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Stacky heights on elliptic curves in characteristic 3

STACKY HEIGHTS ON ELLIPTIC CURVES IN CHARACTERISTIC 3

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ABSTRACT. — We show there are no stacky heights on the moduli stack of stable elliptic curves in characteristic 3 which induce the usual Faltings height, negatively answering a question of Ellenberg, Satriano, and Zureick-Brown.

RÉSUMÉ. — Nous montrons qu'il n'existe pas de hauteur sur le champ de modules des courbes elliptiques en caractéristique 3 qui induit la hauteur de Faltings usuelle. Cela donne une réponse négative à une question posée par Ellenberg, Satriano et Zureick-Brown.

1. Introduction

In this paper, we investigate heights on the compactified moduli stack of elliptic curves in characteristic 3. We show that the notion of stacky height introduced in [2] does not always recover the classical notion of height. Specifically, we show there is no vector bundle whose associated stacky height induces the usual notion of Faltings height for elliptic curves in characteristic 3.

Throughout this paper, we work over a fixed perfect field k of characteristic 3. Let $\overline{\mathscr{M}}_{1,1}$ denote the Deligne-Mumford moduli stack of stable elliptic curves over k. Given a finite field extension K over $k(t)$ and an elliptic curve $E \to \operatorname{Spec} K$, there is a Faltings height on that elliptic curve, which we define as follows.

DEFINITION 1.1. — *Given $E \to \operatorname{Spec} K$ as above, let C be the regular proper connected curve over k whose generic point is $\operatorname{Spec} K$ and let $f : X \to C$ denote the minimal proper regular model of $E \to \operatorname{Spec} K$. The*

Faltings height of E is given by $\deg f_* \omega_{X/C}$. *Thinking of $E \to \operatorname{Spec} K$ as a K point of $\overline{\mathscr{M}}_{1,1}$, given by $x : \operatorname{Spec} K \to \overline{\mathscr{M}}_{1,1}$, we denote this Faltings height by* $\operatorname{ht}(x)$.

Remark 1.2. — The Faltings height is also computable by the formula $\frac{1}{12} \deg(\Delta_{X/C})$ where $\Delta_{X/C}$ is the discriminant of the relative elliptic surface, viewed as a section of $H^0(C, f_* \omega_{X/C}^{\otimes 12})$.

On the other hand, suppose we are given a vector bundle $\mathscr{V}$ on $\overline{\mathscr{M}}_{1,1}$ and a K point $x : \operatorname{Spec} K \to \overline{\mathscr{M}}_{1,1}$, corresponding to a stable elliptic curve $E \to \operatorname{Spec} K$. Ellenberg, Satriano, and Zureick-Brown [2, Definition 2.11] define a notion of height associated to x and $\mathscr{V}$, notated $\operatorname{ht}_{\mathscr{V}}(x)$, see Definition 2.4. Suppose k' is a field of characteristic not 2 or 3, and let ω denote the Hodge bundle over $(\overline{\mathcal{M}}_{1,1})_{k'}$. That is, $\omega := f_* \omega_{\mathscr{E}/(\overline{\mathcal{M}}_{1,1})_{k'}}$ for $f : \mathscr{E} \to (\overline{\mathcal{M}}_{1,1})_{k'}$ the universal stable elliptic curve. Then [2, Proposition 3.11] show that, for K a finite extension of $k'(t)$, and $x : \operatorname{Spec} K \to (\overline{\mathcal{M}}_{1,1})_{k'}$ a point, $\operatorname{ht}_\omega(x) = \operatorname{ht}(x)$, with the latter notion of Faltings height as defined in Definition 1.1. However, as [2, p. 27] observe, for k a field of characteristic 3, it is no longer true that $\operatorname{ht}_\omega(x) = \operatorname{ht}(x)$ for all $x : \operatorname{Spec} K \to \overline{\mathscr{M}}_{1,1}$. In particular, they show cubic twists of the form $y^2 = x^3 - x + f(t)$, for $f(t) \in k[t]$, all have height 0 with respect to the Hodge bundle, even though their Faltings heights can be nonzero.

Moreover, they show there is no line bundle $\mathscr{L}$ on $\overline{\mathscr{M}}_{1,1}$ for which $\operatorname{ht}_{\mathscr{L}}(x) = \operatorname{ht}(x)$ [2, p. 27]. This leads to the following question:

QUESTION 1.3. — *Is there some vector bundle $\mathscr{V}$ (necessarily of rank more than 1) on $\overline{\mathscr{M}}_{1,1}$ so that $\operatorname{ht}_{\mathscr{V}}(x) = \operatorname{ht}(x)$ for every $x : \operatorname{Spec} K \to \overline{\mathscr{M}}_{1,1}$ over a field of characteristic 2 or 3?*

In this note, we show that the answer is "no" when k is a perfect field of characteristic 3. More precisely, we have the following:

THEOREM 1.4. — *Let $\overline{\mathscr{M}}_{1,1}$ denote the Deligne–Mumford stack of stable elliptic curves over a perfect field k of characteristic 3. There is no vector bundle $\mathscr{V}$ on $\overline{\mathscr{M}}_{1,1}$ for which $\operatorname{ht}_{\mathscr{V}}(x) = \operatorname{ht}(x)$ for all points $x : \operatorname{Spec} K \to \overline{\mathscr{M}}_{1,1}$, where K is a finite extension of $k(t)$.*

We deduce this from Theorem 3.2 in Section 3.1.

