ON THE SPECTRAL INSTABILITY OF SOME CNOIDAL AND SNOIDAL WAVES OF THE FULL KLEIN-GORDON-ZAKHAROV SYSTEM

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ABSTRACT. The full Klein-Gordon-Zakharov system is considered in the space periodic context. We construct cnoidal and snoidal type solutions for the fast scale component. It is shown that in parts of the range, the waves are spectrally unstable with respect to co-periodic perturbations.

The result relies on an instability index count for Hamiltonian systems, with self-adjoint portion consisting of a non-standard matrix Hill operator. The spectral analysis of these objects, and in particular the Morse index calculations, is a largely unexplored subject. The method, that we develop herein, might prove useful for other systems and/or second order in time models.

1. INTRODUCTION

Consider the following Klein-Gordon-Zakharov (KGZ) system

$$
\left\{\begin{array}{l} u_{tt} - u_{xx} + u + uv + \beta|u|^2 u = 0, -T \leq x \leq T \quad \text{or} \quad x \in \mathbf{R} \\ v_{tt} - v_{xx} = \frac{1}{2}(|u|^2)_{xx}. \end{array}\right. \tag{1}
$$

The system (1) describes the interaction of a Langmuir wave and an ion acoustic waves in plasma [2, 11, 12]. More precisely, β is a real parameter, u is complex valued functions representing the fast scale component of the electric field, while v is real valued function, measuring the deviation of ion density from equilibrium.

Regarding well-posedness for the whole spaces problem, in [4], global well-posedness on $\mathbf{R}$ was shown, without any small data assumptions. In [9, 3], the 3D version of the system (1) was shown to be well-posed. More recently, using standard methods, we have shown in our previous paper [6], that the Cauchy problem for (1) is locally well-posed for low regularity initial data, both in the periodic and the whole line setting.

The existence and stability properties of standing wave is an important question both from theoretical and practical point of view. in fact, understanding the dynamics around the solitary waves gives an important perspective about the global dynamic properties of the evolution problem at hand. Let us briefly review the literature on this topic. Orbital stability of solitary waves for the KGZ system, posed on the whole line, was studied in [12]. The orbital stability of periodic standing waves for system (1) in case $\beta = 0$ was considered in [6]. Using

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the theory developed in [10] the stability of periodic traveling waves for system (1) is established in [7] for the quadratic case, i.e. $\beta = 0$. Stability of periodic waves of dnoidal type for system 1 was considered in [5].

Our aim of this paper is to stability of periodic standing waves of cnoidal and snoidal type. This turns out to be a rich family of solitons, depending on three parameter[1]. Let us point out that the methods needed for the spectral analysis here is much more sophisticated that the ones presented on the whole line or in the periodic context, but with quadratic non-linearity, i.e. $\beta = 0$. The reason is that, the corresponding matrix linearized operators support two or even three negative eigenvalues, which in the end forces the instability of the waves.

1.1. **Periodic solitary waves.** We now construct the explicit solitons, which we later analyze for stability. Let us mention that as these are Newton's equations, and modulo some constants, these are all periodic solutions that exists. We consider periodic waves of the form $u(t,x) = e^{i\omega t}\varphi(x)$, $v(t,x) = \psi(x)$ for Klein-Gordon-Zakharov system (1). Inserting this ansatz in the system (1), we obtain the following system of ordinary differential equations

$$\left\{ \begin{array}{l} -\varphi'' + (1-\omega^2)\varphi + \varphi\psi + \beta\varphi^3 = 0, \ \ -T \le x \le T, \\ -\psi'' - \frac{1}{2}(\varphi^2)'' = 0. \end{array} \right. \tag{2}$$

Integrating twice in the second equation of (3), we obtain $\psi = -\frac{1}{2}\varphi^2 + Ax + B$. Taking into account the required periodicity of the waves, it must be that $A = 0$. We then consider only the case $B = 0$. Denote $\sigma := 1-\omega^2$, so that (2) takes the form

$$\left\{ \begin{array}{l} -\varphi'' + \sigma\varphi + \left(\beta - \frac{1}{2}\right)\varphi^3 = 0, \ \ -T \le x \le T, \\ \psi = -\frac{1}{2}\varphi^2. \end{array} \right. \tag{3}$$

Multiplying by φ and integrating once the φ equation above, we get after some algebraic manipulations

$$\varphi'^2 = \frac{1-2\beta}{4}\left(-\varphi^4 + \frac{4}{1-2\beta}\sigma\varphi^2 + \frac{4a}{1-2\beta}\right) =: U(\varphi), \tag{4}$$

where a is a constant of integration. This is clearly a particular case of the Newton's equation. We consider two cases.

1.1.1. *Cnoidal solutions (outer case).* Let $1-2\beta > 0$ and $\sigma > 0$. In this case the non-linearity U, can be written in the form

$$U(s) = \frac{1-2\beta}{4}(P^2 - s^2)\left(\frac{4\sigma}{2\beta-1} + P^2 + s^2\right)$$

and solution of (4) is given by

$$\varphi(x) = Pcn(\alpha x, \kappa), -T \le x \le T, \tag{5}$$

where

$$\kappa^2 = \frac{(2\beta-1)P^2}{4\sigma + 2(2\beta-1)P^2}, \ \ \alpha^2 = -\frac{2\sigma + (2\beta-1)P^2}{2} = \frac{\sigma}{2\kappa^2 - 1}. \tag{6}$$

We can then formulate explicitly the conditions on the parameters in the following existence results for cnoidal solutions.

[1] in addition to the usual translation and modulation parameters

Proposition 1. *Let $\sigma > 0$, i.e. $\sigma \in (0,1)$ or $\omega \in (-1,1)$. Let also $\kappa \in \left(\frac{1}{\sqrt{2}}, 1\right)$, $\beta \in (-\infty, \frac{1}{2})$. Then, there exists a cnoidal solution φ of* (4)*, in the form* (5)*, with*

$$P^2 = \frac{4\sigma}{(1-2\beta)(2\kappa^2-1)}, T = \frac{2K(k)}{\alpha} = \frac{2K(k)\sqrt{2\kappa^2-1}}{\sqrt{\sigma}}. \tag{7}$$

Remark: Note that the dependence of φ from the parameter β is only through the amplitude P, in the particular way described in (7).

1.1.2. *Snoidal solutions (truncated pendilum).* Let $1-2\beta < 0$ and $\sigma < 0$. Now, we write $U(s)$ in the form

$$U(s) = \frac{2\beta-1}{4}(P^2-s^2)\left(\frac{4\sigma}{1-2\beta} - P^2 - s^2\right). \tag{8}$$

The solution of (4) is given by

$$\varphi(x) = Psn(\alpha x, \kappa), -T \le x \le T, \tag{9}$$

where

$$\kappa^2 = \frac{(2\beta-1)P^2}{-4\sigma-(2\beta-1)P^2}, \ \alpha^2 = -\frac{4\sigma-(1-2\beta)P^2}{4} = \frac{-\sigma}{1+\kappa^2}. \tag{10}$$

Thus, we can formulate the existence result in the following proposition.

Proposition 2. *Let $\kappa \in (0,1)$, $\sigma < 0$, $\beta > \frac{1}{2}$. Then, there exists a snoidal solution of* (8)*, in the form* (9)*, with*

$$P^2 = \frac{-4\sigma k^2}{(2\beta-1)(1+\kappa^2)}, \ T = \frac{2K(\kappa)}{\alpha} = \frac{2K(\kappa)\sqrt{1+\kappa^2}}{\sqrt{-\sigma}}. \tag{11}$$

Our next order of business is to derive the linearized about the solitary waves equations as they are responsible for the stability of these solitons.

1.2. **Linearized equations and setting up the eigenvalue problems.** We take the perturbation in the form

$$u(t,x) = e^{i\omega t}(\varphi(x) + p(t,x)), \ v(t,x) = \psi(x) + q(t,x), \tag{12}$$

where $p(t,x)$ is complex valued function, $q(t,x)$ is real valued function. Plugging in the system (1), using (3), and ignoring all quadratic and higher order terms yields the following linear equation for (p,q)

$$\begin{cases} p_{tt} + 2i\omega p_t + \sigma p - p_{xx} + \varphi q + (-\frac{1}{2}+\beta)\varphi^2 p + 2\beta\varphi^2 Rep = 0 \\ q_{tt} - q_{xx} - \frac{1}{2}(\varphi^2 + 2\varphi Rep)_{xx} = 0. \end{cases} \tag{13}$$

Introduce the new function $h : \int_{-T}^{T} h(x)dx = 0$, so that $q(t,x) = h_x(t,x)$. Then, we can rewrite (13) as follows

$$\begin{cases} p_{tt} + 2i\omega p_t + \sigma p - p_{xx} + \varphi h_x + (-\frac{1}{2}+\beta)\varphi^2 p + 2\beta\varphi^2 Rep = 0 \\ h_{ttx} - h_{xxx} - \frac{1}{2}(\varphi^2 + 2\varphi Rep)_{xx} = 0. \end{cases} \tag{14}$$

Integrating by x in second equation and taking into account that h is a mean zero function, we get

$$\begin{cases} p_{tt}+2i\omega p_t+\sigma p-p_{xx}+\varphi h_x+(-\frac{1}{2}+\beta)\varphi^2 p+2\beta\varphi^2 Rep=0 \\ h_{tt}-h_{xx}-\varphi' Rep+\varphi Rep_x=0. \end{cases} \tag{15}$$

Spliting real and imaginary parts of complex valued function p and real valued function h as $p=F+iG$ and $h=R$, allows us to rewrite the linearized problem (15) as the following system

$$\begin{cases} F_{tt}-2\omega G_t+\sigma F-F_{xx}+\varphi R_x+(-\frac{1}{2}+\beta)\varphi^2 F+2\beta\varphi^2 F=0 \\ G_{tt}+2\omega F_t+\sigma G-G_{xx}+(-\frac{1}{2}+\beta)\varphi^2 G=0 \\ R_{tt}-R_{xx}-\varphi' F-\varphi F_x=0. \end{cases} \tag{16}$$

Now we can write the system (16) as linearized problem below

$$\vec{\boldsymbol{U}}_{\boldsymbol{tt}}+2\omega J\vec{\boldsymbol{U}}_{\boldsymbol{t}}+H\vec{\boldsymbol{U}}=\boldsymbol{0}, \qquad \vec{\boldsymbol{U}}=\begin{pmatrix} F \\ R \\ G \end{pmatrix} \tag{17}$$

where

$$\begin{aligned} J &= \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad H=\begin{pmatrix} H_1 & A & 0 \\ A^* & H_2 & 0 \\ 0 & 0 & H_3 \end{pmatrix} \\ H_1 &= -\partial_x^2+\sigma+(3\beta-\frac{1}{2})\varphi^2, \quad H_2=-\partial_x^2, \\ H_3 &= -\partial_x^2+\sigma+(\beta-\frac{1}{2})\varphi^2, \\ A &= \varphi\partial_x, \quad A^*=-\varphi\partial_x-\varphi'. \end{aligned}$$

After passing to the eigenvalue ansatz $\vec{\boldsymbol{U}}(t,x)\to e^{\lambda t}\vec{\boldsymbol{U}}(x)$, the stability problem (17) becomes a second order pencil

$$\lambda^2\vec{\boldsymbol{U}}+2\omega J\lambda\vec{\boldsymbol{U}}+H\vec{\boldsymbol{U}}=0. \tag{18}$$

More specifically, instability occurs if (18) has a nontrivial solution $(\lambda,\vec{\boldsymbol{U}})$, with $\Re\lambda>0, \vec{\boldsymbol{U}}\neq 0$. Recall that here the function space is $\vec{\boldsymbol{U}}=\begin{pmatrix} F \\ R \\ G \end{pmatrix}$, with

$$F,G\in H^2_{per}[-T,T], \quad R\in H^2_0[-T,T]:=\left\{f\in H^2_{per}[-T,T]:\int_{-T}^T f(x)dx=0\right\}.$$

It is convenient to introduce a notation for the domain of H, namely

$$X:=D(H)=H^2_{per}[-T,T]\times H^2_0[-T,T]\times H^2_{per}[-T,T].$$

Note that $H:D(H)\to L^2_{per}[-T,T]\times L^2_0\times L^2_{per}[-T,T]$, so in particular the second entry of $\mathcal{H}\vec{U}$ is still mean value zero, whence the base Hilbert space may be conveniently taken in the form $L^2_{per}[-T,T]\times L^2_0\times L^2_{per}[-T,T]$. We further rewrite the pencil (18) in the more familiar Hamiltonian formulation as follows. Upon introducing the operators

$$\mathcal{J}:=\begin{pmatrix} \boldsymbol{0}_3 & \boldsymbol{I}_3 \\ -\boldsymbol{I}_3 & -2\omega J \end{pmatrix}, \quad \mathcal{H}:=\begin{pmatrix} H & 0 \\ 0 & \boldsymbol{I}_3 \end{pmatrix},$$

note $\mathcal{J}^* = -\mathcal{J}, \mathcal{H}^* = \mathcal{H}$, we can write an equivalent to (18) Hamiltonian eigenvalue problem, namely

$$\mathcal{J}\mathcal{H}\vec{V} = \lambda\vec{V}, \quad \vec{V} = \begin{pmatrix} \vec{U}_1 \\ \vec{U}_2 \end{pmatrix} \in X \times X. \tag{19}$$

A direct inspection confirms that $\mathcal{J}\mathcal{H}$ maps the second and the fifth components of the six entries into a mean value zero function, that is, an element of $L^2_0[-T,T]$. Thus, the natural base Hilbert space for the eigenvalue problem in (19) is $(L^2_{per}[-T,T] \times L^2_0 \times L^2_{per}[-T,T])^2$. Due to the bounded interval and periodic boundary conditions, it is easy to check that all spectrum is in fact point spectrum, so (19) is a true eigenvalue problem. This prompts the following definition for spectral stability.

