



An introduction to the algebraic geometry of the Putman–Wieland conjecture

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Abstract

We give algebraic and geometric perspectives on our prior results toward the Putman–Wieland conjecture. This leads to interesting new constructions of families of “origami” curves whose Jacobians have high-dimensional isotrivial isogeny factors. We also explain how a hyperelliptic analogue of the Putman–Wieland conjecture fails, following work of Marković.

Keywords Ivanov’s conjecture · Hodge theory · Canonically embedded curves · Origami curves

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1 Introduction

The goal of this paper is to expand on some of our recent results towards the Putman–Wieland conjecture. Our hope is that this somewhat leisurely exposition will serve as a useful entry point for geometric topologists hoping to use the Hodge-theoretic

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techniques developed in [17] to study mapping class groups. While many of the results of that paper are somewhat stronger than those explained here, our hope is that stripping away some of the technical aspects of the proofs therein will clarify the relevant arguments.

1.1 Review of the Putman–Wieland conjecture

We use $\Sigma_{g,n}$ to denote an orientable topological surface of genus g with n punctures. Given a finite unramified H -cover of punctured topological surfaces $\Sigma_{g',n'} \rightarrow \Sigma_{g,n}$, there is an action of a finite index subgroup Γ of the mapping class group $\text{Mod}_{g,n+1}$ on $H_1(\Sigma_{g'}, \mathbb{C})$, as we now explain. Explicitly, $\text{Mod}_{g,n+1}$ acts on $\pi_1(\Sigma_{g,n}, x)$ for some basepoint x , and we can take Γ to be the stabilizer of the surjection $\phi: \pi_1(\Sigma_{g,n}, x) \twoheadrightarrow H$, where ϕ corresponds to the cover $\Sigma_{g',n'} \rightarrow \Sigma_{g,n}$. Then, for $x' \in \Sigma_{g',n'}$ mapping to x , Γ acts on $\pi_1(\Sigma_{g',n'}, x')$ and hence on $H_1(\Sigma_{g'}, \mathbb{C})$.

Conjecture 1.1 (Putman–Wieland, [24, Conjecture 1.2]) *Fix $g \geq 2$, $n \geq 0$. For any unramified cover $\Sigma_{g',n'} \rightarrow \Sigma_{g,n}$, the vector space $H_1(\Sigma_{g'}, \mathbb{C})$ has no non-zero vectors with finite orbit under the action of Γ .*

Note that the Putman–Wieland conjecture is false when $g = 2$ (see [21] or Proposition 6.7 below). We state an equivalent version of the Putman–Wieland conjecture in Conjecture 2.3.

1.2 Our prior results toward the Putman–Wieland conjecture

One of the main results of [17] implies that $H^1(\Sigma_{g',n'}, \mathbb{C})$ has no non-zero vectors with finite orbit when $g^2 > \#H$. Here is the precise statement.

Theorem 1.2 ([17, Theorem 1.4.1]) *With notation as in Sect. 1.1, for any H cover $\Sigma_{g',n'} \rightarrow \Sigma_{g,n}$, $H^1(\Sigma_{g',n'}, \mathbb{C})$ has no finite orbits under the action of Γ whenever $\#H < g^2$.*

This follows from Corollary 2.7 below, since any irrep of a finite group H has dimension at most $\sqrt{\#H}$.

The proof of Corollary 2.7 given in [17] somewhat obscures what is going on for two reasons. First, there we opted to give an indirect proof which was shorter given our other results in that paper. Second, the proof was complicated due to the technical necessity of dealing with marked points. In Sect. 3, we give a more streamlined account of the idea, which we hope will be easier to digest. We also try to motivate the vector bundle methods described there with an alternative approach to proving Theorem 1.2 via elementary projective geometry in Sect. 4.

Remark 1.3 While Theorem 1.2 shows $H^1(\Sigma_{g',n'}, \mathbb{C})$ contains no non-zero vectors with finite orbit when $\#H < g^2$, the same is not true of $H_1(\Sigma_{g',n'}, \mathbb{C})$, which is dual to $H^1(\Sigma_{g',n'}, \mathbb{C})$ by the universal coefficient theorem. The homology classes corresponding to loops around punctures have finite orbit under the mapping class group. This tells us that when $n' > 1$, $H^1(\Sigma_{g',n'}, \mathbb{C})$ has an $n' - 1$ dimensional space

of coinvariants under the action of a finite index subgroup of the mapping class group, but it has trivial invariants under the action of this subgroup. In particular, the Γ -action on $H^1(\Sigma_{g',n'}, \mathbb{C})$ is not semisimple. We have opted to state most of our results in terms of homology rather than cohomology as the statements are a bit cleaner in this formulation.

Remark 1.4 When the hypotheses of Theorem 1.2 are satisfied, it implies the corresponding case of Conjecture 1.1. Indeed, the action of Γ on $H^1(\Sigma_{g',n'}, \mathbb{C})$ has as a natural subrepresentation its action on $H^1(\Sigma_{g'}, \mathbb{C})$, which hence has no fixed vectors. This action is semisimple, hence the same is true for the dual representation on $H_1(\Sigma_{g'}, \mathbb{C})$.

1.3 Motivation for the Putman–Wieland conjecture: Ivanov’s conjecture

Much of the original motivation for the Putman–Wieland conjecture was as an approach to proving Ivanov’s conjecture, which is a major open question in the study of mapping class groups. Ivanov’s conjecture states that $\text{Mod}_{g,n}$ does not virtually surject onto $\mathbb{Z}$ for $g > 2, n \geq 0$. See [14, Section 7] and [16, Problem 2.11.A].

As a historical note, it appears that Ivanov only posed this as a question, and not a conjecture. However, since then, many sources have referred to this question as Ivanov’s conjecture.

If we use $I_{g,n}$ to denote the statement that Ivanov’s conjecture holds for (g, n) and $\text{PW}_{g,n}$ to denote the statement that the Putman–Wieland conjecture holds for (g, n) , then the relation between Ivanov’s conjecture and the Putman–Wieland conjecture is the following:

Theorem 1.5 ([24, Theorem 1.3]) *For $g \geq 3, n \geq 0$, $\text{PW}_{g-1,n+1} \implies I_{g,n}$ and $I_{g,n+1} \implies \text{PW}_{g,n}$.*

Technically, [24, Theorem 1.3] states a slightly different version of Theorem 1.5. However, the above formulation is equivalent, see Remark 2.5. Therefore, a proof of the Putman–Wieland conjecture would also yield a proof of Ivanov’s conjecture in almost all cases, and vice-versa.

1.4 Motivation for the Putman–Wieland conjecture: big monodromy

Another source of motivation for the Putman–Wieland conjecture comes from a perspective on monodromy groups. Indeed, a typical slogan in much of algebraic geometry and number theory is:

Slogan 1.6 Monodromy groups should be as big as possible.

The Putman–Wieland conjecture [24] is implied by this philosophy. Indeed, if one strongly believes Slogan 1.6, one might imagine that the intersection form and H -action on the cover $\Sigma_{g'}$ of Conjecture 1.1 are the only constraints on the (virtual) action of $\text{Mod}_{g,1}$, and so one might guess the image of the action of a finite index subgroup of $\text{Mod}_{g,1}$ on $H_1(\Sigma_{g'}, \mathbb{C})$ is via a finite-index subgroup of the centralizer

of H in $\mathrm{Sp}_{2g}(\mathbb{Z})$. The Putman–Wieland conjecture merely predicts this action has no non-zero vectors with finite orbit. Even though the Putman–Wieland conjecture is substantially weaker than determining the monodromy, there are no pairs (g, n) with $g \geq 2$ for which it is known to hold.

1.5 Counterexamples in low genus

For $g \leq 1$, it turns out that the intersection form and the H -action are *not* the only constraints on the (virtual) action of $\mathrm{Mod}_{g,n+1}$ on $H_1(\Sigma_{g',n'}, \mathbb{C})$; there are additional constraints arising from Hodge theory. That is, Slogan 1.6 is *false*, if naively interpreted. In Sect. 5 we give a number of new examples where the action of $\mathrm{Mod}_{g,n+1}$ factors through a smaller group than might be expected, for Hodge-theoretic reasons.

For example, we give a number of new counterexamples to the Putman–Wieland conjecture with $g = 1$, $n = 1$, arising from *origami curves*. These lead to interesting families of curves with high-dimensional fixed parts in their Jacobians in Examples 5.8–5.11. Most of our counterexamples are produced via a Hodge-theoretic criterion for the failure of Putman–Wieland, Proposition 5.4. These examples are the primary novelty of this paper. In particular, we give a sequence of families of “origami” curves whose Jacobians have isotrivial isogeny factors of arbitrarily large dimension in Example 5.10, and include a number of open questions in Sect. 5.2. In Sect. 6, we also discuss recent counterexamples of Marković for $g = 2$, which admit a routine generalization to the hyperelliptic setting in any genus. In Remark 6.12, we indicate how this hyperelliptic generalization demonstrates the sharpness of our methods for proving Theorem 1.2.

1.6 Overview

In Sect. 2 we recall our main results toward the Putman–Wieland conjecture. In Sect. 3, we sketch a streamlined version of our original proof, while in Sect. 4 we describe our original, more geometric approach to proving our results toward the Putman–Wieland conjecture. We discuss counterexamples to the Putman–Wieland conjecture in low genus in Sect. 5, and pose related questions in Sect. 5.2. We conclude with Sect. 6, giving an exposition of Marković’s genus 2 counterexample to the Putman–Wieland conjecture, based heavily on a theorem of Bogomolov–Tschinkel. We also explain a sense in which this generalizes to higher genus hyperelliptic curves.