This result leaves open the question as to whether there some vector bundle $\mathscr{W}$ on $\overline{\mathscr{M}}_{1,1}$, over a field of characteristic 3, and some integer n so that $n \operatorname{ht}(x) = \operatorname{ht}_{\mathscr{W}}(x)$. We conjecture the answer is no:

CONJECTURE 1.5. — *There is no vector bundle $\mathscr{W}$ on $\overline{\mathscr{M}}_{1,1}$, over a field of characteristic 3, for which there exists an integer n such that $n \operatorname{ht}(x) = \operatorname{ht}_{\mathscr{W}}(x)$.*

See [5, Remark 9.2.7 and 9.2.8] for some speculation related to this conjecture.

Another related question, originally posed to us by Jordan Ellenberg, is whether there exist Northcott stacky heights on $\overline{\mathscr{M}}_{1,1}$ in characteristic 3. We say a height function on the set of $k(t)$ points of a stack $\mathscr{X}$ satisfies the *Northcott property* if there are finitely many such points of bounded height, cf. [2, p. 4].

QUESTION 1.6 (Ellenberg). — *Does there exist a vector bundle $\mathscr{V}$ on $\overline{\mathscr{M}}_{1,1}$ over a finite field of characteristic 3 whose induced height function $\operatorname{ht}_{\mathscr{V}}$ is Northcott?*

Remark 1.7. — In [5, Theorem 9.2.4], I had claimed there do exist such Northcott bundles. However, the proof of this relies on [5, Lemma 9.7.1], which contains an error where I incorrectly claimed that trigonal curves have a certain minimal form without justification. This leaves the above question open.

1.1. Idea of the proof of Theorem 1.4

We use notation for Kodaira reduction type, as pictured in [7, p. 365]. The idea of the proof of Theorem 1.4 is to show that any $\mathscr{V}$ which induces the correct local stacky height for places of type Kodaira III reduction necessarily induces the incorrect local stacky height for cubic twists.

We now elaborate on the above idea. The substack $BG \subset \overline{\mathscr{M}}_{1,1}$ corresponding to elliptic curves with j-invariant 0 has geometric automorphism group G, where G is the dicyclic group of order 12. When we restrict $\mathscr{V}$ to BG, we obtain a G-representation ρ. We show that some element $g \in G$ of order 4 acts with a codimension 1 fixed space and no eigenvalues equal to -1. This is enough to deduce that ρ is a sum of 1-dimensional representations, and hence factors through the abelianization of G. We then show that any such vector bundle cannot detect nontrivial stacky heights associated to elliptic curves which are isotrivial cyclic cubic twists.

1.2. Overview

The structure of this paper is as follows. We review the notion of stacky heights in Section 2. We then reduce Theorem 1.4 to the statement about

local stacky heights, Theorem 3.2, at the end of Section 3. Finally, we prove Theorem 3.2 at the end of Section 6, using a group theoretic input from Section 4.

Acknowledgements

2. Review of the definition of heights on stacks

We now recall the definition of heights on stacks introduced in [2].

DEFINITION 2.1. — *Let k be a field, let C be a regular proper integral curve over k, and let $K := K(C)$. Let $\mathscr{X}$ be an algebraic stack over k and $x : \operatorname{Spec} K \to \mathscr{X}$ be a K-point. A tuning stack $\mathscr{C}$ for x is a normal algebraic stack $\mathscr{C}$ with finite diagonal together with a map $\bar{x} : \mathscr{C} \to \mathscr{X}$ extending x so that $\pi : \mathscr{C} \to C$ is a birational coarse space map. A tuning stack $(\mathscr{C}, \bar{x}, \pi)$ is a universal tuning stack if it is terminal among all all tuning stacks.*

Remark 2.2. — Although we will not need it, one can also extend Definition 2.1 to the number field case as follows. Let L be a number field, $B = \operatorname{Spec} \mathscr{O}_L$, and let $\mathscr{X}$ be an algebraic stack over B. Let K/L be a finite extension of number fields and $x : \operatorname{Spec} K \to \mathscr{X}$ be a K point. A tuning stack $\mathscr{C}$ for x is then a normal algebraic stack $\mathscr{C}$ with finite diagonal together with a map $\bar{x} : \mathscr{C} \to \mathscr{X}$ extending x so that $\pi : \mathscr{C} \to \operatorname{Spec} \mathscr{O}_K$ is a birational coarse space map.