Definition 1. *We say that the soliton* (φ,ψ) *(that is, the solution of* (2)*), is spectrally stable, if the eigenvalue problem* (19) *does not have solution* $(\lambda,\vec{V})$ *with* $\Re\lambda > 0$ *and* $\vec{V} \neq 0 : \vec{V} \in X \times X$.

In the next section, we provide the basics of the Krein's instability index theory, which often yields good sufficient results for spectral stability/instability of waves.

1.3. **Instability index counting theory.** Suppose that we consider an eigenvalue problem of the form

$$\mathcal{J}\mathcal{L}f = \lambda f, \tag{20}$$

where $\mathcal{L}^* = \mathcal{L}$ is a self-adjoint operator, $\mathcal{J} : \mathcal{J}^* = -\mathcal{J}$ is bounded, skew-symmetric and invertible operator, so that both operators preserve the real subspace[2]. The eigenvalue problem (20) enjoys the Hamiltonian structure, so in particular eigenvalues are symmetric with respect to both the real and the imaginary axes. Let k_r represents the number of positive real eigenvalues of (20), k_c - the number of quadruplets of complex eigenvalues with non-zero real and imaginary parts, while $k_i^{\leq 0}$ is the number of pairs of purely imaginary eigenvalues of non-positive Krein signature. More precisely, *we say that* $\lambda = i\mu$ *is of non-positive Krein signature*, if $v : \mathcal{J}\mathcal{L}v = i\mu v$ (and consequently $\mathcal{J}\mathcal{L}\bar{v} = -i\mu\bar{v}$), then $\langle \mathcal{L}v, v\rangle \leq 0$.

Also of importance in this theory is a finite dimensional matrix $\mathcal{D}$, which is obtained from the adjoint eigenvectors for (20). More specifically, consider the generalized kernel of $\mathcal{J}\mathcal{L}$

$$gKer(\mathcal{J}\mathcal{L}) = span[(Ker(\mathcal{J}\mathcal{L}))^l, l = 1,2,\ldots].$$

Assume that $dim(gKer(\mathcal{J}\mathcal{L})) < \infty$ (note that under minimal Fredholm assumptions on $\mathcal{J},\mathcal{L}$, this is indeed the case). Select an orthonormal basis in $gKer(\mathcal{J}\mathcal{L}) \ominus Ker(\mathcal{J}\mathcal{L}) = span[\eta_j, j = 1,\ldots,N]$. Then $\mathcal{D} \in M_{N\times N}$ is defined via

$$\mathcal{D} := \{\mathcal{D}_{ij}\}_{i,j=1}^N : \mathcal{D}_{ij} = \langle \mathcal{L}\eta_i, \eta_j\rangle.$$

Then, following [8], we have the following formula, relating the number of "instabilities" or Hamiltonian index of the eigenvalue problem (20) and the Morse indices of the operators $\mathcal{L}$ and $\mathcal{D}$

$$k_{Ham} := k_r + 2k_c + 2k_i^{\leq 0} = n(\mathcal{L}) - n(\mathcal{D}). \tag{21}$$

As an easy corollary, based on elementary parity considerations, we conclude that if k_{Ham} is an odd number, there is at least one real unstable eigenmode for (20), while if k_{Ham} is even,

[2] some assumptions are added below, as we go over the requirements and the formulas

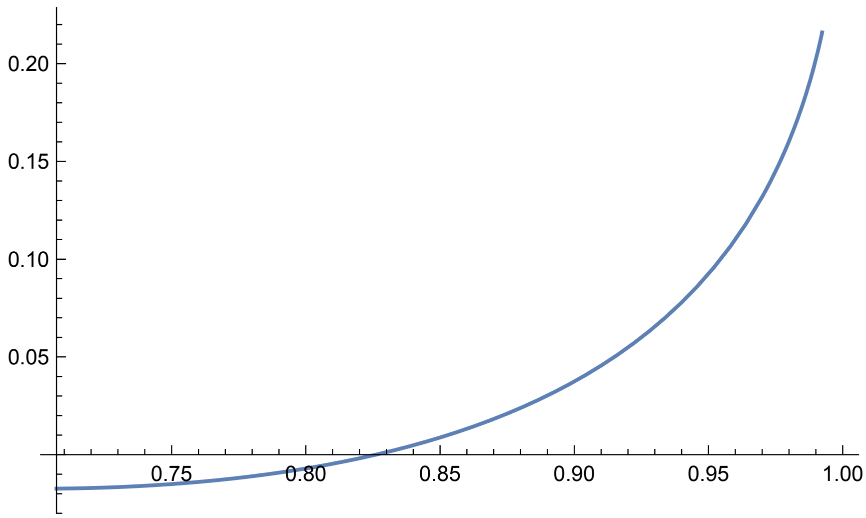

FIGURE 1. Graph of $\beta_0(\kappa)$

one cannot conclude with certainty about instability. For example, as will be the case with various statements below, $k_{Ham} = 2$, one might have two real eigenvalues or a pair of complex eigenvalues $\mu \pm i\nu, \mu > 0$ (i.e. modulational instability) or a pair of marginally stable eigenvalues $\pm i\nu$, with non-positive Krein signature. Clearly, the first two configurations present an unstable scenarios, while the last one is stable.

1.4. **Main results.** We start with the instability of the cnoidal waves. Note that we have a complete description of the unstable spectrum. We refer the reader to Section 1.3 below for the definition of Krein signature of a neutral eigenvalue.

Theorem 1. *(Instability of the cnoidal waves) Let* $\kappa \in (\frac{1}{\sqrt{2}}, 1)$, $\beta \in (-\infty, \frac{1}{2})$, $|\omega| < 1$. *Then, the solitary wave solutions* $(e^{i\omega t}\varphi, -\frac{1}{2}\varphi^2)$ *of the full KGZ system* (1), *described in Proposition 1, are spectrally unstable. More specifically, let*

$$\begin{aligned} M(\kappa) &:= \frac{E^2(\kappa) - 2(1-\kappa^2)E(\kappa)K(\kappa) + (1-\kappa^2)K^2(\kappa)}{(2\kappa^2-1)E(\kappa) + (1-\kappa^2)K(\kappa)} > 0; \\ \beta_0(\kappa) &:= \frac{1}{2} - \frac{M(\kappa)}{K(\kappa)}, \\ \omega_0(\kappa, \beta) &:= \sqrt{\frac{[E(\kappa) - (1-\kappa^2)K(\kappa)][1 - 2\beta - \frac{M(\kappa)}{K(\kappa)}]}{(2\kappa^2-1)(1-2\beta)M(\kappa) + [E(\kappa) - (1-\kappa^2)K(\kappa)][1 - 2\beta - \frac{M(\kappa)}{K(\kappa)}]}}. \end{aligned}$$

Then,

- *If* $\beta < \beta_0(\kappa)$, *and* $|\omega| < \omega_0(\kappa, \beta)$, *then* (19) *has* $k_{Ham} = 3$, *and thus it has at least one one positive eigenvalue.*
- *If* $\frac{1}{2} > \beta > \beta_0(\kappa)$, *then the eigenvalue problem* (19) *has* $k_{Ham} = 2$.
- *If* $\beta < \beta_0(\kappa)$, *and* $1 > |\omega| > \omega_0(\kappa, \beta)$, *then* (19) *also has* $k_{Ham} = 2$.

Our next results concerns the snoidal waves produced in Proposition 2.

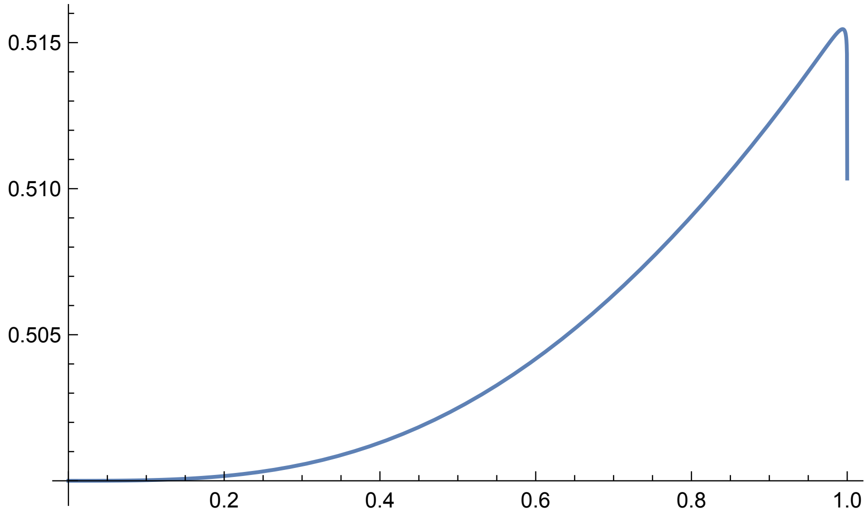

FIGURE 2. Graph of $\beta_1(\kappa)$

Theorem 2. *Let $\kappa \in (0,1)$, $\beta > \frac{1}{2}$ and $|\omega| > 1$. The solitary waves $(e^{i\omega t}\varphi, -\frac{1}{2}\varphi^2)$ of the KGZ system, where φ is as in Proposition 2 are spectrally unstable. Specifically, for $\beta_1 = \beta_1(\kappa)$, $\omega_1(\beta,\kappa)$ as defined in Proposition 11 (see also Figure 2), we have*

- *For $\beta < \beta_1(\kappa)$ and $\omega_1(\beta,\kappa) < |\omega|$, the eigenvalue problem* (19) *has $k_{Ham} = 3$, and hence at least one real unstable eigenvalue.*
- *For $\beta > \beta_1(\kappa)$ and all $|\omega| > 1$, the eigenvalue problem* (19) *has $k_{Ham} = 2$.*
- *For $\beta < \beta_1(\kappa)$ and $\omega_1(\beta,\kappa) > |\omega| > 1$, the eigenvalue problem* (19) *has $k_{Ham} = 2$.*

Our arguments rely on a careful spectral analysis of various self-adjoint operators, both scalar and matrix, as they arise in the eigenvalue problem (19). In order to facilitate our discussion, we introduce a few more notations. Denote

$$H_0 = \begin{pmatrix} H_1 & A \\ A^* & H_2 \end{pmatrix}.$$

Note that H_0 is self-adjoint. Introduce also the second-order differential operator

$$L = -\partial_x^2 + \sigma - 3\left(\frac{1}{2} - \beta\right)\varphi^2, \tag{22}$$

which is the linearized operator naturally appearing in the linearization in the scalar problem (3).

