|\beta|\le k} \operatorname*{ess\,sup}_{(x^1,x^2)\in\Sigma_t} |\partial_x^\beta f|(t,x^1,x^2).$$

2. Define $L^p(\Sigma_t) := W^{0,p}(\Sigma_t)$ and $H^k(\Sigma_t) := W^{k,2}(\Sigma_t)$.

Definition 3.5 *(Fractional Sobolev norms)* For $s \in \mathbb{R} \setminus (\mathbb{N} \cup \{0\})$, define $H^s(\Sigma_t)$ by

$$\|f\|_{H^s(\Sigma_t)} := \|\langle D_x \rangle^s f\|_{L^2(\Sigma_t)}.$$

where $\langle D_x \rangle^s$ is defined via the (spatial) Fourier transform $\mathcal{F}$ (in the x coordinates) by $\mathcal{F}(\langle D_x \rangle^s f) := \langle \xi \rangle^s \mathcal{F}$.

Definition 3.6 *(Weighted norms)*

1. For $k \in \mathbb{N} \cup \{0\}$, $p \in [1, +\infty)$ and $r \in \mathbb{R}$, define the weighted Sobolev norms by

$$\|f\|_{W_r^{k,p}(\Sigma_t)} = \sum_{|\beta| \le k} \left(\int_{\Sigma_t} \langle x \rangle^{p \cdot (r + |\beta|)} |\partial_x^\beta f|^p (t, x^1, x^2) \, dx^1 \, dx^2 \right)^{\frac{1}{p}},$$

   with obvious modifications for $p = \infty$.
2. Define also $L_r^p(\Sigma_t) := W_r^{0,p}(\Sigma_t)$ and $H_r^k(\Sigma_t) := W_r^{k,2}(\Sigma_t)$. Moreover, define $C_r^k(\Sigma_t)$ as the closure of Schwartz functions under the $L_r^\infty(\Sigma_t)$ norm.

Definition 3.7 *(Mixed norms)* We will use mixed Sobolev norms, mostly in the (u_k, θ_k, t_k) coordinates in spacetime or the $(u_k, u_{k'})$ coordinates on Σ_t. Our convention is that the norm on the right is taken first. For instance,

$$\|f\|_{L^2_{u_{k'}} L^\infty_{u_k}(\Sigma_t)} = \left(\int_{u_{k'} \in \mathbb{R}} (\sup_{u_k \in \mathbb{R}} f(t, u_k, u_{k'}))^2 du_{k'} \right)^{\frac{1}{2}},$$

and analogously for other combinations.

Definition 3.8 *(Norms for derivatives)* We combine the notations in Definition 3.1 with those in Definitions 3.4–3.7. For instance, given a scalar function f,

$$\|\partial f\|_{L^2(\Sigma_t)} := \left(\int_{\Sigma_t} \sum_{\beta=0}^{2} |\partial_\beta f|^2 \, dx^1 \, dx^2 \right)^{\frac{1}{2}},$$

and similarly for $\|\partial_x f\|_{L^2(\Sigma_t)}$, $\|\partial \partial_x f\|_{L^2(\Sigma_t)}$, etc.

3.3 The Littlewood–Paley projection and Besov spaces in $(u_k, u_{k'})$ coordinates

Assume for this subsection that $k \neq k'$, so that $(u_k, u_{k'})$ forms a coordinate system on Σ_t.

Definition 3.9 *(Littlewood–Paley projection)* Define the Fourier transform in the $(u_k, u_{k'})$ coordinates by

$$(\mathcal{F}^{u_k, u_{k'}} f)(\xi_k, \xi_{k'}) = \iint_{\mathbb{R}^2} f(u_k, u_{k'}) e^{-2\pi i (u_k \xi_k + u_{k'} \xi_{k'})} \, du_k \, du_{k'}.$$

Let $\varphi : \mathbb{R}^2 \to [0,1]$ be radial, smooth such that $\varphi(\xi) = \begin{cases} 1 & \text{for } |\xi| \leq 1 \\ 0 & \text{for } |\xi| \geq 2 \end{cases}$, where $|\xi| = \sqrt{|\xi_k|^2 + |\xi_{k'}|^2}$.

Define $P_0^{u_k,u_{k'}}$ by

$$P_0^{u_k,u_{k'}} f := (\mathcal{F}^{u_k,u_{k'}})^{-1}(\varphi(\xi)\mathcal{F}^{u_k,u_{k'}} f),$$

and for $q \geq 1$, define $P_q^{u_k,u_{k'}} f$ by

$$P_q^{u_k,u_{k'}} f := (\mathcal{F}^{u_k,u_{k'}})^{-1}((\varphi(2^{-q}\xi) - (\varphi(2^{-q+1}\xi))\mathcal{F}^{u_k,u_{k'}} f(\xi)).$$

Definition 3.10 *(The Besov space $B^{u_k,u_{k'}}_{\infty,1}$)* Define the Besov norm $B^{u_k,u_{k'}}_{\infty,1}(\Sigma_t)$ by

$$\|f\|_{B^{u_k,u_{k'}}_{\infty,1}(\Sigma_t)} := \sum_{q\geq 0} \|P_q^{u_k,u_{k'}} f\|_{L^\infty(\Sigma_t)}.$$

3.4 Lebesgue norms on $C^k_{u_k}$ and $\Sigma_t \cap C^k_{u_k}$

Recall the definition of $C^k_{u_k}$ from Definition 2.8. The L^2 norm on $C^k_{u_k}$ is defined with respect to the measure $d\theta_k\, dt_k$.

Definition 3.11 (L^2 norm on $C^k_{u_k}$) For every fixed u_k, define the $L^2(C^k_{u_k}([0,T)))$ norm by

$$\|f\|_{L^2(C^k_{u_k}([0,T)))} := \left(\int_0^T \int_{\mathbb{R}} |f|^2(u_k,\theta_k,t_k)\, d\theta_k\, dt_k\right)^{\frac{1}{2}}.$$

The L^2 norm $\Sigma_t \cap C^k_{u_k}$ is defined with respect to the measure $d\theta_k$.

Definition 3.12 *(L^2 norm on $\Sigma_t \cap C^k_{u_k}$)* For every fixed t and u_k (and recall $t = t_k$), define the $L^2_{\theta_k}(\Sigma_t \cap C^k_{u_k})$ norm by

$$\|f\|_{L^2_{\theta_k}(\Sigma_t \cap C^k_{u_k})} := \left(\int_{\mathbb{R}} |f|^2(u_k,\theta_k,t_k)\, d\theta_k\right)^{\frac{1}{2}}.$$

4 Main results

4.1 Data assumptions for δ-impulsive waves

Recall that in our companion paper [25], we defined what it means for (ϕ, ϕ', γ, K) to be an **admissible initial data set featuring three δ-impulsive waves with parameters** $(\epsilon, s', s'', R, \kappa_0, \delta)$ ([25], Definition 4.8) for parameters in the ranges $\delta > 0$, $\epsilon > 0$,

$0 < s'' < s' < \frac{1}{2}$, $0 < s' - s'' < \frac{1}{3}$, $R > 10$ and $\kappa_0 > 0$. Here, ϕ, γ and K are the initial data for these quantities, while ϕ' will be the initial data of $\vec{n}\phi$ (where $\vec{n}$ is as in (2.3)).

In particular, we recall that this definition requires that there exists $\phi_{reg}, \phi'_{reg}, \widetilde{\phi}_k, \widetilde{\phi}'_k$ ($k = 1, 2, 3$) such that $\mathrm{supp}(\phi_{reg})$, $\mathrm{supp}(\phi'_{reg})$, $\mathrm{supp}(\widetilde{\phi}_k)$, $\mathrm{supp}(\widetilde{\phi}'_k) \subseteq B(0, \frac{R}{2}) := \{(x^1, x^2) \in \Sigma_0, \sqrt{(x^1)^2 + (x^2)^2} < \frac{R}{2}\}$ and $\mathrm{supp}(\widetilde{\phi}_k) \cup \mathrm{supp}(\widetilde{\phi}'_k) \subseteq \{u_k \geq -\delta\}$ (where u_k solves the equation (2.21), (2.22) with parameters obeying the conditions (2.23), (2.24) for $k = 1, 2, 3$) and moreover that the following (in)equations be satisfied on Σ_0:

$$\phi = \phi_{reg} + \sum_{k=1}^{3} \widetilde{\phi}_k, \qquad \phi' = \phi'_{reg} + \sum_{k=1}^{3} \widetilde{\phi}'_k, \tag{4.1}$$

$$\|\phi_{reg}\|_{H^{2+s'}(\Sigma_0)} + \|\phi'_{reg}\|_{H^{1+s'}(\Sigma_0)} \leq \epsilon, \tag{4.2}$$

$$\|\widetilde{\phi}_k\|_{W^{1,\infty}(\Sigma_0)} + \|\widetilde{\phi}_k\|_{H^{1+s'}(\Sigma_0)} + \|\widetilde{\phi}'_k\|_{L^\infty(\Sigma_0)} + \|\widetilde{\phi}'_k\|_{H^{s'}(\Sigma_0)} \leq \epsilon, \tag{4.3a}$$

$$\|E_k\widetilde{\phi}_k\|_{H^{1+s''}(\Sigma_0)} + \|E_k\widetilde{\phi}'_k\|_{H^{s''}(\Sigma_0)} + \|\widetilde{\phi}'_k - X_k\widetilde{\phi}_k\|_{H^{1+s''}(\Sigma_0)} \leq \epsilon, \tag{4.3b}$$

$$\begin{aligned} &\|\widetilde{\phi}_k\|_{H^2(\Sigma_0)} + \|\widetilde{\phi}'_k\|_{H^1(\Sigma_0)} \\ &\quad + \|E_k\widetilde{\phi}_k\|_{H^2(\Sigma_0)} + \|E_k\widetilde{\phi}'_k\|_{H^1(\Sigma_0)} + \|\widetilde{\phi}'_k - X_k\widetilde{\phi}_k\|_{H^2(\Sigma_0)} \leq \epsilon \cdot \delta^{-\frac{1}{2}}, \end{aligned} \tag{4.4}$$

$$\|\widetilde{\phi}_k\|_{H^2(\Sigma_0 \setminus S^k(-\delta,0))} + \|\widetilde{\phi}'_k\|_{H^1(\Sigma_0 \setminus S^k(-\delta,0))} \leq \epsilon. \tag{4.5}$$

In the sequel, we shall always consider solutions of the system of equations constituted of (4.1), and

$$Ric_{\mu\nu}(g) = 2\partial_\mu\phi\partial_\nu\phi, \tag{4.6}$$

$$\Box_g\tilde{\phi}_1 = \Box_g\tilde{\phi}_2 = \Box_g\tilde{\phi}_3 = \Box_g\phi_{reg} = 0, \tag{4.7}$$

with data on Σ_0 given by an **admissible initial data set featuring three δ-impulsive waves** (ϕ, ϕ', γ, K) **with parameters** $(\epsilon, s', s'', R, \kappa_0, \delta)$ in the above ranges and assuming $0 < \epsilon < \epsilon_0$, $0 < \delta < \delta_0$ with $0 < \delta_0 < \epsilon_0$ sufficiently small.

4.2 Bootstrap assumptions

We will only state the bootstrap assumptions for wave part of the solution. In Part I of our series, we also had bootstrap assumptions for geometric quantities (metric components, Ricci coefficients, etc.), but we then improved all of those assumptions in [25]. In other words, the results in [25] can be rephrased as saying that the bounds for the geometric quantities can be proven under the bootstrap assumptions (4.8)–(4.12c) for the wave part. (The more precise statements from [25] will be recalled below in Propositions 5.2, 5.3 and 5.5 and Lemmas 5.4, 5.6, 5.7.)

In our main estimates (see theorems in Sect. 4.3 below, we will work under the following bootstrap assumptions, where the solution is assumed to remain regular in $[0, T_B)$ for some $T_B \in (0, 1)$.

Energy estimates for ϕ_{reg}.

$$\sup_{0\leq t<T_B} (\|\phi_{reg}\|_{H^{s'}(\Sigma_t)} + \|\partial\phi_{reg}\|_{H^{s'}(\Sigma_t)} + \|\partial^2\phi_{reg}\|_{H^{s'}(\Sigma_t)}) \leq \epsilon^{\frac{3}{4}}. \tag{4.8}$$

Energy estimates for $\widetilde{\phi}_k$.

$$\sup_{0\leq t<T_B} (\|\partial\widetilde{\phi}_k\|_{L^2(\Sigma_t)} + \sum_{Z_k\in\{L_k,\,E_k\}} \|Z_k\partial\widetilde{\phi}_k\|_{L^2(\Sigma_t)}) \leq \epsilon^{\frac{3}{4}}, \tag{4.9a}$$

$$\sup_{0\leq t<T_B} \|\partial^2\widetilde{\phi}_k\|_{L^2(\Sigma_t)} \leq \epsilon^{\frac{3}{4}}\cdot\delta^{-\frac{1}{2}}, \tag{4.9b}$$

$$\sup_{0\leq t<T_B} \|\partial\langle D_x\rangle^{s'}\widetilde{\phi}_k\|_{L^2(\Sigma_t)} \leq \epsilon^{\frac{3}{4}}, \tag{4.9c}$$

$$\sup_{0\leq t<T_B} \|\partial E_k\partial_x\widetilde{\phi}_k\|_{L^2(\Sigma_t)} \leq \epsilon^{\frac{3}{4}}\cdot\delta^{-\frac{1}{2}}. \tag{4.9d}$$

Improved energy estimates for $\widetilde{\phi}_k$.

$$\sup_{0\leq t<T_B} (\|\partial\widetilde{\phi}_k\|_{L^2(\Sigma_t\cap S^k_{2\delta})} + \sum_{Z_k\in\{L_k,\,E_k\}} \|Z_k\partial\widetilde{\phi}_k\|_{L^2(\Sigma_t\cap S^k_{2\delta})}) \leq \epsilon^{\frac{3}{4}}\cdot\delta^{\frac{1}{2}}, \tag{4.10a}$$

$$\sup_{0\leq t<T_B} \|\partial^2\widetilde{\phi}_k\|_{L^2(\Sigma_t\setminus S^k_{\delta})} \leq \epsilon^{\frac{3}{4}}. \tag{4.10b}$$

Flux estimates for the wave variables.

$$\max_k \sup_{u_k\in\mathbb{R}} \sum_{Z_k\in\{L_k,E_k\}} (\|Z_k\partial_x\phi_{reg}\|_{L^2(C^k_{u_k}([0,T_B)))} + \|Z_k\phi_{reg}\|_{L^2(C^k_{u_k}([0,T_B)))}) \leq \epsilon^{\frac{3}{4}}, \tag{4.11a}$$

$$\max_{k,k'} \sup_{u_{k'}\in\mathbb{R}} \sum_{Z_{k'}\in\{L_{k'},\,E_{k'}\}} (\|Z_{k'}\partial_x\widetilde{\phi}_k\|_{L^2(C^{k'}_{u_{k'}}([0,T_B))\setminus S^k_{\delta})} + \|Z_{k'}\widetilde{\phi}_k\|_{L^2(C^{k'}_{u_{k'}}([0,T_B)))}) \leq \epsilon^{\frac{3}{4}}, \tag{4.11b}$$

$$\max_k \sup_{u_k\in\mathbb{R}} (\|L_k\partial_x\widetilde{\phi}_k\|_{L^2(C^k_{u_k}([0,T_B)))} + \|E_k^2\widetilde{\phi}_k\|_{L^2(C^k_{u_k}([0,T_B)))}) \leq \epsilon^{\frac{3}{4}}, \tag{4.11c}$$

$$\max_{k,k'} \sup_{u_{k'}\in\mathbb{R}} \sum_{Z_{k'}\in\{L_{k'},E_{k'}\}} \|Z_{k'}\partial_x\widetilde{\phi}_k\|_{L^2(C^{k'}_{u_{k'}}([0,T_B)))} \leq \epsilon^{\frac{3}{4}}\cdot\delta^{-\frac{1}{2}}. \tag{4.11d}$$

Besov and L^∞ estimates for the wave variables.

$$\sup_{0\leq t<T_B} \max_{(k,k'):k\neq k'} \|\partial\phi_{reg}\|_{B^{u_k,u_{k'}}_{\infty,1}(\Sigma_t)} \leq \epsilon^{\frac{3}{4}}, \tag{4.12a}$$

$$\sup_{0\leq t<T_B} \max_{k':k'\neq k} \|\partial\widetilde{\phi}_k\|_{B^{u_k,u_{k'}}_{\infty,1}(\Sigma_t)} \leq \epsilon^{\frac{3}{4}}, \tag{4.12b}$$

$$\sup_{0\leq t<T_B} (\|\partial\phi_{reg}\|_{L^\infty(\Sigma_t)} + \max_k \|\partial\widetilde{\phi}_k\|_{L^\infty(\Sigma_t)}) \leq \epsilon^{\frac{3}{4}}. \tag{4.12c}$$

4.3 Main wave estimates

The following are the main results for this paper. They are stated and assumed in Part I in order to prove the main existence result for δ-impulsive waves and impulsive waves.

4.3.1 The main Lipschitz and improved Hölder bounds

Definition 4.1 Define $\mathcal{E}(t)$ to be the following norm,

$$\begin{aligned}\mathcal{E}(t) :=\ & \|\partial\langle D_x\rangle^{s'}\widetilde{\phi}_k\|_{L^2(\Sigma_t)} + \|E_k\partial\widetilde{\phi}_k\|_{L^2(\Sigma_t)} + \|\partial E_k\langle D_x\rangle^{s''}\widetilde{\phi}_k\|_{L^2(\Sigma_t)}\\ &+ \delta^{\frac{1}{2}}(\|\partial^2\widetilde{\phi}_k\|_{L^2(\Sigma_t)} + \|\partial E_k\partial\widetilde{\phi}_k\|_{L^2(\Sigma_t)})\\ &+ \delta^{-\frac{1}{2}}\|\partial\widetilde{\phi}_k\|_{L^2(\Sigma_t\cap S^k_{2\delta})} + \delta^{-\frac{1}{2}}\|E_k\partial\widetilde{\phi}_k\|_{L^2(\Sigma_t\cap S^k_{2\delta})}\\ &+ \|\partial^2\widetilde{\phi}_k\|_{L^2(\Sigma_t\setminus S^k_\delta)} + \|\partial^2\langle D_x\rangle^{s'}\phi_{reg}\|_{L^2(\Sigma_t)}.\end{aligned}$$

The following is the main result for obtaining Lipschitz and improved Hölder bounds. It is stated in our previous paper [25] as [25, Theorem 7.3], and will be proven in this paper.

Theorem 4.2 *Let (ϕ,ϕ',γ,K) be an admissible initial data set featuring three δ-impulsive waves with parameters $(\epsilon,s',s'',R,\kappa_0)$ (as defined in [25, Definition 4.3]) for some $0<\delta<\delta_0$, $0<\epsilon<\epsilon_0$, $0<s''<s'<\frac{1}{2}$, $0<s'-s''<\frac{1}{3}$, $R>10$ and $\kappa_0>0$, where $0<\delta_0<\epsilon_0$ are additionally assumed to be sufficiently small.*

Assume the bootstrap assumptions of Sect. 4.2 i.e. (4.8)–(4.12c) hold for some $T_B\in(0,1)$. Then

$$\textit{LHSs of } (4.12a)\text{–}(4.12c) + \sup_{0\leq t<T_B} (\|\partial\phi_{reg}\|_{C^{0,\frac{s''}{2}}(\Sigma_t)} + \|\partial\widetilde{\phi}_k\|_{C^{0,\frac{s''}{2}}(\Sigma_t\cap C^k_{\geq\delta})}) \lesssim \mathcal{E},$$

where the implicit constant in $\lesssim$ depend only on s',s'',R,κ_0.

The proof of Theorem 4.2 will be carried out in Sect. 15.

4.3.2 Energy estimates

The following is the main wave energy estimates stated as [25, Theorem 7.4], which we will prove in this paper.

Theorem 4.3 *Let (ϕ, ϕ', γ, K) be an admissible initial data set featuring three δ-impulsive waves with parameters $(\epsilon, s', s'', R, \kappa_0)$ (as defined in [25, Definition 4.3]) for some $0 < \delta < \delta_0$, $0 < \epsilon < \epsilon_0$, $0 < s'' < s' < \frac{1}{2}$, $0 < s' - s'' < \frac{1}{3}$, $R > 10$ and $\kappa_0 > 0$, where $0 < \delta_0 < \epsilon_0$ are additionally assumed to be sufficiently small.*

Assume the bootstrap assumptions of Sect. 4.2 i.e. (4.8)– (4.12c) hold for some $T_B \in (0, 1)$. Then

1. *there exists $C = C(s', s'', R, \kappa_0) > 0$ such that (4.8)– (4.11d) hold with $C\epsilon$ in place of $\epsilon^{\frac{3}{4}}$,*
2. *the following estimate involving the norm $\mathcal{E}$ is satisfied:*

$$\mathcal{E} \lesssim \epsilon,$$

where the implicit constant in $\lesssim$ depend only on s', s'', R, κ_0.

3. *The following wave energy estimates are satisfied:*

$$\|\phi_{reg}\|_{H^{2+s'}(\Sigma_t)} + \|\partial_t \phi_{reg}\|_{H^{1+s'}(\Sigma_t)} \lesssim \epsilon, \tag{4.13a}$$

$$\|\widetilde{\phi}_k\|_{H^{1+s'}(\Sigma_t)} + \|\partial_t \widetilde{\phi}_k\|_{H^{s'}(\Sigma_t)} \lesssim \epsilon, \tag{4.13b}$$

$$\|L_k \widetilde{\phi}_k\|_{H^{1+s'}(\Sigma_t)} + \|E_k \widetilde{\phi}_k\|_{H^{1+s'}(\Sigma_t)} + \|\partial_t L_k \widetilde{\phi}_k\|_{H^{s'}(\Sigma_t)} + \|\partial_t E_k \widetilde{\phi}_k\|_{H^{s'}(\Sigma_t)} \lesssim \epsilon. \tag{4.13c}$$

$$\|\partial^2 \widetilde{\phi}_k\|_{L^2(\Sigma_t)} + \sum_{\substack{Y_k^{(1)}, Y_k^{(2)}, Y_k^{(3)} \in \{X_k, E_k, L_k\} \\ \exists i, Y_k^{(i)} \neq X_k}} \|Y_k^{(1)} Y_k^{(2)} Y_k^{(3)} \widetilde{\phi}_k\|_{L^2(\Sigma_t)} \lesssim \epsilon \cdot \delta^{-\frac{1}{2}}, \tag{4.14}$$

$$\|\phi\|_{H^3(\Sigma_t)} + \|\vec{n}\phi\|_{H^2(\Sigma_t)} \lesssim \epsilon \cdot \delta^{-\frac{1}{2}} + \|\phi\|_{H^3(\Sigma_0)} + \|\vec{n}\phi\|_{H^2(\Sigma_0)}, \tag{4.15}$$

$$\|\partial^2 \widetilde{\phi}_k\|_{L^2(\Sigma_t \setminus S_\delta^k)} \lesssim \epsilon, \tag{4.16}$$

where, as before, the implicit constant in $\lesssim$ depend only on s', s'', R, κ_0.

The proof of Theorem 4.3 will occupy most of this paper. The conclusion of the proof can be found in Sect. 14.

In view of the parameters that the implicit constants are allowed to depend on in Theorem 4.2 and Theorem 4.3, **from now on, constants $C > 0$ or implicit constants in $\lesssim$ are allowed to depend only on s', s'', R, κ_0. We will also often take ϵ_0 and δ_0 to be sufficiently small without further comments.**

5 Estimates from part I

In this section, we will assume that (ϕ, ϕ', γ, K) constitute an **admissible initial data set featuring three impulsive waves with parameters** $(\epsilon, s', s'', R, \kappa_0, \delta)$ (recall Sect. 4.1) for parameters in the ranges $0 < \delta < \delta_0$, $0 < \epsilon < \epsilon_0$, $0 < s'' < s' < \frac{1}{2}$, $0 < s' - s'' < \frac{1}{3}$, $R > 10$ and $\kappa_0 > 0$, with $0 < \delta_0 < \epsilon_0$ sufficiently small. Moreover, we will assume that the bootstrap assumptions of Sect. 4.2 i.e. (4.8)–(4.12c) hold for some $T_B \in (0, 1)$.

We recall the following results follow from [25, Theorem 7.1] and its proof.

Lemma 5.1 *[25, Lemma 8.1] The following holds on Σ_t for all $t \in [0, T_B)$ and for $k = 1, 2, 3$:*

1. $\mathrm{supp}(\phi_{reg})$, $\mathrm{supp}(\widetilde{\phi_k}) \subseteq B(0, R)$,
2. $\mathrm{supp}(\widetilde{\phi_k}) \subseteq \{(t, x) : u_k(t, x) \geq -\delta\}$.

Next, we collect some estimates for the metric components in the elliptic gauge proven in [25]. The first three statements are directly[9] from [25], while the fourth statement can be easily derived from the first three.

Proposition 5.2 *1. Defining $\alpha = 0.01$, the metric component quantities*

$$\mathfrak{g} \in \{e^{2\gamma} - 1,\ e^{-2\gamma} - 1,\ \beta^j,\ N - 1,\ N^{-1} - 1,\ g_{\nu\beta} - m_{\nu\beta},\ (g^{-1})^{\nu\beta} - m^{\nu\beta}\}$$

(where m is the Minkowski metric) satisfy the following estimates:

$$\sup_{0 \leq t < T_B} (\|\mathfrak{g}\|_{W^{2,\infty}_{-\alpha}(\Sigma_t)} + \|\partial_t \mathfrak{g}\|_{L^\infty_{-\alpha}(\Sigma_t)} + \|\partial_t \mathfrak{g}\|_{W^{1,\frac{2}{s'-s''}}_{-s'+s''-\alpha}(\Sigma_t)}) \lesssim \epsilon^{\frac{3}{2}}. \tag{5.1}$$

2. *[25, Proposition 9.21] Taking $\mathfrak{g}$ as in the previous part,*

$$\sup_{0 \leq t < T_B} \|\partial \partial_x^2 \mathfrak{g}\|_{L^2(\Sigma_t)} \lesssim \epsilon^{\frac{3}{2}} \delta^{-\frac{1}{2}}. \tag{5.2}$$

3. *[25, Propositions 9.8, 9.20] Let ϖ be a cutoff such that $\varpi \equiv 1$ on $B(0, 2R)$ and $\varpi \equiv 0$ on $\mathbb{R}^2 \setminus B(0, 3R)$. Then for $\mathfrak{g}$ as above,*

$$\sup_{0 \leq t < T_B} \|\langle D_x \rangle^{s'} \partial_x^2 \mathfrak{g}\|_{L^2(\Sigma_t)} + \|\langle D_x \rangle^{s'} (\varpi(\partial_x \partial_t \mathfrak{g}))\|_{L^2(\Sigma_t)} \lesssim \epsilon^{\frac{3}{2}}. \tag{5.3}$$

[9] Strictly speaking, to obtain the inequality $\|\langle D_x \rangle^{s'} (\varpi(\partial_x \partial_t \mathfrak{g}))\|_{L^2(\Sigma_t)} \lesssim \epsilon^{\frac{3}{2}}$ requires using Theorem 12.5 in addition to [25, Propositions 9.20]; we omit the straightforward details.

4. *Define* $\Gamma^\lambda = (g^{-1})^{\nu\beta}\Gamma^\lambda_{\nu\beta}$, *where* $\Gamma^\lambda_{\nu\beta}$ *are the Christoffel symbols. Then the following estimates hold:*

$$\begin{aligned}
&\sup_{0\le t<T_B}(\|\Gamma^\lambda\|_{L^\infty_{-3\alpha}(\Sigma_t)} + \|\Gamma^\lambda\|_{W^{1,\frac{2}{s'-s''}}_{-s'+s''-3\alpha}(\Sigma_t)}) \lesssim \epsilon^{\frac{3}{2}},\\
&\sup_{0\le t<T_B}\|\partial_x^2\Gamma^\lambda\|_{L^2(\Sigma_t\cap B(0,3R))} \lesssim \epsilon^{\frac{3}{2}}\delta^{-\frac{1}{2}},\\
&\sup_{0\le t<T_B}\|\langle D_x\rangle^{s'}(\varpi\,\partial_x\Gamma^\lambda)\|_{L^2(\Sigma_t)} \lesssim \epsilon^{\frac{3}{2}}.
\end{aligned} \tag{5.4}$$

Proposition 5.3 *[25, Lemma 8.2, Lemma 8.4, Proposition 10.5] The following estimates hold for* L^β_k, E^i_k *and* X^i_k*:*

$$\sup_{0\le t<T_B}(\|L^\beta_k\|_{L^\infty_{-\epsilon}(\Sigma_t)} + \|E^i_k\|_{L^\infty_{-\epsilon}(\Sigma_t)} + \|X^i_k\|_{L^\infty_{-\epsilon}(\Sigma_t)}) \lesssim 1, \tag{5.5}$$

$$\begin{aligned}
&\sup_{0\le t<T_B}(\|\partial_t L^t_k\|_{L^\infty_{-4\alpha}(\Sigma_t)} + \|\partial_t L^i_k\|_{L^\infty_{1-4\alpha}(\Sigma_t)} + \|\partial_x L^\beta_k\|_{L^\infty_{1-4\alpha}(\Sigma_t)}\\
&\quad + \|\partial X^i_k\|_{L^\infty_{1-4\alpha}(\Sigma_t)} + \|\partial E^i_k\|_{L^\infty_{1-4\alpha}(\Sigma_t)}) \lesssim \epsilon^{\frac{5}{4}},
\end{aligned} \tag{5.6}$$

$$\begin{aligned}
&\sup_{0\le t<T_B}(\|\partial^2 E^i_k\|_{L^2(\Sigma_t\cap B(0,3R))} + \|\partial^2 X^i_k\|_{L^2(\Sigma_t\cap B(0,3R))}\\
&\quad + \|\partial\partial_x L^\nu_k\|_{L^2(\Sigma_t\cap B(0,3R))}) \lesssim \epsilon^{\frac{5}{4}}.
\end{aligned} \tag{5.7}$$

Lemma 5.4 *[25, Lemma 8.3, Lemma 10.4] For any sufficiently regular function* f*, and for all* $(x,t)\in\mathbb{R}^2\times[0,T_B)$*:*

$$|\partial_i f|(x,t) \lesssim \langle x\rangle^\epsilon\,(|E_k f|(x,t) + |X_k f|(x,t)), \tag{5.8}$$

$$|\partial_t f|(x,t) \lesssim \langle x\rangle^\epsilon(|L_k f|(x,t) + |\partial_x f|(x,t)), \tag{5.9}$$

$$|\partial_t f|(x,t) \lesssim \langle x\rangle^\epsilon\,(|L_k f|(x,t) + |X_k f|(x,t)) + \langle x\rangle^{-1+\epsilon}|E_k f|(x,t), \tag{5.10}$$

and for second derivatives, the following estimates hold

$$\begin{aligned}
&\|\partial\partial_x f\|_{L^2(\Sigma_t\cap B(0,3R))}\\
&\lesssim \sum_{Y_k\in\{L_k,X_k,E_k\}}\sum_{Z_k\in\{X_k,E_k\}}\|Y_k Z_k f\|_{L^2(\Sigma_t\cap B(0,3R))} + \|\partial_x f\|_{L^2(\Sigma_t\cap B(0,3R))},
\end{aligned} \tag{5.11}$$

$$\begin{aligned}
&\|\partial^2 f\|_{L^2(\Sigma_t\cap B(0,3R))}\\
&\lesssim \sum_{Y_k,Z_k\in\{L_k,X_k,E_k\}}\|Y_k Z_k f\|_{L^2(\Sigma_t\cap B(0,3R))} + \|\partial f\|_{L^2(\Sigma_t\cap B(0,3R))}.
\end{aligned} \tag{5.12}$$

Proposition 5.5 *[25, Propositions 9.22, 10.1, 10.2, 10.3] The following estimates hold:*

$$\sup_{0\le t<T_B}(\|K\|_{L^\infty_{2-\alpha}(\Sigma_t)} + \|\partial_x K\|_{L^\infty_{2-\alpha}(\Sigma_t)} + \|\partial_t K\|_{L^{\frac{2}{s'-s''}}_{2-s'+s''+\alpha}(\Sigma_t)}) \lesssim \epsilon^{\frac{3}{2}}, \tag{5.13}$$

$$\sup_{0\le t<T_B}(\|\chi_k\|_{L^\infty_{1-\alpha}(\Sigma_t)}+\|\eta_k\|_{L^\infty_{1-\alpha}(\Sigma_t)}+\|L_k\chi_k\|_{L^\infty_{1-\alpha}(\Sigma_t)}+\|L_k\eta_k\|_{L^\infty_{1-\alpha}(\Sigma_t)})\lesssim \epsilon^{\frac{3}{2}}, \tag{5.14}$$

$$\sup_{0\le t<T_B, u_k\in\mathbb{R}}(\|\partial_x\chi_k\|_{L^2_{\theta_k}(\Sigma_t\cap C^k_{u_k})}+\|E_k\eta_k\|_{L^2_{\theta_k}(\Sigma_t\cap C^k_{u_k})})\lesssim \epsilon^{\frac{3}{2}}, \tag{5.15}$$

$$\sup_{0\le t<T_B}(\|L_k\partial_x\chi_k\|_{L^2(\Sigma_t\cap B(0,R))}$$

$$+\|L_kE_k\eta_k\|_{L^2(\Sigma_t\cap B(0,R))}+\sum_{\kappa_k\in\{\chi_k,\eta_k\}}\|L_k^2\kappa_k\|_{L^2(\Sigma_t\cap B(0,R))})\lesssim \epsilon^{\frac{3}{2}}, \tag{5.16}$$

$$\sup_{0\le t<T_B}(\|\log\mu_k-\gamma_{asymp}\omega(|x|)\log|x|\|_{L^\infty_{1-\alpha}(\Sigma_t)}+\|\partial_x\mu_k\|_{L^\infty_{1-\alpha}(\Sigma_t)})\lesssim \epsilon^{\frac{3}{2}}, \tag{5.17}$$

$$\sup_{0\le t<T_B}(\|\log(\Theta_k)$$

$$-\gamma_{asymp}\omega(|x|)\log|x|\|_{L^\infty_{1-2\alpha}(\Sigma_t)}+\|\langle x\rangle^{-\alpha}\partial_x\log\Theta_k\|_{L^2_{\theta_k}(\Sigma_t\cap C^k_{u_k})})\lesssim \epsilon^{\frac{3}{2}}, \tag{5.18}$$

where $\gamma_{asymp}\in[0,C\epsilon)$ is a constant defined by $\lim_{|x|\to\infty}\frac{\gamma|_{\Sigma_0}}{\log|x|}$; see [25, Definition 4.2].

The following lemma gives estimates on various changes of variables:

Lemma 5.6 *1. [25, Corollary 8.6] For any $k\neq k'$, the map $(x^1,x^2)\mapsto(u_k,u_{k'})$ is a C^1-diffeomorphism on Σ_t with entry-wise pointwise estimates independent of δ:*

$$|\partial_i u_k|,\ |\partial_i u_{k'}|\lesssim 1, \tag{5.19}$$

$$1\lesssim\left|\det\begin{bmatrix}\frac{\partial u_k}{\partial x^1} & \frac{\partial u_j}{\partial x^1}\\ \frac{\partial u_k}{\partial x^2} & \frac{\partial u_j}{\partial x^2}\end{bmatrix}\right|\lesssim 1. \tag{5.20}$$

2. [25, Proposition 8.7] For any $k=1,2,3$,

$$|\partial^2_{ij}u_k|\lesssim\epsilon^{\frac{5}{4}}. \tag{5.21}$$

3. [25, (2.11), (2.47), (7.2a), (7.2b), (7.3d), (7.3e)] The Jacobian determinant J_k corresponding to the transformation $(x^1,x^2)\to(u_k,\theta_k)$, defined by $du_k\wedge d\theta_k=J_k\,dx^1\wedge dx^2$, obeys the following estimate

$$\sup_{0\le t<T_B}(\|J_k\|_{L^\infty(\Sigma_t)}+\|J_k^{-1}\|_{L^\infty(\Sigma_t)})\lesssim 1. \tag{5.22}$$

Lastly, as a consequence of (2.24) we have the following estimate:

Lemma 5.7 *[25, (8.13)] For all $k\neq k'$ we have*

$$\frac{\kappa_0}{2}\le|g(E_k,X_{k'})|\le 2. \tag{5.23}$$

6 An integration by parts lemma

In this section, we prove an integration by parts lemma (Proposition 6.2) which will later be useful to control the energy (see Corollary 7.5).

The main purpose of the estimate in Proposition 6.2 will be to handle inhomogeneous terms in wave equations which can be written as an e_0 derivative. In other words suppose $\Box_g v = f_1 + h \cdot e_0 f_2$, we want to get rid of the time derivative $e_0 f_2$ and to replace it by $\partial_x f_2$ using the wave equation (2.12) (see the first term in the right-hand side of (6.3)).

As a first step towards Proposition 6.2, we first prove a simple lemma:

Lemma 6.1 *For any two smooth functions h_1, h_2 which are Schwartz class for every $t \in [0, T_B)$, the following holds for all $T \in (0, T_B)$:*

$$\left| \int_0^T \int_{\Sigma_t} h_2 \cdot e_0 h_1 \, dx dt + \int_0^T \int_{\Sigma_t} h_1 \cdot e_0 h_2 \, dx dt \right| \lesssim \sup_{t \in [0,T)} \|h_1\|_{L^2(\Sigma_t)} \|h_2\|_{L^2(\Sigma_t)}. \tag{6.1}$$

Proof Since $e_0 = \partial_t - \beta^i \partial_i$, an explicit computation gives

$$\begin{aligned} \int_0^T \int_{\Sigma_t} e_0 h_1 \cdot h_2 \, dx dt &= \int_0^T \int_{\Sigma_t} (\partial_i \beta^i \cdot h_1 \cdot h_2 - h_1 \cdot e_0 h_2) \, dx dt \\ &+ \int_{\Sigma_T} h_1 \cdot h_2 \, dx dt - \int_{\Sigma_0} h_1 \cdot h_2 \, dx dt. \end{aligned}$$

Using the L^∞ bound for $\partial_i \beta^i$ in Proposition 5.2, and applying Hölder's inequality, we obtain the desired estimate. □

The following is the main result of the section.

Proposition 6.2 *Let v be a smooth function which is Schwartz on Σ_t. Suppose $\Box_g v = f_1 + h \cdot e_0 f_2$.*

Assume that h satisfies the bounds

$$\|\langle x \rangle^{-\alpha} h\|_{L^\infty(\Sigma_t)} + \|\langle x \rangle^{-\alpha} \partial h\|_{L^\infty(\Sigma_t)} \lesssim 1. \tag{6.2}$$

Then, the following estimate holds for all $r \geq 1$ and all $T \in [0, T_B)$:

$$
\begin{aligned}
&|\int_0^T \int_{\Sigma_t} \langle x\rangle^{-2r} (\Box_g v)(e_0 v) e^{2\gamma}\, dx dt| \\
&\lesssim \sup_{t\in[0,T)} \|\langle x\rangle^{-(r+2\alpha)} \partial v\|_{L^2(\Sigma_t)} \|\langle x\rangle^{-\frac{r}{2}} f_2\|_{L^2(\Sigma_t)} \\
&+ \int_0^T (\|\langle x\rangle^{-\frac{r}{2}} f_1\|^2_{L^2(\Sigma_t)} + \|\langle x\rangle^{-\frac{r}{2}} f_2\|^2_{L^2(\Sigma_t)})\, dt \\
&+ \int_0^T \|\langle x\rangle^{-(r+2\alpha)} \partial v\|_{L^2(\Sigma_t)} \cdot \Big(\|\langle x\rangle^{-\frac{r}{2}} f_1\|_{L^2(\Sigma_t)} + \|\langle x\rangle^{-\frac{r}{2}} f_2\|_{L^2(\Sigma_t)} + \|\langle x\rangle^{-\frac{r}{2}} \partial_x f_2\|_{L^2(\Sigma_t)}\Big)\, dt.
\end{aligned}
\tag{6.3}
$$

Proof We first write

$$
\begin{aligned}
&\int_0^T \int_{\Sigma_t} \langle x\rangle^{-2r} (\Box_g v)(e_0 v) e^{2\gamma}\, dx dt \\
&= \int_0^T \int_{\Sigma_t} \langle x\rangle^{-2r} f_1 (e_0 v) e^{2\gamma}\, dx dt + \int_0^T \int_{\Sigma_t} \langle x\rangle^{-2r} h(e_0 f_2)(e_0 v) e^{2\gamma}\, dx dt \\
&= \int_0^T \int_{\Sigma_t} \langle x\rangle^{-2r} f_1 (e_0 v) e^{2\gamma}\, dx dt - \int_0^T \int_{\Sigma_t} \langle x\rangle^{-\frac{3r}{2}} h(e_0 \langle x\rangle^{-\frac{r}{2}}) f_2 (e_0 v) e^{2\gamma}\, dx dt \\
&\quad + \int_0^T \int_{\Sigma_t} \langle x\rangle^{-\frac{3r}{2}} h[e_0(\langle x\rangle^{-\frac{r}{2}} f_2)](e_0 v) e^{2\gamma}\, dx dt.
\end{aligned}
\tag{6.4}
$$

The first term in (6.4) can be easily controlled as follows, using the Cauchy–Schwarz inequality and the fact that $T \leq T_B \leq 1$:

$$
\Big| \int_0^T \int_{\Sigma_t} \langle x\rangle^{-2r} f_1 \cdot (e_0 v) e^{2\gamma}\, dx dt \Big| \lesssim \sup_{0\leq t\leq T} \|\langle x\rangle^{-\frac{r}{2}} f_1\|_{L^2(\Sigma_t)} \cdot \|\langle x\rangle^{-(r+2\alpha)} \partial v\|_{L^2(\Sigma_t)},
\tag{6.5}
$$

where we have bounded $\|\langle x\rangle^{-\frac{r}{2}+2\alpha} e^{2\gamma}\|_{L^\infty(\Sigma_t)} \lesssim 1$ and $\|\langle x\rangle^{-\frac{r}{2}+2\alpha} \beta^j e^{2\gamma}\|_{L^\infty(\Sigma_t)} \lesssim 1$ using (5.1).

For the second term in (6.4), notice that $e_0 \langle x\rangle^{-\frac{r}{2}} = \frac{r}{2} \beta_i x^i \langle x\rangle^{-\frac{r}{2}-2}$. Hence, by Hölder's inequality, Proposition 5.2 and (6.2), we get

$$
\begin{aligned}
&\Big| \int_0^T \int_{\Sigma_t} \langle x\rangle^{-\frac{3r}{2}} h(e_0 \langle x\rangle^{-\frac{r}{2}}) f_2 (e_0 v) e^{2\gamma}\, dx\, dt \Big| \\
&\lesssim \int_0^T \|\langle x\rangle^{-\alpha} h\|_{L^\infty(\Sigma_t)} \cdot \|\langle x\rangle^{-\frac{r}{2}} f_2\|_{L^2(\Sigma_t)} \cdot \|\langle x\rangle^{-(r+2\alpha)} \partial v\|_{L^2(\Sigma_t)}\, dt \\
&\lesssim \int_0^T \|\langle x\rangle^{-\frac{r}{2}} f_2\|_{L^2(\Sigma_t)} \cdot \|\langle x\rangle^{-(r+2\alpha)} \partial v\|_{L^2(\Sigma_t)}\, dt.
\end{aligned}
\tag{6.6}
$$

For the third term in (6.4), we integrate by parts. Using Lemma 6.1 with $h_1 = \langle x\rangle^{-\frac{3r}{2}} h(e_0 v)e^{2\gamma}$, $h_2 = \langle x\rangle^{-\frac{r}{2}} f_2$, and then using Hölder's inequality, Proposition 5.2 and (6.2), we obtain

$$\begin{aligned}
&\Big|\int_0^T \int_{\Sigma_t} \langle x\rangle^{-\frac{3r}{2}} \cdot h \cdot [e_0(\langle x\rangle^{-\frac{r}{2}} f_2)](e_0 v)e^{2\gamma}\, dx\, dt\Big| \\
&\lesssim \int_0^T [\|\langle x\rangle^{-\alpha} h\|_{L^\infty(\Sigma_t)} + \|\langle x\rangle^{-2\alpha}(e_0 h)\|_{L^\infty(\Sigma_t)}] \cdot \|\langle x\rangle^{-\frac{r}{2}} f_2\|_{L^2(\Sigma_t)} \cdot \|\langle x\rangle^{-(r+2\alpha)} e_0 v\|_{L^2(\Sigma_t)}\, dt \\
&\quad + \Big|\int_0^T \int_{\Sigma_t} \langle x\rangle^{-2r} h \cdot f_2(e_0^2 v)e^{2\gamma}\, dx\, dt\Big| + \sup_{t\in[0,T)} \|\langle x\rangle^{-\frac{3r}{2}} h(e_0 v)e^{2\gamma}\|_{L^2(\Sigma_t)} \|\langle x\rangle^{-\frac{r}{2}} f_2\|_{L^2(\Sigma_t)} \\
&\lesssim \int_0^T \|\langle x\rangle^{-\frac{r}{2}} f_2\|_{L^2(\Sigma_t)} \cdot \|\langle x\rangle^{-(r+2\alpha)} \partial v\|_{L^2(\Sigma_t)}\, dt + \Big|\int_0^T \int_{\Sigma_t} \langle x\rangle^{-2r} \cdot h \cdot f_2(e_0^2 v)e^{2\gamma}\, dx\, dt\Big| \\
&\quad + \sup_{t\in[0,T)} \|\langle x\rangle^{-(r+2\alpha)} \partial v\|_{L^2(\Sigma_t)} \|\langle x\rangle^{-\frac{r}{2}} f_2\|_{L^2(\Sigma_t)}.
\end{aligned} \tag{6.7}$$

Now, we use (2.12) to write $e_0^2 v = -N^2 \Box_g v + e^{-2\gamma} N^2 \delta^{ij} \partial_{ij}^2 v + e_0 \log(N) \cdot e_0 v + e^{-2\gamma} N \delta^{ij} \partial_i N \partial_j v$ so that

$$\begin{aligned}
&\int_0^T \int_{\Sigma_t} \langle x\rangle^{-2r} \cdot e^{2\gamma} \cdot h \cdot f_2 \cdot (e_0^2 v) dx dt \\
&= -\overbrace{\int_0^T \int_{\Sigma_t} \langle x\rangle^{-2r} \cdot e^{2\gamma} \cdot h \cdot f_2 \cdot N^2 \cdot (\Box_g v) dx dt}^{I} + \overbrace{\int_0^T \int_{\Sigma_t} \langle x\rangle^{-2r} \cdot h \cdot f_2 \cdot N^2 \cdot (\delta^{ij} \partial_{ij}^2 v) dx dt}^{II} \\
&\quad + \underbrace{\int_0^T \int_{\Sigma_t} \langle x\rangle^{-2r} \cdot e^{2\gamma} \cdot h \cdot f_2 \cdot e_0 \log(N) \cdot (e_0 v) dx dt}_{III} + \underbrace{\int_0^T \int_{\Sigma_t} \langle x\rangle^{-2r} \cdot h \cdot f_2 \cdot N \cdot (\delta^{ij} \partial_i N \partial_j v) dx dt}_{IV}.
\end{aligned} \tag{6.8}$$

We start with the easiest terms III and IV: an immediate application of the Cauchy–Schwarz inequality, Proposition 5.2 and (6.2) yields:

$$|III| + |IV| \lesssim \epsilon^{\frac{3}{2}} \int_0^T \|\langle x\rangle^{-\frac{r}{2}} f_2\|_{L^2(\Sigma_t)} \cdot \|\langle x\rangle^{-(r+2\alpha)} \partial v\|_{L^2(\Sigma_t)}\, dt. \tag{6.9}$$

For II in (6.8), we integrate by parts in ∂_i, and then use Hölder's inequality, (6.2) and (5.1) to obtain

$$\begin{aligned}
|II| &\lesssim \int_0^T \|\langle x\rangle^{r+2\alpha} \partial_x (\langle x\rangle^{-2r} h f_2 N^2)\|_{L^2(\Sigma_t)} \|\langle x\rangle^{-(r+2\alpha)} \partial_x v\|_{L^2(\Sigma_t)}\, dt \\
&\lesssim \int_0^T (\|\langle x\rangle^{-\frac{r}{2}} f_2\|_{L^2(\Sigma_t)} + \|\langle x\rangle^{-\frac{r}{2}} \partial_x f_2\|_{L^2(\Sigma_t)}) \|\langle x\rangle^{-(r+2\alpha)} \partial_x v\|_{L^2(\Sigma_t)}\, dt.
\end{aligned} \tag{6.10}$$

We now turn to the main term I in (6.8). We write again $\Box_g v = f_1 + h \cdot e_0 f_2$. The term involving f_1 can be estimated directly using Hölder's inequality, (6.2) and Proposition 5.2. For the term involving $e_0(f_2^2)$, we integrate by parts again with Lemma 6.1, and then bound the resulting terms using Hölder's inequality, (6.2) and (5.1). (The weights functions involved in the integration by parts argument can be treated as in (6.6)). We thus obtain

$$\begin{aligned}|I| &\lesssim \int_0^T \|\langle x\rangle^{-\frac{r}{2}} f_1\|_{L^2(\Sigma_t)} \\ &\quad \cdot \|\langle x\rangle^{-\frac{r}{2}} f_2\|_{L^2(\Sigma_t)}\, dt + \left| \int_0^T \int_{\Sigma_t} \langle x\rangle^{-2r} e^{2\gamma} \cdot h^2 \cdot e_0(f_2^2) \cdot N^2 dx dt \right| \\ &\lesssim \int_0^T (\|\langle x\rangle^{-\frac{r}{2}} f_1\|^2_{L^2(\Sigma_t)} + \|\langle x\rangle^{-\frac{r}{2}} f_2\|^2_{L^2(\Sigma_t)})\, dt.\end{aligned} \tag{6.11}$$

Plugging (6.9)–(6.11) into (6.7) and (6.8) gives the desired bounds for the third term in (6.4). Combining this with (6.5) and (6.6) yields the conclusion of the proposition. □

7 Basic energy estimates and commutator estimates

In this section, we prove some basic energy estimates which will be repeatedly used in the later part of the paper.

7.1 Stress-energy-momentum tensor and deformation tensor

Lemma 7.1 *1. Defining $\mathbb{T}_{\mu\nu}[v] = \partial_\mu v \partial_\nu v - \frac{1}{2} g_{\mu\nu} (g^{-1})^{\sigma\beta} \partial_\sigma v \partial_\beta v$, and suppose $\vec{n}$ and (X_k, E_k, L_k) are as in (2.3) and Definition 2.9. Then for $k = 1, 2, 3$,*

$$\mathbb{T}[v](\vec{n}, \vec{n}) = \frac{1}{2} \cdot \left((\vec{n} v)^2 + (X_k v)^2 + (E_k v)^2 \right) = \frac{1}{2} \left((\vec{n} v)^2 + e^{-2\gamma} (\partial_x v)^2 \right), \tag{7.1}$$

$$\mathbb{T}[v](\vec{n}, \partial_i) = (\vec{n} v)(\partial_i v), \quad \mathbb{T}(v)(\partial_i, \partial_j) = (\partial_i v)(\partial_j v) - \frac{1}{2} \delta_{ij} (-e^{2\gamma} (\vec{n} v)^2 + (\partial_x v)^2), \tag{7.2}$$

$$\mathbb{T}[v](L_k, \vec{n}) = \frac{1}{2} \cdot \left((L_k v)^2 + (E_k v)^2 \right). \tag{7.3}$$

2. $\mathbb{T}_{\mu\nu}[v]$ satisfies

$$(g^{-1})^{\sigma\nu} \nabla_\sigma \mathbb{T}_{\mu\nu}[v] = \Box_g v \cdot \partial_\mu v.$$

3. Defining in addition ${}^{(\vec{n})}\pi(Z_1, Z_2) = \frac{1}{2} \left(g(\nabla_{Z_1} \vec{n}, Z_2) + g(\nabla_{Z_2} \vec{n}, Z_1) \right)$, we have

$$\begin{aligned}\mathbb{T}[v]_{\mu\nu} {}^{(\vec{n})}\pi^{\mu\nu} &= -e^{-2\gamma} \delta^{il} (\vec{n} v)(\partial_i v)(\partial_l \log N) \\ &\quad + e^{-4\gamma} \delta^{il} \delta^{jq} K_{lq} [(\partial_i v)(\partial_j v) - \frac{1}{2} \delta_{ij} (-e^{2\gamma} (\vec{n} v)^2 + (\partial_x v)^2)].\end{aligned} \tag{7.4}$$

Proof Parts 1 and 2 are explicit computations.

We turn to 3. By Lemma 2.6, $\nabla_{\vec{n}}\vec{n} = e^{-2\gamma}\delta^{ij}(\partial_i \log N)\partial_j$. Hence, using also (2.6), we have

$$ {}^{(\vec{n})}\pi(\vec{n},\vec{n}) = 0, \quad {}^{(\vec{n})}\pi(\vec{n},\partial_i) = \frac{1}{2}\cdot\partial_i \log N, \quad {}^{(\vec{n})}\pi(\partial_i,\partial_j) = K_{ij}. \tag{7.5}$$

We compute using (2.9) that

$$\begin{aligned} &\mathbb{T}[v]_{\mu\nu}{}^{(\vec{n})}\pi^{\mu\nu} \\ &= \mathbb{T}[v](\vec{n},\vec{n})^{(\vec{n})}\pi(\vec{n},\vec{n}) \\ &\quad - 2e^{-2\gamma}\delta^{il}\mathbb{T}[v](\vec{n},\partial_i)^{(\vec{n})}\pi(\vec{n},\partial_l) + e^{-4\gamma}\delta^{il}\delta^{jq}\mathbb{T}[v](\partial_i,\partial_j)^{(\vec{n})}\pi(\partial_l,\partial_q), \end{aligned} \tag{7.6}$$

which implies the desired conclusion after plugging in (7.1), (7.2) and (7.5). □

7.2 Volume forms

Lemma 7.2 *[25, Lemma 2.14] The spacetime volume form induced by g is given by*

$$dvol = N\cdot e^{2\gamma}\,dx^1 \wedge dx^2 \wedge dt = \mu_k\cdot N\cdot\Theta_k\,dt_k \wedge du_k \wedge d\theta_k.$$

The volume form on Σ_t induced by $\bar{g}$ is given by

$$dvol_{\Sigma_t} = e^{2\gamma}\,dx^1 \wedge dx^2 = \mu_k^2\Theta_k^2\,du_k \wedge d\theta_k. \tag{7.7}$$

Let $dvol_{C^k_{u_k}}$ be the volume form on $C^k_{u_k}$ such that $du_k \wedge dvol_{C_{u_k}} = dvol$. Then

$$dvol_{C^k_{u_k}} = -\mu_k\cdot N\cdot\Theta_k\,dt_k \wedge d\theta_k = \mu_k\cdot N\cdot\Theta_k d\theta_k \wedge dt_k. \tag{7.8}$$

7.3 The main energy estimate

In this subsection, we prove two basic energy estimates.

The first estimate (Proposition 7.3) applies only to compactly supported functions (so that weights can be ignored), and allows for localization in the u_k variable. The second estimate (Proposition 7.4) is a weighted estimate for general (not necessarily compactly supported) functions which does not allows for u_k-localization.[10]

The following is the first general energy estimate.

Proposition 7.3 *Given $k \in \{1,2,3\}$, any $T \in [0,T_B)$ and any $-\infty \le U_0 < U_1 \le +\infty$, define*

$$\mathcal{D}^{(k),T}_{U_0,U_1} := \{(t,x) \in \mathbb{R}\times\mathbb{R}^2 : t \in [0,T],\ u_k(t,x) \in [U_0,U_1]\}. \tag{7.9}$$

[10] One could combine the two energy estimates to obtain a more general proposition, incorporating both weights and u_k-localization. We will not need such a general statement, and therefore only prove the easier estimates.

For any[11] $k' \in \{1, 2, 3\}$, *the following holds for all solutions* v *to* $\Box_g v = f$, *with* $\mathrm{supp}(v),\ \mathrm{supp}(f) \subseteq \{(t, x) : |x| \leq R\}$, *with a constant depending only on R:*

$$\begin{aligned} &\sup_{t\in[0,T)} \|\partial v\|_{L^2(\Sigma_t\cap\mathcal{D}^{(k),T}_{U_0,U_1})} + \sup_{u_{k'}\in\mathbb{R}} \sum_{Z_{k'}\in\{L_{k'},\,E_{k'}\}} \|Z_{k'}v\|_{L^2(C^{k'}_{u_{k'}}\cap\mathcal{D}^{(k),T}_{U_0,U_1})} \\ &\lesssim \|\partial v\|_{L^2(\Sigma_0\cap\mathcal{D}^{(k),T}_{U_0,U_1})} + \sum_{Z_k\in\{L_k,\,E_k\}} \|Z_k v\|_{L^2(C^k_{U_0}\cap\mathcal{D}^{(k),T}_{U_0,U_1})} + \int_0^T \|f\|_{L^2(\Sigma_t\cap\mathcal{D}^{(k),T}_{U_0,U_1})}\,dt. \end{aligned} \tag{7.10}$$

Proof *Step 1: The case* $k' = k$ By Lemma 7.1 (Point 2) and $\Box_g v = f$, we have $\nabla^\nu \mathbb{T}_{\mu\nu}(v) = f\partial_\mu v$. Hence,

$$\nabla^\nu\left(\mathbb{T}_{\mu\nu}[v]\vec{n}^\mu\right) = \mathbb{T}[v]_{\mu\nu}{}^{(\vec{n})}\pi^{\mu\nu} + f\cdot\vec{n}v. \tag{7.11}$$

Fix $T \in [0, T_B)$ and U_0, U_1 as in the statement of the proposition. For every $\tau \in [0, T]$ and $U \in [U_0, U_1]$, define $\mathcal{D}^{(k),\tau}_{U_0,U} := \{(t, x) \in \mathbb{R}\times\mathbb{R}^2 : t \in [0, \tau),\ u_k(t, x) \in [U_0, U)\}$. Note that clearly $\mathcal{D}^{(k),\tau}_{U_0,U} \subseteq \mathcal{D}^{(k),T}_{U_0,U_1}$.