1.7 Notation

Throughout, we work over the complex numbers, unless otherwise stated. For a pointed finite-type scheme or Deligne–Mumford stack (X, x) over $\mathbb{C}$, we will use $\pi_1(X, x)$ to denote the topological fundamental group of the associated complex-analytic space or analytic stack. We use Grothendieck conventions on projective space so that points of $\mathbb{P}V$ correspond to hyperplanes in V , as opposed to lines in V .

2 The Putman–Wieland conjecture

In this section we recall some notation related to the Putman–Wieland conjecture, as well as the main results toward it obtained in [17]. That paper proves some substantially more general results; we explain them here in the special case concerning the Putman–Wieland conjecture. Although we already stated the Putman–Wieland conjecture in Conjecture 1.1, it will be useful to have a slightly different version of it that also incorporates the covering group H . We next introduce notation to state this version precisely.

Notation 2.1 Let Σ_g be an oriented surface of genus $g \geq 0$, and let $\Sigma_{g,n}$ be the complement of $n \geq 0$ disjoint points in Σ_g , so that $\Sigma_{g,n}$ is an oriented genus g surface with n punctures. We let $\text{PMod}_{g,n}$ be the subgroup of the mapping class group of $\Sigma_{g,n}$ fixing the punctures. Fix a basepoint $v_0 \in \Sigma_{g,n}$, which we count as an additional puncture to obtain an action of $\text{PMod}_{g,n+1}$ on $\pi_1(\Sigma_{g,n}, v_0)$. Suppose we are given a finite covering $h: \Sigma_{g',n'} \rightarrow \Sigma_{g,n}$. Let $\Gamma \subset \text{PMod}_{g,n+1}$ denote the finite index subgroup preserving the covering h .

We obtain an action of Γ on $H_1(\Sigma_{g',n'}, \mathbb{C})$. By filling in the punctures, we also obtain an action of Γ on $V_h := H_1(\Sigma_{g'}, \mathbb{C})$, referred to in [24, pp. 80–81] as a *higher Prym representation*.

In the case h is Galois with covering group H , we may view $H_1(\Sigma_{g',n'}, \mathbb{C})$ as an H -representation. If ρ is an irreducible H -representation we let

$$H_1(\Sigma_{g',n'}, \mathbb{C})^\rho := \rho \otimes \text{Hom}_H(\rho, H_1(\Sigma_{g',n'}, \mathbb{C}))$$

denote the ρ -isotypic component. In this case, we obtain an action of Γ on the characteristic subrepresentation $H^1(\Sigma_{g',n'}, \mathbb{C})^\rho \subset H^1(\Sigma_{g',n'}, \mathbb{C})$.

Definition 2.2 Fix non-negative integers g and n and a finite covering $h: \Sigma_{g',n'} \rightarrow \Sigma_{g,n}$. Let $\text{PW}_{g,n}^H$ be the statement that for any Galois H -covering h and $v \in V_h$ any non-zero vector, v has infinite orbit under Γ .

Conjecture 2.3 (Putman–Wieland, [24, Conjecture 1.2]) $\text{PW}_{g,n}^H$ holds for every group H with $g \geq 2$, $n \geq 0$.

Remark 2.4 It is known that the Putman–Wieland conjecture is not in general true for $g = 2$; see [21, Theorem 1.3] or Proposition 6.7 below.

Remark 2.5 We note that there are several differences between the original [24, Conjecture 1.2] and Conjecture 2.3, but the statements are equivalent, see also [17, Remark 6.1.4]. First, [24, Conjecture 1.2] is stated without a group H and with $\mathbb{Q}$ coefficients instead of $\mathbb{C}$ coefficients. Additionally, [24, Conjecture 1.2] keeps track of boundary components in addition to punctures, but it is enough to treat the case of only punctures by [17, Lemma 6.2.5].

We now recall the main result of [17] toward the Putman–Wieland conjecture.

Theorem 2.6 ([17, Theorem 6.2.1]) *With notation as in Notation 2.1, let ρ be an irreducible complex H -representation and let $\Gamma' \subset \Gamma$ be a finite-index subgroup. Then $H^1(\Sigma_{g',n'}, \mathbb{C})^\rho$ has no non-zero Γ' -invariant subrepresentations of dimension strictly less than $2g - 2 \dim \rho$. The same holds for $H_1(\Sigma_{g'}, \mathbb{C})^\rho$ in place of $H^1(\Sigma_{g',n'}, \mathbb{C})^\rho$.*

Because the span of a fixed vector is a 1-dimensional subrepresentation, and $1 < 2g - 2 \dim \rho$ if and only if $\dim \rho < g$, we get the following important corollary:

Corollary 2.7 ([17, Corollary 6.2.3]) *$\text{PW}_{g,n}^H$ holds for every group H such that every irreducible representation ρ of H has dimension $\dim \rho < g$.*

In particular, the hypotheses of Corollary 2.7 are satisfied if $\#H < g^2$, implying Theorem 1.2.

3 The algebraic proof of Corollary 2.7

In this section, we outline the idea of the proof of Corollary 2.7, stripping away some of the additional technical complications from [17].

We restrict to the unramified case, as the ramified case is analogous, once one replaces usual vector bundles with parabolic vector bundles.

To start, given $\phi: \pi_1(\Sigma_g) \rightarrow H$, we set up a parameter space of H -covers. We begin by finding a diagram

$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{f} & \mathcal{C} \\ & \searrow \pi' & \downarrow \pi \\ & & \mathcal{M} \end{array} \quad (3.1)$$

where π is a relative curve of genus g , π' is a relative curve of genus g' , and f is an H -cover, such that the map $\mathcal{M} \rightarrow \mathcal{M}_g$ induced by π is dominant étale, and such that for $m \in \mathcal{M}$, the fiber $f_m: X \rightarrow Y$ is a finite étale connected Galois H -cover corresponding to ϕ .

If $\text{PW}_{g,0}^H$ were false, there would be a non-zero vector $v \in H_1(X, \mathbb{Q})$ fixed by a finite index-subgroup of $\text{Mod}_{g,0} = \pi_1(\mathcal{M}_g)$. After replacing $\mathcal{M}$ with a finite étale cover, we may thus assume v is in fact fixed by $\pi_1(\mathcal{M})$. This implies that the natural map $\nabla: H^0(R^1\pi'_*\mathbb{C} \otimes \mathcal{O}_{\mathcal{M}}) \rightarrow H^0(R^1\pi'_*\mathbb{C} \otimes \Omega_{\mathcal{M}})$ induced by the Gauss–Manin connection has non-zero kernel. Setting $\mathcal{E} = R^1\pi'_*\mathbb{C} \otimes \mathcal{O}_{\mathcal{M}}$, Hodge theory gives a filtration

$$\pi'_*\Omega_{\mathcal{X}/\mathcal{M}} =: F^1\mathcal{E} \subset \mathcal{E} := R^1\pi'_*\mathbb{C} \otimes \mathcal{O}_{\mathcal{M}}$$

with quotient $\mathcal{E}/F^1\mathcal{E} \simeq R^1\pi'_*\mathcal{O}_{\mathcal{X}}$. The theorem of the fixed part tells us that the kernel of the Gauss–Manin connection in fact arises from a sub-Hodge structure of $R^1\pi'_*\mathbb{C}$, and hence meets $F^1\mathcal{E}$ non-trivially. This type of argument is spelled out in more detail

in Lemma 6.10. The connection ∇ , which is only a $\mathbb{C}$ -linear map, induces a $\mathcal{O}_{\mathcal{M}}$ -linear map

$$\bar{\nabla}: \pi'_* \Omega_{\mathcal{X}/\mathcal{M}} \rightarrow R^1 \pi'_* \mathcal{O}_{\mathcal{X}} \otimes \Omega_{\mathcal{M}},$$

which again has a non-trivial kernel, using the theorem of the fixed part.

We now relate the kernel of $\bar{\nabla}$ to the vanishing of a particular map on cohomology. As above, fix a point $m \in \mathcal{M}$, whose fiber yields a finite étale cover $f_m: X \rightarrow Y$. The map $\bar{\nabla}_m$ may be identified as a map

$$\bar{\nabla}_m: H^0(X, \omega_X) \rightarrow H^1(X, \mathcal{O}_X) \otimes H^0(Y, \omega_Y^{\otimes 2}).$$

Using Serre duality and the identification $H^0(X, \omega_X) \simeq H^0(Y, f_{m*} \omega_X)$, this can be identified with the map

$$\begin{aligned} \theta: H^0(Y, f_{m*} \omega_X) &\rightarrow \text{Hom}(H^0(Y, f_{m*} \omega_X), H^0(Y, \omega_Y^{\otimes 2})) \\ s &\mapsto (t \mapsto \text{tr}(s \otimes t)). \end{aligned} \quad (3.2)$$

By assumption θ has a non-zero kernel. Any non-zero element of the kernel yields a non-zero map $f_{m*} \omega_X \rightarrow \omega_Y^{\otimes 2}$, inducing the zero map on global sections.

Finally, we use the vanishing of the above map on cohomology to produce a stable vector bundle of low rank and high slope on Y which is not generically globally generated, furnishing a contradiction. Given a representation $\rho: \pi_1(Y) \rightarrow \text{GL}_r(\mathbb{C})$, we let ρ be the associated local system on Y and set $E^\rho = \rho \otimes \mathcal{O}_Y$. Note that if ρ has finite (or, indeed, unitary) image, then E^ρ is semistable of slope zero. We can identify $f_{m*} \mathcal{O}_X \simeq \bigoplus_{H\text{-irreps } \rho} (E^\rho)^{\oplus \dim \rho}$, by viewing $f_{m*} \mathbb{C}$ as a local system on Y induced by the regular representation of H , and tensoring up with $\mathcal{O}_Y$. Hence by the projection formula,

$$f_{m*} \omega_X = f_{m*} (f_m^* \omega_Y) = \omega_Y \otimes f_{m*} \mathcal{O}_X = \omega_Y \otimes \left(\bigoplus_{H\text{-irreps } \rho} (E^\rho)^{\oplus \dim \rho} \right).$$

As we have produced a non-zero map $f_{m*} \omega_X \rightarrow \omega_Y^{\otimes 2}$, inducing the zero map on global sections, there must be some ρ and a non-zero map $E^\rho \otimes \omega_Y \rightarrow \omega_Y^{\otimes 2}$ which induces the 0 map on global sections. However, $E^\rho \otimes \omega_Y$ is then a semistable vector bundle of slope $2g - 2$, all of whose global sections factor through the subbundle $U := \ker(E^\rho \otimes \omega_Y \rightarrow \omega_Y^{\otimes 2})$. This will force U to have a very high-dimensional space of global sections relative to its rank.