Remark 2.3. — This definition of tuning stack differs slightly from that of [2, Definition 2.1], in that we do not require $\mathscr{X}$ to be a stack over C. In particular, if $\mathscr{X}$ is a stack over a base k, we may discuss tuning stacks and heights of points associated to function fields of transcendence degree 1 over k.

We now recall the notion of stacky height.

DEFINITION 2.4 ([2, Definition 2.11]). — *Let $\mathscr{X}$ be a proper algebraic stack over a field k with finite diagonal, let C be a smooth proper connected curve over k with function field $K(C)$. Let $\mathscr{V}$ be a vector bundle on $\mathscr{X}$ and $x \in \mathscr{X}(K(C))$. If $\mathscr{C}$ is a tuning stack for x and $\bar{x}, \pi$ are the corresponding maps defined in Definition 2.1, then we define the* height *of x with respect to $\mathscr{V}$ as*

$$\mathrm{ht}_{\mathscr{V}}(x) := -\deg_C\left(\pi_*\bar{x}^*\mathscr{V}^\vee\right).$$

We also define the stable height *of x with respect to $\mathscr{V}$ as*

$$\mathrm{ht}_{\mathscr{V}}^{\mathrm{st}}(x) := -\deg_{\mathscr{C}}\left(\bar{x}^*\mathscr{V}^\vee\right).$$

To make sense of this definition, one has to check various properties. For example, one must verify this notion is independent of the choice of tuning stack. These are verified in [2, § 2.2].

One can also define heights in the number field case, but then one has to use metrized line bundles and Arakelov heights as in [2, Appendix A]. We will only be concerned with the function field case, and so do not discuss this further.

Finally, we recall the notion of local stacky height, which will play a crucial role in our proof.

Notation 2.5. — Let L be a field and let K either be a number field or a finite extension of $L(t)$ with regular model C. Here, C is the spectrum of the ring of integers $\mathscr{O}_K$ in the case K is a number field and a regular proper curve over L when K is a function field. Suppose $x : \mathrm{Spec}\, K \to \mathscr{X}$ is a K point of a stack $\mathscr{X}$ locally of finite presentation and with finite diagonal, so that $\mathscr{X}$ possess a coarse moduli space $\alpha : \mathscr{X} \to X$. Let $\mathscr{C}$ denote the universal tuning stack associated to x, as in Definition 2.1 or Remark 2.2. Let C_v denote the localization of C at v and let $\mathscr{C}_v := \mathscr{C} \times_C C_v$. Then, we have a diagram

(2.1)
$$
\begin{array}{ccccc}
& \mathrm{Spec}\, K & & & \\
& \swarrow \qquad \downarrow & & \searrow^{x} & \\
\mathscr{C}_v & \longrightarrow & \mathscr{C} & \xrightarrow{\ \bar{x}\ } & \mathscr{X} \\
\downarrow & \searrow & \downarrow^{\pi} & & \downarrow^{\alpha} \\
C_v & \longrightarrow & C & \longrightarrow & X.
\end{array}
$$

In this setting, for $\mathscr{V}$ a vector bundle on $\mathscr{X}$ and v a place of K, we recall the definition of the local stacky height, which is equivalent to that given in [2, Definition 2.21]. The *local stacky height* associated to x and $\mathscr{V}$ at the

place v as

$$\delta_{\mathscr{V};v}(x) := \deg(\mathrm{coker}\,(\gamma^* \pi^* \pi_* \bar{x}^* \mathscr{V}^\vee \to \gamma^* \bar{x}^* \mathscr{V}^\vee)).$$

3. The reduction to $j = 0$

In this section, we explain how to reduce Theorem 1.4 to another result about the structure of our vector bundle restricted to a particular residual gerbe of $\overline{\mathscr{M}}_{1,1}$. In order to prove our main result, we will need to examine elliptic curves with extra automorphisms. The next remark describes them.

Remark 3.1. — Recall that there are only 2 possibilities for the geometric automorphism group of an elliptic curve in characteristic 3. It is either $\mathbb{Z}/2\mathbb{Z}$ or the dicyclic group of order 12, which we denote G. The following basic facts about G can be found in [4]. This dicyclic group G of order 12 is a semidirect product $G \simeq \mathbb{Z}/3\mathbb{Z} \rtimes \mathbb{Z}/4\mathbb{Z}$ where a generator of $\mathbb{Z}/4\mathbb{Z}$ acts on $\mathbb{Z}/3\mathbb{Z}$ by the nontrivial automorphism. For the remainder, we fix a splitting to identify $\mathbb{Z}/4\mathbb{Z}$ as a subgroup of G.

It will be useful to note that the center of G is $\mathbb{Z}/2\mathbb{Z}$, which can be identified as an index 2 subgroup of $\mathbb{Z}/4\mathbb{Z}$ for any choice of splitting $\mathbb{Z}/4\mathbb{Z} \to G$. The quotient of G by its central $\mathbb{Z}/2\mathbb{Z}$ is isomorphic to S_3.