A road map for the paper is as follows. In Section 2, we establish the required spectral properties of the matrix Hill operator H_0 in the cnoidal case. This is done in several steps. First, we relate the spectral properties of the scalar linearized operators L, H_3 to the standard $\mathcal{L}_\pm$, which arise in the linearization close to the standard cnoidal waves for the cubic NLS. Next, in Section 2.2, we demonstrate that $Ker(H_0)$ is one-dimensional, *except on a two dimensional surface $\beta = \beta_0(\kappa)$ in the three dimensional configuration space*, on which $dim(Ker(H_0)) = 2$. Next, we calculate the Morse index of the operator H_0 - we show that $n(H_0) = 1$ or $n(H_0) = 2$, depending on whether we are above or below the critical surface $\beta = \beta_0(\kappa)$. Here, the crucial ingredients are the particular structure of $Ker(H_0)$ (which confirms that an additional vector enters the kernel exactly at $\beta=\beta_0(\kappa)$), while the main novelty is in Proposition 6, which assures us that a negative eigenvalue of H_0 actually crosses the zero and exits as a positive one, as $\beta\sim\beta_0(\kappa)$. In Section 3, we build up the spectral information necessary for the snoidal solution. This is much simpler, and it implies that $n(H)=3$, while $Ker(H_0)$ is always one dimensional. In Section 4, we first introduce the Hamiltonian instability index theory, following [8]. Then, in Section 4.1, we construct the eigenspace and the generalized eigenspace for the cnoidal solutions, for the grand, six dimensional operators arising in the eigenvalue problem (19). This culminates in the exact calculation of the Hamiltonian instability index $k_{Ham.}$ in Proposition 10. Similarly, in Section 4.2, we calculate the Hamiltonian instability index for the snoidal problem, and its relation to the main result. Theorem 2 is covered in detail in Proposition 11.

2. Spectral theory for the linearized operators in the cnoidal case

We start with the properties of the operators L and H_3.

2.1. Spectral propertie of L and H_3. Using that $sn^2(y)+cn^2(y)=1$ and (6), we get

$$\begin{aligned} L &= -\partial_x^2+\sigma-3(\beta-\tfrac{1}{2})P^2cn^2(\alpha x,\kappa)\\ &= \alpha^2\left[-\partial_y^2+6\kappa^2 sn^2(y,\kappa)-(1+4\kappa^2)\right], \end{aligned}$$

where $y=\alpha x$.

It is well-known that the first four eigenvalues of $\Lambda_1=-\partial_y^2+6k^2sn^2(y,k)$, with periodic boundary conditions on $[0,4K(k)]$ are simple. These eigenvalues and corresponding eigenfunctions are:

$$\begin{cases} \nu_0=2+2k^2-2\sqrt{1-k^2+k^4}, & \phi_0(y)=1-(1+k^2-\sqrt{1-k^2+k^4})sn^2(y,k),\\ \nu_1=1+k^2, & \phi_1(y)=cn(y,k)dn(y,k)=sn'(y,k),\\ \nu_2=1+4k^2, & \phi_2(y)=sn(y,k)dn(y,k)=-cn'(y,k),\\ \nu_3=4+k^2, & \phi_3(y)=sn(y,k)cn(y,k)=-k^{-2}dn'(y,k). \end{cases}$$

It follows that the first five eigenvalues of the operator L, equipped with periodic boundary condition on $[0,4K(k)]$ are simple and zero is the third eigenvalue.

Similarly, for the operator H_3 we have

$$H_3=\alpha^2[-\partial_y^2+2k^2sn^2(y,k)-1].$$

The first three eigenvalues and the corresponding eigenfunctions of the operator $\Lambda_2:=-\partial_y^2+2k^2sn^2(y,k)=1+\alpha^{-2}H_3$, with periodic boundary conditions on $[0,4K(k)]$ are simple and

$$\begin{cases} \epsilon_0=k^2, & \theta_0(y)=dn(y,k),\\ \epsilon_1=1, & \theta_1(y)=cn(y,k),\\ \epsilon_2=1+k^2, & \theta_2(y)=sn(y,k). \end{cases}$$

It follows that zero is an eigenvalue of H_3 and it is the second eigenvalue with corresponding eigenfunction $\varphi(x)$. Thus, we have established the following proposition.

Proposition 3. *The self-adjoint Schrödinger operators L and H_3 have only point spectrum. Moreover, their Morse indices, that is the number of strictly negative eigenvalues, are $n(L)=2$ and $n(H_3)=1$.*

Finally, $Ker(H_3) = span[cn(\alpha x,\kappa)]$ and $Ker(L) = span[cn'(\alpha x,\kappa)]$. In particular, L is invertible on the subspace $\{cn'\}^{\perp}$.

Next, we now compute the Morse index of the matrix Schrödinger operator H_0 as well as its kernel. As is well-known in general, this is a much more difficult task.

2.2. **The operator H_0: structure of $Ker(H_0)$.** We start with a description of $Ker(H_0)$, as this will guide our arguments regarding the number of negative eigenvalues. Recall that

$$D(H_0) = H^2[-T,T] \times H_0^2[-T,T]$$

while the base Hilbert space is $L^2[-T,T] \times L_0^2[-T,T]$.

Proposition 4. *Let $\kappa \in \left(\frac{1}{\sqrt{2}},1\right)$, $\beta < \frac{1}{2}$ and $\sigma > 0$. Define*

$$\begin{aligned} M(\kappa) &:= \frac{E^2(\kappa) - 2(1-\kappa^2)E(\kappa)K(\kappa) + (1-\kappa^2)K^2(\kappa)}{(2\kappa^2-1)E(\kappa) + (1-\kappa^2)K(\kappa)} > 0 \\ \beta_0(\kappa) &:= \frac{1}{2} - \frac{M(\kappa)}{K(\kappa)}. \end{aligned}$$

Then, for all $\beta \neq \beta_0(\kappa)$, the self-adjoint operator H_0 has an eigenvalue at zero, which is simple. In addition, $Ker(H_0) = span[\vec{\psi}_1]$, where

$$\vec{\psi}_1 = \begin{pmatrix} \varphi' \\ -\frac{\varphi^2}{2} + \frac{1}{4T}\int_{-T}^{T}\varphi^2 dx \end{pmatrix}.$$

At $\beta = \beta_0(\kappa)$, there is the identity $\langle L^{-1}\varphi, \varphi\rangle + 2T = 0$. A second eigenfunction exists, given by

$$\vec{\psi}_2 = \begin{pmatrix} -L^{-1}\varphi \\ \partial_x^{-1}[\varphi L^{-1}\varphi + 1] \end{pmatrix}.$$

so that $Ker[H_0] = span[\vec{\psi}_1, \vec{\psi}_2]$.

Remark: Here, $\partial_x^{-1}[\varphi L^{-1}\varphi + 1]$ is the unique mean-zero anti-derivative of (the mean-zero function) $\varphi L^{-1}\varphi + 1$. Note that for $\beta \neq \beta_0(\kappa)$, $\varphi L^{-1}\varphi + 1$ is not mean-zero, and so $\vec{\psi}_2$ is not well-defined.

Proof. Let $\begin{pmatrix} f \\ g \end{pmatrix}$ be an eigenvector corresponding to a zero eigenvalue, that is $H_0\begin{pmatrix} f \\ g \end{pmatrix} = 0$. In other words,

$$\begin{aligned} -f'' + \sigma f + (3\beta - \frac{1}{2})\varphi^2 f + \varphi g' &= 0 \\ -g'' - (\varphi f)' &= 0. \end{aligned} \tag{23}$$

Integrating the second equation implies that for some constant c_0, we have

$$g' = -\varphi f + c_0 \tag{24}$$

whence the equation for f becomes

$$-f''+\sigma f+(3\beta-\frac{3}{2})\varphi^2 f+c_0\varphi=0. \tag{25}$$

Note that (25) is exactly in the form $Lf=-c_0\varphi$. We will show that $c_0=0$ and then, in accordance with the description of $Ker(L)$ from Proposition 3, we conclude that $f=d\varphi'$ for some constant d. Resolving (25) yields

$$f=d\varphi'-c_0L^{-1}\varphi, \tag{26}$$

since $\varphi=cn\perp Ker(L)=span[cn']$. Thus,

$$g'=-d\varphi\varphi'+c_0(\varphi L^{-1}\varphi+1). \tag{27}$$

Integrating the above equation, we get the necessary condition for c_0,

$$c_0(\langle L^{-1}\varphi,\varphi\rangle+2T)=0. \tag{28}$$

Clearly, if $\langle L^{-1}\varphi,\varphi\rangle+2T\neq 0$, then $c_0=0$, so we recover the unique (up to a multiplicative constant) eigenvector $\begin{pmatrix} f\\ g\end{pmatrix}=\begin{pmatrix}\varphi'\\ -\frac{\varphi^2}{2}\end{pmatrix}$.

Otherwise, if $\langle L^{-1}\varphi,\varphi\rangle+2T=0$, then we clearly have another eigenvector in the form $\vec{\psi}_2$, as described in the statement. So, it remains to determine, where it does happen that $\langle L^{-1}\varphi,\varphi\rangle+2T=0$. To this end, we need to compute $\langle L^{-1}\varphi,\varphi\rangle$. We will do it by constructing the Green function for the operator L. We already have $\varphi'\in Ker(L)$. The classical approach is to consider the function

$$\psi(x)=\varphi'(x)\int_0^x\frac{1}{\varphi'^2(s)}ds,\quad \begin{vmatrix}\varphi' & \psi\\ \varphi'' & \psi'\end{vmatrix}=1$$

which is also solution of $L\psi=0$. However, since φ' has zeros, the integral above is not well-defined. Instead, using the identities

$$\frac{1}{sn^2(y,\kappa)}=-\frac{1}{dn(y,\kappa)}\frac{\partial}{\partial_y}\frac{cn(x,\kappa)}{sn(y,\kappa)}$$

and integrating by parts, we get the equivalent formula

$$\psi(x)=\frac{1}{\alpha^2P}\left[cn(\alpha x)-\alpha\kappa^2 sn(\alpha x,\kappa)dn(\alpha x,\kappa)\int_0^x\frac{1+cn^2(\alpha s,\kappa)}{dn^2(\alpha s,\kappa)}ds\right].$$

Thus, we may take the Green function in the form

$$L^{-1}f=\varphi'\int_0^x\psi(s)f(s)ds-\psi(s)\int_0^x\varphi'(s)f(s)s+C_f\psi(x),$$

where C_f is chosen such that $L^{-1}f$ is periodic with same period as $\varphi(x)$. After integrating by parts, we get

$$\langle L^{-1}\varphi,\varphi\rangle=-\langle\varphi^3,\psi\rangle+\frac{\varphi^2(T)+\varphi(0)^2}{2}\langle\varphi,\psi\rangle+C_\varphi\langle\varphi,\psi\rangle. \tag{29}$$

Similarly as in [1], integrating by parts yields

$$\langle\psi'',\varphi\rangle=2\psi'(T)\varphi(T)+\langle\psi,\varphi''\rangle.$$

Using that $L\varphi = -(1-2\beta)\varphi^3$, we get

$$\langle \psi, \varphi^3 \rangle = -\frac{2}{1-2\beta}\psi'(T)\varphi(T).$$

Now, integrating by parts and using the above relations, we get

$$\langle \varphi, \psi \rangle = \tfrac{2}{\alpha^3(1-\kappa^2)} E(\kappa) \tag{30}$$

$$C_\varphi = -\tfrac{\varphi''(T)}{2\psi'(T)} \langle \varphi, \psi \rangle + \tfrac{\varphi^2(T)-\varphi^2(0)}{2}.$$

With this, we obtain

$$\langle L^{-1}\varphi, \varphi \rangle = -\frac{4}{\alpha(1-2\beta)} \frac{E^2(\kappa) - 2(1-\kappa^2)E(\kappa)K(\kappa) + (1-\kappa^2)K^2(\kappa)}{(2\kappa^2-1)E(\kappa) + (1-\kappa^2)K(\kappa)} < 0. \tag{31}$$

Using (9) and the definition of $M(\kappa)$, we get

$$\langle L^{-1}\varphi, \varphi \rangle + 2T = \frac{4}{\alpha}\left[K(\kappa) - \frac{M(\kappa)}{1-2\beta}\right] = \frac{4K(\kappa)}{\alpha(1-2\beta)}\left[1 - 2\beta - \frac{M(\kappa)}{K(\kappa)}\right]. \tag{32}$$

So, if $\beta \neq \beta_0(\kappa)$, then the right side of the above equality is not zero, whence $c_0 = 0$ and $f = d\varphi'$. □

We now turn our attention to the Morse index of H_0. Due to its matrix structure, determining the number of negative eigenvalues is generally a hard task. Nevertheless, we succeed in determining their exact number for all values of the parameters. Some words on the strategy. We use continuity arguments. One important clue provided by our analysis of $Ker(H_0)$ in Proposition 4 is that, if we use β as a bifurcation parameter[3], where $\beta \in (-\infty, \frac{1}{2})$, the kernel is generically one dimensional, unless $\beta = \beta_0(\kappa) < \frac{1}{2}$. Then, there is an additional element $\vec{\psi}_2$ popping up in $Ker(L)$. For the remainder of this section, we consider σ, κ fixed parameters and β moves in $(-\infty, \frac{1}{2})$ as a free parameter.