We now explain why the condition that $H^0(Y, U) = H^0(Y, E^\rho \otimes \omega_Y)$ leads to a contradiction. This is in tension with a version of Clifford's theorem for vector bundles, which says that if U is a vector bundle which has a filtration by semistable bundles of slopes ≥ 0 and $\leq 2g$, then $h^0(Y, U) \leq \frac{\deg U}{2} + \text{rk } U$ [18, Lemma 6.2.1]. This idea is used to prove (without too much difficulty) the more precise [18, Proposition 6.3.1] which tells us that such a U can only exist when $\dim \rho = \text{rk } E^\rho \geq g$. We now spell this proof out in the case $\text{rk } U = \text{rk } E^\rho \otimes \omega_Y - 1$. Note first that $\mu(U) \leq \mu(E^\rho \otimes \omega_Y)$ by

semistability of $E^\rho \otimes \omega_Y$ and so $\mu(U) \leq (2g - 2) \operatorname{rk} U = (2g - 2)(\operatorname{rk} E^\rho \otimes \omega_Y - 1)$. Riemann–Roch, Clifford’s theorem, and the above observation together yield

$$\begin{aligned} \deg E^\rho \otimes \omega_Y + (1 - g) \operatorname{rk} E^\rho \otimes \omega_Y &\leq h^0(Y, E^\rho \otimes \omega_Y) = h^0(U) \\ &\leq \operatorname{rk} U + \deg U/2 \\ &\leq (\operatorname{rk} E^\rho \otimes \omega_Y - 1) + (g - 1)(\operatorname{rk} E^\rho \otimes \omega_Y - 1). \end{aligned}$$

Solving for $\operatorname{rk} E^\rho \otimes \omega_Y$ gives $\dim \rho = \operatorname{rk} E^\rho \otimes \omega_Y \geq g$.

This verifies $\operatorname{PW}_{g,0}^H$, since we assumed H was a group all of whose representations had rank less than g .

Remark 3.1 In fact the methods here can be used to prove a stronger statement, about families of curves that do not necessarily dominate $\mathcal{M}_g$. Indeed, let

$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{f} & \mathcal{C} \\ & \searrow \pi' & \downarrow \pi \\ & & \mathcal{M} \end{array}$$

be as in (3.1), except we now assume that the map $\mathcal{M} \rightarrow \mathcal{M}_g$ has image of codimension δ ; we assume $\mathcal{M} \rightarrow \mathcal{M}_g$ is étale onto its image. Arguing as above, we find that $H^0(E^\rho \otimes \omega_Y) \rightarrow H^0(\omega_Y^{\otimes 2})$ has rank δ , and so replacing the use of [18, Proposition 6.3.1] with [18, Proposition 6.3.6], we obtain:

Theorem 3.2 *With notation as above, suppose $\dim \rho < g - \delta$. Then the action of $\pi_1(\mathcal{M}, m)$ on $H^1(\mathcal{X}_m, \mathbb{C})^\rho$ has no non-zero finite orbits.*

Analogously to Theorem 1.2, one immediately obtains that for $|H| < (g - \delta)^2$, the action of $\pi_1(\mathcal{M}, m)$ on $H^1(\mathcal{X}_m, \mathbb{C})$ has no non-zero vectors with finite orbit. Replacing usual vector bundles with parabolic vector bundles, one may prove an analogue of Theorem 3.2 for ramified covers.

4 A geometric approach to Theorem 2.6

While we were thinking about the Putman–Wieland conjecture, we first came up with a rather different argument for proving Theorem 2.6, which involves analyzing the geometry of the canonical map, and in particular the quadrics which contain the covering curve under its canonical map. We discovered that a certain degeneracy locus naturally appeared in the intersection of these quadrics, which yields a different method to produce the same non-generically globally generated vector bundles that are described in Sect. 3. It was only after we found this geometric approach that we were able to somewhat simplify the presentation via the algebraic argument described in Sect. 3. In this section, we aim to present our original, geometric approach. It is our hope that sharing this more geometric approach may shed some additional light on the problem, and may also be of use in related problems. Since we are only trying

to convey the main idea, as in Sect. 3, we will assume $n = 0$, i.e., the relevant cover $X \rightarrow Y$ is étale.

The first steps in this geometric approach are the same as those described in Sect. 3. Namely, by studying the Gauss–Manin connection, we find that given a counterexample to Putman–Wieland, the natural map

$$H^0(X, \omega_X) \xrightarrow{\theta} \text{Hom}(H^0(X, \omega_X), H^0(Y, \omega_Y^{\otimes 2}))$$

has non-zero kernel. If $X \rightarrow Y$ is an H -cover, we can decompose the above as H -representations, and there must be some irreducible H -representation ρ and a sub-representation $\rho \subset H^0(X, \omega_X)$ in the kernel of θ . Let us now try to understand what this copy of ρ buys us in terms of the canonical map $X \rightarrow \mathbb{P}H^0(X, \omega_X)$. Recall here we are using Grothendieck conventions, so points of $\mathbb{P}H^0(X, \omega_X)$ correspond to hyperplanes in $H^0(X, \omega_X)$. By inspection of (3.2), the map

$$H^0(X, \omega_X) \otimes H^0(X, \omega_X) \rightarrow H^0(Y, \omega_Y^{\otimes 2})$$

adjoint to θ factors through $\text{Sym}^2 H^0(X, \omega_X)$. Hence, restricting to $\rho \otimes H^0(X, \omega_X)$ we obtain that the composition

$$\rho \otimes H^0(X, \omega_X) \xrightarrow{\alpha_1} \text{Sym}^2 H^0(X, \omega_X) \rightarrow H^0(Y, \omega_Y^{\otimes 2}) \quad (4.1)$$

vanishes. Observe that $H^0(Y, \omega_Y^{\otimes 2})$ carries the trivial H -representation, and hence ρ can only pair non-trivially under the above composition with $H^0(X, \omega_X)^{\rho^\vee}$, the $\rho^\vee$ isotypic part of $H^0(X, \omega_X)$. Hence, the vanishing of (4.1) is equivalent to the vanishing of

$$\rho \otimes H^0(X, \omega_X)^{\rho^\vee} \xrightarrow{\alpha_1} \text{Sym}^2 H^0(X, \omega_X) \xrightarrow{\alpha_2} (\text{Sym}^2 H^0(X, \omega_X))^H \rightarrow H^0(X, \omega_X^{\otimes 2})^H = H^0(Y, \omega_Y^{\otimes 2}), \quad (4.2)$$

where α_1 is the multiplication map, α_2 is the averaging map

$$q \mapsto \frac{1}{\#H} \sum_{h \in H} h^* q,$$

and $\alpha := \alpha_2 \circ \alpha_1$. Indeed, viewing $\text{Sym}^2 H^0(X, \omega_X)$ as the space of quadrics on $\mathbb{P}H^0(X, \omega_X)$, any element of the image of α yields a quadric vanishing on the image of X under the canonical map.

Let us derive some consequences from this observation. To get a feeling for what is going on, we work out examples depending on the rank of ρ , so as to motivate the general case. Suppose Y has genus g . By the Chevalley–Weil formula (see the original source [7] or the more expository [23, Theorem 2.1]), $H^0(X, \omega_X)^{\rho^\vee}$ is a direct sum of $(g - 1) \cdot \dim \rho^\vee$ many copies of $\rho^\vee$, and so has dimension $(g - 1) \cdot (\dim \rho)^2$.

4.1 Summary of the remainder of this section

We first see how to use the above observation to rule out 1-dimensional representations in Sect. 4.2. We use $r := \dim \rho$. Following this, we will see how to use the above to produce a vector bundle $E^\rho \otimes \omega_Y$ which is not generically globally generated in the cases $(r = 2, g = 2)$ in Sect. 4.3, $(r = 3, g = 2)$ in Sect. 4.4, and then the general case in Sect. 4.5. We stress that in these examples, we are not trying to prove or disprove the Putman–Wieland conjecture, but merely to indicate a geometric reason for why $E^\rho \otimes \omega_Y$ is not generically globally generated. One can then argue as in Sect. 3 that any vector bundle which is not generically globally generated must have $\text{rank } E^\rho \otimes \omega_Y \geq g$ using Clifford’s Theorem, and so this would lead to a proof of the Putman–Wieland conjecture in the case $\dim \rho < g$. Again, there is a valid alternate argument given in Sect. 3 of why $E^\rho \otimes \omega_Y$ fails to be generically globally generated. The point of this section is to give an alternate, more geometric explanation of that fact.

4.2 The dimension of ρ is 1

We first indicate how to rule out the case that ρ has rank 1. The main idea is that the vanishing of (4.2) yields a quadric of rank 2 containing the image of X under the canonical map. Since rank 2 quadrics vanish on a union of two hyperplanes, this would force the image of X under the canonical map to be contained in a hyperplane, contradicting non-degeneracy of the canonical map. This conclusion is established in Remark 4.3.