Further, a semistable elliptic curve has geometric automorphism group G if and only if its j invariant is 0.

The key to proving Theorem 1.4 is the following:

THEOREM 3.2. — *Suppose k is an algebraically closed field of characteristic 3. Suppose $\mathscr{V}$ is a vector bundle on $\overline{\mathscr{M}}_{1,1}$ for which $\mathrm{ht}_{\mathscr{V}}(x) = \mathrm{ht}(x)$ for all points $x : \mathrm{Spec}\,K \to \overline{\mathscr{M}}_{1,1}$, for K ranging over finite extensions of $k(t)$. Let $\iota : [\mathrm{Spec}\,k/G] \to \overline{\mathscr{M}}_{1,1}$ be the residual gerbe over the point of j-invariant 0, with G as in Remark 3.1. If $\gamma : [\mathrm{Spec}\,k/(\mathbb{Z}/3\mathbb{Z})] \to [\mathrm{Spec}\,k/G]$ denotes the quotient by $\mathbb{Z}/4\mathbb{Z}$, then $\gamma^* \iota^* \mathscr{V}$ is trivial.*

We prove this in Section 6.1.
We next verify Theorem 1.4 assuming Theorem 3.2.

3.1. Proof of Theorem 1.4 assuming Theorem 3.2

To prove Theorem 3.2, that no vector bundle can induce a stacky height which agrees with Faltings height, We first observe we may assume k is

algebraically closed. Indeed, both Faltings height and stacky height are preserved under base change along extensions of k.

By Theorem 3.2, any point $x : \operatorname{Spec} K \to \overline{\mathscr{M}}_{1,1}$ factoring through $B(\mathbb{Z}/3\mathbb{Z})$ at the point of $\overline{\mathscr{M}}_{1,1}$ corresponding to j-invariant 0 must have $\operatorname{ht}_{\mathscr{V}}(x) = 0$. Therefore, it suffices to construct an elliptic curve over $\operatorname{Spec} k(t)$ so that the associated map $x : \operatorname{Spec} k(t) \to \overline{\mathscr{M}}_{1,1}$ factors through $\operatorname{Spec} k(t) \to [\operatorname{Spec} k/(\mathbb{Z}/3\mathbb{Z})] \to [\operatorname{Spec} k/G] \to \overline{\mathscr{M}}_{1,1}$ but so that x has nontrivial Faltings height. Indeed, we can easily construct cubic twists of the form $y^2 = x^3 - x + f(t)$ with nontrivial Faltings height. As a concrete example, we can take $f(t) = t + t^4$ which has additive reduction at infinity and Faltings height 1, as can be verified with a computer. It is therefore enough to verify the associated map $\operatorname{Spec} k(t) \to \overline{\mathscr{M}}_{1,1}$ factors through $[\operatorname{Spec} k/(\mathbb{Z}/3\mathbb{Z})]$. Indeed, the isotrivial elliptic curve $y^2 = x^3 - x + f(t)$ becomes trivial over the $\mathbb{Z}/3\mathbb{Z}$-extension $k(t)[v]/(v^3 - v - f(t))$, as then we can substitute $x - v$ for x to obtain

$$
\begin{aligned}
y^2 &= (x-v)^3 - (x+v) + f(t) \\
&= x^3 - x - (v^3 - v) + f(t) \\
&= x^3 - x - f(t) + f(t) \\
&= x^3 - x.
\end{aligned}
$$

4. Representations of G in characteristic 3

Throughout this section, we work over a field k of characteristic 3 containing all 4th roots of unity. Let G denote the dicyclic group of order 12, as described in Remark 3.1. In order to prove Theorem 3.2, we will need to analyze G-representations in characteristic 3. The only result in this section we will use in the proof of Theorem 3.2 is Proposition 4.5. To begin, we show we can decompose any G-representation into two subrepresentations, depending on how the center of G acts.

LEMMA 4.1. — *Let V be a G representation over a base field k of characteristic 3 containing 4th roots of unity. Then V splits as a direct sum $V_+ \oplus V_-$ where $V_+ \subset V$ is the subspace on which the nontrivial central element $\alpha \in G$ acts by* id *and $V_- \subset V$ is the subspace on which α acts by* $-$ id.

Proof. — This follows from a standard averaging trick. Namely, It is enough to realize V_1 and V_2 as subrepresentations of V. Since 2 is invertible on the base and α has order 2, $\rho(\alpha)$ is a diagonalizable matrix, so $V_1 \oplus V_2 = V$.