Our best guess in such circumstances is that there is a crossing of an eigenvalue at the value of $\beta_0(\kappa)$. That is, we expect that the smallest positive eigenvalue for H_0, when $\beta \in (\beta_0(\kappa), \frac{1}{2})$ crosses the zero at $\beta = \beta_0(\kappa)$ and becomes a negative one, or vice versa, the largest negative eigenvalue of H_0 crosses at $\beta = \beta_0(\kappa)$ and becomes positive one for $\beta < \beta_0(\kappa)$. This is however not guaranteed, as it is possible that a positive eigenvalue decreases to zero, touches it and then bounces back as a positive one.

We need to analyze the problem at hand carefully.

2.3. **The operator H_0: Calculation of the Morse index.**

The next Proposition provides some good starting point in our analysis.

Proposition 5. *Let* $\kappa \in \left(\frac{1}{\sqrt{2}}, 1\right), \sigma > 0$. *Then, the Morse index satisfies*

$$1 \leq n(H_0) \leq 2.$$

Moreover, under the condition

$$\beta < \frac{1}{2} - \frac{2\pi^2\kappa^2}{K(\kappa)[(1+\kappa^2)E(\kappa) - (1-\kappa^2)K(\kappa)]}, \tag{33}$$

[3]Note that all eigenfunctions, eigenvalues etc. depend in a smooth way on all parameters, in particular on β

we have that H_0 has exactly two negative eigenvalues. That is $n(H_0) = 2$.

Before we proceed with the proof of Proposition 5, let us derive an useful corollary. More specifically, and as we shall see below, the condition (33) is not sharp in identifying the set of all β for which $n(H_0) = 2$. On the other hand, it can be checked that

$$M(\kappa) < \frac{2\pi^2\kappa^2}{[(1+\kappa^2)E(\kappa)-(1-\kappa^2)K(\kappa)]},$$

which guarantees that all β satisfying (33) also satisfy $\beta < \beta_0(\kappa)$.

As an easy corollary of this observation, by Proposition 4 and taking into account the continuous dependence of the eigenvalues on all parameters (and β in particular), we may conclude that $n(H_0) = 2$ for all $\beta \in (-\infty, \beta_0(\kappa))$. Indeed, we have two negative eigenvalues for H_0 for all large negative values of β, according to (33). In order for this to change, an eigenvalue must cross zero (either a positive eigenvalue becomes negative, in which case $n(H_0) = 3$ or else a negative eigenvalue becomes positive, in which case $n(H_0) = 1$. Since zero crossing does not occur till $\beta = \beta_0(\kappa)$, see Proposition 4, we have that $n(H_0) = 2$ for all $\beta < \beta_0(\kappa)$.

Corollary 1. *Let $\kappa \in \left(\frac{1}{\sqrt{2}}, 1\right)$, $\sigma > 0$ and $\beta < \beta_0(\kappa)$. Then, $n(H_0) = 2$.*

Proof. (Proposition 5) By direct inspection, one sees the important relation

$$\langle H_0 \begin{pmatrix} u \\ v \end{pmatrix}, \begin{pmatrix} u \\ v \end{pmatrix}\rangle = \langle Lu, u\rangle + \int_{-T}^{T} (v' + \varphi u)^2 dx. \tag{34}$$

Let λ_0 and λ_1 are the negative eigenvalues of L and φ_0 and φ_1, with $\|\varphi_0\| = \|\varphi_1\| = 1$, are the corresponding eigenfunctions. We have

$$\lambda_0 = \alpha^2(\nu_0 - 1 - 4\kappa^2) = -\tfrac{\sigma}{2\kappa^2-1}(2\kappa^2 + 2\sqrt{1-\kappa^2+\kappa^4} - 1) < 0$$

$$\lambda_1 = \alpha^2(\nu_1 - 1 - 4\kappa^2) = -\tfrac{3\sigma\kappa^2}{2\kappa^2-1} < 0$$

and

$$\varphi_0(x) = \tfrac{1}{||\phi_0(\alpha x)||}\phi_0(\alpha x) = \tfrac{1}{||\phi_0(\alpha x)||}[1 - (1 + \kappa^2 - \sqrt{1-\kappa^2+\kappa^4})sn^2(\alpha x, \kappa)],$$

$$\varphi_1(x) = \tfrac{1}{||\phi_1(\alpha x)||}\phi_1(\alpha x) = \tfrac{1}{||\phi_1(\alpha x)||}cn(\alpha x, \kappa)dn(\alpha x\kappa).$$

From the representation (34), we have that

$$\inf_{\begin{pmatrix} u \\ v \end{pmatrix} \perp \begin{pmatrix} \varphi_0 \\ 0 \end{pmatrix}, \begin{pmatrix} \varphi_1 \\ 0 \end{pmatrix}} \langle H_0 \begin{pmatrix} u \\ v \end{pmatrix}, \begin{pmatrix} u \\ v \end{pmatrix}\rangle \geq \inf_{u\perp\varphi_0,\varphi_1} \langle Lu, u\rangle \geq 0.$$

By the Rayleigh-Ritz formulas, it follows that the third smallest eigenvalue of H_0 is non-negative. Equivalently, $n(H_0) \leq 2$.

Next, we show that there is at least one negative eigenvalue, that is $n(H_0) \geq 1$, for all values of β. To this end, take $u_0 = \varphi_0$. Observe that since $\int_0^{4K} cn(y)dy = 0$ and $\int_0^{4K} sn^2(y)cn(y)dy = 0$, we have $\langle \varphi, \varphi_0\rangle = 0$. Then, take $v_0 : v_0' = -\varphi\varphi_0$. Such a periodic v_0 exists, because $\langle \varphi, \varphi_0\rangle = 0$. We obtain

$$\langle H_0 \begin{pmatrix} u_0 \\ v_0 \end{pmatrix}, \begin{pmatrix} u_0 \\ v_0 \end{pmatrix}\rangle = \langle L\varphi_0, \varphi_0\rangle = \lambda_0 < 0.$$

Thus, $n(H_0) \geq 1$.

Finally, we need to show that for all sufficiently small values of β, we have that $n(H_0) = 2$. We will show that under (33), we have indeed $n(H_0) = 2$. In order to establish this, we need to construct a second vector $\begin{pmatrix} u_1 \\ v_1 \end{pmatrix} \perp \begin{pmatrix} u_0 \\ v_0 \end{pmatrix}$, so that $\langle H_0 \begin{pmatrix} u_1 \\ v_1 \end{pmatrix}, \begin{pmatrix} u_1 \\ v_1 \end{pmatrix} \rangle < 0$. This will suffice, by Rayleigh-Ritz minmax formulas, to claim that the second smallest eigenvalue of H_0 is negative, whence $n(H_0) = 2$.

We take the vector in the form $u_1 = \varphi_1 + a\varphi_0$, with a to be determined from the orthogonality condition. Then, select v_1 as follows

$$v_1' = -\varphi u_1 + \frac{\langle \varphi, u_1 \rangle}{2T} = -\varphi(\varphi_1 + a\varphi_0) + \frac{\langle \varphi, \varphi_1 \rangle}{2T}. \tag{35}$$

Note that such periodic v_1 exists, since the function on the right-hand side of (35) has zero mean, by construction. Note that

$$v_1 = a v_0 + \tilde{v}_1, \tilde{v}_1{}' = -\varphi\varphi_1 + \frac{\langle \varphi, \varphi_1 \rangle}{2T}.$$

The orthogonality condition $\begin{pmatrix} u_1 \\ v_1 \end{pmatrix} \perp \begin{pmatrix} u_0 \\ v_0 \end{pmatrix}$ is then equivalent to

$$0 = \langle u_1, u_0 \rangle + \langle v_1, v_0 \rangle = a(1 + \| v_0 \|^2) + \langle v_0, \tilde{v}_1 \rangle,$$

which has the solution

$$a = -\frac{\langle v_0, \tilde{v}_1 \rangle}{(1 + \| v_0 \|^2)}.$$

This is our choice for a, which guarantees $\begin{pmatrix} u_1 \\ v_1 \end{pmatrix} \perp \begin{pmatrix} u_0 \\ v_0 \end{pmatrix}$.

It remains to calculate $\langle H_0 \begin{pmatrix} u_1 \\ v_1 \end{pmatrix}, \begin{pmatrix} u_1 \\ v_1 \end{pmatrix} \rangle$. We have

$$\langle H_0 \begin{pmatrix} u_1 \\ v_1 \end{pmatrix}, \begin{pmatrix} u_1 \\ v_1 \end{pmatrix} \rangle = \langle L(\varphi_1 + a\varphi_0), \varphi_1 + a\varphi_0 \rangle + \frac{\langle \varphi, \varphi_1 \rangle^2}{2T} = a^2 \lambda_0 + \lambda_1 + \frac{\langle \varphi, \varphi_1 \rangle^2}{2T}.$$

Since $a^2 \lambda_0 < 0$, it will suffice to check that

$$\lambda_1 + \frac{\langle \varphi, \varphi_1 \rangle^2}{2T} < 0. \tag{36}$$

Using that

$$\int_0^{2K} sn^2(y, \kappa) dy = \tfrac{2}{\kappa^2} [K(\kappa) - E(\kappa)]$$

$$\int_0^{2K} sn^4(y, \kappa) dy = \tfrac{2}{3\kappa^4} [(2 + \kappa^2) K(\kappa) - 2(1 + \kappa^2) E(\kappa)]$$

$$\int_0^{2K} cn^2(y, \kappa) dn^2(y, \kappa) dy = \tfrac{2}{3\kappa^2} [(1 + \kappa^2) E(\kappa) - (1 - \kappa^2) K(\kappa)],$$

we get

$$||\phi_0(\alpha x)||^2 = \frac{1}{\alpha}[4K(\kappa) - 4(1+\kappa^2 - \sqrt{1-\kappa^2+\kappa^4})\frac{2}{\kappa^2}[K(\kappa)-E(\kappa)]+$$
$$+(1+\kappa^2-\sqrt{1-\kappa^2+\kappa^4})^2\frac{4}{3\kappa^4}[(2+\kappa^2)K(\kappa)-2(1+\kappa^2)E(\kappa)]]$$
$$||\phi_1(\alpha,\kappa)|| = \frac{4}{3\alpha\kappa^2}[(1+\kappa^2)E(\kappa)-(1-\kappa^2)k(\kappa)].$$

Now we will compute the quantity in (36). Using that $\int_0^{4K} cn^2(y)dn(y)dy = \pi$ and that $2T = \frac{4K(\kappa)}{\alpha}$, we get

$$\frac{\langle\varphi,\varphi_1\rangle^2}{2T} = \frac{3\pi^2P^2\kappa^2}{K(\kappa)[(1+\kappa^2)E(\kappa)-(1-\kappa^2)K(\kappa)]}.$$

Combining this with

$$\lambda_1 = \frac{3\sigma\kappa^2}{1-2\kappa^2},\ P^2 = \frac{4\sigma\kappa^2}{(1-2\beta)(2\kappa^2-1)},$$

we get

$$\lambda_1 + \frac{\langle\varphi,\varphi_1\rangle}{2T} = -\frac{3\sigma\kappa^2}{2\kappa^2-1}\left[1+\frac{4\pi^2\kappa^2}{(2\beta-1)K(\kappa)[(1+\kappa^2)E(\kappa)-(1-\kappa^2)K(\kappa)]}\right]. \tag{37}$$

Since $2\kappa^2-1>0$ and for all $\beta: 2\beta < 1 - \frac{4\pi^2\kappa^2}{K(\kappa)[(1+\kappa^2)E(\kappa)-(1-\kappa^2)K(\kappa)]}$, we have that the right side of (37) is negative. □

Our next proposition completes the analysis for the Morse index of H_0. As such, it incorporates and extends the earlier results in this section. On the other hand, they were necessary preliminary steps for it.