Lemma 4.1 *Let α be as in (4.2). Any quadric in $\text{im}(\alpha) \subset (\text{Sym}^2 H^0(X, \omega_X))^H$ has rank at most $2 \dim \rho$.*

Proof Because the multiplication map is H -equivariant, any given map $\rho \otimes H^0(X, \omega_X) \rightarrow H^0(Y, \omega_Y^{\otimes 2})$, where the target has the trivial H -action, must factor through a map

$$\rho \otimes H^0(X, \omega_X)^{\rho^\vee} \rightarrow \rho \otimes \rho^\vee \rightarrow H^0(Y, \omega_Y^{\otimes 2})$$

for $H^0(X, \omega_X)^{\rho^\vee} \rightarrow \rho^\vee$ an irreducible quotient representation isomorphic to the dual of ρ . This in turn factors through $\text{Sym}^2(\rho \oplus \rho^\vee)$, and so the resulting quadric has rank at most $2 \dim \rho = \dim \rho \oplus \rho^\vee$.

The reason for the claimed factorization is that the given map factors through the quotient $\rho \otimes (H^0(X, \omega_X)^{\rho^\vee}/K)$, where K is the subspace of $H^0(X, \omega_X)^{\rho^\vee}$ pairing to zero with ρ ; the quotient $H^0(X, \omega_X)^{\rho^\vee}/K$ is isomorphic as an H -representation to $\rho^\vee$. $\square$

Proposition 4.2 *Suppose that the genus of Y is at least 2. Then there are no ρ of dimension 1 in the kernel of $\overline{\nabla}_m$.*

Proof Any quadric of rank at most 2 is supported on a union of hyperplanes. Since the canonical map is always non-degenerate, its image cannot be contained in a hyperplane.

Since X is smooth and connected, it also cannot be contained in the vanishing locus of a quadric of rank at most 2. But Lemma 4.1 implies that if a 1-dimensional representation was in the kernel of the $\bar{\nabla}_m$, the image of the canonical map would be contained in such a quadric. $\square$

Remark 4.3 As in Sect. 3, Proposition 4.2 immediately implies if $\Sigma_{g'} \rightarrow \Sigma_g$ is an H -cover, with $g \geq 2$, and $\rho: H \rightarrow \mathbb{C}^\times$ is one-dimensional, no non-zero vector in $H^1(\Sigma_{g'}, \mathbb{C})^\rho$ has finite orbit under the (virtual) action of Mod_g . One might say this verifies the “Putman–Wieland conjecture for 1-dimensional representations.” A related, but somewhat stronger statement is originally due to Looijenga [19].

4.3 The dimension of ρ is 2 and $g = 2$

To simplify matters somewhat, we will restrict to the case $g = 2$, where all the main ideas are already present and we further assume that ρ is *not self-dual*. Though see Remark 4.4 for a remark on what happens in the self-dual case.

In the case $\dim \rho = 2$, we have a $2(g - 1) = 2$ dimensional space

$$\text{Hom}_H(\rho^\vee, H^0(X, \omega_X))$$

by Chevalley–Weil, so $H^0(X, \omega_X)^{\rho^\vee} \simeq (\rho^\vee)^{\oplus 2}$. We can consider the $2+2 \cdot 2$ subspace of $H^0(X, \omega_X)$ given by $\rho \oplus H^0(X, \omega_X)^{\rho^\vee}$, where here ρ lies in the kernel of θ from (3.2). From this, we get two quadrics Q_1 and Q_2 of rank 4 vanishing on the image of X under the projection of the canonical map to $\mathbb{P}(\rho \oplus H^0(X, \omega_X)^{\rho^\vee})$. Here, Q_1 and Q_2 are non-zero quadrics, corresponding to the universal quadrics expressing the incidence relation between ρ and the two copies of $\rho^\vee$.

Remark 4.4 This crucially uses that ρ is not self-dual, as in the case ρ is symplectically self-dual, one of these quadrics may be identically the 0 quadric. What changes in the above is that ρ is already contained in $H^0(X, \omega_X)^{\rho^\vee}$, so $\rho \oplus H^0(X, \omega_X)^{\rho^\vee}$ is not a subspace of $H^0(X, \omega_X)$.

To see that one of the quadrics will be 0 in the symplectically self-dual case, we use the fact that there is an isomorphism $\rho \simeq \rho^\vee$ inducing an injection $\wedge^2 \rho \rightarrow \rho \otimes \rho^\vee \simeq \rho \otimes \rho$, (using that we are in characteristic 0 to split the natural surjection) whose projection to $\text{Sym}^2 \rho$ vanishes. This induces the 0 quadric. When ρ is not self-dual, we always obtain two quadrics of rank 4. When ρ is symplectically self-dual, we obtain one quadric of rank 4, while when ρ is orthogonally self-dual, we obtain one quadric of rank 4 and one quadric of rank 3.

To make the situation even more concrete, continue supposing $\dim \rho = 2$ and $g = 2$, give ρ coordinates x, y , and give the two copies of $\rho^\vee$ dual coordinates u, v and s, t . Then, the two quadrics Q_1 and Q_2 can be concretely expressed as $xu - yv = 0$ and $xs - yt = 0$. The variety $Q_1 \cap Q_2$ is a degree 4 variety which has two components: it is the union of the codimension 2 plane P given by $x = y = 0$ and the degree 3 variety V given by $xu - yv = xs - yt = vs - tu = 0$. Since the curve X is irreducible

$$\mathbb{P}(\rho \oplus \rho^\vee \otimes \operatorname{Hom}_H(\rho^\vee, H^0(X, \omega_X)))$$

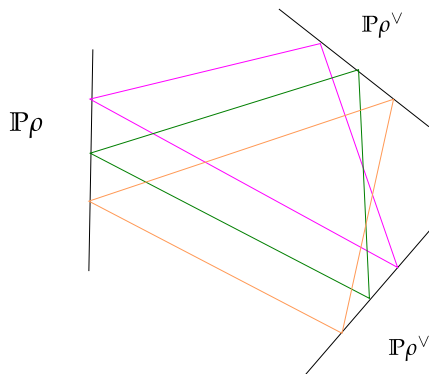


Fig. 1 A picture of the variety V swept out by colored planes, which is a component of the variety $Q_1 \cap Q_2$ cut out by certain quadrics containing the canonical image of X in the case that a 2-dimensional representation lies in the kernel of θ . Each colored plane corresponds to a point $x \in \mathbb{P}\rho$ and is the span x together with the corresponding hyperplanes in each copy of $\mathbb{P}\rho^\vee$ on which x vanishes

and non-degenerate, it cannot lie in the component P and so must be contained in V . We have now managed to produce a new equation containing X , $vs - tu$, which is only a function of the variables in $H^0(X, \omega_X)^{\rho^\vee}$ and does not depend on the ρ component. Moreover, it is the degeneracy locus of the matrix

$$\begin{pmatrix} s & t \\ u & v \end{pmatrix},$$

which already gives a hint as to where the failure of generic global generation of $E^\rho \otimes \omega$ may come from.

There is yet another geometric perspective on the variety $xu - yv = xs - yt = vs - tu = 0$. Starting with a point $[x, y]$ in $\mathbb{P}\rho$, we get corresponding hyperplanes in $\mathbb{P}\rho^\vee$ at which $[x, y]$ vanishes. In the case $\dim \rho = 2$, this hyperplane has codimension 1 in the 1-dimensional $\mathbb{P}\rho^\vee$, and hence corresponds to a point. The variety $xu - yv = xs - yt = vs - tu = 0$ expresses the incidence relation between these two; it parameterizes tuples (x, y, s, t, u, v) so that $[x, y]$ vanishes on both $[s, t]$ and $[u, v]$. We will next see a similar phenomenon when $\dim \rho = 3$.

4.4 The dimension of ρ is 3 and $g = 2$

We next consider the case $\dim \rho = 3$ and $g = 2$. In this subsection, we will explain how to show $E^\rho \otimes \omega_Y$ fails to be generically globally generated, which will not in general lead to a contradiction, but which motivates our arguments in Sect. 3, see Sect. 4.1. As in the dimension 2 case of Sect. 4.3, we obtain a variety cut out by three quadrics expressing the incidence between ρ and $H^0(X, \omega_X)^{\rho^\vee} \simeq (\rho^\vee)^{\oplus 3}$. Viewing this as a subvariety of $\mathbb{P}(\rho \oplus (\rho^\vee)^{\oplus 3})$ we find this degree 8, codimension 3 subvariety is the union of the codimension 3 plane $\rho = 0$ and a certain degree 7 codimension 3

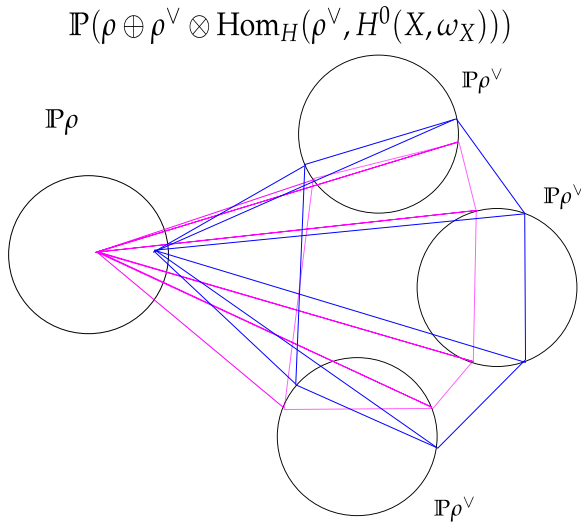


Fig. 2 A picture of the variety V swept out by colored planes, which is contained in the intersection of certain quadrics containing the canonical image of X when a 3-dimensional representation ρ lies in the kernel of θ . Each colored plane corresponds to a point $x \in \mathbb{P}\rho$ and is the span x together with the corresponding hyperplanes in each copy of $\mathbb{P}\rho^\vee$ on which x vanishes

variety, which can be expressed as the closure V of the incidence variety

$$\{(z, a, b, c) \in \rho \oplus (\rho^\vee)^{\oplus 3} : z \neq 0, z(a) = z(b) = z(c) = 0\}.$$

As in the previous case, the image of X is irreducible and non-degenerate, so cannot lie on the codimension 3 plane $\rho = 0$, and hence lies in V .

