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Canonical representations of surface groups

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Abstract

Let $\Sigma_{g,n}$ be an orientable surface of genus g with n punctures. We study actions of the mapping class group $\mathrm{Mod}_{g,n}$ of $\Sigma_{g,n}$ via Hodge-theoretic and arithmetic techniques. We show that if

$$\rho : \pi_1(\Sigma_{g,n}) \to \mathrm{GL}_r(\mathbb{C})$$

is a representation whose conjugacy class has finite orbit under $\mathrm{Mod}_{g,n}$, and $r < \sqrt{g+1}$, then ρ has finite image. This answers questions of Junho Peter Whang and Mark Kisin. We give applications of our methods to the Putman-Wieland conjecture, the Fontaine-Mazur conjecture, and a question of Esnault-Kerz. The proofs rely on non-abelian Hodge theory, our earlier work on semistability of isomonodromic deformations, and recent work of Esnault-Groechenig and Klevdal-Patrikis on Simpson's integrality conjecture for cohomologically rigid local systems.

Contents

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1. Introduction

1.1. *Overview.* Let $\Sigma_{g,n}$ be an orientable topological surface of genus g with n punctures, and let $x \in \Sigma_{g,n}$ be a basepoint. We denote the mapping class group of $\Sigma_{g,n}$ by $\mathrm{Mod}_{g,n}$. There is a natural outer action of $\mathrm{Mod}_{g,n}$ on $\pi_1(\Sigma_{g,n}, x)$, induced by the action of $\mathrm{Homeo}^+(\Sigma_{g,n})$ on $\Sigma_{g,n}$. Thus, $\mathrm{Mod}_{g,n}$ acts on the set of conjugacy classes of representations of $\pi_1(\Sigma_{g,n}, x)$ into $\mathrm{GL}_r(\mathbb{C})$. Our goal is to study the finite orbits of this action via Hodge-theoretic and arithmetic techniques.

Definition 1.1.1. For $g, n, r \geq 0$, a representation $\rho : \pi_1(\Sigma_{g,n}, x) \to \mathrm{GL}_r(\mathbb{C})$ is *MCG-finite* if the conjugacy class of ρ has finite orbit under the action of $\mathrm{Mod}_{g,n}$.

Note here that we are studying *conjugacy classes* of representations. For example, semisimple MCG-finite representations correspond to finite $\mathrm{Mod}_{g,n}$-orbits in the character variety of $\pi_1(\Sigma_{g,n})$, not its representation variety. Indeed, $\mathrm{Mod}_{g,n}$ does not naturally act on the representation variety of $\pi_1(\Sigma_{g,n})$.

Any representation of $\pi_1(\Sigma_{g,n})$ constructed without making choices (for example, without naming a specific curve in $\Sigma_{g,n}$), or by making choices with only a finite amount of indeterminacy, is MCG-finite. So we view MCG-finite representations as the *canonical* representations of $\pi_1(\Sigma_{g,n})$.

The study of MCG-finite representations has a long history. It has a close connection to isomonodromy and the Painlevé VI equation, originally introduced in 1902 by Painlevé [Pai02] and Gambier [Gam10]. For each g, n, r, conjugacy classes of semisimple MCG-finite representations

$$\pi_1(\Sigma_{g,n}) \to \mathrm{GL}_r(\mathbb{C})$$

correspond to algebraic solutions to a certain *isomonodromy* differential equation; see [CH21, Th. A]. Algebraic solutions to the Painlevé VI equation in an appropriate choice of coordinates correspond to MCG-finite representations with trivial determinant in the particular case $g = 0, n = 4, r = 2$ (see [Mah99, §1-§3] and [Dor01, §2]).

There is substantial literature on the search for algebraic solutions to the Painlevé VI equation, [AK02], [Boa05], [Boa06], [Boa07b], [Boa07a], [CL09], [Dub96], [DM00], [Hit95], [Hit03], [Kit05], [Kit06], culminating in the classification given in [LT14]. Boalch [Boa10] pitches the problem of classifying algebraic solutions to Painlevé VI as a natural generalization of Schwarz's list of hypergeometric equations with finite monodromy [Sch73].

We will later see that the classification of MCG-finite representations is closely connected to several major open questions in low-dimensional topology. In particular, it has connections to the Putman-Wieland conjecture [PW13, Conj. 1.2] and Ivanov's conjecture that mapping class groups do not virtually

surject onto $\mathbb{Z}$ [Iva06, §7]. These conjectures in turn have applications to algebraic geometry, as they would determine the first rational homology of finite covers of $\mathscr{M}_{g,n}$.

Many interesting classes of representations are MCG-finite. Examples include the rigid local systems studied by Katz [Kat96], representations constructed via TQFT techniques (e.g., [Kup11], [KS16], [KS18a], [KS18b]), and representations of algebro-geometric interest, such as those constructed via the Kodaira-Parshin trick (e.g., [LL22, Exam. 3.3.1]). We will see later that MCG-finite representations of $\pi_1(\Sigma_{g,n})$ are closely related to representations of finite-index subgroups of $\mathrm{Mod}_{g,n+1}$.

1.2. *Results for surface groups.* Given the difficulty of classifying MCG-finite representations in the $g = 0, n = 4, r = 2$ case (encompassing Painlevé VI), it may be surprising that we are able to obtain a complete and simple characterization of MCG-finite representations when $g \gg r$.

THEOREM 1.2.1. *For $g, n, r \geq 0$, let*

$$\rho : \pi_1(\Sigma_{g,n}) \to \mathrm{GL}_r(\mathbb{C})$$

be a MCG-finite representation. If $r < \sqrt{g+1}$, then ρ has finite image.

This answers a question of Junho Peter Whang [LL22, Question 1.5.3]. We will prove Theorem 1.2.1 in Section 8.7.

While the statement appears to be purely topological or even group-theoretic in nature, the proof relies on (non-abelian and mixed) Hodge theory, and also takes input from the Langlands program, through the work of Esnault-Groechenig [EG18] and Klevdal-Patrikis [KP22] on integrality of cohomologically rigid local systems.

Remark 1.2.2. Some bound on the rank as in Theorem 1.2.1 is necessary. Indeed, there are a number of interesting MCG-finite representations of $\pi_1(\Sigma_{g,n})$ of rank $r \gg g$ with infinite image, as explained in Question 10.2. We have no reason to think that the precise bound of $\sqrt{g+1}$ is optimal in general. That said, the bound cannot be improved too much. It is sharp when $g = 1$, by Remark 10.2.1, but not for $g = 2, 3$, by Remark 10.2.2. For general g, there are MCG-finite representations of $\pi_1(\Sigma_g)$ of rank $2g + 1$ with infinite image (Examples 10.1.6((3))), so the bound cannot be improved to be better than linear in g. We summarize the situation in Figure 1.

As explained in [CH21, Th. A], Theorem 1.2.1 yields a classification result for algebraic solutions to isomonodromy differential equations:

COROLLARY 1.2.3. *Let C be a smooth projective curve of genus g, and let $D \subset C$ be a reduced effective divisor. Let (E, ∇) be a semisimple flat vector*

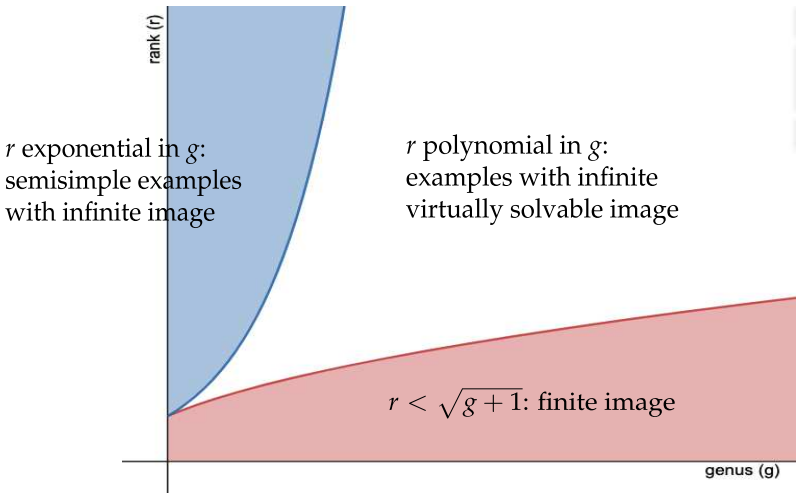


Figure 1. *The geography of known MCG-finite representations*: Theorem 1.2.1 shows that the region $r < \sqrt{g+1}$ contains only MCG-finite representations with finite image. As explained in Question 10.2, there are examples of MCG-finite rank r representations with infinite image for $r \gg g$. For r polynomial in g, all known examples have virtually solvable image. For r exponential in g, there are interesting MCG-finite irreducible representations with infinite image.

bundle on C with regular singularities along D, with $\operatorname{rk} E < \sqrt{g+1}$. *Suppose that the eigenvalues of the residue matrices of (E, ∇) have real parts in $[0, 1)$. Then (E, ∇) has an algebraic universal isomonodromic deformation if and only if (E, ∇) has finite monodromy.*

Note that the condition on residue matrices can always be achieved after a birational gauge transformation, by replacing (E, ∇) with the Deligne canonical extension of $(E, \nabla)|_{C \setminus D}$.

This algebro-geometric reformulation of our main result is not just window dressing. It is the perspective we will take in the proof of Theorem 1.2.1.

1.3. *An application to low rank local systems on $\mathscr{M}_{g,n}$.* We next record a consequence of our main result for local systems of low rank on families of curves. As in Notation 1.10.1, we say a family $\pi : \mathscr{C} \to \mathscr{M}$ of smooth proper genus g curves with geometrically connected fibers, equipped with n disjoint sections $s_i : \mathscr{M} \to \mathscr{C}$, is *versal* if the corresponding map $\mathscr{M} \to \mathscr{M}_{g,n}$ is dominant. We call $\pi^\circ : \mathscr{C}^\circ := \mathscr{C} \setminus \bigcup_i s_i(\mathscr{M}) \to \mathscr{M}$ a *punctured versal family* of genus g curves.

The following is immediate from Theorem 1.2.1, by Proposition 2.1.3 below.

COROLLARY 1.3.1. *Suppose $\pi^\circ : \mathscr{C}^\circ \to \mathscr{M}$ is a punctured versal family of genus g curves, and $\mathbb{V}$ is a local system on $\mathscr{C}^\circ$ of rank $< \sqrt{g+1}$. For any fiber C° of π°, the local system $\mathbb{V}|_{C^\circ}$ has finite monodromy.*

1.4. *The Putman-Wieland conjecture.* Among the main ingredients in the proof of Theorem 1.2.1 are some new results toward the Putman-Wieland conjecture, sketched below and described in detail in Section 7.

We now set some notation to state the Putman-Wieland conjecture. Let $\Sigma_{g,n}$ be a surface, and let $\phi : \pi_1(\Sigma_{g,n}, x) \twoheadrightarrow H$ be a surjection onto a finite group, corresponding to some finite H-cover $\Sigma_{g'} \to \Sigma_g$ branched over n points. Let Γ be the stabilizer of ϕ up to conjugacy in the pure mapping class group of $\Sigma_{g,n+1}$. Then Γ acts naturally acts on $H_1(\Sigma_{g'})$.

We say that *the Putman-Wieland conjecture for (g, n, H) holds* if for every such H-cover $\Sigma_{g'} \to \Sigma_g$, all non-zero vectors in $H_1(\Sigma_{g'})$ have infinite orbit under Γ. The original Putman-Wieland conjecture [PW13, Conj. 1.2] asserts that for any fixed $g \geq 2, n \geq 0$, the Putman-Wieland conjecture for (g, n, H) holds for every H. We prove the Putman-Wieland conjecture for (g, n, H) for any fixed H once g is sufficiently large. In this way, our results are an "asymptotic" version of the Putman-Wieland conjecture.

THEOREM 1.4.1. *For $g \geq 2, n \geq 0$, and any finite group H with $\#H < g^2$, the Putman-Wieland conjecture for (g, n, H) holds.*

Theorem 1.4.1 is proven below as a case of Corollary 7.2.4. More generally, for arbitrary H, we prove that the Putman-Wieland conjecture holds for the subspace of $H_1(\Sigma_{g'})$ spanned by irreducible representations of H of small rank.

THEOREM 1.4.2. *Let ρ be an irreducible representation of H with $\dim \rho < g$, and let $H_1(\Sigma_{g'})^\rho$ be the ρ-isotypic component of $H_1(\Sigma_{g'})$. Then no non-zero vector in $H_1(\Sigma_{g'})^\rho$ has finite orbit under Γ.*

Theorem 1.4.2 is proven below as a special case of Theorem 7.2.1.

As we shall see in the course of the proof of Theorem 1.2.1, especially Lemma 8.6.1, the Putman-Wieland conjecture is connected to the classification of MCG-finite representations with virtually solvable image.

1.5. *Arithmetic applications.* In Section 9, we give a number of applications to questions in arithmetic geometry. For example, we show in Theorem 9.1.2 that low rank representations of the arithmetic étale fundamental group of a generic curve of genus g have finite image when restricted to the geometric fundamental group. This verifies a prediction of the Fontaine-Mazur conjecture; see Remark 9.1.5. These results also answer a question of Esnault-Kerz; see Remark 9.1.3.