Proposition 6. *Let* $\kappa \in \left(\frac{1}{\sqrt{2}},1\right), \sigma > 0$. *Then,*

- $n(H_0) = 2$, *if* $\beta < \beta_0(\kappa)$,
- $n(H_0) = 1$, *if* $\beta_0(\kappa) < \beta < \frac{1}{2}$.

Remark: Before we continue with the proof, we would like to interpret the result in a slightly different form, which will be useful in the sequel. Note that the condition $\beta < \beta_0(\kappa)$ is equivalent, due to the formula (32), to $2T + \langle L^{-1}\varphi,\varphi\rangle > 0$. Thus

$$n(H_0) = \begin{cases} 2 & \text{if } 2T + \langle L^{-1}\varphi,\varphi\rangle > 0 \\ 1 & \text{if } 2T + \langle L^{-1}\varphi,\varphi\rangle < 0. \end{cases} \tag{38}$$

Proof. Due to Corollary 1 and our analysis in Propositions 4 and 5, we have already established most of the claims. We claim that it remains to establish that the second smallest (still negative) eigenvalue of H_0 indeed crosses the zero for $\beta = \beta_0(\kappa)$, to become a positive one, whence the Morse index drops to $n(H_0) = 1$. Indeed, the other alternative, i.e. a positive one crossing zero to become negative cannot happen, since then we would have $n(H_0) = 3$, which cannot be, see Proposition 5. But, we still need to check that crossing of zero (instead of just bouncing of it) does happen.

To this end, we trace the eigenfunction $\vec{\psi}_1$, corresponding to the zero eigenvalue, described in Proposition 4. By the continuous dependence on β, such eigenfunction will serve as eigenfunction of eigenvalues close to zero, for values of β close to $\beta_0(\kappa)$. It will suffice to show that such eigenvalues change sign at $\beta = \beta_0(\kappa)$, which will indicate that zero crossing has indeed happened.

Given the particular dependence on β in (7), introduce a new variable, $m = \sqrt{\frac{1}{1-2\beta}}$, so that the amplitude is now in the form

$$P = m\sqrt{\frac{4\sigma}{2\kappa^2-1}}. \tag{39}$$

With this variable, the wave φ may be written in the form

$$\varphi(x) = m\sqrt{\frac{4\sigma}{2\kappa^2-1}}cn(\alpha x, \kappa) =: m\Psi(x;\sigma,\kappa).$$

Note that the $\Psi = \Psi(\sigma,\kappa)$ and it is independent on m. *Note that the operator L is independent on β and subsequently on m.*

We now track the eigenvalue, corresponding to an eigenfunction $\vec{\psi}_1$. We can do this via the implicit function theorem. Instead, and equivalently, it suffices to determine the quantities

$$\begin{aligned} g(m) &= -L^{-1}\varphi + (m-m_0)g \in H^2[-T,T], \\ h(m) &= \partial_x^{-1}[\varphi L^{-1}\varphi + 1 + (m-m_0)h] \in H_0^2[-T,T] \\ \gamma(m) &= \gamma(m-m_0) + O((m-m_0)^2) \end{aligned}$$

in a neighborhood of $m_0 = \sqrt{\frac{1}{1-2\beta_0(\kappa)}}$. More specifically, we need to solve the system

$$H_0(m)\begin{pmatrix} -L^{-1}\varphi + (m-m_0)g \\ \partial_x^{-1}(\varphi L^{-1}\varphi + 1 + (m-m_0)h) \end{pmatrix} = \gamma(m-m_0)\begin{pmatrix} -L^{-1}\varphi \\ \partial_x^{-1}(\varphi L^{-1}\varphi + 1) \end{pmatrix} + O((m-m_0)^2), \tag{40}$$

in a small neighborhood of m_0. We have

$$H_0(m) = \begin{pmatrix} L + m^2\Psi^2 & m\Psi\partial_x \\ -m\Psi\partial_x - m\Psi' & -\partial_x^2 \end{pmatrix}.$$

However, note that[4]

$$L + m^2\Psi^2 = L + m_0^2\Psi^2 + 2m_0(m-m_0)\Psi^2 + O((m-m_0)^2).$$

Thus, after expanding up to first order in $m - m_0$, we see that the zero order term is satisfied due to the fact that $H_0\vec{\psi}_2 = 0$. The first order term in $m - m_0$ is then the system that we need to solve. Namely,

$$\begin{aligned} &\begin{pmatrix} L + m_0^2\Psi^2 & m_0\Psi\partial_x \\ -m_0(\Psi\partial_x + \Psi') & -\partial_x^2 \end{pmatrix}\begin{pmatrix} g \\ \partial_x^{-1}h \end{pmatrix} + \begin{pmatrix} 2m_0\Psi^2 & \Psi\partial_x \\ -\Psi\partial_x - \Psi' & -\partial_x^2 \end{pmatrix}\begin{pmatrix} -L^{-1}\varphi \\ \partial_x^{-1}[\varphi L^{-1}\varphi + 1] \end{pmatrix} = \\ &= \gamma\begin{pmatrix} -L^{-1}\varphi \\ \partial_x^{-1}(\varphi L^{-1}\varphi + 1) \end{pmatrix} + O((m-m_0)). \end{aligned}$$

Observe that the first matrix operator is self-adjoint and in fact, it is exactly

$$H_0(m_0) = \begin{pmatrix} L + m_0^2\Psi^2 & m_0\Psi\partial_x \\ -m_0(\Psi\partial_x + \Psi') & -\partial_x^2 \end{pmatrix}.$$

[4] recalling that L is independent on m

We now need a solvability condition for this system, which guarantees that $\gamma \neq 0$. Since it is in the form

$$H_0(m_0)\begin{pmatrix} g \\ \partial_x^{-1} h\end{pmatrix} = \gamma \begin{pmatrix} -L^{-1}\varphi \\ \partial_x^{-1}(\varphi L^{-1}\varphi+1)\end{pmatrix} - \begin{pmatrix} 2m_0\Psi^2 & \Psi\partial_x \\ -\Psi\partial_x - \Psi' & -\partial_x^2\end{pmatrix}\begin{pmatrix} -L^{-1}\varphi \\ \partial_x^{-1}[\varphi L^{-1}\varphi+1]\end{pmatrix}, \tag{41}$$

the solvability condition is that the right-hand side is orthogonal to $Ker[H_0] = span[\vec{\psi}_1, \vec{\psi}_2]$. Due to parity considerations, the orthogonality to $\vec{\psi}_1$ is automatic, since the right-hand side of (41) belongs to $L^2_{even} \times L^2_{odd}$, while $\vec{\psi}_1 \in L^2_{odd} \times L^2_{even}$. Thus, the solvability condition is exactly

$$\langle \gamma \begin{pmatrix} -L^{-1}\varphi \\ \partial_x^{-1}(\varphi L^{-1}\varphi+1)\end{pmatrix} - \begin{pmatrix} 2m_0\Psi^2 & \Psi\partial_x \\ -\Psi\partial_x - \Psi' & -\partial_x^2\end{pmatrix}\begin{pmatrix} -L^{-1}\varphi \\ \partial_x^{-1}[\varphi L^{-1}\varphi+1]\end{pmatrix}, \begin{pmatrix} -L^{-1}\varphi \\ \partial_x^{-1}[\varphi L^{-1}\varphi+1]\end{pmatrix}\rangle = 0, \tag{42}$$

which determines γ in a unique way, provided (42) has an unique, non-zero solution for γ. We aim at verifying that for the remainder of the proof. We can rewrite (42) as

$$\gamma \left\| \begin{pmatrix} -L^{-1}\varphi \\ \partial_x^{-1}(\varphi L^{-1}\varphi+1)\end{pmatrix}\right\|^2 = \langle \begin{pmatrix} \Psi - m_0^2\Psi^2 L^{-1}\Psi \\ (m_0-m_0^2)\partial_x(\Psi L^{-1}\Psi)\end{pmatrix}, \begin{pmatrix} -m_0 L^{-1}\Psi \\ \partial_x^{-1}(m_0^2\Psi L^{-1}\Psi+1)\end{pmatrix}\rangle.$$

Note however that $\left\| \begin{pmatrix} -L^{-1}\varphi \\ \partial_x^{-1}(\varphi L^{-1}\varphi+1)\end{pmatrix}\right\|^2 > \|L^{-1}\varphi\|^2 > 0$, while

$$\begin{aligned} I &:= \langle \begin{pmatrix} \Psi - m_0^2\Psi^2 L^{-1}\Psi \\ (m_0-m_0^2)\partial_x(\Psi L^{-1}\Psi)\end{pmatrix}, \begin{pmatrix} -m_0 L^{-1}\Psi \\ \partial_x^{-1}(m_0^2\Psi L^{-1}\Psi+1)\end{pmatrix}\rangle = \\ &= (2m_0^3 - m_0^4)\int \Psi^2 (L^{-1}\Psi)^2 dx - m_0^2\langle L^{-1}\Psi, \Psi\rangle = \left(\frac{2}{m_0} - 1\right)\|\varphi L^{-1}\varphi\|^2 - \langle L^{-1}\varphi, \varphi\rangle. \end{aligned}$$

Recall however that we evaluate this quantity at $\beta = \beta_0(\kappa)$. On this set, $\langle L^{-1}\varphi, \varphi\rangle + 2T = 0$. So, we obtain

$$I = \left(\frac{2}{m_0} - 1\right)\|\varphi L^{-1}\varphi\|^2 + 2T = \left(2\sqrt{1-2\beta_0(\kappa)} - 1\right)\|\varphi L^{-1}\varphi\|^2 + 2T.$$

However, we have plotted the function $2\sqrt{1-2\beta_0(\kappa)} - 1$ and realized that it is positive, see Figure 3. In short, we conclude

$$I = \left(2\sqrt{1-2\beta_0(\kappa)} - 1\right)\|\varphi L^{-1}\varphi\|^2 + 2T > 0.$$

Thus, $\gamma > 0$. Thus, the zero eigenvalue at $\beta = \beta_0(\kappa)$ (which initially, for $\beta < \beta_0(\kappa)$ is the second smallest eigenvalue for H_0) has the asymptotic

$$\lambda_1(H_0) = \gamma(m - m_0) + O((m-m_0)^2),$$

for $m \sim m_0$ or equivalently $\beta \sim \beta_0(\kappa)$. This shows that for $\beta < \beta_0(\kappa)$, $\lambda_1(H_0) < 0$, $\lambda_1(H_0) = 0$ for $\beta = \beta_0(\kappa)$, while $\lambda_1(H_0) > 0$ for $\beta > \beta_0(\kappa)$. This finishes the proof. □

3. Spectral theory for the linearized operators about the snoidal solutions

In this section, we lay out the spectral theory necessary for the stability analysis for the snoidal waves framework. Most of the considerations herein are pretty similar or simpler to