Next, let us try to understand how the non-generically globally generated vector bundle appearing in Sect. 3 is related to this geometric setup. To find it, we project V away from $\mathbb{P}\rho \subset \mathbb{P}(\rho \oplus (\rho^\vee)^{\oplus 3})$. Under this projection, we get that V projects to

$$W := \{(a, b, c) \in (\rho^\vee)^{\oplus 3} : \text{there exists } z \neq 0, z(a) = z(b) = z(c) = 0\}.$$

This can equivalently be expressed as

$$W = \{(a, b, c) \in (\rho^\vee)^{\oplus 3} : (a, b, c) \text{ lie on a common line in } \mathbb{P}\rho\}.$$

The map $\iota: X \rightarrow \mathbb{P}(\rho \oplus \text{Hom}_H(\rho^\vee, H^0(X, \omega_X)))$ is induced by a sub-linear system of ω_X . That is, $\mathcal{O}_{\mathbb{P}(\rho \oplus \text{Hom}_H(\rho^\vee, H^0(X, \omega_X)))}(1)$ pulls back to ω_X , or a subsystem thereof if ω_X has basepoints in $\rho \oplus \text{Hom}_H(\rho^\vee, H^0(X, \omega_X)) \subset H^0(\omega_X)$. Therefore, the above map ι is induced by a map of sheaves $H^0(X, \omega_X)^{\rho^\vee} \otimes \mathcal{O}_X \rightarrow \omega_X$. Since $\rho^\vee \otimes \text{Hom}_H(\rho^\vee, H^0(X, \omega_X)) \simeq H^0(X, \omega_X)^{\rho^\vee}$, this also corresponds to a map $\rho^\vee \otimes \text{Hom}_H(\rho^\vee, H^0(X, \omega_X)) \rightarrow \omega_X$ or equivalently $\psi: \text{Hom}_H(\rho^\vee, H^0(X, \omega_X)) \otimes \mathcal{O}_X \rightarrow \rho \otimes \omega_X$. This is an H -equivariant map, and so descends to a map of vector bundles on Y .

Under the above translation, the condition defining W , that a, b, c are collinear, corresponds to the map ψ not having maximal rank. Here ρ denotes a trivial bundle of rank r on X with a specified H -action, and we use E^ρ to denote the corresponding descent to Y , which pulls back to ρ on X . Note that we can identify H -equivariant maps

$$\mathrm{Hom}_H(\rho^\vee, H^0(X, \omega_X)) \simeq \mathrm{Hom}_H(\rho^\vee, \mathrm{Hom}_X(\mathcal{O}_X, \omega_X)) \simeq \mathrm{Hom}_H(\mathcal{O}_X, \rho \otimes \omega_X).$$

Upon descending to Y , elements of this vector space correspond to elements of $H^0(Y, E^\rho \otimes \omega_Y)$. From this, the map ψ descends to a map of vector bundles on Y ,

$$\xi: H^0(Y, E^\rho \otimes \omega_Y) \otimes \mathcal{O}_Y \rightarrow E^\rho \otimes \omega_Y.$$

Since the pullback $\xi|_X$ factors through a proper sub-bundle of $E^\rho \otimes \omega_Y$, as was shown above, the same holds true of ξ . Further, unwinding the definition of ξ shows it is none other than the natural evaluation map on global sections. This implies that $E^\rho \otimes \omega_Y$ is not generically globally generated.

We note that at this point we have not ruled out the possibility that Putman–Wieland fails in this $r = 3, g = 2$ case because $3 = r \geq g = 2$, see Sect. 4.1. All we have shown that $E^\rho \otimes \omega_Y$ fails to be generically globally generated.

4.5 The general case

There are essentially no new ideas in generalizing the lack of global generation observed above beyond the $\dim \rho = 3, g = 2$ case. We consider only the case ρ is *not self-dual*, but the self-dual case can be handled similarly, with appropriate modifications. See Remark 4.4 for an example of this. We will explain how to show $E^\rho \otimes \omega_Y$ fails to be generically globally generated. By combining this with the argument using Clifford's Theorem for vector bundles in Sect. 3, we find that we must have $\dim \rho \geq g$, see also Sect. 4.1.

Say $\dim \rho = r$ and Y has genus g . By Chevalley–Weil, we obtain a variety cut out by $r(g-1)$ quadrics expressing the incidence between ρ and $H^0(X, \omega_X)^{\rho^\vee} \simeq (\rho^\vee)^{\oplus r(g-1)}$. As in Sect. 4.4, these quadrics cut out a reducible variety in $\mathbb{P}(\rho \oplus (\rho^\vee)^{\oplus r(g-1)})$, one of whose components is the plane $\rho = 0$, with the other being the closure of the subvariety $\{(z, a_1, \dots, a_{r(g-1)}) \in \rho \oplus (\rho^\vee)^{\oplus r(g-1)} : z \neq 0, z(a_i) = 0\}$. The canonical curve cannot lie in the plane, and thus lives in the latter subvariety. Projecting this subvariety away from $\mathbb{P}\rho$ gives a map from X to

$$W := \left\{ (a_1, \dots, a_{r(g-1)}) \in (\rho^\vee)^{\oplus r(g-1)} : \begin{array}{l} (a_1, \dots, a_{r(g-1)}) \\ \text{lie on a common hyperplane in } \mathbb{P}\rho \end{array} \right\}.$$

As in the three dimensional case of Sect. 4.4, the map $X \rightarrow \mathbb{P}H^0(X, \omega_X)^{\rho^\vee}$ gives a map of vector bundles on X , $\mathrm{Hom}_H(\rho^\vee, H^0(X, \omega_X)) \otimes \mathcal{O}_X \rightarrow \rho \otimes \omega_X$. The condition that X factors through W tells us this map of vector bundles drops rank. Letting E^ρ

denote the descent of ρ to Y , the above map descends to Y and we obtain a map of vector bundles $\xi: H^0(Y, E^\rho \otimes \omega_Y) \otimes \mathcal{O}_Y \rightarrow E^\rho \otimes \omega_Y$ which again drops rank and corresponds to the natural evaluation map. Because this evaluation map drops rank, $E^\rho \otimes \omega_Y$ is not generically globally generated. By combining this with the argument using Clifford's Theorem for vector bundles in Sect. 3, we find that we must have $\dim \rho \geq g$, as described in Sect. 4.1.

5 Examples

In this section, we give several counterexamples to the Putman–Wieland conjecture in low genus. Starting in Sect. 5.1, we give a method to produce genus 0 and 1 counterexamples to Putman–Wieland. We give a number of examples, showing that this realizes all the genus 0 cyclic counterexamples which McMullen found [22], as well as the “Eierlegende-Wollmilchsau” genus 1 counterexample in the original Putman–Wieland paper [24, Appendix A]. We also produce many new “origami” counterexamples in genus 1. In particular, Example 5.10 gives an infinite sequence of families of “origami” curves, whose Jacobians have isotrivial isogeny factors of arbitrarily large dimension. We conclude the section with a number of questions about creating families of curves whose Jacobians have isotrivial isogeny factors.

5.1 Genus 0 and 1 counterexamples

There are many cases where $PW_{0,n}$ and $PW_{1,n}$ fail to hold (see e.g. [22, Theorem 8.3] and [24, Appendix A]). From our point of view, these examples admit purely Hodge- and representation-theoretic explanations.

Let us now formally define what it means for a cover to give a counterexample to Putman–Wieland.

Definition 5.1 Suppose $h: \Sigma_{g',n'} \rightarrow \Sigma_{g,n}$ is a finite covering of topological surfaces. Let $\Gamma \subset \text{PMod}_{g,n+1}$ denote the finite index subgroup preserving h . The action of Γ on $\pi_1(\Sigma_{g',n'})$ induces an action on $H_1(\Sigma_{g',n'}, \mathbb{C})$ which preserves the subspace spanned by homology classes of loops around punctures, and hence also induces an action on $H_1(\Sigma_{g'}, \mathbb{C})$. We say h furnishes a counterexample to Putman–Wieland, if there is some non-zero $v \in H_1(\Sigma_{g'}, \mathbb{C})$ with finite orbit under Γ .

Let Y be a compact Riemann surface of genus g with n marked points $p_1, \dots, p_n$. Upon identifying $\pi_1(Y - \{p_1, \dots, p_n\}) \simeq \pi_1(\Sigma_{g,n})$, let $f: X \rightarrow Y$ be the covering of compact Riemann surfaces, ramified at most over $p_1, \dots, p_n$, corresponding to the topological cover h . If h furnishes a counterexample to Putman–Wieland, we also say $f: X \rightarrow Y$ furnishes a counterexample to Putman–Wieland.

Remark 5.2 Even though the Putman–Wieland conjecture assumes $g \geq 2$, we still say $f: X \rightarrow Y$ furnishes a counterexample to Putman–Wieland when Y has genus $g \leq 1$.

To state our criterion for producing counterexamples to Putman–Wieland, we set some notation to describe families of covers of curves.

Notation 5.3 We fix non-negative integers (g, n) so that $n \geq 1$ if $g = 1$ and $n \geq 3$ if $g = 0$, i.e., $\Sigma_{g,n}$ is hyperbolic. Let $\mathcal{M}$ be a connected complex variety. A family of n -pointed curves of genus g over $\mathcal{M}$ is a smooth proper morphism $\pi: \mathcal{C} \rightarrow \mathcal{M}$ of relative dimension one, with geometrically connected genus g fibers, equipped with n sections $s_1, \dots, s_n: \mathcal{M} \rightarrow \mathcal{C}$ with disjoint images. Call such a family *versal* if the induced map $\mathcal{M} \rightarrow \mathcal{M}_{g,n}$ is dominant and étale. Here $\mathcal{M}_{g,n}$ denotes the Deligne–Mumford moduli stack of n -pointed genus g smooth proper curves with geometrically connected fibers.