As a consequence, we construct many residual representations of the geometric fundamental group of a generic curve of genus g that have no lifts to

representations of geometric origin. These examples are related to a conjecture of de Jong [dJ01] (proven by Gaitsgory [Gai07]) and a question of Flach; see Remarks 9.2.5 and 9.2.6.

1.6. *A consequence for free groups.* From Theorem 1.2.1, we deduce an analogous result for free groups.

COROLLARY 1.6.1. *Let F_N be a free group on N generators, with $N = 2g$ or $N = 2g + 1$. Let*

$$\rho : F_N \to \mathrm{GL}_r(\mathbb{C})$$

be a representation whose conjugacy class has finite orbit under $\mathrm{Out}(F_N)$. *If $r < \sqrt{g+1}$, then ρ has finite image.*

Proof. If N is even, choose an isomorphism $F_N \simeq \pi_1(\Sigma_{g,1})$, and if N is odd, choose $F_N \simeq \pi_1(\Sigma_{g,2})$. Any representation of F_N with finite orbit under $\mathrm{Out}(F_N)$ yields a representation of a genus g surface group with finite orbit under the mapping class group. Such a representation has finite image by Theorem 1.2.1. □

Remark 1.6.2. It is natural to ask if the bound on r in Corollary 1.6.1 is sharp. There exist (non-semisimple) representations of F_N of fairly low rank ($r = N + 1$) with infinite image and finite orbit under $\mathrm{Out}(F_N)$ (see Example 10.2.7), so the bound cannot be improved too much. However, in contrast with the case of MCG-finite representations, we do not know any examples of *semisimple* representations of F_N with infinite image and finite orbit under $\mathrm{Out}(F_N)$ as soon as $N \geq 3$. Indeed, a conjecture of Grunewald and Lubotzky [GL09, Conj. in §9.2], combined with the main result of [FH17], implies that no such examples exist when $N \geq 3$. In the interests of provocation, we conjecture (Conjecture 10.2.8) that such examples *do* exist. As evidence, we offer several examples of semisimple MCG-finite representations of surface groups with infinite image; see Question 10.2. Note that there are interesting semisimple representations of F_2 with finite orbit under $\mathrm{Out}(F_2)$; see Example 10.2.7.

Remark 1.6.3. One appeal of Corollary 1.6.1 lies in the fact that it admits a completely elementary reformulation, using Nielsen's description of $\mathrm{Aut}(F_N)$; see [Nie24] and also [MKS04, Th. 3.2, p. 131]. Let $(A_1, \ldots, A_N)$ be an N-tuple of invertible $r \times r$ complex matrices, with $N = 2g$ or $N = 2g + 1$. Consider the following operations on N-tuples of invertible matrices:

(1) the cyclic permutation

$$c : (A_1, A_2, \ldots, A_N) \mapsto (A_2, A_3, \ldots, A_N, A_1);$$

(2) the transposition

$$\tau : (A_1, A_2, A_3, \ldots, A_N) \mapsto (A_2, A_1, A_3, \ldots, A_N);$$

(3) the inversion map

$$\epsilon : (A_1, A_2, \dots, A_N) \mapsto (A_1^{-1}, A_2, \dots, A_N);$$

(4) the Dehn twist

$$d : (A_1, A_2, \dots, A_N) \mapsto (A_1 A_2, A_2, \dots, A_N).$$

We say $(A_1, \dots, A_N)$ and $(A'_1, \dots, A'_N)$ are *conjugate* if there exists some B such that $A_i = BA'_iB^{-1}$ for all i.

Now suppose that the set of N-tuples obtained from $(A_1, \dots, A_N)$ by repeatedly applying c, τ, ϵ, and d only intersect finitely many conjugacy classes of N-tuples $(A_1, \dots, A_N)$. If $r < \sqrt{g+1}$, Corollary 1.6.1 implies that $A_1, \dots, A_N$ generate a finite subgroup of $\mathrm{GL}_r(\mathbb{C})$.

Remark 1.6.4. Similarly to Remark 1.6.3, one could make Theorem 1.2.1 explicit, using any one of the known generating sets of $\mathrm{Mod}_{g,n}$, but the formulas are a bit more involved, as in [CH21, §6]. In our view it would be of great interest to find a *proof* of Theorem 1.2.1 or Corollary 1.6.1 of a similarly explicit nature.

Remark 1.6.5. A result analogous to Corollary 1.6.1 follows immediately for characteristic quotients of free or surface groups. For example, let G be a characteristic quotient of a free group F_N on $N = 2g$ or $N = 2g+1$ generators, and $\rho : G \to \mathrm{GL}_r(\mathbb{C})$ a representation whose conjugacy class has finite orbit under $\mathrm{Out}(G)$. If $r < \sqrt{g+1}$, then ρ has finite image by Corollary 1.6.1.

1.7. *Cohomological results.* One of the new technical inputs we use to achieve Theorem 1.2.1 is an analysis of the cohomology of unitary local systems on families of curves. Our main result, which follows from an analysis of the derivative of the period map associated to the mixed Hodge structure on the cohomology of unitary local systems, is the following. We adapt notation described later in Notation 1.10.1.

THEOREM 1.7.1. *Let $\pi : \mathscr{C} \to \mathscr{M}$ be a smooth proper family of n-pointed curves of genus g with geometrically connected fibers, so that the associated map $\mathscr{M} \to \mathscr{M}_{g,n}$ is dominant étale. Let $\pi^\circ : \mathscr{C}^\circ \to \mathscr{M}$ be the associated family of punctured curves. Let $\mathbb{V}$ be a unitary local system on $\mathscr{C}^\circ$. Then any non-zero sub-local system of $R^1\pi^\circ_*\mathbb{V}$ has rank at least $2g - 2\,\mathrm{rk}\,\mathbb{V}$.*

We prove this in Section 6.1 and give a related cohomological vanishing theorem with milder unitarity hypotheses in Theorem 6.2.1. Note that Theorem 1.7.1 shows that $H^0(\mathscr{M}, R^1\pi^\circ_*\mathbb{V}) = 0$ if $g > \mathrm{rk}\,\mathbb{V}$. This result is used several times in the proof of Theorem 1.2.1, as described in Section 1.9. It is especially interesting when $\mathbb{V}$ has finite monodromy, where it is closely related to the Putman-Wieland conjecture, as described above in Section 1.4 and in more detail in Section 7.

1.8. *Prior work.*

1.8.1. *MCG-finite representations and Painlevé VI.* As mentioned in Section 1.1, there has been a huge amount of effort put forth towards classifying algebraic solutions to the Painlevé VI equation, arguably beginning with work of Riemann [Rie57] and Schwarz [Sch73] and culminating in the classification [LT14]. Calligaris-Mazzocco [CM18] classified algebraic solutions satisfying several conditions in the case $g = 0, n = 5, r = 2$, but the complete classification in this case remains open. See also [Mic06] for a related result in the genus zero case, when the local monodromy matrices are given by reflections; this appears to be the first result showing that MCG-finite representations (satisfying certain conditions) have finite image.

For all the work invested in the algebraic solutions to Painlevé VI, this only addresses the case of classifying MCG-finite representations with $g=0, n=4$, $r=2$, and trivial determinant, as explained in [LT14]. In higher genus, all work that we know of has been on the case of 2-dimensional representations. The beautiful paper [BGMW22] handles the 2-dimensional case ($r=2, g\geq 1, n\geq 0$) with trivial determinant. The non-semisimple case $r = 2, g \geq 1, n \geq 0$ with arbitrary determinant is explained in [CH21, Th. B]. These results appear to depend crucially on the assumption $r = 2$.

Remark 1.8.2. The above-mentioned results in rank 2 led Junho Peter Whang to ask whether, for $g \gg r$, all MCG-finite rank r representations of $\Sigma_{g,n}$ have finite image [LL22, Question 1.5.3]. More generally, motivated by the p-curvature conjecture, Mark Kisin asked whether MCG-finite representations necessarily have finite image; see [BGMW22, p. 3] and [Sin10, p. 1]. (Note that counterexamples are known in general; see [BKMS18, Th. 5.1] for counterexamples coming from TQFT techniques or [LL22, Exam. 3.3.1] for counterexamples coming from the Kodaira-Parshin trick, as well as Question 10.2 of this paper.) In this way, Theorem 1.2.1 answers Whang's question affirmatively, and provides a positive answer to Kisin's question in the regime $g > r^2 - 1$.

There has also been much work put towards finding interesting *examples* of MCG-finite representations, notably [Dor01], [Dia13], [Gir16b], [DL15], [Gir16a].

In a more arithmetic direction, [BGS16] announced striking results on strong approximation for Markoff triples, obtained by analyzing the arithmetic properties of subvarieties of the character variety parametrizing 2-dimensional representations of $\pi_1(\Sigma_{1,1})$. The first step in their analysis is to determine the finite orbits of the mapping class group action on this character variety [BGS16, last paragraph of p. 132]. In this way, our Theorem 1.2.1 can be viewed as a necessary first step toward attempting to generalize their approach to higher rank, higher genus character varieties.

1.8.3. *MCG-finite representations and geometric topology.* In the world of low-dimensional topology a number of authors have studied the dynamics of mapping class group actions on character varieties; finite orbits are the same as semisimple MCG-finite representations. Kasahara [Kas15] relates fixed points of this action corresponding to faithful representations to the well-known question of linearity of the mapping class group. Goldman and many other authors have studied ergodicity of these actions; see [Gol06] and the references therein. Previte-Xia [PX00], [PX02] study the relationship between density of the image of an SU(2)-representation and density of its mapping class group orbit. In some sense our main result is a partial answer to [Gol06, Question 2.7], which asks for necessary and sufficient conditions for a representation to have dense orbit under the mapping class group. We characterize the most extreme possible failure of density, namely the case of finite orbit.

There are also a number of related results on representations of the mapping class group. Farb, Lubotzky, and Minsky [FLM01, Th. 1.6] show there are no faithful linear representations of finite index subgroups of $\mathrm{Mod}_{g,0}$ of dimension $< 2\sqrt{g-1}$. See Remark 10.1.7 for a comparison of our results to theirs. See also [FH13], [Kor02], [Fun11], [KP20] for bounds on the dimension of representations of $\mathrm{Mod}_{g,n}$, although these results only address representations of the full mapping class group, as opposed to representations of finite index subgroups.

1.8.4. *The Putman-Wieland conjecture.* The other main contribution of this paper, towards the Putman-Wieland conjecture, has a number of precursors, notably [Loo97a], [GLLM15], [Loo97b]. Our Hodge-theoretic approach is related to an approach suggested by Looijenga [Loo15]. The strongest previous result towards the Putman-Wieland conjecture is perhaps the main result of [GLLM15], which says that for certain quotients of $\pi_1(\Sigma_g)$, called *ϕ-redundant* quotients, much more than the Putman-Wieland conjecture is true — the monodromy representations considered by the conjecture have very large image, commensurable with an arithmetic group. There has also been interesting recent work related to the Putman-Wieland conjecture by Marković and Marković-Tošić [Mar22], [MT24].

The analogue of the Putman-Wieland conjecture for graphs, as opposed to surfaces, has been proven by Farb and Hensel [FH17].

1.9. *Outline of the proof.* We now sketch the proof of Theorem 1.2.1, which is loosely inspired by Katz's proof of the p-curvature conjecture for the Gauss-Manin connection [Kat72]. Following this, we sketch the proof of the vanishing results we use (Theorem 1.7.1 and its consequence Theorem 6.2.1) in Section 1.9.4, which are used in different ways in each of the three steps listed below of the proof of Theorem 1.2.1. A schematic diagram outlining the main elements of the proof is depicted in Figure 2.

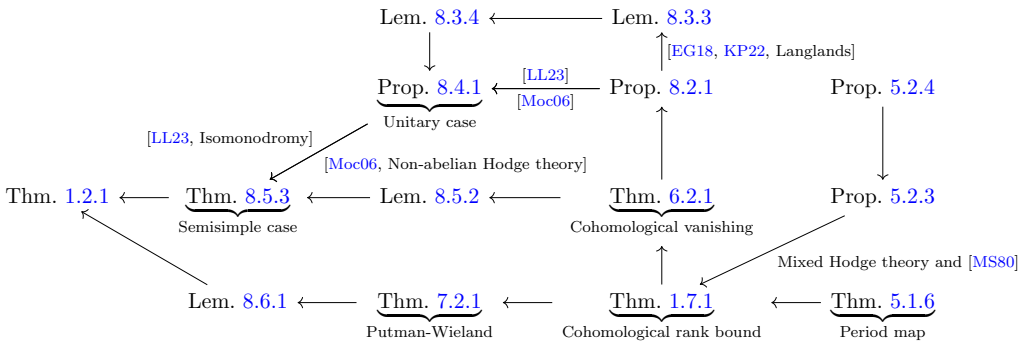


Figure 2. A diagram depicting the structure of the proof of the main result, Theorem 1.2.1, in the shape of a boat.