To check V_1 and V_2 are subrepresentations, it suffices to show that for any $v \in V_i$ and any $g \in G$, $\rho(g)v \in V_i$. We first check this for $i = 1$. Since $\rho(\alpha)|_{V_1} = \text{id}$ and $\rho(\alpha)|_{V_2} = -\text{id}$, we have $\frac{1}{2}(\text{id} + \rho(\alpha))$ is the projector $V \to V_1$ which acts as the identity on V_1. This projector is well defined as 2 is invertible on the base. Then for any $v \in V_1$, using that α is central in G,

$$\rho(g)v = \rho(g)\left(\frac{1}{2}(\text{id} + \rho(\alpha))\,v\right) = \left(\frac{1}{2}(\text{id} + \rho(\alpha))\right)(\rho(g)v) \in V_1$$

The case $i = 2$ is completely analogous, using the projector $\frac{1}{2}(\text{id} - \rho(\alpha))$ in place of $\frac{1}{2}(\text{id} + \rho(\alpha))$. $\qquad\square$

Using the surjection $G \to S_3$, we will want the following elementary fact about S_3 representations.

LEMMA 4.2. — *Let $\rho : S_3 \to \text{GL}(V)$ be a indecomposable representation over a field k of characteristic 3, with dimension at least 2. Let $\tau \in S_3$ be a transposition. Then, $\rho(\tau)$ has at least two distinct eigenvalues.*

Proof. — Because τ has order 2, $\rho(\tau)$ is semisimple, and all eigenvalues are ± 1. Therefore, it is enough to show $\rho(\tau)$ cannot be $\pm\,\text{id}$. After possibly tensoring with the sign representation, it is enough to show $\rho(\tau) \neq \text{id}$. Since transpositions generate S_3, $\rho(\tau) = \text{id}$ implies ρ is trivial. Since $\dim \rho > 1$, it is not indecomposable. $\qquad\square$

Remark 4.3. — According to [3, p. 160], there are precisely 6 indecomposable representations of S_3 in characteristic 3: the trivial representation, the standard representation, the permutation representation, and those three representations tensored with the sign representation, and one can also deduce Lemma 4.2 directly from this classification. I had claimed to give an alternate proof of this classification in [5, Theorem 9.8.2], though the proof there has a number of errors.

Combining the above lemmas, we next show that if an order 4 element has η as an eigenvalue under ρ, it also has $-\eta$ as an eigenvalue.

LEMMA 4.4. — *Let ρ be an indecomposable G-representation of dimension at least 2 over a field k of characteristic 3 containing 4th roots of unity, and $g \in G$ has order 4. If η is a primitive fourth root of unity which is an eigenvalue of $\rho(g)$, then so is $-\eta$.*

Proof. — Let $\alpha = g^2$ so that α generates the center of G. By Lemma 4.1, we have that either $V = V_+$ or $V = V_-$ where α acts on V_+ by id and α acts on V_- by $-\,\text{id}$. After tensoring ρ with a 1 dimensional representation in which g acts by η, we may assume $V = V_+$. Since α generates the kernel

of the surjection $G \to S_3$ and α acts trivially on $V = V_+$, ρ factors through S_3. By Lemma 4.2, $\rho(g)$ must have both ± 1 as an eigenvalue, as we wished to show.   □

We can now use the above lemma to deduce our main group-theoretic result for G-representations.

PROPOSITION 4.5. — *Suppose $\rho : G \to \mathrm{GL}(V)$ is a G-representation over a field k of characteristic 3 containing 4th roots of unity. If there is an order 4 element $g \in G$ so that $\rho(g)$ has a codimension 1 eigenspace with eigenvalue 1 and the remaining eigenvalue is a primitive fourth root of unity η, then ρ is a sum of 1-dimensional representations. In particular, $\rho : G \to \mathrm{GL}(V)$ factors through the abelianization of G, $\rho : G \to \mathbb{Z}/4\mathbb{Z} \to \mathrm{GL}(V)$.*

Proof. — By Lemma 4.4, if ρ is any indecomposable representation of dimension at least 2, then if a primitive fourth root of unity η shows up as an eigenvalue of $\rho(g)$, so does $-\eta$. Therefore, ρ is a sum of 1-dimensional representations. Hence ρ factors through the abelianization of G.   □

5. Review of local Faltings heights

In order to prove our main result, we will need the notion of local Faltings heights, which we now briefly review.

For K a finite extension of $k(t)$ and v a closed point of the proper regular model of K over k and $x : \mathrm{Spec}\, K \to \overline{\mathcal{M}}_{1,1}$, we let $\delta_{\mathscr{V};v}(x)$ denote the local stacky height associated to $\mathscr{V}$ and x at v, as defined in Notation 2.5. Further, [2] define a notion of stable stacky height $\mathrm{ht}^{\mathrm{st}}_{\mathscr{V}}(x)$, see Definition 2.4 which satisfies the relation $\mathrm{ht}^{\mathrm{st}}_{\mathscr{V}}(x) + \sum_v \delta_{\mathscr{V};v}(x) = \mathrm{ht}_{\mathscr{V}}(x)$. We next recall the analogous notion of local and stable Faltings height associated to elliptic curves.