Recall that an irreducible representation ρ is *symplectically self-dual* if $(\wedge^2 \rho)^H \neq 0$, i.e., there is a non-zero map $\text{triv} \rightarrow \wedge^2 \rho$.

Proposition 5.4 *With notation as in Notation 5.3, let $m \in \mathcal{M}$ be a point. Let $X = \mathcal{X}_m$, $Y = \mathcal{C}_m$. Let $V = H^1(X, \mathbb{Q})/H^1(Y, \mathbb{Q})$. Suppose that there exists an irreducible representation*

$$\rho: H \rightarrow \text{GL}_r(\mathbb{Q})$$

such that, letting V^ρ denote the ρ -isotypic piece of V , $V^\rho \otimes \mathbb{C} \simeq \bigoplus_{i=1}^s \rho_i^{n_i}$ with $\rho_1, \dots, \rho_s$ irreducible and pairwise distinct. Suppose that either:

- (1) *the Hodge decomposition of $V^\rho \otimes \mathbb{C}$ is an isotypic decomposition, i.e. there is a subset $S \subset \{1, \dots, s\}$ so that $F^1 V^\rho \otimes \mathbb{C} \simeq \bigoplus_{i \in S} \rho_i^{n_i}$, or*
- (2) *for every i , ρ_i is symplectically self-dual, and ρ_i appears with multiplicity at most 1 in $F^1 V^\rho \otimes \mathbb{C}$.*

Then $f: X \rightarrow Y$ furnishes a counterexample to Putman–Wieland.

Before proceeding with the proof, we comment on some restrictions on the representations that must be satisfied for the conditions to hold.

Remark 5.5 Note that in Proposition 5.4 (1), the representations ρ_i appearing in $F^1 V^\rho$ must all be non-self dual. The reason for this is that, as $V^\rho \otimes \mathbb{C}$ carries a Hodge structure, the quotient $V^\rho \otimes \mathbb{C}/F^1 V^\rho \otimes \mathbb{C}$ is dual to $F^1 V^\rho \otimes \mathbb{C}$. However, if some ρ_i is self dual, it will appear in both the sub $F^1 V^\rho \otimes \mathbb{C}$ and the quotient $V^\rho \otimes \mathbb{C}/F^1 V^\rho \otimes \mathbb{C}$, and then the decomposition will not be isotypic.

Remark 5.6 Note that by the Chevalley–Weil formula theorem [7, Theorem 2.1], as H -representations, $H^0(X, \omega_X) \simeq \text{triv} \oplus \text{reg}^{g-1} \oplus W$, for triv the trivial representation, and reg the regular representation, where W is an auxiliary representation which is 0 if the cover is unramified, and is, in general, explicitly computable in terms of the ramification data of f . Therefore, conditions (1) and (2) of Proposition 5.4 each imply that $g \leq 1$.

As pointed out by a referee, to verify the requirement $g \leq 1$, one only needs the easier fact that the H -module $H^1(X, \mathbb{Q})$ contains $\text{triv}^{\oplus 2} \oplus \text{reg}^{2g-2}$. This is somewhat easier to prove than Chevalley–Weil by considering an H -equivariant triangulation of X , and can be verified with a proof analogous to that of [11, Proposition 1.1].

Remark 5.7 Adding on to Remark 5.6, the Chevalley–Weil formula shows that for faithful representations we can only hope to apply Proposition 5.4 (2) to families

where $g = 1, n = 1$, as if there is more than 1 marked point (and the H -representation is non-trivial at ≥ 2 points) or the genus is more than 1, any such representation will appear with multiplicity greater than 1.

Proof of Proposition 5.4 Let $\mathcal{H}^\rho$ be the ρ -isotypic part of $\mathcal{H} := R^1\pi'_*\mathbb{Q} \otimes \mathcal{O}_{\mathcal{M}}$. The connection on $\mathcal{H}$ induces a connection $\nabla: \mathcal{H}^\rho \rightarrow \mathcal{H}^\rho \otimes \Omega_{\mathcal{M}}^1$. In turn, this induces a $\mathcal{O}_{\mathcal{M}}$ -linear map $\bar{\nabla}: F^1\mathcal{H}^\rho \rightarrow (\mathcal{H}^\rho/F^1\mathcal{H}^\rho) \otimes \Omega_{\mathcal{M}}^1$. We claim that either of the conditions in the theorem statement imply that $\bar{\nabla}$ is identically zero. By e.g. [15, (4.4.2)], this implies $\mathcal{H}^\rho$ has finite monodromy, and so $\text{PW}_{g,n}$ fails.

We now verify $\bar{\nabla} = 0$ in the two cases. Set $V^\rho = (\mathcal{H}^\rho)_m$. In case (1), $\bar{\nabla}$ is an H -equivariant map

$$\bigoplus_{i \in S} \rho_i \otimes \text{Hom}(\rho_i, F^1V^\rho) \rightarrow \bigoplus_{j \notin S} \rho_j \otimes \text{Hom}(\rho_j, V^\rho/F^1V^\rho) \otimes \Omega_{\mathcal{M}}$$

between H -representations with no irreducible subrepresentations in common, and therefore must vanish identically.

To verify (2), observe that an irreducible symplectically self-dual H -representation π satisfies $(\text{Sym}^2 \pi)^H = 0$. Indeed $\text{Hom}(\text{triv}, \wedge^2 \pi)$ is non-zero by assumption. By Schur's lemma, $\text{Hom}(\text{triv}, \pi \otimes \pi)$ has dimension exactly 1 (as π is self-dual). We then find

$$\text{Hom}(\text{triv}, \pi \otimes \pi) = \text{Hom}(\text{triv}, \text{Sym}^2 \pi) \oplus \text{Hom}(\text{triv}, \wedge^2 \pi)$$

and so $\text{Hom}(\text{triv}, \text{Sym}^2 \pi) = 0$.

To check $\bar{\nabla}$ vanishes, we can write it as a sum of maps

$$\bar{\nabla}_i: F^1\mathcal{H}_{\rho_i} \rightarrow (\mathcal{H}_{\rho_i}/F^1\mathcal{H}_{\rho_i}) \otimes \Omega_{\mathcal{M}}^1$$

and show each $\bar{\nabla}_i$ vanishes. If the multiplicity of ρ_i in $F^1\mathcal{H}_{\rho_i}$ is 0, we have $F^1\mathcal{H}_{\rho_i} = 0$, and so $\bar{\nabla}_i = 0$. Otherwise, ρ_i appears in $F^1\mathcal{H}_{\rho_i}$ with multiplicity 1. For α as in (4.1), since ρ_i appears with multiplicity 1 in $H^0(X, \omega_X)$, $\text{im } \alpha \subset \text{Sym}^2 H^0(X, \omega_X)^H$ factors through a copy of $\text{Sym}^2(\rho_1)^H = 0$. Hence α vanishes, so $\bar{\nabla}_i$ vanishes. $\square$

We now apply Proposition 5.4 to some examples. We first consider the ‘‘Eierlegende-Wollmilchsau’’ example discussed in the paper of Putman and Wieland where the Putman–Wieland conjecture was originally formulated.

Example 5.8 ([24, Appendix A]) Let Q_8 be the quaternion group (of order 8). Putman and Wieland describe in [24, Appendix A] a family of Q_8 covers of genus one curves $\mathcal{X} \rightarrow \mathbb{C}$, such that the vector space V of Proposition 5.4 is $\mathbb{Q}$ -irreducible as a Q_8 -representation, with $F^1V_{\mathbb{C}}$ irreducible and symplectically self-dual. Hence Proposition 5.4(2) applies.

Example 5.9 ([22, Theorem 8.3]) The eight examples appearing in [22, Table 10] all satisfy Proposition 5.4(1), by the displayed equation in the proof of [22, Theorem 8.3].

For example, this includes a family of $\mathbb{Z}/4\mathbb{Z}$ -covers of $\mathbb{P}^1$ which are totally ramified over four points. This family may be expressed in terms of equations as $y^4 = x(x - 1)(x - \lambda)$, for $\lambda \in \mathbb{A}_\lambda^1 - \{0, 1\}$ the parameter of the family.

We can also make new examples using Proposition 5.4.

Example 5.10 Let H be any non-abelian finite subgroup of $\mathrm{SL}_2(\mathbb{C})$, with $\rho: H \hookrightarrow \mathrm{SL}_2(\mathbb{C})$ the given embedding. Suppose H is generated by two elements, say γ and δ . Then we claim there is a cover with Galois group H , furnishing a counterexample to $\mathrm{PW}_{1,1}$.

Indeed, since H is non-abelian and generated by two elements, any two generators of H cannot commute, so their commutator c is non-trivial. Moreover ρ is automatically irreducible, as H is non-abelian. Finally, note that ρ is symplectically self-dual, as the determinant yields a symplectic form preserved by ρ . Since ρ is two-dimensional and semisimple, we obtain that $\rho(c)$ has two eigenvalues which are inverse to each other and neither is equal to 1. Now, let Y be a once punctured genus 1 curve. Identify $\pi_1(Y, y) \simeq F_2$, and let $X \rightarrow Y$ be the cover associated to the representation $F_2 \rightarrow H$ sending the first generator to γ and the second to δ . It follows from the Chevalley–Weil formula, as in [23, Theorem 2.1] that ρ appears with multiplicity 1 in $H^0(X, \omega_X)$. Further, any Galois conjugate of ρ is also faithful, and by the same reasoning, it also appears with multiplicity 1 in $H^0(X, \omega_X)$. Therefore, Proposition 5.4(2) applies and $f: X \rightarrow Y$ furnishes a counterexample to Putman–Wieland.