Integrating $\nabla^\nu\left(\mathbb{T}_{\mu\nu}[v]\vec{n}^\mu\right)$ on the spacetime region $\mathcal{D}^\tau_{U_0,U}$ and using Stokes' theorem, we obtain (for $\mathbb{T}_{\mu\nu} = \mathbb{T}_{\mu\nu}[v]$)

$$\begin{aligned} &\int_{\Sigma_\tau\cap\mathcal{D}^{(k),\tau}_{U_0,U}} \mathbb{T}(\vec{n},\vec{n})\,dvol_{\Sigma_t} \\ &\quad - \int_{\Sigma_0\cap\mathcal{D}^{(k),\tau}_{U_0,U}} \mathbb{T}(\vec{n},\vec{n})\,dvol_{\Sigma_0} + \int_{C^k_U\cap\mathcal{D}^{(k),\tau}_{U_0,U}} \mathbb{T}(\vec{n},(-du_k)^\sharp)\,dvol_{C^k_U} \\ &= \int_{\mathcal{D}^{(k),\tau}_{U_0,U}} \nabla^\nu\left(\mathbb{T}_{\mu\nu}\vec{n}^\mu\right)\,dvol. \end{aligned} \tag{7.12}$$

Using $(-du_k)^\sharp = \mu_k^{-1}L_k$ (by (2.28) and (2.29)), the computations for $\mathbb{T}$ in (7.1), (7.3), the computations for the volume forms in Lemma 7.2, and the computations for $\nabla^\nu\left(\mathbb{T}_{\mu\nu}[v]\vec{n}^\mu\right)$ in (7.11) and (7.4), we obtain using (7.12) that

$$\begin{aligned} &\underbrace{\frac{1}{2}\int_{\Sigma_\tau\cap\mathcal{D}^{(k),\tau}_{U_0,U}} [e^{2\gamma}(\vec{n}v)^2 + (\partial_x v)^2]\,dx^1\,dx^2}_{=:I} + \underbrace{\int_{C^k_U\cap\mathcal{D}^{(k),\tau}_{U_0,U}} \frac{\Theta_k N}{2}\cdot[(L_k v)^2 + (E_k v)^2]\,dt_k\,d\theta_k}_{=:II} \\ &- \underbrace{\int_{\mathcal{D}^{(k),\tau}_{U_0,U}} \left[-e^{-2\gamma}\delta^{il}(\vec{n}v)(\partial_i v)(\partial_l \log N)\right]\cdot Ne^{2\gamma}\,dx^1\,dx^2\,dt}_{=:III} \end{aligned}$$

[11] In particular, k' could be the same as k, and could also be different from k. The same comment applies to Propositions 8.9, 8.11 and 8.13.

$$
\begin{aligned}
&\underbrace{-\int_{\mathcal{D}^{(k),\tau}_{U_0,U}}\left[e^{-4\gamma}\delta^{il}\delta^{jq}K_{lq}[(\partial_i v)(\partial_j v)-\frac{1}{2}\delta_{ij}(-e^{2\gamma}(\vec{n}v)^2+(\partial_x v)^2)]+f\cdot\vec{n}v\right]\cdot Ne^{2\gamma}\,dx^1\,dx^2\,dt}_{=:IV}\\
&=\underbrace{\frac{1}{2}\int_{\Sigma_0\cap\mathcal{D}^{(k),\tau}_{U_0,U}}[e^{2\gamma}(\vec{n}v)^2+(\partial_x v)^2]\,dx^1\,dx^2}_{=:V}+\underbrace{\int_{C^k_{U_0}\cap\mathcal{D}^{(k),\tau}_{U_0,U}}\frac{\Theta_k N}{2}\cdot[(L_k v)^2+(E_k v)^2]\,dt_k\,d\theta_k}_{=:VI}\,.
\end{aligned}
\tag{7.13}
$$

By Proposition 5.2, Proposition 5.5 and support properties of v,

$$
I+II\gtrsim\|\partial v\|^2_{L^2(\Sigma_\tau\cap\mathcal{D}^{(k),\tau}_{U_0,U})}+\sum_{Z_k\in\{L_k,\,E_k\}}\|Z_{k'}v\|^2_{L^2(C^k_U\cap\mathcal{D}^{(k),\tau}_{U_0,U})}. \tag{7.14}
$$

Using Proposition 5.2, Proposition 5.5, the Cauchy–Schwarz inequality and Young's inequality, we get that

$$
\begin{aligned}
|III|+|IV|&\leq C\epsilon^{\frac{3}{2}}\int_0^\tau\|\partial v\|^2_{L^2(\Sigma_t\cap\mathcal{D}^{(k),\tau}_{U_0,U})}\,dt+C\int_0^\tau\|\partial v\|_{L^2(\Sigma_t\cap\mathcal{D}^{(k),\tau}_{U_0,U})}\|f\|_{L^2(\Sigma_t\cap\mathcal{D}^{(k),\tau}_{U_0,U})}\,dt\\
&\leq\left(\frac{1}{2}+C\epsilon^{\frac{3}{2}}\right)\sup_{t\in[0,T]}\|\partial v\|^2_{L^2(\Sigma_t\cap\mathcal{D}^{(k),\tau}_{U_0,U})}+C\cdot\left(\int_0^T\|f\|_{L^2(\Sigma_t\cap\mathcal{D}^{(k),\tau}_{U_0,U})}\,dt\right)^2.
\end{aligned}
\tag{7.15}
$$

Finally, the data terms can be controlled using Proposition 5.2 and Proposition 5.5 applied at $t=0$:

$$
|V|+|VI|\lesssim\|\partial v\|^2_{L^2(\Sigma_0\cap\mathcal{D}^{(k),T}_{U_0,U})}+\sum_{Z_k\in\{L_k,\,E_k\}}\|Z_k v\|^2_{L^2(C^k_{U_0}\cap\mathcal{D}^{(k),T}_{U_0,U})}. \tag{7.16}
$$

Plugging the estimates (7.14)–(7.16) into (7.13), and taking supremum over all $\tau\in[0,T]$ and $U\in[U_0,U_1]$, we obtain

$$
\begin{aligned}
&\sup_{t\in[0,T]}\|\partial v\|^2_{L^2(\Sigma_t\cap\mathcal{D}^{(k),T}_{U_0,U})}+\sup_{u_k\in\mathbb{R}}\sum_{Z_k\in\{L_k,\,E_k\}}\|Z_k v\|^2_{L^2(C^k_{u_k}\cap\mathcal{D}^{(k),T}_{U_0,U})}\\
&\leq\left(\frac{1}{2}+C\epsilon^{\frac{3}{2}}\right)\sup_{t\in[0,T]}\|\partial v\|^2_{L^2(\Sigma_t\cap\mathcal{D}^{(k),T}_{U_0,U})}+C\|\partial v\|^2_{L^2(\Sigma_0\cap\mathcal{D}^{(k),T}_{U_0,U})}\\
&\quad+C\sum_{Z_k\in\{L_k,\,E_k\}}\|Z_k v\|^2_{L^2(C^k_{U_0}\cap\mathcal{D}^{(k),T}_{U_0,U})}\\
&\quad+C\left(\int_0^T\|f\|_{L^2(\Sigma_t\cap\mathcal{D}^{(k),T}_{U_0,U})}\,dt\right)^2.
\end{aligned}
\tag{7.17}
$$

Note that while the supremum at first only gives $\sup_{u_k\in[U_0,U_1]}$ for the second term on the left-hand side of (7.17), we can change this to $\sup_{u_k\in\mathbb{R}}$ after noticing that $C^k_{u_k}\cap\mathcal{D}=\emptyset$ if $u_k\in\mathbb{R}\setminus[U_0,U_1]$.

The first terms on the right-hand side of (7.17) can be absorbed to the left-hand side for ϵ_0 sufficiently small, giving

$$\begin{aligned} &\sup_{t\in[0,T]} \|\partial v\|^2_{L^2(\Sigma_t\cap\mathcal{D}^{(k),T}_{U_0,U_1})} + \sup_{u_k\in\mathbb{R}} \sum_{Z_k\in\{L_k,\,E_k\}} \|Z_k v\|^2_{L^2(C^k_{u_k}\cap\mathcal{D}^{(k),T}_{U_0,U_1})} \\ &\lesssim \|\partial v\|^2_{L^2(\Sigma_0\cap\mathcal{D}^{(k),T}_{U_0,U})} + \sum_{Z_k\in\{L_k,\,E_k\}} \|Z_k v\|^2_{L^2(C^k_{U_0}\cap\mathcal{D}^{(k),T}_{U_0,U})} + \left(\int_0^T \|f\|_{L^2(\Sigma_t\cap\mathcal{D}^{(k),T}_{U_0,U})}\,dt\right)^2. \end{aligned} \tag{7.18}$$

The bound (7.18) gives the control of the first term in (7.10), and of the second term in (7.10) when $k' = k$.

Step 2: The general case To complete the proof of (7.10), we need to bound the second term on the left-hand side of (7.10), corresponding to the flux on $C^{k'}_{u_{k'}}$ in the case $k' \neq k$. Fix $k' \neq k$ and $U' \in \mathbb{R}$, integrate the same quantity $\nabla^\nu\left(\mathbb{T}_{\mu\nu}[v]\vec{n}^\mu\right)$ but now on $\mathcal{D}' := \mathcal{D}^{(k),T}_{U_0,U} \cap \mathcal{D}^{(k'),T}_{-\infty,U'} = \mathcal{D}^{(k),T}_{U_0,U} \cap \{(t,x) : u_{k'}(t,x) \leq U'\}$, and use Stokes' theorem. We then obtain an analogue of (7.13), except with $\mathcal{D}^{(k),T}_{U_0,U}$ replaced by $\mathcal{D}'$, and with an additional flux term $\int_{C^{k'}_{U'}\cap\mathcal{D}'} \frac{\Theta_{k'}N}{2}\cdot[(L_{k'}v)^2 + (E_{k'}v)^2]\,dt_{k'}\,d\theta_{k'}$ on the left-hand side.

We now control the bulk terms (i.e. terms corresponding to III and IV in (7.13)) in the same manner as in Step 1. Since $\mathcal{D}' \subseteq \mathcal{D}^{(k),T}_{U_0,U}$, we obtain an analogue of (7.18), but with the control of an addition flux term on the left-hand side:

$$\begin{aligned} &\sup_{t\in[0,T]} \|\partial v\|^2_{L^2(\Sigma_t\cap\mathcal{D}')} + \sup_{u_k\in\mathbb{R}} \sum_{Z_k\in\{L_k,\,E_k\}} \|Z_k v\|^2_{L^2(C^k_{u_k}\cap\mathcal{D}')} + \sum_{Z_{k'}\in\{L_{k'},\,E_{k'}\}} \|Z_{k'} v\|^2_{L^2(C^{k'}_{U'}\cap\mathcal{D}')} \\ &\lesssim \|\partial v\|^2_{L^2(\Sigma_0\cap\mathcal{D}^{(k),T}_{U_0,U})} + \sum_{Z_k\in\{L_k,\,E_k\}} \|Z_k v\|^2_{L^2(C^k_{U_0}\cap\mathcal{D}^{(k),T}_{U_0,U})} + \left(\int_0^T \|f\|_{L^2(\Sigma_t\cap\mathcal{D}^{(k),T}_{U_0,U})}\,dt\right)^2. \end{aligned} \tag{7.19}$$

We now take supremum over all $U' \in \mathbb{R}$. Noting that $C^{k'}_{U'} \cap \mathcal{D}' = C^{k'}_{U'} \cap \mathcal{D}^{(k),T}_{U_0,U}$, we deduce from (7.19) that

$$\begin{aligned} &\sup_{u_{k'}\in\mathbb{R}} \sum_{Z_{k'}\in\{L_{k'},\,E_{k'}\}} \|Z_{k'} v\|^2_{L^2(C^{k'}_{u_{k'}}\cap\mathcal{D}^{(k),T}_{U_0,U})} \\ &\lesssim \|\partial v\|^2_{L^2(\Sigma_0\cap\mathcal{D}^{(k),T}_{U_0,U})} + \sum_{Z_k\in\{L_k,\,E_k\}} \|Z_k v\|^2_{L^2(C^k_{U_0}\cap\mathcal{D}^{(k),T}_{U_0,U})} + \left(\int_0^T \|f\|_{L^2(\Sigma_t\cap\mathcal{D}^{(k),T}_{U_0,U})}\,dt\right)^2. \end{aligned} \tag{7.20}$$

(7.20) thus bounds the second term in (7.10) when $k' \neq k$. Combining this with (7.18) concludes the proof. □

Finally, we prove a version of Proposition 7.3 with weights. (Note that the weights can clearly be improved, but will not be relevant for later applications.)

Proposition 7.4 *Let v be a smooth function which is Schwartz for on Σ_t for all $0 \leq t < T_B$.*

Then for all $r \geq 1$, the following holds for any $T \in [0, T_B)$, with a constant depending only on r:

$$\begin{aligned} \sup_{t\in[0,T)} \|\langle x\rangle^{-(r+2\alpha)}\partial v\|^2_{L^2(\Sigma_t)} &\lesssim \|\langle x\rangle^{-\frac{r}{2}}\partial v\|^2_{L^2(\Sigma_0)} \\ &+ \sup_{t\in[0,T)} \left| \int_0^t \int_{\Sigma_\tau} \langle x\rangle^{-2r}(e_0 v)(\Box_g v)\, e^{2\gamma}\, dx\, d\tau \right|. \end{aligned} \tag{7.21}$$

Proof Using the multiplier $\langle x\rangle^{-2r}\vec{n}$, we have, by (7.4),

$$\begin{aligned} \nabla^\nu\left(\mathbb{T}_{\mu\nu}[v]\langle x\rangle^{-2r}\vec{n}^\mu\right) = &- \langle x\rangle^{-2r}e^{-2\gamma}\delta^{il}(\vec{n}v)(\partial_i v)(\partial_l \log N) \\ &+ \langle x\rangle^{-2r}e^{-4\gamma}\delta^{il}\delta^{jq}K_{lq}[(\partial_i v)(\partial_j v) \\ &- \frac{1}{2}\delta_{ij}(-e^{2\gamma}(\vec{n}v)^2 + (\partial_x v)^2)] \\ &+ \langle x\rangle^{-2r}(\Box_g v)\cdot\vec{n}v + (g^{-1})^{\nu\sigma}\mathbb{T}_{\mu\nu}[v](\partial_\sigma\langle x\rangle^{-2r})\vec{n}^\mu. \end{aligned} \tag{7.22}$$

Using (2.9), it can be computed that

$$(g^{-1})^{\nu\sigma}(\partial_\sigma\langle x\rangle^{-2r})\partial_\nu = \frac{-2r\langle x\rangle^{-2r-2}}{N^2}\left(\beta^i x_i\partial_t + (N^2e^{-2\gamma}\delta^{ij} - \beta^i\beta^j)x_j\partial_i\right),$$

which in particular implies, by the metric estimates of Proposition 5.2 and the bound in Lemma 5.4, that

$$|(g^{-1})^{\nu\sigma}\mathbb{T}_{\mu\nu}[v](\partial_\sigma\langle x\rangle^{-2r})\vec{n}^\mu| \lesssim \langle x\rangle^{-2r-1+10\alpha}[(\vec{n}v)^2 + (X_k v)^2 + (E_k v)^2],$$

where the implicit constant is allowed to depend on r.

Then we use Proposition 5.2 (specifically $e^{-2\gamma} \lesssim \langle x\rangle^\alpha$, $|\partial_x \log N| \lesssim \epsilon\langle x\rangle^{-1+\alpha}$, $|K| \lesssim \epsilon\langle x\rangle^{-1}$ and $|T_{\mu\nu}[v]| \lesssim \langle x\rangle^{2\alpha}\cdot|\partial v|^2$, where the indices μ and ν are in the coordinate system (t, x^1, x^2)) and Lemma 5.4, and repeating the argument of Proposition 7.3 we get, integrating (7.22) on $\{0 \leq t' \leq t\}$:

$$\begin{aligned} &\int_{\Sigma_t} e^{2\gamma}\langle x\rangle^{-2r}[(\vec{n}v)^2 + (X_k v)^2 + (E_k v)^2]\, dx^1 dx^2 \\ &\lesssim \underbrace{\int_{\Sigma_0} e^{2\gamma}\langle x\rangle^{-2r}[(\vec{n}v)^2 + (X_k v)^2 + (E_k v)^2]\, dx^1 dx^2}_{=:I} \end{aligned}$$

$$+\underbrace{\int_0^t\int_{\Sigma_{t'}}\langle x\rangle^{-2r-1+10\alpha}[(\vec{n}v)^2+(X_kv)^2+(E_kv)^2]\,dx^1dx^2dt'}_{=:II}$$

$$+\underbrace{\left|\int_0^t\int_{\Sigma_{t'}}\langle x\rangle^{-2r}(\Box_g v)\cdot\vec{n}v\cdot Ne^{2\gamma}\,dx^1dx^2dt'\right|}_{=:III}. \tag{7.23}$$

We now bound each term on the right-hand side of (7.23).

1. For term I, we note that by Proposition 5.2 and Proposition 5.3, $e^{2\gamma}\langle x\rangle^{-2r}[(\vec{n}v)^2+(X_kv)^2+(E_kv)^2]\lesssim\left|\langle x\rangle^{-\frac{r}{2}}\partial v\right|^2$, and thus $I\lesssim\|\langle x\rangle^{-\frac{r}{2}}\partial v\|^2_{L^2(\Sigma_0)}$.
2. For term II, since $10\alpha=0.1<1-\alpha$, it can be absorbed to the left-hand side using Grönwall's inequality.
3. Finally, we just keep the term III as it is (which is allowed on the right-hand side of (7.21)), since $N\vec{n}=e_0$.

Combining the above bounds, it follows that

$$\int_{\Sigma_t}e^{2\gamma}\langle x\rangle^{-2r}[(\vec{n}v)^2+(X_kv)^2+(E_kv)^2]\,dx^1dx^2\lesssim\text{RHS of (7.21)}.$$

Finally, notice that by (5.8), (5.10) and Proposition 5.2,

$$\|\langle x\rangle^{-(r+2\alpha)}\partial v\|^2_{L^2(\Sigma_t)}\lesssim\int_{\Sigma_t}e^{2\gamma}\langle x\rangle^{-2r}[(\vec{n}v)^2+(X_kv)^2+(E_kv)^2]\,dx^1dx^2$$

which therefore gives the desired result. □

Corollary 7.5 *Let v be a smooth function which is Schwartz on Σ_t for all $0\le t<T_B$. Suppose $\Box_g v=f_1+h\cdot e_0f_2$, where f_1, f_2 and h are all smooth and Schwartz on Σ_t for all $0\le t<T_B$, and h satisfies (6.2).*

Then for all $r\ge 1$, the following holds for any $T\in[0,T_B)$, with a constant depending only on r:

$$\begin{aligned}&\sup_{t\in[0,T)}\|\langle x\rangle^{-(r+2\alpha)}\partial v\|^2_{L^2(\Sigma_t)}\\&\lesssim\|\langle x\rangle^{-\frac{r}{2}}\partial v\|^2_{L^2(\Sigma_0)}+\sup_{t\in[0,T)}\|\langle x\rangle^{-\frac{r}{2}}f_2\|^2_{L^2(\Sigma_t)}\\&\quad+\int_0^T\left(\|\langle x\rangle^{-\frac{r}{2}}f_1\|^2_{L^2(\Sigma_t)}+\|\langle x\rangle^{-\frac{r}{2}}f_2\|^2_{L^2(\Sigma_t)}+\|\langle x\rangle^{-\frac{r}{2}}\partial_xf_2\|^2_{L^2(\Sigma_t)}\right)dt.\end{aligned}$$

Proof We first apply Proposition 7.4 so that

$$\begin{aligned}\sup_{t\in[0,T)}\|\langle x\rangle^{-(r+2\alpha)}\partial v\|^2_{L^2(\Sigma_t)}&\lesssim\|\langle x\rangle^{-\frac{r}{2}}\partial v\|^2_{L^2(\Sigma_0)}\\&+\sup_{t\in[0,T)}\left|\int_0^t\int_{\Sigma_\tau}\langle x\rangle^{-2r}(e_0v)\cdot(\Box_g v)\,e^{2\gamma}\,dx\,d\tau\right|.\end{aligned}$$

Controlling the last term by Proposition 6.2, we obtain

$$
\begin{aligned}
&\sup_{t\in[0,T)} \|\langle x\rangle^{-(r+2\alpha)}\partial v\|^2_{L^2(\Sigma_t)}\\
&\lesssim \|\langle x\rangle^{-\frac{r}{2}}\partial v\|^2_{L^2(\Sigma_0)} + \int_0^T (\|\langle x\rangle^{-\frac{r}{2}} f_1\|^2_{L^2(\Sigma_t)} + \|\langle x\rangle^{-\frac{r}{2}} f_2\|^2_{L^2(\Sigma_t)})\, dt\\
&\quad + \sup_{t\in[0,T)} \|\langle x\rangle^{-(r+2\alpha)}\partial v\|_{L^2(\Sigma_t)} \|\langle x\rangle^{-\frac{r}{2}} f_2\|_{L^2(\Sigma_t)}\\
&\quad + \int_0^T \|\langle x\rangle^{-(r+2\alpha)}\partial v\|_{L^2(\Sigma_t)}\\
&\quad \cdot \Big(\|\langle x\rangle^{-\frac{r}{2}} f_1\|_{L^2(\Sigma_t)} + \|\langle x\rangle^{-\frac{r}{2}} f_2\|_{L^2(\Sigma_t)} + \|\langle x\rangle^{-\frac{r}{2}} \partial_x f_2\|_{L^2(\Sigma_t)}\Big)\, dt.
\end{aligned}
$$

For the terms on the last two lines, we use Young's inequality and absorb $\sup_{t\in[0,T)} \|\langle x\rangle^{-(r+2\alpha)}\partial v\|_{L^2(\Sigma_t)}$ to the left-hand side. For the terms on the last line, we additionally use the Cauchy–Schwarz inequality in t, giving the desired inequality. □

8 Basic estimates for the commutations with the wave operator

8.1 Two auxiliary estimates

To streamline the later exposition in this section, before we even consider the commutations with the wave operator, we first prove in this subsection two auxiliary estimates in Propositions 8.2 and 8.3. They concern second derivatives of the metric (in the geometric coordinates or in the elliptic gauge coordinates) which are not bounded in L^∞.

The estimates in this subsection apply either to Σ_t or to a half space in the u_k variable. In the remainder of this subsection, for a fixed k, we will use $\mathcal{D}$ to denote one of the following sets:

$$
\begin{aligned}
\mathcal{D} &= \{(t,x) \in [0,T_B) \times \mathbb{R}^2 : u_k(t,x) \geq U_0\},\\
\mathcal{D} &= \{(t,x) \in [0,T_B) \times \mathbb{R}^2 : u_k(t,x) \leq U_1\},
\end{aligned} \tag{8.1}
$$

where $U_0 \in [-\infty,\infty)$, $U_1 \in (-\infty,\infty]$. Notice that either $\Sigma_t \cap \mathcal{D} = \Sigma_t$ (when $U_0 = -\infty$ or $U_1 = \infty$) or $\Sigma_t \cap \mathcal{D}$ is a half-space in u_k (when U_0 or U_1 is finite).

Before we proceed to the first auxiliary estimate in Proposition 8.2, we first need the following simple lemma.

Lemma 8.1 *Let $k \neq k'$, and let $\mathcal{D}$ be one of the sets in (8.1). For all f which is sufficiently regular,*

$$
\|f\|_{L^2_{u_k} L^\infty_{u_{k'}}(\Sigma_t\cap\mathcal{D})} \lesssim \|f\|_{L^2(\Sigma_t\cap\mathcal{D})} + \|E_k f\|_{L^2_{\frac{1}{16}}(\Sigma_t\cap\mathcal{D})}. \tag{8.2}
$$

Proof First, by the standard 1-dimensional Sobolev embedding,

$$\|f\|_{L^2_{u_k}L^\infty_{u_{k'}}(\Sigma_t\cap\mathcal{D})} \lesssim \|f\|_{L^2_{u_k,u_{k'}}(\Sigma_t\cap\mathcal{D})} + \|\not\partial_{u_{k'}} f\|_{L^2_{u_k,u_{k'}}(\Sigma_t\cap\mathcal{D})}. \tag{8.3}$$

Finally, by (2.55), (5.17), (5.23) and Lemma 5.6, we obtain (8.2). (Note that we need a small positive weight, e.g., $\langle x\rangle^{\frac{1}{16}}$, here because there is a $\mu_{k'}$ factor in (2.55), which grows slowly at infinity according to (5.17).) □

Proposition 8.2 *Fix k and let $\mathcal{D}$ be one of the sets in (8.1). The following holds for any sufficiently regular f:*

$$\|\partial_x\chi_k\cdot f\|_{L^2_{-\frac14}(\Sigma_t\cap\mathcal{D})} + \|E_k\eta_k\cdot f\|_{L^2_{-\frac14}(\Sigma_t\cap\mathcal{D})} \lesssim \epsilon^{\frac32}\cdot(\|f\|_{L^2_{-\frac18}(\Sigma_t\cap\mathcal{D})} + \|E_k f\|_{L^2_{-\frac18}(\Sigma_t\cap\mathcal{D})}). \tag{8.4}$$

Proof By (5.22), we can bound the $L^2(\Sigma_t)$ norm in either the (x^1, x^2) or the (u_k, θ_k) coordinates. Denoting $h \in \{\partial_1\chi_k, \partial_2\chi_k, E_k\eta_k\}$, we first use Hölder's inequality in the (u_k, θ_k) coordinate system to obtain

$$\begin{aligned}\|\langle x\rangle^{-\frac14}h\cdot f\|_{L^2(\Sigma_t\cap\mathcal{D})} &\lesssim (\sup_{u_k\in\mathbb{R}}\|h\|_{L^2_{\theta_k}((\Sigma_t\cap\mathcal{D})\cap C_{u_k})})\cdot\|\langle x\rangle^{-\frac14}f\|_{L^2_{u_k}L^\infty_{\theta_k}(\Sigma_t\cap\mathcal{D})}\\ &\lesssim \epsilon^{\frac32}\cdot\|\langle x\rangle^{-\frac14}f\|_{L^2_{u_k}L^\infty_{\theta_k}(\Sigma_t\cap\mathcal{D})}.\end{aligned} \tag{8.5}$$

Notice now that for a fixed $k' \neq k$, the $L^2_{u_k}L^\infty_{\theta_k}$ norm is equal to the $L^2_{u_k}L^\infty_{u_{k'}}$ norm. Hence, by (8.2),

$$\begin{aligned}\|\langle x\rangle^{-\frac14}f\|_{L^2_{u_k}L^\infty_{\theta_k}(\Sigma_t\cap\mathcal{D})} &\lesssim \|\langle x\rangle^{-\frac14}f\|_{L^2(\Sigma_t\cap\mathcal{D})} + \|\langle x\rangle^{-\frac18}E_k f\|_{L^2(\Sigma_t\cap\mathcal{D})}\\ &\quad + \|\langle x\rangle^{\frac1{16}}(E_k\langle x\rangle^{-\frac14})f\|_{L^2(\Sigma_t\cap\mathcal{D})}\\ &\lesssim \|\langle x\rangle^{-\frac18}f\|_{L^2(\Sigma_t\cap\mathcal{D})} + \|\langle x\rangle^{-\frac18}E_k f\|_{L^2(\Sigma_t\cap\mathcal{D})},\end{aligned} \tag{8.6}$$

where we have used $|E_k\langle x\rangle^{-\frac14}| \lesssim \langle x\rangle^{-\frac54+\epsilon}$ (by Proposition 5.3).

Combining (8.5) and (8.6) then yields (8.4). □

We now turn to our second auxiliary estimate.

Proposition 8.3 *Let $\mathcal{D}$ be one of the sets in (8.1) for some k. For all smooth function f which is Schwartz class on Σ_t for all $t \in [0, T_B)$, the following estimate holds for all $t \in [0, T_B)$, where $\mathfrak{g} \in \{e^{2\gamma}-1,\ e^{-2\gamma}-1,\ \beta^j,\ N-1,\ N^{-1}-1,\ g_{\nu\beta},\ (g^{-1})^{\nu\beta}\}$:*

$$\|\partial\partial_x\mathfrak{g}\cdot f\|_{L^2(\Sigma_t\cap\mathcal{D})} \lesssim \epsilon^{\frac32}\|f\|_{H^1(\Sigma_t\cap\mathcal{D})}. \tag{8.7}$$

Proof We use Hölder's inequality to obtain

$$\|\partial\partial_x \mathfrak{g}\cdot f\|_{L^2(\Sigma_t\cap\mathcal{D})}\lesssim \|\partial\partial_x \mathfrak{g}\|_{L^4(\Sigma_t\cap\mathcal{D})}\cdot\|f\|_{L^4(\Sigma_t\cap\mathcal{D})}.$$

Since $s'-s''<\frac{1}{2}$, Proposition 5.2 implies an L^4 estimate for $\partial\partial_x\mathfrak{g}$. On the other hand, f can be controlled using the Sobolev embedding $H^1(\mathbb{R}^2)\hookrightarrow L^4(\mathbb{R}^2)$ or $H^1(\mathbb{R}^2_+)\hookrightarrow L^4(\mathbb{R}^2_+)$. (In the case where $\Sigma_t\cap\mathcal{D}$ is a half-space in u_k, we perform Sobolev embedding for the half-space in the $(u_k,u_{k'})$ coordinates (for $k'\neq k$), and note that the $H^1_{x^1,x^2}(\Sigma_t)$, $H^1_{u_k,u_{k'}}(\Sigma_t)$ norms, or the $L^4_{x^1,x^2}(\Sigma_t)$, $L^4_{u_k,u_{k'}}(\Sigma_t)$ norms, are equivalent by (5.19)–(5.20)). Hence,

$$\|\partial\partial_x \mathfrak{g}\cdot f\|_{L^2(\Sigma_t\cap\mathcal{D})}\lesssim \epsilon^{\frac{3}{2}}\cdot\|f\|_{L^4(\Sigma_t\cap\mathcal{D})}\lesssim \epsilon^{\frac{3}{2}}\|f\|_{H^1(\Sigma_t\cap\mathcal{D})}.$$

□

8.2 Computations of the commutators

8.2.1 The wave operator

Lemma 8.4 *For any C^2 function v:*

$$\begin{aligned}\Box_g v=&-L_k^2v-2X_kL_kv+E_k^2v+2\eta_kE_kv-(X_k\log(N))L_kv-\chi_kX_kv\\ =&-L_k^2v-2L_kX_kv+E_k^2v+2(E_k\log(N)-K(E_k,E_k))E_kv\\ &+(X_k\log(N))L_kv+(-\chi_k-2K(X_k,X_k)+2X_k\log(N))X_kv.\end{aligned}\tag{8.8}$$

Proof By (2.34), we have

$$\Box_g v=-L_k^2v+(\nabla_{L_k}L_k)v-2X_k(L_kv)+2(\nabla_{X_k}L_k)v+E_k^2v-(\nabla_{E_k}E_k)v.$$

Hence, by (2.48), (2.44) and (2.46), we get

$$\begin{aligned}\Box_g v=&-L_k^2v-2X_kL_kv+E_k^2v+2\eta_kE_kv\\ &-(K(X_k,X_k)+K(E_k,E_k)+X_k\log(N))L_kv-\chi_kX_kv.\end{aligned}$$

Now, by (2.7) and the fact that (X_k,E_k) is a g-orthonormal frame, we have $K(X_k,X_k)+K(E_k,E_k)=0$, proving the first equality of (8.8). The second equality follows from the first combined with (2.45). □

In the three subsubsections below, we compute the commutator of $\Box_g$ with ∂_l, E_k and L_k respectively. We introduce the following conventions: for each commutator we divide into three types of terms; see the statements of Lemmas 8.5, 8.6 and 8.7. I has second derivatives of metric and first derivatives of v; II has first derivatives of metric and second derivatives of v; III contains at most one derivative of the metric or v.

8.2.2 Commuting the wave operator with ∂_l

Lemma 8.5 *For any C^3 function v and $l = 1, 2$,*

$$\left[\partial_l, \Box_g\right] v = I(\partial_l)(v) + II(\partial_l)(v) + III(\partial_l)(v),$$

where

$$I(\partial_l)(v) = + \frac{e_0(\partial_l \beta^j)}{N^2}\partial_j v + \frac{\partial_l e_0 \log(N)}{N^2} e_0 v + e^{-2\gamma}\delta^{ij}\partial^2_{il}\log(N)\partial_j v, \tag{8.9}$$

$$II(\partial_l)(v) = -2\partial_l \log N \cdot \Box_g v + \frac{2\partial_l \beta^j}{N^2} e_0 \partial_j v + 2\partial_l \log N e^{-2\gamma}\delta^{ij}\partial^2_{ij} v - 2\partial_l \gamma \cdot e^{-2\gamma}\delta^{ij}\partial^2_{ij} v, \tag{8.10}$$

$$\begin{aligned} III(\partial_l)(v) =& 2\partial_l \log N \cdot \partial_i \log(N) e^{-2\gamma}\delta^{ij}\partial_j v - \frac{\partial_l \beta^j \partial_j \beta^q}{N^2}\partial_q v \\ & - \frac{e_0 N}{N^3}\partial_l \beta^j \partial_j v - 2\partial_l \gamma \cdot e^{-2\gamma}\delta^{ij}\partial_i \log(N)\partial_j v. \end{aligned} \tag{8.11}$$

Proof First, from the definition of $e_0 = \partial_t - \beta^j \partial_j$, we get the commutator identity

$$[\partial_l, e_0] = -\partial_l \beta^j \partial_j.$$

We now compute the commutator. By (2.12) there are four terms to control. The first term is

$$\begin{aligned} \left[\partial_l, \frac{-e_0^2}{N^2}\right] v =& \frac{2\partial_l \log N}{N^2} e_0^2 v - \frac{1}{N^2}\left[\partial_l, e_0^2\right] v \\ =& \frac{2\partial_l \log N}{N^2} e_0^2 v - \frac{1}{N^2}[\partial_l, e_0] e_0 v - \frac{1}{N^2} e_0([\partial_l, e_0] v) \\ =& \frac{2\partial_l \log N}{N^2} e_0^2 v + \frac{\partial_l \beta^j}{N^2}\partial_j e_0 v + \frac{e_0(\partial_l \beta^j)}{N^2}\partial_j v + \frac{\partial_l \beta^j}{N^2} e_0 \partial_j v \\ =& \frac{2\partial_l \log N}{N^2} e_0^2 v + \frac{2\partial_l \beta^j}{N^2} e_0 \partial_j v + \frac{\partial_l \beta^j}{N^2}\left[\partial_j, e_0\right] v + \frac{e_0(\partial_l \beta^j)}{N^2}\partial_j v \\ =& 2\partial_l \log N \left(-\Box_g v + e^{-2\gamma}\delta^{ij}\partial^2_{ij} v + \frac{e_0 N}{N^3} e_0 v + \frac{e^{-2\gamma}}{N}\delta^{ij}\partial_i N \partial_j v\right) \\ & + \frac{2\partial_l \beta^j}{N^2} e_0 \partial_j v - \frac{\partial_l \beta^j \partial_j \beta^q}{N^2}\partial_q v + \frac{e_0(\partial_l \beta^j)}{N^2}\partial_j v, \end{aligned}$$

where in the last line we expressed $e_0^2 v$ in terms of the other derivatives and $\Box_g v$ using (2.12).

The second term is

$$\left[\partial_l, e^{-2\gamma}\delta^{ij}\partial^2_{ij}\right] v = -2\partial_l \gamma \cdot e^{-2\gamma}\delta^{ij}\partial^2_{ij} v.$$

The third term is

$$\left[\partial_l, \frac{e_0 N}{N^3} e_0\right] v = \partial_l (\frac{e_0 N}{N^3}) e_0 v - \frac{e_0 N}{N^3} \partial_l \beta^j \partial_j v$$
$$= \frac{\partial_l e_0 \log(N)}{N^2} e_0 v - \frac{2\partial_l \log(N) e_0 \log(N)}{N^2} e_0 v - \frac{e_0 N}{N^3} \partial_l \beta^j \partial_j v,$$

and the fourth term is

$$\left[\partial_l, \frac{e^{-2\gamma}}{N} \delta^{ij} \partial_i N \partial_j\right] v = \partial_l (\frac{e^{-2\gamma}}{N} \partial_i N) \delta^{ij} \partial_j v$$
$$= -2\partial_l \gamma \cdot e^{-2\gamma} \delta^{ij} \partial_i \log(N) \partial_j v + e^{-2\gamma} \delta^{ij} \partial^2_{il} \log(N) \partial_j v.$$

Finally, we regroup according to our convention described above, noticing that the $\frac{2\partial_l \log(N) e_0 \log(N)}{N^2} e_0 v$ terms cancel in $III(\partial_l)$. □

8.2.3 Commuting the wave operator with E_k

Lemma 8.6 *For any C^3 function v,*

$$\left[E_k, \Box_g\right] v = I(E_k)(v) + II(E_k)(v) + III(E_k)(v),$$

where

$$I(E_k)(v) = (-2X_k \chi_k + 2E_k \eta_k - L_k \chi_k) \cdot E_k v - E_k \chi_k \cdot X_k v \\ + (L_k E_k + 2X_k E_k - E_k X_k) \log(N) \cdot L_k v, \tag{8.12}$$

$$II(E_k)(v) = 2E_k \log(N) \cdot L_k^2 v + 2(E_k \log(N) + \eta_k - K(E_k, X_k)) \cdot X_k L_k v \\ - 2\chi_k \cdot X_k E_k v - 2K(E_k, E_k) \cdot E_k L_k v, \tag{8.13}$$

$$III(E_k)(v) = \chi_k \cdot (\eta_k - K(E_k, X_k)) \cdot X_k v \\ + \chi_k \cdot (2\chi_k - X_k \log N - K(E_k, E_k)) \cdot E_k v \\ + E_k \log(N) \cdot (X_k \log(N) - \chi_k) \cdot L_k v. \tag{8.14}$$

Proof *Step 1: The main computation.* From (8.8), we see that

$$\begin{aligned} &\left[E_k, \Box_g\right] v \\ &= L_k([L_k, E_k]v) + [L_k, E_k] L_k v + 2X_k([L_k, E_k]v) + 2[X_k, E_k] L_k v \\ &\quad + 2E_k \eta_k \cdot E_k v - E_k \chi_k \cdot X_k v - \chi_k \cdot [E_k, X_k] v \\ &\quad - (X_k \log(N)) \cdot [E_k, L_k] v - E_k X_k \log(N) \cdot L_k v. \end{aligned} \tag{8.15}$$

We deal with all the terms one by one. For the first commutator, notice using (2.39) that

$$L_k([L_k, E_k]v) = E_k \log(N) \cdot L_k^2 v - \chi_k \cdot L_k E_k v$$

$$- L_k \chi_k \cdot E_k v + L_k E_k \log(N) \cdot L_k v, \tag{8.16}$$

Similarly,

$$[L_k, E_k] L_k v = E_k \log(N) \cdot L_k^2 v - \chi_k \cdot E_k L_k v. \tag{8.17}$$

Now using (2.39) and (2.42), we obtain

$$\begin{aligned} X_k([L_k, E_k]v) &= E_k \log(N) \cdot X_k L_k v - \chi_k \cdot X_k E_k v \\ &\quad - X_k \chi_k \cdot E_k v + X_k E_k \log(N) \cdot L_k v, && (8.18) \\ [X_k, E_k] L_k v &= (\eta_k - K(E_k, X_k)) \cdot X_k L_k v \\ &\quad + (\chi_k - K(E_k, E_k)) \cdot E_k L_k v. && (8.19) \end{aligned}$$

Step 2: Rewriting some terms We rearrange the term $-\chi_k L_k E_k v$ from (8.16), which we write by (2.39) as

$$\begin{aligned} -\chi_k L_k E_k v &= -\chi_k E_k L_k v - \chi_k [L_k, E_k] v \\ &= -\chi_k (E_k L_k v - \chi_k \cdot E_k v + E_k \log N \cdot L_k v). \end{aligned} \tag{8.20}$$

Notice that all instances of $\chi_k \cdot E_k L_k v$ in (8.16) + (8.17) + 2 × (8.18) + 2 × (8.19) then cancel.

Finally, we conclude the proof by plugging (8.16)–(8.19) into (8.15), expanding the remaining terms using Lemma 2.11, and finally substituting in (8.20). Note that we split into the terms I, II and III according to our convention described in Sect. 8.2.1 above. □

8.2.4 Commuting the wave operator with L_k

Lemma 8.7 *For any C^3 function v,*

$$\left[L_k, \Box_g\right] v = I(L_k)(v) + II(L_k)(v) + III(L_k)(v),$$

where

$$\begin{aligned} I(L_k)(v) = &- L_k \chi_k \cdot X_k v + (2 L_k \eta_k - E_k \chi_k) \cdot E_k v \\ &+ \left(E_k^2 \log(N) - L_k X_k \log(N)\right) \cdot L_k v, \end{aligned} \tag{8.21}$$

$$\begin{aligned} II(L_k)(v) = &\, 2 (K(E_k, X_k) + \eta_k) \cdot E_k L_k v + 2 (K(X_k, X_k) - X_k \log(N)) \cdot X_k L_k v \\ &- 2 X_k \log(N) \cdot L_k^2 v - 2 \chi_k \cdot E_k^2 v, \end{aligned} \tag{8.22}$$

$$\begin{aligned} III(L_k)(v) = &\, \chi_k \cdot (K(X_k, X_k) - X_k \log(N)) \cdot X_k v \\ &+ \chi_k \cdot (K(E_k, X_k) - 2 E_k \log(N) - \eta_k) \cdot E_k v \\ &+ (2 \eta_k \cdot E_k \log(N) + (E_k \log(N))^2 - \chi_k \cdot X_k \log(N)) \cdot L_k v. \end{aligned} \tag{8.23}$$

Proof By (8.8), we have

$$
\begin{aligned}
&\left[L_k, \Box_g\right] v \\
&= \overbrace{-2[L_k, X_k]L_k v}^{A(L_k)} + \overbrace{[L_k, E_k]E_k v + E_k([L_k, E_k]v)}^{B(L_k)} + \overbrace{2L_k\eta_k \cdot E_k v - L_k\chi_k \cdot X_k v}^{C(L_k)} \\
&\underbrace{-L_k X_k \log(N) L_k v}_{D_1(L_k)} + \underbrace{2\eta_k[L_k, E_k]v - \chi_k[L_k, X_k]v}_{D_2(L_k)} .
\end{aligned}
$$

We treat each term separately. We start with $A(L_k)$ and using (2.45) we obtain

$$
\begin{aligned}
A(L_k) = {}& 2(K(E_k, X_k) - E_k \log N + \eta_k) \cdot E_k L_k v + 2(K(X_k, X_k) \\
& - X_k \log(N)) \cdot X_k L_k v - 2X_k \log(N) \cdot L_k^2 v.
\end{aligned}
$$

Now we handle $B(L_k)$ using (2.39):

$$
\begin{aligned}
& B(L_k) \\
={}& -2\chi_k \cdot E_k^2 v - E_k\chi_k \cdot E_k v + E_k \log(N) \cdot L_k E_k v + E_k \log(N) \cdot E_k L_k v \\
& + E_k^2 \log(N) \cdot L_k v \\
={}& -2\chi_k \cdot E_k^2 v - E_k\chi_k \cdot E_k v + 2E_k \log(N) \cdot E_k L_k v + E_k^2 \log(N) \cdot L_k v \\
& - E_k \log N(\chi_k \cdot E_k v - E_k \log N \cdot L_k v).
\end{aligned}
$$

For $C(L_k)$ and $D_1(L_k)$, there is nothing to do.
Finally, for $D_2(L_k)$ we use (2.39) and (2.45) to get

$$
\begin{aligned}
D_2(L_k) = {}& \chi_k \cdot (-\eta_k + K(E_k, X_k) - E_k \log(N)) \cdot E_k v \\
& + \chi_k \cdot (K(X_k, X_k) - X_k \log(N)) \cdot X_k v \\
& + (2\eta_k E_k \log(N) - \chi_k \cdot X_k \log(N)) \cdot L_k v.
\end{aligned}
$$

Rearranging the terms according to conventions in Sect. 8.2.1 for I, II and III yields the conclusion. □

8.3 Estimating the commutator $[\Box_g, \partial_i]$

In the remainder of this section, we bound the commutators of $\Box_g$ with different vector fields. Once we bound the commutators, we also obtain an energy estimate for the commuted quantity using Proposition 7.3.

In this subsection, we begin with the commutator $[\Box_g, \partial_i]$. We recall that this commutator is computed in Lemma 8.5.

Proposition 8.8 *For any $k \in \{1, 2, 3\}$, define $\mathcal{D}$ as one of the sets in (8.1). Then the following holds for all solutions v to $\Box_g v = f$, with* $\mathrm{supp}(v)$, $\mathrm{supp}(f) \subseteq \{(t, x) : |x| \leq R\}$*:*

$$\|[\Box_g, \partial_i]v\|_{L^1_t([0,T_B),L^2(\Sigma_t\cap\mathcal{D}))} \lesssim \epsilon^{\frac{3}{2}} \cdot (\|\partial_x \partial v\|_{L^1_t([0,T_B),L^2(\Sigma_t\cap\mathcal{D}))} + \|f\|_{L^1_t([0,T_B),L^2(\Sigma_t\cap\mathcal{D}))}).$$

Proof We control each term in Lemma 8.5. Controlling the metric terms using (5.1), we immediately obtain

$$\|II(\partial_l)(v)\|_{L^2(\Sigma_t\cap\mathcal{D})} \lesssim \epsilon^{\frac{3}{2}} \cdot (\|\partial_x \partial v\|_{L^2(\Sigma_t\cap\mathcal{D})} + \|\Box_g v\|_{L^2(\Sigma_t\cap\mathcal{D})}),$$
$$\|III(\partial_l)(v)\|_{L^2(\Sigma_t\cap\mathcal{D})} \lesssim \epsilon^3 \cdot \|\partial v\|_{L^2(\Sigma_t\cap\mathcal{D})} \lesssim \epsilon^3 \cdot \|\partial_x \partial v\|_{L^2(\Sigma_t\cap\mathcal{D})},$$

where in the last inequality, we used $\mathrm{supp}(v) \subseteq B(0, R)$ and Poincaré's inequality.

For $I(\partial_l)(v)$, notice that after using (5.1) and the support properties, each term is bounded above by $|\partial\partial_x \mathfrak{g}\cdot\partial v|$, where $\partial\partial_x \mathfrak{g}$ is as in Proposition 8.3. Thus, Proposition 8.3 implies

$$\|I(\partial_l)(v)\|_{L^2(\Sigma_t\cap\mathcal{D})} \lesssim \epsilon^{\frac{3}{2}} \cdot \|\partial v\|_{H^1(\Sigma_t\cap\mathcal{D})} \lesssim \epsilon^{\frac{3}{2}} \cdot \|\partial_x \partial v\|_{L^2(\Sigma_t\cap\mathcal{D})},$$

where we again used Poincaré's inequality in the last inequality.

Taking the L^1_t norm of these three inequalities, and using $\Box_g v = f$, yields the claimed estimate. □

Now, we use the commutator estimate in Proposition 8.8 to control the energy for the commuted function:

Proposition 8.9 *Suppose $\Box_g v = f$ with v and f both smooth and compactly supported in $B(0, R)$ for every $t \in [0, T_B)$. Let $-\infty \le U_0 \le U_1 \le +\infty$, with either $U_0 = -\infty$ or $U_1 = \infty$ (or both). Let $\mathcal{D} = \mathcal{D}^{(k),T_B}_{U_0,U_1}$, where $\mathcal{D}^{(k),T_B}_{U_0,U_1}$ is given by (7.9).*

Then, for any k',

$$\sup_{0\le t<T_B} \|\partial\partial_x v\|_{L^2(\Sigma_t\cap\mathcal{D})} + \sup_{u_{k'}\in[U_0,U_1)} \sum_{Z_{k'}\in\{L_{k'},E_{k'}\}} \|Z_{k'}\partial_x v\|_{L^2(C^{k'}_{u_{k'}}\cap\mathcal{D})}$$
$$\lesssim \|\partial\partial_x v\|_{L^2(\Sigma_0\cap\mathcal{D})} + \sum_{Z_k\in\{L_k,E_k\}} \|Z_k\partial_x v\|_{L^2(C^k_{U_0})} + \|\partial_x f\|_{L^1_t([0,T_B),L^2(\Sigma_t\cap\mathcal{D}))}.$$

Proof We apply the energy estimate in Proposition 7.3, but to $\partial_i v$ instead of v. Notice that $\Box_g(\partial_i v) = [\Box_g, \partial_i]v + \partial_i f$. Hence, combining the energy estimate in Proposition 7.3 with the bound for the commutator $[\Box_g, \partial_i]$ in Proposition 8.8, we obtain

$$\sup_{0\le t<T_B} \|\partial\partial_x v\|_{L^2(\Sigma_t\cap\mathcal{D})} + \sup_{u_{k'}\in[U_0,U_1)} \sum_{Z_{k'}\in\{L_{k'},E_{k'}\}} \|Z_{k'}\partial_x v\|_{L^2(C^{k'}_{u_{k'}}\cap\mathcal{D})}$$
$$\lesssim \|\partial\partial_x v\|_{L^2(\Sigma_0\cap\mathcal{D})} + \sum_{Z_k\in\{L_k,E_k\}} \|Z_k\partial_x v\|_{L^2(C^k_{U_0})} + \epsilon^{\frac{3}{2}} \cdot \|\partial_x \partial v\|_{L^1_t([0,T_B),L^2(\Sigma_t\cap\mathcal{D}))}$$
$$+ \|f\|_{L^1_t([0,T_B),L^2(\Sigma_t\cap\mathcal{D}))} + \|\partial_x f\|_{L^1_t([0,T_B),L^2(\Sigma_t\cap\mathcal{D}))}.$$

Now notice that by $T_B \le 1$, $\epsilon^{\frac{3}{2}}\|\partial_x \partial v\|_{L^1_t([0,T_B),L^2(\Sigma_t\cap\mathcal{D}))} \lesssim \epsilon^{\frac{3}{2}} \sup_{0\le t<T_B} \|\partial\partial_x v\|_{L^2(\Sigma_t\cap\mathcal{D})}$, and hence this term can be absorbed by the left-hand

side. Moreover, using that $\mathrm{supp}(f) \subseteq B(0,R)$, we have $\|f\|_{L^1_t([0,T_B),L^2(\Sigma_t\cap\mathcal{D}))} \lesssim \|\partial_x f\|_{L^1_t([0,T_B),L^2(\Sigma_t\cap\mathcal{D}))}$ by Poincaré's inequality. Combining all these observations yields the desired estimate. □

8.4 Estimating the commutator $[\Box_g, E_k]$

Next, we turn to the commutator $[\Box_g, E_k]$. Unlike the estimates in Sect. 8.3, when we bound $[\Box_g, E_k]v$, we will not assume v to be compactly supported. (We remark that such bounds for non-compactly supported v are needed for the applications in Sect. 12.)

Proposition 8.10 *Fix k and define $\mathcal{D}$ as one of the sets in (8.1). Let v be smooth function which is in Schwartz class for every $t \in [0, T_B)$. Then, for all $r \geq 1$, the following holds for all $t \in [0, T_B)$:*

$$\|\langle x\rangle^{-\frac{r}{2}}[\Box_g, E_k]v\|_{L^2(\Sigma_t\cap\mathcal{D})} \lesssim \epsilon^{\frac{3}{2}} \cdot (\|\partial v\|_{L^2(\Sigma_t\cap\mathcal{D})} + \sum_{Z_k\in\{E_k,L_k\}} \|\partial Z_k v\|_{L^2(\Sigma_t\cap\mathcal{D})}),$$

where the implicit constant is allowed to depend on r.

Proof Recall the computation of $[\Box_g, E_k]$ in Lemma 8.6. We now bound the terms $I(E_k)$, $II(E_k)$, $III(E_k)$ from Lemma 8.6.

We first recall the definition of $I(E_k)(v)$ in (8.12). By the L^∞ bound for $L_k\chi_k$ in (5.14), the L^∞ estimates for the geometric vector fields in Proposition 5.3, and the L^∞ estimates for the metric coefficients and their derivatives in (5.1), we get (recalling $\alpha = 0.01$)

$$|I(E_k)|(v) \lesssim \underbrace{\epsilon^{\frac{3}{2}}|\partial v|}_{=:A} + \underbrace{\langle x\rangle^{\alpha}\,(|\partial_x\chi_k| + |E_k\eta_k|)\,|E_k v|}_{=:B} + \underbrace{\langle x\rangle^{\nu}|\partial_x\partial N|\cdot|L_k v|}_{=:D}. \tag{8.24}$$

We control the $L^2_{-\frac{r}{2}}(\Sigma_t)$ norm of each term. The term A obviously satisfies

$$\|A\|_{L^2_{-\frac{r}{2}}(\Sigma_t\cap\mathcal{D})} \lesssim \epsilon^{\frac{3}{2}}\|\partial v\|_{L^2(\Sigma_t\cap\mathcal{D})}. \tag{8.25}$$

For B, we use Proposition 8.2, $r \geq 1$ and Proposition 5.3 to obtain

$$\begin{aligned}\|B\|_{L^2_{-\frac{r}{2}}(\Sigma_t\cap\mathcal{D})} &\lesssim \|(|\partial_x\chi_k| + |E_k\eta_k|)\cdot E_k v\|_{L^2_{-\frac{1}{4}}(\Sigma_t\cap\mathcal{D})}\\ &\lesssim \epsilon^{\frac{3}{2}}\|E_k v\|_{L^2_{-\frac{1}{8}}(\Sigma_t\cap\mathcal{D})} + \epsilon^{\frac{3}{2}}\|E_k^2 v\|_{L^2_{-\frac{1}{8}}(\Sigma_t\cap\mathcal{D})}\\ &\lesssim \epsilon^{\frac{3}{2}}\|\partial v\|_{L^2_{-\frac{1}{16}}(\Sigma_t\cap\mathcal{D})} + \epsilon^{\frac{3}{2}}\|\partial E_k v\|_{L^2_{-\frac{1}{16}}(\Sigma_t\cap\mathcal{D})}\\ &\lesssim \epsilon^{\frac{3}{2}}\|\partial v\|_{L^2(\Sigma_t)} + \epsilon^{\frac{3}{2}}\|\partial E_k v\|_{L^2(\Sigma_t\cap\mathcal{D})}.\end{aligned} \tag{8.26}$$

The term D can be handled by Proposition 8.3, giving

$$\begin{aligned} &\|\langle x\rangle^{-\frac{r}{2}+\alpha}|\partial\partial_x N|\cdot|L_k v|\|_{L^2(\Sigma_t\cap\mathcal{D})} \\ &\quad\lesssim \epsilon^{\frac{3}{2}}(\|\partial v\|_{L^2(\Sigma_t\cap\mathcal{D})}+\|\partial L_k v\|_{L^2(\Sigma_t\cap\mathcal{D})}). \end{aligned} \tag{8.27}$$

Combining (8.24)–(8.27), we obtain

$$\begin{aligned} &\|\langle x\rangle^{-\frac{r}{2}} I(E_k)(v)\|_{L^2(\Sigma_t\cap\mathcal{D})} \\ &\quad\lesssim \epsilon^{\frac{3}{2}}\cdot\Big(\|\partial v\|_{L^2(\Sigma_t\cap\mathcal{D})}+\sum_{Z_k\in\{E_k,L_k\}}\|\partial Z_k v\|_{L^2(\Sigma_t\cap\mathcal{D})}\Big). \end{aligned} \tag{8.28}$$

By (8.13) and the estimates in Propositions 5.2, 5.3 and 5.5, we have the pointwise estimate[12]

$$|II(E_k)(v)|\lesssim \epsilon^{\frac{3}{2}}\left(|\partial L_k v|+|\partial E_k v|\right). \tag{8.29}$$

In a similar manner, but starting with (8.14), we also obtain the pointwise estimate

$$|III(E_k)(v)|\lesssim \epsilon^3|\partial v|. \tag{8.30}$$

By (8.29) and (8.30), it follows immediately that

$$\begin{aligned} &\|\langle x\rangle^{-\frac{r}{2}} II(E_k)(v)\|_{L^2(\Sigma_t\cap\mathcal{D})}+\|\langle x\rangle^{-\frac{r}{2}} III(E_k)(v)\|_{L^2(\Sigma_t\cap\mathcal{D})} \\ &\lesssim \epsilon^{\frac{3}{2}}\cdot(\|\partial v\|_{L^2(\Sigma_t\cap\mathcal{D})}+\sum_{Z_k\in\{E_k,L_k\}}\|\partial Z_k v\|_{L^2(\Sigma_t\cap\mathcal{D})}). \end{aligned} \tag{8.31}$$

Combining Lemma 8.6, (8.28) and (8.31) yields the conclusion. □

In the next proposition, we are going to use Proposition 8.10 to estimate the energy commuted with the vector field E_k, this time for compactly supported functions (so that the spatial weights become irrelevant).

Proposition 8.11 *Suppose $\Box_g v = f$ with v and f both smooth and compactly supported in $B(0, R)$ for every $t \in [0, T_B)$. Let $-\infty \le U_0 \le U_1 \le +\infty$, with either $U_0 = -\infty$ or $U_1 = \infty$ (or both). Let $\mathcal{D} = \mathcal{D}^{(k),T_B}_{U_0,U_1}$, where $\mathcal{D}^{(k),T_B}_{U_0,U_1}$ is given by (7.9).*

Then, for any k',

[12] Notice that one could even put in additional decaying weights of $\langle x\rangle$ in this estimate, but this will not be necessary.

$$
\begin{aligned}
&\sup_{0\le t<T_B} \|\partial E_k v\|_{L^2(\Sigma_t\cap\mathcal{D})} + \sup_{u_{k'}\in\mathbb{R}} \sum_{Z_{k'}\in\{L_{k'},\, E_{k'}\}} \|Z_{k'} E_k v\|_{L^2(C^{k'}_{u_{k'}}\cap\mathcal{D})} \\
&\lesssim \|\partial E_k v\|_{L^2(\Sigma_0\cap\mathcal{D})} + \sum_{Z_k\in\{L_k,\, E_k\}} \|Z_k E_k v\|_{L^2(C^k_{U_0})} \\
&\quad + \epsilon^{\frac{3}{2}} (\|\partial v\|_{L^1_t([0,T_B),L^2(\Sigma_t\cap\mathcal{D}))} + \sum_{Z_k\in\{E_k,L_k\}} \|\partial Z_k v\|_{L^1_t([0,T_B),L^2(\Sigma_t\cap\mathcal{D}))}) \\
&\quad + \|E_k f\|_{L^1_t([0,T_B),L^2(\Sigma_t\cap\mathcal{D}))}.
\end{aligned}
$$

Proof This is an immediate consequence of the combination of Proposition 7.3 and Proposition 8.10, since

$$\Box_g(E_k v) = [\Box_g, E_k]v + E_k f.$$

□

8.5 Estimating the commutator $[\Box_g, L_k]$

The final commutator to estimate is $[\Box_g, L_k]$. We will prove analogues of Propositions 8.10 and 8.11 with E_k replaced by L_k.