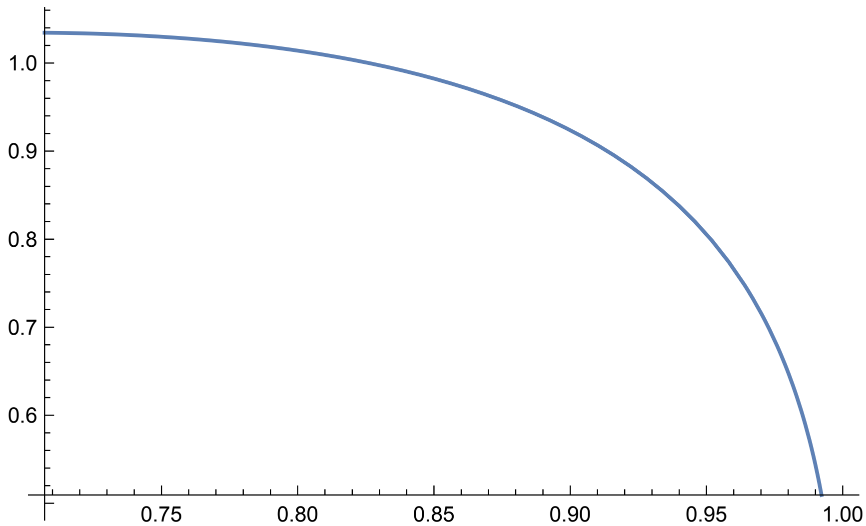

FIGURE 3. Graph of $\kappa \to 2\sqrt{1-2\beta_0(\kappa)}-1$, $\kappa \in (\sqrt{\frac{1}{2}}, 1)$

the ones introduced in Section 2. We start with the operator L. We have

$$\begin{aligned} L &= -\partial_x^2 + \sigma - 3(\beta - \frac{1}{2})P^2 sn^2(\alpha x, \kappa) = \\ &= \alpha^2 \left[-\partial_y^2 + 6\kappa^2 sn^2(y,\kappa) - (1+\kappa^2) \right] = \\ &= \alpha^2 [\Lambda_1 - (1+\kappa^2)]. \end{aligned}$$

where $y = \alpha x$, and the operator Λ_1, along with its first few eigenvalues and eigenfunctions was introduced in Section 2. It follows that L has simple negative eigenvalue with corresponding eigenfunction $\phi_0(\alpha x)$ and zero is the second eigenvalue of L with corresponding eigenfunction φ'. For the operator H_3 we have

$$H_3 = \alpha^2[-\partial_y^2 + 2k^2 sn^2(y,k) - (1+\kappa^2)] = \alpha^2[\Lambda_2 - (1+\kappa^2)],$$

where again the eigenvalues and the first few eigenvalues and eigenvectors were described in detail in Section 2. It follows that zero is an eigenvalue of H_3 and it is the third eigenvalue with corresponding eigenfunction $\varphi(x)$. Thus, $n(L) = 1$ and $n(H_3) = 2$. We summarize the results in the following proposition.

Proposition 7. *The self-adjoint operator H_3, L, have Morse indices $n(H_3) = 2$, $n(L) = 1$ respectively. Moreover, $Ker(H_3) = span[\varphi]$, whereas $Ker(L) = span[\varphi']$.*

Now we turn our attention to determining the Morse index of H. We have the following result.

Proposition 8. *For all values of the parameters, $n(H) = 3$.*

Proof. By the block structure of H, see (17), we know that $n(H) = n(H_3) + n(H_0)$. We have just established, see Proposition 7, that $n(H_3) = 2$, so it remains to prove that $n(H_0) = 1$. To this end, observe that the formula (34) is still valid. From it, and the fact that $L|_{span[\phi_0(\alpha\cdot)]^\perp} \geq 0$, it is clear that

$$\begin{aligned}\inf_{\begin{pmatrix} u \\ v \end{pmatrix} \perp \begin{pmatrix} \phi_0(\alpha\cdot) \\ 0 \end{pmatrix}} \langle H_0 \begin{pmatrix} u \\ v \end{pmatrix}, \begin{pmatrix} u \\ v \end{pmatrix} \rangle &= \inf_{\begin{pmatrix} u \\ v \end{pmatrix} \perp \begin{pmatrix} \phi_0(\alpha\cdot) \\ 0 \end{pmatrix}} \left[\langle Lu, u \rangle + \int_{-T}^{T} (v' + \varphi u)^2 dx \right] \geq \\ &\geq \inf_{u \perp \phi_0(\alpha\cdot)} \langle Lu, u \rangle \geq 0.\end{aligned}$$

Thus, from the min-max principle, $n(H_0) \leq 1$.

On the other hand, notice that $\phi_0(\alpha x) \perp \varphi(x)$, since $\phi_0(\alpha x) \in span[1, sn^2(\alpha x, \kappa)] \perp sn(\alpha x, \kappa)$. Thus, one can take $v_0 : v_0' = -\varphi\phi_0(\alpha\cdot)$. By (34),

$$\langle H_0 \begin{pmatrix} \phi_0(\alpha\cdot) \\ v_0 \end{pmatrix}, \begin{pmatrix} \phi_0(\alpha\cdot) \\ v_0 \end{pmatrix} \rangle = \langle L\phi_0(\alpha\cdot), \phi_0(\alpha\cdot) \rangle < 0.$$

It follows that $n(H_0) = 1$. □

Next, we need to analyze the kernel of H. We have the following proposition.

Proposition 9. *Let $\kappa \in (0,1)$, $\beta > \frac{1}{2}$ and $\sigma < 0$. Then, the self-adjoint operator H_0 has an eigenvalue at zero, which is simple. In addition, $Ker(H_0) = span[\vec{\psi}_1]$, where*

$$\vec{\psi}_1 = \begin{pmatrix} \varphi' \\ -\frac{\varphi^2}{2} + \frac{1}{4T} \int_{-T}^{T} \varphi^2 dx \end{pmatrix}.$$

Proof. The proof of this proposition is completely analogous to the proof of Proposition 4, in fact the arguments there apply here as well, which is how the vector $\vec{\psi}_1$ is identified. The only point of difference is at (28), where one needs to decide whether or not there is an additional element of $Ker(H_0)$. This depends on whether the quantity $\langle L^{-1}\varphi, \varphi \rangle + 2T = 0$. This turns out to be impossible here, since $\varphi \perp span\{\phi_0(\alpha\cdot), \varphi'\}$, whence, according to Proposition 7, we have $\langle L^{-1}\varphi, \varphi \rangle > 0$(and hence $\langle L^{-1}\varphi, \varphi \rangle + 2T > 2T > 0$). □

4. Stability analysis for the waves

We start with an analysis of the cnoidal waves.

4.1. Stability analysis of the cnoidal waves.

We apply the instability index count (21) to the eigenvalue problem (19). First of all, observe that $\mathscr{J}$ is invertible and in fact

$$\mathscr{J}^{-1} = \begin{pmatrix} -2\omega J & -\mathbf{I}_3 \\ \mathbf{I}_3 & \mathbf{0}_3 \end{pmatrix}.$$

Consequently, $Ker(\mathscr{J}\mathscr{H}) = Ker(\mathscr{H})$, which was described in Proposition 4. As a result, the following two vectors span $Ker(\mathscr{H})$

$$\vec{\xi}_1 = \begin{pmatrix} \varphi' \\ -\frac{\varphi^2}{2} + \frac{1}{4T} \int_{-T}^{T} \varphi^2 dx \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \vec{\xi}_2 = \begin{pmatrix} 0 \\ 0 \\ \varphi \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

We now proceed to find the generalized kernel, i.e. the adjoint eigenvectors.

4.1.1. *Generalized kernel of $\mathcal{J}\mathcal{H}$.* Recall that we are only interested in those outside $Ker(\mathcal{H})$. A direct computation yields

$$\vec{\eta}_1 = \mathcal{H}^{-1}\mathcal{J}^{-1}\xi_1 = \mathcal{H}^{-1}\begin{pmatrix} 0 \\ 0 \\ -2\omega\varphi' \\ \varphi' \\ -\frac{\varphi^2}{2}+\frac{1}{4T}\int_{-T}^{T}\varphi^2 dx \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -2\omega H_3^{-1}\varphi' \\ \varphi' \\ -\frac{\varphi^2}{2}+\frac{1}{4T}\int_{-T}^{T}\varphi^2 dx \\ 0 \end{pmatrix},$$

since $\mathcal{H}^{-1}: Ker(\mathcal{H})^{\perp} \to Ker(\mathcal{H})^{\perp}$.

As we are looking to exhaust the set of possible generalized eigenvectors, we need to look further at a second order adjoints, that is solutions of $\mathcal{J}\mathcal{H}\vec{\eta} = \vec{\eta}_1$. Equivalently, we need to solve $\mathcal{H}\vec{\eta} = \mathcal{J}^{-1}\vec{\eta}_1$. A necessary condition for the solvability of this last problem is $\mathcal{J}^{-1}\vec{\eta}_1 \perp \vec{\xi}_1$. But this condition is violated[5] due to the following calculation

$$\langle \mathcal{J}^{-1}\vec{\eta}_1, \vec{\xi}_1\rangle = \langle \begin{pmatrix} -4\omega^2 H_3^{-1}[\varphi'] - \varphi' \\ \frac{\varphi^2}{2} - \frac{1}{4T}\int_{-T}^{T}\varphi^2 dx \\ * \end{pmatrix}, \vec{\xi}_1\rangle = -(4\omega^2\langle H_3^{-1}\varphi', \varphi'\rangle + \|\varphi'\|^2 + \|\frac{\varphi^2}{2} - \frac{1}{4T}\int_{-T}^{T}\varphi^2 dx\|^2) < 0,$$

since $\langle H_3^{-1}\varphi', \varphi'\rangle > 0$. Indeed, a direct check shows that $\varphi' \perp \{\varphi, \theta_0(\alpha x)\}$ (here $\theta_0(\alpha x)$ is the eigenfunction corresponding negative eigenvalue of H_3, while $Ker(H_3) = span[\varphi]$), whence $H_3^{-1}|_{\{\varphi,\theta_0(\alpha x)\}^{\perp}} > 0$.

Next, we find the generalized eigenvectors associated to $\vec{\xi}_2$. Note that the existence of η_2 is not guaranteed, unless some solvability conditions are met! To begin with, we have

$$\vec{\eta}_2 = \mathcal{H}^{-1}\mathcal{J}^{-1}\xi_2 = \mathcal{H}^{-1}\begin{pmatrix} 2\omega\varphi \\ 0 \\ 0 \\ 0 \\ 0 \\ \varphi \end{pmatrix} = \begin{pmatrix} 2\omega H^{-1}\begin{pmatrix} \varphi \\ 0 \\ 0 \end{pmatrix} \\ 0 \\ 0 \\ \varphi \end{pmatrix} = \begin{pmatrix} 2\omega H_0^{-1}\begin{pmatrix} \varphi \\ 0 \end{pmatrix} \\ 0 \\ 0 \\ 0 \\ \varphi \end{pmatrix}.$$

Clearly, for the existence of $\vec{\eta}_2$, we need that $\begin{pmatrix} \varphi \\ 0 \end{pmatrix} \perp Ker(H_0)$. According to Proposition 4, we have that $\begin{pmatrix} \varphi \\ 0 \end{pmatrix} \perp \vec{\psi}_1$, but for the case $\beta = \beta_0(\kappa)$, it is clearly not orthogonal to $\vec{\psi}_2$. Thus, we are working exclusively on $\beta \neq \beta_0(\kappa)$.

Denote $\begin{pmatrix} f_2 \\ g_2 \end{pmatrix} = H_0^{-1}\begin{pmatrix} \varphi \\ 0 \end{pmatrix}$. We shall need to determine f_2, g_2 in our further analysis, but for now we take them in this abstract form. Looking for a further adjoint vectors, we need to solve $\mathcal{J}\mathcal{H}\vec{\eta} = \vec{\eta}_2$. This as before leads to the necessary (and in fact sufficient) condition

[5] and hence $\vec{\eta}_1$ is the only generalized eigenvector associated to $\vec{\xi}_1$

$\mathcal{J}^{-1}\vec{\eta}_2 \perp \vec{\xi}_2$ - note that the other necessary condition $\mathcal{J}^{-1}\vec{\eta}_2 \perp \vec{\xi}_1$ is satisfied. In any case, we obtain

$$\langle \mathcal{J}^{-1}\vec{\eta}_2, \vec{\xi}_2\rangle = -4\omega^2\langle f_2, \varphi\rangle - \|\varphi\|^2.$$

So, under the condition

$$4\omega^2\langle f_2, \varphi\rangle + \|\varphi\|^2 \neq 0, \tag{43}$$

we have no further elements of $gKer(\mathcal{J}\mathcal{H}) \ominus Ker(\mathcal{H})$ and indeed,

$$gKer(\mathcal{J}\mathcal{H}) \ominus Ker(\mathcal{H}) = span[\vec{\eta}_1, \vec{\eta}_2].$$

Remark: Note however, that if (43) is violated, that is if $4\omega^2\langle f_2, \varphi\rangle + \|\varphi\|^2 = 0$, we have an additional generalized eigenvector above $\vec{\eta}_2$. In other words, we have a zero crossing at the points in the parameter space, where $4\omega^2\langle f_2, \varphi\rangle + \|\varphi\|^2 = 0$. By the Hamiltonian structure of the spectrum of $\mathcal{J}\mathcal{H}$, there are exactly two additional generalized eigenvectors at these points.