The non-abelian finite subgroups of $\mathrm{SL}_2(\mathbb{C})$ are the dicyclic groups and three exceptional groups (the binary octahedral, binary icosahedral, and binary tetrahedral groups), see [25]. One can directly check that each such group is generated by two elements. Hence, for each such group, we can find a cover furnishing a counterexample to Putman–Wieland with $g = 1$ and $n = 1$.

The cover of $\Sigma_{1,1}$ associated to an embedding of the order $4n$ dicyclic group Dic_n into $\mathrm{SL}_2(\mathbb{C})$ as above has genus $2n - 1$. By analyzing the representation theory of dicyclic groups, one may verify the dimension of the fixed part is n if n is even and $n - 1$ if n is odd. In particular, the dimension of this fixed part tends to ∞ as $n \rightarrow \infty$.

Recall that a cover of a curve of genus one, branched over one point as in Example 5.10, is called an *origami curve*.

We can also make examples of representations of dimension more than 2. We expect there to be an unwieldy collection of these larger-dimensional examples, and it is an enjoyable exercise to look for them.

Example 5.11 Let H be the order 32 group given in terms of generators and relations by

$$H = \left\langle a, b, c \mid \begin{array}{l} a^4 = 1, b^4 = a^2, c^2 = b \cdot a \cdot b^{-1} = a^{-1} \\ a \cdot c = c \cdot a, c \cdot b \cdot c^{-1} = a^{-1} b^3 \end{array} \right\rangle. \quad (5.1)$$

As described at [12], the commutator subgroup of H is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^2$. Since $c^2 = a^{-1}$, this group is generated by b and c . One may directly verify that $cbc^{-1}b^{-1}$ is

not central. Let Y be a once punctured genus 1 curve. Upon identifying $\pi_1(Y, y) \simeq F_2$, let $X \rightarrow Y$ be the cover associated to the representation $F_2 \rightarrow H$ sending the first generator to c and the second to b . There is a unique 4-dimensional symplectic representation ρ of H , and under this representation, the character $\text{tr } \rho(cbc^{-1}b^{-1}) = 0$. Since $cbc^{-1}b^{-1}$ has order 2, its eigenvalues are ± 1 , and as they sum to 0, two must be 1 and two must be -1 . The Chevalley–Weil formula as in [23, Theorem 2.1] implies ρ appears with multiplicity 1 in $H^0(X, \omega_X)$. Since ρ is the only symplectically self-dual representation of H , it has no non-isomorphic Galois conjugates so we conclude $s = 1$ in Proposition 5.4. We therefore conclude by Proposition 5.4(2) that $f: X \rightarrow Y$ furnishes a counterexample to Putman–Wieland, where here $g = 1$ and $n = 1$.

5.2 Questions on families with large isotrivial isogeny factor

Examples 5.8–5.11 above seem interesting applications of Proposition 4.2, but they only scratch the surface. We were essentially able to characterize all 2-dimensional counterexamples to Putman–Wieland coming from Proposition 5.4(2) in Example 5.10, while we gave only a single 4-dimensional counterexample in Example 5.11 via a non-systematic examination of 4-dimensional symplectic representations. Note such counterexamples can only occur when $g = 1$ and $n = 1$, as remarked in Remark 5.7. We suspect there are infinitely many more counterexamples obtainable in this way, but do not have any systematic approach to searching for them.

Question 5.12 Other than the example in Example 5.11, are there more 4-dimensional counterexamples to Putman–Wieland obtainable via Proposition 5.4? Are there infinitely many such? Can one classify them?

Moreover, generalizing from 4-dimensional representations to higher dimensional representations, we ask the following.

Question 5.13 Can one obtain families of origami curves with isotrivial isogeny factor coming from Proposition 5.4(2) via representations of arbitrarily large dimension? For each such dimension, are there infinitely many? Can one classify them?

In Example 5.10, we showed there are families of origami curves of arbitrarily large genus g , whose Jacobians have an isotrivial isogeny factor of dimension more than $g/2$.

Question 5.14 What is the supremum over all positive real numbers c so that, for arbitrarily large g , there is a non-isotrivial family of genus g curves whose Jacobians have an isotrivial isogeny factor of dimension at least cg ? What about the analogous question for families of origami curves?

As mentioned above, by Example 5.10, $c \geq 1/2$, and so $1/2 \leq c \leq 1$.

Even deciding whether $c < 1$ seems quite difficult, which leads to our next question. Möller showed in [20, Theorem 5.1] (using also [20, Corollary 3.3]), that for $g \geq 6$, there are no families of genus g curves whose Jacobian has an isotrivial isogeny factor of dimension $g - 1$. More recently, Aulicino and Norton showed moreover that no such curves exist when $g = 5$ [1].

Question 5.15 For any fixed positive integer m , are there families of curves with an isotrivial isogeny factor of dimension $g - m$ for arbitrarily large g ?

As noted above, we find that the answer is “no” for $m = 1$. If the answer to Question 5.15 is “yes” for any m , then $c = 1$ in Question 5.14.

Finally, we note that many of the examples of isotrivial isogeny factors we found above were isotypic in the following sense: consider a family of curves $\mathcal{C} \rightarrow B$ with a finite group H acting on $\mathcal{C}$. For any $b \in B$, the resulting H action on $H^1(\mathcal{C}_b, \mathcal{O}_{\mathcal{C}_b})$ decomposes as a sum of ρ -isotypic components $H^1(\mathcal{C}_b, \mathcal{O}_{\mathcal{C}_b})^\rho$ over H -irreps ρ . We say the isotrivial isogeny factor of the Jacobian of $\mathcal{C} \rightarrow B$ is *isotypic* if, for any $b \in B$, the tangent space to the maximal isotrivial isogeny factor is identified with a direct sum of ρ -isotypic components of $H^1(\mathcal{C}_b, \mathcal{O}_{\mathcal{C}_b})$.

Question 5.16 Can one produce non-isotrivial families of curves $\mathcal{C} \rightarrow B$ with an H action as above whose Jacobian has an isotypic isotrivial isogeny factor and so that the quotient $\mathcal{C}/H$ has genus at least 2? Can one produce such families where $\mathcal{C}/H$ has arbitrarily large genus? Can one produce such families of curves with an isotypic isotrivial isogeny factor so that the base B has large dimension?

As a partial answer to the last question above, in a forthcoming paper, we will show it is impossible to have isotypic isotrivial isogeny factors whenever the quotient curve $\mathcal{C}/H$ has genus h and dominates $\mathcal{M}_h$ if $h \geq 3$. On the other hand, it would be quite interesting to find such an example when $h = 2$. If such a family exists, it would give a counterexample to Putman–Wieland in genus 2.

6 Counterexamples for hyperelliptic curves

We now explain that the Putman–Wieland conjecture fails if we restrict our attention to the hyperelliptic mapping class group, following Marković [21]. In particular, this implies that the Putman–Wieland conjecture fails in genus 2.

Definition 6.1 Choose a hyperelliptic involution ι acting on S_g and let the *hyperelliptic mapping class group* $\text{HMod}_g := \text{Mod}_g^\iota \subset \text{Mod}_g$ denote the centralizer of ι in Mod_g . Let $\text{HMod}_{g,n} \subset \text{Mod}_{g,n}$ denote the preimage of $\text{HMod}_g \subset \text{Mod}_g$ under the surjection $\text{Mod}_{g,n} \rightarrow \text{Mod}_g$.

Remark 6.2 Any two such hyperelliptic involutions $\iota \in \text{Mod}_g$ are conjugate by [9, Proposition 7.15].

We next recall a foundational result on topology of the moduli stack of hyperelliptic curves. For a reference, we recommend the survey article [10, Proposition 1], and the reader may also consult [13, Theorem 2].

Lemma 6.3 Let $g \geq 2$, $n \geq 0$ and let $\mathcal{H}_{g,n} \subset \mathcal{M}_{g,n}$ denote the closed substack parameterizing families of smooth proper hyperelliptic curves of genus g with geometrically connected fibers and n disjoint sections. Under the identification of $\text{Mod}(\Sigma_{g,n})$ with $\pi_1(\mathcal{M}_{g,n})$, one may recover $\text{HMod}_{g,n}$ as the image of $\pi_1(\mathcal{H}_{g,n}) \rightarrow \pi_1(\mathcal{M}_{g,n})$. Moreover $\mathcal{H}_{g,n}$ is a $K(\pi_1(\mathcal{H}_{g,n}), 1)$.

Remark 6.4 The hyperelliptic mapping class group is sometimes also called the symmetric mapping class group [9, Section 9.4].

Definition 6.5 In the setup of Definition 2.2, let $\text{HPW}_{g,n}$ be the statement that for every finite index characteristic subgroup $K \subset \pi_1(\Sigma_{g,n}, v_0)$, there are no non-zero vectors $v \in V_h := H_1(\Sigma_{g'}, \mathbb{C})$ with finite orbit under the action of $\text{HMod}_{g,n+1}$.

Problem 6.6 (Hyperelliptic Putman–Wieland problem, cf. [4, Problem 3.4]) Suppose $g \geq 2, n \geq 0$. Does $\text{HPW}_{g,n}$ hold?

The following result is essentially [21, Theorem 1.3] where the case $g = 2$ is proven. We observe here that an analogous proof works for hyperelliptic curves in all genera. Recall we use $\mathcal{H}_{g,n}$ for the moduli stack of smooth hyperelliptic curves of genus g with n marked points, as in Lemma 6.3.