DEFINITION 5.1. — *Let $x : \mathrm{Spec}\, K \to \overline{\mathcal{M}}_{1,1}$ be a point. For each closed point v, let $\mathrm{Spec}\, L \to \mathrm{Spec}\, K$ be a finite extension over which x acquires semistable reduction at v. Define the local stable Faltings height of x at v, notated $\mathrm{ht}^{\mathrm{st}}_v(x)$, to be $\mathrm{ht}^{\mathrm{st}}_v(x) := \frac{1}{12} \sum_{w|v} \frac{1}{\deg(L/K)} \deg(\Delta_w)$, for Δ_w the discriminant of w restricted to the local ring at w, for w ranging over closed points of L over v. Define the* stable Faltings height *by $\mathrm{ht}^{\mathrm{st}}(x) := \sum_v \mathrm{ht}^{\mathrm{st}}_v(x)$.*

Also, define the local Faltings height *of $x : \mathrm{Spec}\, K \to \overline{\mathcal{M}}_{1,1}$ at a closed point v, notated $\mathrm{ht}_v(x)$, to be $\frac{1}{12}(\deg \Delta_v) - \mathrm{ht}^{\mathrm{st}}_v(x)$.*

Using uniqueness of semistable models, one may verify the above notion of stable Faltings height is well defined, and can be computed explicitly in terms of the discriminant at various closed points v. In what follows, we use Kodaira's notation for reduction type of elliptic curves. See, for example, [7, IV Section 9], especially the chart on [7, p. 365].

Example 5.2. — When the fiber at a k-rational closed point v has reduction type I_n^*, we have $\deg \Delta_v = n + 6$ by [7, p. 365]. The valuation of the j-invariant of a given Kodaira reduction type is $\geqslant 0$ if and only if that curve has potentially good reduction after a base change, and is equal to $-n$ if that curve has I_n reduction after a base change. From [7, p. 365], we therefore find that $\mathrm{ht}_v^{\mathrm{st}}(x) = \frac{1}{12}n$, and hence $\mathrm{ht}_v(x) = \frac{1}{12}(n+6) - \frac{1}{12}n = 1/2$.

6. Computing heights on $\overline{\mathscr{M}}_{1,1}$

To conclude the proof of Theorem 1.4, it remains to prove Theorem 3.2. We do so at the end of this section. The basic idea is to compute what the restriction of $\mathscr{V}$ to BG has to be using Proposition 4.5 and studying the action of a certain order 4 element of G. This is done via Lemma 6.3, whose hypothesis is verified in Lemma 6.6.

Remark 6.1. — If we have an elliptic curve with reduction II, III, IV, II*, III*, or IV* at v, it has potentially good reduction, with corresponding j invariant 0 by [6, Theorem 2.1]. Therefore, if $x : \operatorname{Spec} K \to \overline{\mathscr{M}}_{1,1}$ is a map with one of the above reduction types at v, we must have that v maps to the point of $\overline{\mathscr{M}}_{1,1}$ lying over $j = 0$ in the coarse moduli space, for $j : \overline{\mathscr{M}}_{1,1} \to \mathbb{P}^1$ the coarse moduli space of $\overline{\mathscr{M}}_{1,1}$.

We next introduce some notation used heavily in the remainder of the proof.

Notation 6.2. — We will assume $k = \bar{k}$. Suppose we have some vector bundle $\mathscr{V}$ on $\overline{\mathscr{M}}_{1,1}$, a point $x : \operatorname{Spec} K \to \overline{\mathscr{M}}_{1,1}$, and a place v of K so that $\delta_{\mathscr{V};v}(x) = \mathrm{ht}_v(x)$. Note that $\overline{\mathscr{M}}_{1,1}$ has coarse space given by the j-invariant map $j : \overline{\mathscr{M}}_{1,1} \to \mathbb{P}^1_k$. Construct the universal tuning stack $\mathscr{C} \to \overline{\mathscr{M}}_{1,1}$. Let $B(G_v)$ denote the residual gerbe of $\mathscr{C}$ at the place v, so that we obtain an induced map $B(G_v) \to BG \xrightarrow{\iota} \overline{\mathscr{M}}_{1,1}$ inducing an injection $G_v \to G$ on inertia groups. Let s denote the geometric point over $BG \to \overline{\mathscr{M}}_{1,1}$. Then, under the identification between vector bundles on BG and G representations, $\iota^* \mathscr{V}$ can be viewed as a G-representation $\rho : G \to \mathrm{GL}(\mathscr{V}|_s)$.

Now, recall that in Remark 3.1, we chose a splitting $\mathbb{Z}/4\mathbb{Z} \to G$. Fix a generator 1 of $\mathbb{Z}/4\mathbb{Z}$ in G. This abelian group then has a diagonalizable action on $\mathscr{V}|_s$. When we restrict ρ to a $\mathbb{Z}/4\mathbb{Z}$ representation, we obtain a decomposition $\rho|_{\mathbb{Z}/4\mathbb{Z}} \simeq \oplus_{i=0}^3 \chi_i^{\oplus b_i}$ where χ_i are the four characters of $\mathbb{Z}/4\mathbb{Z}$ given by $\chi_i(1) = \zeta^i$ for ζ a fixed primitive 4th root of unity.