1.9.1. *Step* 1. *The unitary case.* Every unitary representation is a direct sum of irreducible unitary representations. Therefore, we suppose ρ is unitary, irreducible, and MCG-finite, with $\operatorname{rk}\rho < \sqrt{g+1}$. We construct from ρ a finite étale cover $\mathscr{M}$ of $\mathscr{M}_{g,n}$, with associated family of punctured curves $\pi^\circ : \mathscr{C}^\circ \to \mathscr{M}$, and a projective unitary local system $\mathbb{V}$ on $\mathscr{C}^\circ$ whose restriction to a fiber C° of π° has monodromy given by ρ. Applying Proposition 8.2.1, (which is a fairly straightforward consequence of our cohomological vanishing result, Theorem 6.2.1, applied to $\operatorname{ad}\mathbb{V}$), shows that $\mathbb{V}$ is cohomologically rigid. The main result of [KP22] then gives that ρ is defined over the ring of integers $\mathscr{O}_K$ of some number field K. Moreover, using Proposition 2.3.4, by replacing $\mathscr{M}$ by a dominant étale scheme over it, along which certain cohomological lifting obstructions vanish, we can assume $\mathbb{V}$ lifts from a projective local system to a bona fide local system.

By compactness of the unitary group and discreteness of $\mathscr{O}_K$, it suffices to show that for each embedding $\iota : \mathscr{O}_K \hookrightarrow \mathbb{C}$, $\rho \otimes_{\mathscr{O}_K,\iota} \mathbb{C}$ is unitary. (We know this unitarity for one such ι by assumption, but not the others.) The rigidity of $\mathbb{V}$ implies by non-abelian Hodge theory that these $\rho \otimes_{\mathscr{O}_K,\iota} \mathbb{C}$ underlie complex polarizable variations of Hodge structure for any complex structure on $\Sigma_{g,n}$. Hence, given their low rank, these local systems are unitary by [LL23, Th. 1.2.12].

1.9.2. *Step* 2. *The semisimple case.* Now suppose ρ is an arbitrary semisimple, MCG-finite representation with $\operatorname{rk}\rho < \sqrt{g+1}$. Again, we associate to ρ a local system $\mathbb{V}$ on a family of curves $\pi^\circ : \mathscr{C}^\circ \to \mathscr{M}$, so that $\mathscr{M}$ has a dominant étale map to $\mathscr{M}_{g,n}$ and whose fibral monodromy is given by ρ. By non-abelian Hodge theory, we may deform $\mathbb{V}$ to a local system $\mathbb{V}_0$ underlying a complex polarizable variation of Hodge structure. By [LL23, Th. 1.2.12], $\mathbb{V}_0$ has unitary monodromy when restricted to a fiber of π°. By the unitary case, Section 1.9.1, $\mathbb{V}_0$ thus has finite monodromy when restricted to a fiber of π°.

Note that even if $n = 0$, i.e., π° is proper, we here need to use non-abelian Hodge theory for non-proper varieties, as the total space $\mathscr{C}^\circ$ will not be proper.

It remains to argue that the restriction $\mathbb{V}_0|_{C^\circ}$ of $\mathbb{V}_0$ to a fiber C° of π° agrees with the restriction $\mathbb{V}|_{C^\circ}$, corresponding to ρ. Recall that $\mathbb{V}_0$ and $\mathbb{V}$ were only deformation equivalent, so it may be surprising that they necessarily restrict to the same local system on fibers. We verify this agreement in Lemma 8.5.2 through another application of Theorem 6.2.1, which which tells us that since the fibral monodromy $\mathbb{V}_0|_{C^\circ}$ is unitary of low rank, it does not admit non-trivial MCG-finite deformations.

Note that $\mathbb{V}_0$ may not a priori be unitary; we only know that $\mathbb{V}_0|_{C^\circ}$ is unitary. In particular, it is not clear whether $\mathbb{V}_0$ is necessarily cohomologically rigid.

1.9.3. *Step* 3. *The general case.* The crucial input for dealing with non-semisimple representations is our work towards the Putman-Wieland conjecture. By the above it is enough to show that low-rank MCG-finite representations are semisimple, i.e., we wish to verify that certain extensions split. For simplicity, let us suppose for the purpose of this sketch that ρ is an extension of two irreducible MCG-finite representations ρ_1 and ρ_2. By the previous step, ρ_1 and ρ_2 have finite monodromy, so after passing to a finite cover $\Sigma_{g',n'}$ of $\Sigma_{g,n}$, we may assume ρ_1 and ρ_2 have trivial monodromy. The splitting of this extension of ρ_2 by ρ_1 corresponds to the vanishing of a certain element in $\mathrm{Ext}^1_{\pi_1(\Sigma_{g',n'})}(\rho_1, \rho_2)$. Because we arranged that ρ_1 and ρ_2 have trivial monodromy on $\pi_1(\Sigma_{g',n'})$, the above extension class corresponds to a map $\pi_1(\Sigma_{g',n'}) \to \rho_1^\vee \otimes \rho_2$, with unipotent abelian image. Hence, it factors through $H_1(\Sigma_{g',n'})$, and so defines a low rank subspace of $H^1(\Sigma_{g',n'})$, stable under a finite index subgroup of $\mathrm{Mod}_{g,n+1}$. We verify in Lemma 8.6.1, using our main result toward the Putman-Wieland conjecture, Theorem 7.2.1, that such a subspace cannot exist. Theorem 7.2.1 itself follows more or less immediately from Theorem 1.7.1.

1.9.4. *The proof of our cohomological vanishing results.* Note that we used Theorem 1.7.1, or its immediate consequence, Theorem 6.2.1, in every step above. Before explaining the idea of the proof of these results, let us recall the setting. We begin with a scheme $\mathscr{M}$ with a dominant étale map $\mathscr{M} \to \mathscr{M}_{g,n}$. We denote the associated family of punctured curves by $\mathscr{C}^\circ \to \mathscr{M}$. We are given a local system $\mathbb{V}$ on $\mathscr{C}^\circ$. Theorem 1.7.1 says that if $\mathbb{V}$ is unitary, there are no sub-local systems of $R^1\pi^\circ_*\mathbb{V}$ of low rank (that is, rank less than $2g - 2\,\mathrm{rk}\,\mathbb{V}$). Let C° be a fiber of π°. We deduce Theorem 6.2.1, which says that when $\mathbb{V}|_{C^\circ}$ is unitary and has low rank (less than g), then $R^1\pi^\circ_*\mathbb{V}$ has no global sections. The benefit of Theorem 6.2.1 is that we only need check unitarity on fibers, but the cost is that we are only able to rule out global sections, i.e., trivial sub-local systems, instead of arbitrary low rank sub-local systems.

First, we deduce Theorem 6.2.1 from Theorem 1.7.1. The idea is to use the assumption that $\mathbb{V}$ is unitary on fibers to reduce via Lemma 2.4.2 to the case that $\mathbb{V}$ is a tensor product $\mathbb{U} \otimes (\pi^\circ)^*\mathbb{W}$, with $\mathbb{U}$ unitary. A non-zero global section of $R^1\pi^\circ_*(\mathbb{U} \otimes (\pi^\circ)^*\mathbb{W}) = (R^1\pi^\circ_*\mathbb{U}) \otimes \mathbb{W}$ yields a non-zero map $\mathbb{W}^\vee \to R^1\pi^\circ_*\mathbb{U}$, and hence a low rank sub-local system of $R^1\pi^\circ_*\mathbb{U}$, contradicting Theorem 1.7.1.

The proof of Theorem 1.7.1 boils down to an analysis of the derivative of the period map associated to the complex variation of mixed Hodge structure on $R^1\pi^\circ_*\mathbb{V}$ for $\mathbb{V}$ unitary. We identify this derivative with a natural multiplication map

$$H^0(E \otimes \omega_C(D)) \otimes H^0(E^\vee \otimes \omega_C) \to H^0(\omega_C^{\otimes 2}(D))$$

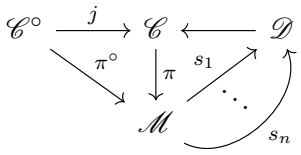
in Theorem 5.1.6, where C is the smooth projective compactification of C° with boundary D, and E is the vector bundle on C associated to $\mathbb{V}|_{C^\circ}$ by the Mehta-Seshadri correspondence [MS80]. By the theorem of the fixed part, the existence of a low rank sub-local system of $R^1\pi^\circ_*\mathbb{V}$ places restrictions on this map, which we rule out using vector bundle methods, as developed in [LL23, §6], ultimately relying on Clifford's theorem for vector bundles.

1.10. *Notation.* Unless otherwise stated, we will work over the field of complex numbers $\mathbb{C}$. In particular, we will lift this restriction in Section 9.

Notation 1.10.1. We fix non-negative integers (g, n) so that $n \geq 1$ if $g = 1$ and $n \geq 3$ if $g = 0$, i.e., $\Sigma_{g,n}$ is hyperbolic. Let $\mathscr{M}$ be a connected complex variety. A *family of n-pointed curves of genus g over $\mathscr{M}$* is a smooth proper morphism $\pi : \mathscr{C} \to \mathscr{M}$ of relative dimension one, with geometrically connected genus g fibers, equipped with n sections $s_1, \ldots, s_n : \mathscr{M} \to \mathscr{C}$ with disjoint images. Call such a family *versal* if the induced map $\mathscr{M} \to \mathscr{M}_{g,n}$ is dominant and étale. Here $\mathscr{M}_{g,n}$ denotes the Deligne-Mumford moduli stack of n-pointed genus g smooth proper curves with geometrically connected fibers.

If $\pi : \mathscr{C} \to \mathscr{M}$ is a family of n-pointed curves, we let $\mathscr{D} := \coprod_{i=1}^n \mathrm{im}(s_i)$ denote the images of the sections, which is finite étale of degree n over $\mathscr{M}$. Also let $\mathscr{C}^\circ := \mathscr{C} \setminus \bigcup_i \mathrm{im}(s_i)$, let $j : \mathscr{C}^\circ \hookrightarrow \mathscr{C}$ be the natural inclusion, and let $\pi^\circ := \pi \circ j : \mathscr{C}^\circ \to \mathscr{M}$ denote the composition. We will refer to $\pi^\circ : \mathscr{C}^\circ \to \mathscr{M}$ as the *associated family of punctured curves*. If π° arises as the family of punctured curves associated to a versal family of n-pointed curves, we will call it a *punctured versal family*. We will frequently use $m \in \mathscr{M}$ as a basepoint, and $c \in \mathscr{C}^\circ$ as a basepoint with $\pi^\circ(c) = m$. We use C° as notation to denote the fiber $(\pi^\circ)^{-1}(m) = \mathscr{C}^\circ_m$:

(1.1)



Notation 1.10.2. For G an algebraic group with derived subgroup G^{der} and corresponding Lie algebra $\mathfrak{g}^{\mathrm{der}}$, we use $\mathrm{Ad} : G \to \mathrm{GL}(\mathfrak{g}^{\mathrm{der}})$ to denote the natural action of G on $\mathfrak{g}^{\mathrm{der}}$ by conjugation. Given a representation $\rho : \pi_1(X,x) \to G(\mathbb{C})$, let $\mathrm{ad}(\rho) := \rho \circ \mathrm{Ad} : \pi_1(X,x) \to \mathrm{GL}(\mathfrak{g}^{\mathrm{der}})$. In particular, given $\rho : \pi_1(X,x) \to \mathrm{GL}_r(\mathbb{C})$ or $\rho : \pi_1(X,x) \to \mathrm{PGL}_r(\mathbb{C})$ we use $\mathrm{ad}(\rho)$ to denote the composite map $\mathrm{ad}(\rho) : \pi_1(X,x) \to \mathrm{GL}(\mathfrak{pgl}_r(\mathbb{C}))$. Under the identification between local systems on a connected space and representations of the fundamental group, if $\mathbb{W}$ is a local system on X corresponding to some representation ρ, we use $\mathrm{ad}\,\mathbb{W}$ to denote the local system corresponding to $\mathrm{ad}\,\rho$.

A representation $\pi_1(X,x) \to \mathrm{GL}_r(\mathbb{C})$ is *unitary* if its image has compact closure. A complex local system is unitary if its monodromy representation is unitary. By an averaging argument, unitary representations are exactly those that preserve a positive-definite Hermitian form on $\mathbb{C}^n$; i.e., they are conjugate to a representation that factors through the unitary group $\mathrm{U}(n)$.

Notation 1.10.3. For a pointed finite-type scheme or Deligne-Mumford stack (X,x) over $\mathbb{C}$, we will use $\pi_1(X,x)$ to denote the topological fundamental group of the associated complex-analytic space or analytic stack. Similarly, for a local system Λ on X, we use $H^i(X,\Lambda)$ to denote the singular cohomology of the associated complex-analytic space or stack, unless otherwise stated. We will use $\mathrm{Mod}_{g,n}$ to denote the mapping class group of an orientable surface of genus g with n punctures, and $\mathrm{PMod}_{g,n} = \pi_1(\mathscr{M}_{g,n})$ (see Lemma 2.0.1) to denote pure mapping class group, i.e., the subgroup of $\mathrm{Mod}_{g,n}$ of index $n!$ preserving the punctures pointwise.