Proposition 8.12 *Fix k and define $\mathcal{D}$ as one of the sets in (8.1). Let v be smooth function which is in Schwartz class for every $t \in [0, T_B)$. Then, for all $r \ge 1$, the following holds for all $t \in [0, T_B)$:*

$$\|\langle x\rangle^{-\frac{r}{2}} [\Box_g, L_k] v\|_{L^2(\Sigma_t\cap\mathcal{D})} \lesssim \epsilon^{\frac{3}{2}} \cdot (\|\partial v\|_{L^2(\Sigma_t\cap\mathcal{D})} + \sum_{Z_k\in\{E_k,L_k\}} \|\partial Z_k v\|_{L^2(\Sigma_t\cap\mathcal{D})}),$$

where the implicit constant is allowed to depend on r.

Proof We bound the terms $I(L_k)$, $II(L_k)$, $III(L_k)$ from Lemma 8.7, following the same lines of reasoning as for Proposition 8.10. We get (recall $\alpha = 0.01$):

$$
\begin{aligned}
&|I(L_k)(v)| \lesssim \epsilon^{\frac{3}{2}} |\partial v| + \langle x\rangle^{2\alpha} (|\partial_x \chi_k| + |\partial\partial_x N|) \cdot |\partial v|, \\
&|II(L_k)(v)| \lesssim \epsilon^{\frac{3}{2}} |\partial L_k v|, \\
&|III(L_k)(v)| \lesssim \epsilon^3 |\partial v|.
\end{aligned}
$$

These terms are exactly those in Proposition 8.10, and therefore can be treated in exactly the same manner. □

The next proposition is analogous to Proposition 8.11:

Proposition 8.13 *Suppose $\Box_g v = f$ with v and f both smooth and compactly supported in $B(0, R)$ for every $t \in [0, T_B)$. Let $-\infty \le U_0 \le U_1 \le +\infty$, with either $U_0 = -\infty$ or $U_1 = \infty$ (or both). Let $\mathcal{D} = \mathcal{D}^{(k),T_B}_{U_0,U_1}$, where $\mathcal{D}^{(k),T_B}_{U_0,U_1}$ is given by (7.9).*

Then, for any k'

$$\sup_{0\leq t<T_B}\|\partial L_k v\|_{L^2(\Sigma_t\cap\mathcal{D})}+\sup_{u_{k'}\in\mathbb{R}}\sum_{Z_{k'}\in\{L_{k'},E_{k'}\}}\|Z_{k'}L_k v\|_{L^2(C^{k'}_{u_{k'}}\cap\mathcal{D})}$$
$$\lesssim\|\partial L_k v\|_{L^2(\Sigma_0)}+\epsilon^{\frac{3}{2}}\cdot(\|\partial v\|_{L^1_t([0,T_B),L^2(\Sigma_t\cap\mathcal{D}))}+\sum_{Z_k\in\{E_k,L_k\}}\|\partial Z_k v\|_{L^1_t([0,T_B),L^2(\Sigma_t\cap\mathcal{D}))})$$
$$+\|L_k f\|_{L^1_t([0,T_B),L^2(\Sigma_t\cap\mathcal{D}))}.$$

Proof Noting $\Box_g(L_k v)=[\Box_g,L_k]v+L_k f$, this is an immediate consequence of Propositions 7.3 and 8.12. ◻

9 Energy estimates for $\widetilde{\phi}_k$ up to two derivatives I: the basic estimates

The goal of this section is to obtain energy estimates, for the scalar field commuted with zero or one derivative on the whole of Σ_t. When there is no commutations or one commutation with a good derivative, we bound the energy uniformly in δ, while if there is one commutation with a general spatial derivative, we allow the energy to grow in δ^{-1}.

We note already that some of these estimates will be later improved in Sect. 10, by localizing on different regions of the spacetime.

The main result of this section is the next proposition:

Proposition 9.1 *The following energy estimate holds for the lowest order energy:*

$$\sup_{0\leq t<T_B}\|\partial\widetilde{\phi}_k\|_{L^2(\Sigma_t)}+\sup_{u_{k'}\in\mathbb{R}}\sum_{Z_{k'}\in\{L_{k'},E_{k'}\}}\|Z_{k'}\widetilde{\phi}_k\|_{L^2(C^{k'}_{u_{k'}}([0,T_B)))}\lesssim\epsilon. \tag{9.1}$$

The following energy estimate holds after commutation with one good vector field:

$$\sum_{Z_k\in\{L_k,E_k\}}\Big(\sup_{0\leq t<T_B}\|\partial Z_k\widetilde{\phi}_k\|_{L^2(\Sigma_t)}+\sup_{u_k\in\mathbb{R}}\sum_{Y_{k'}\in\{L_{k'},E_{k'}\}}\|Y_{k'}Z_k\widetilde{\phi}_k\|_{L^2(C^{k'}_{u_{k'}}([0,T_B)))}\Big)\lesssim\epsilon. \tag{9.2}$$

Finally, the following energy estimate holds for more general second derivatives of $\widetilde{\phi}_k$:

$$\sup_{0\leq t<T_B}\|\partial^2\widetilde{\phi}_k\|_{L^2(\Sigma_t)}+\sup_{u_{k'}\in\mathbb{R}}\sum_{Z_{k'}\in\{L_{k'},E_{k'}\}}\|Z_{k'}\partial_x\widetilde{\phi}_k\|_{L^2(C^{k'}_{u_{k'}}([0,T_B)))}\lesssim\epsilon\cdot\delta^{-\frac{1}{2}}. \tag{9.3}$$

Proof *Step 1: Proof of* (9.1) This is an immediate consequence of the energy estimate in Proposition 7.3 (with $v=\widetilde{\phi}_k$, $f=0$, $U_0=-\infty$, and $U_1=\infty$), and the initial data bound in (4.3a).

Step 2: Proof of (9.2) Summing the estimates in Propositions 8.11 and 8.13 with $v = \widetilde{\phi}_k$ (so that $f = \Box_g v = \Box_g \widetilde{\phi}_k = 0$), we have

$$\begin{aligned}&\sum_{Z_k \in \{L_k, E_k\}} \Big(\sup_{0 \le t < T_B} \|\partial Z_k \widetilde{\phi}_k\|_{L^2(\Sigma_t)} + \sup_{u_k \in \mathbb{R}} \sum_{Y_{k'} \in \{L_{k'}, E_{k'}\}} \|Y_{k'} Z_k \widetilde{\phi}_k\|_{L^2(C^{k'}_{u_{k'}}([0,T_B)))} \Big) \\ &\lesssim \underbrace{\sum_{Z_k \in \{L_k, E_k\}} \|\partial Z_k \widetilde{\phi}_k\|_{L^2(\Sigma_0)}}_{=:I} + \underbrace{\epsilon^{\frac{3}{2}} \|\partial \widetilde{\phi}_k\|_{L^1_t([0,T_B), L^2(\Sigma_t))}}_{=:II} + \underbrace{\epsilon^{\frac{3}{2}} \sum_{Z_k \in \{E_k, L_k\}} \|\partial Z_k \widetilde{\phi}_k\|_{L^1_t([0,T_B), L^2(\Sigma_t))}}_{=:III}.\end{aligned} \tag{9.4}$$

The data term can be controlled using (4.3a) and (4.3b) by $I \lesssim \epsilon$. The term $II \lesssim \epsilon^{\frac{3}{2}} \cdot \epsilon \lesssim \epsilon^{\frac{5}{2}}$ by (9.1). For the term III, we can absorb it by the first term on the left-hand side, after choosing ϵ_0 smaller if necessary. Putting all these together gives (9.2).

Step 3: Proof of (9.3) Using Proposition 8.9 with $v = \widetilde{\phi}_k$, $U_0 = -\infty$ and $U_1 = \infty$, we obtain

$$\begin{aligned}&\sup_{t \in [0,T_B)} \|\partial \partial_x \widetilde{\phi}_k\|_{L^2(\Sigma_t)} + \sup_{u_{k'} \in \mathbb{R}} \sum_{Z_{k'} \in \{L_{k'}, E_{k'}\}} \|Z_{k'} \partial_x \widetilde{\phi}_k\|_{L^2(C^{k'}_{u_{k'}}([0,T_B)))} \\ &\lesssim \|\partial \partial_x \widetilde{\phi}_k\|_{L^2(\Sigma_0)} \lesssim \epsilon \delta^{-\frac{1}{2}},\end{aligned} \tag{9.5}$$

where we used (4.9b) in the final inequality. In particular, this controls every term in (9.3), with the only exception being the term $\sup_{t \in [0,T_B)} \|\partial^2_{tt} \widetilde{\phi}_k\|_{L^2(\Sigma_t)}$.

In order to bound $\sup_{t \in [0,T_B)} \|\partial^2_{tt} \widetilde{\phi}_k\|_{L^2(\Sigma_t)}$, we write $\partial^2_{tt} = \partial_t(\beta^i \partial_i + N \cdot L_k + N \cdot X_k)$. Then, using the bounds for the metric in Proposition 5.2, together with Proposition 5.3, we have

$$\sup_{t \in [0,T_B)} \|\partial^2_{tt} \widetilde{\phi}_k\|_{L^2(\Sigma_t)} \lesssim \|\partial \widetilde{\phi}_k\|_{L^2(\Sigma_t)} + \|\partial \partial_x \widetilde{\phi}_k\|_{L^2(\Sigma_t)} + \sum_{Z_k \in \{E_k, L_k\}} \|\partial Z_k \widetilde{\phi}_k\|_{L^2(\Sigma_t)} \lesssim \epsilon \delta^{-\frac{1}{2}},$$

where at the end we used (9.1), (9.2) and (9.5). Putting everything together gives (9.3). □

10 Energy estimates for $\widetilde{\phi}_k$ up to two derivatives II: the improved estimates

In this section, we derive improved estimates for the first and second derivatives of $\widetilde{\phi}_k$. We will obtain two improvements:

- In the (slightly enlarged) singular region $S^k_{2\delta}$, $\partial \widetilde{\phi}_k$ and $\partial Z_k \widetilde{\phi}_k$ satisfy *smallness* (in terms of δ) bounds in energy. (See (1.21) and (1.22) in the introduction, and Proposition 10.2 below.)

- Away from the singular region, i.e. in $\Sigma_t \setminus S^k_{2\delta}$, the L^2 norm of $\partial^2\widetilde{\phi}_k$ is bounded independently of δ^{-1}, in contrast to the global bound in (9.3). (See (1.24) in the introduction, and Proposition 10.3 below.)

These two improved bounds are highly related: indeed, in order to obtain the latter estimate, we use the former estimate together with a slice-picking argument.

We begin with the localized estimate *restricted to the initial data*.

Proposition 10.1 *The following estimates hold on the initial hypersurface Σ_0:*

$$\|\partial\widetilde{\phi}_k\|_{L^2(\Sigma_0\cap S^k_{2\delta})} \lesssim \epsilon\cdot\delta^{\frac{1}{2}}, \tag{10.1}$$

$$\sum_{Z_k\in\{L_k,\,E_k\}} \|\partial Z_k\widetilde{\phi}_k\|_{L^2(\Sigma_0\cap S^k_{2\delta})} \lesssim \epsilon\cdot\delta^{\frac{1}{2}}. \tag{10.2}$$

Proof Recall that on the initial hypersurface Σ_0, (u_k,θ_k) are affine functions of (x^1,x^2); see (2.22) and (2.50). Therefore, in all the following estimates, we can easily bound $|\partial_{u_k} f|\lesssim|\partial_x f|$, as well as pass between $L^2_{x^1,x^2}(\Sigma_0)$ and $L^2_{u_k,\theta_k}(\Sigma_0)$.

Given any $f:\Sigma_0\to\mathbb{R}$ such that $\mathrm{supp}(f)\subseteq B(0,R)\cap\{u_k\geq-\delta\}$, the fundamental theorem of calculus, the Minkowski inequality and the Cauchy–Schwarz inequality imply that for every $u_k\geq-\delta$,

$$\|f\|_{L^2_{\theta_k}(\Sigma_0\cap C^k_{u_k})} \lesssim \int_{-\delta}^{u_k} \|\partial_{u_k} f\|_{L^2_{\theta_k}(\Sigma_0\cap C^k_{\tilde{u}_k})}\, d\tilde{u}_k \lesssim |u_k+\delta|^{\frac{1}{2}}\|\partial_x f\|_{L^2(\Sigma_0)}. \tag{10.3}$$

We now apply (10.3) to $f=\partial\widetilde{\phi}_k$ and $f=\partial Z_k\widetilde{\phi}_k$ (for $Z_k\in\{E_k,L_k\}$). First, by (4.4), we have

$$\|\partial_x\partial\widetilde{\phi}_k\|_{L^2(\Sigma_0)} + \|\partial_x Z_k\partial\widetilde{\phi}_k\|_{L^2(\Sigma_0)} \lesssim \epsilon\cdot\delta^{-\frac{1}{2}}.$$

Hence, using (10.3), we have

$$\sup_{u_k\in[-2\delta,2\delta]}\Big(\|\partial\widetilde{\phi}_k\|_{L^2_{\theta_k}(\Sigma_0\cap C^k_{u_k})} + \sum_{Z_k\in\{L_k,\,E_k\}}\|\partial Z_k\widetilde{\phi}_k\|_{L^2_\theta(\Sigma_0\cap C^k_{u_k})}\Big) \lesssim \epsilon. \tag{10.4}$$

Finally, Hölder's inequality implies that

$$\begin{aligned}
&\|\partial\widetilde{\phi}_k\|_{L^2(\Sigma_0\cap S^k_{2\delta})} + \sum_{Z_k\in\{L_k,\,E_k\}}\|\partial Z_k\widetilde{\phi}_k\|_{L^2(\Sigma_0\cap S^k_{2\delta})}\\
&\lesssim \delta^{\frac{1}{2}}\sup_{u_k\in[-2\delta,2\delta]}\Big(\|\partial\widetilde{\phi}_k\|_{L^2_{\theta_k}(\Sigma_0\cap C^k_{u_k})} + \sum_{Z_k\in\{L_k,\,E_k\}}\|\partial Z_k\widetilde{\phi}_k\|_{L^2_{\theta_k}(\Sigma_0\cap C^k_{u_k})}\Big) \lesssim \epsilon\cdot\delta^{\frac{1}{2}}.
\end{aligned} \tag{10.5}$$

□

It is now straightforward to use the energy estimates in Propositions 7.3, 8.11 and 8.13 to propagate the initial data bounds (10.1) and (10.2) to all future times. This gives our first improved energy estimate.

Proposition 10.2

$$\sup_{0\leq t<T_B} \|\partial\widetilde{\phi}_k\|_{L^2(\Sigma_t\cap S^k_{2\delta})} \lesssim \epsilon\cdot\delta^{\frac{1}{2}}, \tag{10.6}$$

$$\sup_{0\leq t<T_B} \sum_{Z_k\in\{L_k,\, E_k\}} \|\partial Z_k\widetilde{\phi}_k\|_{L^2(\Sigma_t\cap S^k_{2\delta})} \lesssim \epsilon\cdot\delta^{\frac{1}{2}}. \tag{10.7}$$

Proof Applying Proposition 7.3 with $v=\widetilde{\phi}_k$, $f=0$, $U_0=-2\delta$, $U_1=2\delta$, and bounding the initial data terms by Proposition 10.1 and Lemma 5.1, we obtain (10.6).

Next, we apply Propositions 8.11 and 8.13 with $v=\widetilde{\phi}_k$, $f=0$, $U_0=-\infty$, $U_1=2\delta$. (Note that even though we apply the propositions with $U_0=-\infty$, since $\widetilde{\phi}_k$ is supported only in $\{u_k\geq-\delta\}$ (by Lemma 5.1), we indeed obtain an estimate which is integrated over $\Sigma_t\cap S^k_{2\delta}$.) We thus obtain

$$\begin{aligned}
&\sup_{0\leq t<T_B}\sum_{Z_k\in\{L_k,E_k\}}\|\partial Z_k v\|_{L^2(\Sigma_t\cap S^k_{2\delta})}\\
\lesssim&\sum_{Z_k\in\{L_k,E_k\}}\|\partial Z_k v\|_{L^2(\Sigma_0\cap S^k_{2\delta})}\\
&+\epsilon^{\frac{3}{2}}\Big(\|\partial v\|_{L^1_t([0,T_B),L^2(\Sigma_t\cap\mathcal{D}))}+\sum_{Z_k\in\{E_k,L_k\}}\|\partial Z_k v\|_{L^1_t([0,T_B),L^2(\Sigma_t\cap S^k_{2\delta}))}\Big).
\end{aligned}\tag{10.8}$$

The first term in (10.8) is bounded $\lesssim\epsilon\delta^{\frac{1}{2}}$ by Proposition 10.1. The second term is $\lesssim\epsilon^{\frac{5}{2}}\delta^{\frac{1}{2}}$ by the estimate (10.6) that we just proved. Finally, the last term obeys

$$\epsilon^{\frac{3}{2}}\sum_{Z_k\in\{E_k,L_k\}}\|\partial Z_k v\|_{L^1_t([0,T_B),L^2(\Sigma_t\cap S^k_{2\delta}))}\lesssim\epsilon^{\frac{3}{2}}\sup_{0\leq t<T_B}\sum_{Z_k\in\{L_k,E_k\}}\|\partial Z_k v\|_{L^2(\Sigma_t\cap S^k_{2\delta})}.$$

This can thus be absorbed by the left-hand side. This concludes the proof of (10.7). □

We now turn to the second improved energy estimate, which is an improved estimate after the singular zone.

Proposition 10.3 *The following away-from-the-singular-zone estimate holds:*

$$\begin{aligned}
&\sup_{t\in[0,T_B)}\|\partial^2\widetilde{\phi}_k\|_{L^2(\Sigma_t\cap C^k_{\geq\delta})}\\
&\quad+\sup_{u_k\in[\delta,+\infty)}\sum_{Z_{k'}\in\{L_{k'},\,E_{k'}\}}\|Z_{k'}\partial_x\widetilde{\phi}_k\|_{L^2(C^{k'}_{u_{k'}}([0,T_B))\setminus S^k_\delta)}\lesssim\epsilon.
\end{aligned}\tag{10.9}$$

Proof *Step 1: Finding a good slice* We square (10.7) and we integrate on $[0, T_B)$ to obtain on $\mathcal{D} := \{0 \le t < T_B, 0 \le u_k \le \delta\}$:

$$\sum_{Z_k \in \{L_k, E_k\}} \int_{\mathcal{D}} |\partial Z_k \widetilde{\phi}_k|^2 dx^1 dx^2 dt \lesssim \epsilon^2 \cdot \delta.$$

Controlling the commutator $[\partial, Z_k]\widetilde{\phi}_k$ using Proposition 5.3 and (10.6), we obtain

$$\sum_{Z_k \in \{L_k, E_k\}} \int_{\mathcal{D}} |Z_k \partial \widetilde{\phi}_k|^2 dx^1 dx^2 dt \lesssim \epsilon^2 \cdot \delta.$$

Since the volume measures $dx^1\, dx^2\, dt$ and $du_k\, d\theta_k\, dt_k$ are comparable (by (5.22)), it follows that

$$\sum_{Z_k \in \{L_k, E_k\}} \int_0^{\delta} \int_0^{T_B} \int_{-\infty}^{\infty} |Z_k \partial \widetilde{\phi}_k|^2 (u_k, \theta_k, t_k)\, d\theta_k\, dt_k\, du_k \lesssim \epsilon^2 \cdot \delta.$$

By the mean value theorem, there exists $u_k^* \in [0, \delta]$ such that

$$\begin{aligned} &\sum_{Z_k \in \{L_k, E_k\}} \|Z_k \partial \widetilde{\phi}_k\|^2_{L^2(C^k_{u_k^*}([0,T_B)))} \\ &\quad \lesssim \sum_{Z_k \in \{L_k, E_k\}} \int_0^{T_B} \int_{-\infty}^{\infty} |Z_k \partial \widetilde{\phi}_k|^2 (u_k^*, \theta_k, t_k)\, d\theta_k\, dt_k \lesssim \epsilon^2. \end{aligned} \tag{10.10}$$

Notice that for this special value u_k^*, the estimate (10.10) is better than the bound provided by Proposition 9.1, which would have $\epsilon^2\delta^{-1}$ on the right-hand side instead of ϵ^2.

Step 2: Applying an energy estimate in the regular region The key point now is that we can apply an energy estimate again, but only in the region where $u_k \ge u_k*$. The initial data for this new problem has two parts: the energy on the hypersurface $C^k_{u_k^*}$ is good (i.e. δ-independent) thanks to (10.10), while the energy on the restriction of the initial hypersurface $\Sigma_0 \cap \{u_k \ge u_*\}$ is good by assumption on the data since $u_k^* \ge 0$.

More precisely, we apply Proposition 8.9 with $v = \widetilde{\phi}_k$, $f = 0$, $U_0 = u_k^*$ and $U_1 = +\infty$. Note in particular that $\mathcal{D}$ corresponds to $C^k_{\ge u_k^*}$.

$$\begin{aligned} &\sup_{t \in [0, T_B)} \|\partial \partial_x \widetilde{\phi}_k\|_{L^2(\Sigma_t \cap C^k_{\ge u_k^*})} + \sup_{u_{k'} \in \mathbb{R}} \sum_{Z_{k'} \in \{L_{k'}, E_{k'}\}} \|Z_k \partial_x \widetilde{\phi}_k\|_{L^2(C^{k'}_{u_{k'}}([0,T_B)) \cap C^k_{\ge u_k^*})} \\ &\lesssim \|\partial^2 \widetilde{\phi}_k\|_{L^2(\Sigma_0 \cap C^k_{\ge u_k^*})} + \sum_{Z_k \in \{L_k, E_k\}} \|Z_k \partial \widetilde{\phi}_k\|_{L^2(C^k_{u_k^*}([0,T_B)))} \lesssim \epsilon, \end{aligned} \tag{10.11}$$

where in the last inequality we used (4.5) and (10.10).

Notice that (10.11) bounds every term in (10.9) except for $\|\partial_{tt}^2\widetilde{\phi}_k\|_{L^2(\Sigma_t\cap C^k_{\geq u_k^*})}$. In order to bound this term, we write $\partial_{tt}^2 = \partial_t(\beta^i\partial_i + N\cdot L_k + N\cdot X_k)$ and use the estimates (5.1), (9.1), (9.2) together with the bound (10.11) that we just established. □

11 Energy estimates for the third derivatives

In this section, we prove energy estimates for the third derivatives of $\widetilde{\phi}_k$ and ϕ_{reg}.

There are two different estimates that we prove. The first type are estimates that concern $\widetilde{\phi}_k$. These are third derivative estimates where among the three derivatives on $\widetilde{\phi}_k$, there is at least one good derivative L_k or E_k; see Proposition 11.7 for a precise statement. As we discussed in Sect. 1.1.6, these derivatives will be proven using specially chosen commutators $E_k\partial_q$ and L_kL_k. It will be shown that the estimates for $\|\partial E_k\partial_q\widetilde{\phi}_k\|_{L^2(\Sigma_t)}$ and $\|\partial L_k^2\widetilde{\phi}_k\|_{L^2(\Sigma_t)}$ will indeed be sufficient to deduce the remaining desired bounds for third derivatives for $\widetilde{\phi}_k$. This will occupy Sects. 11.1–11.4. (Notice that this type of anisotropic third derivative estimates can also be derived for ϕ_{reg}, but it is unnecessary and will not be derived. The fact that this is unnecessary is because $\phi_{reg}\in H^{2+s'}$ uniformly in δ; see Sect. 13.)

The second type of estimates we derive in this section concerns third derivatives for ϕ, where none of the derivatives are required to be good. This includes bounding both $\widetilde{\phi}_k$ and ϕ_{reg}. These estimates will be proven in Sect. 11.5; see Proposition 11.8. These estimates are easier to obtain because we allow the bound to be very large in terms of δ^{-1}.

11.1 Commutations of the three derivatives

We first show that it suffices to control specific combination of order of commutators, namely that we only have to bound $\|\partial E_k\partial_x\widetilde{\phi}_k\|_{L^2(\Sigma_t)}$ and $\|\partial L_k^2\widetilde{\phi}_k\|_{L^2(\Sigma_t)}$; see Corollary 11.3. This is particularly important because $E_k\partial_q$ and L_k^2 have better properties when commuted with $\Box_g$, thus allowing us to obtain the desired estimate.

We first prove the following commutation estimate.

Lemma 11.1 *Let $\sigma\in S_3$ be a permutation, and let $Y^{(1)}$, $Y^{(2)}$ and $Y^{(3)}$ be three (possibly non-distinct) vector fields from the set $\{L_k, E_k, X_k, \vec{n}, \partial_1, \partial_2\}$. Then*

$$\|Y^{(1)}Y^{(2)}Y^{(3)}\widetilde{\phi}_k\|_{L^2(\Sigma_t)} \lesssim \epsilon\cdot\delta^{-\frac{1}{2}} + \|Y^{(\sigma(1))}Y^{(\sigma(2))}Y^{(\sigma(3))}\widetilde{\phi}_k\|_{L^2(\Sigma_t)}.$$

Proof Clearly, it suffices to control

$\|[Y^{(i)}, Y^{(j)}]Y^{(l)}\widetilde{\phi}_k\|_{L^2(\Sigma_t)}$ and $\|Y^{(i)}[Y^{(j)}, Y^{(l)}]\widetilde{\phi}_k\|_{L^2(\Sigma_t)}$. Observe that since $\vec{n} = L_k + X_k$ (by (2.32)), we can assume that $Y^{(1)}, Y^{(2)}, Y^{(3)}\in\{L_k, E_k, X_k, \partial_1, \partial_2\}$.

We begin with $\|[Y^{(i)}, Y^{(j)}]Y^{(l)}\widetilde{\phi}_k\|_{L^2(\Sigma_t)}$. Using Proposition 5.3, we see that L_k^μ, E_k^i and X_k^i obey C^1 bounds on $B(0,R)$. Hence, using Hölder's inequality and (9.1), (9.3), we obtain

$$\begin{aligned} &\|[Y^{(i)}, Y^{(j)}]Y^{(l)}\widetilde{\phi}_k\|_{L^2(\Sigma_t)} \\ &\lesssim \Big(\sum_{Y_k\in\{L_k,E_k,X_k\}} \|\partial Y_k^{\mu}\|_{L^\infty(\Sigma_t\cap B(0,R))}\Big)\Big[\|\partial^2\widetilde{\phi}_k\|_{L^2(\Sigma_t)} \\ &\quad + \Big(\sum_{Y_k\in\{L_k,E_k,X_k\}} \|\partial Y_k^{\mu}\|_{L^\infty(\Sigma_t\cap B(0,R))}\Big)\|\partial\widetilde{\phi}_k\|_{L^2(\Sigma_t)}\Big] \\ &\lesssim \epsilon^{\frac54}(\|\partial^2\widetilde{\phi}_k\|_{L^2(\Sigma_t)} + \|\partial\widetilde{\phi}_k\|_{L^2(\Sigma_t)}) \lesssim \epsilon^{\frac54}\cdot(\epsilon\delta^{-\frac12}) \lesssim \epsilon\delta^{-\frac12}. \end{aligned} \tag{11.1}$$

To bound $\|Y^{(i)}[Y^{(j)}, Y^{(l)}]\widetilde{\phi}_k\|_{L^2(\Sigma_t)}$, we first observe that Proposition 5.3 does <u>not</u> give $L^2(\Sigma_t)$ control of all second (spacetime) derivatives of L_k^{μ}, E_k^i and X_k^i on $B(0,R)$. Nonetheless, the only second derivative that is not controlled is the term $\partial_{tt}^2 L_k^t$.

Next, observe that in the set $\{L_k, E_k, X_k, \partial_1, \partial_2\}$, the only vector field with a ∂_t component in the $\{\partial_t, \partial_1, \partial_2\}$ basis is L_k. Since $[L_k, L_k] = 0$, $[Y^{(j)}, Y^{(l)}]$ cannot generate a $\partial_t L^t$ term. As a result, using Hölder's inequality, Proposition 5.3, and (9.1), (9.3) together with the bootstrap assumption (4.12c), we obtain

$$\begin{aligned} &\|Y^{(i)}[Y^{(j)}, Y^{(l)}]\widetilde{\phi}_k\|_{L^2(\Sigma_t)} \\ &\lesssim \Big(\sum_{Y_k\in\{E_k,X_k\}} \|\partial^2 Y_k^i\|_{L^2(\Sigma_t\cap B(0,R))} + \|\partial\partial_x L_k^{\mu}\|_{L^2(\Sigma_t\cap B(0,R))}\Big)\|\partial\widetilde{\phi}_k\|_{L^\infty(\Sigma_t)} \\ &\quad + \sum_{Y_k\in\{E_k,X_k,L_k\}} \|\partial Y_k^{\mu}\|_{L^\infty(\Sigma_t\cap B(0,R))}\|\partial^2\widetilde{\phi}_k\|_{L^2(\Sigma_t)} \\ &\lesssim \epsilon^{\frac54}(\|\partial\widetilde{\phi}_k\|_{L^\infty(\Sigma_t)} + \|\partial^2\widetilde{\phi}_k\|_{L^2(\Sigma_t)}) \lesssim \epsilon^{\frac54}\cdot(\epsilon\delta^{-\frac12}) \lesssim \epsilon\delta^{-\frac12}. \end{aligned} \tag{11.2}$$

Combining (11.1) and (11.2) yields the conclusion. □

Proposition 11.2 *The following holds for all $t \in [0, T_B)$:*

$$\sum_{\substack{Y_k^{(1)},Y_k^{(2)},Y_k^{(3)}\in\{X_k,E_k,L_k\} \\ \exists i, Y_k^{(i)}\neq X_k}} \|Y_k^{(1)}Y_k^{(2)}Y_k^{(3)}\widetilde{\phi}_k\|_{L^2(\Sigma_t)} \lesssim \epsilon\cdot\delta^{-\frac12} + \|\partial E_k\partial_x\widetilde{\phi}_k\|_{L^2(\Sigma_t)} + \|\partial L_k^2\widetilde{\phi}_k\|_{L^2(\Sigma_t)}.$$

Proof By Lemma 11.1, it suffices to control $Y_k^{(1)}Y_k^{(2)}Y_k^{(3)}\widetilde{\phi}_k$ with any order of $Y_k^{(1)}$, $Y_k^{(2)}$ and $Y_k^{(3)}$. We consider all possible cases below. (We will silently use that $\mathrm{supp}(\widetilde{\phi}_k) \subseteq B(0,R)$ so that we do not need to be concerned about the weights at infinity.)
Case 1: At least one of $Y_k^{(i)} = E_k$ There are two subcases: 1(a) there is at least one other spatial vector field E_k or X_k, and 1(b) $Y_k^{(1)}Y_k^{(2)}Y_k^{(3)}$ is some commutation of $E_kL_kL_k$. In case 1(a), we assume $Y_k^{(2)} = E_k$ and $Y_k^{(3)} \in \{E_k, X_k\}$. Expanding $Y_k^{(3)}$ in terms of ∂_i, and using the bounds in Proposition 5.3, we can control the term by $\epsilon\cdot\delta^{-\frac12} + \|\partial E_k\partial_x\widetilde{\phi}_k\|_{L^2(\Sigma_t)}$. In case 1(b), the term is trivially controlled by $\epsilon\cdot\delta^{-\frac12} + \|\partial L_k^2\widetilde{\phi}_k\|_{L^2(\Sigma_t)}$.

Case 2: At least one of $Y_k^{(i)} = L_k$, and none of them is E_k The three vector fields must therefore be (commutations of) 2(a) $L_k L_k L_k$, 2(b) $X_k L_k L_k$, or 2(c) $X_k L_k X_k$. In cases 2(a) and 2(b), clearly we have (using Proposition 5.3)

$$\|L_k^3 \widetilde{\phi}_k\|_{L^2(\Sigma_t)} + \|X_k L_k^2 \widetilde{\phi}_k\|_{L^2(\Sigma_t)} \lesssim \|\partial L_k^2 \widetilde{\phi}_k\|_{L^2(\Sigma_t)},$$

which is acceptable. In case 2(c), we use the wave equation $\Box_g \widetilde{\phi}_k = 0$ and the expression (8.8), as well as the bounds in Propositions 5.2–5.5 to obtain

$$\begin{aligned}
&\|X_k L_k X_k \widetilde{\phi}_k\|_{L^2(\Sigma_t)} \\
&\lesssim \|X_k L_k^2 \widetilde{\phi}_k\|_{L^2(\Sigma_t)} + \|X_k E_k^2 \widetilde{\phi}_k\|_{L^2(\Sigma_t)} + \epsilon^{\frac{3}{2}} (\|\partial^2 \widetilde{\phi}_k\|_{L^2(\Sigma_t)} + \|\partial \widetilde{\phi}_k\|_{L^\infty(\Sigma_t)}).
\end{aligned}$$

The first two terms are other combinations of $Y_k^{(1)} Y_k^{(2)} Y_k^{(3)} \widetilde{\phi}_k$ which we have controlled above, while the last two terms are bounded above by $\epsilon \cdot \delta^{-\frac{1}{2}}$ using (9.3) and the bootstrap assumption (4.12c). $\square$

In fact, we can slightly strengthen Proposition 11.2 to include $\vec{n}$ and ∂_q derivatives.

Corollary 11.3 *The following estimates hold for all* $t \in [0, T_B)$*:*

$$\begin{aligned}
&\sum_{\substack{Y_k^{(1)}, Y_k^{(2)}, Y_k^{(3)} \in \{X_k, E_k, L_k, \partial_q, \vec{n}\} \\ \exists i, Y_k^{(i)} = E_k \text{ or } Y_k^{(i)} = L_k}} \|Y_k^{(1)} Y_k^{(2)} Y_k^{(3)} \widetilde{\phi}_k\|_{L^2(\Sigma_t)} \\
&\lesssim \epsilon \cdot \delta^{-\frac{1}{2}} + \|\partial E_k \partial_x \widetilde{\phi}_k\|_{L^2(\Sigma_t)} + \|\partial L_k^2 \widetilde{\phi}_k\|_{L^2(\Sigma_t)},
\end{aligned} \tag{11.3}$$

and

$$\begin{aligned}
&\sum_{\substack{Y_k^{(1)}, Y_k^{(2)} \in \{X_k, E_k, L_k, \partial_q, \vec{n}\} \\ \exists i, Y_k^{(i)} = E_k \text{ or } Y_k^{(i)} = L_k}} \|\partial Y_k^{(1)} Y_k^{(2)} \widetilde{\phi}_k\|_{L^2(\Sigma_t)} \\
&\lesssim \epsilon \cdot \delta^{-\frac{1}{2}} + \|\partial E_k \partial_x \widetilde{\phi}_k\|_{L^2(\Sigma_t)} + \|\partial L_k^2 \widetilde{\phi}_k\|_{L^2(\Sigma_t)},
\end{aligned} \tag{11.4}$$

Proof Clearly it suffices to prove (11.3), since (11.4) follows from using (11.3) together with (5.8) (5.10).

Comparing to Proposition 11.2, the only new vector fields in (11.3) are $\vec{n}$ and ∂_q.

- Since $\vec{n} = L_k + X_k$ (by (2.32)), if we have the vector field $\vec{n}$ (but not ∂_i), we can reduce directly to Proposition 11.2.
- Suppose now among $Y_k^{(1)}, Y_k^{(2)}, Y_k^{(3)}$, there is at least one ∂_q and one $Z_k \in \{L_k, E_k\}$. Then using Lemma 11.1 to commute the vector fields, it suffices to bound

$$\sum_{Z_k \in \{L_k, E_k\}} \|\partial \partial_x Z_k \widetilde{\phi}_k\|_{L^2(\Sigma_t)}.$$

After using (5.11), this can in turn be reduced to a term as in Proposition 11.2 and plus another term $\sum_{Z_k\in\{L_k,E_k\}} \|\partial_x Z_k\widetilde{\phi}_k\|_{L^2(\Sigma_t)}$. The latter term can be controlled using Proposition 9.1.

□

Remark 11.4 Note that despite Corollary 11.3, we do not control a term such as $\|\partial_t L_k \partial_t\widetilde{\phi}_k\|_{L^2(\Sigma_t)}$. This is due to a lack of control of $\partial_t^2 L_k^\nu$ from Proposition 5.3.

11.2 Controlling $\|\partial E_k\partial_x\widetilde{\phi}_k\|_{L^2(\Sigma_t)}$

Proposition 11.5

$$\sup_{t\in[0,T_B)} \|\partial E_k\partial_x\widetilde{\phi}_k\|_{L^2(\Sigma_t)} \lesssim \epsilon\cdot\delta^{-\frac12} + \epsilon^{\frac32} \sup_{t\in[0,T_B)} \|\partial L_k^2\widetilde{\phi}_k\|_{L^2(\Sigma_t)}. \tag{11.5}$$

Proof We apply energy estimates for $E_k\partial_q\widetilde{\phi}_k$. First, we write

$$\Box_g(E_k\partial_q\widetilde{\phi}_k) = [\Box_g, E_k]\partial_q\widetilde{\phi}_k + E_k([\Box_g,\partial_q]\widetilde{\phi}_k).$$

Step 1: Controlling $[\Box_g, E_k]\partial_q\widetilde{\phi}_k$ By Proposition 8.10 and the support properties of $\widetilde{\phi}_k$, we obtain

$$\|[\Box_g, E_k]\partial_q\widetilde{\phi}_k\|_{L^2(\Sigma_t)} \lesssim \epsilon^{\frac32}\cdot(\|\partial^2\widetilde{\phi}_k\|_{L^2(\Sigma_t)} + \sum_{Z_k\in\{E_k,L_k\}} \|\partial Z_k\partial_q\widetilde{\phi}_k\|_{L^2(\Sigma_t)})$$
$$\lesssim \epsilon\cdot\delta^{-\frac12} + \epsilon^{\frac32}\|\partial E_k\partial_x\widetilde{\phi}_k\|_{L^2(\Sigma_t)} + \epsilon^{\frac32}\|\partial L_k^2\widetilde{\phi}_k\|_{L^2(\Sigma_t)} \lesssim \epsilon\cdot\delta^{-\frac12} + \epsilon^{\frac32}\|\partial L_k^2\widetilde{\phi}_k\|_{L^2(\Sigma_t)},$$

where in the last line we additionally used (9.3), Corollary 11.3 and bootstrap assumption (4.9d).

Step 2: Controlling $\|E_k([\Box_g,\partial_q]\widetilde{\phi}_k)\|_{L^2(\Sigma_t)}$ By Lemma 8.5, $[\Box_g,\partial_q]\widetilde{\phi}_k$ can be written as a sum of terms of the schematic form

$$\Omega(\mathfrak{g})\cdot\partial\partial_x g\cdot\partial\widetilde{\phi}_k,\quad \Omega(\mathfrak{g})\cdot\partial_x\mathfrak{g}\cdot\partial_x\partial_x\widetilde{\phi}_k,\quad \Omega(\mathfrak{g})\cdot\partial_x\mathfrak{g}\cdot\vec{n}\partial_x\widetilde{\phi}_k,\quad \Omega(\mathfrak{g})\cdot(\partial\mathfrak{g})^2\cdot\partial\widetilde{\phi}_k,$$

with $\mathfrak{g}\in\{N,\beta,\gamma\}$ and $\Omega(\mathfrak{g})$ a smooth function of the metric coefficients. (The important feature to notice here,[13] other than the number of derivatives, is that there are no terms with $\partial_t^2\mathfrak{g}$ or $\partial_t\partial_x\mathfrak{g}$. It is also useful to note that there are no $\Omega(\mathfrak{g})\cdot\partial_t\mathfrak{g}\cdot\partial_x\partial_x\widetilde{\phi}_k$ or $\Omega(\mathfrak{g})\cdot\partial_t\mathfrak{g}\cdot\vec{n}\partial_x\widetilde{\phi}_k$ terms.)

Therefore, using

- that E_k is a spatial derivatives, satisfying (5.5),
- that $|\Omega(\mathfrak{g})|\lesssim 1$, $|\partial\mathfrak{g}|\lesssim\epsilon^{\frac32}$, $|\partial_x^2\mathfrak{g}|\lesssim\epsilon^{\frac32}$ on the support of $\widetilde{\phi}_k$ (by (5.1)), and
- that $|\partial\widetilde{\phi}_k|\lesssim\epsilon^{\frac34}\varpi$ by (4.12c) and Lemma 5.1, where $\varpi\in C_c^\infty$ is a cutoff such that $\varpi\equiv 1$ on $B(0,2R)$ and $\mathrm{supp}(\varpi)\subseteq B(0,3R)$,

[13] One may observe that there are also no terms with $\vec{n}^2\widetilde{\phi}_k$, but this is irrelevant for the argument below.

we obtain

$$
\begin{aligned}
&|E_k([\Box_g, \partial_q]\widetilde{\phi}_k)| \\
&\lesssim \underbrace{\epsilon^{\frac{3}{4}}\varpi(|\partial\partial_x^2\mathfrak{g}| + |\partial\partial_x\mathfrak{g}|)}_{A} + \underbrace{\epsilon^{\frac{3}{4}}|\partial^2\widetilde{\phi}_k|}_{B} + \underbrace{|\partial\partial_x\mathfrak{g}|(|E_k\partial_x\widetilde{\phi}_k| + |E_k\vec{n}\widetilde{\phi}_k|)}_{D} \\
&\quad + \underbrace{\epsilon^{\frac{3}{2}}(|E_k\partial_x^2\widetilde{\phi}_k| + |E_k\vec{n}\partial_x\widetilde{\phi}_k|)}_{F}.
\end{aligned}
\tag{11.6}
$$

We now control each term in (11.6). The term A can be bounded using (5.1) and (5.2):

$$\|A\|_{L^2(\Sigma_t)} \lesssim \epsilon^{\frac{3}{4}} \cdot \epsilon^{\frac{3}{2}} \cdot \delta^{-\frac{1}{2}} \lesssim \epsilon^{\frac{9}{4}}\delta^{-\frac{1}{2}}.$$

The term B can be bounded using (9.3):

$$\|B\|_{L^2(\Sigma_t)} \lesssim \epsilon^{\frac{3}{4}} \cdot \epsilon \cdot \delta^{-\frac{1}{2}} \lesssim \epsilon^{\frac{7}{4}}\delta^{-\frac{1}{2}}.$$

The term D can be bounded by first using Proposition 8.3 and then using (9.2) and Corollary 11.3:

$$
\begin{aligned}
\|D\|_{L^2(\Sigma_t)} &\lesssim \epsilon^{\frac{3}{2}}\Big(\|E_k\partial_x\widetilde{\phi}_k\|_{L^2(\Sigma_t)} + \|E_k\vec{n}\widetilde{\phi}_k\|_{L^2(\Sigma_t)} + \|\partial_x E_k\partial_x\widetilde{\phi}_k\|_{L^2(\Sigma_t)} + \|\partial_x E_k\vec{n}\widetilde{\phi}_k\|_{L^2(\Sigma_t)}\Big) \\
&\lesssim \epsilon^{\frac{5}{2}} + \epsilon^{\frac{5}{2}} \cdot \delta^{-\frac{1}{2}} + \epsilon^{\frac{3}{2}}\|\partial E_k\partial_x\widetilde{\phi}_k\|_{L^2(\Sigma_t)} + \epsilon^{\frac{3}{2}}\|\partial L_k^2\widetilde{\phi}_k\|_{L^2(\Sigma_t)}.
\end{aligned}
$$

Finally, for the term F, we use Corollary 11.3 to obtain

$$\|F\|_{L^2(\Sigma_t)} \lesssim \epsilon^{\frac{5}{2}} \cdot \delta^{-\frac{1}{2}} + \epsilon^{\frac{3}{2}}\|\partial E_k\partial_x\widetilde{\phi}_k\|_{L^2(\Sigma_t)} + \epsilon^{\frac{3}{2}}\|\partial L_k^2\widetilde{\phi}_k\|_{L^2(\Sigma_t)}.$$

Putting all these together, we obtain

$$\|E_k([\Box_g, \partial_q]\widetilde{\phi}_k)\|_{L^1([0,T_B);L^2(\Sigma_t))} \lesssim \epsilon \cdot \delta^{-\frac{1}{2}} + \epsilon^{\frac{3}{2}}\|\partial E_k\partial_x\widetilde{\phi}_k\|_{L^2(\Sigma_t)} + \epsilon^{\frac{3}{2}}\|\partial L_k^2\widetilde{\phi}_k\|_{L^2(\Sigma_t)}.$$

Step 3: Putting everything together Combining the estimates in Steps 1 and 2, we obtain

$$\|\Box_g(E_k\partial_q\widetilde{\phi}_k)\|_{L^2(\Sigma_t)} \lesssim \epsilon \cdot \delta^{-\frac{1}{2}} + \epsilon^{\frac{3}{2}}\|\partial E_k\partial_x\widetilde{\phi}_k\|_{L^2(\Sigma_t)} + \epsilon^{\frac{3}{2}}\|\partial L_k^2\widetilde{\phi}_k\|_{L^2(\Sigma_t)}.$$

Therefore, applying Proposition 7.3 with $v = E_k\partial_q\widetilde{\phi}_k$, $U_0 = -\infty$ and $U_1 = +\infty$, and bounding the initial data by (4.4), we obtain

$$\sup_{t\in[0,T_B)}\|\partial E_k\partial_x\widetilde{\phi}_k\|_{L^2(\Sigma_t)} \lesssim \epsilon \cdot \delta^{-\frac{1}{2}} + \epsilon^{\frac{3}{2}}\sup_{t\in[0,T_B)}(\|\partial E_k\partial_x\widetilde{\phi}_k\|_{L^2(\Sigma_t)} + \|\partial L_k^2\widetilde{\phi}_k\|_{L^2(\Sigma_t)}).$$

Choosing ϵ sufficiently small, we can absorb $\epsilon^{\frac{3}{2}} \sup_{t\in[0,T_B)} \|\partial E_k \partial_x \widetilde{\phi}_k\|_{L^2(\Sigma_t)}$ by the term on the left-hand side, thus concluding the proof. □

11.3 Controlling $\|\partial L_k^2 \widetilde{\phi}_k\|_{L^2(\Sigma_t)}$

Proposition 11.6

$$\|\partial L_k^2 \widetilde{\phi}_k\|_{L^2(\Sigma_t)} \lesssim \epsilon \cdot \delta^{-\frac{1}{2}}. \tag{11.7}$$

Proof We apply energy estimates to $L_k^2 \widetilde{\phi}_k$. First, we expand

$$\Box_g (L_k^2 \widetilde{\phi}_k) = [\Box_g, L_k] L_k \widetilde{\phi}_k + L_k([\Box_g, L_k]\widetilde{\phi}_k). \tag{11.8}$$

The first term will be controlled in Step 1 and the second term will be controlled in Steps 2–3 below, after which we carry out the energy estimates in Step 4.

Step 1: Controlling $[\Box_g, L_k] L_k \widetilde{\phi}_k$ We use Proposition 8.12 and $\mathrm{supp}(\widetilde{\phi}_k) \subseteq B(0, R)$ to obtain

$$\begin{aligned}
&\|[\Box_g, L_k] L_k \widetilde{\phi}_k\|_{L^2(\Sigma_t)} \lesssim \epsilon^{\frac{3}{2}} \cdot (\|\partial^2 \widetilde{\phi}_k\|_{L^2(\Sigma_t)} + \sum_{Z_k \in \{E_k, L_k\}} \|\partial Z_k L_k \widetilde{\phi}_k\|_{L^2(\Sigma_t)}) \\
&\lesssim \epsilon^{\frac{5}{2}} \cdot \delta^{-\frac{1}{2}} + \epsilon^{\frac{3}{2}} \|\partial E_k \partial_x \widetilde{\phi}_k\|_{L^2(\Sigma_t)} + \epsilon^{\frac{3}{2}} \|\partial L_k^2 \widetilde{\phi}_k\|_{L^2(\Sigma_t)} \lesssim \epsilon^{\frac{9}{4}} \cdot \delta^{-\frac{1}{2}} + \epsilon^{\frac{3}{2}} \|\partial L_k^2 \widetilde{\phi}_k\|_{L^2(\Sigma_t)},
\end{aligned} \tag{11.9}$$

where in the last line we additionally used (9.3), Corollary 11.3 and the bootstrap assumption (4.9d).

Step 2: Controlling $L_k[\Box_g, L_k]\widetilde{\phi}_k$ *except for one term* Recalling the notations from Lemma 8.7, we need to handle $L_k[\Box_g, L_k]\widetilde{\phi}_k = L_k(I(L_k)\widetilde{\phi}_k) + L_k(II(L_k)\widetilde{\phi}_k) + L_k(III(L_k)\widetilde{\phi}_k)$.

Using the L^∞ bounds in (5.1), (5.5), (5.6), (5.14), and (4.12c), it is easy to deduce from (8.21), that

$$\begin{aligned}
&|L_k(I(L_k\widetilde{\phi}_k) + L_k X_k \log(N) \cdot L_k \widetilde{\phi}_k)| \\
&\lesssim \epsilon^{\frac{9}{4}} \varpi + \epsilon^{\frac{3}{2}} \cdot |\partial^2 \widetilde{\phi}_k| + \epsilon^{\frac{3}{4}} \cdot (|\partial^2 E_k^i| + |\partial^2 X_k^i|) \cdot \varpi \\
&\quad + (\epsilon^{\frac{9}{4}} + |\partial L_k \widetilde{\phi}_k|) \cdot |\partial \partial_x \mathfrak{g}| \varpi + \epsilon^{\frac{3}{4}} |\partial \partial_x^2 \mathfrak{g}| \varpi \\
&\quad + |\partial_x \chi_k| \cdot (\epsilon^{\frac{9}{4}} \varpi + |L_k E_k \widetilde{\phi}_k|) + \epsilon^{\frac{3}{4}} \cdot (|L_k \partial_x \chi_k| + |L_k^2 \chi_k| + |L_k^2 \eta_k|) \varpi,
\end{aligned} \tag{11.10}$$

where, as in Proposition 11.5, $\mathfrak{g} \in \{\beta^i, N, \gamma\}$ and ϖ is a smooth cutoff with $\mathrm{supp}(\varpi) \subseteq B(0, 3R)$. We isolated the term $L_k(L_k X_k \log(N) \cdot L_k \widetilde{\phi}_k)$ on the left-hand side of (11.10), which will be treated in later steps.

Arguing similarly, but starting with (8.22), (8.23), we obtain

$$|L_k(II(L_k)\widetilde{\phi}_k)| \lesssim \epsilon^{\frac{3}{2}} \cdot (|\partial L_k^2\widetilde{\phi}_k| + |L_k E_k^2\widetilde{\phi}_k| + |\partial^2\widetilde{\phi}_k|) + (\epsilon^3 + |\partial\partial_x \mathfrak{g}| + |\partial K|) \cdot |\partial L_k\widetilde{\phi}_k|, \tag{11.11}$$

$$|L_k(III(L_k)\widetilde{\phi}_k)| \lesssim \epsilon^3 \cdot |\partial^2\phi| + \epsilon^{\frac{9}{4}} \cdot (|\partial\partial_x \mathfrak{g}| + |\partial K|) \cdot \varpi, \tag{11.12}$$

where $\mathfrak{g}$ and ϖ are as in (11.10). (Note that in (11.11) we have used $L_k X_k L_k \widetilde{\phi}_k = X_k^i \partial_i L_k^2\widetilde{\phi}_k + (X^\nu\partial_\sigma L^\sigma - L^\nu\partial_\sigma X^\sigma)\partial_\nu L_k\widetilde{\phi}_k$ combined with (5.5), (5.6); similarly for $L_k E_k L_k\widetilde{\phi}_k$.)

We now control the terms (11.10), (11.11) and (11.12). We begin with the right-hand side of (11.10). First, using the bootstrap assumption (4.9b) and (5.1), (5.2), (5.7), (5.14), (5.15), (5.16), we handle all the linear terms to obtain

$$\begin{aligned} &\|L_k(I(L_k\widetilde{\phi}_k + L_k X_k \log(N)) \cdot L_k\widetilde{\phi}_k)\|_{L^2(\Sigma_t)} \\ &\lesssim \epsilon^{\frac{9}{4}} \cdot \delta^{-\frac{1}{2}} + \|\partial L_k\widetilde{\phi}_k \cdot \partial\partial_x \mathfrak{g}\|_{L^2(\Sigma_t)} + \|\partial_x \chi_k \cdot L_k E_k\widetilde{\phi}_k\|_{L^2(\Sigma_t)}. \end{aligned} \tag{11.13}$$

For the $\partial L_k\widetilde{\phi}_k \cdot \partial\partial_x \mathfrak{g}$ term in (11.13), we first use Lemma 5.4 (for $|\partial L_k\widetilde{\phi}_k|$) and then apply (8.7), Corollary 11.3, Proposition 5.3, (9.1), (9.3) and the bootstrap assumption (4.9d) to obtain

$$\begin{aligned} &\|\partial L_k\widetilde{\phi}_k \cdot \partial\partial_x \mathfrak{g}\|_{L^2(\Sigma_t)} \lesssim \sum_{Y_k \in \{X_k, E_k, L_k\}} \|Y_k L_k\widetilde{\phi}_k \cdot \partial\partial_x g\|_{L^2(\Sigma_t)} \\ &\lesssim \epsilon^{\frac{3}{2}} \sum_{Y_k \in \{X_k, E_k, L_k\}} (\|Y_k L_k\widetilde{\phi}_k\|_{L^2(\Sigma_t)} + \|\partial Y_k L_k\widetilde{\phi}_k\|_{L^2(\Sigma_t)}) \\ &\lesssim \epsilon^{\frac{5}{2}} \cdot \delta^{-\frac{1}{2}} + \epsilon^{\frac{3}{2}}(\|\partial^2\widetilde{\phi}_k\|_{L^2(\Sigma_t)} + \|\partial E_k\partial_x\widetilde{\phi}_k\|_{L^2(\Sigma_t)} + \|\partial L_k^2\widetilde{\phi}_k\|_{L^2(\Sigma_t)}) \\ &\lesssim \epsilon^{\frac{9}{4}} \cdot \delta^{-\frac{1}{2}} + \epsilon^{\frac{3}{2}}\|\partial L_k^2\widetilde{\phi}_k\|_{L^2(\Sigma_t)}. \end{aligned} \tag{11.14}$$

For the $\partial_x \chi_k \cdot L_k E_k\widetilde{\phi}_k$ term in (11.13), we use (8.4) and then Proposition 5.3, Corollary 11.3 together with (9.1), (9.3) and the bootstrap assumption (4.9d) to obtain

$$\begin{aligned} &\|\partial_x \chi_k \cdot L_k E_k\widetilde{\phi}_k\|_{L^2(\Sigma_t)} \\ &\lesssim \|L_k E_k\widetilde{\phi}_k\|_{L^2(\Sigma_t)} + \|E_k L_k E_k\widetilde{\phi}_k\|_{L^2(\Sigma_t)} \\ &\lesssim \epsilon^{\frac{5}{2}} \cdot \delta^{-\frac{1}{2}} + \epsilon^{\frac{3}{2}}(\|\partial^2\widetilde{\phi}_k\|_{L^2(\Sigma_t)} + \|\partial E_k\partial_x\widetilde{\phi}_k\|_{L^2(\Sigma_t)} + \|\partial L_k^2\widetilde{\phi}_k\|_{L^2(\Sigma_t)}) \\ &\lesssim \epsilon^{\frac{9}{4}} \cdot \delta^{-\frac{1}{2}} + \epsilon^{\frac{3}{2}}\|\partial L_k^2\widetilde{\phi}_k\|_{L^2(\Sigma_t)}. \end{aligned} \tag{11.15}$$

Plugging (11.14)–(11.15) into (11.13), we obtain

$$\|L_k(I(L_k\widetilde{\phi}_k + L_k X_k \log(N)) \cdot L_k\widetilde{\phi}_k)\|_{L^2(\Sigma_t)} \lesssim \epsilon^{\frac{9}{4}} \cdot \delta^{-\frac{1}{2}} + \epsilon^{\frac{3}{2}}\|\partial L_k^2\widetilde{\phi}_k\|_{L^2(\Sigma_t)}. \tag{11.16}$$

For the term in (11.11), we use the estimates (5.1), (5.13), (5.2), (9.1), (9.3) and Corollary 11.3 together with (8.7) to get

$$
\begin{aligned}
\|L_k(II(L_k)\widetilde{\phi}_k)\|_{L^2(\Sigma_t)} &\lesssim \epsilon^{\frac{9}{4}} \cdot \delta^{-\frac{1}{2}} + \|\partial\partial_x \mathfrak{g} \cdot \partial L_k \widetilde{\phi}_k\|_{L^2(\Sigma_t)} \\
&\quad + \epsilon^{\frac{3}{2}} \|\partial E_k \partial_x \widetilde{\phi}_k\|_{L^2(\Sigma_t)} + \epsilon^{\frac{3}{2}} \|\partial L_k^2 \widetilde{\phi}_k\|_{L^2(\Sigma_t)} \\
&\lesssim \epsilon^{\frac{9}{4}} \cdot \delta^{-\frac{1}{2}} + \epsilon^{\frac{3}{2}} \|\partial L_k \widetilde{\phi}_k\|_{H^1(\Sigma_t)} \\
&\quad + \epsilon^{\frac{3}{2}} \|\partial E_k \partial_x \widetilde{\phi}_k\|_{L^2(\Sigma_t)} + \epsilon^{\frac{3}{2}} \|\partial L_k^2 \widetilde{\phi}_k\|_{L^2(\Sigma_t)} \\
&\lesssim \epsilon^{\frac{9}{4}} \cdot \delta^{-\frac{1}{2}} + \epsilon^{\frac{3}{2}} \|\partial L_k^2 \widetilde{\phi}_k\|_{L^2(\Sigma_t)}.
\end{aligned}
\tag{11.17}
$$

where we have used (2.14) to rewrite ∂K in terms of $\partial\partial_x \mathfrak{g}$, and in the last inequality we have used (11.14) and the bootstrap assumption (4.9d).