We can now characterize the b_i appearing in the above decomposition of $\rho|_{\mathbb{Z}/4\mathbb{Z}}$. For the proof, it will be useful to recall the notation for local heights introduced in Section 5.

LEMMA 6.3. — *Suppose $\mathscr{V}$ satisfies $\delta_{\mathscr{V};v}(x) = \mathrm{ht}_v(x)$ for some place v at which x has type III reduction. Under the above decomposition of $\rho|_{\mathbb{Z}/4\mathbb{Z}} \simeq \oplus_{i=0}^3 \chi_i^{\oplus b_i}$ from Notation 6.2, after possibly modifying our choice of generator 1 for $\mathbb{Z}/4\mathbb{Z}$, we have $b_0 = \dim \rho - 1$, $b_1 = 1$ and $b_2 = b_3 = 0$.*

Proof. — We first set up notation to describe the $\mathbb{Z}/4\mathbb{Z}$ representation. From the explicit computation of the discriminant for an elliptic curve $x : \operatorname{Spec} K \to \overline{\mathscr{M}}_{1,1}$ with a closed point v of type III reduction, we know $\mathrm{ht}_v(x) = \frac{1}{4}$, by [7, p. 365]. Therefore, we must have $\delta_{\mathscr{V};v}(x) = 1/4$. Now, x induces a tuning stack map $\bar{x} : \mathscr{C} \to \overline{\mathscr{M}}_{1,1}$ with coarse space $\pi : \mathscr{C} \to C$. Let C_v be the local scheme at $v \in C$ and let $\mathscr{C}_v = \pi^{-1}(C_v)$. Because v acquires semistable reduction after a degree 4 cyclic extension, but not after any smaller extension, the residual gerbe of $\mathscr{C}$ at v is $G_v := \mathbb{Z}/4\mathbb{Z}$ and the induced map on residual gerbes $B(G_v) \to BG$ is obtained from an injection $\mathbb{Z}/4\mathbb{Z} = G_v \to G$. Denote by $\alpha : B(G_v) \to BG \to \overline{\mathscr{M}}_{1,1}$ the composite map above. We may view $\alpha^* \mathscr{V}^\vee$ as a $\mathbb{Z}/4\mathbb{Z}$ representation.

We claim that under the above identification, we may view $\alpha^* \mathscr{V}^\vee$ as the direct sum of a 1-dimensional faithful representation of $\mathbb{Z}/4\mathbb{Z}$ and a codimension 1 trivial representation. This will complete the proof because after modifying the generator, we can assume the faithful representation is χ_1, in which case $b_0 = \dim \rho - 1$, $b_1 = 1$ and $b_2 = b_3 = 0$.

To prove our claim, we will need the following assertion: In general, if $\mathscr{W}$ is a vector bundle on $\mathscr{C}_v$, we assert one can read off the degree of $\mathscr{W}$ from the μ_4 representation corresponding to the restriction of $\mathscr{W}$ to the residual μ_4 gerbe over v. We now explain how this assertion follows from [1, Théorème 3.13], which shows there is a correspondence between vector bundles on the tame stacky curve $\mathscr{C}_v$ and parabolic bundles on C_v. This correspondence preserves degree, which is shown for proper curves in [1, Théorème 4.3], but the proof works equally well for localizations by restricting from the proper case. However, the definition of parabolic degree

on C_v, as in [1, Définition 4.1] is then given in terms of the corresponding μ_4 representation obtained by restricting the vector bundle on $\mathscr{C}_v$ to v.

We next describe how to compute the degree of the vector bundle $\mathscr{W}$ from the previous paragraph. Choose an isomorphism $\mu_4 \simeq \mathbb{Z}/4\mathbb{Z}$, which is possible since $k = \bar{k}$. Suppose the vector bundle $\mathscr{W}$ on $\mathscr{C}_v$ corresponds to a $\mathbb{Z}/4\mathbb{Z}$-representation ρ'. It follows from the definition of parabolic degree that we can express the degree of $\mathscr{W}$ as follows: Let ζ denote a fixed primitive fourth root of unity as in Notation 6.2. There is some generator $g \in \mathbb{Z}/4\mathbb{Z}$ so that $\rho'(g)$ has a b_i dimensional eigenspace with eigenvalue ζ^i, and $\deg \mathscr{W} = \sum_{i=1}^{3} \frac{b_i}{4}$.

We now return to proving our claim. If g is a generator of $\mathbb{Z}/4\mathbb{Z}$, we have seen any nontrivial eigenvalue of $\rho(g)$ contributes at least $1/4$ to the degree $\delta_{\mathscr{V};v}(x)$. Since $\delta_{\mathscr{V};v}(x) = 1/4$, we find that $\alpha^*\mathscr{V}^\vee$ must be the direct sum of a codimension 1 trivial representation and a 1-dimensional nontrivial representation. Moreover, that 1-dimensional representation must be faithful, as otherwise $\delta_{\mathscr{V};v}(x) = 1/4$ would be a multiple of $1/2$.   □

In order to apply Lemma 6.3 to prove Theorem 3.2, we need to verify its hypothesis holds. We verify this in Lemma 6.6 below. The specific x we will use is the following elliptic curve.