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2. Representation-theoretic preliminaries

In this section, we give group-theoretic constructions that we will use to analyze representations with finite mapping class group orbit. In Section 2.1, we verify basic properties of MCG-finite representations. In Section 2.2, given an irreducible representation $\rho : \pi_1(\Sigma_{g,n}, x) \to \mathrm{GL}_r(\mathbb{C})$ whose conjugacy class is fixed by a finite-index subgroup $\Gamma \subset \mathrm{PMod}_{g,n+1}$, we construct a representation $\widetilde{\rho} : \Gamma \to \mathrm{PGL}_r(\mathbb{C})$, which will be analyzed throughout this paper. In Section 2.3, we show how to lift certain projective representations of the fundamental groups of families of curves to honest representations into $\mathrm{GL}_r(\mathbb{C})$, after passing to a suitable cover. Finally, in Section 2.4, we prove some structural results about the representations we have constructed.

We will use the following lemma throughout, to connect properties of mapping class groups to geometry.

LEMMA 2.0.1. *For $m \in \mathscr{M}_{g,n}$ a basepoint and $m' \in \mathscr{M}_{g,n+1}$ a lift of m, there are isomorphisms $\pi_1(\mathscr{M}_{g,n}, m) \simeq \mathrm{PMod}_{g,n}$, $\pi_1(\mathscr{M}_{g,n+1}, m') \simeq \mathrm{PMod}_{g,n}$ such that the diagram*

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \pi_1(\Sigma_{g,n}) & \longrightarrow & \pi_1(\mathscr{M}_{g,n+1}, m') & \longrightarrow & \pi_1(\mathscr{M}_{g,n}, m) & \longrightarrow & 1 \\ & & \| & & \downarrow\sim & & \downarrow\sim & & \\ 1 & \longrightarrow & \pi_1(\Sigma_{g,n}) & \longrightarrow & \mathrm{PMod}_{g,n+1} & \longrightarrow & \mathrm{PMod}_{g,n} & \longrightarrow & 1 \end{array}$$

commutes, where the vertical maps are given by these isomorphisms, the top row is the exact sequence of homotopy groups induced by the forgetful map $\mathscr{M}_{g,n+1} \to \mathscr{M}_{g,n}$, and the bottom row is the Birman exact sequence [FM12, p. 98].

Proof. This follows from the contractibility of the universal cover of $\mathscr{M}_{g,n}$, and the fact that $\mathscr{M}_{g,n}$ is the quotient of its universal cover by the properly discontinuous action of $\mathrm{PMod}_{g,n}$. See [FM12, §10.6.3 and p. 353]. There is a choice involved here, namely, if C is the Riemann surface associated to the point $m \in \mathscr{M}_{g,n}$, one must choose a homeomorphism between C and our reference surface $\Sigma_{g,n}$, and similarly with m'. Changing this homeomorphism replaces the vertical isomorphisms by conjugate isomorphisms. □

The subgroup $\pi_1(\Sigma_{g,n}) \subset \mathrm{PMod}_{g,n+1}$ is often referred to as the *point-pushing subgroup*.

2.1. *Basic properties of MCG-finite representations.*

To acquaint the reader with MCG-finite representation, we now spell out some of their basic properties.

PROPOSITION 2.1.1. *The direct sum and tensor product of two MCG-finite representations of $\pi_1(\Sigma_{g,n})$ is MCG-finite. Any semisimple subquotient of an MCG-finite representation is MCG-finite.*

Proof. The first statement, about direct sums and tensor products, is clear. The second is immediate from [LL22, Lemma 2.2.1]. □

We next show that restrictions of representations of $\mathrm{Mod}_{g,n+1}$ to the point-pushing subgroup are MCG-finite. See also [Kas15, Lemma 2.2] for a related result with analogous proof.

PROPOSITION 2.1.2. *Let $\Gamma \subset \mathrm{Mod}_{g,n+1}$ be a finite index subgroup containing the point-pushing subgroup $\pi_1(\Sigma_{g,n}) \subset \mathrm{Mod}_{g,n+1}$. Let*

$$\rho : \Gamma \to \mathrm{GL}_r(\mathbb{C})$$

be a representation. Then $\rho|_{\pi_1(\Sigma_{g,n})}$ is MCG-finite.

Proof. Replacing Γ with $\Gamma \cap \mathrm{PMod}_{g,n+1}$, we may assume $\Gamma \subset \mathrm{PMod}_{g,n+1}$, the pure mapping class group. We claim that the image of Γ in $\mathrm{PMod}_{g,n}$ (under the map $\mathrm{PMod}_{g,n+1} \to \mathrm{PMod}_{g,n}$ arising from the Birman exact sequence) stabilizes the conjugacy class of ρ. It suffices to show that for each $\gamma \in \Gamma$,

$$\begin{aligned} \rho^\gamma : \pi_1(\Sigma_{g,n}) &\to \mathrm{GL}_r(\mathbb{C}), \\ g &\mapsto \rho(\gamma g \gamma^{-1}) \end{aligned}$$

is conjugate to ρ. Indeed, $\rho^\gamma(h) = \rho(\gamma)\rho(h)\rho(\gamma)^{-1}$. □

We later give a partial converse to Proposition 2.1.2, for irreducible representations, in Corollary 2.3.5. We now give a geometric counterpart to Proposition 2.1.2, which is closely related to [CH21, Th. A1].

PROPOSITION 2.1.3. *Let $\pi^\circ : \mathscr{C}^\circ \to \mathscr{M}$ be a punctured versal family of n-pointed genus g curves, as in Notation 1.10.1. In particular, $\mathscr{M} \to \mathscr{M}_{g,n}$ is dominant and étale. Let C° be a fiber of π°. If*

$$\rho : \pi_1(\mathscr{C}^\circ) \to \mathrm{GL}_r(\mathbb{C})$$

is a representation, $\rho|_{\pi_1(C^\circ)}$ is MCG-finite.

We prove Proposition 2.1.3 in Section 2.1.6, but we first require some well-known lemmas about families of curves.

LEMMA 2.1.4. *For $\mathscr{M}$ a scheme and any dominant étale map $\mathscr{M} \to \mathscr{M}_{g,n}$, $\mathrm{im}(\pi_1(\mathscr{M}) \to \pi_1(\mathscr{M}_{g,n}))$ has finite index in $\pi_1(\mathscr{M}_{g,n}) \simeq \mathrm{PMod}_{g,n}$.*

Proof. After pulling back to a finite étale cover $\mathscr{N} \to \mathscr{M}_{g,n}$ that is a scheme (such exists by, e.g., [PdJ95, Prop. 2.3.4]), we may assume there is a dominant étale map of schemes $\alpha : \mathscr{M} \to \mathscr{N}$. It is enough to show $\mathrm{im}(\pi_1(\mathscr{M}) \to \pi_1(\mathscr{N}))$ has finite index in $\pi_1(\mathscr{N})$, which follows from [Deb01, Lemma 4.19]. The basic idea here is to pass to opens over which the map is finite flat, and use that passing to opens will preserve the property that the image of the map of fundamental groups has finite index. □

LEMMA 2.1.5. *If $\mathscr{C} \to \mathscr{M}$ is a versal family of n-pointed curves of genus g, with $3g-3+n \geq 0$, as in Notation 1.10.1, so that C° is a fiber of π°, the sequence*

$$1 \longrightarrow \pi_1(C^\circ) \longrightarrow \pi_1(\mathscr{C}^\circ) \longrightarrow \pi_1(\mathscr{M}) \longrightarrow 1 \tag{2.1}$$

is exact.

Proof. Except for injectivity, the result is immediate from the long exact sequence in homotopy groups and the fact that C° is connected. Injectivity holds in the universal case where $\mathscr{M} = \mathscr{M}_{g,n}$ is the Deligne-Mumford moduli stack of curves and $\mathscr{C}^\circ = \mathscr{C}^\circ_{g,n}$ is the universal punctured curve, as the universal cover of $\mathscr{M}_{g,n}$ is contractible. We next show the map $\pi_1(C^\circ) \to \pi_1(\mathscr{C}^\circ)$ is injective in the general case via pullback. The composite map $C^\circ \to \mathscr{C} \to \mathscr{C}^\circ_{g,n}$ is the natural inclusion of C° as a fiber of $\mathscr{C}^\circ_{g,n}$ over $\mathscr{M}_{g,n}$. Hence, the composition $\pi_1(C^\circ) \to \pi_1(\mathscr{C}^\circ) \to \pi_1(\mathscr{C}^\circ_{g,n})$ is injective, and so the first map is injective. □

2.1.6. *Proof of Proposition 2.1.3.* The exact sequence (2.1) induces an outer action of $\pi_1(\mathscr{M})$ on $\pi_1(C^\circ)$, which factors through the outer action of $\mathrm{PMod}_{g,n}$ on $\pi_1(C^\circ)$ induced by the Birman exact sequence. Specifically, this outer action sends $\gamma \in \mathrm{PMod}_{g,n}$ to the action of conjugation by $\widetilde{\gamma}$ on $\pi_1(C^\circ)$ for $\widetilde{\gamma} \in \mathrm{PMod}_{g,n+1}$ a lift of γ. By construction, this outer action is compatible with the outer action of $\mathrm{Mod}_{g,n}$ on $\pi_1(C^\circ)$ of Definition 1.1.1.

To show $\rho|_{\pi_1(C^\circ)}$ is MCG-finite, it therefore suffices to show the conjugacy class of $\rho|_{\pi_1(C^\circ)}$ is stable under the image Γ of

$$\pi_1(\mathscr{C}^\circ) \twoheadrightarrow \pi_1(\mathscr{M}) \to \pi_1(\mathscr{M}_{g,n}) \simeq \mathrm{PMod}_{g,n}\,.$$

This image has finite index by Lemma 2.1.4. The conjugacy class of $\rho|_{\pi_1(C^\circ)}$ is stable under this image because, if $\widetilde{\gamma} \in \pi_1(\mathscr{C}^\circ)$ is any lift of $\gamma \in \Gamma$, then $\rho^{\widetilde{\gamma}}$ is conjugate to ρ by $\rho(\widetilde{\gamma})$, as in the proof of Proposition 2.1.2. □

2.2. *Projective local systems associated to MCG-finite representations.* We next give a construction lifting an irreducible MCG-finite representation of $\pi_1(\Sigma_{g,n})$ to a projective representation of a finite index subgroup of $\mathrm{Mod}_{g,n+1}$.

Notation 2.2.1. Suppose ρ is an irreducible MCG-finite representation of $\pi_1(\Sigma_{g,n}, x)$. Let $\Gamma \subset \mathrm{PMod}_{g,n}$ denote the finite index stabilizer of the conjugacy class of ρ, and let $\widetilde{\Gamma} \subset \mathrm{PMod}_{g,n+1}$ denote the preimage of Γ under the surjection $\mathrm{PMod}_{g,n+1} \to \mathrm{PMod}_{g,n}$ coming from the Birman exact sequence.

LEMMA 2.2.2. *In the setup of Notation 2.2.1, there exists a unique representation $\widetilde{\rho} : \widetilde{\Gamma} \to \mathrm{PGL}_r(\mathbb{C})$ so that*

$$\begin{array}{ccc} \pi_1(\Sigma_{g,n}, x) & \xrightarrow{\rho} & \mathrm{GL}_r(\mathbb{C}) \\ \downarrow & & \downarrow \\ \widetilde{\Gamma} & \xrightarrow{\widetilde{\rho}} & \mathrm{PGL}_r(\mathbb{C}) \end{array} \tag{2.2}$$

commutes, where $\pi_1(\Sigma_{g,n}, x) \subset \widetilde{\Gamma} \subset \mathrm{PMod}_{g,n+1}$ *is the inclusion of the point-pushing subgroup from the Birman exact sequence.*

Proof. First, we construct the representation $\widetilde{\rho}$. By identifying $\mathrm{PMod}_{g,n+1} \subset \mathrm{Aut}(\pi_1(\Sigma_{g,n}, x))$, we obtain an action of $\mathrm{PMod}_{g,n+1}$ on r-dimensional representations of $\pi_1(\Sigma_{g,n}, x)$. Since $\widetilde{\Gamma}$ preserves ρ up to conjugacy, for every $\gamma \in \widetilde{\Gamma}$, there is some matrix $M_\gamma \in \mathrm{GL}_r(\mathbb{C})$ so that $M_\gamma \rho M_\gamma^{-1} = \gamma(\rho)$. Since ρ is irreducible, M_γ is unique up to scaling by Schur's lemma. We let $\overline{M}_\gamma$ denote the image of M_γ in $\mathrm{PGL}_r(\mathbb{C})$; $\overline{M}_\gamma$ is well defined by the previous sentence. Define $\widetilde{\rho}$ by

$$\widetilde{\rho} : \widetilde{\Gamma} \to \mathrm{PGL}_r(\mathbb{C}),$$
$$\gamma \mapsto \overline{M}_\gamma.$$

Uniqueness of $\overline{M}_\gamma \in \mathrm{PGL}_r(\mathbb{C})$ implies that $\widetilde{\rho}$ is a representation, as $\overline{M}_\gamma \overline{M}_{\gamma'} = \overline{M}_{\gamma\gamma'}$.