Proposition 6.7 *Let $g \geq 2, n \geq 0$ be integers. There exists a finite étale map $\mathcal{H} \rightarrow \mathcal{H}_{g,n}$, so that if $\mathcal{C}$ is the corresponding relative curve over $\mathcal{H}$, there is a finite étale cover $\mathcal{X}$ of $\mathcal{C}$ of degree 36 such that:*

- (1) *the composite map $\mathcal{X} \rightarrow \mathcal{C} \rightarrow \mathcal{H}$ has geometrically connected fibers,*
- (2) *the relative Jacobian $\text{Pic}_{\mathcal{X}/\mathcal{H}}^0$ has an isotrivial isogeny factor.*

For $x \in \mathcal{H}$ a geometric point, the finite index subgroup of the hyperelliptic mapping class group $\pi_1(\mathcal{H}, x)$ has a vector with finite orbit under its action on $H^1(\mathcal{X}_x, \mathbb{Q})$ and so the answer to the hyperelliptic Putman–Wieland problem is negative for every $g \geq 2, n \geq 0$. In particular, $\text{PW}_{2,n}$ is false for all $n \geq 0$.

We will prove Proposition 6.7 below in Sect. 6.1. We will use the following result of Bogomolov–Tschinkel:

Theorem 6.8 ([5, Proposition 3.8, Remark 3.10]) *Let $g \geq 2$ be an integer. Let C_0 be the genus 2 curve defined by $y^6 = x(x-1)$ and E_0 be the genus 1 curve defined by $y^3 = x(x-1)$. Let $Z \in \{C_0, E_0\}$. There exists a finite étale cover $\mathcal{H}_Z$ of $\mathcal{H}_g$, and a finite étale cover $\mathcal{X}_Z$ of $\mathcal{C}_Z := \mathcal{C}_g \times_{\mathcal{H}_g} \mathcal{H}_Z$ such that the composite map $\pi'_Z: \mathcal{X}_Z \rightarrow \mathcal{C}_Z \rightarrow \mathcal{H}_Z$ has geometrically connected fibers, and such that every fiber of π'_Z admits a non-constant map to the smooth projective curve Z . If $Z = C_0$, the cover $\mathcal{X}_Z \rightarrow \mathcal{C}_Z$ has degree 648 while if $Z = E_0$ this map has degree 36. The non-constant map $\mathcal{X}_Z \rightarrow Z \times \mathcal{H}_Z$ has degree 4.*

Proof Strictly speaking, in [5, Proposition 3.8, Remark 3.10], it is only shown that a hyperelliptic curve C admits a finite étale cover X with a non-constant map to C_0 ; however, their construction works in families, hence the statement. From their construction, the map is a composite of 5 maps of respective degrees 2, 9, 2, 9, and 2, so has degree $2 \cdot 9 \cdot 2 \cdot 9 \cdot 2 = 648$. (This was mistakenly claimed to be 72 in [5, Remark 3.10] but updated in a later version [6, Remark 3.10].) An intermediate step in the proof of [5, Proposition 3.8] shows that the hyperelliptic curve C admits a degree 36 finite étale cover X with a degree 4 non-constant map to E_0 . This cover $X \rightarrow C$ is a composite of the first three maps of the above-mentioned 5 maps, which have degrees 2, 9, and 2. Hence, this cover has degree 36. $\square$

Remark 6.9 Note that there is a small typo in [5, Example 3.7], which claims that every hyperelliptic curve admits a non-constant map to an elliptic curve satisfying some extra conditions; this claim is evidently false. However, given a hyperelliptic curve C , one may construct a finite étale cover X of C mapping to an elliptic curve and satisfying the desired conditions as follows; such an X suffices for the rest of their argument to go through. Let $f: C \rightarrow \mathbb{P}^1$ be the hyperelliptic double cover, which we may without loss of generality assume is branched over $0, 1, \infty$. Let λ be another branch point of f , and let E_λ be the elliptic curve defined by

$$E_\lambda: y^2 = x(x-1)(x-\lambda).$$

Then any component X of the normalized fiber product $\widetilde{E \times_{\mathbb{P}^1} C}$ satisfies the desired properties.

In order to prove Proposition 6.7, we will also use the following equivalences. This is a fairly standard Hodge-theoretic argument, the key input being the theorem of the fixed part.

Lemma 6.10 *Suppose $\mathcal{M}'$ is a scheme and we have a relative smooth proper curve $\pi': \mathcal{C}' \rightarrow \mathcal{M}'$ with geometrically connected fibers. The following properties are equivalent:*

- (1) *For $x \in \mathcal{M}'$, the vector space $H_1(\mathcal{C}'_x, \mathbb{C})$ contains no non-zero vector with finite orbit under the action of $\pi_1(\mathcal{M}', x)$.*
- (2) *The local system $R^1\pi'_*\mathbb{C}$ contains no sub-local system with finite monodromy.*
- (3) *The relative Jacobian $\text{Pic}^0_{\mathcal{C}'/\mathcal{M}'}$ has no isotrivial isogeny factor.*

The following properties are also equivalent:

- (1') *For $x \in \mathcal{M}'$, the vector space $H_1(\mathcal{C}'_x, \mathbb{C})$ contains no non-zero fixed vector under the action of $\pi_1(\mathcal{M}', x)$.*
- (2') *The local system $R^1\pi'_*\mathbb{C}$ contains no trivial sub-local system.*
- (3') *The relative Jacobian $\text{Pic}^0_{\mathcal{C}'/\mathcal{M}'}$ has no constant isogeny factor.*

Proof The equivalence of (1), (2), and (3) follow from that of (1'), (2'), and (3') by passing to a suitable connected finite étale cover of $\mathcal{M}'$ upon which the vector with finite orbit in (1) becomes a fixed vector, the sub-local system with finite monodromy in (2) becomes trivial, and the isotrivial factor in (3) becomes trivial.

It remains to verify the equivalence of (1'), (2'), and (3'). Under the equivalence between $\mathbb{C}$ representations of $\pi_1(\mathcal{M}', x)$ and local systems of $\mathbb{C}$ vector spaces on $\mathcal{M}$, a non-zero fixed vector in $H_1(\mathcal{C}'_x, \mathbb{C})$ corresponds to a non-trivial sub-local system, which gives the equivalence between (1') and (2'). Note here we are using Poincaré duality to identify $H_1(\mathcal{C}'_x, \mathbb{C})$ and $H^1(\mathcal{C}'_x, \mathbb{C})$.

We next verify (3') $\implies$ (2'). To this end, let $\phi: \text{Pic}^0_{\mathcal{C}'/\mathcal{M}'} \rightarrow \mathcal{M}'$ denote the structure map. A trivial isogeny factor $\iota: A \rightarrow \text{Pic}^0_{\mathcal{C}'/\mathcal{M}'}$, gives a trivial sub-local system $R^1(\phi \circ \iota)_*\mathbb{C} \subset R^1\pi'_*\mathbb{C}$.

We conclude by demonstrating (2') $\implies$ (3'). Suppose we begin with a trivial sub-local system of $\mathbb{W} \subset R^1\pi'_*\mathbb{C}$, the theorem of the fixed part [8, Corollaire 4.1.2] implies

that $\mathbb{W}$ is also a variation of Hodge structure. Let $\mathbb{W}_{\mathbb{Z}} \subset \mathbb{W}$ denote the trivial $\mathbb{Z}$ local system on $\mathcal{M}$ for which $\mathbb{W}_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{C} \simeq \mathbb{W}$. We can then recover the trivial isogeny factor as the quotient $F^1(\mathbb{W} \otimes_{\mathbb{C}} \mathcal{O}_{\mathcal{M}'})^{\vee} / \mathbb{W}_{\mathbb{Z}}^{\vee}$. (On fibers, this corresponds to the fact that an abelian variety A can be written as the quotient $H^0(A, \Omega_A)^{\vee} / H^1(A, \mathbb{Z})$.) $\square$

6.1 Proof of Proposition 6.7

Let $\mathcal{H} := \mathcal{H}_{E_0}$, $\mathcal{C} := \mathcal{C}_{E_0}$, $\mathcal{X} := \mathcal{X}_{E_0}$ be as defined in Theorem 6.8. As shown there, the cover $\mathcal{X} \rightarrow \mathcal{C}$ has degree 36. The Jacobian of E_0 is an isogeny factor of $\text{Pic}_{\mathcal{X}/\mathcal{H}}^0$.

Having produced a relative Jacobian with a constant isogeny factor, the equivalence of Lemma 6.10(1) and Lemma 6.10(3) together with the identification of $\pi_1(\mathcal{H}_{g,n})$ with $\text{HMod}_{g,n}$ from Lemma 6.3, implies the answer to Problem 6.6 is negative for every $g \geq 2$, $n \geq 0$.

Finally, $\text{PW}_{2,n}$ fails because the natural map $\mathcal{H}_2 \rightarrow \mathcal{M}_2$ is an isomorphism, and hence in genus 2 the hyperelliptic mapping class group is identified with the entire mapping class group by Lemma 6.3. $\square$

Remark 6.11 Proposition 6.7 was originally claimed in [3]; unfortunately the proof there was incorrect (see [4, Remark 3.17]). We would like to thank Marco Boggi for his extremely helpful email correspondence regarding this issue, which was useful to our understanding of the Putman–Wieland conjecture. The reader may also consult [2, Corollary 6.11].

Remark 6.12 In some sense Proposition 6.7 shows that our proof of Corollary 2.7 and Theorem 3.2 is sharp. Indeed, Theorem 6.8 shows that for each g there exists a diagram

$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{f} & \mathcal{C} \\ & \searrow \pi' & \downarrow \pi \\ & & \mathcal{H} \end{array}$$

with π a relative curve of genus g , so that the induced map $\mathcal{H} \rightarrow \mathcal{M}_g$ dominates the hyperelliptic locus, and moreover such that f has degree bounded independently of g , yielding a counterexample to $\text{HPW}_{g,0}$. The hyperelliptic locus has codimension $g - 2$ in $\mathcal{M}_g$, so this does not contradict Theorem 3.2; but that theorem shows that such an example is impossible with families of curves whose codimension δ_g in $\mathcal{M}_g$ grows such that $g - \delta_g \rightarrow \infty$ with g .

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Declarations

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