Example 6.4. — The magma code

```
F<t> := FunctionField(GF(3));
E := EllipticCurve([0,t+t^2,0,t+t^2 +t^3,t^2+t^4+t^5]);
LocalInformation(E);
```

shows that the elliptic curve $y^2 = x^3 + (t + t^2)x^2z + (t + t^2 + t^3)xz^2 + (t^2 + t^4 + t^5)z^3$ has a unique place of additive reduction (t). Further, at (t), this curve has reduction type III and discriminant of valuation 3.

We will use the following criterion for local stacky height to agree with local Faltings height.

LEMMA 6.5. — *Suppose $\mathscr{V}$ is a vector bundle on $\overline{\mathscr{M}}_{1,1}$ for which $\mathrm{ht}_{\mathscr{V}}(x) = \mathrm{ht}(x)$ for all points $x : \operatorname{Spec} K \to \overline{\mathscr{M}}_{1,1}$, for K a finite extension of $k(t)$. Any point $y : \operatorname{Spec} k(t) \to \overline{\mathscr{M}}_{1,1}$ which has a unique place v of additive reduction, i.e., a unique place with nontrivial local height, gives an example of a point for which $\delta_{\mathscr{V};v}(x) = \mathrm{ht}_v(x)$.*

Proof. — Let $K/k(t)$ denote an extension on which y acquires semistable reduction and let $z : \operatorname{Spec} K \to \overline{\mathscr{M}}_{1,1}$ denote the corresponding point. Because $\mathrm{ht}_{\mathscr{V}}(z) = \mathrm{ht}(z)$, and both $\mathrm{ht}_{\mathscr{V}}(z) = \mathrm{ht}_{\mathscr{V}}^{\mathrm{st}}(z) = \deg(K/k(t)) \, \mathrm{ht}_{\mathscr{V}}^{\mathrm{st}}(y)$ and $\mathrm{ht}(z) = \mathrm{ht}^{\mathrm{st}}(z) = \deg(K/k(t)) \, \mathrm{ht}^{\mathrm{st}}(y)$ [2, Proposition 2.14], we conclude $\mathrm{ht}_{\mathscr{V}}^{\mathrm{st}}(y) = \mathrm{ht}^{\mathrm{st}}(y)$. Because we are assuming $\mathrm{ht}_{\mathscr{V}}^{\mathrm{st}}(y) + \delta_{\mathscr{V};v}(y) =$

$\mathrm{ht}_{\mathscr{V}}(y) = \mathrm{ht}(y) = \mathrm{ht}^{\mathrm{st}}(y) + \mathrm{ht}_v(y)$ and we have shown $\mathrm{ht}^{\mathrm{st}}_{\mathscr{V}}(y) = \mathrm{ht}^{\mathrm{st}}(y)$, we conclude $\delta_{\mathscr{V};v}(y) = \mathrm{ht}_v(y)$.   □

The next lemma verifies the hypothesis of Lemma 6.3.

LEMMA 6.6. — *Suppose $\mathscr{V}$ is a vector bundle on $\overline{\mathscr{M}}_{1,1}$ for which $\mathrm{ht}_{\mathscr{V}}(x) = \mathrm{ht}(x)$ for all points $x : \operatorname{Spec} K \to \overline{\mathscr{M}}_{1,1}$, for K a finite extension of $k(t)$. The point $y : \operatorname{Spec} k(t) \to \overline{\mathscr{M}}_{1,1}$ of Example 6.4 (base changed from $\mathbb{F}_3$ to k) gives an example of a point for which $\delta_{\mathscr{V};v}(x) = \mathrm{ht}_v(x)$ at the place (t) of additive type III reduction.*

Proof. — Let $v = (t)$ denote the place of $k(t)$. Note this is the unique place of $k(t)$ at which y has additive reduction, by Example 6.4. The lemma then follows from Lemma 6.5.   □

We can now prove the main result, Theorem 3.2.

6.1. Proof of Theorem 3.2

Retain notation from Notation 6.2. By Lemma 6.6, the hypothesis of Lemma 6.3 holds, and so an order 4 element of G has a 1-dimensional ζ eigenspace and a codimension one 1-eigenspace. By Proposition 4.5, $\iota^*\mathscr{V}$ corresponds to a representation $\rho : G \to \mathrm{GL}(V|_s)$ that splits as the direct sum of 1-dimensional representations. factoring through $\mathbb{Z}/4\mathbb{Z}$. Therefore, $\rho|_{\mathbb{Z}/3\mathbb{Z}}$ is trivial.   □

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