Commutativity of (2.2) follows from the definition of the inclusion $\iota : \pi_1(\Sigma_{g,n}, x) \hookrightarrow \mathrm{PMod}_{g,n+1} \subset \mathrm{Aut}(\pi_1(\Sigma_{g,n}, x))$ as the point-pushing subgroup, realizing the set of inner automorphisms. Indeed, for $\eta \in \pi_1(\Sigma_{g,n}, x)$, ι sends η to the automorphism of $\pi_1(\Sigma_{g,n}, x)$ given by $\beta \mapsto \eta\beta\eta^{-1}$. Hence we may take $M_{\iota(\eta)} = \rho(\eta)$, implying (2.2) commutes.

Finally, uniqueness of $\widetilde{\rho}$ follows from commutativity of (2.2) and uniqueness of $\overline{M}_\gamma$. Indeed, for any $\gamma \in \widetilde{\Gamma}$ and for all $\eta \in \pi_1(\Sigma_{g,n}, x)$, we must have

$$\widetilde{\rho}(\gamma\eta\gamma^{-1}) = \widetilde{\rho}(\gamma)\widetilde{\rho}(\eta)\widetilde{\rho}(\gamma)^{-1},$$

and hence $\widetilde{\rho}(\gamma)$ must equal $\overline{M}_\gamma$. □

See, e.g., the proof of [Sim92, Th. 4] for a similar argument.

We will also need a geometric variant of Lemma 2.2.2. We give the geometric rephrasing of Lemma 2.2.2, implicitly using Lemma 2.0.1. Below $\mathscr{M}$ is the finite étale cover of $\mathscr{M}_{g,n}$ corresponding to a finite-index subgroup of $\Gamma \subset \mathrm{PMod}_{g,n}$, chosen so that $\mathscr{M}$ is a scheme (as opposed to a Deligne-Mumford stack).

LEMMA 2.2.3. *Let* $\rho : \pi_1(\Sigma_{g,n}, x) \to \mathrm{GL}_r(\mathbb{C})$ *be an irreducible MCG-finite representation. There exists a scheme* $\mathscr{M}$ *with a finite étale map* $\mathscr{M} \to \mathscr{M}_{g,n}$, *with associated family of* n*-times punctured curves* $\pi^\circ : \mathscr{C}^\circ \to \mathscr{M}$, *a point* $c \in \mathscr{C}^\circ$ *with* $m := \pi^\circ(c)$, *and a representation* $\widetilde{\rho} : \pi_1(\mathscr{C}^\circ, c) \to \mathrm{PGL}_r(\mathbb{C})$ *so that the following holds. Upon identifying* $\pi_1(\Sigma_{g,n}, x) \simeq \pi_1(\mathscr{C}^\circ_m, c)$, *the diagram*

$$\begin{array}{ccc} \pi_1(\mathscr{C}^\circ_m, c) & \xrightarrow{\rho} & \mathrm{GL}_r(\mathbb{C}) \\ \downarrow & & \downarrow \\ \pi_1(\mathscr{C}^\circ, c) & \xrightarrow{\widetilde{\rho}} & \mathrm{PGL}_r(\mathbb{C}) \end{array} \tag{2.3}$$

commutes.

2.3. *Lifting Projective representations.* The main goal of this section is to prove Proposition 2.3.4, which says that, étale locally on the base $\mathscr{M}$, we can lift the representation $\widetilde{\rho}$ into $\mathrm{PGL}_r(\mathbb{C})$ constructed in Lemma 2.2.3 to a representation ρ' into $\mathrm{GL}_r(\mathbb{C})$.

Because the obstruction to lifting is related to $\mathbb{C}^\times = \ker(\mathrm{GL}_r(\mathbb{C}) \to \mathrm{PGL}_r(\mathbb{C}))$, we will use the following classification when $r = 1$.

PROPOSITION 2.3.1 ([BGMW22, Lemma 3.2]). *Let $g \geq 1, n \geq 0$. Then any MCG-finite representation*

$$\rho : \pi_1(\Sigma_{g,n}) \to \mathbb{C}^\times$$

has finite image.

For the reader's benefit, we recall the idea of the proof of Proposition 2.3.1. The idea is to show that if any standard generator of $\pi_1(\Sigma_{g,n})$ corresponding to a loop γ in $\Sigma_{g,n}$ has infinite order under ρ, then acting on ρ by powers of the Dehn twist about γ gives an infinite collection of distinct representations.

In order to lift $\mathrm{PGL}_r(\mathbb{C})$-reps to $\mathrm{GL}_r(\mathbb{C})$-reps, we will need to know that the image of ρ has finite intersection with the center of $\mathrm{GL}_r(\mathbb{C})$.

LEMMA 2.3.2. *Let $g \geq 1$. Suppose $\rho : \pi_1(\Sigma_{g,n}) \to \mathrm{GL}_r(\mathbb{C})$ is an MCG-finite representation. Then ρ has finite determinant. Moreover, ρ factors through a subgroup G with $\mathrm{SL}_r(\mathbb{C}) \subset G \subset \mathrm{GL}_r(\mathbb{C})$ and $\mu := G \cap \ker(\mathrm{GL}_r(\mathbb{C}) \to \mathrm{PGL}_r(\mathbb{C}))$ a finite group.*

Proof. By Proposition 2.3.1, the composition $\pi_1(\Sigma_{g,n}) \xrightarrow{\rho} \mathrm{GL}_r(\mathbb{C}) \xrightarrow{\det} \mathbb{C}^\times$ has finite image H. We can take $G := \det^{-1}(H) \subset \mathrm{GL}_r(\mathbb{C})$, which indeed has finite intersection with the center of $\mathrm{GL}_r(\mathbb{C})$, and hence finite intersection with $\ker(\mathrm{GL}_r(\mathbb{C}) \to \mathrm{PGL}_r(\mathbb{C}))$. □

In order to lift representations from $\mathrm{PGL}_r(\mathbb{C})$ to $\mathrm{GL}_r(\mathbb{C})$, we will need to kill certain cohomological obstructions using dominant étale maps.

LEMMA 2.3.3. *Let $i > 0$ and $\mathscr{M}$ be a complex variety. Suppose μ is a finite abelian group and $\sigma \in H^i(\pi_1(\mathscr{M}, m), \mu)$ a cohomology class. Then there is a dominant étale map $\mathscr{M}' \to \mathscr{M}$ so that $\sigma|_{\mathscr{M}'} = 0 \in H^i(\pi_1(\mathscr{M}'), \mu)$.*

Proof. First, we claim that for any class $\alpha \in H^i(\mathscr{M}, \mu)$, there is a dominant étale map $\mathscr{N} \to \mathscr{M}$ so that $\alpha|_{\mathscr{N}} = 0$. Since μ is finite, we can use the comparison between singular and étale cohomology to reduce to showing that for any $\alpha \in H^i_{\text{ét}}(\mathscr{M}, \mu)$, there is a dominant étale map $\mathscr{N} \to \mathscr{M}$ for which $\alpha|_{\mathscr{N}} = 0$. Choose an injective resolution $I^\bullet$ of μ, and represent α by some $\bar{\alpha} \in \ker(I^i(\mathscr{M}) \to I^{i+1}(\mathscr{M}))$. Because $I^\bullet$ is exact, there exists for each $x \in \mathscr{M}$ an étale neighborhood U_x of x so that $\alpha|_{U_x}$ is in the image of $I^{i-1}(U_x) \to I^i(U_x)$. Taking x to be the generic point of $\mathscr{M}$ gives the claim.

Now, as above let α denote the image of σ under the natural map

$$H^i(\pi_1(\mathscr{M}, m), \mu) \to H^i(\mathscr{M}, \mu),$$

and take $\mathscr{N}$ as above so that $\alpha|_{\mathscr{N}} = 0$. By [SGA73, Exp. XI, 4.6] we can find a Zariski open $\mathscr{M}' \subset \mathscr{N}$ that is a $K(\pi_1(\mathscr{M}', m'), 1)$ Eilenberg-Maclane space. This implies the natural map $H^i(\pi_1(\mathscr{M}', m'), \mu) \to H^i(\mathscr{M}', \mu)$ is an isomorphism. Therefore, since $\alpha|_{\mathscr{N}} = 0$, we also have $\alpha|_{\mathscr{M}'} = 0$, and hence $\sigma|_{\pi_1(\mathscr{M}')} = 0$, as desired. □

We now combine the above results to show that the projective representation $\widetilde{\rho}$ constructed in Lemma 2.2.3 can be lifted to a linear representation.

PROPOSITION 2.3.4. *Suppose $\mathscr{C} \to \mathscr{M}$ is a versal family of n-pointed curves of genus $g \geq 1$, with associated family of punctured curves $\pi^\circ : \mathscr{C}^\circ \to \mathscr{M}$. Let $c \in \mathscr{C}^\circ$ be a point and $m = \pi^\circ(c)$. Suppose we have representations $\rho : \pi_1(\mathscr{C}^\circ_m, c) \to \mathrm{GL}_r(\mathbb{C})$ and $\widetilde{\rho} : \pi_1(\mathscr{C}^\circ, c) \to \mathrm{PGL}_r(\mathbb{C})$ so that the diagram*

$$\begin{array}{ccc} \pi_1(\mathscr{C}^\circ_m, c) & \xrightarrow{\rho} & \mathrm{GL}_r(\mathbb{C}) \\ \downarrow & & \downarrow \\ \pi_1(\mathscr{C}^\circ, c) & \xrightarrow{\widetilde{\rho}} & \mathrm{PGL}_r(\mathbb{C}) \end{array} \tag{2.4}$$

commutes. Moreover, assume ρ is MCG-finite. Then, there exist

(1) *a dominant étale map $\mathscr{M}' \to \mathscr{M}$,*
(2) *corresponding relative curve $\mathscr{C}' := \mathscr{M}' \times_{\mathscr{M}} \mathscr{C}$ with associated family of punctured curves $\mathscr{C}'^\circ$, and*
(3) *a representation $\rho' : \pi_1(\mathscr{C}'^\circ, c') \to \mathrm{GL}_r(\mathbb{C})$ for some basepoint $c' \in \mathscr{C}'^\circ$ lying over c*

so that, upon choosing some m' over m with $\mathscr{C}_{m'} \simeq \mathscr{C}_m$,

$$\begin{array}{ccccc} & & \overset{\rho}{\frown} & & \\ \pi_1(\mathscr{C}^\circ_{m'}, c') & \longrightarrow & \pi_1(\mathscr{C}'^\circ, c') & \xrightarrow{\rho'} & \mathrm{GL}_r(\mathbb{C}) \\ & & \downarrow & & \downarrow \\ & & \pi_1(\mathscr{C}^\circ, c) & \xrightarrow{\widetilde{\rho}} & \mathrm{PGL}_r(\mathbb{C}) \end{array} \tag{2.5}$$

commutes. Moreover, $\operatorname{im}\rho'$ is contained in a subgroup G with $\mathrm{SL}_r(\mathbb{C}) \subset G \subset \mathrm{GL}_r(\mathbb{C})$, such that $\mathrm{SL}_r(\mathbb{C}) \subset G$ has finite index.

Proof. Taking G and μ as in Lemma 2.3.2, there is a commuting diagram

$$\begin{array}{ccc} H^1(\pi_1(\mathscr{C}^\circ, c), \mu) & \xrightarrow{\lambda} & H^1(\pi_1(\mathscr{C}^\circ_m, c), \mu) \\ \downarrow{\scriptstyle \nu} & & \downarrow{\scriptstyle \xi} \\ \mathrm{Hom}(\pi_1(\mathscr{C}^\circ, c), G) & \xrightarrow{\alpha} & \mathrm{Hom}(\pi_1(\mathscr{C}^\circ_m, c), G) \\ \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \gamma} \\ \mathrm{Hom}(\pi_1(\mathscr{C}^\circ, c), \mathrm{PGL}_r(\mathbb{C})) & \xrightarrow{\delta} & \mathrm{Hom}(\pi_1(\mathscr{C}^\circ_m, c), \mathrm{PGL}_r(\mathbb{C})) \\ \downarrow{\scriptstyle \varepsilon} & & \downarrow{\scriptstyle \zeta} \\ H^2(\pi_1(\mathscr{C}^\circ, c), \mu) & \xrightarrow{\eta} & H^2(\pi_1(\mathscr{C}^\circ_m, c), \mu), \end{array} \tag{2.6}$$

where the columns are exact sequences of sets. We start with a representation $\widetilde{\rho} \in \mathrm{Hom}(\pi_1(\mathscr{C}^\circ, c), \mathrm{PGL}_r(\mathbb{C}))$ and $\rho \in \mathrm{Hom}(\pi_1(\mathscr{C}^\circ_m, c), G)$. The commutativity of (2.4) can be equivalently expressed as the statement that $\delta(\widetilde{\rho}) = \gamma(\rho)$. We seek to construct ρ' such that (2.5) commutes, meaning that, after replacing $\mathscr{M}$ with a dominant étale $\mathscr{M}'$ and $\mathscr{C}$ with $\mathscr{C}'$, there is some ρ' with $\beta(\rho') = \widetilde{\rho}$ and $\alpha(\rho') = \rho$.