Finally, for the term in (11.12), we simply use (9.3) and the estimates (5.1) and (5.13) to obtain

$$
\|L_k(III(L_k)\widetilde{\phi}_k)\|_{L^2(\Sigma_t)} \lesssim \epsilon^4 \cdot \delta^{-\frac{1}{2}} \lesssim \epsilon \cdot \delta^{-\frac{1}{2}}. \tag{11.18}
$$

Step 3: Controlling $L_k X_k \log(N) \cdot L_k \widetilde{\phi}_k$ and $\partial_x(L_k X_k \log(N) \cdot L_k \widetilde{\phi}_k)$ Combining (11.9) in Step 1 and (11.16)–(11.18) in Step 2 (together with Lemma 8.7), we have proven

$$
\|\Box_g(L_k^2 \widetilde{\phi}_k) + L_k(L_k X_k \log(N) \cdot L_k \widetilde{\phi}_k)\|_{L^2(\Sigma_t)} \lesssim \epsilon^{\frac{9}{4}} \cdot \delta^{-\frac{1}{2}} + \epsilon^{\frac{3}{2}} \|\partial L_k^2 \widetilde{\phi}_k\|_{L^2(\Sigma_t)}. \tag{11.19}
$$

We will not directly estimate the term $L_k(L_k X_k \log(N) \cdot L_k \widetilde{\phi}_k)$. Instead, we rely an integration by parts argument using Corollary 7.5. In preparation of the integration by parts argument, we estimate $L_k X_k \log(N) \cdot L_k \widetilde{\phi}_k$ and $\partial_x(L_k X_k \log(N) \cdot L_k \widetilde{\phi}_k)$.

First, for $L_k X_k \log(N) \cdot L_k \widetilde{\phi}_k$, we use the bootstrap assumption (4.12c) and the estimates (5.1), (5.5), (5.6) to obtain

$$
\|L_k X_k \log(N) \cdot L_k \widetilde{\phi}_k\|_{L^2(\Sigma_t)} \lesssim \epsilon^{\frac{9}{4}}. \tag{11.20}
$$

As for the derivative $\partial_x(L_k X_k \log(N) \cdot L_k \widetilde{\phi}_k)$, the Leibniz rule generates two terms: if ∂_x falls on $L_k \widetilde{\phi}_k$ we get

$$
\|L_k X_k \log(N) \cdot \partial_x L_k \widetilde{\phi}_k\|_{L^2(\Sigma_t)} \lesssim \epsilon^{\frac{9}{4}} \cdot \delta^{-\frac{1}{2}} + \epsilon^{\frac{3}{2}} \|\partial L_k^2 \widetilde{\phi}_k\|_{L^2(\Sigma_t)},
$$

where we used (8.7) and Corollary 11.3 together with (5.5), (5.6), (5.1) and the bootstrap assumption (4.9d).

If, instead, ∂_x falls on $L_k X_k \log(N)$, we get

$$
\|\partial_x L_k X_k \log(N) \cdot L_k \widetilde{\phi}_k\|_{L^2(\Sigma_t)} \lesssim \|\partial_x L_k X_k \log(N)\|_{L^2(\Sigma_t \cap B(0,R))} \|\partial \widetilde{\phi}_k\|_{L^\infty(\Sigma_t)} \lesssim \epsilon^{\frac{9}{4}} \cdot \delta^{-\frac{1}{2}},
$$

where we used the bootstrap assumption (4.12c) together with (5.1), (5.2), (5.5), (5.6), (5.7). Therefore, using (5.5) again, we prove that

$$\begin{aligned}\|X_k[L_kX_k\log(N)\cdot L_k\widetilde{\phi}_k]\|_{L^2(\Sigma_t)} &\lesssim \|\partial_x(L_kX_k\log(N)\cdot L_k\widetilde{\phi}_k)\|_{L^2(\Sigma_t)}\\ &\lesssim \epsilon^{\frac{9}{4}}\cdot\delta^{-\frac{1}{2}}+\epsilon^{\frac{3}{2}}\|\partial L_k^2\widetilde{\phi}_k\|_{L^2(\Sigma_t)}.\end{aligned} \tag{11.21}$$

Step 4: An integration by parts argument and putting everything together Therefore, writing the decomposition $L_k=-X_k+N^{-1}e_0$ (by (2.32), (2.5)) and combining the estimates in (11.19) and (11.21), we get

$$\|\Box_g(L_k^2\widetilde{\phi}_k)-N^{-1}e_0(L_kX_k\log(N)\cdot L_k\widetilde{\phi}_k)\|_{L^2(\Sigma_t)}\lesssim \epsilon^{\frac{9}{4}}\cdot\delta^{-\frac{1}{2}}+\epsilon^{\frac{3}{2}}\|\partial L_k^2\widetilde{\phi}_k\|_{L^2(\Sigma_t)}.$$

Writing $\Box_g(L_k^2\widetilde{\phi}_k)=f_1+N^{-1}e_0f_2$ with $f_2=L_kX_k\log(N)\cdot L_k\widetilde{\phi}_k$, we have therefore proved that

$$\|\Box_g(L_k^2\widetilde{\phi}_k)-N^{-1}e_0f_2\|_{L^2(\Sigma_t)}=\|f_1\|_{L^2(\Sigma_t)}\lesssim\epsilon^{\frac{3}{2}}\cdot(\delta^{-\frac{1}{2}}+\|\partial L_k^2\widetilde{\phi}_k\|_{L^2(\Sigma_t)}). \tag{11.22}$$

On the other hand, (11.20) and (11.21) give

$$\|f_2\|_{L^2(\Sigma_t)}+\|\partial_xf_2\|_{L^2(\Sigma_t)}\lesssim\epsilon^{\frac{3}{2}}\cdot\delta^{-\frac{1}{2}}+\epsilon^{\frac{3}{2}}\|\partial L_k^2\widetilde{\phi}_k\|_{L^2(\Sigma_t)}. \tag{11.23}$$

By (11.22) and (11.23), applying Corollary 7.5 we get

$$\begin{aligned}&\|\partial L_k^2\widetilde{\phi}_k\|^2_{L^2(\Sigma_t)}\\ &\lesssim \|\partial L_k^2\widetilde{\phi}_k\|^2_{L^2(\Sigma_0)}+\epsilon^{\frac{3}{2}}\cdot\delta^{-1}+\epsilon^{\frac{3}{2}}\int_0^t\|\partial L_k^2\widetilde{\phi}_k\|^2_{L^2(\Sigma_\tau)}d\tau\\ &\quad+\sup_{0\le\tau\le t}\|\langle x\rangle^{-r}L_kX_k\log(N)\cdot L_k\widetilde{\phi}_k\|_{L^2(\Sigma_\tau)}\\ &\lesssim\epsilon^{\frac{3}{2}}\cdot\delta^{-1}+\epsilon^{\frac{3}{2}}\sup_{0\le\tau\le t}\|\partial L_k^2\widetilde{\phi}_k\|^2_{L^2(\Sigma_\tau)},\end{aligned}$$

where for the last inequality we have used

- the assumption on the data (4.4) (recall indeed that $\widetilde{\phi}'_k-X_k\widetilde{\phi}_k=L_k\widetilde{\phi}_k$ on Σ_0) together with (5.5), (5.6) to control the data term; and
- (5.1) and the bootstrap assumption (4.12c) with the Hölder's inequality to bound the last term.

Taking the supremum over $t\in[0,T_B)$, and absorbing $\sup_{0\le\tau<T_B}\|\partial L_k^2\widetilde{\phi}_k\|^2_{L^2(\Sigma_\tau)}$ to the left-hand side, we obtain (11.7). This concludes the proof of the proposition. □

11.4 Energy estimates for three derivatives of $\widetilde{\phi}_k$ with at least one good derivative

We finally obtain our main result regarding the energy estimates for three derivatives of $\widetilde{\phi}_k$, where at least one of the three derivatives is E_k or L_k.

Proposition 11.7 *The following estimate holds for all $t \in [0, T_B)$:*

$$\sum_{\substack{Y_k^{(1)},Y_k^{(2)},Y_k^{(3)} \in \{X_k, E_k, L_k, \partial_q, \vec{n}\} \\ \exists i, Y_k^{(i)}=E_k \text{ or } Y_k^{(i)}=L_k}} \|Y_k^{(1)} Y_k^{(2)} Y_k^{(3)} \widetilde{\phi}_k\|_{L^2(\Sigma_t)} \lesssim \epsilon \cdot \delta^{-\frac{1}{2}}. \tag{11.24}$$

In particular, the bootstrap assumption (4.9d) holds with $\epsilon^{\frac{3}{4}}$ replaced by $C\epsilon$.

Proof This follows immediately from the combination of Corollary 11.3, Proposition 11.5 and Proposition 11.6. □

11.5 General third derivatives of ϕ

We end this section with an estimate for general third derivatives of ϕ. We will not require any derivative to be good. In fact, the estimate we prove applies to the full ϕ, and not just $\widetilde{\phi}_k$. Notice that the proposition only bounds $\|\partial\phi\|_{H^2(\Sigma_t)}$ in terms of the H^3 norm of the initial data. In particular, while the right-hand side is finite for each fixed $\delta > 0$, it is allowed to blow up very rapidly as $\delta \to 0$.

Proposition 11.8 *The following estimate holds for all $t \in [0, T_B)$:*

$$\|\partial\phi\|_{H^2(\Sigma_t)} \lesssim \epsilon \cdot \delta^{-\frac{1}{2}} + \|\phi\|_{H^3(\Sigma_0)} + \|\vec{n}\phi\|_{H^2(\Sigma_0)}. \tag{11.25}$$

Proof Letting $\Gamma^\lambda := (g^{-1})^{\nu\beta}\Gamma^\lambda_{\nu\beta}$, we write $\Box_g f = (g^{-1})^{\nu\beta}\partial^2_{\nu\beta} f + \Gamma^\lambda\partial_\lambda f$. Hence, using the support properties of ϕ, the estimates for the derivatives for g^{-1} and Γ in Proposition 5.2, and the bootstrap assumption (4.12c), we can bound the commutator $[\Box_g, \partial^2_{ij}]$ as follows:

$$\begin{aligned} |[\Box_g, \partial^2_{ij}]\phi| &= |(\partial^2_{ij}(g^{-1})^{\nu\beta})\partial^2_{\nu\beta}\phi + 2(\partial_{(i}(g^{-1})^{\nu\beta})\partial^3_{j)\nu\beta}\phi + (\partial^2_{ij}\Gamma^\lambda)\partial_\lambda\phi + 2(\partial_{(i}\Gamma^\lambda)\partial^2_{j)\lambda}\phi| \\ &\lesssim \underbrace{\epsilon^{\frac{3}{2}}|\partial^2\phi|}_{=:I} + \underbrace{\epsilon^{\frac{3}{2}}|\partial_x\partial^2\phi|}_{=:II} + \underbrace{\epsilon^{\frac{3}{4}}|\varpi\partial_x^2\Gamma^\lambda|}_{=:III} + \underbrace{|\partial_x\Gamma^\lambda||\partial\partial_x\phi|}_{=:IV}, \end{aligned} \tag{11.26}$$

where $\varpi \in C_c^\infty$ is a cutoff such that $\varpi = 1$ on $B(0, R)$ and $\text{supp}(\varpi) \subseteq B(0, 3R)$.

The terms I in (11.26) can be estimated directly using (4.8) and (9.3) so that

$$\|I\|_{L^2(\Sigma_t)} \lesssim \epsilon^{\frac{3}{2}}\|\partial^2\phi\|_{L^2(\Sigma_t)} \lesssim \epsilon^{\frac{9}{4}} \cdot \delta^{-\frac{1}{2}}. \tag{11.27}$$

We turn to term II in (11.26). This term is bounded by $\epsilon^{\frac{3}{2}}|\partial\partial_x^2\phi|$ unless we have two ∂_t derivatives, in which case we need to control $\partial_x\partial_t\partial_t\phi$. Since $(g^{-1})^{tt} = -\frac{1}{N^2}$, we have $\partial_{tt}^2\phi = N^2(2(g^{-1})^{tj}\partial_{tj}^2\phi + (g^{-1})^{ij}\partial_{ij}^2\phi + \Gamma^\lambda\partial_\lambda\phi)$ (using $\Box_g\phi = 0$). Therefore, using Proposition 5.2, (9.3) and the bootstrap assumptions (4.12c), (4.8), we have

$$\begin{aligned}\|\partial_x\partial^2\phi\|_{L^2(\Sigma_t)} &\lesssim \|\partial\partial_x^2\phi\|_{L^2(\Sigma_t)} + \epsilon^{\frac{3}{4}}\|\partial_x\Gamma^\lambda\|_{L^2(\Sigma_t)} + \epsilon^{\frac{3}{2}}\|\partial^2\phi\|_{L^2(\Sigma_t)}\\ &\lesssim \|\partial\partial_x^2\phi\|_{L^2(\Sigma_t)} + \epsilon^{\frac{9}{4}}\delta^{-\frac{1}{2}}.\end{aligned}$$

Putting all these together, we thus obtain

$$\|II\|_{L^2(\Sigma_t)} \lesssim \epsilon^{\frac{15}{4}} \cdot \delta^{-\frac{1}{2}} + \epsilon^{\frac{3}{2}}\|\partial\partial_x^2\phi\|_{L^2(\Sigma_t)}. \tag{11.28}$$

We bound the remaining term in (11.26). By (5.4), III in (11.26) can be bounded by

$$\|III\|_{L^2(\Sigma_t)} \lesssim \epsilon^{\frac{3}{4}}\|\partial_x^2\Gamma^\lambda\|_{L^2(\Sigma_t\cap B(0,3R))} \lesssim \epsilon^{\frac{9}{4}} \cdot \delta^{-\frac{1}{2}}. \tag{11.29}$$

To handle term IV in (11.26), first observe that we can use Proposition 5.2 to bound $|\partial_x\Gamma^\lambda| \lesssim \epsilon^{\frac{3}{2}} + |\partial_x\partial_t\mathfrak{g}|$ on $B(0, 3R)$, where $\mathfrak{g}$ is as in Proposition 8.3. Hence, using Proposition 8.3 together with (4.8) and (9.3), we obtain

$$\begin{aligned}\|IV\|_{L^2(\Sigma_t)} &\lesssim \epsilon^{\frac{3}{2}}\|\partial^2\phi\|_{L^2(\Sigma_t)} + \||\partial\partial_x\mathfrak{g}| \cdot |\partial\partial_x\phi|\|_{L^2(\Sigma_t)} \lesssim \epsilon^{\frac{9}{4}}\delta^{-\frac{1}{2}} + \epsilon^{\frac{3}{2}}\|\partial\partial_x\phi\|_{H^1(\Sigma_t)}\\ &\lesssim \epsilon^{\frac{9}{4}}\delta^{-\frac{1}{2}} + \epsilon^{\frac{3}{2}}\|\partial\partial_x^2\phi\|_{L^2(\Sigma_t)},\end{aligned} \tag{11.30}$$

where in the last line we also used that $\text{supp}(\phi) \subseteq B(0, R)$ and applied Poincaré's inequality.

Combining (11.26)–(11.30), we obtain

$$\|[\Box_g, \partial_{ij}^2]\phi\|_{L^2(\Sigma_t)} \lesssim \epsilon^{\frac{9}{4}} \cdot \delta^{-\frac{1}{2}} + \epsilon^{\frac{3}{2}}\|\partial\partial_x^2\phi\|_{L^2(\Sigma_t)}. \tag{11.31}$$

Since $\Box_g\partial_{ij}^2\phi = [\Box_g, \partial_{ij}^2]\phi$, applying the energy estimates in Proposition 7.3 (for $v = \partial_{ij}^2\phi$, $U_0 = -\infty$, $U_1 = \infty$) with (11.31), we have

$$\sup_{t\in[0,T_B)}\|\partial\partial_x^2\phi\|_{L^2(\Sigma_t)} \lesssim \|\partial\partial_x^2\phi\|_{L^2(\Sigma_0)} + \epsilon^{\frac{9}{4}} \cdot \delta^{-\frac{1}{2}} + \epsilon^{\frac{3}{2}}\sup_{t\in[0,T_B)}\|\partial\partial_x^2\phi\|_{L^2(\Sigma_t)}.$$

To obtain the desired estimate, we absorb the last term to the left-hand side, and bound the data term as follows: using (2.3) we get

$$\begin{aligned}\|\partial\partial_x^2\phi\|_{L^2(\Sigma_0)} &\lesssim \|\phi\|_{H^3(\Sigma_0)} + \|N(\vec{n}\phi + \beta^i\partial_i\phi)\|_{H^2(\Sigma_0)}\\ &\lesssim \epsilon\delta^{-\frac{1}{2}} + \|\phi\|_{H^3(\Sigma_0)} + \|\vec{n}\phi\|_{H^2(\Sigma_0)},\end{aligned}$$

where in the last inequality we have used (9.1), (9.3), and the metric estimates of Proposition 5.2. □

12 Fractional energy estimates for $\widetilde{\phi}_k$

In this section, we prove the energy estimates for $\widetilde{\phi}_k$ that involve fractional derivatives. These include

- bounds for $\|\partial \langle D_x\rangle^{s'}\widetilde{\phi}_k\|_{L^2(\Sigma_t)}$ to be proven in Sect. 12.5, and
- bounds for $\|\partial E_k \langle D_x\rangle^{s''}\widetilde{\phi}_k\|_{L^2(\Sigma_t)}$ and $\|\partial L_k \langle D_x\rangle^{s''}\widetilde{\phi}_k\|_{L^2(\Sigma_t)}$ to be proven respectively in Sect. 12.6 and Sect. 12.7. (See also some auxiliary estimates in Sect. 12.8.)

These estimates are the most technical ones in this paper, as they involve simultaneously the geometric vector fields, the weights at spatial infinity and fractional derivatives. Some of the main preliminaries regarding the fractional derivatives and the weights can be found in Sect. 12.1 and Sect. 12.4, respectively. We also refer the reader back to Sect. 1.1.6 for some comments on the analysis.

12.1 Fractional derivative commutator estimates

Definition 12.1 Define the Fourier transform in the $x = (x^1, x^2)$ coordinates with the following normalization: for all $f \in L^2(\mathbb{R}^2)$

$$\hat{f}(\xi) = (\mathcal{F} f)(\xi) := \iint_{\mathbb{R}^2} f(x) e^{-2\pi i x\cdot\xi}\, dx^1\, dx^2,$$

and denote by $\mathcal{F}^{-1}$ the corresponding inverse Fourier transform.

Let $\varphi : \mathbb{R}^2 \to [0, 1]$ be radial, smooth such that $\varphi(\xi) = \begin{cases} 1 & \text{for } |\xi| \leq 1 \\ 0 & \text{for } |\xi| \geq 2 \end{cases}$, where $|\xi| = \sqrt{|\xi_1|^2 + |\xi_2|^2}$.

Define P_0 by

$$P_0 f := (\mathcal{F})^{-1}(\varphi(\xi)\mathcal{F} f),$$

and for $q \geq 1$, define $P_q f$ by

$$P_q f := (\mathcal{F})^{-1}(\tilde{\varphi}_q(\xi)\mathcal{F} f(\xi)),$$

where we introduced $\tilde{\varphi}_q(\xi) := \varphi(2^{-q}\xi) - \varphi(2^{-q+1}\xi)$.

Definition 12.2 With the notations of Definition 12.1, we further define

$$\varphi_{hh}(\sigma, \xi) := \sum_j \sum_{k:|k-j|\leq 1} \widetilde{\varphi}_k(\sigma)\widetilde{\varphi}_j(\xi),$$

$$\varphi_{hl}(\sigma,\xi) := \sum_j \sum_{k:k>j+1} \widetilde{\varphi}_k(\sigma)\widetilde{\varphi}_j(\xi),$$
$$\varphi_{lh}(\sigma,\xi) := \sum_j \sum_{k:k<j-1} \widetilde{\varphi}_k(\sigma)\widetilde{\varphi}_j(\xi).$$

Note that since $\sum_{k\geq 0}\widetilde{\varphi}_k = 1$, we also have $\varphi_{hh}+\varphi_{hl}+\varphi_{lh} = 1$.

Definition 12.3 For any multiplier $m(\sigma,\xi)$ real-valued function on $\mathbb{R}^2$, we define the para-product

$$T_m(f_1,f_2)(x) = \int_{\xi\in\mathbb{R}^2,\sigma\in\mathbb{R}^2} e^{2\pi i(\xi+\sigma)\cdot x} m(\sigma,\xi)\hat{f}_1(\sigma)\hat{f}_2(\xi)d\sigma d\xi,$$

We also define the high-high m-para-product of f_1 and f_2

$$\begin{aligned}\Pi_{hh}(m)(f_1,f_2)(x) &= T_{\varphi_{hh}\cdot m}(f_1,f_2)(x)\\ &= \int_{\xi\in\mathbb{R}^2,\sigma\in\mathbb{R}^2} e^{2\pi i(\xi+\sigma)\cdot x}\varphi_{hh}(\sigma,\xi)m(\sigma,\xi)\hat{f}_1(\sigma)\hat{f}_2(\xi)d\sigma d\xi,\end{aligned}$$

the high-low m-para-product of f_1 and f_2

$$\begin{aligned}\Pi_{hl}(m)(f_1,f_2)(x) &= T_{\varphi_{hl}\cdot m}(f_1,f_2)(x)\\ &= \int_{\xi\in\mathbb{R}^2,\sigma\in\mathbb{R}^2} e^{2\pi i(\xi+\sigma)\cdot x}\varphi_{hl}(\sigma,\xi)m(\sigma,\xi)\hat{f}_1(\sigma)\hat{f}_2(\xi)d\sigma d\xi,\end{aligned}$$

and finally the low-high m-para-product of f_1 and f_2

$$\begin{aligned}\Pi_{lh}(m)(f_1,f_2)(x) &= T_{\varphi_{lh}\cdot m}(f_1,f_2)(x)\\ &= \int_{\xi\in\mathbb{R}^2,\sigma\in\mathbb{R}^2} e^{2\pi i(\xi+\sigma)\cdot x}\varphi_{lh}(\sigma,\xi)m(\sigma,\xi)\hat{f}_1(\sigma)\hat{f}_2(\xi)d\sigma d\xi.\end{aligned}$$

Since $\varphi_{hl},\varphi_{hh},\varphi_{lh}$ form a partition of unity, note that

$$T_m(f_1,f_2) = \Pi_{hh}(m)(f_1,f_2) + \Pi_{hl}(m)(f_1,f_2) + \Pi_{lh}(m)(f_1,f_2).$$

We also denote $\Pi_{hh}(f_1,f_2) := \Pi_{hh}(1)(f_1,f_2)$, $\Pi_{hl}(f_1,f_2) := \Pi_{hl}(1)(f_1,f_2)$, $\Pi_{lh}(f_1,f_2) := \Pi_{lh}(1)(f_1,f_2)$.

Next, we recall the Coifman–Meyer theorem. This can be found for instance in [33]:

Theorem 12.4 (Coifman–Meyer) *Let $m(\xi,\eta)$ be a smooth function on $\mathbb{R}^2$ which obeys the following bounds for any multi-indices ν, β:*

$$|\partial_\xi^\nu\partial_\sigma^\beta m(\sigma,\xi)| \lesssim_{\nu,\beta} (\langle\xi\rangle+\langle\sigma\rangle)^{-|\nu|-|\beta|}.$$

We say that m is a Coifman–Meyer multiplier.

Then for any $1 \le p, q, r \le \infty$ *such that* $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$ *and* $(p, q) \ne (\infty, \infty), (\infty, 1), (1, \infty)$*, we have,*

$$\|T_m(f_1, f_2)\|_{L^r(\mathbb{R}^2)} \lesssim \|f_1\|_{L^p(\mathbb{R}^2)} \|f_2\|_{L^q(\mathbb{R}^2)}.$$

Moreover, if m is a high-high Coifman–Meyer multiplier in the sense that $\operatorname{supp} m(\sigma, \xi) \subseteq \{(\sigma, \xi) \in \mathbb{R}^2 \times \mathbb{R}^2 : 10^{-1}|\sigma| \le |\xi| \le 10|\sigma|\}$*, then the following end-point estimate holds:*

$$\|T_m(f_1, f_2)\|_{L^2(\mathbb{R}^2)} \lesssim \|f_1\|_{BMO(\mathbb{R}^2)} \|f_2\|_{L^2(\mathbb{R}^2)}.$$

We will need several different Kato–Ponce type commutator estimates to estimate $\langle D_x\rangle^s(fh) - f\langle D_x\rangle^s h$; see Theorem 12.5–Corollary 12.10. The difference among these propositions is essentially the number of derivatives that is put on f.

Theorem 12.5 (Li [20, Theorem 1.1]) *For all* $0 < \theta \le 1$ *and* $1 < p < \infty$ *and* $1 < p_1, p_2 \le \infty$ *with* $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$*, the following holds for all* $f, h \in \mathcal{S}(\mathbb{R}^2)$ *with a constant depending only on* p_1, p_2 *and* θ*:*

$$\|\langle D_x\rangle^\theta(fh) - f(\langle D_x\rangle^\theta h)\|_{L^p(\mathbb{R}^2)} \lesssim \|\langle D_x\rangle^\theta f\|_{L^{p_1}(\mathbb{R}^2)} \|h\|_{L^{p_2}(\mathbb{R}^2)}.$$

An easy consequence of Theorem 12.5 is the following estimate[14]:

Lemma 12.6 *For any* $0 < \theta \le 1$ *and* $2 \le p_1, p_1', p_2, p_2' \le +\infty$ *such that* $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{2} = \frac{1}{p_1'} + \frac{1}{p_2'}$*,*

$$\|\langle D_x\rangle^\theta(fh)\|_{L^2(\mathbb{R}^2)} \lesssim \|\langle D_x\rangle^\theta f\|_{L^{p_1}(\mathbb{R}^2)} \|h\|_{L^{p_2}(\mathbb{R}^2)} + \|f\|_{L^{p_1'}(\mathbb{R}^2)} \|\langle D_x\rangle^\theta h\|_{L^{p_2'}(\mathbb{R}^2)}.$$

Proposition 12.7 *Let* $\theta \ge 0$ *and* $p \in [2, +\infty]$*. Then the following holds for any* $f, h \in \mathcal{S}(\mathbb{R}^2)$ *with an implicit constant depending only on* θ *and* p*:*

$$\|\langle D_x\rangle^\theta(fh) - f(\langle D_x\rangle^\theta h)\|_{L^2(\mathbb{R}^2)} \lesssim \|f\|_{W^{1,p}(\mathbb{R}^2)} \|\langle D_x\rangle^{\theta-1} h\|_{L^{\frac{2p}{p-2}}(\mathbb{R}^2)}.$$

Proof First, by the Plancherel formula we have

$$\begin{aligned}
&\|\langle D_x\rangle^\theta(fh) - f(\langle D_x\rangle^\theta h)\|_{L^2(\mathbb{R}^2)} \\
&\quad = \left\| \int_{\xi \in \mathbb{R}^2, \sigma \in \mathbb{R}^2} e^{2\pi i(\xi+\sigma)\cdot x} \left(\langle 2\pi(\xi+\sigma)\rangle^\theta - \langle 2\pi\xi\rangle^\theta\right) \hat{f}(\sigma)\hat{h}(\xi)\, d\sigma\, d\xi \right\|_{L^2(\mathbb{R}^2_x)}.
\end{aligned}$$

We will now write $1 = \varphi_{lh}(\sigma, \xi) + \varphi_{hh}(\sigma, \xi) + \varphi_{hl}(\sigma, \xi)$ and analyze each term separately.

[14] Remark that this estimate can also be derived directly, and is in fact much easier than Theorem 12.5.

Step 1: The low-high term To handle this term, we need to exploit the commutator structure to obtain the sought estimate.

$$\begin{aligned} &\varphi_{lh}(\sigma,\xi)(\langle 2\pi(\xi+\sigma)\rangle^{\theta}-\langle 2\pi\xi\rangle^{\theta}) \\ &\quad =(4\pi^2\theta)\varphi_{lh}(\sigma,\xi)\int_{t=0}^{1}\langle 2\pi(\xi+t\sigma)\rangle^{\theta-2}(\sigma\cdot(\xi+t\sigma))\,dt. \end{aligned} \tag{12.1}$$

Note that $\langle\xi\rangle^{-\theta+1}\varphi_{lh}(\sigma,\xi)\int_{t=0}^{1}\langle 2\pi(\xi+t\sigma)\rangle^{\theta-2}(\xi+t\sigma)_i\,dt$ is a Coifman–Meyer multiplier (for each i). Indeed, on the support of φ_{lh}, $\langle\xi+t\sigma\rangle$ and $\langle\xi\rangle$ are comparable when $t\in[0,1]$. Hence it is enough to show that for any ν, β:

$$\left|\partial_\xi^\nu\partial_\sigma^\beta\Big(\langle\xi\rangle^{-\theta+1}\int_{t=0}^{1}\langle\xi+t\sigma\rangle^{\theta-2}(\xi+t\sigma)_i\,dt\Big)\right|\lesssim\langle\xi\rangle^{-|\nu|+|\beta|},$$

which is an elementary computation. It follows from Theorem 12.4 that

$$\begin{aligned} &\|\langle D_x\rangle^{\theta}(\Pi_{lh}(f,h))-\Pi_{lh}(f,\langle D_x\rangle^{\theta}h)\|_{L^2(\mathbb{R}^2)} \\ &\quad\lesssim\|f\|_{W^{1,p}(\mathbb{R}^2)}\|\langle D_x\rangle^{\theta-1}h\|_{L^{\frac{2p}{p-2}}(\mathbb{R}^2)}. \end{aligned}$$

Step 2: The high-high terms We do not need the commutator structure. In other words, we bound $\langle D_x\rangle^{\theta}(\Pi_{hh}(f,h))$ and $\Pi_{hh}(f,\langle D_x\rangle^{\theta}h)$ separately.

We begin with the term $\langle D_x\rangle^{\theta}(\Pi_{hh}(f,h))$:

$$\begin{aligned} &\|\langle D_x\rangle^{\theta}(\Pi_{hh}(f,h))\|_{L^2(\mathbb{R}^2)}^2\lesssim\sum_k\|P_k\langle D_x\rangle^{\theta}(\Pi_{hh}(f,h))\|_{L^2(\mathbb{R}^2)}^2 \\ &\lesssim\sum_k\sum_{k':k'\geq k-3}\sum_{k'':|k''-k'|\leq 3}2^{2\theta k}\|P_{k'}f\|_{L^p(\mathbb{R}^2)}^2\|P_{k''}h\|_{L^{\frac{2p}{p-2}}(\mathbb{R}^2)}^2 \\ &\lesssim\sum_{k'}\sum_{k'':|k''-k'|\leq 3}(\sum_{k:k\leq k'+3}2^{2\theta k})\|P_{k'}f\|_{L^p(\mathbb{R}^2)}^2\|P_{k''}h\|_{L^{\frac{2p}{p-2}}(\mathbb{R}^2)}^2 \\ &\lesssim\sum_{k'}\sum_{k'':|k''-k'|\leq 3}(2^{2k'}\|P_{k'}f\|_{L^p(\mathbb{R}^2)}^2)(2^{2(\theta-1)k''}\|P_{k''}h\|_{L^{\frac{2p}{p-2}}(\mathbb{R}^2)}^2) \\ &\lesssim(\sup_{k'}2^{2k'}\|P_{k'}f\|_{L^p(\mathbb{R}^2)}^2)(\sum_{k''}2^{2(\theta-1)k''}\|P_{k''}h\|_{L^{\frac{2p}{p-2}}(\mathbb{R}^2)}^2) \\ &\lesssim\|f\|_{W^{1,p}(\mathbb{R}^2)}^2\|\langle D_x\rangle^{\theta-1}h\|_{L^{\frac{2p}{p-2}}(\mathbb{R}^2)}^2. \end{aligned}$$

Note that for any fixed k, we sum (k',k'') over $\{(k',k''),\ k'\geq k-3,\ |k''-k'|\leq 3\}$ because the support of the $\tilde{\varphi}_i$ (only) overlaps with the supports of $\tilde{\varphi}_{i-1}$ and $\tilde{\varphi}_{i+1}$.

To handle the term $\Pi_{hh}(f,\langle D_x\rangle^{\theta}h)$, we rely on the Coifman–Meyer theorem. First, it is easy to check that $\frac{\langle\xi\rangle}{\langle\sigma\rangle}\varphi_{hh}(\sigma,\xi)$ is a Coifman–Meyer multiplier. Therefore, for

$p \in [2, +\infty)$, the Coifman–Meyer theorem gives

$$\begin{aligned}\|\Pi_{hh}(f, \langle D_x\rangle^\theta h)\|_{L^2(\mathbb{R}^2)} &\lesssim \|\langle D_x\rangle f\|_{L^p(\mathbb{R}^2)}\|\langle D_x\rangle^{\theta-1}h\|_{L^{\frac{2p}{p-2}}(\mathbb{R}^2)}\\ &\lesssim \|f\|_{W^{1,p}(\mathbb{R}^2)}\|\langle D_x\rangle^{\theta-1}h\|_{L^{\frac{2p}{p-2}}(\mathbb{R}^2)}.\end{aligned}$$

When $p = +\infty$, we use moreover that we have a high-high multiplier so that the BMO endpoint holds[15]. Combining this with the estimate $\|\langle D_x\rangle f\|_{BMO(\mathbb{R}^2)} \lesssim \|f\|_{W^{1,\infty}(\mathbb{R}^2)}$, we obtain

$$\begin{aligned}\|\Pi_{hh}(f, \langle D_x\rangle^\theta h)\|_{L^2(\mathbb{R}^2)} &\lesssim \|\langle D_x\rangle f\|_{BMO(\mathbb{R}^2)}\|\langle D_x\rangle^{\theta-1}h\|_{L^2(\mathbb{R}^2)}\\ &\lesssim \|f\|_{W^{1,\infty}(\mathbb{R}^2)}\|\langle D_x\rangle^{\theta-1}h\|_{L^2(\mathbb{R}^2)}.\end{aligned}$$

Step 3: The high-low terms In a similar manner as in Step 2, we estimate $\langle D_x\rangle^\theta(\Pi_{hl}(f,h))$ and $\Pi_{hl}(f, \langle D_x\rangle^\theta h)$ separately.

The $\langle D_x\rangle^\theta(\Pi_{hl}(f,h))$ term can be estimated as follows:

$$\begin{aligned}&\|\langle D_x\rangle^\theta(\Pi_{hl}(f,h))\|^2_{L^2(\mathbb{R}^2)} \lesssim \sum_k \|P_k\langle D_x\rangle^\theta(\Pi_{hl}(f,h))\|^2_{L^2(\mathbb{R}^2)}\\ &\lesssim \sum_k \sum_{k':|k'-k|\le 3}\sum_{k'':k''\le k+3} 2^{2\theta k}\|P_{k'}f\|^2_{L^p(\mathbb{R}^2)}\|P_{k''}h\|^2_{L^{\frac{2p}{p-2}}(\mathbb{R}^2)}\\ &\lesssim \sum_{k''}\sum_{k':k'\ge k''-6} 2^{2\theta k'}\|P_{k'}f\|^2_{L^p(\mathbb{R}^2)}\|P_{k''}h\|^2_{L^{\frac{2p}{p-2}}(\mathbb{R}^2)}\\ &\lesssim \sum_{k''}\Big(\sum_{k':k'\ge k''-6} 2^{2(\theta-1)k'}\Big)\Big(\sup_{\tilde{k}'} 2^{2\tilde{k}'}\|P_{\tilde{k}'}f\|^2_{L^p(\mathbb{R}^2)}\Big)\|(P_{k''}h)\|^2_{L^{\frac{2p}{p-2}}(\mathbb{R}^2)}\\ &\lesssim \Big(\sup_{\tilde{k}'} 2^{2\tilde{k}'}\|P_{\tilde{k}'}f\|^2_{L^p(\mathbb{R}^2)}\Big)\Big(\sum_{k''}2^{2(\theta-1)k''}\|P_{k''}h\|^2_{L^{\frac{2p}{p-2}}(\mathbb{R}^2)}\Big) \lesssim \|f\|^2_{W^{1,p}(\mathbb{R}^2)}\|\langle D_x\rangle^{\theta-1}h\|^2_{L^{\frac{2p}{p-2}}(\mathbb{R}^2)}.\end{aligned} \tag{12.2}$$

For the $\Pi_{hl}(f, \langle D_x\rangle^\theta h)$ term, we begin with the trivial estimate

$$\begin{aligned}\|\Pi_{hl}(f, \langle D_x\rangle^\theta h)\|^2_{L^2(\mathbb{R}^2)} &\lesssim \sum_k \|P_k\Pi_{hl}(f, \langle D_x\rangle^\theta h)\|^2_{L^2(\mathbb{R}^2)}\\ &\lesssim \sum_k \sum_{k':|k'-k|\le 3}\sum_{k'':k''\le k+3} 2^{2\theta k''}\|P_{k'}f\|^2_{L^p(\mathbb{R}^2)}\|P_{k''}h\|^2_{L^{\frac{2p}{p-2}}(\mathbb{R}^2)}\\ &\lesssim \sum_k \sum_{k':|k'-k|\le 3}\sum_{k'':k''\le k+3} 2^{2\theta k}\|P_{k'}f\|^2_{L^p(\mathbb{R}^2)}\|P_{k''}h\|^2_{L^{\frac{2p}{p-2}}(\mathbb{R}^2)}.\end{aligned}$$

This coincides with the second line of (12.2) and we can argue in exactly the same way. □

[15] Note that we need to use the BMO estimate here because the Riesz transform $\langle D_x\rangle^{-1}\partial_x$ is not bounded on L^∞. Instead, we rely on the fact that $\|\langle D_x\rangle f\|_{BMO(\mathbb{R}^2)} \lesssim \|\partial_x f\|_{BMO(\mathbb{R}^2)} \lesssim \|f\|_{W^{1,\infty}(\mathbb{R}^2)}$.

Corollary 12.8 *Let $p \in (2, +\infty]$ and $0 < \theta_2 < \theta_1 \leq 1$ such $p \geq \frac{2}{\theta_1 - \theta_2}$. Then the following holds for any $f, h \in \mathcal{S}(\mathbb{R}^2)$ with an implicit constant depending only on p, θ_1 and θ_2:*

$$\|\langle D_x\rangle^{\theta_2}(fh) - f(\langle D_x\rangle^{\theta_2} h)\|_{L^2(\mathbb{R}^2)} \lesssim \|f\|_{W^{1,p}(\mathbb{R}^2)} \|\langle D_x\rangle^{\theta_1 - 1} h\|_{L^2(\mathbb{R}^2)}.$$

Proof This follows from Proposition 12.7 and the Sobolev embedding $H^{\theta_1-\theta_2}(\mathbb{R}^2) \hookrightarrow L^{\frac{2p}{p-2}}(\mathbb{R}^2)$. □

Next, we need a more precise commutator estimate which essentially gives the "main term" of the commutator up to some residual error satisfying better estimates. To set up the notation, for any $f, h \in \mathcal{S}(\mathbb{R}^2)$, define

$$T^\theta_{\text{res}}(f, h) := \langle D_x\rangle^\theta (fh) - f(\langle D_x\rangle^\theta h) - \theta\delta^{ij}(\partial_i f)(\partial_j \langle D_x\rangle^{\theta-2} h), \tag{12.3}$$

Define also $\Pi_{lh} T^\theta_{\text{res}}(f, h)$ by

$$\Pi_{lh} T^\theta_{\text{res}}(f, h) := \langle D_x\rangle^\theta \Pi_{lh}(f, h) - \Pi_{lh}(f, \langle D_x\rangle^\theta h) - \theta\delta^{ij}\Pi_{lh}(\partial_i f, \partial_j \langle D_x\rangle^{\theta-2} h)$$

and similarly for $\Pi_{hh} T^\theta_{\text{res}}(f, h)$ and $\Pi_{hl} T^\theta_{\text{res}}(f, h)$. Then the following estimate holds.

Proposition 12.9 *Let $\theta \geq 0$ and $p \in (2, +\infty]$. Then the following holds for any $f, h \in \mathcal{S}(\mathbb{R}^2)$ with an implicit constant depending only on θ and p:*

$$\|T^\theta_{\text{res}}(f, h)\|_{L^2(\mathbb{R}^2)} \lesssim \|f\|_{W^{2,p}(\mathbb{R}^2)} \|\langle D_x\rangle^{\theta-2} h\|_{L^{\frac{2p}{p-2}}(\mathbb{R}^2)}. \tag{12.4}$$

Proof Note that the only difficulty concerns the low-high term. For the high-high and high-low interactions, it is easy to check as in the proof of Proposition 12.7 that each of the terms $\|\langle D_x\rangle^\theta \Pi_{hh}(f, h)\|_{L^2(\mathbb{R}^2)}$, $\|\Pi_{hh}(f, \langle D_x\rangle^\theta h)\|_{L^2(\mathbb{R}^2)}$, $\|\Pi_{hh}(\partial_i f, \partial_j \langle D_x\rangle^{\theta-2} h)\|_{L^2(\mathbb{R}^2)}$, $\|\langle D_x\rangle^\theta \Pi_{hl}(f, h)\|_{L^2(\mathbb{R}^2)}$, $\|\Pi_{hl}(f, \langle D_x\rangle^\theta h)\|_{L^2(\mathbb{R}^2)}$ and $\|\Pi_{hl}(\partial_i f, \partial_j \langle D_x\rangle^{\theta-2} h)\|_{L^2(\mathbb{R}^2)}$ is bounded by the right-hand side of (12.4); we omit the details.

For the low-high term, we continue the computation of (12.1). More precisely, we use

$$r(1) = r(0) + r'(0) + \int_{t=0}^1 (1-t) r''(t)\, \mathrm{d}t$$

with $r(t) = \langle 2\pi(\xi + t\sigma)\rangle^\theta$ to obtain

$$\begin{aligned} &(\langle 2\pi(\xi+\sigma)\rangle^\theta - \langle 2\pi\xi\rangle^\theta) - 4\pi^2\theta\langle 2\pi\xi\rangle^{\theta-2}(\xi\cdot\sigma) \\ &= 4\pi^2\theta \int_{t=0}^1 (1-t)\{4\pi^2(\theta-2)\langle 2\pi(\xi+t\sigma)\rangle^{\theta-4}(\sigma\cdot(\xi+t\sigma))^2 \\ &\quad + \langle 2\pi(\xi+t\sigma)\rangle^{\theta-2}|\sigma|^2\}\, \mathrm{d}t \\ &=: m(\sigma, \xi). \end{aligned} \tag{12.5}$$

Notice that

$$m(\sigma,\xi) = |\sigma|^2 m_A(\sigma,\xi) + \sum_{i,j} \sigma_i\sigma_j (m_B)_{ij}(\sigma,\xi),$$

where m_A and m_B are defined by

$$\begin{aligned} m_A(\sigma,\xi) &:= 4\pi^2\theta \int_{t=0}^1 (1-t)\{4\pi^2(\theta-2)\langle 2\pi(\xi+t\sigma)\rangle^{\theta-4}(t^2|\sigma|^2 + 2t(\sigma\cdot\xi)) \\ &\quad + \langle 2\pi(\xi+t\sigma)\rangle^{\theta-2}\}\,\mathrm{d}t, \end{aligned}$$

and

$$(m_B)_{ij}(\sigma,\xi) := 16\pi^4\theta(\theta-2)\xi_i\xi_j \int_{t=0}^1 (1-t)\langle 2\pi(\xi+t\sigma)\rangle^{\theta-4}\,dt.$$

It is easy to check that $\langle\xi\rangle^{-\theta+2}\varphi_{lh}(\sigma,\xi)m_A(\sigma,\xi)$ and $\langle\xi\rangle^{-\theta+2}\varphi_{lh}(\sigma,\xi)(m_B)_{ij}(\sigma,\xi)$ are both Coifman–Meyer multipliers.

The computation (12.5) implies that

$$\begin{aligned} &\|\Pi_{lh}T^{\theta}_{\mathrm{res}}(f,h)\|_{L^2(\mathbb{R}^2)} \\ &\quad = \|\langle D_x\rangle^{\theta}\Pi_{lh}(f,h) - \Pi_{lh}(f,\langle D_x\rangle^{\theta}h) - \theta\delta^{ij}\Pi_{lh}(\partial_i f, \partial_j\langle D_x\rangle^{\theta-2}h)\|_{L^2(\mathbb{R}^2)} \\ &\quad \lesssim \|T_{\varphi_{lh}m_A}(\Delta f,h)\|_{L^2(\mathbb{R}^2)} + \sum_{i,j}\|T_{\varphi_{lh}(m_B)_{ij}}(\partial^2_{ij}f,h)\|_{L^2(\mathbb{R}^2)} \\ &\quad \lesssim \|f\|_{W^{2,p}(\Sigma_t)}\|\langle D_x\rangle^{\theta-2}h\|_{L^{\frac{2p}{p-2}}(\mathbb{R}^2)}, \end{aligned}$$

where in the last line we have used the Coifman–Meyer theorem (Theorem 12.4).

This gives the desired estimates for the low-high interaction. As described in the beginning the high-high and high-low are easier, and we have therefore completed the proof of the proposition. □

We record another easy but useful way to estimate the term in Proposition 12.9:

Corollary 12.10 *Let $T^{\theta_2}_{\mathrm{res}}$ be as in (12.3). Let $p \in (2,+\infty]$ and $0 < \theta_2 < \theta_1$ such $p \geq \frac{2}{\theta_1-\theta_2}$. Then the following hold for any $f, h \in \mathcal{S}(\mathbb{R}^2)$ with implicit constant depending only on p, θ_1 and θ_2:*

$$\begin{aligned} &\|T^{\theta_2}_{\mathrm{res}}(f,h)\|_{L^2(\mathbb{R}^2)} \\ &\quad \lesssim \min\{\|f\|_{W^{1,p}(\mathbb{R}^2)}\|\langle D_x\rangle^{\theta_1-1}h\|_{L^2(\mathbb{R}^2)},\ \|f\|_{W^{2,p}(\mathbb{R}^2)}\|\langle D_x\rangle^{\theta_1-2}h\|_{L^2(\mathbb{R}^2)}\}. \end{aligned}$$

Proof On the one hand, by the triangle inequality, Corollary 12.8 and Hölder's inequality,

$$\|T^{\theta_2}_{\mathrm{res}}(f,h)\|_{L^2(\mathbb{R}^2)} \lesssim \|f\|_{W^{1,p}(\mathbb{R}^2)}\|\langle D_x\rangle^{\theta_1-1}h\|_{L^2(\mathbb{R}^2)}.$$

On the other hand, by the triangle inequality, Proposition 12.9 and the Sobolev embedding $H^{\theta_1-\theta_2}(\mathbb{R}^2) \hookrightarrow L^{\frac{2p}{p-2}}(\mathbb{R}^2)$,

$$\|T^{\theta_2}_{\mathrm{res}}(f,h)\|_{L^2(\mathbb{R}^2)} \lesssim \|f\|_{W^{2,p}(\mathbb{R}^2)}\|\langle D_x\rangle^{\theta_1-2}h\|_{L^2(\mathbb{R}^2)}.$$

Combining yields the result. □

Finally, we need an auxiliary commutation lemma concerning the commutation of a vector field with the (inhomogeneous) Riesz transform.

Lemma 12.11 *Let $Y^i\partial_i$ be a vector field on $\mathbb{R}^2$ such that $Y^i \in W^{1,\infty}(\mathbb{R}^2)$ and $f \in L^2(\mathbb{R}^2)$. Denoting $R_j = \partial_j\langle D_x\rangle^{-1}$, we have*

$$\big\|\big[Y, R_j\big] f\big\|_{L^2(\mathbb{R}^2)} \lesssim \max_{i=1,2} \|Y^i\|_{W^{1,\infty}(\mathbb{R}^2)} \cdot \|f\|_{L^2(\mathbb{R}^2)}.$$

Proof By the Calderón commutator estimate (see [32, Corollary on p.309]),

$$\|\partial_i R_j(Y^i f) - Y^i\partial_i R_j f\|_{L^2(\mathbb{R}^2)} \lesssim \max_{i=1,2} \|Y^i\|_{W^{1,\infty}(\mathbb{R}^2)} \cdot \|f\|_{L^2(\mathbb{R}^2)}.$$

Hence, by the triangle inequality and the L^2-boundedness of R_j,

$$\begin{aligned}\big\|\big[Y, R_j\big] f\big\|_{L^2(\mathbb{R}^2)} &\lesssim \|\partial_i R_j(Y^i f) - Y^i\partial_i R_j f\|_{L^2(\mathbb{R}^2)} + \|R_j[(\partial_i Y^i)f]\|_{L^2(\mathbb{R}^2)}\\ &\lesssim \max_{i=1,2} \|Y^i\|_{W^{1,\infty}(\mathbb{R}^2)} \cdot \|f\|_{L^2(\mathbb{R}^2)}.\end{aligned}$$

□

12.2 Notations for this section

We now define some notations that will be useful for the remainder of the section.

From now on, fix a cutoff function $\varpi \in C^\infty_c$ such that $\varpi \equiv 1$ on $B(0,2R)$ and $\mathrm{supp}(\varpi) \subseteq B(0,3R)$.

We also introduce the following notations for the wave equation.

Let $\Box^2$ and $\Box^1$ be operators defined by

$$\Box_g f = \overbrace{(g^{-1})^{\nu\beta}\partial^2_{\nu\beta} f}^{:=\Box^2(f)} - \overbrace{\Gamma^\lambda \cdot \partial_\lambda f}^{:=\Box^1(f)}, \tag{12.6}$$

where $(g^{-1})^{\nu\beta}$ are the components of the inverse matrix of $g_{\nu\beta}$, as expressed in (2.2) and

$$\Gamma^\lambda := (g^{-1})^{\nu\beta}\Gamma^\lambda_{\nu\beta}, \quad \Gamma^\lambda_{\nu\beta} := \frac{1}{2}(g^{-1})^{\lambda\sigma}(\partial_\nu g_{\sigma\beta} + \partial_\beta g_{\sigma\nu} - \partial_\sigma g_{\nu\beta}) \tag{12.7}$$

(in the coordinate system (t, x^1, x^2) of (2.2)).

Finally, define

$$\breve{g}^{ij} := \frac{(g^{-1})^{ij}}{(g^{-1})^{tt}}, \quad \breve{g}^{it} := \frac{2(g^{-1})^{it}}{(g^{-1})^{tt}} \tag{12.8}$$

so that

$$\partial_{tt}^2 = \frac{1}{(g^{-1})^{tt}}\Box_g - \breve{g}^{i\lambda}\partial_{i\lambda}^2 + \frac{\Gamma^\lambda}{(g^{-1})^{tt}}\partial_\lambda. \tag{12.9}$$

12.3 Preliminary estimates

The following basic estimate will be repeatedly used. (Recall the notation for ϖ defined in the beginning of Sect. 12.2.)

Lemma 12.12 *Let v be a smooth, compactly supported function on $B(0, R)$ and f be a smooth function. Then*

$$\|\langle D_x\rangle^{s'}(fv)\|_{L^2(\Sigma_t)} \lesssim \|\varpi f\|_{L^\infty\cap W^{1,2}(\Sigma_t)}\|\langle D_x\rangle^{s'}v\|_{L^2(\Sigma_t)}.$$

Proof Note that $fv = \varpi fv$. Hence, by Theorem 12.5 (with $p = 2$, $p_1 = \frac{2}{s'}$, $p_2 = \frac{2}{1-s'}$) and Hölder's inequality,

$$\begin{aligned}\|\langle D_x\rangle^{s'}(fv)\|_{L^2(\Sigma_t)} &\lesssim \|\varpi f\|_{L^\infty(\Sigma_t)}\|v\|_{H^{s'}(\Sigma_t)} + \|\langle D_x\rangle^{s'}(\varpi f)\|_{L^{\frac{2}{s'}}(\Sigma_t)}\|v\|_{L^{\frac{2}{1-s'}}(\Sigma_t)}\\ &\lesssim \|\varpi f\|_{L^\infty\cap W^{1,2}(\Sigma_t)}\|v\|_{H^{s'}(\Sigma_t)},\end{aligned}$$

where in the last inequality we have used Sobolev embeddings $H^1(\Sigma_t) \hookrightarrow W^{s',\frac{2}{s'}}(\Sigma_t)$ and $H^{s'}(\Sigma_t) \hookrightarrow L^{\frac{2}{1-s'}}(\Sigma_t)$. □

We apply Lemma 12.12 in the special case where $v = \partial_\lambda\widetilde{\phi}_k$.

Lemma 12.13 *Let f be a smooth function satisfying*

$$\|\varpi f\|_{L^\infty\cap W^{1,2}(\Sigma_t)} \lesssim 1.$$

Then

$$\|\langle D_x\rangle^{s'}(f\partial_\lambda\widetilde{\phi}_k)\|_{L^2(\Sigma_t)} \lesssim \|\partial\langle D_x\rangle^{s'}\widetilde{\phi}_k\|_{L^2(\Sigma_t)}.$$

Proof Using the support properties in Lemma 5.1, the result follows from Lemma 12.12 with $v = \partial_\lambda\widetilde{\phi}_k$. □

We will derive a few of consequences of Lemma 12.13.

Lemma 12.14 *Let f be a smooth function satisfying*

$$\|\varpi f\|_{L^\infty\cap W^{1,2}(\Sigma_t)} \lesssim 1.$$

Then (recall the notation in (12.8)):

$$\|\langle D_x\rangle^{s'}(f\breve{g}^{j\lambda}\partial_\lambda\widetilde{\phi}_k)\|_{L^2(\Sigma_t)} \lesssim \|\partial\langle D_x\rangle^{s'}\widetilde{\phi}_k\|_{L^2(\Sigma_t)}, \tag{12.10}$$

$$\|\langle D_x\rangle^{s'}[f(\partial_j\breve{g}^{j\lambda})(\partial_\lambda\widetilde{\phi}_k)]\|_{L^2(\Sigma_t)} \lesssim \|\partial\langle D_x\rangle^{s'}\widetilde{\phi}_k\|_{L^2(\Sigma_t)}, \tag{12.11}$$

$$\|\langle D_x\rangle^{s'}[f\frac{\Gamma^\lambda\partial_\lambda\widetilde{\phi}_k}{(g^{-1})^{tt}}]\|_{L^2(\Sigma_t)} \lesssim \|\partial\langle D_x\rangle^{s'}\widetilde{\phi}_k\|_{L^2(\Sigma_t)}. \tag{12.12}$$

Proof The estimates follow immediately from Lemma 12.13 and the estimates (5.1). □

Another consequence of Lemma 12.13 is that we can control negative fractional derivatives of $\partial^2\widetilde{\phi}_k$, i.e. terms of the form $\langle D_x\rangle^{s'-1}\partial^2\widetilde{\phi}_k$. We start with a more general lemma, before turning to $\langle D_x\rangle^{s'-1}\partial^2\widetilde{\phi}_k$ in Lemma 12.16.

Lemma 12.15 *Let f be a smooth function satisfying*[16]

$$\|\varpi f\|_{L^\infty\cap W^{1,2}(\Sigma_t)} + \|\varpi\partial_i f\|_{L^4(\Sigma_t)} \lesssim 1, \tag{12.13}$$

and v be a smooth, compactly supported function on $B(0,R)$. Then

$$\|\langle D_x\rangle^{s'-1}(f\partial^2_{i\lambda}v)\|_{L^2(\Sigma_t)} \lesssim \|\partial\langle D_x\rangle^{s'}v\|_{L^2(\Sigma_t)} \tag{12.14}$$

and

$$\|\langle D_x\rangle^{s'-1}(f\partial^2_{tt}v)\|_{L^2(\Sigma_t)} \lesssim \|\partial\langle D_x\rangle^{s'}v\|_{L^2(\Sigma_t)} + \|\Box_g v\|_{L^2(\Sigma_t)}. \tag{12.15}$$

Proof For (12.14), note that

$$\begin{aligned}
&\|\langle D_x\rangle^{s'-1}(f\partial^2_{i\lambda}v)\|_{L^2(\Sigma_t)}\\
&\lesssim \|\langle D_x\rangle^{s'-1}\partial_i(f\partial_\lambda v)\|_{L^2(\Sigma_t)} + \|\langle D_x\rangle^{s'-1}[(\partial_i f)(\partial_\lambda v)]\|_{L^2(\Sigma_t)}\\
&\lesssim \|\langle D_x\rangle^{s'}(f\partial_\lambda v)\|_{L^2(\Sigma_t)} + \|\varpi\partial_i f\|_{L^4(\Sigma_t)}\|\partial_\lambda v\|_{L^2(\Sigma_t)} \lesssim \|\partial\langle D_x\rangle^{s'}v\|_{L^2(\Sigma_t)},
\end{aligned}$$

where in the penultimate estimate we have used Sobolev embedding $\langle D_x\rangle^{s'-1}$: $L^{\frac{4}{3}}(\Sigma_t) \to L^2(\Sigma_t)$ (which is true since $s' < \frac{1}{2}$) and Hölder's inequality, and in the last estimate we have used Lemma 12.13.

[16] It should be noted that the $W^{1,2}$ bound on ϖf in (12.13) is extraneous, as it is implied by the other bounds. We state the assumption as in (12.13) so as to make the application of Lemma 12.14 more obvious.