4.1.2. *Calculation of the instability index for the cnoidal waves.* We will now need to calculate the elements of the matrix $\mathcal{D} \in M_{2\times 2}$. Let us first show that the off-diagonal element are zero. We have

$$\mathcal{D}_{12} = \mathcal{D}_{21} = \frac{1}{\|\vec{\eta}_1\|\|\vec{\eta}_2\|}\langle \mathcal{H}\vec{\eta}_1, \vec{\eta}_2\rangle = \langle \begin{pmatrix} 0 \\ 0 \\ -2\omega\varphi' \\ \varphi' \\ -\frac{\varphi^2}{2} + \frac{1}{4T}\int_{-T}^{T}\varphi^2 dx \\ 0 \end{pmatrix}, \vec{\eta}_2\rangle = 0.$$

So, $\mathcal{D}$ is a diagonal matrix. It follows that $n(\mathcal{D}) = n(\mathcal{D}_{11}) + n(\mathcal{D}_{22}) = n(\langle \mathcal{H}\vec{\eta}_1, \vec{\eta}_1\rangle) + n(\langle \mathcal{H}\vec{\eta}_2, \vec{\eta}_2\rangle)$.

$$\begin{aligned} \langle \mathcal{H}\vec{\eta}_1, \vec{\eta}_1\rangle &= \langle \begin{pmatrix} 0 \\ 0 \\ -2\omega\varphi' \\ \varphi' \\ -\frac{\varphi^2}{2} + \frac{1}{4T}\int_{-T}^{T}\varphi^2 dx \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ -2\omega H_3^{-1}[\varphi'] \\ \varphi' \\ -\frac{\varphi^2}{2} + \frac{1}{4T}\int_{-T}^{T}\varphi^2 dx \\ 0 \end{pmatrix}\rangle = \\ &= 4\omega^2\langle H_3^{-1}\varphi', \varphi'\rangle + \left\| \begin{pmatrix} \varphi' \\ -\frac{\varphi^2}{2} + \frac{1}{4T}\int_{-T}^{T}\varphi^2 dx \end{pmatrix} \right\|^2 > 0, \end{aligned}$$

by recalling that $\langle H_3^{-1}\varphi', \varphi'\rangle > 0$.

Finally, we discuss $\langle \mathcal{H}\vec{\eta}_2, \vec{\eta}_2\rangle$. We have

$$\langle \mathcal{H}\vec{\eta}_2, \vec{\eta}_2\rangle = \langle \begin{pmatrix} 2\omega\varphi \\ 0 \\ 0 \\ 0 \\ 0 \\ \varphi \end{pmatrix}, \begin{pmatrix} 2\omega f_2 \\ 2\omega g_2 \\ 0 \\ 0 \\ 0 \\ \varphi \end{pmatrix} \rangle = 4\omega^2 \langle f_2, \varphi\rangle + \|\varphi\|^2.$$

Observe that this the condition (43) is exactly equivalent to $\mathcal{D}_{22} \neq 0$. Recall that this was necessary for the non-existence of an additional generalized eigenvector associated to $\vec{\xi}_2$. It remains to compute the sign of this expression.

$$\left| \begin{array}{l} H_1 f_2 + A g_2 = \varphi \\ A^* f_2 + H_2 g_2 = 0. \end{array} \right.$$

Obviously, we need to find only f_2 in order to find D_{22}. From the first and second equations of the above system, we have

$$\left| \begin{array}{l} -f_2'' + \sigma f_2 + (-\frac{1}{2} + 3\beta)\varphi^2 f_2 + \varphi g_2' = \varphi \\ -g_2'' - (\varphi f_2)' = 0. \end{array} \right.$$

Integrating in the second equation, we get $g_2' = -\varphi f_2 + c_2$. Hence for f_2, we get the equation

$$L f_2 = (1 - c_2)\varphi,$$

which is

$$f_2 = (1 - c_2) L^{-1}\varphi. \tag{44}$$

Now, we have

$$g_2' = -(1 - c_2)\varphi L^{-1}\varphi + c_2$$

and integrating, we get[6]

$$c_2 = \frac{\langle L^{-1}\varphi, \varphi\rangle}{2T + \langle L^{-1}\varphi, \varphi\rangle}.$$

whence

$$\langle \mathcal{H}\vec{\eta}_2, \vec{\eta}_2\rangle = 4\omega^2\langle f_2, \varphi\rangle + \|\varphi\|^2 = \|\varphi\|^2 \left[1 + \frac{4\omega^2}{\|\varphi\|^2} \frac{2T\langle L^{-1}\varphi, \varphi\rangle}{2T + \langle L^{-1}\varphi, \varphi\rangle}\right]. \tag{45}$$

Using that

$$\|\varphi\|^2 = \frac{4P^2}{\alpha\kappa^2}[E(\kappa) - (1-\kappa^2)K(\kappa)] = \frac{16\sigma}{\alpha(1-2\beta)(2\kappa^2-1)}[E(\kappa) - (1-\kappa^2)K(\kappa)],$$

we get, recalling that $\sigma = 1 - \omega^2$,

$$\langle \mathcal{H}\vec{\eta}_2, \vec{\eta}_2\rangle = \|\varphi\|^2 \left[1 - \frac{\omega^2(2\kappa^2-1)(1-2\beta)M(\kappa)}{(1-\omega^2)[E(\kappa) - (1-\kappa^2)K(\kappa)][1 - 2\beta - \frac{M(\kappa)}{K(\kappa)}]}\right]. \tag{46}$$

For the calculation of the index, both formulas (45) and (46) will be useful. Indeed, we have already confirmed that $\langle L^{-1}\varphi, \varphi\rangle < 0$, see (31). Note that the points in the parameter range for which $2T + \langle L^{-1}\varphi, \varphi\rangle = 0$ represent a vertical asymptote for the function $(\kappa, \beta) \to$

[6]note the solvability condition $2T + \langle L^{-1}\varphi, \varphi\rangle \neq 0$, appearing naturally in the process of determining c_2!

$\langle \mathcal{H}\vec{\eta}_2, \vec{\eta}_2\rangle$. How do we explain that? Recall that at $2T + \langle L^{-1}\varphi, \varphi\rangle = 0$, an additional element appears in $Ker(\mathcal{H})$. This is why the quantity $\langle \mathcal{H}\vec{\eta}_2, \vec{\eta}_2\rangle = \langle \mathcal{J}^{-1}\xi_2, \mathcal{H}^{-1}\mathcal{J}^{-1}\xi_2\rangle$ is no longer well-defined, which is a consequence of the fact that $\mathcal{H}^{-1}\mathcal{J}^{-1}\xi_2$ no longer makes sense (due to the appearance of an additional element in the generalized kernel).

For values of $\beta : \beta > \beta_0(\kappa)$, we have that $2T + \langle L^{-1}\varphi, \varphi\rangle < 0$ and hence, $\langle \mathcal{H}\vec{\eta}_2, \vec{\eta}_2\rangle > 0$ for all values of ω, from the formula (45). Hence, $n(\mathcal{D}) = 0$. Incidentally, in the same range, according to Proposition 6, we have that $n(H_0) = 1$, and hence $n(\mathcal{H}) = 2$. It follows from (21) that $k_{Ham.} = 2$.

For values of $\beta : \beta < \beta_0(\kappa)$, we have that $2T + \langle L^{-1}\varphi, \varphi\rangle > 0$. Recall that in this range, by Proposition 6, we have that $n(\mathcal{H}) = n(H_0) + n(H_3) = 2 + 1 = 3$. But, looking at the formula (46), we see that $\langle \mathcal{H}\vec{\eta}_2, \vec{\eta}_2\rangle$ could be positive or negative. More precisely, if

$$1 - \frac{\omega^2(2\kappa^2-1)(1-2\beta)M(\kappa)}{(1-\omega^2)[E(\kappa)-(1-\kappa^2)K(\kappa)][1-2\beta-\frac{M(\kappa)}{K(\kappa)}]} < 0, \tag{47}$$

we conclude that $\langle \mathcal{H}\vec{\eta}_2, \vec{\eta}_2\rangle < 0$, whence $n(\mathcal{D}) = 1$, whence $k_{Ham} = 3 - 1 = 2$. If however, the opposite inequality to (47) holds, we have that $n(\mathcal{D}) = 0$, whence $k_{Ham} = 3$.

Thus, we are ready to formulate the definite result for the value of k_{Ham}, where we provide the explicit solutions of (47), in terms of ω.

Proposition 10. *Let $\kappa \in (\frac{1}{\sqrt{2}}, 1)$, $\beta < \frac{1}{2}$ and $|\omega| < 1$. Then,*

- *If $\beta > \beta_0(\kappa)$, then $k_{Ham} = 2$.*
- *If $\beta < \beta_0(\kappa)$ and*

$$1 > |\omega| > \sqrt{\frac{[E(\kappa)-(1-\kappa^2)K(\kappa)][1-2\beta-\frac{M(\kappa)}{K(\kappa)}]}{(2\kappa^2-1)(1-2\beta)M(\kappa)+[E(\kappa)-(1-\kappa^2)K(\kappa)][1-2\beta-\frac{M(\kappa)}{K(\kappa)}]}} \tag{48}$$

  then, $k_{Ham} = 2$.
- *If $\beta < \beta_0(\kappa)$ and*

$$|\omega| < \sqrt{\frac{[E(\kappa)-(1-\kappa^2)K(\kappa)][1-2\beta-\frac{M(\kappa)}{K(\kappa)}]}{(2\kappa^2-1)(1-2\beta)M(\kappa)+[E(\kappa)-(1-\kappa^2)K(\kappa)][1-2\beta-\frac{M(\kappa)}{K(\kappa)}]}}$$

  then, $k_{Ham} = 3$.

Remark: The inequality in (48) always has solutions ω as the expression on the right is less than one, as long as $\beta < \beta_0(\kappa)$.

4.2. **Stability analysis for the snoidal waves.** This section is also very reminiscent of the previous one. Namely, due to Proposition 7 and Proposition 9, we have exactly two generating vectors in $Ker(\mathcal{H})$, namely $\vec{\xi}_1.\vec{\xi}_2$ as described in Section 4.1. We now proceed to identify an orthonormal basis for $gKer(\mathcal{J}\mathcal{H}) \ominus Ker(\mathcal{J}\mathcal{H})$.