First, we argue that we can pass to some $\mathscr{M}'$ so that $\widetilde{\rho}$ lies in the image of β. It is equivalent to arrange that $\varepsilon(\widetilde{\rho}) = 0$. As a first step, we claim $\eta(\varepsilon(\widetilde{\rho})) = 0$. This holds because $\eta(\varepsilon(\widetilde{\rho})) = \zeta(\delta(\widetilde{\rho}))$, and $\delta(\widetilde{\rho}) = \gamma(\rho)$ lies in the image of γ. We therefore have that $\varepsilon(\widetilde{\rho}) \in \ker \eta$. The Hochschild-Serre spectral sequence associated to the short exact sequence

$$1 \to \pi_1(\mathscr{C}^\circ_m) \to \pi_1(\mathscr{C}^\circ) \to \pi_1(\mathscr{M}) \to 1$$

of Lemma 2.1.5 yields a short exact sequence

$$0 \to H^{2,0} \to \ker \eta \xrightarrow{\iota} H^{1,1} \to 0,$$

where $H^{i,j}$ is a subquotient of $H^i(\pi_1(\mathscr{M}, m), H^j(\pi_1(\mathscr{C}^\circ_m, c), \mu))$.

By passing to a finite étale cover $\mathscr{M}_1$ of $\mathscr{M}$, we can assume $\pi_1(\mathscr{M}, m)$ acts trivially on the finite group $H^1(\pi_1(\mathscr{C}^\circ_m, c), \mu)$. Now as

$$H^i(\pi_1(\mathscr{M}, m), H^j(\pi_1(\mathscr{C}^\circ_m, c), \mu))$$

is finite, by Lemma 2.3.3 we may pass to a further dominant étale scheme over $\mathscr{M}$ killing every element of $H^{i,j}$, $(i, j) = (2, 0)$ or $(1, 1)$, and hence every element of $\ker \eta$. Thus after passing to any component $\mathscr{M}_2$ of this dominant étale scheme over $\mathscr{M}_2$, we have $\epsilon(\widetilde{\rho}) = 0$. Let $\mathscr{C}^\circ_2$ be the pullback of $\mathscr{C}^\circ$ to $\mathscr{M}_2$.

Let $\overline{\rho} \in H^1(\pi_1(\mathscr{C}^\circ_2, c), G)$ be an element with $\beta(\overline{\rho}) = \widetilde{\rho}$; such a $\overline{\rho}$ exists as we have arranged $\epsilon(\widetilde{\rho}) = 0$. We wish to arrange $\alpha(\overline{\rho}) = \rho$. This may not be the case, but it is enough to show that after passing to a further dominant étale $\mathscr{M}'$, we can modify $\overline{\rho}$ by an element of $\mathrm{im}\, \nu$ so that this does hold. More precisely, note that $\alpha(\overline{\rho})$ differs from ρ by an element of $\sigma \in H^1(\pi_1(\mathscr{C}^\circ_m, c), \mu)$, as by construction $\gamma(\alpha(\overline{\rho})) = \gamma(\rho)$. It suffices to show that, after passing to

a dominant étale scheme over $\mathscr{M}_2$, σ is in the image of λ. After passing to a finite étale cover of $\mathscr{M}_2$, we may assume that $H^1(\pi_1(\mathscr{C}_m^\circ, c), \mu)$ is fixed by $\pi_1(\mathscr{M}, m)$. The Hochschild-Serre spectral sequence gives a map

$$H^0(\pi_1(\mathscr{M}, m), H^1(\pi_1(\mathscr{C}_m^\circ, c), \mu)) \xrightarrow{\chi} H^2(\pi_1(\mathscr{M}, m), H^0(\pi_1(\mathscr{C}_m^\circ, c), \mu))$$

such that $\chi(\sigma) = 0$ if an only if $\sigma \in \operatorname{im} \lambda$. Since $H^0(\pi_1(\mathscr{C}_m^\circ, c), \mu) = \mu$ is finite, we can apply Lemma 2.3.3 so as to assume, after replacing $\mathscr{M}$ by a dominant étale $\mathscr{M}'$, that $\chi(\sigma) = 0$. This implies that $\sigma \in \operatorname{im} \lambda$, as desired. □

We can now give a partial converse to Proposition 2.1.3.

COROLLARY 2.3.5. *Let $\rho : \pi_1(\Sigma_{g,n}) \to \mathrm{GL}_r(\mathbb{C})$ be an irreducible MCG-finite representation, with $g \geq 1$. There are a punctured versal family of curves $\mathscr{C}^\circ \to \mathscr{M}$ and a representation ρ' of $\pi_1(\mathscr{C}^\circ)$ whose determinant has finite order, with the following property: For C° a fiber of π°, there is an identification $\pi_1(\Sigma_{g,n}) \simeq \pi_1(C^\circ)$ such that, under this identification, $\rho'|_{\pi_1(C^\circ)} = \rho$.*

Proof. This follows by combining Lemma 2.2.3 and Proposition 2.3.4. □

Remark 2.3.6. Combining Corollary 2.3.5 with Proposition 2.1.3 gives a proof of [CH21, Th. A] in the case that ρ is semisimple and the flat vector bundle (E, ∇_0) of [CH21, Th. A] is taken to be the Deligne canonical extension of $(\rho|_{C^\circ} \otimes \mathscr{O}, \mathrm{id} \otimes d)$ to C.

2.4. *Representations that are unitary on fibers.* The main goal of this section is to verify Lemma 2.4.2, which allows us to write local systems on a family of punctured curves $\mathscr{C}^\circ \to \mathscr{M}$ in terms of unitary local systems and local systems pulled back from $\mathscr{M}$, after passing to a dominant étale map.

The following criterion for unitarity and finiteness will be useful when analyzing the representations produced by Lemma 2.2.2 and Proposition 2.3.4.

LEMMA 2.4.1. *Let G be a group and $H \trianglelefteq G$ a normal subgroup. Let*

$$\rho : G \to \mathrm{GL}_r(\mathbb{C})$$

be a representation such that $\det \rho$ has finite order, and suppose $\rho|_H$ is irreducible.

(1) *If $\rho|_H$ has finite image, then ρ has finite image.*
(2) *If $\rho|_H$ is unitary, then ρ is unitary.*

Proof. We first prove (1). As $\det \rho$ has finite order, it suffices to show that the projectivization

$$\mathbb{P}\rho : G \xrightarrow{\rho} \mathrm{GL}_r(\mathbb{C}) \to \mathrm{PGL}_r(\mathbb{C})$$

has finite image. Let $t = \#\operatorname{im}(\rho|_H)$. For each $g \in G$, $\mathbb{P}\rho(g)$ is the unique (by Schur's lemma) element of $\mathrm{PGL}_r(\mathbb{C})$ such that

$$\mathbb{P}\rho(g)\mathbb{P}\rho(h)\mathbb{P}\rho(g)^{-1} = \mathbb{P}\rho(ghg^{-1})$$

for all $h \in H$. Since $\rho(g)$ acts by conjugation on the order t set $\mathrm{im}(\rho|_H)$, its action has order dividing $t!$, so $\rho(h) = \rho(g^{t!})\rho(h)\rho(g^{-t!})$. Hence we have $\mathbb{P}\rho(g^{t!}) = \mathrm{id}$ by uniqueness. Thus the image of $\mathbb{P}\rho$ has exponent dividing $t!$. But a linear group with finite exponent is finite [Bur05].

We now turn to the unitary case and verify (2). Let h be a positive-definite Hermitian inner product preserved by $\rho|_H$. We first check that $\mathrm{im}\,\rho$ lies in the general unitary group; i.e., it preserves h up to scaling. Let V be the underlying vector space of our representation ρ. The invariant inner product $h : V \times V \to \mathbb{C}$ corresponds to a $\mathbb{C}$-linear isomorphism of H-representations $\phi : V \to \overline{V}^{\vee}$. The linear map ϕ, and hence h, is unique up to scaling by Schur's lemma. For any $g \in G$,

$$h^g : (v, w) \mapsto h(\rho(g)v, \rho(g)w)$$

is another H-invariant Hermitian form, and hence $h^g = c \cdot h$ for some $c \in \mathbb{C}^\times$. Equivalently, $\mathrm{im}(\rho) \subset GU(h)$. Since ρ has finite determinant, we moreover obtain $\mathrm{im}(\rho) \subset U(h)$. □

We conclude the section by giving a convenient description of local systems $\mathbb{V}$ on a family of punctured curves $\pi^\circ : \mathscr{C}^\circ \to \mathscr{M}$, with unitary fibral monodromy. It may be useful for the reader to consider the case $A = \mathbb{C}$.

As in Notation 1.10.1, let $\pi : \mathscr{C} \to \mathscr{M}$ be a versal family of n-pointed curves, and let $\mathscr{C}^\circ \to \mathscr{M}$ be the associated family of punctured curves. Let $m \in \mathscr{M}$ be a point.

Lemma 2.4.2. *Let A be an Artin local $\mathbb{C}$-algebra with residue field $\mathbb{C}$. Suppose $\mathbb{V}$ is a local system of free A-modules on $\mathscr{C}^\circ$ such that $\mathbb{V}|_{\mathscr{C}^\circ_m}$ is a constant deformation of a unitary local system; that is, there exists a unitary $\mathbb{C}$-local system $\mathbb{V}_0$ on $\mathscr{C}^\circ_m$ such that $\mathbb{V}|_{\mathscr{C}^\circ_m} \simeq \mathbb{V}_0 \otimes_{\mathbb{C}} A$.*

There is a dominant étale map $\mathscr{M}' \to \mathscr{M}$, with $\mathscr{C}' = \mathscr{M}' \times_{\mathscr{M}} \mathscr{C}$ and $\pi'^\circ : \mathscr{C}'^\circ \to \mathscr{M}'$ the associated family of punctured curves, over which

$$\mathbb{V}|_{\mathscr{C}'^\circ} \simeq \oplus_{i=1}^s \mathbb{U}_i \otimes (\pi'^\circ)^* \mathbb{W}_i,$$

where the $\mathbb{W}_i$ are locally constant sheaves of free A-modules on $\mathscr{M}'$ and the $\mathbb{U}_i$ are unitary local systems on $\mathscr{C}'^\circ$. Moreover, for C'° a fiber of $\mathscr{C}'^\circ \to \mathscr{M}'$, each $\mathbb{U}_i|_{C'^\circ}$ is irreducible, the $\mathbb{U}_i|_{C'^\circ}$ are pairwise non-isomorphic, and $\mathbb{W}_i \simeq \pi'^\circ_* \mathrm{Hom}(\mathbb{U}_i, \mathbb{V}|_{\mathscr{C}'^\circ})$.

Proof. Since $\mathbb{V}_0$ is unitary, we can express it as as a sum of irreducible unitary local systems, $\mathbb{V}_0 \simeq \oplus_{i=1}^s \mathbb{S}_i^{\oplus n_i}$, with the $\mathbb{S}_i$ pairwise non-isomorphic. Let $\rho : \pi_1(\mathscr{C}^\circ_m) \to \mathrm{GL}_r(\mathbb{C})$ be the monodromy representation associated to $\mathbb{V}_0$, and let ρ_i be the (irreducible) representation associated to $\mathbb{S}_i$. By Proposition 2.1.3, ρ is MCG-finite. Hence each ρ_i is MCG-finite by Proposition 2.1.1.

By repeatedly applying Corollary 2.3.5, there are a dominant étale map $\mathscr{M}' \to \mathscr{M}$ and representations $\rho_i' : \pi_1(\mathscr{C}'^{\circ}) \to \mathrm{GL}_r(\mathbb{C})$ with finite determinant so that for any $m' \in \mathscr{M}'$, ρ_i' restricts to a representation $\pi_1(\mathscr{C}'^{\circ}_{m'}) \to \mathrm{GL}_r(\mathbb{C})$ identified with ρ_i. Let $\mathbb{U}_i$ denote the local system on $\mathscr{C}'^{\circ}$ corresponding to ρ_i'. Each $\mathbb{U}_i$ is unitary by Lemma 2.4.1. The $\mathbb{U}_i$ are irreducible and pairwise non-isomorphic because the same holds for the ρ_i.