We now prove (12.15). We rewrite $\partial_{tt}^2 v$ in terms of $\Box_g v$ using (12.9) and apply the triangle inequality to obtain

$$\begin{aligned}
&\|\langle D_x\rangle^{s'-1}(f\partial_{tt}^2 v)\|_{L^2(\Sigma_t)}\\
&\lesssim \|\langle D_x\rangle^{s'-1}[f(\frac{\Box_g v}{(g^{-1})^{tt}} - \breve{g}^{i\lambda}\partial_{i\lambda}^2 v + \frac{\Gamma^\lambda\partial_\lambda v}{(g^{-1})^{tt}})]\|_{L^2(\Sigma_t)}\\
&\lesssim \|R_i\langle D_x\rangle^{s'}(f\breve{g}^{i\lambda}\partial_\lambda v)\|_{L^2(\Sigma_t)} + \|\langle D_x\rangle^{s'-1}[(\partial_i(f\breve{g}^{i\lambda}))(\partial_\lambda v)]\|_{L^2(\Sigma_t)}\\
&\quad + \|\langle D_x\rangle^{s'-1}(\frac{f\Gamma^\lambda\partial_\lambda v}{(g^{-1})^{tt}})\|_{L^2(\Sigma_t)} + \|\langle D_x\rangle^{s'-1}(f\frac{\Box_g v}{(g^{-1})^{tt}})\|_{L^2(\Sigma_t)},
\end{aligned}$$

where $R_i = \partial_i\langle D_x\rangle^{-1}$ as before.

For the first term, we use that $R_i : L^2(\Sigma_t) \to L^2(\Sigma_t)$ is bounded and then use Lemma 12.12, (12.13) and (5.1) to bound it by $\lesssim \|\partial\langle D_x\rangle^{s'} v\|_{L^2(\Sigma_t)}$. For the second and third terms, we use in addition the Sobolev embedding $\langle D_x\rangle^{s'-1} : L^{\frac{4}{3}}(\Sigma_t) \to L^2(\Sigma_t)$. For instance, for the second term we have (using $v = \varpi v$ and Hölder's inequality)

$$\begin{aligned}
&\|\langle D_x\rangle^{s'-1}[(\partial_i(f\breve{g}^{i\lambda}))(\partial_\lambda v)]\|_{L^2(\Sigma_t)} \lesssim \|[(\partial_i(f\breve{g}^{i\lambda}))(\partial_\lambda v)]\|_{L^{\frac{4}{3}}(\Sigma_t)}\\
&\lesssim \left(\|\varpi\partial_i f\|_{L^4(\Sigma_t)}\|\varpi\breve{g}^\lambda\|_{L^\infty(\Sigma_t)} + \|\varpi f\|_{L^\infty(\Sigma_t)}\|\varpi\partial_i\breve{g}^\lambda\|_{L^4(\Sigma_t)}\right)\|\partial_\lambda v\|_{L^2(\Sigma_t)}\\
&\lesssim \|\partial v\|_{L^2(\Sigma_t)},
\end{aligned}$$

where in the last estimate we used (12.13) together with (5.1). The third term is similar and omitted. For the fourth term, we use the Sobolev embedding $\langle D_x\rangle^{s'-1} : L^{\frac{4}{3}}(\Sigma_t) \to L^2(\Sigma_t)$ and the estimates (12.13), (5.1) to get

$$\|\langle D_x\rangle^{s'-1}(f(\frac{\Box_g v}{(g^{-1})^{tt}}))\|_{L^2(\Sigma_t)} \lesssim \|\Box_g v\|_{L^2(\Sigma_t)}.$$

We have thus obtained (12.15). □

Lemma 12.16 *Let f be a smooth function satisfying*

$$\|\varpi f\|_{L^\infty\cap W^{1,2}(\Sigma_t)} + \|\varpi\partial_i f\|_{L^4(\Sigma_t)} \lesssim 1. \tag{12.16}$$

Then

$$\|\langle D_x\rangle^{s'-1}(f\partial_{\beta\lambda}^2\widetilde{\phi}_k)\|_{L^2(\Sigma_t)} \lesssim \|\partial\langle D_x\rangle^{s'}\widetilde{\phi}_k\|_{L^2(\Sigma_t)}. \tag{12.17}$$

Proof This follows immediately from an application of Lemma 12.15, with $v = \widetilde{\phi}_k$ since $\widetilde{\phi}_k$ is smooth compactly supported in $B(0, R)$ and $\Box_g\widetilde{\phi}_k = 0$. □

12.4 Weighted estimates and cutoffs

12.4.1 Gaining weights in the estimate for $\partial\langle D_x\rangle^{s'}\widetilde{\phi}_k$

Lemma 12.17 *Let $\varpi \in C_c^\infty$ be a cutoff function such that $\varpi \equiv 1$ on $B(0,2R)$ and* $\mathrm{supp}(\varpi) \subset B(0,3R)$*; and $\varpi' \in C_c^\infty$ such that $\varpi' \equiv 1$ on $B(0,R)$ and* $\mathrm{supp}(\varpi') \subset B(0,2R)$. *Let P be a fixed pseudo-differential operator (of arbitrary order). Then*

$$[\varpi, P]\varpi' \text{ is a pseudo-differential operator of order } -\infty.$$

In particular, for any $\sigma \in \mathbb{R}$, the following estimate holds:

$$\|[P,\varpi]f\|_{H^\sigma(\mathbb{R}^2)} \lesssim \|f\|_{H^{-2}(\mathbb{R}^2)}, \tag{12.18}$$

where the implicit constant depends only on P, R and σ.

Proof Since $\partial_x\varpi$ and ϖ' have disjoint support, the desired conclusion follows from the usual symbolic calculus for pseudo-differential operators; see for instance [32, Theorem 2 on p.237]. □

Proposition 12.18 *Let v be a smooth, compactly supported function on $B(0,R)$. Then v satisfies the following*

$$\|\partial\langle D_x\rangle^{s'}v\|_{L^2(\Sigma_t)} \lesssim \|\langle x\rangle^{-r}\partial\langle D_x\rangle^{s'}v\|_{L^2(\Sigma_t)} + \|\partial v\|_{L^2(\Sigma_t)}. \tag{12.19}$$

Proof Let $\varpi \in C_c^\infty(\mathbb{R}^2;\mathbb{R})$ be as in Lemma 12.17.

Using the fact $v = \varpi v$, we compute

$$\partial\langle D_x\rangle^{s'}v = \varpi\partial\langle D_x\rangle^{s'}v - [\varpi, \langle D_x\rangle^{s'}]\partial v.$$

Since ϖ is compactly supported, we can bound

$\|\varpi\partial\langle D_x\rangle^{s'}v\|_{L^2(\Sigma_t)} \lesssim \|\langle x\rangle^{-r}\partial\langle D_x\rangle^{s'}v\|_{L^2(\Sigma_t)}$. On the other hand, $[\varpi, \langle D_x\rangle^{s'}] : L^2 \to L^2$ is bounded by Lemma 12.17. Hence (12.19) follows. □

Proposition 12.19

$$\|\partial\langle D_x\rangle^{s'}\widetilde{\phi}_k\|_{L^2(\Sigma_t)} \lesssim \|\langle x\rangle^{-r}\partial\langle D_x\rangle^{s'}\widetilde{\phi}_k\|_{L^2(\Sigma_t)} + \|\partial\widetilde{\phi}_k\|_{L^2(\Sigma_t)}. \tag{12.20}$$

Proof After we recall that $\widetilde{\phi}_k$ is supported in $B(0,R)$ for every t, (12.20) is obtained as an immediate application of Proposition 12.18 with $v = \widetilde{\phi}_k$. □

12.4.2 Gaining weights in the estimates for $\partial E_k\langle D_x\rangle^{s''}\widetilde{\phi}_k$ and $\partial L_k\langle D_x\rangle^{s''}\widetilde{\phi}_k$

Our next goal will be to prove an analogue of Proposition 12.19, but with also commutations with E_k and L_k; see already Proposition 12.23. In order to achieve this, we need to understand weighted bounds and derivative bounds involving $[\varpi, \langle D_x\rangle^{s''}]$. This will be achieved in the next three lemmas, before we finally turn to Proposition 12.23.

Note that we use $\langle D_x\rangle^{s''}$ here instead of $\langle D_x\rangle^{s'}$ since we will only estimate $\partial E_k \langle D_x\rangle^{s''}\widetilde{\phi}_k$ and $\partial L_k \langle D_x\rangle^{s''}\widetilde{\phi}_k$ (in terms of $\partial\langle D_x\rangle^{s'}\widetilde{\phi}_k$) where we recall that $0 < s'' < s' < \frac{1}{2}$. It is important to comment that we cannot estimate $\partial E_k \langle D_x\rangle^{s'}\widetilde{\phi}_k$ and $\partial L_k \langle D_x\rangle^{s'}\widetilde{\phi}_k$ due to the low regularity of the metric; see the second bullet point in the explanation of (1.31) in Sect. 1.1.6.

Lemma 12.20 *Let f be a smooth function which is supported in $B(0,R)$ for each t. Then*

$$\|\langle x\rangle[\varpi, \langle D_x\rangle^{s''}]f\|_{L^2(\Sigma_t)} \lesssim \|f\|_{L^2(\Sigma_t)}.$$

Proof *Step 1: An easy reduction* Obviously,

$$\|\langle x\rangle[\varpi, \langle D_x\rangle^{s''}]f\|^2_{L^2(\Sigma_t)} = \|[\varpi, \langle D_x\rangle^{s''}]f\|^2_{L^2(\Sigma_t)} + \sum_{\ell=1}^{2} \|x^\ell[\varpi, \langle D_x\rangle^{s''}]f\|^2_{L^2(\Sigma_t)}.$$

Since $\|[\varpi, \langle D_x\rangle^{s''}]f\|_{L^2(\Sigma_t)} \lesssim \|\langle D_x\rangle^{s''-2} f\|_{L^2(\Sigma_t)}$ by Lemma 12.17, it suffices to show that for $\ell = 1, 2$,

$$\|x^\ell[\varpi, \langle D_x\rangle^{s''}]f\|_{L^2(\Sigma_t)} \lesssim \|\langle D_x\rangle^{s''-1} f\|_{L^2(\Sigma_t)} \lesssim \|f\|_{L^2(\Sigma_t)}. \tag{12.21}$$

Step 2: Proof of (12.21) We compute

$$x^\ell[\varpi, \langle D_x\rangle^{s''}]f = \underbrace{[\varpi, \langle D_x\rangle^{s''}](x^\ell f)}_{=:I} + \underbrace{\varpi[x^\ell, \langle D_x\rangle^{s''}]f}_{=:II} \underbrace{-[x^\ell, \langle D_x\rangle^{s''}](\varpi f)}_{=:III} \tag{12.22}$$

We have by Lemma 12.17 and the support property of f that

$$\|I\|_{L^2(\Sigma_t)} \lesssim \|\langle D_x\rangle^{s''-2}(x^\ell f)\|_{L^2(\Sigma_t)} \lesssim \|x^\ell f\|_{L^2(\Sigma_t)} \lesssim \|f\|_{L^2(\Sigma_t)}.$$

Before handling II and III, first note that using the Fourier transform, it can easily be checked that for any Schwartz function f,

$$[x^\ell, \langle D_x\rangle^{s''}]f = s''\langle D_x\rangle^{s''-2}\partial_\ell f. \tag{12.23}$$

From (12.23), it immediately follows that

$$\|II\|_{L^2(\Sigma_t)} + \|III\|_{L^2(\Sigma_t)} \lesssim \|\langle D_x\rangle^{s''-1} f\|_{L^2(\Sigma_t)} \lesssim \|f\|_{L^2(\Sigma_t)}.$$

Combining the estimates for I, II and III, we have thus proven (12.21), which by Step 1 implies the desired estimate. □

Lemma 12.21 *Let f be a smooth function which is supported in $B(0,R)$ for each t. Then, for every index ν,*

$$\|\langle x\rangle \partial_\nu [\varpi, \langle D_x\rangle^{s''}] f\|_{L^2(\Sigma_t)} \lesssim \|\partial f\|_{L^2(\Sigma_t)}.$$

Moreover, if $\nu = i$ is a spatial index, we have the improved estimate[17]

$$\|\langle x\rangle \partial_i [\varpi, \langle D_x\rangle^{s''}] f\|_{L^2(\Sigma_t)} \lesssim \|\langle D_x\rangle^{s''} f\|_{L^2(\Sigma_t)}. \tag{12.24}$$

Proof *Step 0: An easy reduction*

By Lemma 12.17 and the Poincaré inequality (since f is compactly supported in $B(0,R)$),

$$\|\partial_\nu [\varpi, \langle D_x\rangle^{s''}] f\|_{L^2(\Sigma_t)} \lesssim \|f\|_{L^2(\Sigma_t)} + \|\partial_t f\|_{L^2(\Sigma_t)} \lesssim \|\partial f\|_{L^2(\Sigma_t)}.$$

Hence, by a reduction similar to Step 1 of Lemma 12.20, it suffices to prove that for $\ell = 1, 2$,

$$\begin{aligned} \|x^\ell \partial_t [\varpi, \langle D_x\rangle^{s''}] f\|_{L^2(\Sigma_t)} &\lesssim \|\partial f\|_{L^2(\Sigma_t)}, \\ \|x^\ell \partial_i [\varpi, \langle D_x\rangle^{s''}] f\|_{L^2(\Sigma_t)} &\lesssim \|\langle D_x\rangle^{s''} f\|_{L^2(\Sigma_t)}. \end{aligned} \tag{12.25}$$

Step 1: Estimates for general ν We compute

$$\begin{aligned} &x^\ell \partial_\nu [\varpi, \langle D_x\rangle^{s''}] f = x^\ell \partial_\nu (\varpi \langle D_x\rangle^{s''} f) - x^\ell \partial_\nu \langle D_x\rangle^{s''} (\varpi f) \\ &= \underbrace{\partial_\nu [\varpi, \langle D_x\rangle^{s''}](x^\ell f)}_{=:I} + \underbrace{\partial_\nu (\varpi [x^\ell, \langle D_x\rangle^{s''}] f)}_{=:II} \underbrace{- \partial_\nu ([x^\ell, \langle D_x\rangle^{s''}](\varpi f))}_{=:III} \underbrace{- (\partial_\nu x^\ell)[\varpi, \langle D_x\rangle^{s''}] f}_{=:IV}. \end{aligned} \tag{12.26}$$

The term I needs to be treated differently for $\nu = i$ and $\nu = t$; see Steps 2 and 3 below.

For the terms II and III,
we use (12.23) and the L^2-boundedness of the $\langle D_x\rangle^{-1}\partial_k$ to obtain

$$\begin{aligned} \|II\|_{L^2(\Sigma_t)} \leq \|\partial_\nu (\varpi \langle D_x\rangle^{s''-2} \partial_k f)\|_{L^2(\Sigma_t)} &\lesssim \|\langle D_x\rangle^{s''-1} f\|_{L^2(\Sigma_t)} \\ &\quad + \|\langle D_x\rangle^{s''-1} \partial_\nu f\|_{L^2(\Sigma_t)} \end{aligned} \tag{12.27}$$

and

$$\|III\|_{L^2(\Sigma_t)} \leq \|\langle D_x\rangle^{s''-2} \partial^2_{k\nu} (\varpi f)\|_{L^2(\Sigma_t)} \lesssim \|\langle D_x\rangle^{s''-1} \partial_\nu f\|_{L^2(\Sigma_t)}. \tag{12.28}$$

[17] Note that this is indeed an improvement since f is compactly supported in $B(0,R)$ and we can apply the Poincaré inequality.

The term IV is the simplest. Since $\partial_\nu x^\ell$ is bounded (in L^∞), we have by Lemma 12.17

$$\|IV\|_{L^2(\Sigma_t)} \lesssim \|\langle D_x\rangle^{s''-2} f\|_{L^2(\Sigma_t)} \lesssim \|f\|_{L^2(\Sigma_t)}. \tag{12.29}$$

Step 2: Estimates when $\nu = t$ We handle term I in (12.26) when $\nu = t$. Note that $[\partial_t, [\varpi, \langle D_x\rangle^{s''}]x^\ell] = 0$. Hence, using Lemma 12.17, we obtain

$$\begin{aligned}
\|I\|_{L^2(\Sigma_t)} &= \|\partial_t[\varpi, \langle D_x\rangle^{s''}](x^\ell f)\|_{L^2(\Sigma_t)} \\
&= \|[\varpi, \langle D_x\rangle^{s''}](x^\ell \partial_t f)\|_{L^2(\Sigma_t)} \\
&\lesssim \|\langle D_x\rangle^{s''-2}(x^\ell \partial_t f)\|_{L^2(\Sigma_t)} \\
&\lesssim \|x^\ell \partial_t f\|_{L^2(\Sigma_t)} \lesssim \|\partial f\|_{L^2(\Sigma_t)},
\end{aligned} \tag{12.30}$$

where we have used the support property of f.

On the other hand, by (12.27)–(12.29) (and the L^2-boundedness of $\langle D_x\rangle^{s''-1}$ as well as the Poincaré inequality), we have

$$\|II\|_{L^2(\Sigma_t)} + \|III\|_{L^2(\Sigma_t)} + \|IV\|_{L^2(\Sigma_t)} \lesssim \|\partial f\|_{L^2(\Sigma_t)}. \tag{12.31}$$

Combining (12.30) and (12.31), we have thus proven the first estimate in (12.25).
Step 3: Estimates when $\nu = i$ If $\nu = i$, by Lemma 12.17, we have

$$\begin{aligned}
\|I\|_{L^2(\Sigma_t)} &= \|\partial_i[\varpi, \langle D_x\rangle^{s''}](x^\ell f)\|_{L^2(\Sigma_t)} \\
&\lesssim \|\langle D_x\rangle^{s''-1}(x^\ell f)\|_{L^2(\Sigma_t)} \lesssim \|x^\ell f\|_{L^2(\Sigma_t)} \\
&\lesssim \|f\|_{L^2(\Sigma_t)} \lesssim \|\langle D_x\rangle^{s''} f\|_{L^2(\Sigma_t)},
\end{aligned} \tag{12.32}$$

where we have used the support property of f.

On the other hand, by (12.27)–(12.29) (and the L^2-boundedness of $\langle D_x\rangle^{-1}\partial_i$), we have

$$\|II\|_{L^2(\Sigma_t)} + \|III\|_{L^2(\Sigma_t)} + \|IV\|_{L^2(\Sigma_t)} \lesssim \|\langle D_x\rangle^{s''} f\|_{L^2(\Sigma_t)}.$$

Together with Step 2, we have thus completed the proof of (12.25), which then implies the lemma. □

Lemma 12.22 *For every index* ν, β,

$$\|\langle x\rangle \partial^2_{\nu\beta}[\varpi, \langle D_x\rangle^{s''}]\widetilde{\phi}_k\|_{L^2(\Sigma_t)} \lesssim \|\partial\langle D_x\rangle^{s'}\widetilde{\phi}_k\|_{L^2(\Sigma_t)}.$$

Proof *Step 0: Preliminary computations* Using Lemma 12.17, if $(\nu, \beta) \neq (t, t)$, we have

$$\|\partial^2_{\nu\beta}[\varpi, \langle D_x\rangle^{s''}]\widetilde{\phi}_k\|_{L^2(\Sigma_t)} \lesssim \|\partial\langle D_x\rangle^{s''}\widetilde{\phi}_k\|_{L^2(\Sigma_t)}. \tag{12.33}$$

In the case $(\nu,\beta)=(t,t)$, we know that $[\partial^2_{tt},[\varpi,\langle D_x\rangle^{s''}]]=0$ and hence by Lemma 12.17 and Lemma 12.16, we have

$$\|\partial^2_{tt}[\varpi,\langle D_x\rangle^{s''}]\widetilde{\phi}_k\|_{L^2(\Sigma_t)}\lesssim\|\langle D_x\rangle^{s''-2}\partial^2_{tt}\widetilde{\phi}_k\|_{L^2(\Sigma_t)}\lesssim\|\partial\langle D_x\rangle^{s'}\widetilde{\phi}_k\|_{L^2(\Sigma_t)}. \tag{12.34}$$

Using (12.33) and (12.34) and arguing as in Step 1 of Lemma 12.21, it suffices to prove

$$\|x^\ell\partial^2_{\nu\beta}[\varpi,\langle D_x\rangle^{s''}]\widetilde{\phi}_k\|_{L^2(\Sigma_t)}\lesssim\|\partial\langle D_x\rangle^{s'}\widetilde{\phi}_k\|_{L^2(\Sigma_t)}. \tag{12.35}$$

We then compute using (12.22) that

$$\begin{aligned}
&x^\ell\partial^2_{\nu\beta}[\varpi,\langle D_x\rangle^{s''}]\widetilde{\phi}_k\\
&=\partial^2_{\nu\beta}(x^\ell[\varpi,\langle D_x\rangle^{s''}]\widetilde{\phi}_k)-(\partial_\nu x^\ell)(\partial_\beta([\varpi,\langle D_x\rangle^{s''}]\widetilde{\phi}_k))-(\partial_\beta x^\ell)(\partial_\nu([\varpi,\langle D_x\rangle^{s''}]\widetilde{\phi}_k))\\
&=\underbrace{\partial^2_{\nu\beta}([\varpi,\langle D_x\rangle^{s''}](x^\ell\widetilde{\phi}_k))}_{=:I}\underbrace{+\partial^2_{\nu\beta}(\varpi[x^\ell,\langle D_x\rangle^{s''}]\widetilde{\phi}_k)}_{=:II}\underbrace{-\partial^2_{\nu\beta}([x^\ell,\langle D_x\rangle^{s''}]\varpi\widetilde{\phi}_k)}_{=:III}\\
&\quad\underbrace{-(\partial_\nu x^\ell)(\partial_\beta([\varpi,\langle D_x\rangle^{s''}]\widetilde{\phi}_k))}_{=:IV}\underbrace{-(\partial_\beta x^\ell)(\partial_\nu([\varpi,\langle D_x\rangle^{s''}]\widetilde{\phi}_k))}_{=:V}.
\end{aligned}\tag{12.36}$$

We control each term in (12.36) in the steps below.
Step 1: Term I We separate into three cases. When $(\nu,\beta)=(i,j)$, by Lemma 12.17 and Poincaré's inequality (since $\mathrm{supp}(\widetilde{\phi}_k)\subset B(0,R)$),

$$\|\partial^2_{ij}([\varpi,\langle D_x\rangle^{s''}](x^\ell\widetilde{\phi}_k))\|_{L^2(\Sigma_t)}\lesssim\|\langle D_x\rangle^{s''}(x^\ell\widetilde{\phi}_k))\|_{L^2(\Sigma_t)}\lesssim\|\partial\widetilde{\phi}_k\|_{L^2(\Sigma_t)}.$$

When $(\nu,\beta)=(i,t)$, since $[\partial_t,[\varpi,\langle D_x\rangle^{s''}]x^\ell]=0$, we use Lemma 12.17 and then Lemma 12.13 to obtain

$$\|\partial^2_{it}([\varpi,\langle D_x\rangle^{s''}](x^\ell\widetilde{\phi}_k))\|_{L^2(\Sigma_t)}\lesssim\|\langle D_x\rangle^{s''-1}(x^\ell\partial_t\widetilde{\phi}_k)\|_{L^2(\Sigma_t)}\lesssim\|\partial\langle D_x\rangle^{s'}\widetilde{\phi}_k\|_{L^2(\Sigma_t)}.$$

Finally, when $(\nu,\beta)=(t,t)$, since $[\partial^2_{tt},[\varpi,\langle D_x\rangle^{s''}]x^\ell]=0$, we use Lemma 12.17 and then Lemma 12.16 to get

$$\|\partial^2_{tt}([\varpi,\langle D_x\rangle^{s''}](x^\ell\widetilde{\phi}_k))\|_{L^2(\Sigma_t)}\lesssim\|\langle D_x\rangle^{s''-2}(x^\ell\partial^2_{tt}\widetilde{\phi}_k)\|_{L^2(\Sigma_t)}\lesssim\|\partial\langle D_x\rangle^{s'}\widetilde{\phi}_k\|_{L^2(\Sigma_t)}.$$

Thus in all cases

$$\|I\|_{L^2(\Sigma_t)}\lesssim\|\partial\langle D_x\rangle^{s'}\widetilde{\phi}_k\|_{L^2(\Sigma_t)}.$$

Step 2: Terms II and III Terms II and III are very similar — we will only treat II. We use the formula (12.23). When one of ν or β is a spatial derivative, it follows easily from (12.23) and the support properties of $\widetilde{\phi}_k$ that

$$\|\partial^2_{\nu\beta}(\varpi[x^\ell, \langle D_x\rangle^{s''}]\widetilde{\phi}_k)\|_{L^2(\Sigma_t)} \lesssim \|\partial\widetilde{\phi}_k\|_{L^2(\Sigma_t)}.$$

If $(\nu, \beta) = (t, t)$, we use $[\partial^2_{tt}, \varpi[x^\ell, \langle D_x\rangle^{s''}]] = 0$, the equation (12.23) and then Lemma 12.16 to obtain

$$\begin{aligned}&\|\partial^2_{tt}(\varpi[x^\ell, \langle D_x\rangle^{s''}]\widetilde{\phi}_k)\|_{L^2(\Sigma_t)}\\ &\lesssim \|\varpi\langle D_x\rangle^{s''-2}\partial_\ell\partial^2_{tt}\widetilde{\phi}_k\|_{L^2(\Sigma_t)} \lesssim \|\langle D_x\rangle^{s''-1}\partial^2_{tt}\widetilde{\phi}_k\|_{L^2(\Sigma_t)} \lesssim \|\partial\langle D_x\rangle^{s'}\widetilde{\phi}_k\|_{L^2(\Sigma_t)}.\end{aligned}$$

In either case

$$\|II\|_{L^2(\Sigma_t)} \lesssim \|\partial\langle D_x\rangle^{s'}\widetilde{\phi}_k\|_{L^2(\Sigma_t)}.$$

The same holds for III in a similar manner; we omit the details.
Step 3: Terms IV and V Since $\partial_\nu x^\ell$ and $\partial_\beta x^\ell$ are both bounded, we have, by Lemma 12.17,

$$\|IV\|_{L^2(\Sigma_t)} + \|V\|_{L^2(\Sigma_t)} \lesssim \|\partial[\varpi, \langle D_x\rangle^{s''}]\widetilde{\phi}_k\|_{L^2(\Sigma_t)} \lesssim \|\partial\langle D_x\rangle^{s'}\widetilde{\phi}_k\|_{L^2(\Sigma_t)}.$$

We have thus estimated every term on the right-hand side of (12.36) and proven (12.35). As argued in Step 0, this gives the lemma. □

Proposition 12.23 *For any $r \geq 0$, the following holds (with implicit constants depending on r)*[18]*:*

$$\|\partial E_k\langle D_x\rangle^{s''}\widetilde{\phi}_k\|_{L^2(\Sigma_t)} \lesssim \|\langle x\rangle^{-r}\partial E_k\langle D_x\rangle^{s''}\widetilde{\phi}_k\|_{L^2(\Sigma_t)} + \|\partial\langle D_x\rangle^{s'}\widetilde{\phi}_k\|_{L^2(\Sigma_t)}, \tag{12.37}$$

$$\|\partial L_k\langle D_x\rangle^{s''}\widetilde{\phi}_k\|_{L^2(\Sigma_t)} \lesssim \|\langle x\rangle^{-r}\partial L_k\langle D_x\rangle^{s''}\widetilde{\phi}_k\|_{L^2(\Sigma_t)} + \|\partial\langle D_x\rangle^{s'}\widetilde{\phi}_k\|_{L^2(\Sigma_t)}. \tag{12.38}$$

Proof We will only prove (12.37) in detail; (12.38) is similar.
Since $\partial E_k\langle D_x\rangle^{s''}\widetilde{\phi}_k = \partial E_k\langle D_x\rangle^{s''}(\varpi\widetilde{\phi}_k)$, we compute

$$\begin{aligned}\partial E_k\langle D_x\rangle^{s''}\widetilde{\phi}_k &= \varpi\partial E_k\langle D_x\rangle^{s''}\widetilde{\phi}_k + \partial[(E_k\varpi)\langle D_x\rangle^{s''}\widetilde{\phi}_k]\\ &\quad - \partial E_k[\varpi, \langle D_x\rangle^{s''}]\widetilde{\phi}_k + (\partial\varpi)E_k\langle D_x\rangle^{s''}\widetilde{\phi}_k.\end{aligned}$$

In particular, by the bounds of E_k in (5.5), (5.6) and the support properties of ϖ,

$$\begin{aligned}\|\partial E_k\langle D_x\rangle^{s''}\widetilde{\phi}_k\|_{L^2(\Sigma_t)} &\lesssim \underbrace{\|\partial E_k\langle D_x\rangle^{s''}\widetilde{\phi}_k\|_{L^2(\Sigma_t\cap B(0,3R))}}_{=:I} + \underbrace{\|\partial\langle D_x\rangle^{s''}\widetilde{\phi}_k\|_{L^2(\Sigma_t\cap B(0,3R))}}_{=:II}\\ &\quad + \underbrace{\|\langle D_x\rangle^{s''}\widetilde{\phi}_k\|_{L^2(\Sigma_t)}}_{=:III} + \underbrace{\|\partial E_k[\varpi, \langle D_x\rangle^{s''}]\widetilde{\phi}_k\|_{L^2(\Sigma_t)}}_{=:IV}.\end{aligned}$$

[18] We remark that the estimates hold with $\|\partial\langle D_x\rangle^{s'}\widetilde{\phi}_k\|_{L^2(\Sigma_t)}$ replaced by $\|\partial\langle D_x\rangle^{s''}\widetilde{\phi}_k\|_{L^2(\Sigma_t)}$ on the right-hand side, but the exposition becomes slightly less convenient when citing earlier lemmas.

The terms I and II are obviously bounded by the right-hand side of (12.37).

For the term III, we use interpolation and then Poincaré's inequality to obtain

$$\|\langle D_x\rangle^{s''}\widetilde{\phi}_k\|_{L^2(\Sigma_t)} \lesssim \|\widetilde{\phi}_k\|_{L^2(\Sigma_t)} + \|\partial\widetilde{\phi}_k\|_{L^2(\Sigma_t)} \lesssim \|\partial\widetilde{\phi}_k\|_{L^2(\Sigma_t)} \lesssim \|\partial\langle D_x\rangle^{s'}\widetilde{\phi}_k\|_{L^2(\Sigma_t)}. \tag{12.39}$$

For the term IV, we use the bounds for E_k^μ in (5.5), (5.6), Hölder's inequality and Lemmas 12.21 and 12.22 to obtain

$$\begin{aligned}
&\|\partial_\nu E_k[\varpi, \langle D_x\rangle^{s''}]\widetilde{\phi}_k\|_{L^2(\Sigma_t)} \\
\lesssim\ & \|\langle x\rangle^{-1}\partial_\nu E_k^\mu\|_{L^\infty(\Sigma_t)}\|\langle x\rangle\partial_\mu[\varpi, \langle D_x\rangle^{s''}]\widetilde{\phi}_k\|_{L^2(\Sigma_t)} \\
&+ \|\langle x\rangle^{-1}E_k^\mu\|_{L^\infty(\Sigma_t)}\|\langle x\rangle\partial^2_{\nu\mu}[\varpi, \langle D_x\rangle^{s''}]\widetilde{\phi}_k\|_{L^2(\Sigma_t)} \\
\lesssim\ & \|\partial\langle D_x\rangle^{s'}\widetilde{\phi}_k\|_{L^2(\Sigma_t)} + \|\partial\widetilde{\phi}_k\|_{L^2(\Sigma_t)} \lesssim \|\partial\langle D_x\rangle^{s'}\widetilde{\phi}_k\|_{L^2(\Sigma_t)}.
\end{aligned}$$

□

12.4.3 Auxiliary weighted estimates for commutator of a vector field and Riesz transform

Lemma 12.24 *Let f, h be smooth functions such that*

$$\begin{aligned}
&\|\langle x\rangle^{-1}h\|_{L^\infty(\Sigma_t)} + \|\varpi h\|_{W^{1,\infty}\cap W^{2,4}(\Sigma_t)} \lesssim 1, \\
&\|\varpi f\|_{L^\infty\cap W^{1,4}(\Sigma_t)} + \|\varpi\partial_i f\|_{L^4(\Sigma_t)} \lesssim 1.
\end{aligned}$$

Then, for $R_j = \partial_j\langle D_x\rangle^{-1}$, we have

$$\|\langle x\rangle^{-1}[h\partial_i, R_l R_q]\langle D_x\rangle^{s''}(f\partial_\lambda\widetilde{\phi}_k)\|_{L^2(\Sigma_t)} \lesssim \|\partial\langle D_x\rangle^{s'}\widetilde{\phi}_k\|_{L^2(\Sigma_t)}, \tag{12.40}$$

and

$$\|\langle x\rangle^{-1}[h\partial_i, R_l R_q\langle D_x\rangle^{s''}](f\partial_\lambda\widetilde{\phi}_k)\|_{L^2(\Sigma_t)} \lesssim \|\partial\langle D_x\rangle^{s'}\widetilde{\phi}_k\|_{L^2(\Sigma_t)}. \tag{12.41}$$

Proof *Step 1: Proof of* (12.40) We first compute

$$\begin{aligned}
&[h\partial_i, R_l R_q]\langle D_x\rangle^{s''}(f\partial_\lambda\widetilde{\phi}_k) \\
=\ & \underbrace{[(\varpi h)\partial_i, R_l R_q]\langle D_x\rangle^{s''}(f\partial_\lambda\widetilde{\phi}_k)}_{=:I} + \underbrace{h[\partial_i R_l R_q\langle D_x\rangle^{s''}, \varpi](f\partial_\lambda\widetilde{\phi}_k)}_{=:II} \\
& \underbrace{-R_l R_q\{h\partial_i[\langle D_x\rangle^{s''}, \varpi](f\partial_\lambda\widetilde{\phi}_k)\}}_{=:III} \underbrace{-R_l R_q\{h(\partial_i\varpi)\langle D_x\rangle^{s''}(f\partial_\lambda\widetilde{\phi}_k)\}}_{=:IV}.
\end{aligned}$$

Term I can be bounded using Lemmas 12.11 (with $Y = h\partial_i$), Lemma 12.13 and the identity $[Y, R_l R_q] = [Y, R_l]R_q + R_l[Y, R_q]$ so that

$$\|I\|_{L^2(\Sigma_t)} \lesssim \|\varpi h\|_{W^{1,\infty}(\Sigma_t)}\|\langle D_x\rangle^{s''}(f\partial_\lambda\widetilde{\phi}_k)\|_{L^2(\Sigma_t)} \lesssim \|\partial\langle D_x\rangle^{s'}\widetilde{\phi}_k\|_{L^2(\Sigma_t)}.$$

For term II, we use Hölder's inequality, Lemmas 12.17 and 12.13 to obtain

$$\begin{aligned}\|\langle x\rangle^{-1}II\|_{L^2(\Sigma_t)} &\lesssim \|\langle x\rangle^{-1}h\|_{L^\infty}\|[\partial_i R_l R_q\langle D_x\rangle^{s''}, \varpi](f\partial_\lambda\widetilde{\phi}_k)\|_{L^2(\Sigma_t)}\\ &\lesssim \|\langle D_x\rangle^{s''-1}(f\partial_\lambda\widetilde{\phi}_k)\|_{L^2(\Sigma_t)} \lesssim \|\partial\langle D_x\rangle^{s'}\widetilde{\phi}_k\|_{L^2(\Sigma_t)}.\end{aligned}$$

For the term III, we use the boundedness of the Riesz transform, Hölder's inequality, the improved estimate (12.24) in Lemma 12.21 and then Lemma 12.13 to obtain

$$\begin{aligned}\|III\|_{L^2(\Sigma_t)} &\lesssim \|h\partial_i[\langle D_x\rangle^{s''}, \varpi](f\partial_\lambda\widetilde{\phi}_k)\|_{L^2(\Sigma_t)}\\ &\lesssim \|\langle x\rangle^{-1}h\|_{L^\infty(\Sigma_t)}\|\langle x\rangle\partial_i[\langle D_x\rangle^{s''}, \varpi](f\partial_\lambda\widetilde{\phi}_k)\|_{L^2(\Sigma_t)}\\ &\lesssim \|\langle D_x\rangle^{s''}(f\partial_\lambda\widetilde{\phi}_k)\|_{L^2(\Sigma_t)} \lesssim \|\partial\langle D_x\rangle^{s'}\widetilde{\phi}_k\|_{L^2(\Sigma_t)}.\end{aligned}$$

Finally, for the term IV, we use the L^2-boundedness of the Riesz transform, Hölder's inequality and Lemma 12.13 to obtain

$$\|IV\|_{L^2(\Sigma_t)} \lesssim \|h\partial_i\varpi\|_{L^\infty(\Sigma_t)}\|\langle D_x\rangle^{s''}(f\partial_\lambda\widetilde{\phi}_k)\|_{L^2(\Sigma_t)} \lesssim \|\partial\langle D_x\rangle^{s'}\widetilde{\phi}_k\|_{L^2(\Sigma_t)}.$$

Step 2: Proof of (12.41) We first notice that

$$[h\partial_i, R_l R_q\langle D_x\rangle^{s''}](f\partial_\lambda\widetilde{\phi}_k) = [h\partial_i, R_l R_q]\langle D_x\rangle^{s''}(f\partial_\lambda\widetilde{\phi}_k) + R_l R_q[h\partial_i, \langle D_x\rangle^{s''}](f\partial_\lambda\widetilde{\phi}_k).$$

In view of (12.40) (which controls the first term) and the boundedness of the Riesz transform, it therefore suffices to prove

$$\|[h\partial_i, \langle D_x\rangle^{s''}](f\partial_\lambda\widetilde{\phi}_k)\|_{L^2(\Sigma_t)} \lesssim \|\partial\langle D_x\rangle^{s'}\widetilde{\phi}_k\|_{L^2(\Sigma_t)}.$$

On the other hand,

$$\begin{aligned}&[h\partial_i, \langle D_x\rangle^{s''}](f\partial_\lambda\widetilde{\phi}_k) = [h\partial_i, \langle D_x\rangle^{s''}](\varpi f\partial_\lambda\widetilde{\phi}_k)\\ &\quad= \underbrace{h\partial_i[\langle D_x\rangle^{s''}, \varpi](f\partial_\lambda\widetilde{\phi}_k)}_{=:I} + \underbrace{[(\varpi h)\partial_i, \langle D_x\rangle^{s''}](f\partial_\lambda\widetilde{\phi}_k)}_{=:II}\\ &\qquad- \underbrace{\langle D_x\rangle^{s''}[(\partial_i\varpi)hf(\partial_\lambda\widetilde{\phi}_k)]}_{=:III} + \underbrace{h(\partial_i\varpi)\langle D_x\rangle^{s''}(f\partial_\lambda\widetilde{\phi}_k)}_{=:IV}.\end{aligned}$$

For term I, we use Hölder's inequality, Lemmas 12.17 and 12.13 to obtain

$$\begin{aligned}\|\langle x\rangle^{-1}I\|_{L^2(\Sigma_t)} &\lesssim \|\langle x\rangle^{-1}h\|_{L^\infty(\Sigma_t)}\|\partial_i[\langle D_x\rangle^{s''}, \varpi](f\partial_\lambda\widetilde{\phi}_k)\|_{L^2(\Sigma_t)}\\ &\lesssim \|\langle D_x\rangle^{s''-1}(f\partial_\lambda\widetilde{\phi}_k)\|_{L^2(\Sigma_t)} \lesssim \|\partial\langle D_x\rangle^{s'}\widetilde{\phi}_k\|_{L^2(\Sigma_t)}.\end{aligned}$$

For term II, we use Proposition 12.7 (with $p = \infty$), the L^2-boundedness of $\partial_i \langle D_x \rangle^{-1}$ and Lemma 12.13 to obtain

$$\|II\|_{L^2(\Sigma_t)} \lesssim \|\varpi h\|_{W^{1,\infty}} \|\langle D_x \rangle^{s''-1} \partial_i (f \partial_\lambda \widetilde{\phi}_k)\|_{L^2(\Sigma_t)}$$
$$\lesssim \|\langle D_x \rangle^{s''} (f \partial_\lambda \widetilde{\phi}_k)\|_{L^2(\Sigma_t)} \lesssim \|\partial \langle D_x \rangle^{s'} \widetilde{\phi}_k\|_{L^2(\Sigma_t)}.$$

For term III, note that $\varpi \equiv 1$ on $\mathrm{supp}(\widetilde{\phi}_k)$, hence $(\partial_i \varpi)\partial_\lambda \widetilde{\phi}_k \equiv 0$ which implies $III = 0$.

Finally, term IV can be treated exactly as term IV in the proof of (12.40) so that

$$\|IV\|_{L^2(\Sigma_t)} \lesssim \|\partial \langle D_x \rangle^{s'} \widetilde{\phi}_k\|_{L^2(\Sigma_t)}.$$

□

12.5 Estimates for $\partial \langle D_x \rangle^{s'} \widetilde{\phi}_k$

In this section, we bound $\|\partial \langle D_x \rangle^{s'} \widetilde{\phi}_k\|_{L^2(\Sigma_t)}$. We will give the main result in Proposition 12.25 and give a high level proof. The main estimates that are used in the proof will be proven in Propositions 12.26 and 12.27 below.

Proposition 12.25

$$\sup_{t \in [0, T_B)} \|\partial \langle D_x \rangle^{s'} \widetilde{\phi}_k\|_{L^2(\Sigma_t)} \lesssim \epsilon. \tag{12.42}$$

Proof By Corollary 7.5 (with $v = \langle D_x \rangle^{s'} \widetilde{\phi}_k$, $f_1 = \Box_g \langle D_x \rangle^{s'} \widetilde{\phi}_k$, $f_2 = 0$) and Proposition 12.19, for every $T \in [0, T_B)$,

$$\begin{aligned}
&\sup_{t \in [0, T)} \|\partial \langle D_x \rangle^{s'} \widetilde{\phi}_k\|^2_{L^2(\Sigma_t)} \\
&\lesssim \|\langle x \rangle^{-\frac{r}{2}} \partial \langle D_x \rangle^{s'} \widetilde{\phi}_k\|^2_{L^2(\Sigma_0)} + \sup_{t \in [0, T)} \|\partial \widetilde{\phi}_k\|^2_{L^2(\Sigma_t)} \\
&\quad + \int_0^T \|\partial \langle D_x \rangle^{s'} \widetilde{\phi}_k\|^2_{L^2(\Sigma_\tau)} \, d\tau + \int_0^T \|\langle x \rangle^{-\frac{r}{2}} \Box_g \langle D_x \rangle^{s'} \widetilde{\phi}_k\|^2_{L^2(\Sigma_\tau)} \, d\tau \\
&\lesssim \epsilon^2 + \int_0^T \|\partial \langle D_x \rangle^{s'} \widetilde{\phi}_k\|^2_{L^2(\Sigma_\tau)} \, d\tau + \int_0^T \|\langle x \rangle^{-\frac{r}{2}} \Box_g \langle D_x \rangle^{s'} \widetilde{\phi}_k\|^2_{L^2(\Sigma_\tau)} \, d\tau,
\end{aligned}$$

where in the last line we have used the data bound (4.3a) and the estimate (9.1).

Clearly, $\Box_g \langle D_x \rangle^{s'} \widetilde{\phi}_k = [\Box_g, \langle D_x \rangle^{s'}] \widetilde{\phi}_k$. Using (12.6), we in fact have

$$\Box_g \langle D_x \rangle^{s'} \widetilde{\phi}_k = [\Box^2, \langle D_x \rangle^{s'}] \widetilde{\phi}_k - [\Box^1, \langle D_x \rangle^{s'}] \widetilde{\phi}_k.$$

In Propositions 12.26 and 12.27 below, we will prove respectively that for $r \geq 1$,

$$\|\langle x \rangle^{-\frac{r}{2}} [\Box^2, \langle D_x \rangle^{s'}] \widetilde{\phi}_k\|^2_{L^2(\Sigma_t)} \lesssim \|\partial \langle D_x \rangle^{s'} \widetilde{\phi}_k\|^2_{L^2(\Sigma_t)}, \tag{12.43}$$

and

$$\|\langle x\rangle^{-\frac{r}{2}}[\Box^1, \langle D_x\rangle^{s'}]\widetilde{\phi}_k\|^2_{L^2(\Sigma_t)} \lesssim \|\partial\langle D_x\rangle^{s'}\widetilde{\phi}_k\|^2_{L^2(\Sigma_t)}. \tag{12.44}$$

Hence,

$$\sup_{t\in[0,T)} \|\partial\langle D_x\rangle^{s'}\widetilde{\phi}_k\|^2_{L^2(\Sigma_t)} \lesssim \epsilon^2 + \int_0^T \|\partial\langle D_x\rangle^{s'}\widetilde{\phi}_k\|^2_{L^2(\Sigma_t)}\, dt.$$

The desired estimate therefore follows from Grönwall's inequality. □

Given the proof of Proposition 12.25 above, we need to prove the commutator estimates (12.43) and (12.44). For both of these bounds, we also prove corresponding commutator estimates for more general functions. (These more general commutator bounds will be useful later in Sect. 13.)

Proposition 12.26 *Let v be a smooth, compactly supported function on $B(0, R)$. Then for $r \geq 1$,*

$$\|\langle x\rangle^{-\frac{r}{2}}[\Box^2, \langle D_x\rangle^{s'}]v\|_{L^2(\Sigma_t)} \lesssim \|\partial\langle D_x\rangle^{s'}v\|_{L^2(\Sigma_t)} + \|\Box_g v\|_{L^2(\Sigma_t)}. \tag{12.45}$$

As a result, (12.43) holds.

Proof Recall from (12.6) that $\Box^2 = (g^{-1})^{\nu\beta}\partial^2_{\nu\beta}$. By the support properties of v, we have $\partial^2_{\nu\beta}v = \varpi\partial^2_{\nu\beta}v$. Hence,

$$\begin{aligned}
&\|\langle x\rangle^{-\frac{r}{2}}[\Box^2, \langle D_x\rangle^{s'}]v\|_{L^2(\Sigma_t)} = \|\langle x\rangle^{-\frac{r}{2}}[\langle D_x\rangle^{s'}((g^{-1})^{\nu\beta}\partial^2_{\nu\beta}v) - (g^{-1})^{\nu\beta}\langle D_x\rangle^{s'}\partial^2_{\nu\beta}v]\|_{L^2(\Sigma_t)} \\
&\lesssim \underbrace{\|\langle D_x\rangle^{s'}(\varpi(g^{-1})^{\nu\beta}\partial^2_{\nu\beta}v) - \varpi(g^{-1})^{\nu\beta}\langle D_x\rangle^{s'}\partial^2_{\nu\beta}v\|_{L^2(\Sigma_t)}}_{=:I} \\
&\quad + \underbrace{\|\langle x\rangle^{-\frac{r}{2}}(g^{-1})^{\nu\beta}[\langle D_x\rangle^{s'}, \varpi]\partial^2_{\nu\beta}v\|_{L^2(\Sigma_t)}}_{=:II}.
\end{aligned} \tag{12.46}$$

By Proposition 12.7 with $p = \infty$, the estimates for the metric components in (5.1), and Lemma 12.15, I in (12.46) is bounded by

$$\begin{aligned}
&\|\varpi(g^{-1})^{\nu\beta}\partial^2_{\nu\beta}\langle D_x\rangle^{s'}v - \langle D_x\rangle^{s'}(\varpi(g^{-1})^{i\lambda}\partial^2_{i\lambda}v)\|_{L^2(\Sigma_t)} \\
&\lesssim \|\varpi(g^{-1})^{\nu\beta}\|_{W^{1,\infty}(\Sigma_t)}\|\langle D_x\rangle^{s'-1}\partial^2_{\nu\beta}v\|_{L^2(\Sigma_t)} \lesssim \|\partial\langle D_x\rangle^{s'}v\|_{L^2(\Sigma_t)} + \|\Box_g v\|_{L^2(\Sigma_t)}.
\end{aligned}$$

By Hölder's inequality, (5.1), Lemmas 12.17 and 12.15, II in (12.46) can be controlled by

$$\begin{aligned}
II &\lesssim \|\langle x\rangle^{-\frac{r}{2}}(g^{-1})^{\nu\beta}\|_{L^\infty(\Sigma_t)}\|[\langle D_x\rangle^{s'}, \varpi]\partial^2_{\nu\beta}v\|_{L^2(\Sigma_t)} \\
&\lesssim \|\langle D_x\rangle^{s'-2}\partial^2_{\nu\beta}v\|_{L^2(\Sigma_t)} \lesssim \|\partial\langle D_x\rangle^{s'}v\|_{L^2(\Sigma_t)} + \|\Box_g v\|_{L^2(\Sigma_t)}.
\end{aligned}$$

Combining the above estimates gives (12.45).

Finally, since $\widetilde{\phi}_k$ is supported in $B(0, R)$ for every t, and that $\Box_g \widetilde{\phi}_k = 0$, (12.43) follows from (12.45). □

Proposition 12.27 *Let v be a smooth, compactly supported function on $B(0, R)$. Then for $r \geq 1$,*

$$\|\langle x\rangle^{-\frac{r}{2}}[\Box^1, \langle D_x\rangle^{s'}]v\|_{L^2(\Sigma_t)} \lesssim \|\partial\langle D_x\rangle^{s'} v\|_{L^2(\Sigma_t)}. \tag{12.47}$$

As a result, (12.44) holds.

Proof Recall that $\Box^1 = \Gamma^\lambda\partial_\lambda$. We will in fact not need the commutator structure and bound each term separately. By Hölder's inequality and the estimates for Γ^λ in (5.4), we have

$$\begin{aligned}\|\langle x\rangle^{-\frac{r}{2}}\Box^1(\langle D_x\rangle^{s'} v)\|_{L^2(\Sigma_t)} &= \|\langle x\rangle^{-\frac{r}{2}}\Gamma^\lambda\partial_\lambda\langle D_x\rangle^{s'} v\|_{L^2(\Sigma_t)} \\ &\lesssim \|\langle x\rangle^{-\frac{r}{2}}\Gamma^\lambda\|_{L^\infty(\Sigma_t)}\|\partial\langle D_x\rangle^{s'} v\|_{L^2(\Sigma_t)}.\end{aligned}$$

On the other hand, by (5.4), we have $\|\varpi\Gamma^\lambda\|_{L^\infty\cap W^{1,2}(\Sigma_t)} \lesssim \epsilon^{\frac{3}{2}}$. Hence, by Lemma 12.12, we obtain

$$\|\langle x\rangle^{-\frac{r}{2}}\langle D_x\rangle^{s'}(\Box^1 v)\|_{L^2(\Sigma_t)} = \|\langle D_x\rangle^{s'}(\Gamma^\lambda\partial_\lambda v)\|_{L^2(\Sigma_t)} \lesssim \|\partial\langle D_x\rangle^{s'} v\|_{L^2(\Sigma_t)}.$$

Combining the above estimates gives (12.47). We then conclude (12.44) by using (12.47), $\Box_g\widetilde{\phi}_k = 0$ and the support property of $\widetilde{\phi}_k$. □

12.6 Estimates for $\partial E_k\langle D_x\rangle^{s''}\widetilde{\phi}_k$

Similarly as in Sect. 12.5, let us first give a high level proof of our main estimate. The main steps will be postponed to a number of propositions below.

Proposition 12.28 *The following estimate holds for all $t \in [0, T_B)$:*

$$\|\partial E_k\langle D_x\rangle^{s''}\widetilde{\phi}_k\|_{L^2(\Sigma_t)} \lesssim \epsilon + \epsilon^{\frac{3}{2}}\|\partial L_k\langle D_x\rangle^{s''}\widetilde{\phi}_k\|_{L^2(\Sigma_t)}. \tag{12.48}$$

Proof Take $r \geq 2$. By Corollary 7.5 (with $v = E_k\langle D_x\rangle^{s''}\widetilde{\phi}_k$, $f_1 = \Box_g(E_k\langle D_x\rangle^{s''}\widetilde{\phi}_k)$ and $f_2 = 0$),

$$\begin{aligned}&\sup_{t\in[0,T)} \|\langle x\rangle^{-r-2\nu}\partial E_k\langle D_x\rangle^{s''}\widetilde{\phi}_k\|^2_{L^2(\Sigma_t)} \\ &\lesssim \|\langle x\rangle^{-\frac{r}{2}}\partial E_k\langle D_x\rangle^{s''}\widetilde{\phi}_k\|^2_{L^2(\Sigma_0)} + \int_0^T \|\langle x\rangle^{-\frac{r}{2}}\Box_g E_k\langle D_x\rangle^{s''}\widetilde{\phi}_k\|^2_{L^2(\Sigma_\tau)}\,\mathrm{d}\tau \qquad (12.49)\\ &\lesssim \epsilon^2 + \int_0^T \|\langle x\rangle^{-\frac{r}{2}}\Box_g E_k\langle D_x\rangle^{s''}\widetilde{\phi}_k\|^2_{L^2(\Sigma_\tau)}\,\mathrm{d}\tau,\end{aligned}$$

where we have bound the initial data term as $\|\langle x\rangle^{-\frac{r}{2}}\partial E_k\langle D_x\rangle^{s''}\widetilde{\phi}_k\|^2_{L^2(\Sigma_0)}\lesssim\epsilon^2$ using the assumptions (4.3a) and (4.3b) on Σ_0 (using additionally (2.3), (5.5), (5.6), (5.1) and Proposition 12.7 to address commutator terms involving fractional derivatives).

Using (12.37) and (12.42), we can gain weights on the left-hand side of (12.49) as follows

$$\begin{aligned}
&\sup_{t\in[0,T)}\|\partial E_k\langle D_x\rangle^{s''}\widetilde{\phi}_k\|^2_{L^2(\Sigma_t)}\\
&\lesssim\epsilon^2+\sup_{t\in[0,T)}\|\partial\langle D_x\rangle^{s'}\widetilde{\phi}_k\|^2_{L^2(\Sigma_t)}+\int_0^T\|\langle x\rangle^{-\frac{r}{2}}\Box_g E_k\langle D_x\rangle^{s''}\widetilde{\phi}_k\|^2_{L^2(\Sigma_\tau)}\,\mathrm{d}\tau\\
&\lesssim\epsilon^2+\int_0^T\|\langle x\rangle^{-\frac{r}{2}}\Box_g E_k\langle D_x\rangle^{s''}\widetilde{\phi}_k\|^2_{L^2(\Sigma_\tau)}\,\mathrm{d}\tau.
\end{aligned}$$

Thus, in order to prove the bound (12.48), it suffices to show

$$\begin{aligned}
\sup_{t\in[0,T_B)}\|\langle x\rangle^{-\frac{r}{2}}\Box_g E_k\langle D_x\rangle^{s''}\widetilde{\phi}_k\|_{L^2(\Sigma_t)}&\lesssim\epsilon\\
+\epsilon^{\frac{3}{2}}\sum_{Z_k\in\{E_k,L_k\}}\|\partial Z_k\langle D_x\rangle^{s''}\widetilde{\phi}_k\|_{L^2(\Sigma_t)},&
\end{aligned}\tag{12.50}$$

since we can then absorb the $\epsilon^{\frac{3}{2}}\|\partial E_k\langle D_x\rangle^{s''}\widetilde{\phi}_k\|_{L^2(\Sigma_t)}$ terms into the left-hand side of (12.50) using the smallness of ϵ.