4.2.1. *Generalized kernel of $\mathcal{J}\mathcal{H}$.* This is of course again very similar to Section 4.1, but there are different relationships between the concrete quantities. Let us start with $\vec{\eta}_2$, the associated vector to $\vec{\xi}_2$. We have calculated it to be, in Section 4.1 (and this is still valid)

$$\vec{\eta}_2 = \begin{pmatrix} 2\omega H_0^{-1}\begin{pmatrix} \varphi \\ 0 \end{pmatrix} \\ 0 \\ 0 \\ 0 \\ \varphi \end{pmatrix} = \begin{pmatrix} 2\omega f_2 \\ 2\omega g_2 \\ 0 \\ 0 \\ 0 \\ \varphi \end{pmatrix},$$

where by (44), recall that for the snoidal waves $2T + \langle L^{-1}\varphi,\varphi\rangle > 2T > 0$,

$$f_2 = (1-c_2)L^{-1}\varphi = \frac{2T}{2T+\langle L^{-1}\varphi,\varphi\rangle}L^{-1}\varphi.$$

Thus, the necessary condition for non-existence of a further adjoint vectors, (43), is verified since

$$4\omega^2\langle f_2,\varphi\rangle + \|\varphi\|^2 = \frac{2T\langle L^{-1}\varphi,\varphi\rangle}{2T+\langle L^{-1}\varphi,\varphi\rangle} + \|\varphi\|^2 > 0. \tag{49}$$

We now find the adjoint vectors associated to $\vec{\xi}_1$. We can proceed, with an identical argument to Section 4.1 to find $\vec{\eta}_1$, which is given by

$$\vec{\eta}_1 = \begin{pmatrix} 0 \\ 0 \\ -2\omega H_3^{-1}\varphi' \\ \varphi' \\ -\frac{\varphi^2}{2} + \frac{1}{4T}\int_{-T}^{T}\varphi^2 dx \\ 0 \end{pmatrix}.$$

The necessary and sufficient condition[7] is given by, identically to the Section 4.1 as

$$4\omega^2\langle H_3^{-1}\varphi',\varphi'\rangle + \|\varphi'\|^2 + \|\frac{\varphi^2}{2} - \frac{1}{4T}\int_{-T}^{T}\varphi^2 dx\|^2 \neq 0. \tag{50}$$

This is not so easy to be dismissed now (as in Section 4.1), since it is possible that $\langle H_3^{-1}\varphi',\varphi'\rangle < 0$. So, it is conceivable that (50) is violated for some points in the parameter space. Nevertheless, let us assume (50) and we proceed. We have that

$$gKer(\mathcal{J}\mathcal{H}) \ominus Ker(\mathcal{J}\mathcal{H}) = span[\vec{\eta}_1,\vec{\eta}_2].$$

4.2.2. *Calculating the instability index for the snoidal waves.* As in Section 4.1, we see that $\mathcal{D}_{12} = \mathcal{D}_{21} = 0$. Thus, $n(\mathcal{D}) = n(\langle\mathcal{H}\vec{\eta}_1,\vec{\eta}_1\rangle) + n(\langle\mathcal{H}\vec{\eta}_2,\vec{\eta}_2\rangle)$. First, with the same computation as in Section 4.1,

$$\langle\mathcal{H}\vec{\eta}_2,\vec{\eta}_2\rangle = 4\omega^2\langle f_2,\varphi\rangle + \|\varphi\|^2 > 0,$$

as already seen, (49).

[7]There is another condition, which is satisfied automatically, so (50) remains as a necessary and sufficient for npn-solvability for the non-existence of further adjoints

Regarding $\langle \mathcal{H}\vec{\eta}_1, \vec{\eta}_1\rangle$, we have, as in Section 4.1,

$$\langle \mathcal{H}\vec{\eta}_1, \vec{\eta}_1\rangle = 4\omega^2 \langle H_3^{-1}\varphi', \varphi'\rangle + \|\varphi'\|^2 + \|\frac{\varphi^2}{2} - \frac{1}{4T}\int_{-T}^{T} \varphi^2 dx\|^2,$$

which is exactly the what appears in (50) as well. So, we need to compute the last quantity.

We have $H_3\varphi = 0$ and $\psi = \varphi \int^x \frac{1}{\varphi^2} ds$ is also solution of $H_3\psi = 0$. Using the identities

$$\frac{1}{sn^2(y,\kappa)} = -\frac{1}{\alpha dn(y,\kappa)} \frac{\partial}{\partial_y} \frac{cn(x,\kappa)}{sn(y,\kappa)}$$

and integrating by parts we get

$$\psi(x) = -\frac{1}{\alpha P}\left[\frac{cn(\alpha x)}{dn(\alpha x)} - \alpha\kappa^2 sn(\alpha x,\kappa)\int_0^x \frac{cn^2(\alpha s,\kappa)}{dn^2(\alpha s,\kappa)} ds\right].$$

Using that φ is odd function and ψ is even function, we get

$$\langle H_3^{-1}\varphi', \varphi'\rangle = -\frac{1}{3}\int_{-T}^{T} \varphi^2\varphi'\psi dx + C_{\varphi'}\int_{-T}^{T} \varphi'\psi dx$$

and

$$C_{\varphi'} = -\frac{\varphi'(T)}{2\psi'(T)}\int_{-T}^{T} \varphi'\psi dx.$$

Hence

$$\langle H_3^{-1}\varphi', \varphi'\rangle = -\frac{1}{3}\int_{-T}^{T} \varphi^2\varphi'\psi dx - \frac{\varphi'(T)}{2\psi'(T)}\left(\int_{-T}^{T} \varphi'\psi dx\right)^2.$$

By direct computations, we have

$$\begin{cases} \int_{-T}^{T} \varphi'\psi dx = -\frac{1}{\alpha}\left[2\int_0^{2K} cn^2(x)dx + \kappa^2\int_{-T}^{T} \frac{sn^2(x)cn^2(x)}{dn^2(x)} dx\right] \\ \frac{\varphi'(T)}{2\psi'(T)} = \frac{\alpha P^2}{4[K(\kappa)-E(\kappa)]} \\ \int_{-T}^{T} \varphi^2\varphi'\psi dx = -\frac{P^2}{\alpha}\left[2\int_0^{2K} sn^2(x)cn^2(x)dx + \frac{\kappa^2}{2}\int_0^{2K} \frac{sn^4(x)cn^2(x)}{dn^2(x)} dx\right] \\ P^2 = \frac{-4\sigma\kappa^2}{(2\beta-1)(1+\kappa^2)}, \ \ \alpha^2 = \frac{-\sigma}{1+\kappa^2}. \end{cases}$$

Putting all this together, we get

$$\begin{aligned} \langle H_3^{-1}\varphi', \varphi'\rangle &= \frac{P^2}{\alpha}\left[\frac{2}{3}\int_0^{2K} sn^2(x)cn^2(x)dx + \frac{\kappa^2}{6}\int_0^{2K} \frac{sn^4(x)cn^2(x)}{dn^2(x)} dx\right] - \\ &- \frac{P^2}{\alpha}\frac{1}{4[K(\kappa)-E(\kappa)]}\left(2\int_0^{2K} cn^2(x)dx + \kappa^2\int_0^{2K} \frac{sn^2(x)cn^2(x)}{dn^2(x)} dx\right)^2 =: \frac{P^2}{\alpha}R(\kappa). \end{aligned}$$

Using Mathematica, we have computed this expression

$$R(\kappa) = \frac{(-5k^2+16k-8)K(k) + 9((k^2-4k+2)K(k) + 2(k-1)E(k))^2 + (k^2-4k-8)E(k)}{9k^2(E(k)-K(k))}.$$

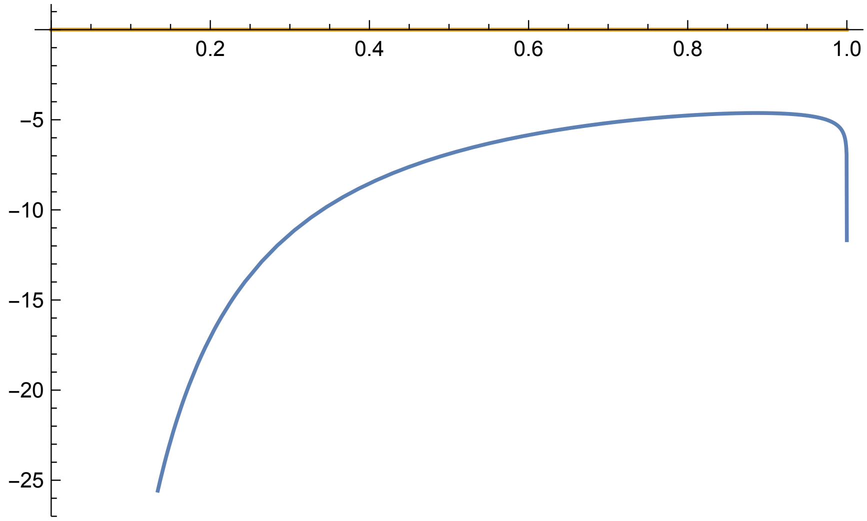

FIGURE 4. Graph of $R(\kappa)$

One can also see, by plotting the above function in terms of κ, that $\langle H_3^{-1}\varphi', \varphi'\rangle = \frac{P^2}{\alpha} R(\kappa) < 0$, for all values of κ, see the Figure 4 below. Next, we have that (by using Mathematica for the integrals)

$$\begin{aligned}\int_{-T}^{T} \varphi'^2 dx &= 2\alpha P^2 \int_0^{2K} cn^2(x) dn^2(x) dx = \\ &= 2\alpha P^2 \frac{2((k-1)K(k) + (k+1)E(k))}{3k} =: 2\alpha P^2 Q(\kappa)\end{aligned}$$

and (again, with the help of Mathematica)

$$\begin{aligned}\| \frac{\varphi^2}{2} - \frac{1}{4T}\int_{-T}^{T} \varphi^2 \|^2 &= \frac{1}{2}\left[\int_0^T \varphi^4 - \frac{1}{T}\left(\int_0^T \varphi^2\right)^2\right] = \\ &= \frac{P^4}{\alpha}\left[\frac{1}{2}\int_0^{2K} sn^4(x) dx - \frac{1}{4K(\kappa)}\left(\int_0^{2K} sn^2(x) dx\right)^2\right] = \\ &= \frac{P^4}{\alpha} \frac{(k-1)K^2(k) + 2(2-k)K(k)E(k) - 3E^2(k)}{3k^2 K(k)} =: \frac{P^4}{\alpha} Z(\kappa).\end{aligned}$$

Putting all of these formula together, we see that $\langle \mathcal{H}\vec{\eta}_1, \vec{\eta}_1\rangle < 0$ exactly when

$$\omega^2 > \frac{2\alpha^2 Q(\kappa) + P^2 Z(\kappa)}{-4R(\kappa)} = (\omega^2 - 1)\frac{2Q(\kappa) + \frac{4\kappa^2}{(2\beta-1)} Z(\kappa)}{-4(1+\kappa^2)R(\kappa)}.$$

Solving in terms of ω, we obtain that $\langle \mathcal{H}\vec{\eta}_1, \vec{\eta}_1\rangle < 0$ if and only if

$$\frac{2Q(\kappa) + \frac{4\kappa^2}{(2\beta-1)} Z(\kappa)}{-4(1+\kappa^2)R(\kappa)} > \omega^2 \left(\frac{2Q(\kappa) + \frac{4\kappa^2}{(2\beta-1)} Z(\kappa) + 4(1+\kappa^2)R(\kappa)}{-4(1+\kappa^2)R(\kappa)}\right). \tag{51}$$

Knowing that $R(\kappa) < 0$ for all κ and in fact, by plotting with Mathematica one could see that $2Q(\kappa)+4(1+\kappa^2)R(\kappa) < 0$, we have that if $\frac{1}{2} < \beta < \beta_1(\kappa)$, where

$$\beta_1(\kappa) := \frac{1}{2} - \frac{2\kappa^2 Z(\kappa)}{2Q(\kappa)+4(1+\kappa^2)R(\kappa)} \tag{52}$$

then (51) will have a solution, recall that we are in the regime $0 > \sigma = 1-\omega^2$,

$$\omega_1(\beta,\kappa) = \sqrt{\frac{2(2\beta-1)Q(\kappa)+4\kappa^2 Z(\kappa)}{2(2\beta-1)Q(\kappa)+4\kappa^2 Z(\kappa)+4(2\beta-1)(1+\kappa^2)M(\kappa)}} > |\omega| > 1. \tag{53}$$

If $\beta > \beta_1(\kappa)$, we have that the right-hand side of (51) is negative, hence it will be satisfied for all allowable values of $\omega : |\omega| > 1$. Thus, we have established the following proposition.

Proposition 11. *Let* $\kappa \in (0,1)$. *Then, there exist functions* $\beta_1(\kappa), \omega_1(\beta,\kappa)$, *introduced in* (52) *and* (53) *respectively, so that the Hamiltonian index is computed according to the following law*

$$k_{Ham} = \begin{cases} 2 & \beta > \beta_1(\kappa) \ \& \ |\omega| > 1 \\ 2 & \beta < \beta_1(\kappa) \ \& \ \omega_1(\beta,\kappa) > |\omega| > 1 \\ 3 & \beta < \beta_1(\kappa) \ \& \ \omega_1(\beta,\kappa) < |\omega|. \end{cases} \tag{54}$$

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