Let $\mathbb{W}_i = \pi_*'^{\circ} \mathrm{Hom}(\mathbb{U}_i, \mathbb{V}|_{\mathscr{C}'^{\circ}})$, as in the statement. The $\mathbb{W}_i$ are locally constant sheaves of free A-modules by the hypothesis that $\mathbb{V}|_{\mathscr{C}^{\circ}_m} \simeq \mathbb{V}_0 \otimes_{\mathbb{C}} A$. There is a natural evaluation map

$$\psi : \oplus_{i=1}^{s} (\pi'^{\circ})^* \mathbb{W}_i \otimes \mathbb{U}_i \to \mathbb{V}|_{\mathscr{C}'^{\circ}}.$$

It only remains to show that ψ is an isomorphism. We do so after restriction to any fiber of $\mathscr{C}'^{\circ} \to \mathscr{M}'$, where ψ is identified with the isomorphism

$$\oplus_{i=1}^{s} \mathrm{Hom}(\rho_i, \rho \otimes A) \otimes \rho_i \to \rho \otimes A,$$
$$\sum_{i=1}^{s} \alpha_1 \otimes v_i \mapsto \sum_{i=1}^{s} \alpha_i(v_i). \qquad \square$$

3. Deformation-theoretic preliminaries

In this section we introduce some deformation theory of group representations. In Section 8, we will use the theory developed here in conjunction with the vanishing results proven in Section 6 to show that MCG-finite representations have certain rigidity properties.

3.1. *Deformations of representations.* We begin by recalling a standard description of deformations of representations in terms of cohomological data.

Recall Notation 1.10.2: for R a ring and a representation $\rho : G \to \mathrm{GL}_r(R)$, we use $\mathrm{ad}(\rho)$ to denote the adjoint G-representation obtained by composing ρ with $\mathrm{GL}_r(R) \to \mathrm{GL}(\mathfrak{pgl}_r(R))$.

Fix a group G and a representation $\rho_0 : G \to \mathrm{GL}_r(\mathbb{C})$. Let Art be the category of local Artin $\mathbb{C}$-algebras with residue field $\mathbb{C}$. If $A \in \mathrm{Art}$, with maximal ideal $\mathfrak{m}_A$, we say that a representation $\rho_A : G \to \mathrm{GL}_r(A)$ is a deformation of ρ_0 if $\rho_A = \rho_0 \bmod \mathfrak{m}_A$. The *constant* deformation is the deformation obtained via the composition

$$G \xrightarrow{\rho_0} GL_r(\mathbb{C}) \to \mathrm{GL}_r(A).$$

We say a representation $\rho : G \to \mathrm{GL}_r(A)$ is *conjugate to a constant representation* if there exists a matrix $M \in \mathrm{GL}_r(A)$ such that $M\rho M^{-1}$ factors through $\mathrm{GL}_r(\mathbb{C})$. Say a deformation ρ_A of ρ_0 has *constant determinant* if $\det \rho_A = \det \rho_0$, regarded as a map $G \to A^{\times}$ via the natural inclusion $\mathbb{C}^{\times} \hookrightarrow A^{\times}$.

We now define deformations of representations over square-zero extensions in Art. Given a square-zero extension

$$0 \to I \to B \to A \to 0$$

with $A, B \in \text{Art}$, and a representation $\rho_A : G \to \text{GL}_r(A)$, a deformation of ρ_A is a representation $\rho_B : G \to \text{GL}_r(B)$ with $\rho_B = \rho_A \bmod I$. Two such deformations ρ_B, ρ'_B are *equivalent* if there exists a matrix $M \in \text{GL}_r(B)$ with $\rho_B = M\rho'_B M^{-1}$, such that $M = \text{id} \bmod I$.

LEMMA 3.1.1. *Fix a group G and a representation $\rho_0 : G \to \text{GL}_r(\mathbb{C})$. Consider*

$$0 \to \epsilon A \to A[\epsilon]/\epsilon^2 \to A \to 0$$

with $A \in \text{Art}$, and let $\rho_A : G \to \text{GL}_r(A)$ be a deformation of ρ_0 over A with constant determinant. The set of equivalence classes of deformations of ρ_A to an $A[\epsilon]/\epsilon^2$-representation with constant determinant is naturally in bijection with $H^1(G, \text{ad}(\rho_A))$. Moreover, this description is functorial in G.

Proof. See [Maz97, Ch. V, §21, Prop. 1]. □

LEMMA 3.1.2. *Let*

$$1 \to N \to G \to Q \to 1$$

be a short exact sequence of groups (i.e., N is normal in G and $Q = G/N$). Let $A \in \text{Art}$ and

$$\rho : G \to \text{GL}_r(A)$$

be a representation, and suppose that $H^0(Q, H^1(N, \text{ad}(\rho)|_N)) = 0$. If ρ_ϵ is any deformation of ρ to $A[\epsilon]/\epsilon^2$ with constant determinant, then $\rho_\epsilon|_N$ is equivalent to $\rho \otimes_A A[\epsilon]/\epsilon^2|_N$.

Proof. By Lemma 3.1.1, the set of equivalence classes of deformations of ρ to $A[\epsilon]/\epsilon^2$ is naturally in bijection with $H^1(G, \text{ad}(\rho))$, and the set of equivalence classes of deformations of $\rho|_N$ is naturally in bijection with $H^1(N, \text{ad}(\rho)|_N)$. By functoriality, the restriction map is given by the natural map

$$H^1(G, \text{ad}(\rho)) \to H^1(N, \text{ad}(\rho)|_N),$$

which, by the five-term exact sequence from the Hochschild-Serre spectral sequence, factors through $H^0(Q, H^1(N, \text{ad}(\rho)|_N)) = 0$. Hence $\rho_\epsilon|_N$ is equivalent to $\rho \otimes_A A[\epsilon]/\epsilon^2|_N$ as desired. □

3.2. *A criterion for constancy.* The next proposition gives a criterion for a deformation of a group representation to induce the trivial deformation on a normal subgroup. This will be used in the proof of the semisimple version of our main theorem, Theorem 8.5.3.

We define

$$B_n = \mathbb{C}[t]/t^{n+1},$$
$$R_n = B_{n-1}[\epsilon]/\epsilon^2 = \mathbb{C}[t,\epsilon]/(t^n,\epsilon^2).$$

Let $d_n : B_n \to R_n$ be the map given by

$$d_n : f(t) \mapsto f(t) + \epsilon f'(t),$$

where $f'(t)$ is the derivative of $f(t)$. Note that d_n is injective.

Below we say that an element of $\mathrm{GL}_r(B_n)$ is constant if it lies in $\mathrm{GL}_r(\mathbb{C}) \subset \mathrm{GL}_r(B_n)$. We say that an element of $\mathrm{GL}_r(\mathbb{C}[[t]])$ is constant mod t^{n+1} if its image in $\mathrm{GL}_r(B_n)$ is constant. Likewise, a representation ρ into $\mathrm{GL}_r(B_n)$ is constant if it factors through $\mathrm{GL}_r(\mathbb{C})$; it is conjugate to a constant representation if there exists $M \in \mathrm{GL}_r(B_n)$ such that $M\rho M^{-1}$ is constant. Similarly, a representation into $\mathrm{GL}_r(\mathbb{C}[[t]])$ is constant mod t^{n+1} if its image in $\mathrm{GL}_r(B_n)$ is constant.

PROPOSITION 3.2.1. *Let*

$$1 \to N \to G \to Q \to 1$$

be a short exact sequence of groups. Let

$$\rho_0 : G \to \mathrm{GL}_r(\mathbb{C})$$

be a representation, and suppose that for all n and all deformations $\gamma : G \to \mathrm{GL}_r(B_n)$ of ρ_0 with $\gamma|_N$ constant, we have $H^0(Q, H^1(N, \mathrm{ad}(\gamma)|_N)) = 0$. If

$$\rho_\infty : G \to \mathrm{GL}_r(\mathbb{C}[[t]])$$

is any representation with $\rho_\infty = \rho_0 \bmod t$ and constant determinant, then $\rho_\infty|_N$ is conjugate to a constant representation. That is, there exists $M_\infty \in \mathrm{GL}_r(\mathbb{C}[[t]])$ such that $M_\infty \rho_\infty|_N M_\infty^{-1}$ factors through $\mathrm{GL}_r(\mathbb{C})$.

Proof. We wish to construct $M_\infty \in \mathrm{GL}_r(\mathbb{C}[[t]])$ such that $M_\infty \rho_\infty|_N M_\infty^{-1}$ is constant, i.e., factors through $\mathrm{GL}_r(\mathbb{C})$. We do so by successive approximation. For all $m \geq 0$, set $\rho_m : G \to \mathrm{GL}_r(B_m)$ to be $\rho_\infty \bmod t^{m+1}$.

Set $S_1 = \mathrm{id}$. Suppose we have found $S_n \in \mathrm{GL}_r(\mathbb{C}[[t]])$ such that $S_n \rho_\infty|_N S_n^{-1}$ is constant modulo t^n. We claim it suffices to construct an element $M_n \in \mathrm{GL}_r(B_n), M_n = \mathrm{id} \bmod t^n$, such that $M_n S_n \rho_n|_N S_n^{-1} M_n^{-1}$ is constant. Indeed, let $M_1 = \mathrm{id}$ and for $n \geq 1$, let $\widetilde{M_n}$ be an arbitrary lift of M_n to $\mathrm{GL}_r(\mathbb{C}[[t]])$. Then, the representation $\widetilde{M_n} S_n \rho_\infty|_N S_n^{-1} \widetilde{M_n}^{-1}$ is constant mod t^{n+1}. Set $S_{n+1} = \widetilde{M_n} S_n$, so that $S_{n+1} = \widetilde{M_n} \cdot \widetilde{M_{n-1}} \cdots \widetilde{M_1}$ by induction. Then, setting

$$M_\infty := \lim_{n\to\infty} \widetilde{M_n} \cdot \widetilde{M_{n-1}} \cdots \widetilde{M_1} = \lim_{n\to\infty} S_n,$$

we claim that the limit converges as $\widetilde{M_n} \to \mathrm{id}$, and $M_\infty \rho_\infty|_N M_\infty^{-1}$ is constant. This claim holds because $M_\infty = S_n \bmod t^n$ for all n, so we have

$$M_\infty \rho_\infty|_N M_\infty^{-1} \bmod t^n = S_n \rho_\infty|_N S_n^{-1} \bmod t^n,$$

which is constant for all n by construction.

We now construct the desired matrix $M_n \in \mathrm{GL}_r(B_n)$. Replacing ρ_∞ by $S_n \rho_\infty S_n^{-1}$, we may assume $\rho_\infty|_N$ is constant modulo t^n, i.e., $\rho_{n-1}|_N$ is constant. We wish to find $M_n \in \mathrm{GL}_r(B_n)$ such that

(1) $M_n = \mathrm{id} \bmod t^n$, and
(2) $M_n \rho_n|_N M_n^{-1}$ is constant.

Given $g \in N$, we let $\rho_n(g)' \in \mathrm{Mat}_{r\times r}(B_{n-1})$ be the matrix obtained by differentiating the entries of $\rho_n(g)$. The representation $d_n \circ \rho_n|_N$ is given by

$$g \mapsto \rho_0(g) + \epsilon \rho_n(g)',$$

as $\rho_n|_N$ is constant mod t^n.

Note that $d_n \circ \rho_n|_N : N \to \mathrm{GL}_r(R_n)$ is constant mod ϵ (as it is equal to $\rho_{n-1}|_N$ mod ϵ). By hypothesis,

$$H^0(Q, H^1(N, \mathrm{ad}(\rho_{n-1})|_N)) = 0,$$

and so by Lemma 3.1.2, $d_n \circ \rho_n|_N$ is conjugate to the constant representation

$$\rho_{n-1}|_N \otimes_{B_{n-1}} R_n = \rho_0|_N \otimes_{\mathbb{C}} R_n$$

by some matrix

$$\mathrm{id} + \epsilon \sum_{i=0}^{n-1} C_i t^i \in \mathrm{GL}_r(R_n),$$

where the $C_i \in \mathrm{Mat}_{r\times r}(\mathbb{C})$. The idea of the remainder of the proof is to view the matrix above as a vector field that we may flow along to make $\rho_n|_N$ constant; we find the desired conjugating matrix M_n by "integrating" this vector field.

We compute that for $g \in N$,

$$\begin{aligned}
(\rho_0 \otimes_{\mathbb{C}} R_n)(g) &= (\rho_{n-1} \otimes_{B_{n-1}} R_n)(g) \\
&= \left(\mathrm{id} + \epsilon \sum_{i=0}^{n-1} C_i t^i\right) (d_n \circ \rho_n(g)) \left(\mathrm{id} - \epsilon \sum_{i=0}^{n-1} C_i t^i\right) \\
&= \left(\mathrm{id} + \epsilon \sum_{i=0}^{n-1} C_i t^i\right) (\rho_0(g) + \epsilon \rho_n(g)') \left(\mathrm{id} - \epsilon \sum_{i=0}^{n-1} C_i t^i\right) \\
&= \rho_0(g) + \epsilon \rho_n(g)' + \epsilon \sum_{i=0}^{n-1} [C_i, \rho_0(g)] t^i.
\end{aligned}$$

As $\rho(g)$ is constant mod t^n, $\rho(g)'$ is 0 mod t^{n-1}. Hence equating coefficients, we find

$$[C_i, \rho_0(g)] = 0 \text{ for } i < n-1$$

and

$$\rho_n(g)' = [\rho_0(g), C_{n-1}]t^{n-1}.$$

Now set

$$M_n = \mathrm{id} + \frac{C_{n-1}}{n}t^n \in \mathrm{GL}_r(B_n).$$

We claim $M_n\rho_n|_N M_n^{-1}$is constant. Indeed,

$$\begin{aligned} M_n\rho_n(g)M_n^{-1} &= \left(\mathrm{id} + \frac{C_{n-1}}{n}t^n\right)\rho_n(g)\left(\mathrm{id} - \frac{C_{n-1}}{n}t^n\right) \\ &= \rho_n(g) + \frac{t}{n}[t^{n-1}C_{n-1}, \rho_n(g)] \\ &= \rho_n(g) + \frac{t}{n}[t^{n-1}C_{n-1}, \rho_0(g)] \\ &= \rho_n(g) - \frac{t}{n}\rho_n(g)'. \end{aligned}$$

But the only non-vanishing coefficient of $\rho(g)'$ is the coefficient of t^{n-1}, so the above expression is constant as desired. □

4. **Hodge-theoretic preliminaries**

In this section we recall some preliminaries on polarizable complex variations of Hodge structure (complex PVHS), mixed Hodge theory, and Simpson-Mochizuki's non-abelian Hodge theory.