To prove (12.50), we need to control (recall the notation in (12.6))

$$\begin{aligned}
\Box_g E_k\langle D_x\rangle^{s''}\widetilde{\phi}_k&=[\Box_g,E_k\langle D_x\rangle^{s''}]\widetilde{\phi}_k\\
&=[\Box_g,E_k]\langle D_x\rangle^{s''}\widetilde{\phi}_k+E_k[\Box^1,\langle D_x\rangle^{s''}]\widetilde{\phi}_k+E_k[\Box^2,\langle D_x\rangle^{s''}]\widetilde{\phi}_k.
\end{aligned}\tag{12.51}$$

We further expand the last term in (12.51) using the product rule. First, we will introduce the notation

$$(\bar{g}^{-1})^{\nu\beta}:=(g^{-1})^{\nu\beta}-m^{\nu\beta},\tag{12.52}$$

where m is the Minkowski metric. Note moreover that

$$\begin{aligned}
[\Box^2,\langle D_x\rangle^{s''}]\widetilde{\phi}_k&=[(g^{-1})^{\nu\beta}\partial^2_{\nu\beta},\langle D_x\rangle^{s''}]\widetilde{\phi}_k\\
&=[(\bar{g}^{-1})^{\nu\beta}\partial^2_{\nu\beta},\langle D_x\rangle^{s''}]\widetilde{\phi}_k,\quad\partial_\lambda[(\bar{g}^{-1})^{\nu\beta}]=\partial_\lambda[(g^{-1})^{\nu\beta}].
\end{aligned}\tag{12.53}$$

Now we compute the last term in (12.51) using the product rule and (12.53) (recall here (12.3)):

$$
\begin{aligned}
E_k[\Box^2, \langle D_x\rangle^{s''}]\widetilde{\phi}_k &= E_k^i \partial_i [\Box^2, \langle D_x\rangle^{s''}]\widetilde{\phi}_k \\
&= E_k^i(\partial_i (g^{-1})^{\nu\beta})[\langle D_x\rangle^{s''}, \varpi]\partial^2_{\nu\beta}\widetilde{\phi}_k \\
&\quad + \varpi E_k^i[(\partial_i (g^{-1})^{\nu\beta})\partial^2_{\nu\beta}\langle D_x\rangle^{s''}\widetilde{\phi}_k] \\
&\quad - E_k^i\langle D_x\rangle^{s''}[\varpi(\partial_i (g^{-1})^{\nu\beta})\partial^2_{\nu\beta}\widetilde{\phi}_k] \\
&\quad + E_k^i[(\bar{g}^{-1})^{\nu\beta}[\langle D_x\rangle^{s''}, \varpi]\partial^3_{i\nu\beta}\widetilde{\phi}_k] \\
&\quad + \varpi E_k^i[(\bar{g}^{-1})^{\nu\beta}\partial^3_{i\nu\beta}\langle D_x\rangle^{s''}\widetilde{\phi}_k] \\
&\quad - E_k^i\langle D_x\rangle^{s''}[\varpi(\bar{g}^{-1})^{\nu\beta}\partial^3_{i\nu\beta}\widetilde{\phi}_k] \\
&= E_k^i[\varpi(\partial_i (g^{-1})^{\nu\beta})\partial^2_{\nu\beta}\langle D_x\rangle^{s''}\widetilde{\phi}_k] \\
&\quad - E_k^i\langle D_x\rangle^{s''}[\varpi(\partial_i (g^{-1})^{\nu\beta})\partial^2_{\nu\beta}\widetilde{\phi}_k] \\
&\quad + E_k^i(\partial_i (g^{-1})^{\nu\beta})[\langle D_x\rangle^{s''}, \varpi]\partial^2_{\nu\beta}\widetilde{\phi}_k \\
&\quad + E_k^i[(\bar{g}^{-1})^{\nu\beta}[\langle D_x\rangle^{s''}, \varpi]\partial^3_{i\nu\beta}\widetilde{\phi}_k] \\
&\quad - s''\delta^{jq}[(\partial_j(\varpi(\bar{g}^{-1})^{\nu\beta}))E_k\partial^3_{q\nu\beta}\langle D_x\rangle^{s''-2}\widetilde{\phi}_k] \\
&\quad - E_k^i T^{s''}_{\mathrm{res}}(\varpi(\bar{g}^{-1})^{\nu\beta}, \partial^3_{i\nu\beta}\langle D_x\rangle^{s''}\widetilde{\phi}_k).
\end{aligned}
\tag{12.54}
$$

Therefore, by (12.51) and (12.54), in order to obtain (12.50), it suffices to prove

$$
\begin{aligned}
&\|\langle x\rangle^{-\frac{r}{2}}[\Box_g, E_k]\langle D_x\rangle^{s''}\widetilde{\phi}_k\|_{L^2(\Sigma_t)} \\
&\quad \lesssim \epsilon + \epsilon^{\frac{3}{2}} \sum_{Z_k\in\{E_k, L_k\}} \|\partial Z_k\langle D_x\rangle^{s''}\widetilde{\phi}_k\|_{L^2(\Sigma_t)},
\end{aligned}
\tag{12.55}
$$

$$
\|\langle x\rangle^{-\frac{r}{2}}E_k[\Box^1, \langle D_x\rangle^{s''}]\widetilde{\phi}_k\|_{L^2(\Sigma_t)} \lesssim \epsilon,
\tag{12.56}
$$

$$
\begin{aligned}
&\|\langle x\rangle^{-\frac{r}{2}}E_k^i(\partial_i (g^{-1})^{\nu\beta})[\langle D_x\rangle^{s''}, \varpi]\partial^2_{\nu\beta}\widetilde{\phi}_k\|_{L^2(\Sigma_t)} \\
&\quad + \|\langle x\rangle^{-\frac{r}{2}}E_k^i(\bar{g}^{-1})^{\nu\beta}[\langle D_x\rangle^{s''}, \varpi]\partial^3_{i\nu\beta}\widetilde{\phi}_k\|_{L^2(\Sigma_t)} \lesssim \epsilon,
\end{aligned}
\tag{12.57}
$$

$$
\begin{aligned}
&\|\langle x\rangle^{-\frac{r}{2}}\{E_k^i[\varpi(\partial_i (g^{-1})^{\nu\beta})\partial^2_{\nu\beta}\langle D_x\rangle^{s''}\widetilde{\phi}_k] \\
&\quad - E_k^i\langle D_x\rangle^{s''}[\varpi(\partial_i (g^{-1})^{\nu\beta})\partial^2_{\nu\beta}\widetilde{\phi}_k]\}\|_{L^2(\Sigma_t)} \lesssim \epsilon,
\end{aligned}
\tag{12.58}
$$

$$
\begin{aligned}
&\|\langle x\rangle^{-\frac{r}{2}}s''\delta^{jq}[(\partial_j(\varpi(\bar{g}^{-1})^{\nu\beta})E_k\partial^3_{q\nu\beta}\langle D_x\rangle^{s''-2}\widetilde{\phi}_k]\|_{L^2(\Sigma_t)} \\
&\quad \lesssim \epsilon + \epsilon^{\frac{3}{2}}\|\partial E_k\langle D_x\rangle^{s''}\widetilde{\phi}_k\|_{L^2(\Sigma_t)},
\end{aligned}
\tag{12.59}
$$

$$
\|E_k^i T^{s''}_{\mathrm{res}}(\varpi(\bar{g}^{-1})^{\nu\beta}, \partial^3_{i\nu\beta}\langle D_x\rangle^{s''}\widetilde{\phi}_k)\|_{L^2(\Sigma_t)} \lesssim \epsilon.
\tag{12.60}
$$

The above six estimates will respectively be proven in Propositions 12.29, 12.30, 12.31, 12.32, 12.34 and 12.35 below, for $r \geq 2$. □

Proposition 12.29 *For $r \geq 2$, the estimate (12.55) holds, i.e.*

$$\begin{aligned}\|\langle x\rangle^{-\frac{r}{2}}\left[\Box_g, E_k\right]\langle D_x\rangle^{s''}\widetilde{\phi}_k\|_{L^2(\Sigma_t)} &\lesssim \epsilon^{\frac{5}{2}} \\ &+\epsilon^{\frac{3}{2}}\sum_{Z_k\in\{E_k,L_k\}}\|\partial Z_k\langle D_x\rangle^{s''}\widetilde{\phi}_k\|_{L^2(\Sigma_t)}.\end{aligned}$$

Proof By Proposition 8.10, we obtain

$$\begin{aligned}\|\langle x\rangle^{-\frac{r}{2}}\left[\Box_g, E_k\right]\langle D_x\rangle^{s''}\widetilde{\phi}_k\|_{L^2(\Sigma_t)} &\lesssim \epsilon^{\frac{3}{2}}(\|\partial\langle D_x\rangle^{s''}\widetilde{\phi}_k\|_{L^2(\Sigma_t)} \\ &+\epsilon^{\frac{3}{2}}\sum_{Z_k\in\{E_k,L_k\}}\|\partial Z_k\langle D_x\rangle^{s''}\widetilde{\phi}_k\|_{L^2(\Sigma_t)}).\end{aligned}$$

To conclude, we control the first term by (12.42). □

Proposition 12.30 *Let $\Box^1 := \Gamma^\lambda\partial_\lambda$ as in (12.6). Then, for $r \geq 2$,*

$$\left\|\langle x\rangle^{-\frac{r}{2}}\partial_i\left[\Box^1, \langle D_x\rangle^{s''}\right]\widetilde{\phi}_k\right\|_{L^2(\Sigma_t)} \lesssim \epsilon^{\frac{9}{4}}. \tag{12.61}$$

In particular, (12.56) holds.

Proof That (12.56) holds is immediate from (12.61) and the estimate (5.5) for E_k^i. From now on we focus on (12.61).

Using the product rule and the fact that $\varpi\widetilde{\phi}_k = \widetilde{\phi}_k$,

$$\begin{aligned}\partial_i\left[\langle D_x\rangle^{s''}, \Box^1\right]\widetilde{\phi}_k &= \langle D_x\rangle^{s''}(\varpi\Gamma^\lambda\partial_\lambda\partial_i\widetilde{\phi}_k) - \varpi\Gamma^\lambda\partial_\lambda\langle D_x\rangle^{s''}\partial_i\widetilde{\phi}_k + \Gamma^\lambda[\varpi, \langle D_x\rangle^{s''}]\partial_\lambda\partial_i\widetilde{\phi}_k \\ &+ \langle D_x\rangle^{s''}(\varpi(\partial_i\Gamma^\lambda)\partial_\lambda\widetilde{\phi}_k) - \varpi(\partial_i\Gamma^\lambda)\langle D_x\rangle^{s''}\partial_\lambda\widetilde{\phi}_k + (\partial_i\Gamma^\lambda)[\varpi, \langle D_x\rangle^{s''}]\partial_\lambda\widetilde{\phi}_k \\ &= \underbrace{[\langle D_x\rangle^{s''}, \varpi\Gamma^\lambda\partial_\lambda]\partial_i\widetilde{\phi}_k}_{=:I} + \underbrace{\Gamma^\lambda[\varpi, \langle D_x\rangle^{s''}]\partial_\lambda\partial_i\widetilde{\phi}_k}_{=:II} \\ &+ \underbrace{[\langle D_x\rangle^{s''}, \varpi\partial_i\Gamma^\lambda]\partial_\lambda\widetilde{\phi}_k}_{=:III} \\ &+ \underbrace{(\partial_i\Gamma^\lambda)[\varpi, \langle D_x\rangle^{s''}]\partial_\lambda\widetilde{\phi}_k}_{=:IV}.\end{aligned}$$

By Corollary 12.8 we have

$$\|I\|_{L^2(\Sigma_t)} \lesssim \|\varpi\Gamma\|_{W^{1,\frac{2}{s'-s''}}(\Sigma_t)}\|\partial_\lambda\widetilde{\phi}_k\|_{H^{s'}(\Sigma_t)} \lesssim \epsilon^{\frac{9}{4}} \tag{12.62}$$

using (12.42) and the estimate (5.4).

For II, we use Lemma 12.17, Hölder's inequality and (5.4), (9.1) to obtain

$$\begin{aligned}\|\langle x\rangle^{-\frac{r}{2}}II\|_{L^2(\Sigma_t)} &\lesssim \|\langle x\rangle^{-\frac{r}{2}}\Gamma^\lambda\|_{L^\infty(\Sigma_t)}\|[\langle D_x\rangle^{s''}, \varpi]\partial^2_{i\lambda}\widetilde{\phi}_k\|_{L^2(\Sigma_t)} \\ &\lesssim \epsilon^{\frac{3}{2}}\|\langle D_x\rangle^{s''-2}\partial^2_{i\lambda}\widetilde{\phi}_k\|_{L^2(\Sigma_t)} \lesssim \epsilon^{\frac{3}{2}}\|\partial_\lambda\widetilde{\phi}_k\|_{L^2(\Sigma_t)} \lesssim \epsilon^{\frac{5}{2}}.\end{aligned} \tag{12.63}$$

For III, we apply the commutator estimate in Theorem 12.5 with $p_1 = \infty$, $p = p_2 = 2$ to obtain

$$\|III\|_{L^2(\Sigma_t)} \lesssim \|\langle D_x\rangle^{s''}(\varpi\,\partial_i\Gamma)\|_{L^2(\Sigma_t)} \cdot \|\partial\widetilde{\phi}_k\|_{L^\infty(\Sigma_t)} \lesssim \epsilon^{\frac{9}{4}}, \tag{12.64}$$

using (5.4) to control $\|\langle D_x\rangle^{s''}(\varpi\,\partial_i\Gamma)\|_{L^2(\Sigma_t)}$ and the bootstrap assumption (4.12c).

For IV, we use Lemma 12.17, Sobolev embedding ($H^{\frac{3}{2}-s''}(\mathbb{R}^2) \hookrightarrow L^\infty(\mathbb{R}^2)$) and Hölder's inequality to obtain

$$\begin{aligned}
\|\langle x\rangle^{-\frac{r}{2}} IV\|_{L^2(\Sigma_t)} &\lesssim \|\langle x\rangle^{-\frac{r}{2}}\partial_i\Gamma^\lambda\|_{L^2(\Sigma_t)}\|[\langle D_x\rangle^{s''}, \varpi]\partial\widetilde{\phi}_k\|_{L^\infty(\Sigma_t)} \\
&\lesssim \epsilon^{\frac{3}{2}}\|[\langle D_x\rangle^{s''}, \varpi]\partial\phi\|_{H^{\frac{3}{2}-s''}(\Sigma_t)} \lesssim \epsilon^{\frac{3}{2}}\|\partial\phi\|_{H^{-\frac{1}{2}}(\Sigma_t)} \\
&\lesssim \epsilon^{\frac{3}{2}}\|\partial\phi\|_{L^2(\Sigma_t)} \lesssim \epsilon^{\frac{5}{2}},
\end{aligned} \tag{12.65}$$

where in the second line we have used (5.4) and (9.1). Combining (12.62)–(12.65) yields the proposition. □

Proposition 12.31 *For any indices (ν, β),*

$$\|[\langle D_x\rangle^{s''}, \varpi]\partial^3_{i\nu\beta}\widetilde{\phi}_k\|_{L^2(\Sigma_t)} \lesssim \epsilon \tag{12.66}$$

and

$$\|[\langle D_x\rangle^{s''}, \varpi]\partial^2_{\nu\beta}\widetilde{\phi}_k\|_{L^2(\Sigma_t)} \lesssim \epsilon. \tag{12.67}$$

As a consequence, (12.57) holds.

Proof Assuming (12.66) and (12.67), it follows from Hölder's inequality, and the estimates (5.5), (5.1) that

$$\begin{aligned}
&\|\langle x\rangle^{-\frac{r}{2}} E^i_k(\partial_i(g^{-1})^{\nu\beta})[\langle D_x\rangle^{s''}, \varpi]\partial^2_{\nu\beta}\widetilde{\phi}_k\|_{L^2(\Sigma_t)} \\
&\quad + \|\langle x\rangle^{-\frac{r}{2}} E^i_k[(\bar{g}^{-1})^{\nu\beta}[\langle D_x\rangle^{s''}, \varpi]\partial^3_{i\nu\beta}\widetilde{\phi}_k]\|_{L^2(\Sigma_t)} \\
&\lesssim \|\langle x\rangle^{-\frac{r}{2}} E^i_k(\partial_i(g^{-1})^{\nu\beta})\|_{L^\infty(\Sigma_t)}\|[\langle D_x\rangle^{s''}, \varpi]\partial^2_{\nu\beta}\widetilde{\phi}_k\|_{L^2(\Sigma_t)} \\
&\quad + \|\langle x\rangle^{-\frac{r}{2}} E^i_k(\bar{g}^{-1})^{\nu\beta}\|_{L^\infty(\Sigma_t)}\|[\langle D_x\rangle^{s''}, \varpi]\partial^3_{i\nu\beta}\widetilde{\phi}_k\|_{L^2(\Sigma_t)} \lesssim \epsilon,
\end{aligned}$$

i.e. (12.57) holds (recall the notation $(\bar{g}^{-1})^{\nu\beta}$ from (12.52)).

To obtain (12.66), we note that by Lemma 12.17, the L^2-boundedness of $\partial_i\langle D_x\rangle^{-1}$, Lemma 12.16, and Proposition 12.25,

$$\|[\langle D_x\rangle^{s''}, \varpi]\partial^3_{i\nu\beta}\widetilde{\phi}_k\|_{L^2(\Sigma_t)} \lesssim \|\langle D_x\rangle^{s'-1}\partial^2_{\nu\beta}\widetilde{\phi}_k\|_{L^2(\Sigma_t)} \lesssim \|\partial\langle D_x\rangle^{s'}\widetilde{\phi}_k\|_{L^2(\Sigma_t)} \lesssim \epsilon.$$

The estimate (12.67) is even simpler and can be obtained similarly. □

Proposition 12.32 *For any index (ν, β) and any f satisfying*

$$\|f\|_{W^{1,\frac{2}{s'-s''}}(\Sigma_t)} \lesssim 1, \tag{12.68}$$

we have

$$\|f\langle D_x\rangle^{s''}\partial^2_{\nu\beta}\widetilde{\phi}_k - \langle D_x\rangle^{s''}(f\partial^2_{\nu\beta}\widetilde{\phi}_k)\|_{L^2(\Sigma_t)} \lesssim \epsilon. \tag{12.69}$$

As a consequence, for any index σ,

$$\|\varpi(\partial_\sigma(g^{-1})^{\nu\beta})\partial^2_{\nu\beta}\langle D_x\rangle^{s''}\widetilde{\phi}_k - \langle D_x\rangle^{s''}[\varpi(\partial_\sigma(g^{-1})^{\nu\beta})\partial^2_{\nu\beta}\widetilde{\phi}_k]\|_{L^2(\Sigma_t)} \lesssim \epsilon. \tag{12.70}$$

In particular, (12.58) holds.

Proof *Step 1: Proof of* (12.69) By Corollary 12.8 and (12.68),

$$\begin{aligned}\text{LHS of (12.69)} &\lesssim \|f\|_{W^{1,\frac{2}{s'-s''}}(\Sigma_t)}\|\langle D_x\rangle^{s'-1}\partial^2_{\nu\beta}\widetilde{\phi}_k\|_{L^2(\Sigma_t)}\\ &\lesssim \|\langle D_x\rangle^{s'-1}\partial^2_{\nu\beta}\widetilde{\phi}_k\|_{L^2(\Sigma_t)}.\end{aligned} \tag{12.71}$$

The estimate (12.69) thus follows from Lemma 12.16 and Proposition 12.25.
Step 2: Proof of (12.70) *and* (12.58) By (12.69), to establish (12.70) requires only that

$$\|\varpi(\partial_\sigma(g^{-1})^{\nu\beta})\|_{W^{1,\frac{2}{s'-s''}}(\Sigma_t)} \lesssim 1,$$

which in turn follows from (5.1).

Using (12.70), Hölder's inequality and (5.5), we obtain

$$\begin{aligned}&\|\langle x\rangle^{-\frac{r}{2}}\{E^i_k[\varpi(\partial_i(g^{-1})^{\nu\beta})\partial^2_{\nu\beta}\langle D_x\rangle^{s''}\widetilde{\phi}_k] - E^i_k\langle D_x\rangle^{s''}[\varpi(\partial_i(g^{-1})^{\nu\beta})\partial^2_{\nu\beta}\widetilde{\phi}_k]\}\|_{L^2(\Sigma_t)}\\ &\lesssim \|\langle x\rangle^{-\frac{r}{2}}E^i_k\|_{L^\infty(\Sigma_t)}\|[\varpi(\partial_i(g^{-1})^{\nu\beta})\partial^2_{\nu\beta}\langle D_x\rangle^{s''}\widetilde{\phi}_k]\\ &\quad - E^i_k\langle D_x\rangle^{s''}[\varpi(\partial_i(g^{-1})^{\nu\beta})\partial^2_{\nu\beta}\widetilde{\phi}_k]\|_{L^2(\Sigma_t)} \lesssim \epsilon,\end{aligned}$$

which establishes (12.58). □

Next, we consider the term (12.59); the main estimate will be obtained in Proposition 12.34 below. To ease the exposition, we prove an important but technically involved commutator estimate in the following lemma:

Lemma 12.33 *Let Y^t be a smooth function satisfying*

$$\|\langle x\rangle^{-1}Y^t\|_{L^\infty(\Sigma_t)} \lesssim 1, \quad \|\varpi Y^t\|_{W^{1,\infty}(\Sigma_t)\cap W^{2,4}(\Sigma_t)} \lesssim 1.$$

Then

$$\|\langle x\rangle^{-1}\{Y^t R_q R_i \langle D_x\rangle^{s''}(\breve{g}^{i\lambda}\partial^2_{t\lambda}\widetilde{\phi}_k) - R_i R_q \langle D_x\rangle^{s''}(\breve{g}^{i\lambda} Y^t \partial^2_{t\lambda}\widetilde{\phi}_k)\}\|_{L^2(\Sigma_t)} \lesssim \epsilon. \tag{12.72}$$

Proof We consider separately the cases when we sum λ over the spatial indices and when $\lambda = t$.
Step 1: Summing λ over spatial indices We compute

$$\begin{aligned} & Y^t R_q R_i \langle D_x\rangle^{s''}(\breve{g}^{ij}\partial^2_{jt}\widetilde{\phi}_k) - R_q R_i \langle D_x\rangle^{s''}(\breve{g}^{ij} Y^t \partial^2_{tj}\widetilde{\phi}_k) \\ &= \underbrace{[Y^t\partial_j, R_q R_i \langle D_x\rangle^{s''}](\breve{g}^{ij}\partial_t\widetilde{\phi}_k)}_{=:I} \underbrace{- Y^t R_q R_i \langle D_x\rangle^{s''}[(\partial_j \breve{g}^{ij})(\partial_t\widetilde{\phi}_k)]}_{=:II} \\ &\quad + \underbrace{R_q R_i \langle D_x\rangle^{s''}[Y^t(\partial_j \breve{g}^{ij})(\partial_t\widetilde{\phi}_k)]}_{=:III}. \end{aligned} \tag{12.73}$$

The term $\langle x\rangle^{-1}I$ is bounded in $L^2(\Sigma_t)$ by ϵ using (12.41) (with $h = Y^t$, $f = \breve{g}^{ij}$), (5.1) and Proposition 12.25. $\langle x\rangle^{-1}II$ and $\langle x\rangle^{-1}III$ are both bounded in $L^2(\Sigma_t)$ by ϵ by the assumed estimates on Y^t, the L^2-boundedness of R_i, (5.1), Lemma 12.13 and Proposition 12.25. Combining these observations give (12.72) when λ is only summed over the spatial indices.
Step 2: The $\lambda = t$ case It is notationally more convenient to prove more generally that for f satisfying $\|\varpi f\|_{W^{2,\infty}(\Sigma_t)} \lesssim 1$, we have

$$\|\langle x\rangle^{-1}\{Y^t R_q R_i \langle D_x\rangle^{s''}(f\partial^2_{tt}\widetilde{\phi}_k) - R_i R_q \langle D_x\rangle^{s''}(f Y^t \partial^2_{tt}\widetilde{\phi}_k)\}\|_{L^2(\Sigma_t)} \lesssim \epsilon. \tag{12.74}$$

First we compute, using the wave equation for $\widetilde{\phi}_k$ (see (12.9)), that

$$\begin{aligned} & - Y^t R_q R_i \langle D_x\rangle^{s''}(f\partial^2_{tt}\widetilde{\phi}_k) \\ &= Y^t \partial_q R_i \langle D_x\rangle^{s''-1}(f\breve{g}^{j\lambda}\partial^2_{j\lambda}\widetilde{\phi}_k + \frac{f\Gamma^\lambda\partial_\lambda\widetilde{\phi}_k}{(g^{-1})^{tt}}) \\ &= \underbrace{[Y^t\partial_q, R_i R_j \langle D_x\rangle^{s''}](f\breve{g}^{j\lambda}\partial_\lambda\widetilde{\phi}_k)}_{=:I} \\ &\quad + \underbrace{R_i R_j \langle D_x\rangle^{s''}[Y^t(\partial_q(f\breve{g}^{j\lambda}))(\partial_\lambda\widetilde{\phi}_k)]}_{=:II} + \underbrace{R_i R_j \langle D_x\rangle^{s''}(Y^t f\breve{g}^{j\lambda}\partial^2_{q\lambda}\widetilde{\phi}_k)}_{=:III} \\ &\quad \underbrace{- Y^t R_q R_i \langle D_x\rangle^{s''}[(\partial_j(f\breve{g}^{j\lambda}))(\partial_\lambda\widetilde{\phi}_k)]}_{=:IV} + \underbrace{Y^t R_q R_i \langle D_x\rangle^{s''}(\frac{f\Gamma^\lambda\partial_\lambda\widetilde{\phi}_k}{(g^{-1})^{tt}})}_{=:V}, \end{aligned} \tag{12.75}$$

where the term $Y^t\partial_q R_i\langle D_x\rangle^{s''-1}(f\breve{g}^{j\lambda}\partial^2_{j\lambda}\widetilde{\phi}_k)$ is computed in a way similar to (12.73). The main term in (12.75) is III. Indeed, term I can be controlled by (12.41) in

Lemma 12.24 (combined with (5.1)), while terms II, IV and V can be handled using Lemma 12.13 (again combined with (5.1)) so that

$$\|I\|_{L^2(\Sigma_t)} + \|II\|_{L^2(\Sigma_t)} + \|IV\|_{L^2(\Sigma_t)} + \|V\|_{L^2(\Sigma_t)} \lesssim \epsilon^{\frac{3}{4}}. \tag{12.76}$$

Finally, for the term III in (12.75), we shuffle the ∂_j and ∂_q derivatives and once again use the wave equation for $\widetilde{\phi}_k$ (recall again (12.9)) to obtain

$$\begin{aligned} III &= R_i R_j \langle D_x\rangle^{s''}(Y^t f \breve{g}^{j\lambda}\partial^2_{q\lambda}\widetilde{\phi}_k) \\ &= R_i R_q \langle D_x\rangle^{s''}(f\breve{g}^{j\lambda}Y^t\partial^2_{j\lambda}\widetilde{\phi}_k) + R_i R_q\langle D_x\rangle^{s''}[(\partial_j(fY^t\breve{g}^{j\lambda}))(\partial_\lambda\widetilde{\phi}_k)] \\ &\quad - R_i R_j\langle D_x\rangle^{s''}[(\partial_q(fY^t\breve{g}^{j\lambda}))(\partial_\lambda\widetilde{\phi}_k)] \\ &= \underbrace{-R_i R_q\langle D_x\rangle^{s''}(fY^t\partial^2_{tt}\widetilde{\phi}_k)}_{=:III_1}\underbrace{-R_iR_q\langle D_x\rangle^{s''}(\frac{fY^t\Gamma^\lambda\partial_\lambda\widetilde{\phi}_k}{(g^{-1})^{tt}})}_{=:III_2} \\ &\quad + \underbrace{R_iR_q\langle D_x\rangle^{s''}[(\partial_j(fY^t\breve{g}^{j\lambda}))(\partial_\lambda\widetilde{\phi}_k)]}_{=:III_3}\underbrace{-R_iR_j\langle D_x\rangle^{s''}[(\partial_q(fY^t\breve{g}^{j\lambda}))(\partial_\lambda\widetilde{\phi}_k)]}_{=:III_4}. \end{aligned} \tag{12.77}$$

The main term here is III_1 (i.e. it is included in the main term on the left-hand side of (12.74)). Using again Lemma 12.13 (or obvious modifications), we have

$$\|III_2\|_{L^2(\Sigma_t)} + \|III_3\|_{L^2(\Sigma_t)} + \|III_4\|_{L^2(\Sigma_t)} \lesssim \epsilon^{\frac{3}{4}}. \tag{12.78}$$

Combining (12.75)–(12.78), we obtain the desired estimate. □

Proposition 12.34 *Let Y be a spacetime vector field satisfying*

$$\begin{aligned} &\|\langle x\rangle^{-1}Y^\nu\|_{L^\infty(\Sigma_t)} + \|\partial_x Y^\nu\|_{L^\infty(\Sigma_t)} + \|\langle x\rangle^{-1}\partial_t Y^\nu\|_{L^\infty(\Sigma_t)} \\ &\quad + \|\varpi Y^\nu\|_{W^{1,\infty}\cap W^{2,4}(\Sigma_t)} \lesssim 1. \end{aligned} \tag{12.79}$$

Then for any spacetime index $(\nu, \beta) \neq (t, t)$ and $r \geq 2$,

$$\|\langle x\rangle^{-\frac{r}{2}}Y\langle D_x\rangle^{s''-2}\partial^3_{q\nu\beta}\widetilde{\phi}_k\|_{L^2(\Sigma_t)} \lesssim \|\partial Y\langle D_x\rangle^{s''}\widetilde{\phi}_k\|_{L^2(\Sigma_t)} + \epsilon. \tag{12.80}$$

Also,

$$\begin{aligned} &\|\langle x\rangle^{-\frac{r}{2}}\{Y\langle D_x\rangle^{s''-2}\partial^3_{qtt}\widetilde{\phi}_k + Y^t\partial_t\langle D_x\rangle^{s''-2}\partial_q(\frac{\Gamma^\lambda\partial_\lambda\widetilde{\phi}_k}{(g^{-1})^{tt}})\}\|_{L^2(\Sigma_t)} \\ &\lesssim \|\partial Y\langle D_x\rangle^{s''}\widetilde{\phi}_k\|_{L^2(\Sigma_t)} + \epsilon. \end{aligned} \tag{12.81}$$

In particular, (12.59) holds.

Proof Once we obtain (12.80), (12.59) follows easily from the fact that E_k has only spatial components, the estimates (5.1) and (5.5), (5.6). More precisely, taking advantage of the compact support of ϖ, we obtain

$$\begin{aligned}
&\|s''\delta^{jq}[(\partial_j(\varpi(\bar{g}^{-1})^{\nu\beta}))E_k\partial^3_{q\nu\beta}\langle D_x\rangle^{s''-2}\widetilde{\phi}_k]\|_{L^2(\Sigma_t)}\\
&\lesssim \|\langle x\rangle^{\frac{r}{2}}\partial_j(\varpi(\bar{g}^{-1})^{\nu\beta})\|_{L^\infty(\Sigma_t)}\|\langle x\rangle^{-\frac{r}{2}}E_k\partial^3_{q\nu\beta}\langle D_x\rangle^{s''-2}\widetilde{\phi}_k\|_{L^2(\Sigma_t)}\\
&\lesssim \|\partial_j(\varpi(\bar{g}^{-1})^{\nu\beta})\|_{L^\infty(\Sigma_t)}\|\langle x\rangle^{-\frac{r}{2}}E_k\partial^3_{q\nu\beta}\langle D_x\rangle^{s''-2}\widetilde{\phi}_k\|_{L^2(\Sigma_t)}\\
&\lesssim \epsilon^{\frac{3}{2}}(\|\partial E_k\langle D_x\rangle^{s''}\widetilde{\phi}_k\|_{L^2(\Sigma_t)}+\epsilon).
\end{aligned}$$

In the remainder of the proof we focus on proving (12.80) and (12.81): (12.80) will be proven in Steps 1–2 and (12.81) will be established in Step 3. The proof of (12.80) will be further split into two cases: the $(\nu,\beta)=(i,j)$ case will be treated in Step 1; the $(\nu,\beta)=(i,t)$ case will be treated in Step 2.

Step 1: Proof of (12.80) *when* $(\nu,\beta)=(i,j)$ In this case,

$$\begin{aligned}
Y\langle D_x\rangle^{s''-2}\partial^3_{qij}\widetilde{\phi}_k &= Y^\sigma\partial_q R_iR_j\partial_\sigma\langle D_x\rangle^{s''}\widetilde{\phi}_k\\
&=[Y^\sigma\partial_q, R_iR_j]\partial_\sigma\langle D_x\rangle^{s''}\widetilde{\phi}_k - R_iR_j[(\partial_qY^\sigma)(\partial_\sigma\langle D_x\rangle^{s''}\widetilde{\phi}_k)]\\
&\quad + R_iR_j(\partial_q(Y\langle D_x\rangle^{s''}\widetilde{\phi}_k)).
\end{aligned}$$

Using Lemma 12.24 for the first term, and using the L^2-boundedness of R_i (and Hölder's inequality) for the second and third terms, we thus obtain

$$\begin{aligned}
&\|\langle x\rangle^{-\frac{r}{2}}Y\langle D_x\rangle^{s''-2}\partial^3_{qij}\widetilde{\phi}_k\|_{L^2(\Sigma_t)}\\
&\lesssim \|\partial\langle D_x\rangle^{s''}\widetilde{\phi}_k\|_{L^2(\Sigma_t)} + \|\partial_\ell Y^\sigma\|_{L^\infty(\Sigma_t)}\|\partial\langle D_x\rangle^{s''}\widetilde{\phi}_k\|_{L^2(\Sigma_t)}\\
&\quad + \|\partial Y\langle D_x\rangle^{s''}\widetilde{\phi}_k\|_{L^2(\Sigma_t)}\\
&\lesssim \|\partial Y\langle D_x\rangle^{s''}\widetilde{\phi}_k\|_{L^2(\Sigma_t)} + \epsilon,
\end{aligned}$$

where we have used (12.79) and Proposition 12.25.

Step 2: Proof of (12.80) *when* $(\nu,\beta)=(i,t)$ Decompose $Y=Y^\ell\partial_\ell+Y^t\partial_t$ and commute $Y^\ell\partial_\ell$ with $R_qR_i\langle D_x\rangle^{s''}$. We then obtain

$$\begin{aligned}
Y\langle D_x\rangle^{s''-2}\partial^3_{qit}\widetilde{\phi}_k &= [Y^\ell\partial_\ell, R_qR_i]\langle D_x\rangle^{s''}\partial_t\widetilde{\phi}_k + R_qR_i(Y^\ell\partial_\ell\langle D_x\rangle^{s''}\partial_t\widetilde{\phi}_k)\\
&\quad + Y^t\partial_qR_i\langle D_x\rangle^{s''-1}\partial^2_{tt}\widetilde{\phi}_k.
\end{aligned} \tag{12.82}$$

Rearranging, this implies

$$\begin{aligned}
&Y\langle D_x\rangle^{s''-2}\partial^3_{qit}\widetilde{\phi}_k - R_qR_i(Y\langle D_x\rangle^{s''}\partial_t\widetilde{\phi}_k)\\
&=\underbrace{[Y^\ell\partial_\ell, R_qR_i]\langle D_x\rangle^{s''}\partial_t\widetilde{\phi}_k}_{=:I}+\underbrace{(Y^tR_qR_i\langle D_x\rangle^{s''}\partial^2_{tt}\widetilde{\phi}_k - R_qR_i(Y^t\langle D_x\rangle^{s''}\partial^2_{tt}\widetilde{\phi}_k))}_{=:II}.
\end{aligned} \tag{12.83}$$

We begin with the main term on the left-hand side of (12.83). We compute

$$
\begin{aligned}
Y\langle D_x\rangle^{s''}\partial_t\widetilde{\phi}_k &= \partial_t(Y\langle D_x\rangle^{s''}\widetilde{\phi}_k) - \varpi(\partial_t Y^\nu)\partial_\nu\langle D_x\rangle^{s''}\widetilde{\phi}_k \\
&\quad -(\partial_t Y^\nu)\partial_\nu[\langle D_x\rangle^{s''},\varpi]\widetilde{\phi}_k - (\partial_t Y^\nu)(\partial_\nu\varpi)(\langle D_x\rangle^{s''}\widetilde{\phi}_k).
\end{aligned}
$$

Hence, by L^2-boundedness of $R_q R_i$, Hölder's inequality, (12.79), (9.1), Lemma 12.21, (12.39) and Proposition 12.25,

$$
\begin{aligned}
&\|R_q R_i(Y\langle D_x\rangle^{s''}\partial_t\widetilde{\phi}_k)\|_{L^2(\Sigma_t)} \\
&\lesssim \|\partial_t Y\langle D_x\rangle^{s''}\widetilde{\phi}_k\|_{L^2(\Sigma_t)} + \|\varpi\partial_t Y^\nu\|_{L^\infty(\Sigma_t)}\|\partial_\nu\langle D_x\rangle^{s''}\widetilde{\phi}_k\|_{L^2(\Sigma_t)} \\
&\quad + \|\langle x\rangle^{-1}\partial_t Y^\nu\|_{L^\infty(\Sigma_t)}(\|\langle x\rangle\partial[\langle D_x\rangle^{s''},\varpi]\widetilde{\phi}_k\|_{L^2(\Sigma_t)} \\
&\quad + \|\langle x\rangle\partial_\nu\varpi\|_{L^\infty(\Sigma_t)}\|\langle D_x\rangle^{s''}\widetilde{\phi}_k\|_{L^2(\Sigma_t)}) \\
&\lesssim \|\partial_t Y\langle D_x\rangle^{s''}\widetilde{\phi}_k\|_{L^2(\Sigma_t)} + \epsilon.
\end{aligned}
\tag{12.84}
$$

By Lemma 12.24 and Proposition 12.25, the term I in (12.83) can be bounded as follows,

$$
\|\langle x\rangle^{-\frac{r}{2}} I\|_{L^2(\Sigma_t)} \lesssim \|\langle D_x\rangle^{s''}\partial_t\widetilde{\phi}_k\|_{L^2(\Sigma_t)} \lesssim \epsilon. \tag{12.85}
$$

For the commutator term II, since in general $[Y^t, R_q R_i]$ is only bounded $L^2(\Sigma_t)\to L^2(\Sigma_t)$ (instead of gaining one derivative), we need to use the wave equation for $\widetilde{\phi}_k$ and then exploit the gain given by Lemma 12.24. More precisely, we compute using (12.9) and $\partial_\lambda\widetilde{\phi}_k = \varpi\partial_\lambda\widetilde{\phi}_k$ that

$$
\begin{aligned}
&Y^t R_q R_i\langle D_x\rangle^{s''}\partial^2_{tt}\widetilde{\phi}_k \\
&= -Y^t R_q R_i\langle D_x\rangle^{s''}(\breve{g}^{j\lambda}\partial^2_{j\lambda}\widetilde{\phi}_k) - Y^t R_q R_i\langle D_x\rangle^{s''}(\frac{\Gamma^\lambda\partial_\lambda\widetilde{\phi}_k}{(g^{-1})^{tt}}) \\
&= -R_q R_i Y^t\langle D_x\rangle^{s''}(\breve{g}^{j\lambda}\partial^2_{j\lambda}\widetilde{\phi}_k) - [Y^t\partial_j, R_q R_i]\langle D_x\rangle^{s''}(\breve{g}^{j\lambda}\partial_\lambda\widetilde{\phi}_k) \\
&\quad -R_q R_i Y^t\langle D_x\rangle^{s''}[(\partial_j\breve{g}^{j\lambda})(\partial_\lambda\widetilde{\phi}_k)] \\
&\quad +Y^t R_q R_i\langle D_x\rangle^{s''}[(\partial_j\breve{g}^{j\lambda})(\partial_\lambda\widetilde{\phi}_k)] \\
&\quad -Y^t R_q R_i\langle D_x\rangle^{s''}(\frac{\Gamma^\lambda\partial_\lambda\widetilde{\phi}_k}{(g^{-1})^{tt}}) \\
&= R_q R_i Y^t\langle D_x\rangle^{s''}\partial^2_{tt}\widetilde{\phi}_k \underbrace{-[Y^t\partial_j, R_q R_i]\langle D_x\rangle^{s''}(\breve{g}^{j\lambda}\partial_\lambda\widetilde{\phi}_k)}_{=:II_1} \\
&\quad \underbrace{-R_q R_i\{\varpi Y^t\langle D_x\rangle^{s''}[(\partial_j\breve{g}^{j\lambda})(\partial_\lambda\widetilde{\phi}_k)]\}}_{=:II_2} \\
&\quad -\underbrace{R_q R_i\{Y^t[\langle D_x\rangle^{s''},\varpi]((\partial_j\breve{g}^{j\lambda})(\partial_\lambda\widetilde{\phi}_k))\}}_{=:II_3}
\end{aligned}
$$

$$+\underbrace{Y^t R_q R_i \langle D_x\rangle^{s''}[(\partial_j \breve{g}^{j\lambda})(\partial_\lambda \widetilde{\phi}_k)]}_{=:II_4} \underbrace{-Y^t R_q R_i \langle D_x\rangle^{s''}(\frac{\Gamma^\lambda \partial_\lambda \widetilde{\phi}_k}{(g^{-1})^{tt}})}_{=:II_5}$$

$$+\underbrace{R_q R_i[\varpi Y^t \langle D_x\rangle^{s''}(\frac{\Gamma^\lambda \partial_\lambda \widetilde{\phi}_k}{(g^{-1})^{tt}}])}_{=:II_6} + \underbrace{R_q R_i Y^t [\langle D_x\rangle^{s''}, \varpi](\frac{\Gamma^\lambda \partial_\lambda \widetilde{\phi}_k}{(g^{-1})^{tt}})}_{=:II_7}. \qquad (12.86)$$

The first term on the right-hand side of (12.86) is the main term (recall again term II in (12.83)). We will bound all the other terms. First, II_1 can be bounded by Lemmas 12.24, and the estimates (5.1) and Proposition 12.25,

$$\|II_1\|_{L^2(\Sigma_t)} \lesssim \|\partial \langle D_x\rangle^{s'} \widetilde{\phi}_k\|_{L^2(\Sigma_t)} \lesssim \epsilon. \qquad (12.87)$$

The terms II_2, II_4, II_5 and II_6 are easier: We use the L^2-boundedness of R_i, Hölder's inequality, the assumption (12.79), Lemma 12.14 and Proposition 12.25 to obtain

$$\|II_2\|_{L^2(\Sigma_t)} + \|\langle x\rangle^{-\frac{r}{2}} II_4\|_{L^2(\Sigma_t)} + \|\langle x\rangle^{-\frac{r}{2}} II_5\|_{L^2(\Sigma_t)} + \|II_6\|_{L^2(\Sigma_t)} \lesssim \epsilon. \qquad (12.88)$$

For II_3 and II_7, we use the L^2-boundedness of R_i, Hölder's inequality, the assumption (12.79), Lemma 12.20, (9.1), (5.1), (5.4) to obtain

$$\begin{aligned} &\|II_3\|_{L^2(\Sigma_t)} + \|II_7\|_{L^2(\Sigma_t)} \\ &\lesssim \|\langle x\rangle^{-1} Y^t\|_{L^\infty(\Sigma_t)} (\|\langle x\rangle [\langle D_x\rangle^{s''}, \varpi]((\partial_j \breve{g}^{j\lambda})(\partial_\lambda \widetilde{\phi}_k))\|_{L^2(\Sigma_t)} \\ &\quad + \|\langle x\rangle [\langle D_x\rangle^{s''}, \varpi](\frac{\Gamma^\lambda \partial_\lambda \widetilde{\phi}_k}{(g^{-1})^{tt}})\|_{L^2(\Sigma_t)}) \\ &\lesssim \|(\partial_j \breve{g}^{j\lambda})(\partial_\lambda \widetilde{\phi}_k)\|_{L^2(\Sigma_t)} + \|\frac{\Gamma^\lambda \partial_\lambda \widetilde{\phi}_k}{(g^{-1})^{tt}}\|_{L^2(\Sigma_t)} \lesssim \epsilon. \end{aligned} \qquad (12.89)$$

Plugging in the estimates (12.87)–(12.89) into (12.86), we obtain

$$\|II\|_{L^2(\Sigma_t)} = \|Y^t R_q R_i \langle D_x\rangle^{s''} \partial^2_{tt} \widetilde{\phi}_k - R_q R_i Y^t \langle D_x\rangle^{s''} \partial^2_{tt} \widetilde{\phi}_k\|_{L^2(\Sigma_t)} \lesssim \epsilon. \qquad (12.90)$$

Combining (12.84), (12.85) and (12.90), and returning to (12.83), we thus obtain

$$\begin{aligned} \|Y\langle D_x\rangle^{s''-2} \partial^3_{qit} \widetilde{\phi}_k\|_{L^2(\Sigma_t)} &\lesssim \|R_q R_i (Y \langle D_x\rangle^{s''} \partial_t \widetilde{\phi}_k)\|_{L^2(\Sigma_t)} + \|I\|_{L^2(\Sigma_t)} + \|II\|_{L^2(\Sigma_t)} \\ &\lesssim \|\partial Y \langle D_x\rangle^{s''} \widetilde{\phi}_k\|_{L^2(\Sigma_t)} + \epsilon, \end{aligned}$$

as desired.

Step 3: Proof of (12.81) We begin with an application of the wave equation (recall (12.9)):

$$\begin{aligned} Y\langle D_x\rangle^{s''-2}\partial^3_{qtt}\widetilde{\phi}_k = & \underbrace{-YR_qR_i\langle D_x\rangle^{s''}(\breve{g}^{i\lambda}\partial_\lambda\widetilde{\phi}_k)}_{=:I} \\ & + \underbrace{YR_q\langle D_x\rangle^{s''-1}[(\partial_i\breve{g}^{i\lambda})(\partial_\lambda\widetilde{\phi}_k)]}_{II} \underbrace{-Y^\ell R_\ell R_q\langle D_x\rangle^{s''}(\frac{\Gamma^\lambda\partial_\lambda\widetilde{\phi}_k}{(g^{-1})^{tt}})}_{III} \\ & -Y^t\partial_tR_q\langle D_x\rangle^{s''-1}(\frac{\Gamma^\lambda\partial_\lambda\widetilde{\phi}_k}{(g^{-1})^{tt}}). \end{aligned} \tag{12.91}$$

Note that the last term $-Y^t\partial_tR_q\langle D_x\rangle^{s''-1}(\frac{\Gamma^\lambda\partial_\lambda\widetilde{\phi}_k}{(g^{-1})^{tt}})$ is present in the statement of (12.81). It is therefore sufficient to control each of I, II and III in (12.91). This will be carried out in Steps 3(a)–3(c) below.

Step 3(a): Term I in (12.91) As in Step 2, we write $Y = Y^\ell\partial_\ell + Y^t\partial_t$. We compute

$$\begin{aligned} & YR_qR_i\langle D_x\rangle^{s''}(\breve{g}^{i\lambda}\partial_\lambda\widetilde{\phi}_k) \\ & = \underbrace{[Y^\ell\partial_\ell, R_qR_i\langle D_x\rangle^{s''}](\breve{g}^{i\lambda}\partial_\lambda\widetilde{\phi}_k)}_{=:I_1} + \underbrace{R_qR_i\langle D_x\rangle^{s''}[Y^\ell(\partial_\ell\breve{g}^{i\lambda})(\partial_\lambda\widetilde{\phi}_k)]}_{=:I_2} \\ & \quad + \underbrace{R_qR_i\langle D_x\rangle^{s''}[\breve{g}^{i\lambda}(Y^\ell\partial_\ell\partial_\lambda\widetilde{\phi}_k)]}_{=:I_3} \\ & \quad + \underbrace{Y^tR_qR_i\langle D_x\rangle^{s''}[(\partial_t\breve{g}^{i\lambda})(\partial_\lambda\widetilde{\phi}_k)]}_{=:I_4} + \underbrace{Y^tR_qR_i\langle D_x\rangle^{s''}(\breve{g}^{i\lambda}\partial^2_{\lambda t}\widetilde{\phi}_k)}_{=:I_5}. \end{aligned} \tag{12.92}$$

I_1, I_2, I_4 are easier error terms. Indeed, using Lemma 12.24, (12.79) and (5.1) for I_1, using Lemma 12.14 for I_2 and I_4, and then applying Proposition 12.25, we have the estimates

$$\|I_1\|_{L^2(\Sigma_t)} + \|I_2\|_{L^2(\Sigma_t)} + \|I_4\|_{L^2(\Sigma_t)} \lesssim \epsilon, \tag{12.93}$$

we skip the details.

We will combine I_3 and I_5 in (12.92). First, by Lemma 12.33,

$$\|\langle x\rangle^{-1}(I_5 - R_qR_i\langle D_x\rangle^{s''}(\breve{g}^{i\lambda}Y^t\partial^2_{t\lambda}\widetilde{\phi}_k))\|_{L^2(\Sigma_t)} \lesssim \epsilon. \tag{12.94}$$

Therefore, recalling the definition of I_3 and I_5 in (12.92) and then using (12.94), we obtain

$$\|I_3 + I_5 - R_qR_i\langle D_x\rangle^{s''}[\breve{g}^{i\lambda}(Y\partial_\lambda\widetilde{\phi}_k)]\|_{L^2(\Sigma_t)} \lesssim \epsilon. \tag{12.95}$$

We next estimate the term $R_q R_i \langle D_x\rangle^{s''}[\breve{g}^{i\lambda}(Y\partial_\lambda\widetilde{\phi}_k)]$ (appearing in (12.95)). By the L^2-boundedness of $R_q R_i$ and $Y\partial_\lambda\widetilde{\phi}_k = \varpi Y\partial_\lambda\widetilde{\phi}_k$, Proposition 12.7 and (5.1)

$$
\begin{aligned}
&\|R_q R_i \langle D_x\rangle^{s''}[\breve{g}^{i\lambda}(Y\partial_\lambda\widetilde{\phi}_k)]\|_{L^2(\Sigma_t)} \lesssim \|\langle D_x\rangle^{s''}[\varpi\breve{g}^{i\lambda}(Y\partial_\lambda\widetilde{\phi}_k)]\|_{L^2(\Sigma_t)}\\
&\lesssim \|\varpi\breve{g}^{i\lambda}\|_{L^\infty(\Sigma_t)}\|\langle D_x\rangle^{s''}Y\partial_\lambda\widetilde{\phi}_k\|_{L^2(\Sigma_t)}\\
&\quad + \|\varpi\breve{g}^{i\lambda}\|_{W^{1,\infty}(\Sigma_t)}\|\langle D_x\rangle^{s''-1}Y\partial_\lambda\widetilde{\phi}_k\|_{L^2(\Sigma_t)}\\
&\lesssim \|\langle D_x\rangle^{s''}Y\partial_\lambda\widetilde{\phi}_k\|_{L^2(\Sigma_t)} + \|\langle D_x\rangle^{s''-1}Y^\nu\partial^2_{\nu\lambda}\widetilde{\phi}_k\|_{L^2(\Sigma_t)}
\end{aligned}
\tag{12.96}
$$

We then bound each term on the right-hand side of (12.96). For the first term, we use $\partial^2_{\nu\lambda}\widetilde{\phi}_k = \varpi\partial^2_{\nu\lambda}\widetilde{\phi}_k$, Proposition 12.7, (12.79), Lemma 12.16 and Proposition 12.25 to obtain

$$
\begin{aligned}
&\|\langle D_x\rangle^{s''}(Y^\nu\partial^2_{\nu\lambda}\widetilde{\phi}_k)\|_{L^2(\Sigma_t)} = \|\langle D_x\rangle^{s''}(\varpi Y^\nu\partial^2_{\nu\lambda}\widetilde{\phi}_k)\|_{L^2(\Sigma_t)}\\
&\quad\lesssim \|\varpi Y^\nu\partial^2_{\nu\lambda}\langle D_x\rangle^{s''}\widetilde{\phi}_k\|_{L^2(\Sigma_t)}\\
&\qquad + \|\varpi Y^\nu\|_{W^{1,\infty}(\Sigma_t)}\|\langle D_x\rangle^{s''-1}\partial^2_{\nu\lambda}\widetilde{\phi}_k\|_{L^2(\Sigma_t)}\\
&\quad\lesssim \|\varpi\partial_\lambda(Y\langle D_x\rangle^{s''}\widetilde{\phi}_k)\|_{L^2(\Sigma_t)} + \|\partial_\lambda(\varpi Y^\nu)\|_{L^\infty(\Sigma_t)}\|\partial_\nu\langle D_x\rangle^{s''}\widetilde{\phi}_k\|_{L^2(\Sigma_t)}\\
&\qquad + \|\langle D_x\rangle^{s''-1}\partial^2_{\nu\lambda}\widetilde{\phi}_k\|_{L^2(\Sigma_t)}\\
&\quad\lesssim \|\partial Y\langle D_x\rangle^{s''}\widetilde{\phi}_k\|_{L^2(\Sigma_t)} + \epsilon.
\end{aligned}
\tag{12.97}
$$

For the second term on the right-hand side of (12.96), we directly use Lemma 12.16 and Proposition 12.25 to obtain

$$
\|\langle D_x\rangle^{s''-1}Y^\nu\partial^2_{\nu\lambda}\widetilde{\phi}_k\|_{L^2(\Sigma_t)} \lesssim \|\partial\langle D_x\rangle^{s'}\widetilde{\phi}_k\|_{L^2(\Sigma_t)} \lesssim \epsilon. \tag{12.98}
$$

Plugging (12.97) and (12.98) into (12.96), we thus obtain

$$
\|R_q R_i\langle D_x\rangle^{s''}[\breve{g}^{i\lambda}(Y\partial_\lambda\widetilde{\phi}_k)]\|_{L^2(\Sigma_t)} \lesssim \|\partial Y\langle D_x\rangle^{s''}\widetilde{\phi}_k\|_{L^2(\Sigma_t)} + \epsilon. \tag{12.99}
$$

Finally, we combine (12.93), (12.95) and (12.99) to obtain

$$
\|I\|_{L^2(\Sigma_t)} \lesssim \|R_q R_i\langle D_x\rangle^{s''}[\breve{g}^{i\lambda}(Y\partial_\lambda\widetilde{\phi}_k)]\|_{L^2(\Sigma_t)} + \epsilon \lesssim \|\partial Y\langle D_x\rangle^{s''}\widetilde{\phi}_k\|_{L^2(\Sigma_t)} + \epsilon. \tag{12.100}
$$

Step 3(b): Term II in (12.91) We again write $Y = Y^\ell \partial_\ell + Y^t \partial_t$ and expand as follows:

$$\begin{aligned} II &= Y R_q \langle D_x \rangle^{s''-1} [(\partial_i \breve{g}^{i\lambda})(\partial_\lambda \widetilde{\phi}_k)] \\ &= \underbrace{Y^\ell R_q R_\ell \langle D_x \rangle^{s''} [(\partial_i \breve{g}^{i\lambda})(\partial_\lambda \widetilde{\phi}_k)]}_{=:II_1} \\ &\quad + \underbrace{Y^t R_q \langle D_x \rangle^{s''-1} [(\partial^2_{it} \breve{g}^{i\lambda})(\partial_\lambda \widetilde{\phi}_k)]}_{=:II_2} \\ &\quad + \underbrace{Y^t R_q \langle D_x \rangle^{s''-1} [(\partial_i \breve{g}^{i\lambda})(\partial^2_{t\lambda} \widetilde{\phi}_k)]}_{=:II_3}. \end{aligned} \tag{12.101}$$

The term II_1 can be directly handled by (12.79), Lemma 12.14 and Proposition 12.25 so that

$$\| \langle x \rangle^{-\frac{r}{2}} II_1 \|_{L^2(\Sigma_t)} \lesssim \epsilon. \tag{12.102}$$

II_2 is even easier: we use the L^2-boundedness of $R_q \langle D_x \rangle^{s''-1}$, (12.79), (5.1) and the bootstrap assumption (4.12c) to obtain

$$\begin{aligned} \| \langle x \rangle^{-\frac{r}{2}} II_2 \|_{L^2(\Sigma_t)} &\lesssim \| \langle x \rangle^{-1} Y^t \|_{L^\infty(\Sigma_t)} \| (\partial^2_{it} \breve{g}^{i\lambda})(\partial_\lambda \widetilde{\phi}_k) \|_{L^2(\Sigma_t)} \\ &\lesssim \| (\partial^2_{it} \breve{g}^{i\lambda})(\partial_\lambda \widetilde{\phi}_k) \|_{L^2(\Sigma_t)} \lesssim \| \partial^2_{it} \breve{g}^{i\lambda} \|_{L^2(\Sigma_t)} \| \partial \widetilde{\phi}_k \|_{L^\infty(\Sigma_t)} \lesssim \epsilon^{\frac{9}{4}}. \end{aligned}$$

For the term II_3 in (12.101), we use (12.79), Lemma 12.16, (5.1) and Proposition 12.25 to obtain

$$\| \langle x \rangle^{-\frac{r}{2}} II_3 \|_{L^2(\Sigma_t)} \lesssim \epsilon. \tag{12.103}$$

Finally, combining (12.102) and (12.103) yields

$$\| \langle x \rangle^{-\frac{r}{2}} II \|_{L^2(\Sigma_t)} \lesssim \epsilon. \tag{12.104}$$

Step 3(c): Term III in (12.91) The very final term III in (12.91) is simple. Indeed, by Hölder's inequality, (12.79), Lemma 12.14 and Proposition 12.25,

$$\| \langle x \rangle^{-\frac{r}{2}} III \|_{L^2(\Sigma_t)} \lesssim \| \langle x \rangle^{-1} Y^t \|_{L^\infty(\Sigma_t)} \| \langle D_x \rangle^{s''} (\frac{\Gamma^\lambda \partial_\lambda \widetilde{\phi}_k}{(g^{-1})^{tt}}) \|_{L^2(\Sigma_t)} \lesssim \epsilon. \tag{12.105}$$

Finally, we plug the estimates (12.100), (12.104) and (12.105) into (12.91) to obtain (12.81). □

Proposition 12.35 *When $(\nu, \beta, \sigma) \neq (t, t, t)$,*

$$\| T^{s''}_{\text{res}} (\varpi (\bar{g}^{-1})^{\nu\beta}, \partial^3_{\sigma\nu\beta} \langle D_x \rangle^{s''} \widetilde{\phi}_k) \|_{L^2(\Sigma_t)} \lesssim \epsilon^{\frac{5}{2}}, \tag{12.106}$$

where we recall the notation $(\bar{g}^{-1})^{\nu\beta}$ from (12.52).
In particular, (12.60) holds.

Proof That (12.60) holds is immediate from (12.106) and the fact that E_k does not have a t component. From now on we focus on the proof of (12.106).

By Corollary 12.10 and (5.1), the left-hand side of (12.106) is bounded by

$$\begin{aligned}\text{LHS of (12.106)} &\lesssim \|\varpi(\bar{g}^{-1})^{\nu\beta}\|_{W^{2,\frac{2}{s'-s''}}(\Sigma_t)}\|\langle D_x\rangle^{s'-2}\partial^3_{\sigma\nu\beta}\widetilde{\phi}_k\|_{L^2(\Sigma_t)}\\ &\lesssim \epsilon^{\frac{3}{2}}\|\langle D_x\rangle^{s'-2}\partial^3_{\sigma\nu\beta}\widetilde{\phi}_k\|_{L^2(\Sigma_t)}.\end{aligned}$$

Note that if two of σ, ν, β are spatial, we use Proposition 12.25 to get

$$\|\langle D_x\rangle^{s'-2}\partial^3_{\sigma\nu\beta}\widetilde{\phi}_k\|_{L^2(\Sigma_t)} \lesssim \|\partial\langle D_x\rangle^{s'}\widetilde{\phi}_k\|_{L^2(\Sigma_t)} \lesssim \epsilon,$$

which gives the desired estimate.

The only remaining case to consider (after relabeling) is $(\nu, \beta, \sigma) = (t, t, i)$. By the L^2-boundedness of $\langle D_x\rangle^{-1}\partial_i$, Lemma 12.16 and Proposition 12.25,

$$\|\langle D_x\rangle^{s'-2}\partial^3_{itt}\widetilde{\phi}_k\|_{L^2(\Sigma_t)} \lesssim \|\langle D_x\rangle^{s'-1}\partial^2_{tt}\widetilde{\phi}_k\|_{L^2(\Sigma_t)} \lesssim \epsilon,$$

which again gives the desired estimate. □

12.7 Estimates for $\partial L_k\langle D_x\rangle^{s''}\widetilde{\phi}_k$

As in Sect. 12.6, we begin with a high level proof, leaving the main estimates in later propositions.

Proposition 12.36

$$\|\partial L_k\langle D_x\rangle^{s''}\widetilde{\phi}_k\|_{L^2(\Sigma_t)} \lesssim \epsilon. \tag{12.107}$$

Moreover, combining (12.48) with (12.107), we obtain

$$\|\partial E_k\langle D_x\rangle^{s''}\widetilde{\phi}_k\|_{L^2(\Sigma_t)} \lesssim \epsilon. \tag{12.108}$$

Proof In order to prove $\|\partial L_k\langle D_x\rangle^{s''}\widetilde{\phi}_k\|_{L^2(\Sigma_t)} \lesssim \epsilon$, we will derive and use the wave equation for $L_k\langle D_x\rangle^{s''}\widetilde{\phi}_k$.

Our main strategy is to decompose

$$\Box_g L_k\langle D_x\rangle^{s''}\widetilde{\phi}_k = L^t_k\partial_t C + F \tag{12.109}$$

for appropriate C and F (Step 1). The term F will be bounded in $L^2(\Sigma_t)$, while the term $L^t_k\partial_t C$ will be treated with an additional integration by parts in t. The relevant estimates will be treated in Step 2 below.

Step 1: Achieving the decomposition (12.109) We first write (recall the notation in (12.6))

$$\Box_g L_k \langle D_x\rangle^{s''}\widetilde{\phi}_k = [\Box_g, L_k\langle D_x\rangle^{s''}]\widetilde{\phi}_k = [\Box_g, L_k]\langle D_x\rangle^{s''}\widetilde{\phi}_k + L_k[\Box^1, \langle D_x\rangle^{s''}]\widetilde{\phi}_k + L_k[\Box^2, \langle D_x\rangle^{s''}]\widetilde{\phi}_k. \tag{12.110}$$

Now, recalling the notation in (12.3) and (12.52), the last term in (12.110) can be further computed using the product rule as follows, by the same computation that led to (12.54):

$$\begin{aligned} &L_k[\Box^2, \langle D_x\rangle^{s''}]\widetilde{\phi}_k = L_k^\mu \partial_\mu[\Box^2, \langle D_x\rangle^{s''}]\widetilde{\phi}_k \\ &= L_k^\mu[\varpi(\partial_\mu(g^{-1})^{\nu\beta})\partial^2_{\nu\beta}\langle D_x\rangle^{s''}\widetilde{\phi}_k] - L_k^\mu\langle D_x\rangle^{s''}[\varpi(\partial_\mu(g^{-1})^{\nu\beta})\partial^2_{\nu\beta}\widetilde{\phi}_k] \\ &\quad + L_k^\mu(\partial_\mu(g^{-1})^{\nu\beta})[\langle D_x\rangle^{s''}, \varpi]\partial^2_{\nu\beta}\widetilde{\phi}_k + L_k^\mu[(\bar{g}^{-1})^{\nu\beta}[\langle D_x\rangle^{s''}, \varpi]\partial^3_{\mu\nu\beta}\widetilde{\phi}_k] \\ &\quad - s''\delta^{jq}(\partial_j(\varpi(\bar{g}^{-1})^{\nu\beta}))(L_k\partial^3_{q\nu\beta}\langle D_x\rangle^{s''-2}\widetilde{\phi}_k) - L_k^\mu T^{s''}_{\mathrm{res}}(\varpi(\bar{g}^{-1})^{\nu\beta}, \partial^3_{\mu\nu\beta}\widetilde{\phi}_k). \end{aligned} \tag{12.111}$$

There are a few terms in (12.110) and (12.111) which cannot be estimated directly and have to be separated out. First, there are the following terms:

$$\begin{aligned} &L_k^t\partial_t[\Box^1, \langle D_x\rangle^{s''}]\widetilde{\phi}_k, \quad L_k^t(\bar{g}^{-1})^{tt}[\langle D_x\rangle^{s''}, \varpi]\partial^3_{ttt}\widetilde{\phi}_k, \\ &- L_k^t T^{s''}_{\mathrm{res}}(\varpi(\bar{g}^{-1})^{tt}, \partial^3_{ttt}\widetilde{\phi}_k). \end{aligned} \tag{12.112}$$

The term $-s''\delta^{jq}[(\partial_j(\varpi(\bar{g}^{-1})^{\nu\beta}))L_k\partial^3_{q\nu\beta}\langle D_x\rangle^{s''-2}\widetilde{\phi}_k]$ also cannot be controlled directly. It can be shown (see (12.123) below) that up to controllable error, this term is essentially the following:

$$s''\delta^{jq}(\partial_j(\varpi(\bar{g}^{-1})^{tt}))L_k^t\partial_t\langle D_x\rangle^{s''-2}\partial_q(\frac{\Gamma^\lambda\partial_\lambda\widetilde{\phi}_k}{(g^{-1})^{tt}}). \tag{12.113}$$

In order to handle the terms in (12.112) and (12.113), we define

$$\begin{aligned} C :=\ &[\Box^1, \langle D_x\rangle^{s''}]\widetilde{\phi}_k + s''\delta^{jq}(\partial_j(\varpi(\bar{g}^{-1})^{tt}))\langle D_x\rangle^{s''-2}\partial_q(\frac{\Gamma^\lambda\partial_\lambda\widetilde{\phi}_k}{(g^{-1})^{tt}}) \\ &+ (\bar{g}^{-1})^{tt}[\langle D_x\rangle^{s''}, \varpi]\partial^2_{tt}\widetilde{\phi}_k - T^{s''}_{\mathrm{res}}(\varpi(\bar{g}^{-1})^{tt}, \partial^2_{tt}\widetilde{\phi}_k). \end{aligned} \tag{12.114}$$

It is easy to check that

$$\begin{aligned} L_k^t\partial_t C =\ &L_k^t\partial_t[\Box^1, \langle D_x\rangle^{s''}]\widetilde{\phi}_k + s''\delta^{jq}(\partial_j(\varpi(\bar{g}^{-1})^{tt}))L_k^t\partial_t\langle D_x\rangle^{s''-2}\partial_q(\frac{\Gamma^\lambda\partial_\lambda\widetilde{\phi}_k}{(g^{-1})^{tt}}) \\ &+ L_k^t(g^{-1})^{tt}[\langle D_x\rangle^{s''}, \varpi]\partial^3_{ttt}\widetilde{\phi}_k - L_k^t T^{s''}_{\mathrm{res}}(\varpi(\bar{g}^{-1})^{tt}, \partial^3_{ttt}\widetilde{\phi}_k) \\ &+ s''\delta^{jq}(\partial^2_{tj}(\varpi(\bar{g}^{-1})^{tt}))\langle D_x\rangle^{s''-2}\partial_q(\frac{\Gamma^\lambda\partial_\lambda\widetilde{\phi}_k}{(g^{-1})^{tt}}) \\ &+ (\partial_t(g^{-1})^{tt})[\langle D_x\rangle^{s''}, \varpi]\partial^2_{tt}\widetilde{\phi}_k - L_k^t T^{s''}_{\mathrm{res}}(\partial_t(\varpi(\bar{g}^{-1})^{tt}), \partial^2_{tt}\widetilde{\phi}_k) \end{aligned} \tag{12.115}$$

so that the first four terms are exactly the uncontrollable terms in (12.112) and (12.113), and the last three terms are error terms generated in this process.

Finally, we define F as follows so that (12.109) holds by (12.110), (12.111), (12.114) and (12.115):

$$\begin{aligned}
F &:= [\Box_g, L_k]\langle D_x\rangle^{s''}\widetilde{\phi}_k \\
&+L_k^i\partial_i[\Box^1, \langle D_x\rangle^{s''}]\widetilde{\phi}_k + L_k^\mu[(\partial_\mu(g^{-1})^{\nu\beta})[\langle D_x\rangle^{s''}, \varpi]\partial^2_{\nu\beta}\widetilde{\phi}_k] \\
&+L_k^\mu[(\bar{g}^{-1})^{\nu\beta}[\langle D_x\rangle^{s''}, \varpi]\partial^3_{\mu\nu\beta}\widetilde{\phi}_k] - L_k^t(\bar{g}^{-1})^{tt}[\langle D_x\rangle^{s''}, \varpi]\partial^3_{ttt}\widetilde{\phi}_k \\
&+L_k^\mu[\varpi(\partial_\mu(g^{-1})^{\nu\beta})\partial^2_{\nu\beta}\langle D_x\rangle^{s''}\widetilde{\phi}_k] - L_k^\mu\langle D_x\rangle^{s''}[\varpi(\partial_\mu(g^{-1})^{\nu\beta})\partial^2_{\nu\beta}\widetilde{\phi}_k] \\
&-s''\delta^{jq}[(\partial_j(\varpi(\bar{g}^{-1})^{\nu\beta}))L_k\partial^3_{q\nu\beta}\langle D_x\rangle^{s''-2}\widetilde{\phi}_k] \\
&-s''\delta^{jq}(\partial_j(\varpi(\bar{g}^{-1})^{\nu\beta}))L_k^t\partial_t\langle D_x\rangle^{s''-2}\partial_q(\frac{\Gamma^\lambda\partial_\lambda\widetilde{\phi}_k}{(g^{-1})^{tt}}) \\
&-L_k^\mu T^{s''}_{\text{res}}(\varpi(\bar{g}^{-1})^{\nu\beta},\ \partial^3_{\mu\nu\beta}\widetilde{\phi}_k) + L_k^t T^{s''}_{\text{res}}(\varpi(\bar{g}^{-1})^{tt},\ \partial^3_{ttt}\widetilde{\phi}_k) \\
&-s''\delta^{jq}(\partial^2_{tj}(\varpi(\bar{g}^{-1})^{tt}))\langle D_x\rangle^{s''-2}\partial_q(\frac{\Gamma^\lambda\partial_\lambda\widetilde{\phi}_k}{(g^{-1})^{tt}}) - (\partial_t(g^{-1})^{tt})[\langle D_x\rangle^{s''}, \varpi]\partial^2_{tt}\widetilde{\phi}_k \\
&+L_k^t T^{s''}_{\text{res}}(\partial_t(\varpi(\bar{g}^{-1})^{tt}),\ \partial^2_{tt}\widetilde{\phi}_k).
\end{aligned} \tag{12.116}$$

Step 2: The estimates We will handle the F term and the $L^t\partial_t C$ term separately.

For the F term, we will prove that for all $r \geq 2$

$$\|\langle x\rangle^{-\frac{r}{2}}F\|_{L^2(\Sigma_t)} \lesssim \epsilon + \epsilon^{\frac{3}{2}}\sum_{Z_k\in\{E_k,L_k\}}\|\partial Z_k\langle D_x\rangle^{s''}\widetilde{\phi}_k\|_{L^2(\Sigma_t)}. \tag{12.117}$$

In view of (12.116), the following estimates together imply (12.117) (note in particular the similarity of (12.118)–(12.124) with (12.55)–(12.60)):

$$\|\langle x\rangle^{-\frac{r}{2}}[\Box_g, L_k]\langle D_x\rangle^{s''}\widetilde{\phi}_k\|_{L^2(\Sigma_t)} \lesssim \epsilon + \epsilon^{\frac{3}{2}}\sum_{Z_k\in\{E_k,L_k\}}\|\partial Z_k\langle D_x\rangle^{s''}\widetilde{\phi}_k\|_{L^2(\Sigma_t)}, \tag{12.118}$$

$$\|\langle x\rangle^{-\frac{r}{2}}L_k^i\partial_i[\Box^1, \langle D_x\rangle^{s''}]\widetilde{\phi}_k\|_{L^2(\Sigma_t)} \lesssim \epsilon, \tag{12.119}$$

$$\|\langle x\rangle^{-\frac{r}{2}}L_k^\mu[(\partial_\mu(g^{-1})^{\nu\beta})[\langle D_x\rangle^{s''}, \varpi]\partial^2_{\nu\beta}\widetilde{\phi}_k]\|_{L^2(\Sigma_t)} \lesssim \epsilon, \tag{12.120}$$

$$\begin{aligned}
&\|\langle x\rangle^{-\frac{r}{2}}\{L_k^\mu[(\bar{g}^{-1})^{\nu\beta}[\langle D_x\rangle^{s''}, \varpi]\partial^3_{\mu\nu\beta}\widetilde{\phi}_k] \\
&\quad -L_k^t(\bar{g}^{-1})^{tt}[\langle D_x\rangle^{s''}, \varpi]\partial^3_{ttt}\widetilde{\phi}_k\}\|_{L^2(\Sigma_t)} \lesssim \epsilon,
\end{aligned} \tag{12.121}$$

$$\begin{aligned}
&\|\langle x\rangle^{-\frac{r}{2}}\{L_k^\mu[\varpi(\partial_\mu(g^{-1})^{\nu\beta})\partial^2_{\nu\beta}\langle D_x\rangle^{s''}\widetilde{\phi}_k] \\
&\quad -L_k^\mu\langle D_x\rangle^{s''}[\varpi(\partial_\mu(g^{-1})^{\nu\beta})\partial^2_{\nu\beta}\widetilde{\phi}_k]\}\|_{L^2(\Sigma_t)} \lesssim \epsilon,
\end{aligned} \tag{12.122}$$

$$\begin{aligned}
&\|\langle x\rangle^{-\frac{r}{2}}s''\delta^{jq}\{(\partial_j(\varpi(\bar{g}^{-1})^{\nu\beta}))L_k\partial^3_{q\nu\beta}\langle D_x\rangle^{s''-2}\widetilde{\phi}_k \\
&\quad +L_k^t(\partial_j(\varpi(\bar{g}^{-1})^{tt}))\partial_t\langle D_x\rangle^{s''-2}\partial_q(\frac{\Gamma^\lambda\partial_\lambda\widetilde{\phi}_k}{(g^{-1})^{tt}})\}\|_{L^2(\Sigma_t)} \lesssim \epsilon,
\end{aligned} \tag{12.123}$$

$$\|\langle x\rangle^{-\frac{r}{2}}[L_k^\mu T_{\rm res}^{s''}(\varpi(\bar{g}^{-1})^{\nu\beta},\ \partial^3_{\mu\nu\beta}\widetilde{\phi}_k)-L_k^t T_{\rm res}^{s''}(\varpi(\bar{g}^{-1})^{tt},\ \partial^3_{ttt}\widetilde{\phi}_k)]\|_{L^2(\Sigma_t)}\lesssim\epsilon, \tag{12.124}$$

$$\|\langle x\rangle^{-\frac{r}{2}}s''\delta^{jq}(\partial^2_{tj}(\varpi(\bar{g}^{-1})^{tt}))\langle D_x\rangle^{s''-2}\partial_q(\frac{\Gamma^\lambda\partial_\lambda\widetilde{\phi}_k}{(g^{-1})^{tt}})\|_{L^2(\Sigma_t)}\lesssim\epsilon, \tag{12.125}$$

$$\|\langle x\rangle^{-\frac{r}{2}}(\partial_t(g^{-1})^{tt})[\langle D_x\rangle^{s''},\varpi]\partial^2_{tt}\widetilde{\phi}_k\|_{L^2(\Sigma_t)}\lesssim\epsilon. \tag{12.126}$$

$$\|\langle x\rangle^{-\frac{r}{2}}L_k^t T_{\rm res}^{s''}(\partial_t(\varpi(\bar{g}^{-1})^{tt}),\ \partial^2_{tt}\widetilde{\phi}_k)\|_{L^2(\Sigma_t)}\lesssim\epsilon. \tag{12.127}$$

The estimates (12.118)–(12.127) will be proven in Proposition 12.37.

On the other hand, we will prove in Proposition 12.38 below that the term C in (12.114) can be bounded as follows for all $r'\geq 1$:

$$\|\langle x\rangle^{-\frac{r'}{2}}C\|_{L^2(\Sigma_t)}+\|\langle x\rangle^{-\frac{r'}{2}}\partial_i C\|_{L^2(\Sigma_t)}\lesssim\epsilon. \tag{12.128}$$

Step 3: Putting everything together We rewrite $L^t\partial_t C=\frac{1}{N}e_0 C+\frac{\beta^i}{N}\partial_i C$ using (2.5) and (2.31). We now apply Corollary 7.5 with $v=L_k\langle D_x\rangle^{s''}\widetilde{\phi}_k$, $f_1=F+\frac{\beta^i}{N}\partial_i C$, $f_2=C$, $h=\frac{1}{N}$ and $r\geq 2$. Note that the bound (6.2) holds for $h=\frac{1}{N}$ thanks to (5.1). Thus

$$\begin{aligned}
&\sup_{t\in[0,T)}\|\langle x\rangle^{-(r+2\alpha)}\partial L_k\langle D_x\rangle^{s''}\widetilde{\phi}_k\|^2_{L^2(\Sigma_t)}\\
&\lesssim\|\langle x\rangle^{-\frac{r}{2}}\partial L_k\langle D_x\rangle^{s''}\widetilde{\phi}_k\|^2_{L^2(\Sigma_0)}+\sup_{t\in[0,T)}\|\langle x\rangle^{-\frac{r}{2}}C\|^2_{L^2(\Sigma_t)}\\
&\quad+\int_0^T(\|\langle x\rangle^{-\frac{r}{2}}f_1\|^2_{L^2(\Sigma_t)}+\|\langle x\rangle^{-\frac{r}{2}}C\|^2_{L^2(\Sigma_t)}+\|\langle x\rangle^{-\frac{r}{2}}\partial_x C\|^2_{L^2(\Sigma_t)})\,dt\\
&\lesssim\epsilon^2+\int_0^T\|\langle x\rangle^{-\frac{r}{2}}f_1\|^2_{L^2(\Sigma_t)}\,dt,
\end{aligned} \tag{12.129}$$

where in the last line, we have used the initial data bounds in (4.3a) and (4.3b), as well as controlled C using (12.128).

Notice that by choosing $r'=r-1$, (12.128) and (5.1) imply that $\|\langle x\rangle^{-\frac{r}{2}}\frac{\beta^i}{N}\partial_i C\|_{L^2(\Sigma_t)}\lesssim\epsilon$. Combining this with (12.117), we thus obtain

$$\|\langle x\rangle^{-\frac{r}{2}}f_1\|_{L^2(\Sigma_t)}\lesssim\epsilon+\epsilon^{\frac{3}{2}}\sum_{Z_k\in\{E_k,L_k\}}\|\partial Z_k\langle D_x\rangle^{s''}\widetilde{\phi}_k\|_{L^2(\Sigma_t)}.$$

Plugging this into (12.129), and then using (12.23) and Proposition 12.25, we thus obtain

$$\|\partial L_k\langle D_x\rangle^{s''}\widetilde{\phi}_k\|_{L^2(\Sigma_t)}\lesssim\epsilon+\epsilon^{\frac{3}{2}}\sum_{Z_k\in\{E_k,L_k\}}\|\partial Z_k\langle D_x\rangle^{s''}\widetilde{\phi}_k\|_{L^2(\Sigma_t)}.$$

Then we use (12.48) and the smallness of ϵ to absorb the $\epsilon^{\frac{3}{2}}\sum_{Z_k\in\{E_k,L_k\}}\|\partial Z_k\langle D_x\rangle^{s''}\widetilde{\phi}_k\|_{L^2(\Sigma_t)}$ terms into the left-hand side: both (12.107) and (12.108) follow immediately. □

Proposition 12.37 *The estimates (12.118)–(12.127) hold for $r\geq 2$.*

Proof *Step 1: Proof of* (12.118) This follows from Proposition 8.12 applied with $v=\langle D_x\rangle^{s''}\widetilde{\phi}_k$ and Proposition 12.25.
Step 2: Proof of (12.119) This follows from Proposition 12.30 and (5.5).
Step 3: Proof of (12.120) This follows from Proposition 12.31, Hölder's inequality and (5.5), (5.1).
Step 4: Proof of (12.121) By Hölder's inequality, (5.5), (5.1), we get

$$\begin{aligned}
&\|\langle x\rangle^{-\frac{r}{2}}\{L_k^{\mu}[(g^{-1})^{\nu\beta}[\langle D_x\rangle^{s''},\varpi]\partial^3_{\mu\nu\beta}\widetilde{\phi}_k]-L_k^t(g^{-1})^{tt}[\langle D_x\rangle^{s''},\varpi]\partial^3_{ttt}\widetilde{\phi}_k\}\|_{L^2(\Sigma_t)}\\
&\quad\lesssim\sum_{(\nu,\beta,\mu)\neq(t,t,t)}\|\langle x\rangle^{-\frac{r}{2}}L_k^{\mu}(g^{-1})^{\nu\beta}\|_{L^\infty(\Sigma_t)}\|[\langle D_x\rangle^{s''},\varpi]\partial^3_{\nu\beta\mu}\widetilde{\phi}_k\|\|_{L^2(\Sigma_t)}\\
&\quad\lesssim\epsilon^{\frac{3}{2}}\sum_{(\nu,\beta,\mu)\neq(t,t,t)}\|[\langle D_x\rangle^{s''},\varpi]\partial^3_{\nu\beta\mu}\widetilde{\phi}_k\|_{L^2(\Sigma_t)}\lesssim\epsilon^{\frac{5}{2}}.
\end{aligned}$$

(Note that $-L_k^t(g^{-1})^{tt}[\langle D_x\rangle^{s''},\varpi]\partial^3_{ttt}\widetilde{\phi}_k$ exactly removes the $(\nu,\beta,\mu)=(t,t,t)$ contribution from the first term.) Note also that in the final inequality, we have used (12.66) from Proposition 12.31 (since there is no $(\nu,\beta,\mu)=(t,t,t)$ term).
Step 5: Proof of (12.122) This follows from (12.70) in Proposition 12.32 and (5.5).
Step 6: Proof of (12.123) First, using Hölder's inequality and (5.1), we obtain

$$\begin{aligned}
&\|\langle x\rangle^{-\frac{r}{2}}s''\delta^{jq}\{(\partial_j(g^{-1})^{\nu\beta})L_k\partial^3_{q\nu\beta}\langle D_x\rangle^{s''-2}\widetilde{\phi}_k+L_k^t(\partial_j(g^{-1})^{tt})\partial_t\langle D_x\rangle^{s''-2}\partial_q(\frac{\Gamma^\lambda\partial_\lambda\widetilde{\phi}_k}{(g^{-1})^{tt}})\}\|_{L^2(\Sigma_t)}\\
&\lesssim\sum_{(\nu,\beta)\neq(t,t)}\|\partial_j(g^{-1})^{\nu\beta}\|_{L^\infty(\Sigma_t)}\|\langle x\rangle^{-\frac{r}{2}}L_k\partial^3_{q\nu\beta}\langle D_x\rangle^{s''-2}\widetilde{\phi}_k\|_{L^2(\Sigma_t)}\\
&\quad+\|\partial_j(g^{-1})^{tt}\|_{L^\infty(\Sigma_t)}\|\langle x\rangle^{-\frac{r}{2}}\{L_k\partial^3_{qtt}\langle D_x\rangle^{s''-2}\widetilde{\phi}_k+L_k^t\partial_t\langle D_x\rangle^{s''-2}\partial_q(\frac{\Gamma^\lambda\partial_\lambda\widetilde{\phi}_k}{(g^{-1})^{tt}})\}\|_{L^2(\Sigma_t)}\\
&\lesssim\epsilon^{\frac{3}{2}}\sum_{(\nu,\beta)\neq(t,t)}\|\langle x\rangle^{-\frac{r}{2}}L_k\partial^3_{q\nu\beta}\langle D_x\rangle^{s''-2}\widetilde{\phi}_k\|_{L^2(\Sigma_t)}\\
&\quad+\epsilon^{\frac{3}{2}}\|\langle x\rangle^{-\frac{r}{2}}\{L_k\partial^3_{qtt}\langle D_x\rangle^{s''-2}\widetilde{\phi}_k+L_k^t\partial_t\langle D_x\rangle^{s''-2}\partial_q(\frac{\Gamma^\lambda\partial_\lambda\widetilde{\phi}_k}{(g^{-1})^{tt}})\}\|_{L^2(\Sigma_t)}.
\end{aligned}$$

To conclude, we apply Proposition 12.34 with $Y=L_k$ (where the bounds (12.79) are given by (5.5), (5.6)): more specifically we use (12.80) for the first term and (12.81) for the second term.
Step 7: Proof of (12.124) This follows from Hölder's inequality, (5.5) and Proposition 12.35. (Note that $L_k^tT^{s''}_{\mathrm{res}}(\varpi(\bar{g}^{-1})^{tt},\ \partial^3_{ttt}\widetilde{\phi}_k)$ exactly removes the contribution in $L_k^{\mu}T^{s''}_{\mathrm{res}}(\varpi(\bar{g}^{-1})^{\nu\beta},\ \partial^3_{\mu\nu\beta}\widetilde{\phi}_k)$ where $(\mu,\nu,\beta)=(t,t,t)$.)

Step 8: Proof of (12.125) By Sobolev embedding, $\langle D_x\rangle^{-2}\partial_q : L^4(\Sigma_t) \to L^2(\Sigma_t)$ is bounded. Thus by Hölder's inequality, Lemma 12.14 and Proposition 12.25,

$$\begin{aligned}&\|s''\delta^{jq}(\partial^2_{tj}(\varpi(\bar{g}^{-1})^{tt}))\langle D_x\rangle^{s''-2}\partial_q(\frac{\Gamma^\lambda\partial_\lambda\widetilde{\phi}_k}{(g^{-1})^{tt}})\|_{L^2(\Sigma_t)}\\ &\lesssim \|\partial^2_{tj}(\varpi(\bar{g}^{-1})^{tt})\|_{L^4(\Sigma_t)}\|\langle D_x\rangle^{s''}\frac{\Gamma^\lambda\partial_\lambda\widetilde{\phi}_k}{(g^{-1})^{tt}}\|_{L^2(\Sigma_t)} \lesssim \epsilon^{\frac{3}{2}}\|\partial\langle D_x\rangle^{s'}\widetilde{\phi}_k\|_{L^2(\Sigma_t)} \lesssim \epsilon^{\frac{5}{2}},\end{aligned}$$

where we have used Hölder's inequality and the condition $0 < s' - s'' < \frac{1}{3}$ to deduce $\|\partial^2_{tj}(g^{-1})^{tt}\|_{L^4(\Sigma_t)} \lesssim \epsilon^{\frac{3}{2}}$ from (5.1).
Step 9: Proof of (12.126) By Hölder's inequality, (5.1), Lemma 12.17, Lemma 12.16 and Proposition 12.25,

$$\begin{aligned}&\|\langle x\rangle^{-\frac{r}{2}}(\partial_t(g^{-1})^{tt})[\langle D_x\rangle^{s''},\varpi]\partial^2_{tt}\widetilde{\phi}_k\|_{L^2(\Sigma_t)}\\ &\lesssim \|\langle x\rangle^{-\frac{r}{2}}(\partial_t(g^{-1})^{tt})\|_{L^\infty(\Sigma_t)}\|\langle D_x\rangle^{s''-2}\partial^2_{tt}\widetilde{\phi}_k\|_{L^2(\Sigma_t)} \lesssim \epsilon^{\frac{5}{2}}.\end{aligned}$$

Step 10: Proof of (12.127) By Hölder's inequality and Corollary 12.10 (with $\theta_1 = s'$, $\theta_2 = s''$, $p = \frac{2}{s'-s''}$), (5.1), (5.5), Lemma 12.16 and Proposition 12.25,

$$\begin{aligned}&\|\langle x\rangle^{-\frac{r}{2}}L^t_k T^{s''}_{\mathrm{res}}(\partial_t(\varpi(\bar{g}^{-1})^{tt}),\ \partial^2_{tt}\widetilde{\phi}_k)\|_{L^2(\Sigma_t)}\\ &\lesssim \|\langle x\rangle^{-\frac{r}{2}}L^t_k\|_{L^\infty(\Sigma_t)}\|\partial_t(\varpi(\bar{g}^{-1})^{tt})\|_{W^{1,\frac{2}{s'-s''}}(\Sigma_t)}\|\langle D_x\rangle^{s'-1}\partial^2_{tt}\widetilde{\phi}_k\|_{L^2(\Sigma_t)}\\ &\lesssim \epsilon^{\frac{3}{2}}\|\langle D_x\rangle^{s'-1}\partial^2_{tt}\widetilde{\phi}_k\|_{L^2(\Sigma_t)} \lesssim \epsilon^{\frac{3}{2}}\|\partial\langle D_x\rangle^{s'}\widetilde{\phi}_k\|_{L^2(\Sigma_t)} \lesssim \epsilon^{\frac{5}{2}}.\end{aligned}$$

□

Proposition 12.38 *Let C be as in (12.114). Then for all $r \geq 1$*

$$\|\langle x\rangle^{-\frac{r}{2}}C\|_{L^2(\Sigma_t)} + \|\langle x\rangle^{-\frac{r}{2}}\partial_i C\|_{L^2(\Sigma_t)} \lesssim \epsilon.$$

Proof We consider the four terms in (12.114) respectively in Steps 1–4 below.
Step 1: $[\Box^1, \langle D_x\rangle^{s''}]\widetilde{\phi}_k$
That $\|\langle x\rangle^{-\frac{r}{2}}[\Box^1, \langle D_x\rangle^{s''}]\widetilde{\phi}_k\|_{L^2(\Sigma_t)} \lesssim \epsilon$ follows from Proposition 12.27 and Proposition 12.25; that $\|\langle x\rangle^{-\frac{r}{2}}\partial_i[\Box^1, \langle D_x\rangle^{s''}]\widetilde{\phi}_k\|_{L^2(\Sigma_t)} \lesssim \epsilon$ is a consequence of Proposition 12.30.
Step 2: $s''\delta^{jq}(\partial_j(\varpi(\bar{g}^{-1})^{tt}))\langle D_x\rangle^{s''-2}\partial_q(\frac{\Gamma^\lambda\partial_\lambda\widetilde{\phi}_k}{(g^{-1})^{tt}})$
By (5.1), $\|\langle x\rangle^{-\frac{r}{2}}\partial_j(\varpi(\bar{g}^{-1})^{tt})\|_{L^\infty(\Sigma_t)} \lesssim \epsilon^{\frac{3}{2}}$. Hence, using Hölder's inequality, Lemma 12.14, the L^2-boundedness of $\langle D_x\rangle^{-2}\partial_q$, and Proposition 12.25, we obtain

$$\|\langle x\rangle^{-\frac{r}{2}}s''\delta^{jq}(\partial_j(\varpi(\bar{g}^{-1})^{tt}))\langle D_x\rangle^{s''-2}\partial_q(\frac{\Gamma^\lambda\partial_\lambda\widetilde{\phi}_k}{(g^{-1})^{tt}})\|_{L^2(\Sigma_t)} \lesssim \epsilon^{\frac{3}{2}}\|\langle D_x\rangle^{s''}(\frac{\Gamma^\lambda\partial_\lambda\widetilde{\phi}_k}{(g^{-1})^{tt}})\|_{L^2(\Sigma_t)} \lesssim \epsilon^{\frac{5}{2}}.$$

To estimate the derivative, we use also the product rule, Hölder's inequality, (5.1), Lemma 12.14 and Proposition 12.25 to obtain

$$\begin{aligned}
&\left\| \langle x\rangle^{-\frac{r}{2}} \partial_i \left(s''\delta^{jq}(\partial_j(\varpi(\bar{g}^{-1})^{tt}))\langle D_x\rangle^{s''-2}\partial_q(\frac{\Gamma^\lambda\partial_\lambda\widetilde{\phi}_k}{(g^{-1})^{tt}}) \right) \right\|_{L^2(\Sigma_t)} \\
&\lesssim \|\partial_j(\varpi(\bar{g}^{-1})^{tt})\|_{L^\infty(\Sigma_t)}\|\langle D_x\rangle^{s''}(\frac{\Gamma^\lambda\partial_\lambda\widetilde{\phi}_k}{(g^{-1})^{tt}})\|_{L^2(\Sigma_t)} \\
&\quad +\|\partial^2_{ij}(\varpi(\bar{g}^{-1})^{tt})\|_{L^\infty(\Sigma_t)}\|\langle D_x\rangle^{s''-1}(\frac{\Gamma^\lambda\partial_\lambda\widetilde{\phi}_k}{(g^{-1})^{tt}})\|_{L^2(\Sigma_t)} \\
&\lesssim \epsilon^{\frac{3}{2}}\|\langle D_x\rangle^{s''}(\frac{\Gamma^\lambda\partial_\lambda\widetilde{\phi}_k}{(g^{-1})^{tt}})\|_{L^2(\Sigma_t)} \lesssim \epsilon^{\frac{5}{2}}.
\end{aligned}$$

Step 3: $(g^{-1})^{tt}[\langle D_x\rangle^{s''},\varpi]\partial^2_{tt}\widetilde{\phi}_k$ To bound $(g^{-1})^{tt}[\langle D_x\rangle^{s''},\varpi]\partial^2_{tt}\widetilde{\phi}_k$ itself, we use Hölder's inequality, the estimate (5.1), Lemma 12.17, Lemma 12.16 and Proposition 12.25 to obtain

$$\begin{aligned}
&\|\langle x\rangle^{-\frac{r}{2}}(g^{-1})^{tt}[\langle D_x\rangle^{s''},\varpi]\partial^2_{tt}\widetilde{\phi}_k\|_{L^2(\Sigma_t)} \\
&\lesssim \|\langle x\rangle^{-\frac{r}{2}}(g^{-1})^{tt}\|_{L^\infty(\Sigma_t)}\|\langle D_x\rangle^{s''-2}\partial^2_{tt}\widetilde{\phi}_k\|_{L^2(\Sigma_t)} \lesssim \epsilon.
\end{aligned}$$

For the derivative, we first use the product rule to distribute the ∂_i derivative and then argue in a similar way as above, i.e.

$$\begin{aligned}
&\|\langle x\rangle^{-\frac{r}{2}}\partial_i\{(g^{-1})^{tt}[\langle D_x\rangle^{s''},\varpi]\partial^2_{tt}\widetilde{\phi}_k\}\|_{L^2(\Sigma_t)} \\
&\lesssim \|\langle x\rangle^{-\frac{r}{2}}\partial_i(g^{-1})^{tt}\|_{L^\infty(\Sigma_t)}\|\langle D_x\rangle^{s''-2}\partial^2_{tt}\widetilde{\phi}_k\|_{L^2(\Sigma_t)} \\
&\quad + \|\langle x\rangle^{-\frac{r}{2}}(g^{-1})^{tt}\|_{L^\infty(\Sigma_t)}\|\langle D_x\rangle^{s''-1}\partial^2_{tt}\widetilde{\phi}_k\|_{L^2(\Sigma_t)} \lesssim \epsilon.
\end{aligned}$$

Step 4: $T^{s''}_{\mathrm{res}}(\varpi(\bar{g}^{-1})^{tt},\,\partial^2_{tt}\widetilde{\phi}_k)$ For $T^{s''}_{\mathrm{res}}(\varpi(\bar{g}^{-1})^{tt},\,\partial^2_{tt}\widetilde{\phi}_k)$ itself, we use Corollary 12.10, the estimate (5.1), Lemma 12.16 and Proposition 12.25 to obtain

$$\|T^{s''}_{\mathrm{res}}(\varpi(\bar{g}^{-1})^{tt},\,\partial^2_{tt}\widetilde{\phi}_k)\|_{L^2(\Sigma_t)} \lesssim \|\varpi(\bar{g}^{-1})^{tt}\|_{W^{1,\infty}(\Sigma_t)}\|\langle D_x\rangle^{s''-1}\partial^2_{tt}\widetilde{\phi}_k\|_{L^2(\Sigma_t)} \lesssim \epsilon.$$

For the derivative, we first use product rule to obtain

$$\begin{aligned}
&\|\partial_i T^{s''}_{\mathrm{res}}(\varpi(\bar{g}^{-1})^{tt},\,\partial^2_{tt}\widetilde{\phi}_k)\|_{L^2(\Sigma_t)} \\
&\lesssim \|T^{s''}_{\mathrm{res}}(\partial_i(\varpi(\bar{g}^{-1})^{tt}),\,\partial^2_{tt}\widetilde{\phi}_k)\|_{L^2(\Sigma_t)} + \|T^{s''}_{\mathrm{res}}(\varpi(\bar{g}^{-1})^{tt},\,\partial^3_{itt}\widetilde{\phi}_k)\|_{L^2(\Sigma_t)}.
\end{aligned} \tag{12.130}$$

The first term in (12.130) can be estimated using Corollary 12.10, the estimate (5.1), Lemma 12.16 and Proposition 12.25 to obtain

$$
\begin{aligned}
&\|T^{s''}_{\mathrm{res}}(\partial_i(\varpi(\bar{g}^{-1})^{tt}),\ \partial^2_{tt}\widetilde{\phi}_k)\|_{L^2(\Sigma_t)}\\
&\quad\lesssim \|\partial_i(\varpi(\bar{g}^{-1})^{tt})\|_{W^{1,\frac{2}{s'-s''}}(\Sigma_t)}\|\langle D_x\rangle^{s'-1}\partial^2_{tt}\widetilde{\phi}_k\|_{L^2(\Sigma_t)}\lesssim\epsilon.
\end{aligned}
$$

The second term in (12.130) can also be estimated as follows using Corollary 12.10, the estimate (5.1), Lemma 12.16 and Proposition 12.25

$$
\|T^{s''}_{\mathrm{res}}(\varpi(\bar{g}^{-1})^{tt},\ \partial^3_{itt}\widetilde{\phi}_k)\|_{L^2(\Sigma_t)}\lesssim\|\varpi(\bar{g}^{-1})^{tt}\|_{W^{2,\frac{2}{s'-s''}}(\Sigma_t)}\|\langle D_x\rangle^{s'-1}\partial^2_{tt}\widetilde{\phi}_k\|_{L^2(\Sigma_t)}\lesssim\epsilon.
$$

□

12.8 Control of $E_k\widetilde{\phi}_k$ and $L_k\widetilde{\phi}_k$ in $H^{1+s'}$

We turn to the estimates that are analogous to (12.107) and (12.108) but with vector fields and fractional derivatives taken in a slightly different order.

Proposition 12.39 *The following estimates are satisfied:*

$$
\sum_{Z_k\in\{E_k,L_k\}}\|Z_k\widetilde{\phi}_k\|_{H^{1+s'}(\Sigma_t)}\lesssim\epsilon, \tag{12.131}
$$

$$
\sum_{Z_k\in\{E_k,L_k\}}\|\partial_t Z_k\widetilde{\phi}_k\|_{H^{s'}(\Sigma_t)}\lesssim\epsilon. \tag{12.132}
$$

Proof Take $Z_k\in\{L_k,E_k\}$: the goal is to show $\|\langle D_x\rangle^{s''}(\partial_\nu Z_k\widetilde{\phi}_k)\|_{L^2(\Sigma_t)}\lesssim\epsilon$. We write the following identity

$$
\begin{aligned}
\langle D_x\rangle^{s''}(\partial_\nu Z_k\widetilde{\phi}_k) &= \langle D_x\rangle^{s''}\left((\partial_\nu Z^i_k)\partial_i\widetilde{\phi}_k\right)+\langle D_x\rangle^{s''}\left(Z^i_k\partial_\nu\partial_i\widetilde{\phi}_k\right)\\
&=\langle D_x\rangle^{s''}\left((\partial_\nu Z^i_k)\partial_i\widetilde{\phi}_k\right)+\varpi Z_k\partial_\nu\langle D_x\rangle^{s''}\widetilde{\phi}_k+[\langle D_x\rangle^{s''},\varpi Z^i_k]\partial_\nu\partial_i\widetilde{\phi}_k\\
&=\underbrace{\langle D_x\rangle^{s''}\left(\varpi(\partial_\nu Z^i_k)\partial_i\widetilde{\phi}_k\right)}_{I}+\underbrace{\varpi\partial_\nu Z_k\langle D_x\rangle^{s''}\widetilde{\phi}_k}_{II}-\underbrace{\varpi(\partial_\nu Z^i_k)\partial_i\langle D_x\rangle^{s''}\widetilde{\phi}_k}_{III}\\
&\quad+\underbrace{[\langle D_x\rangle^{s''},\varpi Z^i_k]\partial_\nu\partial_i\widetilde{\phi}_k}_{IV}.
\end{aligned}
$$

We will treat each term separately.

For I, we use Lemma 12.6

$$
\begin{aligned}
\|I\|_{L^2(\Sigma_t)}&\lesssim\|\varpi(\partial Z^i_k)\|_{L^\infty(\Sigma_t)}\|\partial_i\widetilde{\phi}_k\|_{H^{s'}(\Sigma_t)}+\|\varpi(\partial Z^i_k)\|_{H^{s'}(\Sigma_t)}\|\partial_i\widetilde{\phi}_k\|_{L^\infty(\Sigma_t)}\\
&\lesssim\epsilon^{\frac{5}{2}}+\|\varpi(\partial Z^i_k)\|_{H^1(\Sigma_t)}\|\partial_i\widetilde{\phi}_k\|_{L^\infty(\Sigma_t)}\lesssim\epsilon,
\end{aligned}
$$

where for the second inequality we used (5.6) and Proposition 12.25 and for the last one we used (5.6), (5.7) and the bootstrap assumption (4.12c).

For II, we use (12.107) and (12.108) and we get directly

$$\|II\|_{L^2(\Sigma_t)} \lesssim \epsilon.$$

For III, we use (5.6) and Proposition 12.25 to obtain

$$\|IV\|_{L^2(\Sigma_t)} \lesssim \|\partial Z_k^i\|_{L^\infty(\Sigma_t\cap B(0,3R))}\|\partial_i \langle D_x\rangle^{s''}\widetilde{\phi}_k\|_{L^2(\Sigma_t)} \lesssim \epsilon.$$

For IV, we use Proposition 12.7 with $f = Z_k^i \varpi$, $h = \partial_\nu \partial_i \widetilde{\phi}_k$, $\theta = s''$ and $p = \infty$:

$$\begin{aligned}\|IV\|_{L^2(\Sigma_t)} &\lesssim \|Z_k^i\|_{W^{1,\infty}(\Sigma_t\cap B(0,3R))}\|\langle D_x\rangle^{-1}\partial_i\partial_\nu\langle D_x\rangle^{s''}\widetilde{\phi}_k\|_{L^2(\Sigma_t)}\\ &\lesssim \|\partial\langle D_x\rangle^{s''}\widetilde{\phi}_k\|_{L^2(\Sigma_t)} \lesssim \epsilon,\end{aligned}$$

where for the second inequality we have used (5.5), (5.6) and the L^2-boundedness of the operator $\langle D_x\rangle^{-1}\partial_i$, and for the last inequality we have used Proposition 12.25. □

13 Energy estimates for ϕ_{reg}

In this section, we prove energy estimates for ϕ_{reg}. We will prove that ϕ_{reg} is bounded in $H^{2+s'}(\Sigma)$, uniformly in δ. Since ϕ_{reg} is initially more regular, the proof of the energy estimates for ϕ_{reg} is also considerably easier than the higher order energy estimates for $\widetilde{\phi}_k$.

We begin with the energy estimates for up to the second derivative of ϕ_{reg}. These bounds follow easily from the general energy estimates derived in Sects. 7 and 8.3.

Proposition 13.1 *The following estimates hold:*

$$\sup_{t\in[0,T_B)} (\|\partial\phi_{reg}\|_{L^2(\Sigma_t)} + \|\partial^2\phi_{reg}\|_{L^2(\Sigma_t)}) \lesssim \epsilon, \tag{13.1}$$

$$\max_k \sup_{u_k\in\mathbb{R}} \sum_{Z_k\in\{L_k,E_k\}} (\|Z_k\partial_x\phi_{reg}\|_{L^2(C^k_{u_k}([0,T_B)))} + \|Z_k\phi_{reg}\|_{L^2(C^k_{u_k}([0,T_B)))}) \lesssim \epsilon. \tag{13.2}$$

Proof We recall that ϕ_{reg} is compactly supported on $B(0,R)$ and satisfies $\Box_g\phi_{reg} = 0$. Thus we can apply Proposition 7.3 and Proposition 8.9 simultaneously with $U_0 = -\infty$ and $U_1 = +\infty$ to get

$$\sup_{t\in[0,T_B)} (\|\partial\phi_{reg}\|_{L^2(\Sigma_t)} + \|\partial\partial_x\phi_{reg}\|_{L^2(\Sigma_t)}) \lesssim \epsilon, \tag{13.3}$$

$$\max_k \sup_{u_k\in\mathbb{R}} \sum_{Z_k\in\{L_k,E_k\}} (\|Z_k\partial_x\phi_{reg}\|_{L^2(C^k_{u_k}([0,T_B)))} + \|Z_k\phi_{reg}\|_{L^2(C^k_{u_k}([0,T_B)))}) \lesssim \epsilon, \tag{13.4}$$

after using (4.2) to bound the initial data term.

Compared with the desired estimates (13.1) and (13.2), the only thing missing is a bound on $\|\partial_{tt}^2\phi_{reg}\|_{L^2(\Sigma_t)}$. Using the wave equation $\Box_g\phi_{reg} = 0$, we can rewrite $\partial_{tt}^2\phi_{reg}$ by (12.9) as terms which can be bounded using (13.3) above together with (5.1) and (5.4), yielding the desired estimate. □

We then turn to the energy estimates for the $2+s'$ derivatives of ϕ_{reg}. For this we will also use the commutator estimates with fractional derivatives proven in Sect. 12.

Proposition 13.2 *The following estimate holds for all $t \in [0, T_B)$:*

$$\|\langle D_x\rangle^{s'}\partial^2\phi_{reg}\|_{H^{s'}(\Sigma_t)} \lesssim \epsilon. \tag{13.5}$$

Proof *Step 1: Using Corollary 7.5* We recall that ϕ_{reg} is compactly supported on $B(0, R)$ and satisfies $\Box_g\phi_{reg} = 0$. We apply the energy estimates in Corollary 7.5 with $v = \langle D_x\rangle^{s'}\partial_i\phi_{reg}$, $f_2 = 0$ and

$$f_1 := \Box_g\langle D_x\rangle^{s'}\partial_i\phi_{reg} = [\Box_g, \langle D_x\rangle^{s'}]\partial_i\phi_{reg} + \langle D_x\rangle^{s'}[\Box_g, \partial_i]\phi_{reg}$$

to get that for every $T \in (0, T_B)$,

$$\begin{aligned}
&\sup_{t\in[0,T)} \|\langle x\rangle^{-r+2\alpha}\partial\langle D_x\rangle^{s'}\partial_i\phi_{reg}\|^2_{L^2(\Sigma_t)} \\
&\lesssim \|\langle x\rangle^{-\frac{r}{2}}\partial\langle D_x\rangle^{s'}\partial_i\phi_{reg}\|^2_{L^2(\Sigma_0)} + \int_0^T (\|\langle x\rangle^{-\frac{r}{2}}([\Box_g, \langle D_x\rangle^{s'}]\partial_i\phi_{reg} \\
&\quad + \langle D_x\rangle^{s'}[\Box_g, \partial_i]\phi_{reg})\|^2_{L^2(\Sigma_\tau)} \\
&\lesssim \epsilon^2 + \int_0^T \|\langle x\rangle^{-\frac{r}{2}}[\Box_g, \langle D_x\rangle^{s'}]\partial_i\phi_{reg}\|^2_{L^2(\Sigma_\tau)}\,d\tau \\
&\quad + \int_0^T \|\langle x\rangle^{-\frac{r}{2}}\langle D_x\rangle^{s'}[\Box_g, \partial_i]\phi_{reg}\|^2_{L^2(\Sigma_\tau)}\,d\tau,
\end{aligned} \tag{13.6}$$

where in the last line we have controlled the data term using (4.2).

Step 2: Bounding $[\Box_g, \langle D_x\rangle^{s'}]\partial_i\phi_{reg}$ Recall the decomposition $\Box_g = -\Box^1 + \Box^2$ from (12.6). By (12.47) (applied to $v = \partial_i\phi_{reg}$), we obtain

$$\|\langle x\rangle^{-\frac{r}{2}}[\Box^1, \langle D_x\rangle^{s'}]\partial_i\phi_{reg}\|_{L^2(\Sigma_t)} \lesssim \|\partial\langle D_x\rangle^{s'}\partial_i\phi_{reg}\|_{L^2(\Sigma_t)}.$$

By (12.45) (applied to $v = \partial_i\phi_{reg}$), we get

$$\|\langle x\rangle^{-\frac{r}{2}}[\Box^2, \langle D_x\rangle^{s'}]\partial_i\phi_{reg}\|_{L^2(\Sigma_t)} \lesssim \|\partial\langle D_x\rangle^{s'}\partial\phi_{reg}\|_{L^2(\Sigma_t)} + \|[\Box_g, \partial_i]\phi_{reg}\|_{L^2(\Sigma_t)}.$$

Using Lemma 8.5 together with the metric estimates in (5.1), we have, using also (13.1):

$$\|[\Box_g, \partial_i]\phi_{reg}\|_{L^2(\Sigma_t)} \lesssim \epsilon.$$

Putting these bounds together, we obtain

$$\|\langle x\rangle^{-\frac{r}{2}}[\Box_g, \langle D_x\rangle^{s'}]\partial_i\phi_{reg}\|_{L^2(\Sigma_t)} \lesssim \epsilon + \|\partial\langle D_x\rangle^{s'}\partial_i\phi_{reg}\|_{L^2(\Sigma_t)}. \tag{13.7}$$

Step 3: Bounding $\partial_i[\Box_g, \langle D_x\rangle^{s'}]\phi_{reg}$ Using again the decomposition in (12.6), we have

$$\langle D_x\rangle^{s'}[\Box_g, \partial_i]\phi_{reg} = \underbrace{-\langle D_x\rangle^{s'}[(\partial_i(g^{-1})^{\nu\beta})\partial^2_{\nu\beta}\phi_{reg}]}_{=:I} + \underbrace{\langle D_x\rangle^{s'}[(\partial_i\Gamma^\lambda)(\partial_\lambda\phi_{reg})]}_{=:II}. \tag{13.8}$$

By Hölder's inequality, Lemma 12.6, (5.1) and (5.3),

$$\begin{aligned}\|I\|_{L^2(\Sigma_t)} &\lesssim \|\langle D_x\rangle^{s'}(\partial_i(g^{-1})^{\nu\beta})\|_{L^\infty(\Sigma_t)}\|\partial^2_{\nu\beta}\phi_{reg}\|_{L^2(\Sigma_t)}\\ &\quad + \|\partial_i(g^{-1})^{\nu\beta}\|_{L^\infty(\Sigma_t)}\|\langle D_x\rangle^{s'}\partial^2_{\nu\beta}\phi_{reg}\|_{L^2(\Sigma_t)}\\ &\lesssim \epsilon^{\frac{3}{2}}(\|\partial^2\phi_{reg}\|_{L^2(\Sigma_t)} + \|\langle D_x\rangle^{s'}\partial^2\phi_{reg}\|_{L^2(\Sigma_t)})\\ &\lesssim \epsilon^{\frac{3}{2}}\|\langle D_x\rangle^{s'}\partial^2\phi_{reg}\|_{L^2(\Sigma_t)}\\ &\lesssim \epsilon^{\frac{3}{2}}\|\langle D_x\rangle^{s'}\partial\partial_x\phi_{reg}\|_{L^2(\Sigma_t)},\end{aligned} \tag{13.9}$$

where in the last line we used $\|\langle D_x\rangle^{s'}\partial^2\phi_{reg}\|_{L^2(\Sigma_t)} \lesssim \|\langle D_x\rangle^{s'}\partial\partial_x\phi_{reg}\|_{L^2(\Sigma_t)}$, which in turn follow from the wave equation. More precisely, since $\Box_g\phi_{reg} = 0$, we use (12.9), Lemma 12.6, (5.1) and (5.4) to obtain

$$\begin{aligned}&\|\langle D_x\rangle^{s'}(\partial^2_{tt}\phi_{reg})\|_{L^2(\Sigma_t)}\\ &\quad\lesssim \|\langle D_x\rangle^{s'}(\frac{\Box_g\phi_{reg}}{(g^{-1})^{tt}} - \breve{g}^{i\lambda}\partial^2_{i\lambda}\phi_{reg} + \frac{\Gamma^\lambda\partial_\lambda\phi_{reg}}{(g^{-1})^{tt}})\|_{L^2(\Sigma_t)}\\ &\quad\lesssim \|\partial_i\langle D_x\rangle^{s'}(\breve{g}^{i\lambda}\partial_\lambda\phi_{reg})\|_{L^2(\Sigma_t)} + \|\langle D_x\rangle^{s'}[(\partial_i\breve{g}^{i\lambda})(\partial_\lambda\phi_{reg})]\|_{L^2(\Sigma_t)}\\ &\qquad + \|\langle D_x\rangle^{s'}(\frac{\Gamma^\lambda\partial_\lambda\phi_{reg}}{(g^{-1})^{tt}})\|_{L^2(\Sigma_t)}\\ &\quad\lesssim \|\langle D_x\rangle^{s'}\partial\partial_x\phi_{reg}\|_{L^2(\Sigma_t)}.\end{aligned} \tag{13.10}$$

The term II in (13.8) can be treated similarly. Using Lemma 12.6, (5.4), we obtain

$$\begin{aligned}\|II\|_{L^2(\Sigma_t)} &\lesssim \|\langle D_x\rangle^{s'}(\varpi\partial_i\Gamma^\lambda)\|_{L^2(\Sigma_t)}\|\partial_\lambda\phi_{reg}\|_{L^\infty(\Sigma_t)}\\ &\quad + \|\varpi\partial_i\Gamma^\lambda\|_{L^\infty(\Sigma_t)}\|\langle D_x\rangle^{s'}\partial_\lambda\phi_{reg}\|_{L^2(\Sigma_t)}\\ &\lesssim \epsilon^{\frac{3}{2}}\|\partial_\lambda\phi_{reg}\|_{L^\infty(\Sigma_t)} + \epsilon^{\frac{3}{2}}\|\langle D_x\rangle^{s'}\partial_\lambda\phi_{reg}\|_{L^2(\Sigma_t)}\\ &\lesssim \epsilon^{\frac{3}{2}}\|\partial\phi_{reg}\|_{L^\infty(\Sigma_t)} + \epsilon^{\frac{3}{2}}\|\partial\phi_{reg}\|_{L^2(\Sigma_t)} + \epsilon^{\frac{3}{2}}\|\partial\langle D_x\rangle^{s'}\partial_i\phi_{reg}\|_{L^2(\Sigma_t)}\\ &\lesssim \epsilon^{\frac{9}{4}} + \epsilon^{\frac{3}{2}}\|\partial\langle D_x\rangle^{s'}\partial_i\phi_{reg}\|_{L^2(\Sigma_t)},\end{aligned} \tag{13.11}$$

where we controlled $\|\langle D_x\rangle^{s'}\partial_\lambda\phi_{reg}\|_{L^2(\Sigma_t)}$ by interpolating between $\|\partial_\lambda\phi_{reg}\|_{L^2(\Sigma_t)}$ and $\|\partial_\lambda\langle D_x\rangle^{s'}\partial_i\phi_{reg}\|_{L^2(\Sigma_t)}$ (for instance using Plancherel's theorem), and finally we used (4.12c) to control both $\|\partial\phi_{reg}\|_{L^\infty(\Sigma_t)}$ and $\|\partial\phi_{reg}\|_{L^2(\Sigma_t)}$

Putting together (13.8), (13.9) and (13.11), we obtain

$$\|\langle x\rangle^{-\frac{r}{2}}\langle D_x\rangle^{s'}[\Box_g,\partial_i]\phi_{reg}\|_{L^2(\Sigma_t)} \lesssim \epsilon + \|\partial\langle D_x\rangle^{s'}\partial_i\phi_{reg}\|_{L^2(\Sigma_t)}. \tag{13.12}$$

Step 4: Putting everything together Combining the estimates in (13.6), (13.7) and (13.12), we obtain

$$\sup_{t\in[0,T)}\|\langle x\rangle^{-r-2\alpha}\partial\langle D_x\rangle^{s'}\partial_x\phi_{reg}\|^2_{L^2(\Sigma_t)} \lesssim \epsilon^2 + \int_0^T \|\partial\langle D_x\rangle^{s'}\partial_x\phi_{reg}\|^2_{L^2(\Sigma_\tau)}\,d\tau.$$

By Proposition 12.18 (applied to $v = \partial_x\phi_{reg}$), we can strengthen the weights on the left-hand side, i.e.

$$\sup_{t\in[0,T)}\|\partial\langle D_x\rangle^{s'}\partial_x\phi_{reg}\|^2_{L^2(\Sigma_t)} \lesssim \epsilon^2 + \int_0^T \|\partial\langle D_x\rangle^{s'}\partial_x\phi_{reg}\|^2_{L^2(\Sigma_\tau)}\,d\tau.$$

By Grönwall's inequality, we obtain

$$\sup_{t\in[0,T_B)}\|\partial\langle D_x\rangle^{s'}\partial_x\phi_{reg}\|^2_{L^2(\Sigma_t)} \lesssim \epsilon.$$

Combining this with (13.10) yields the desired conclusion (13.5). □

14 Conclusion of the proof of Theorem 4.3

In this section, we conclude the proof of Theorem 4.3. Theorem 4.3 consists of parts 1, 2, and 3, which will be proven, respectively, in Proposition 14.1, Proposition 14.2 and Proposition 14.3. (For part 2, we recall the definition of $\mathcal{E}$ in Definition 4.1.)

Proposition 14.1 (Statement 1 of Theorem 4.3) *There exists $C = C(s', s'', R, \kappa_0) > 0$ such that (4.8)– (4.11d) hold with $C\epsilon$ in place of $\epsilon^{\frac{3}{4}}$.*

Proof We look at each of the bootstrap assumptions (4.8)– (4.11d). We point out the precise locations in the earlier sections which improve these bounds from $\epsilon^{\frac{3}{4}}$ to $C\epsilon$.

- Improvement of (4.8): it follows directly from (13.5) in Proposition 13.2 and (13.1) in Proposition 13.1.
- Improvement of (4.9a): it follows from (9.1) and (9.2) in Proposition 9.1, using also Proposition 5.3 to address the commutator term involving $[\partial, Z_k]$.
- Improvement of (4.9b): it follows directly from (9.3) in Proposition 9.1.
- Improvement of (4.9c): it follows directly from (12.42) in Proposition 12.25.
- Improvement of (4.9d): it was already stated and proven in Proposition 11.7.

- Improvement of (4.10a): the $\|\partial\widetilde{\phi}_k\|_{L^2(\Sigma_t\cap S^k_{2\delta})}$ term follows directly from (10.6) in Proposition 10.2. The $\|Z_k\partial\widetilde{\phi}_k\|_{L^2(\Sigma_t\cap S^k_{2\delta})}$ term follows (10.7) and the commutation of Z_k with ∂, using (10.6), (5.5), (5.6).
- Improvement of (4.10b): this follows directly from (10.9) in Proposition 10.3.
- Improvement of (4.11a): it follows directly from (13.2) in Proposition 13.1.
- Improvement of (4.11b): the first term in (4.11b) is bounded by (10.9) in Proposition 10.3, while the second term in (4.11b) is bounded by (9.1) in Proposition 9.1.
- Improvement of (4.11c): the second term in (4.11c) is bounded by (9.2) in Proposition 9.1.
  To control the first term in (4.11c), i.e. to bound $\sup_{u_k\in\mathbb{R}}\|L_k\partial_x\widetilde{\phi}_k\|_{L^2(C^k_{u_k}([0,T_B]))}$, we first notice that it suffices to bound $\sup_{u_k\in\mathbb{R}}\|\partial_x L_k\widetilde{\phi}_k\|_{L^2(C^k_{u_k}([0,T_B]))}$ since the commutator can be estimated with the help of Proposition 5.3 and (4.12c). This latter term can in turn be bounded by $\|E_kL_k\widetilde{\phi}_k\|_{L^2(C^k_{u_k}([0,T_B]))}$ and $\|X_kL_k\widetilde{\phi}_k\|_{L^2(C^k_{u_k}([0,T_B]))}$ thanks to Lemma 5.4 (and the support properties of $\widetilde{\phi}_k$). The estimate for $\|E_kL_k\widetilde{\phi}_k\|_{L^2(C^k_{u_k}([0,T_B]))}$ follows directly from (9.2), while the estimate for $\|X_kL_k\widetilde{\phi}_k\|_{L^2(C^k_{u_k}([0,T_B]))}$ follows from using the wave equation (8.8) and applying the estimates from Proposition 5.2, Proposition 5.5, (4.12c) and (9.2).
- Improvement of (4.11d): it follows directly from (9.3) in Proposition 9.1. □

Proposition 14.2 (Statement 2 of Theorem 4.3) *The following estimate holds:*

$$\mathcal{E}\lesssim\epsilon.$$

Proof $\mathcal{E}$ is composed of a sum of terms given in Definition 4.1. We treat each of these terms one by one. Most of these bounds have already been stated in the proof of Proposition 14.1.

- $\|\partial\langle D_x\rangle^{s'}\widetilde{\phi}_k\|_{L^2(\Sigma_t)}\lesssim\epsilon$: already obtained with the improvement of (4.9c).
- $\|E_k\partial\widetilde{\phi}_k\|_{L^2(\Sigma_t)}\lesssim\epsilon$: already obtained with the improvement of (4.9a).
- $\|\partial E_k\langle D_x\rangle^{s''}\widetilde{\phi}_k\|_{L^2(\Sigma_t)}\lesssim\epsilon$: this follows directly from (12.108) in Proposition 12.36.
- $\delta^{\frac12}\|\partial^2\widetilde{\phi}_k\|_{L^2(\Sigma_t)}\lesssim\epsilon$: already obtained with the improvement of (4.9b).
- $\delta^{\frac12}\|\partial E_k\partial\widetilde{\phi}_k\|_{L^2(\Sigma_t)}\lesssim\epsilon$: the bound $\delta^{\frac12}\|\partial E_k\partial_x\widetilde{\phi}_k\|_{L^2(\Sigma_t)}\lesssim\epsilon$ (i.e. the particular case where the first derivative is spatial) follows directly from (11.24) in Proposition 11.7. To address the remaining term $\delta^{\frac12}\|\partial E_k\partial_t\widetilde{\phi}_k\|_{L^2(\Sigma_t)}$, we use (2.3) as

$$\begin{aligned}\partial E_k\partial_t\widetilde{\phi}_k&=\partial E_k\left(N(\vec{n}\widetilde{\phi}_k+\beta^i\partial_i\widetilde{\phi}_k)\right)=\partial\left((E_kN)(\vec{n}\widetilde{\phi}_k+\beta^i\partial_i\widetilde{\phi}_k)\right)\\&\quad+\partial\left(N(E_k\vec{n}\widetilde{\phi}_k+\beta^iE_k\partial_i\widetilde{\phi}_k)\right)\\&=\underbrace{(\partial E_kN)(\vec{n}\widetilde{\phi}_k+\beta^i\partial_i\widetilde{\phi}_k)}_{I}+\underbrace{(E_kN)\partial(\vec{n}\widetilde{\phi}_k+\beta^i\partial_i\widetilde{\phi}_k)}_{II}\\&\quad+\underbrace{(\partial N)(E_k\vec{n}\widetilde{\phi}_k+\beta^iE_k\partial_i\widetilde{\phi}_k)}_{III}\end{aligned}$$

$$+ \underbrace{N(\partial E_k \vec{n}\widetilde{\phi}_k + \beta^i \partial E_k \partial_i \widetilde{\phi}_k)}_{IV} + \underbrace{(N\partial\beta^i) E_k \partial_i \widetilde{\phi}_k}_{V}.$$

We treat each term individually:

- Term I: by (5.1), (2.3), and the bootstrap assumption (4.12c), we have

$$\begin{aligned}\delta^{\frac{1}{2}} \|I\|_{L^2(\Sigma_t)} &\lesssim \delta^{\frac{1}{2}} \|\partial E_k N\|_{L^2(B(0,R))} \|\vec{n}\widetilde{\phi}_k + \beta^i \partial_i \widetilde{\phi}_k\|_{L^\infty(\Sigma_t)} \\ &\lesssim \delta^{\frac{1}{2}} \|\partial E_k N\|_{L^2(B(0,R))} \|\partial \widetilde{\phi}_k\|_{L^\infty(\Sigma_t)} \lesssim \delta^{\frac{1}{2}} \epsilon^{\frac{9}{4}} \lesssim \epsilon.\end{aligned}$$

- Term II: by (5.1), (2.3) and (9.1), (9.3) we have

$$\begin{aligned}\delta^{\frac{1}{2}} \|II\|_{L^2(\Sigma_t)} &\lesssim \delta^{\frac{1}{2}} \|E_k N\|_{L^\infty(B(0,R))} \|\partial(\vec{n}\widetilde{\phi}_k + \beta^i \partial_i \widetilde{\phi}_k)\|_{L^2(\Sigma_t)} \\ &\lesssim \delta^{\frac{1}{2}} \epsilon^{\frac{5}{2}} \delta^{-\frac{1}{2}} \lesssim \epsilon.\end{aligned}$$

- Term III: by (5.1), (2.3) and (9.1), (9.3) we have

$$\begin{aligned}\delta^{\frac{1}{2}} \|III\|_{L^2(\Sigma_t)} &\lesssim \delta^{\frac{1}{2}} \|\partial N\|_{L^\infty(B(0,R))} \|E_k \vec{n}\widetilde{\phi}_k + \beta^i E_k \partial_i \widetilde{\phi}_k\|_{L^2(\Sigma_t)} \\ &\lesssim \delta^{\frac{1}{2}} \epsilon^{\frac{5}{2}} \delta^{-\frac{1}{2}} \lesssim \epsilon.\end{aligned}$$

- Term IV: by (5.1) and (11.24) in Proposition 11.7, we have

$$\delta^{\frac{1}{2}} \|IV\|_{L^2(\Sigma_t)} \lesssim \delta^{\frac{1}{2}} (\|\partial E_k \vec{n}\widetilde{\phi}_k\|_{L^2(\Sigma_t)} + \|\partial E_k \partial_x \widetilde{\phi}_k\|_{L^2(\Sigma_t)}) \lesssim \delta^{\frac{1}{2}} \epsilon \delta^{-\frac{1}{2}} \lesssim \epsilon.$$

- Term V: by (5.1), (5.5) and (9.3) we have

$$\delta^{\frac{1}{2}} \|V\|_{L^2(\Sigma_t)} \lesssim \delta^{\frac{1}{2}} \|N\partial\beta^i\|_{L^\infty(B(0,R)} \|E_k \partial_x \widetilde{\phi}_k\|_{L^2(\Sigma_t)} \lesssim \delta^{\frac{1}{2}} \epsilon \delta^{-\frac{1}{2}} \lesssim \epsilon.$$

- $\delta^{-\frac{1}{2}} \|\partial \widetilde{\phi}_k\|_{L^2(\Sigma_t \cap S^k_{2\delta})} \lesssim \epsilon$ and $\delta^{-\frac{1}{2}} \|E_k \partial \widetilde{\phi}_k\|_{L^2(\Sigma_t \cap S^k_{2\delta})} \lesssim \epsilon$: already obtained with the improvement of (4.10a).
- $\|\partial^2 \widetilde{\phi}_k\|_{L^2(\Sigma_t \setminus S^k_\delta)} \lesssim \epsilon$: already obtained in the improvement of (4.10b).
- $\|\partial^2 \langle D_x \rangle^{s'} \phi_{reg}\|_{L^2(\Sigma_t)} \lesssim \epsilon$: already obtained in the improvement of (4.8).

□

Proposition 14.3 *(Statement 3 of Theorem 4.3) The estimates (4.13a)–(4.16) are satisfied.*

Proof We prove each of the estimates (4.13a)–(4.16) individually. Some of these estimates are already obtained in the proof of Proposition 14.1 or Proposition 14.2.

- Proof of (4.13a): it follows directly from (13.5) in Proposition 13.2 and (13.1) in Proposition 13.1.
- Proof of (4.13b): it was already proven with the improvement of (4.9c).

- Proof of (4.13c): It follows directly from (12.131) and (12.132) in Proposition 12.39.
- Proof of (4.14): the first inequality $\|\partial^2\widetilde{\phi}_k\|_{L^2(\Sigma_t)} \lesssim \epsilon \cdot \delta^{-\frac{1}{2}}$ was already proven with the improvement of (4.9b). The inequality $\sum_{\substack{Y_k^{(1)},Y_k^{(2)},Y_k^{(3)}\in\{X_k,E_k,L_k\} \\ \exists i, Y_k^{(i)}\neq X_k}} \|Y_k^{(1)}Y_k^{(2)}Y_k^{(3)}\widetilde{\phi}_k\|_{L^2(\Sigma_t)} \lesssim \epsilon \cdot \delta^{-\frac{1}{2}}$ follows directly from (11.24) in Proposition 11.7.
- Proof of (4.15): this follows directly from (11.25) in Proposition 11.8.
- Proof of (4.16): it was already proven with the improvement of (4.10b).

□

15 Lipschitz estimates and improved Hölder bounds for ϕ

In this section, we prove Lipschitz estimates for $\widetilde{\phi}_k$, as well as improved $C^{0,\frac{s''}{2}}$ Hölder estimates for $\partial\phi_{reg}$ and for $\partial\widetilde{\phi}_k$ away from the singular zone.

While proving the Lipschitz estimates, we will prove stronger Besov type estimates (recall Sect. 1.1.5 and [25, Section 1.1.4]). When combined with the energy estimates that we have already obtained, the result in this section improves the bootstrap assumptions (4.12a), (4.12b) and (4.12c).

The following is the main result of this section (recall the definition of $\mathcal{E}$ in Definition 4.1):

Theorem 15.1 (Main Lipschitz and improved Hölder estimates) *Let $\rho_k(u_k,\theta_k,t) = \widetilde{\rho}(\frac{u_k}{\delta})$ be a cutoff function, where $\widetilde{\rho}:\mathbb{R}\to[0,1]$ is smooth function with $\widetilde{\rho}\equiv 0$ on $[2,\infty)$, and $\widetilde{\rho}\equiv 1$ on $(-\infty,1]$.*

The following estimates hold for all $t\in[0,T_B)$ (recall the definition of the Besov space $B^{u_k,u_{k'}}_{\infty,1}(\Sigma_t)$ in Definition 3.10):

1. *For $k\in\{1,2,3\}$, $\partial\widetilde{\phi}_k$ obeys the following estimate near the singular zone for any $k'\neq k$:*

$$\|\rho_k\cdot\partial\widetilde{\phi}_k\|_{B^{u_k,u_{k'}}_{\infty,1}(\Sigma_t)} \lesssim \mathcal{E}. \tag{15.1}$$

2. *For $k\in\{1,2,3\}$, $\partial\widetilde{\phi}_k$ obeys the following estimate away from the singular zone for any $k'\neq k$:*

$$\|\partial\widetilde{\phi}_k\|_{C^{0,\frac{s''}{2}}(\Sigma_t\cap C^k_{\geq\delta})} \lesssim \mathcal{E}, \tag{15.2}$$

$$\|(1-\rho_k)\cdot\partial\widetilde{\phi}_k\|_{B^{u_k,u_{k'}}_{\infty,1}(\Sigma_t)} \lesssim \mathcal{E}. \tag{15.3}$$

3. *The regular part ϕ_{reg} of ϕ satisfies the following estimate for any k,k' with $k\neq k'$:*

$$\|\partial\phi_{reg}\|_{C^{0,\frac{s''}{2}}(\Sigma_t)} \lesssim \mathcal{E}, \quad \|\partial\phi_{reg}\|_{B^{u_k,u_{k'}}_{\infty,1}(\Sigma_t)} \lesssim \mathcal{E}. \tag{15.4}$$

4. *As a consequence, the following estimate holds for ϕ:*

$$\|\partial\phi\|_{L^\infty(\Sigma_t)} \lesssim \mathcal{E}. \tag{15.5}$$

15.1 Localized or anisotropic Sobolev embeddings

In this subsection, we prove two general embedding results, namely Theorem 15.3 and Theorem 15.5. These are the functional bounds (1.28) and (1.27) discussed in the introduction. They will be applied in the later subsections to $\partial\widetilde{\phi}_k$, $\partial\phi_{reg}$ (or appropriately localized versions) to prove Theorem 15.1.

Notice that all the general embedding results derived in this subsection will be applied in the $(u_k, u_{k'})$ coordinates. In order to keep the exposition general, and also distinguish the coordinates here from the (x^1, x^2) coordinates in the elliptic gauge, **we will use (y^1, y^2) to denote a general coordinate system on $\mathbb{R}^2$**. In the following estimates, ∂_{y^2} can be thought of as a good derivative, and in applications it corresponds to $\not{\partial}_{u_{k'}}$.

Before we turn to the actual embedding results, introduce the notations regarding Fourier transform, and an anisotropic Littlewood–Paley theory decomposition.

Definition 15.2 1. Given $f = f(y_1, y_2) \in \mathcal{S}(\mathbb{R}^2)$, denote by $\hat{f} = \hat{f}(\xi_1, \xi_2)$, or $(\mathcal{F}\{) = (\mathcal{F}\{)(\xi_1, \xi_2)$, the Fourier transform of f.
2. Let $s > 0$. Fractional derivatives are defined as in Definition 3.5, except now in (y^1, y^2) coordinates, i.e. $\langle D_y\rangle^s f := \mathcal{F}^{-1}(\langle\xi\rangle^s\mathcal{F}(f))$. Define also a homogeneous version by $|D_y|^s f := \mathcal{F}^{-1}(|\xi|^s\mathcal{F}(f))$.
3. Let $\{P_k\}_{k\in\mathbb{N}\cup\{0\}}$ be the Littlewood–Paley projections as in Definition 3.9, except with (y^1, y^2) in place of $(u_k, u_{k'})$.
4. Define the anisotropic Littlewood–Paley projections $\{P_{kl}\}_{k\in\mathbb{N}\cup\{0\},\, l\in\mathbb{Z}}$ as follows. Take $\varphi : \mathbb{R} \to [0, 1]$ be even, smooth and such that $\underline{\varphi}(\eta) = \begin{cases} 1 & \text{if } |\eta| \le 1 \\ 0 & \text{if } |\eta| \ge 2 \end{cases}$. For each $l \in \mathbb{Z}$, define $\underline{P}_l$ by

$$\underline{P}_l f := \mathcal{F}^{-1}\Big[(\underline{\varphi}(2^{-l}\xi_2) - \underline{\varphi}(2^{-l+1}\xi_2))\mathcal{F}f\Big].$$

Then, for $k \in \mathbb{N}\cup\{0\}$, $l \in \mathbb{Z}$, define

$$P_{kl} := P_k \circ \underline{P}_l,$$

where P_k is as in point 3 above.
5. Define also the notation that

$$f_k := P_k(f), \quad f_{kl} := P_{kl}(f).$$

15.1.1 Anisotropic localized Sobolev embedding

Theorem 15.3 *Let $\sigma > \frac{1}{2}$. The following holds for all Schwartz function f with an implicit constant depending only on σ:*

$$\|f\|_{L^\infty(\mathbb{R}^2)},\ \|f\|_{B_{\infty,1}(\mathbb{R}^2)} \lesssim \inf_{\delta>0}(\delta^{-\frac{1}{2}}\|f\|_{L^2(\mathbb{R}^2)} + \delta^{\sigma-\frac{1}{2}}\|\partial_{y^2}|D_y|^\sigma f\|_{L^2(\mathbb{R}^2)}$$
$$+\delta^{\frac{1}{2}}\|\partial_y f\|_{L^2(\mathbb{R}^2)} + \delta^{-\frac{1}{2}}\|\partial_{y^2} f\|_{L^2(\mathbb{R}^2)}).$$

Here, $B_{\infty,1}(\mathbb{R}^2)$ is the Besov norm as in Definition 3.10, except with (y^1, y^2) in place of $(u_k, u_{k'})$.

Proof By the triangle inequality, $\|f\|_{L^\infty(\mathbb{R}^2)} \lesssim \sum_{k\geq 0}\|f_k\|_{L^\infty(\mathbb{R}^2)} = \|f\|_{B_{\infty,1}(\mathbb{R}^2)}$. It suffices therefore to bound $\|f\|_{B_{\infty,1}(\mathbb{R}^2)}$.

By scaling, it suffices to show $\|f\|_{B_{\infty,1}(\mathbb{R}^2)} \lesssim 1$ (with an implicit constant independent of δ), assuming there exists $\delta > 0$ such that

$$\|f\|_{L^2(\mathbb{R}^2)} \leq \delta^{\frac{1}{2}}, \tag{15.6}$$
$$\|\partial_{y^2}|D_y|^\sigma f\|_{L^2(\mathbb{R}^2)} \leq \delta^{-\sigma+\frac{1}{2}}, \tag{15.7}$$
$$\|\partial_y f\|_{L^2(\mathbb{R}^2)} \leq \delta^{-\frac{1}{2}}, \tag{15.8}$$
$$\|\partial_{y^2} f\|_{L^2(\mathbb{R}^2)} \leq \delta^{\frac{1}{2}}. \tag{15.9}$$

Now we estimate, using the Cauchy–Schwarz inequality, the Plancherel identity, and the easy volume estimate $|\{|\xi| \sim 2^k, |\xi_2| \sim 2^l\}| \sim 2^k \cdot 2^l$, that

$$\|\hat{f}_{kl}\|_{L^1(\mathbb{R}^2)} \lesssim \|\hat{f}_{kl}\|_{L^2(\mathbb{R}^2)} \cdot 2^{\frac{k}{2}} \cdot 2^{\frac{l}{2}} \lesssim \|f_{kl}\|_{L^2(\mathbb{R}^2)} \cdot 2^{\frac{k}{2}} \cdot 2^{\frac{l}{2}}.$$

It then follows from (15.6)–(15.9) and the support properties of the Littlewood–Paley pieces that

$$\|\hat{f}_{kl}\|_{L^1(\mathbb{R}^2)} \lesssim \|f_{kl}\|_{L^2(\mathbb{R}^2)} \cdot 2^{\frac{k}{2}} \cdot 2^{\frac{l}{2}} \lesssim \delta^{\frac{1}{2}} \cdot 2^{\frac{k}{2}} \cdot 2^{-\frac{l}{2}}, \tag{15.10}$$
$$\|\hat{f}_{kl}\|_{L^1(\mathbb{R}^2)} \lesssim \|\partial_{y^2}|D_y|^\sigma f_{kl}\|_{L^2(\mathbb{R}^2)} \cdot 2^{k\cdot(\frac{1}{2}-\sigma)} \cdot 2^{-\frac{l}{2}} \lesssim \delta^{-\sigma+\frac{1}{2}} \cdot 2^{k\cdot(\frac{1}{2}-\sigma)} \cdot 2^{-\frac{l}{2}}, \tag{15.11}$$
$$\|\hat{f}_{kl}\|_{L^1(\mathbb{R}^2)} \lesssim \|\partial_y f_{kl}\|_{L^2(\mathbb{R}^2)} \cdot 2^{-\frac{k}{2}} \cdot 2^{\frac{l}{2}} \lesssim \delta^{-\frac{1}{2}} \cdot 2^{-\frac{k}{2}} \cdot 2^{\frac{l}{2}}, \tag{15.12}$$
$$\|\hat{f}_{kl}\|_{L^1(\mathbb{R}^2)} \lesssim \|\partial_{y^2} f_{kl}\|_{L^2(\mathbb{R}^2)} \cdot 2^{\frac{k}{2}} \cdot 2^{-\frac{l}{2}} \lesssim \delta^{\frac{1}{2}} \cdot 2^{\frac{k}{2}} \cdot 2^{-\frac{l}{2}}. \tag{15.13}$$

We divide the sum into four cases, depending on the values of k and l:

1. When $\delta^{-1} \lesssim 2^k$ and $l \leq 0$, we use (15.12) to obtain

$$\sum_{\delta^{-1}\lesssim 2^k, l\leq 0} \|\hat{f}_{kl}\|_{L^1(\mathbb{R}^2)} \lesssim 1. \tag{15.14}$$

2. When $\delta^{-1} \lesssim 2^k$ and $l \geq 0$, we use (15.11) to obtain

$$\sum_{\delta^{-1} \lesssim 2^k, l \geq 0} \|\hat{f}_{kl}\|_{L^1(\mathbb{R}^2)} \lesssim 1. \tag{15.15}$$

3. When $2^k \lesssim \delta^{-1}$ and $l \leq 0$, we use (15.10) to obtain

$$\sum_{\delta^{-1} \gtrsim 2^k, l \leq 0} \|\hat{f}_{kl}\|_{L^1(\mathbb{R}^2)} \lesssim 1. \tag{15.16}$$

4. When $2^k \lesssim \delta^{-1}$ and $l \geq 0$, we use (15.13) to obtain

$$\sum_{\delta^{-1} \gtrsim 2^k, l \geq 0} \|\hat{f}_{kl}\|_{L^1(\mathbb{R}^2)} \lesssim 1. \tag{15.17}$$

Now, combining (15.14)–(15.17), it is clear by the triangle inequality that

$$\|f\|_{B_{\infty,1}(\mathbb{R}^2)} \leq \sum_{k \in \mathbb{N} \cup \{0\}} \|\hat{f}_k\|_{L^1(\mathbb{R}^2)} \leq \sum_{k \in \mathbb{N} \cup \{0\},\, l \in \mathbb{Z}} \|\hat{f}_{kl}\|_{L^1(\mathbb{R}^2)} \lesssim 1.$$

□

15.1.2 Anisotropic Sobolev embedding into Hölder spaces

Our next goal is an anisotropic Sobolev embedding which maps into Hölder spaces on a half space. The main result is given in Theorem 15.5 below. We will start with the following lemma, which is a variant of the desired estimate, but on all of $\mathbb{R}^2$.

Lemma 15.4 *Let $s \in (0, \frac{1}{2})$. The following estimate holds for all sufficiently regular functions f (with an implicit constant depending on s):*

$$\|f\|_{C^{0,\frac{s}{2}}(\mathbb{R}^2)} \lesssim \|f\|_{H^1(\mathbb{R}^2)} + \|\partial_{y^2} |D_y|^s f\|_{L^2(\mathbb{R}^2)}.$$

Proof As in the proof of Theorem 15.3, we bound the L^∞ norm of each Littlewood–Paley piece in different ways using the Hausdorff–Young, Cauchy–Schwarz inequalities, Plancherel's theorem and the volume estimate in Fourier space. Hence, denoting $\|f\| := \|f\|_{H^1(\mathbb{R}^2)} + \|\partial_{y^2} |D_y|^s f\|_{L^2(\mathbb{R}^2)}$, we have

$$\begin{aligned}
\|f_{kl}\|_{L^\infty(\mathbb{R}^2)} &\lesssim \|f_{kl}\|_{L^2(\mathbb{R}^2)} \cdot 2^{\frac{k}{2}} \cdot 2^{\frac{l}{2}} \lesssim \|f\| \cdot 2^{\frac{k}{2}} \cdot 2^{\frac{l}{2}}, \\
\|f_{kl}\|_{L^\infty(\mathbb{R}^2)} &\lesssim \|\partial_{y^2} |D_y|^s f_{kl}\|_{L^2(\mathbb{R}^2)} \cdot 2^{k \cdot (\frac{1}{2} - s)} \cdot 2^{-\frac{l}{2}} \lesssim \|f\| \cdot 2^{k \cdot (\frac{1}{2} - s)} \cdot 2^{-\frac{l}{2}}, \\
\|f_{kl}\|_{L^\infty(\mathbb{R}^2)} &\lesssim \|\partial_y f_{kl}\|_{L^2(\mathbb{R}^2)} \cdot 2^{-\frac{k}{2}} \cdot 2^{\frac{l}{2}} \lesssim \|f\| \cdot 2^{-\frac{k}{2}} \cdot 2^{\frac{l}{2}}, \\
\|f_{kl}\|_{L^\infty(\mathbb{R}^2)} &\lesssim \|\partial_{y^2} f_{kl}\|_{L^2(\mathbb{R}^2)} \cdot 2^{\frac{k}{2}} \cdot 2^{-\frac{l}{2}} \lesssim \|f\| \cdot 2^{\frac{k}{2}} \cdot 2^{-\frac{l}{2}}.
\end{aligned}$$

Without loss of generality, we take $\|f\| = 1$. Thus,

$$\|f_{kl}\|_{L^\infty(\mathbb{R}^2)} \lesssim 2^{\frac{k}{2}} \cdot 2^{\frac{l}{2}}, \tag{15.18}$$

$$2^{\frac{ks}{2}} \cdot \|f_{kl}\|_{L^\infty(\mathbb{R}^2)} \lesssim 2^{k\cdot(\frac{1}{2}-\frac{s}{2})} \cdot 2^{-\frac{l}{2}}, \tag{15.19}$$

$$2^{\frac{ks}{2}} \cdot \|f_{kl}\|_{L^\infty(\mathbb{R}^2)} \lesssim 2^{k\cdot(\frac{s}{2}-\frac{1}{2})} \cdot 2^{\frac{l}{2}}, \tag{15.20}$$

$$\|f_{kl}\|_{L^\infty(\mathbb{R}^2)} \lesssim 2^{\frac{k}{2}} \cdot 2^{-\frac{l}{2}}. \tag{15.21}$$

For all $k \geq 0$, we use (15.19) and (15.20) respectively to sum $l \geq (1-s)k$ and $l \leq (1-s)k$ to obtain

$$\sum_{l\in\mathbb{Z}} 2^{\frac{ks}{2}} \cdot \|f_{kl}\|_{L^\infty(\mathbb{R}^2)} \lesssim \sum_{l\geq(1-s)k} 2^{\frac{ks}{2}} \cdot \|f_{kl}\|_{L^\infty(\mathbb{R}^2)} + \sum_{l\leq(1-s)k} 2^{\frac{ks}{2}} \cdot \|f_{kl}\|_{L^\infty(\mathbb{R}^2)} \lesssim 1.$$

Recalling that $f_k = \sum_{l\in\mathbb{Z}} f_{kl}$, the above inequalities and the triangle inequality thus implies

$$\|f\|_{C^{0,\frac{s}{2}}(\mathbb{R}^2)} \sim \sup_{k\geq 0} 2^{\frac{sk}{2}} \cdot \|f_k\|_{L^\infty(\mathbb{R}^2)} \lesssim 1$$

where we have used the Littlewood–Paley characterization of the Hölder space. □

Using the above lemma, we obtain the main result of this subsubsection via a reflection argument.

Theorem 15.5 *Let $s \in (0, \frac{1}{2})$, $a \in \mathbb{R}$ and $\Omega_L := (-\infty, a) \times \mathbb{R}$ be the open left half plane.*

Then the following holds for all $v \in \mathcal{S}(\mathbb{R}^2)$ with an implicit constant depending only on s:

$$\|v\|_{C^{0,\frac{s}{2}}(\Omega_L)} \lesssim \|v_{|\Omega_L}\|_{H^1(\Omega_L)} + \|\partial_{y^2}|D_y|^s v\|_{L^2(\mathbb{R}^2)}.$$

Moreover, $v_{|\Omega_L}$ can be extended to a $C^{0,\frac{s}{2}}(\mathbb{R}^2)$ function $Rv : \mathbb{R}^2 \to \mathbb{R}$ such that

$$\|Rv\|_{C^{0,\frac{s}{2}}(\mathbb{R}^2)} \lesssim \|v_{|\Omega_L}\|_{H^1(\Omega_L)} + \|\partial_{y^2}|D_y|^s v\|_{L^2(\mathbb{R}^2)}.$$

Proof Our strategy is to extend $v_{|\Omega_L}$ into a function $Rv : \mathbb{R}^2 \to \mathbb{R}$, which may differ from v, but for which we can prove boundedness using Lemma 15.4.

By a standard Sobolev extension result (see [1, Theorem 5.19]), there exists a bounded linear extension operator $E : H^1(\Omega_L) \to H^1(\mathbb{R}^2)$ satisfying $Ef_{|\Omega_L} = f$ (which is also bounded $E : L^2(\Omega_L) \to L^2(\mathbb{R}^2)$).

As a consequence, defining $Rf = E(f_{|\Omega_L})$, we have

$$\|Rv\|_{H^1(\mathbb{R}^2)} \lesssim \|v_{|\Omega_L}\|_{H^1(\Omega_L)}; \tag{15.22}$$

and using moreover that ∂_{y^2} is tangential to the boundary together with an interpolation argument, we obtain

$$\|\partial_{y^2}|D_y|^s(Rv)\|_{L^2(\mathbb{R}^2)} \lesssim \|\partial_{y^2}|D_y|^s v\|_{L^2(\mathbb{R}^2)}. \tag{15.23}$$

Since $Rv_{|\Omega_L} = v$, by (15.22), (15.23) and Lemma 15.4,

$$\begin{aligned}\|v\|_{C^{0,\frac{s}{2}}(\Omega_L)} &\lesssim \|Rv\|_{C^{0,\frac{s}{2}}(\mathbb{R}^2)} \lesssim \|Rv\|_{H^1(\mathbb{R}^2)} + \|\partial_{y^2}|D_y|^s(Rv)\|_{L^2(\mathbb{R}^2)}\\ &\lesssim \|v_{|\Omega_L}\|_{H^1(\Omega_L)} + \|\partial_{y^2}|D_y|^s v\|_{L^2(\mathbb{R}^2)}.\end{aligned}$$

□

15.2 Converting the estimates into $(u_k, u_{k'})$ coordinates

In this subsection, we convert the L^2 bounds in $\mathcal{E}$ (see Definition 4.1), which are defined with respect to the (x^1, x^2) coordinate system in the elliptic gauge and with the geometric vector field E_k, into estimates in the $(u_k, u_{k'})$ coordinate system. This will later allow us to apply the embedding results obtained in Sect. 15.1 in the $(u_k, u_{k'})$ coordinate system.

In the remainder of the section, recall the coordinate system $(u_k, u_{k'})$ and the notations introduced in Sect. 2.4.2. In particular, recall that $(\not\partial_{u_k}, \not\partial_{u_{k'}})$ denote the coordinate partial derivatives in the $(u_k, u_{k'})$ coordinates.

15.2.1 Equivalence of L^p and $W^{1,p}$ norms

Lemma 15.6 *For any $p \in [1, \infty]$,*

$$\begin{aligned}\|f\|_{L^p_{x^1,x^2}(\Sigma_t)} \lesssim \|f\|_{L^p_{u_k,u_{k'}}(\Sigma_t)} \lesssim \|f\|_{L^p_{x^1,x^2}(\Sigma_t)},\\ \|f\|_{W^{1,p}_{x^1,x^2}(\Sigma_t)} \lesssim \|f\|_{W^{1,p}_{u_k,u_{k'}}(\Sigma_t)} \lesssim \|f\|_{W^{1,p}_{x^1,x^2}(\Sigma_t)}.\end{aligned}$$

Similar estimates hold when the L^p and $W^{1,p}$ norms are taken over subsets of Σ_t.

Proof This is an immediate consequence of (5.19)–(5.20). □

Because of the above lemma, **for the remainder of the section, we will write** $L^p(\Sigma_t)$, **etc. without precisely indicating whether the coordinate system** (x^1, x^2) **or** $(u_k, u_{k'})$ **is used.**

15.2.2 L^2 estimates involving E_k

Lemma 15.6 controls the change of variables for isotropic L^p or $W^{1,p}$ spaces. In this subsection, we translate some estimates in $\mathcal{E}$ that involve the good derivative E_k, and write them in terms of $\not\partial_{u_{k'}}$; see Lemma 15.8.

We begin with a simple lemma.

Lemma 15.7 *For $k \neq k'$, the following holds for all $t \in [0, T_B)$:*

$$\|\mu_k \cdot g(E_k, X_{k'})^{-1}\|_{W^{1,\infty}(\Sigma_t \cap B(0,R))} \lesssim 1. \tag{15.24}$$

Proof This follows from (5.17), (5.1), (5.5), (5.6) and (5.23). □

Lemma 15.8 (L^2 estimates under coordinate change) *For $k \neq k'$, the following holds for all $t \in [0, T_B)$:*

$$\|\partial\!\!\!/_{u_{k'}} \partial\widetilde{\phi}_k\|_{L^2(\Sigma_t \cap S^k_{2\delta})} \lesssim \delta^{\frac{1}{2}} \cdot \mathcal{E}. \tag{15.25}$$

$$\|\partial\!\!\!/^2_{u_k u_{k'}} \partial\widetilde{\phi}_k\|_{L^2(\Sigma_t)} + \|\partial\!\!\!/^2_{u_{k'} u_{k'}} \partial\widetilde{\phi}_k\|_{L^2(\Sigma_t)} \lesssim \delta^{-\frac{1}{2}} \cdot \mathcal{E}, \tag{15.26}$$

Proof The bound (15.25) follows from (2.55), Lemma 15.6 and the definition of $\mathcal{E}$ (Definition 4.1).

For (15.26), we first control the $\partial\!\!\!/_{u_{k'}}$ or $\partial\!\!\!/_{u_k}$ derivative by ∂_x using Lemma 15.6. We then write $\partial\!\!\!/_{u_{k'}}$ in terms of E_k using (2.55). Finally, applying the product rule and (15.24), and using Definition 4.1, we obtain

$$\begin{aligned}
&\|\partial\!\!\!/^2_{u_k u_{k'}} \partial\widetilde{\phi}_k\|_{L^2(\Sigma_t)} + \|\partial\!\!\!/^2_{u_{k'} u_{k'}} \partial\widetilde{\phi}_k\|_{L^2(\Sigma_t)} \lesssim \|\partial_x \partial\!\!\!/_{u_{k'}} \partial\widetilde{\phi}_k\|_{L^2(\Sigma_t)} \\
&\quad = \|\partial_x[\mu_{k'} \cdot g(E_k, X_{k'})^{-1} E_k \partial\widetilde{\phi}_k]\|_{L^2(\Sigma_t)} \\
&\quad \lesssim \|\mu_k \cdot g(E_k, X_{k'})^{-1}\|_{W^{1,\infty}(\Sigma_t \cap B(0,R))} (\|E_k \partial\widetilde{\phi}_k\|_{L^2(\Sigma_t)} + \|\partial_x E_k \partial\widetilde{\phi}_k\|_{L^2(\Sigma_t)}) \\
&\quad \lesssim \delta^{-\frac{1}{2}} \cdot \mathcal{E}.
\end{aligned}$$

□

15.2.3 L^2 estimates involving fractional derivatives

Lemma 15.9 *Let (y_1, y_2), (z_1, z_2) be two systems of coordinates on $\mathbb{R}^2$ such that*

$$1 \lesssim \left| \det \left(\frac{\partial z_i}{\partial y_j} \right)_{ij} \right| \lesssim 1, \quad \left| \frac{\partial z_i}{\partial y_j} \right| \lesssim 1. \tag{15.27}$$

Then, for every $0 < \sigma < 1$, the following holds for all $f \in \mathcal{S}(\mathbb{R}^2)$:

$$\|\langle D_{z_1,z_2} \rangle^\sigma f\|_{L^2_z(\mathbb{R}^2)} \lesssim \|\langle D_{y_1,y_2} \rangle^\sigma f\|_{L^2_y(\mathbb{R}^2)} \lesssim \|\langle D_{z_1,z_2} \rangle^\sigma f\|_{L^2_z(\mathbb{R}^2)}. \tag{15.28}$$

Proof Define the change of variable map $y : z \in \mathbb{R}^2 \to y(z) \in \mathbb{R}^2$ and define the linear map $\Phi_y : f \in L^2(\mathbb{R}^2) \to \Phi_y(f) := f \circ y \in L^2(\mathbb{R}^2)$.

The bounds (15.27) obviously imply that $\Phi_y : L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$, $\Phi_y : H^1(\mathbb{R}^2) \to H^1(\mathbb{R}^2)$ are bounded maps with bounded inverses. The desired conclusion thus follows from interpolation. □

Returning to our setting, this implies by Lemma 5.6 that

Lemma 15.10 *For $s \in \{s', s''\}$, the following estimate holds for all Schwartz function f:*

$$\|\langle D_x\rangle^s f\|_{L^2(\Sigma_t)} \lesssim \|\langle D_{u_k,u_{k'}}\rangle^s f\|_{L^2_{u_k,u_{k'}}(\Sigma_t)} \lesssim \|\langle D_x\rangle^s f\|_{L^2(\Sigma_t)}.$$

After the above preliminaries, we are ready to translate the control for $\mathcal{E}$ into L^2 estimates on the derivatives of $\partial\widetilde{\phi}_k$ in the $(u_k, u_{k'})$ coordinate system. We begin with the

Lemma 15.11 *For $k \neq k'$, the following holds for all $t \in [0, T_B)$:*

$$\|\not\partial_{u_{k'}}\langle D_{u_k,u_{k'}}\rangle^{s''}\partial\widetilde{\phi}_k\|_{L^2_{u_k,u_{k'}}(\Sigma_t)} \lesssim \mathcal{E}.$$

Proof Consider the following chain of estimates:

$$\|\not\partial_{u_{k'}}\langle D_{u_k,u_{k'}}\rangle^{s''}\partial\widetilde{\phi}_k\|_{L^2(\Sigma_t)} = \|\langle D_{u_k,u_{k'}}\rangle^{s''}\not\partial_{u_{k'}}\partial\widetilde{\phi}_k\|_{L^2(\Sigma_t)} \tag{15.29}$$

$$\lesssim \|\langle D_x\rangle^{s''}\not\partial_{u_{k'}}\partial\widetilde{\phi}_k\|_{L^2(\Sigma_t)} = \|\langle D_x\rangle^{s''}(\mu_{k'}g(E_k, X_{k'})^{-1}E_k\partial\widetilde{\phi}_k)\|_{L^2(\Sigma_t)} \tag{15.30}$$

$$\lesssim \|\langle D_x\rangle^{s''}(\varpi\mu_{k'}g(E_k, X_{k'})^{-1})\|_{L^\infty(\Sigma_t)}\|\langle D_x\rangle^{s''}(E_k\partial\widetilde{\phi}_k)\|_{L^2(\Sigma_t)}$$

$$\lesssim \|\langle D_x\rangle^{s''}E_k\partial\widetilde{\phi}_k\|_{L^2(\Sigma_t)} \tag{15.31}$$

$$\lesssim \|E_k\langle D_x\rangle^{s''}\partial\widetilde{\phi}_k\|_{L^2(\Sigma_t)} + \|\langle D_x\rangle^{s''-1}\partial_x\partial\widetilde{\phi}_k\|_{L^2(\Sigma_t)} \tag{15.32}$$

$$\lesssim \|\partial E_k\langle D_x\rangle^{s''}\widetilde{\phi}_k\|_{L^2(\Sigma_t)} + \|\langle D_x\rangle^{s''}\partial\widetilde{\phi}_k\|_{L^2(\Sigma_t)}. \tag{15.33}$$

For (15.30), we first use Lemmas 15.10 and 15.6, and then (2.55). To obtain the first inequality in (15.31), we use Lemma 12.6 (and $\mathrm{supp}(\widetilde{\phi}_k) \subseteq B(0, R)$). In the second inequality, $\|\mu_{k'}g(E_k, X_{k'})^{-1}\|_{W^{1,\infty}(B(0,3R))}$ is bounded using (15.24). For (15.32), we write $E_k = E_k^i\partial_i$ and use Proposition 12.7 and (5.6) to estimate the commutator $[\langle D_x\rangle^{s''}, E_k^i]$. For (15.33), the first term is obtained after commuting $[\partial, E_k]$, and using (5.6) again; while the second term is obtained by the L^2-boundedness of the inhomogeneous Riesz transform $\langle D_x\rangle^{-1}\partial_x$. Finally, note that both terms on (15.33) are controlled by $\mathcal{E}$, which concludes the proof. □

Lemma 15.12 *For $k \neq k'$, the following holds for all $t \in [0, T_B)$:*

$$\|\not\partial_{u_{k'}}\langle D_{u_k,u_{k'}}\rangle^{s''}\partial\phi_{reg}\|_{L^2_{u_k,u_{k'}}(\Sigma_t)} \lesssim \mathcal{E}.$$

Proof This is similar to Lemma 15.11 except it is much easier because we control $\|\partial\partial_x\langle D_x\rangle^{s''}\phi_{reg}\|_{L^2(\Sigma_t)}$ and $\|\langle D_x\rangle^{s''}\partial\phi_{reg}\|_{L^2(\Sigma_t)}$ (instead of only $\|\partial E_k\langle D_x\rangle^{s''}\phi_{reg}\|_{L^2(\Sigma_t)}$ and $\|\langle D_x\rangle^{s''}\partial\phi_{reg}\|_{L^2(\Sigma_t)}$); we omit the details. □

15.3 Boundedness of $\partial\widetilde{\phi}_k$ in the singular region: proof of (15.1)

In the next few subsections, we will prove the estimates asserted in Theorem 15.1; see the conclusion of the proof in Sect. 15.6. We begin with (15.1).

Proof of (15.1) We apply Theorem 15.3 to $f = \rho_k \cdot \partial\widetilde{\phi}_k$ in the coordinate system $(y^1, y^2) = (u_k, u_{k'})$ and with $\sigma = 1$. Note that ∂_{y^2} in the notations of Theorem 15.3 corresponds to $\not\partial_{u_{k'}}$. In order to use Theorem 15.3, it suffices to show that

$$\underbrace{\delta^{-\frac{1}{2}}\|\rho_k\partial\widetilde{\phi}_k\|_{L^2(\Sigma_t)}}_{=:I},\ \underbrace{\delta^{\frac{1}{2}}(\|\not\partial^2_{u_{k'}u_{k'}}(\rho_k\partial\widetilde{\phi}_k)\|_{L^2(\Sigma_t)} + \|\not\partial^2_{u_k u_{k'}}(\rho_k\partial\widetilde{\phi}_k)\|_{L^2(\Sigma_t)})}_{=:II},$$
$$\underbrace{\delta^{\frac{1}{2}}\left(\|\not\partial_{u_k}(\rho_k\partial\widetilde{\phi}_k)\|_{L^2(\Sigma_t)} + \|\not\partial_{u_{k'}}(\rho_k\partial\widetilde{\phi}_k)\|_{L^2(\Sigma_t)}\right)}_{=:III},\ \underbrace{\delta^{-\frac{1}{2}}\|\not\partial_{u_{k'}}(\rho_k\partial\widetilde{\phi}_k)\|_{L^2(\Sigma_t)}}_{=:IV} \lesssim \mathcal{E}.$$

To control the terms I, II, III and IV above, first note that the cutoff function ρ_k satisfies

$$|\rho_k| \lesssim 1, \quad \not\partial_{u_{k'}}\rho_k = 0, \quad |\not\partial_{u_k}\rho_k| \lesssim \delta^{-1}, \quad \mathrm{supp}(\rho_k) \subseteq S^k_{2\delta}. \tag{15.34}$$

We first bound term I, using (15.34) and the $\delta^{-\frac{1}{2}}\|\partial\widetilde{\phi}_k\|_{L^2(\Sigma_t\cap S^k_{2\delta})}$ term in $\mathcal{E}$.

To bound term II, we use (15.34) together with the estimates in Lemma 15.8.

To estimate III, we use (15.34) and the bounds for $\delta^{-\frac{1}{2}}\|\partial\widetilde{\phi}_k\|_{L^2(\Sigma_t\cap S^k_{2\delta})}$ and $\delta^{\frac{1}{2}}\|\partial^2\widetilde{\phi}_k\|_{L^2(\Sigma_t)}$ in $\mathcal{E}$.

Finally, the bound for term IV can be obtained using (15.34) and Lemma 15.8. □

15.4 Hölder estimates for $\partial\widetilde{\phi}_k$ away from the singular zone: proof of (15.2) and (15.3)

Even though we are interested in the Hölder estimates in the coordinates of the elliptic gauge (see (15.2), (15.3) and Definition 3.3), in order to make use of the good derivative, we will apply Theorem 15.5 in the $(u_k, u_{k'})$ coordinate system. Nevertheless, it is easy to check that the Hölder norms in these two coordinate systems are equivalent as we will state in the following lemma.

Lemma 15.13 *For any $\sigma \in (0,1)$, and any open domain $\Omega \subseteq \Sigma_t$ with a Lipschitz boundary, the following holds for all Schwartz functions f:*

$$\|f\|_{C^\sigma_{u_k,u_{k'}}(\Sigma_t\cap\Omega)} \lesssim \|f\|_{C^\sigma(\Sigma_t\cap\Omega)} \lesssim \|f\|_{C^\sigma_{u_k,u_{k'}}(\Sigma_t\cap\Omega)}.$$

Proof This is an immediate consequence of Lemma 5.22. □

We are now ready to prove (15.2) and (15.3).

Proof of (15.2) and (15.3) *Step 1: Proof of* (15.2) In view of Lemma 15.13, it suffices to prove Hölder estimates in the $(u_k, u_{k'})$ coordinates. We apply Theorem 15.5 to $v = \partial\widetilde{\phi}_k$, $\Omega_L := \{u_k \geq \delta,\ u_{k'} \in \mathbb{R}\} \subseteq \Sigma_t$, $s = s''$ in the coordinate system $(u_k, u_{k'})$. Note (as in Sect. 15.3) that $\partial_{y^2} = \not\partial_{u_{k'}}$.

By Theorem 15.5, we know that (15.2) holds as long as we can verify

$$\|(\partial\widetilde{\phi}_k)_{|\Omega_L}\|_{H^1(\Omega_L)} + \|\eth_{u_{k'}}\langle D_{u_k.u_{k'}}\rangle^{s''}\partial\widetilde{\phi}_k\|_{L^2(\Sigma_t)} \lesssim \mathcal{E}. \tag{15.35}$$

Now the first term in (15.35) can be controlled using the $\|\partial\langle D_x\rangle^{s'}\widetilde{\phi}_k\|_{L^2(\Sigma_t)}$ and $\|\partial^2\widetilde{\phi}_k\|_{L^2(\Sigma_t\setminus S^k_\delta)}$ terms in the definition of $\mathcal{E}$ (Definition 4.1); while the second term is controlled by Lemma 15.11.

Step 2: Proof of (15.3) First, notice that the application of Theorem 15.5 in Step 1 in fact gives a stronger result: namely, $\partial\widetilde{\phi}_k$ admits an extension $R\partial\widetilde{\phi}_k$ defined on the whole Σ_t so that $\|R\partial\widetilde{\phi}_k\|_{C^{0,\frac{s}{2}}_{u_k,u_{k'}}(\Sigma_t)} \lesssim \mathcal{E}$.

Let ϖ be the cutoff function as in the beginning of Sect. 12.2. It can be checked explicitly that $\|\varpi\cdot(1-\rho_k)\|_{B^{u_k,u_{k'}}_{\infty,1}} \lesssim 1$. (If the reader prefers not to carry out the explicit estimate for the corresponding oscillatory integral, one can more easily check that $\varpi\cdot(1-\rho_k)$ obeys the assumptions of Theorem 15.3, and apply Theorem 15.3 to deduce the Besov bound.)

Using the fact that $\|f_1 f_2\|_{B^{u_k,u_{k'}}_{\infty,1}} \lesssim \|f_1\|_{B^{u_k,u_{k'}}_{\infty,1}}\|f_2\|_{B^{u_k,u_{k'}}_{\infty,1}}$, we have $\|\varpi\cdot(1-\rho_k)\cdot R\partial\widetilde{\phi}_k\|_{B^{u_k,u_{k'}}_{\infty,1}} \lesssim \mathcal{E}$. Finally, one checks that the support properties for ϖ, $1-\rho_k$ and $\widetilde{\phi}_k$ imply that $(1-\rho_k)\cdot\partial\widetilde{\phi}_k = \varpi\cdot(1-\rho_k)\cdot R\partial\widetilde{\phi}_k$. This concludes the proof of (15.3). □

15.5 Hölder estimates for the regular part: proof of (15.4)

Having completed the estimates for $\widetilde{\phi}_k$, we now turn to the estimate for ϕ_{reg}.

Proof of (15.4) We begin with the first estimate in (15.4). Pick any $k' \neq k$. By the equivalence of the Hölder norms (Lemma 15.13), it suffices to prove that $\partial\phi_{reg}$ is in $C^{0,\frac{s''}{2}}_{u_k,u_{k'}}(\Sigma_t)$. For this, we apply Lemma 15.4 in the $(u_k, u_{k'})$ coordinate system.

It suffices to check

$$\begin{aligned}&\|\partial\phi_{reg}\|_{L^2(\Sigma_t)} + \|\eth_{u_k}\partial\phi_{reg}\|_{L^2(\Sigma_t)} + \|\eth_{u_{k'}}\partial\phi_{reg}\|_{L^2(\Sigma_t)}\\ &\quad+\|\eth_{u_k}\langle D_{u_k,u_{k'}}\rangle^{s'}\partial\phi_{reg}\|_{L^2(\Sigma_t)} \lesssim \mathcal{E}.\end{aligned}$$

The bounds for the first three terms follow directly from the definition of $\mathcal{E}$ (Definition 4.1) and Lemma 15.6, while the last term is controlled in Lemma 15.12.

Finally, since $C^{0,\frac{s''}{2}}_{u_k,u_{k'}} \subseteq B^{u_k,u_{k'}}_{\infty,1}(\Sigma_t)$ continuously, we obtain the second estimate in (15.4). □

15.6 Conclusion of the proof of Theorem 15.1: proof of (15.5)

In view of the estimates derived in Sects. 15.3–15.5, in order to conclude the proof of Theorem 15.1, it suffices to prove (15.5).

Proof of (15.5) In the following, we use that $B^{u_k,u_{k'}}_{\infty,1}(\Sigma_t)$ embeds continuously into $L^\infty(\Sigma_t)$ whenever $k \neq k'$.

- Combining (15.1) and (15.3), and using the triangle inequality, we have, for every k and every $k' \neq k$,

$$\|\partial\widetilde{\phi}_k\|_{L^\infty(\Sigma_t)} \lesssim \|\partial\widetilde{\phi}_k\|_{B^{u_k,u_{k'}}_{\infty,1}(\Sigma_t)} \lesssim \mathcal{E}. \tag{15.36}$$

- The second inequality in (15.4) implies that for any choice of $k \neq k'$,

$$\|\partial\phi_{reg}\|_{L^\infty(\Sigma_t)} \lesssim \|\partial\phi_{reg}\|_{B^{u_k,u_{k'}}_{\infty,1}(\Sigma_t)} \lesssim \mathcal{E}. \tag{15.37}$$

Combining (15.36), (15.37), and using the triangle inequality, we have

$$\|\partial\phi\|_{L^\infty(\Sigma_t)} \leq \|\partial\phi_{reg}\|_{L^\infty(\Sigma_t)} + \sum_{k=1}^{3} \|\partial\widetilde{\phi}_k\|_{L^\infty(\Sigma_t)} \lesssim \mathcal{E}.$$

□

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References

1. Adams, Robert A., Fournier, John J. F.: Sobolev spaces, volume 140 of Pure and Applied Mathematics (Amsterdam). Elsevier/Academic Press, Amsterdam, second edition (2003)
2. Alinhac, S.: Interaction d'ondes simples pour des équations complètement non-linéaires. Ann. Sci. École Norm. Sup. (4) **21**(1), 91–132 (1988)
3. Alinhac, Serge: Temps de vie des solutions régulières des équations d'Euler compressibles axisymétriques en dimension deux. Invent. Math. **111**(3), 627–670 (1993)
4. Barreto, Antônio Sá: Interactions of Semilinear Progressing Waves in Two or More Space Dimensions. (2020) arXiv:2001.11061, preprint
5. Barreto, Antônio Sá, Wang, Yiran: Singularities generated by the triple interaction of semilinear conormal waves. (2018) arXiv:1809.09253, preprint
6. Beals, Michael: Self-spreading and strength of singularities for solutions to semilinear wave equations. Ann. of Math. (2) **118**(1), 187–214 (1983)
7. Beals, Michael: Singularities of conormal radially smooth solutions to nonlinear wave equations. Comm. Partial Differential Equations **13**(11), 1355–1382 (1988)
8. Bony, J.-M.: Interaction des singularités pour les équations de Klein-Gordon non linéaires. In Goulaouic-Meyer-Schwartz seminar, 1983–1984, pages Exp. No. 10, 28. École Polytech., Palaiseau (1984)
9. Bony, Jean-Michel: Second microlocalization and propagation of singularities for semilinear hyperbolic equations. In Hyperbolic equations and related topics (Katata/Kyoto, 1984), pages 11–49. Academic Press, Boston, MA (1986)
10. Chemin, J.Y.: Interaction de trois ondes dans les equations semi-linears strictement hyperboliques d'ordre 2. Communications in Partial Differential Equations **12**, 1203–1225 (1987)
11. Christodoulou, Demetrios: The Formation of Black holes in General Relativity. EMS Monographs in Mathematics. European Mathematical Society (EMS), Zürich (2009)
12. Disconzi, Marcelo M., Luo, Chenyun, Mazzone, Giusy, Speck, Jared: Rough sound waves in 3D compressible Euler flow with vorticity. (2019) arXiv:1909.02550, preprint

13. Hassell, Andrew, Rozendaal, Jan: L^p and $\mathcal{H}^p_{FIO}$ regularity for wave equations with rough coefficients, Part I. (2020) arXiv:2010.13761, preprint
14. Huneau, Cécile., Luk, Jonathan: High-frequency backreaction for the Einstein equations under polarized $\mathbb{U}(1)$-symmetry. Duke Math. J. **167**(18), 3315–3402 (2018)
15. Klainerman, Sergiu: A commuting vectorfields approach to Strichartz-type inequalities and applications to quasi-linear wave equations. Internat. Math. Res. Notices **5**, 221–274 (2001)
16. Klainerman, Sergiu, Rodnianski, Igor: Improved local well-posedness for quasilinear wave equations in dimension three. Duke Math. J. **117**(1), 1–124 (2003)
17. Klainerman, Sergiu, Rodnianski, Igor, Szeftel, Jeremie: The bounded L^2 curvature conjecture. Invent. Math. **202**(1), 91–216 (2015)
18. Kurylev, Yaroslav, Lassas, Matti, Uhlmann, Gunther: Inverse problems for Lorentzian manifolds and non-linear hyperbolic equations. Invent. Math. **212**(3), 781–857 (2018)
19. Lassas, Matti, Uhlmann, Gunther, Wang, Yiran: Inverse problems for semilinear wave equations on Lorentzian manifolds. Comm. Math. Phys. **360**(2), 555–609 (2018)
20. Li, Dong: On Kato-Ponce and fractional Leibniz. Rev. Mat. Iberoam. **35**(1), 23–100 (2019)
21. Luk, Jonathan: Weak null singularities in general relativity. J. Amer. Math. Soc. **31**(1), 1–63 (2018)
22. Luk, Jonathan, Rodnianski, Igor: Local propagation of impulsive gravitational waves. Comm. Pure Appl. Math. **68**(4), 511–624 (2015)
23. Luk, Jonathan, Rodnianski, Igor: Nonlinear interaction of impulsive gravitational waves for the vacuum Einstein equations. Camb. J. Math. **5**(4), 435–570 (2017)
24. Luk, Jonathan, Rodnianski, Igor: High-frequency limits and null dust shell solutions in general relativity. (2020) arXiv:2009.08968, preprint
25. Luk, Jonathan, Moortel, Maxime Van de: Nonlinear interaction of three impulsive gravitational waves I: Main result and the geometric estimates. (2021) arXiv:2101.08353, preprint
26. Melrose, Richard, Ritter, Niles: Interaction of nonlinear progressing waves for semilinear wave equations. Ann. of Math. (2) **121**(1), 187–213 (1985)
27. Rauch, Jeffrey, Reed, Michael C.: Propagation of singularities for semilinear hyperbolic equations in one space variable. Annals of Mathematics **111**(3), 531–552 (1980)
28. Rauch, Jeffrey, Reed, Michael C.: Singularities produced by the nonlinear interaction of three progressing waves; examples. Comm. Partial Differential Equations **7**(9), 1117–1133 (1982)
29. Barreto, Antônio Sá.: Interactions of conormal waves for fully semilinear wave equations. J. Funct. Anal. **89**(2), 233–273 (1990)
30. Klainerman, Sergiu: Geometric and fourier methods for nonlinear wave equations. Unpublished; available at www.math.princeton.edu/ seri/homepage/papers/ucla1.pdf, pages 1–58 (2003)
31. Smith, Hart F., Tataru, Daniel: Sharp local well-posedness results for the nonlinear wave equation. Ann. of Math. (2) **162**(1), 291–366 (2005)
32. Stein, Elias M.: Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals, volume 43 of Princeton Mathematical Series. Princeton University Press, Princeton, NJ. With the assistance of Timothy S. Murphy, Monographs in Harmonic Analysis, III (1993)
33. Tao, Terence: Lecture notes 6 for 247B. Available online at https://www.math.ucla.edu/~tao/247b.1.07w/notes6.pdf
34. Tataru, Daniel: Strichartz estimates for operators with nonsmooth coefficients and the nonlinear wave equation. Amer. J. Math. **122**(2), 349–376 (2000)
35. Tataru, Daniel, Geba, Dan-Andrei.: Dispersive estimates for wave equations. Comm. Partial Differential Equations **30**(4–6), 849–880 (2005)
36. Uhlmann, Gunther, Wang, Yiran: Determination of space-time structures from gravitational perturbations. (2018) arXiv:1806.06461, preprint
37. Wang, Qian: A geometric approach for sharp Local well-posedness of quasilinear wave equations. ArXiv e-prints (2014)
38. Zworski, Maciej: An example of new singularities in the semi-linear interaction of a cusp and a plane. Comm. Partial Differential Equations **19**(5–6), 901–909 (1994)