4.1. *Variations of mixed Hodge structure.* Our main goal in this section is Theorem 4.1.1, which describes the variation of mixed Hodge structure on the cohomology of a unitary local system on a family of punctured curves. We then describe the resulting bigrading explicitly in Lemma 4.1.2.

We now review pertinent notation. We refer the reader to [LL23, §5] for basic definitions related to complex PVHS. Let $\mathbb{V}$ be a unitary local system on a smooth quasi-projective variety. We say that $\mathbb{V}$ *has a real structure* if there exists a real orthogonal local system $\mathbb{V}_{\mathbb{R}}$ such that $\mathbb{V} \simeq \mathbb{V}_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$. Note that for any unitary local system $\mathbb{V}$, the local system $\mathbb{V} \oplus \mathbb{V}^\vee$ has a natural real structure.

Let $\pi : \mathscr{C} \to \mathscr{M}$ be a family of n-pointed curves as in Notation 1.10.1. Further assume that $\mathscr{M}$ is smooth. Let $\pi^\circ : \mathscr{C}^\circ \to \mathscr{M}$ be the associated family of punctured curves, $j : \mathscr{C}^\circ \to \mathscr{C}$ the inclusion, $\mathscr{D} = \mathscr{C} \setminus \mathscr{C}^\circ$, and $\mathbb{V}$ a unitary local system on $\mathscr{C}^\circ$. Below we will use the notion of the Deligne canonical extension of a flat vector bundle from $\mathscr{C}^\circ$ to $\mathscr{C}$, as described in [Del70, Rems. 5.5(i)] or [LL23, Def. 4.1.2].

See [PS08, Defs. 14.44, 14.45, and 14.49] for the definitions of variations of mixed Hodge structure, graded-polarizability and admissibility in the rational case; the real case is analogous and discussed in [SZ85, Def. 3.4].

The following is well known but perhaps difficult to extract from the literature.

THEOREM 4.1.1. *Suppose* $\mathbb{V}$ *is a unitary local system on* $\mathscr{C}^\circ$ *with real structure* $\mathbb{V}_\mathbb{R}$. *Then,* $R^1\pi^\circ_*\mathbb{V}_\mathbb{R}$ *underlies an admissible graded-polarizable real variation of mixed Hodge structure with weights in* $[1,2]$, *and weight filtration given by*

$$W^1R^1\pi^\circ_*\mathbb{V}_\mathbb{R} := R^1\pi_*j_*\mathbb{V}_\mathbb{R} \subset R^1\pi^\circ_*\mathbb{V}_\mathbb{R} =: W^2R^1\pi^\circ_*\mathbb{V}_\mathbb{R}.$$

Let $(\mathscr{E},\nabla)$ *denote the Deligne canonical extension of* $(V\otimes\mathscr{O}_{\mathscr{C}^\circ}, \mathrm{id}\otimes d)$ *to* $\mathscr{C}$. *The Hodge filtration on* $R^1\pi^\circ_*\mathbb{V}\otimes\mathscr{O}_\mathscr{M}$ *is given, under the canonical identification*

$$R^1\pi^\circ_*\mathbb{V}\otimes\mathscr{O}_\mathscr{M} \xrightarrow{\sim} R^1\pi_*(\mathscr{E}\xrightarrow{\nabla}\mathscr{E}\otimes\Omega^1_{\mathscr{C}/\mathscr{M}}(\log\mathscr{D})),$$

by the filtration induced by the filtration bête *on the relative de Rham complex* $\mathscr{E}\xrightarrow{\nabla}\mathscr{E}\otimes\Omega^1_{\mathscr{C}/\mathscr{M}}(\log\mathscr{D})$. *That is,*

$$F^1R^1\pi^\circ_*\mathbb{V}\otimes\mathscr{O}_\mathscr{M} := \mathrm{im}(\pi_*(\mathscr{E}\otimes\Omega^1_{\mathscr{C}/\mathscr{M}}(\log\mathscr{D}))\to R^1\pi^\circ_*\mathbb{V}\otimes\mathscr{O}_\mathscr{M}).$$

Proof. First, Griffiths transversality holds vacuously, as the Hodge filtration only has two steps. To verify the remaining properties of a variation of mixed Hodge structure we may do so pointwise. Over a point, the verification that $R^1\pi^\circ_*\mathbb{V}_\mathbb{R}$ underlies a real variation of mixed Hodge structure is carried out in [Tim87, Theorem, p. 152]. The description of the weight filtration follows from [Tim87, Lemma 6.2]. The description of the Hodge filtration follows from the definition of the Hodge filtration and the degeneration of the Hodge-de Rham spectral sequence [Tim87, Th. 7.1(a)].

It remains only to verify graded-polarizability and admissibility. This follows from Saito's theory [Sai90] (also see [Sch19]) as we now explain. As $\mathbb{V}_\mathbb{R}$ is orthogonal, it underlies a real PVHS with trivial Hodge filtration and polarization arising from the orthogonal structure. Hence, $Rj_*(\mathbb{V}_\mathbb{R})$ underlies an object of the derived category of (real) mixed Hodge modules on $\mathscr{C}$. Hence $R\pi^\circ_*\mathbb{V}_\mathbb{R} = R\pi_*Rj_*\mathbb{V}_\mathbb{R}$ underlies an object of the derived category of (real) mixed Hodge modules on $\mathscr{M}$. As π° is a fiber bundle and hence $R^1\pi^\circ_*\mathbb{V}_\mathbb{R}$ is locally constant, this implies (by [Sch19, Th. 21.1]) that $R^1\pi^\circ_*\mathbb{V}_\mathbb{R}$ in fact underlies a graded-polarizable admissible variation of mixed Hodge structure. It remains only to compare it with the structure described in the theorem. But this follows by compatibility when $\mathscr{M}$ is a point, by base change. □

Let $\mathbb{V}$ be a unitary local system on $\mathscr{C}^\circ$, and let $\mathscr{H}_\mathbb{V} := R^1\pi^\circ_*\mathbb{V}\otimes\mathscr{O}_\mathscr{M}$. If $\mathbb{V}$ has a real structure $\mathbb{V}_\mathbb{R}$, as in Theorem 4.1.1, then $R^1\pi^\circ_*\mathbb{V}_\mathbb{R}$ underlies a natural

polarized variation of $\mathbb{R}$-mixed Hodge structure. In general, we let $W^\bullet, F^\bullet$ be the weight and Hodge filtrations respectively, and we let $\overline{F}^\bullet$ be the conjugate Hodge filtration induced by the natural isomorphism

$$\mathscr{H}_{\mathbb{V}} \simeq \overline{\mathscr{H}_{\mathbb{V}^\vee}}.$$

Define

$$\mathscr{H}_{\mathbb{V}}^{1,0} := F^1 \cap W^1,$$
$$\mathscr{H}_{\mathbb{V}}^{0,1} := \overline{F^1} \cap W^1,$$

and

$$\mathscr{H}_{\mathbb{V}}^{1,1} = F^1 \cap \overline{F^1}.$$

We give a schematic picture of the structures on $\mathscr{H}_{\mathbb{V}}$ in Figure 3.

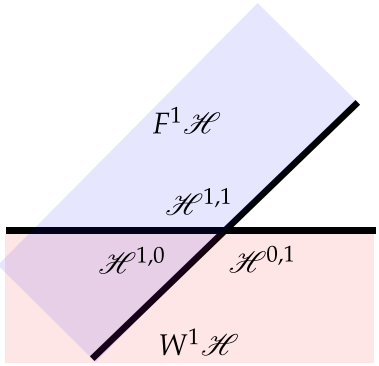

Figure 3. A schematic diagram of the mixed Hodge structure on the cohomology of a unitary local system.

LEMMA 4.1.2. *For $\mathbb{V}$ a unitary local system on $\mathscr{C}^\circ$, there is a natural direct sum decomposition*

$$\mathscr{H}_{\mathbb{V}} = \mathscr{H}_{\mathbb{V}}^{1,0} \oplus \mathscr{H}_{\mathbb{V}}^{0,1} \oplus \mathscr{H}_{\mathbb{V}}^{1,1}. \tag{4.1}$$

Further, under the natural isomorphism $\overline{\mathscr{H}_{\mathbb{V}}} = \mathscr{H}_{\mathbb{V}^\vee}$,

$$\overline{\mathscr{H}_{\mathbb{V}}^{i,j}} = \mathscr{H}_{\mathbb{V}^\vee}^{j,i}. \tag{4.2}$$

Proof. One may check both statements hold by doing so on fibers, in which case it is a straightforward verification using the definitions of the weight and Hodge filtrations. □

4.2. *Variations on the theorem of the fixed part.* Our main goal here is to prove Proposition 4.2.2, which is a variant of the theorem of the fixed part.

THEOREM 4.2.1 (Theorem of the fixed part [SZ85, 4.19], [PS08, 14.52]). *Let $\mathscr{X}$ be a smooth quasiprojective complex variety, and let $\mathbb{W}$ be an admissible*

graded-polarizable real variation of mixed Hodge structure on $\mathscr{X}$. Then there is a natural real Hodge structure on $H^0(\mathscr{X}, \mathbb{W})$ such that the natural inclusion

$$H^0(\mathscr{X}, \mathbb{W}) \otimes \underline{\mathbb{R}} \to \mathbb{W}$$

is a morphism of real variations of mixed Hodge structure.

For our main results, we will need a variant of the theorem of the fixed part for irreducible representations appearing the cohomology of a unitary local system on a family of curves. Given a complex local system $\mathbb{L}$ on a variety, we define the local system $\widetilde{\mathbb{L}}$ by

$$\widetilde{\mathbb{L}} := \begin{cases} \mathbb{L} & \text{if } \mathbb{L} \text{ admits a real structure,} \\ \mathbb{L} \oplus \overline{\mathbb{L}} & \text{otherwise.} \end{cases} \tag{4.3}$$

Note that $\widetilde{\mathbb{L}}$ admits a natural real structure. Note that if $\mathbb{L}$ is an irreducible local system underlying a complex variation of Hodge structure (necessarily unique up to reindexing), and if $\mathbb{L}$ admits a (necessarily unique) real structure, then $\mathbb{L}$ in fact underlies a real variation of Hodge structure. Indeed, the complex variation carried by $\mathbb{L}$ must be preserved (up to reindexing) by complex conjugation, as otherwise $\mathbb{L}$ would carry two distinct complex variations.

PROPOSITION 4.2.2. *Let $\mathscr{X}$ be a smooth quasiprojective complex variety, and let $\mathbb{W}$ be an admissible graded-polarizable real variation of mixed Hodge structure on $\mathscr{X}$. Let $\mathbb{L}$ be an irreducible complex local system on $\mathscr{X}$ such that* $\mathrm{Hom}(\mathbb{L}, \mathbb{W}_{\mathbb{C}}) \neq 0$. *Then, $\widetilde{\mathbb{L}}$ underlies a polarizable real variation of Hodge structure, and there exist a non-zero* constant *real mixed Hodge structure Q and a non-zero map of variations of mixed Hodge structure*

$$Q \otimes \widetilde{\mathbb{L}} \to \mathbb{W}.$$

Proof. First, let i be an integer for which there exists a non-zero map $\mathbb{L} \to \mathrm{gr}^i_W \mathbb{W}_{\mathbb{C}}$. Then

$$\mathrm{gr}^i_W \mathbb{W}_{\mathbb{C}} = \oplus_j \mathbb{V}_j \otimes W_j,$$

where the $\mathbb{V}_j$ are distinct irreducible local systems carrying polarizable complex variations of Hodge structure, and the W_j are constant complex variations [LL23, Prop. 4.1.4(2)]. By assumption, $\mathbb{L}$ occurs among the $\mathbb{V}_j$, and hence carries a complex PVHS. Hence $\widetilde{\mathbb{L}}$ carries a real PVHS.

Let $Q := \mathrm{Hom}(\widetilde{\mathbb{L}}, \mathbb{W}) = H^0(\mathscr{X}, \widetilde{\mathbb{L}}^\vee \otimes \mathbb{W})$. The real variation $\widetilde{\mathbb{L}}^\vee \otimes \mathbb{W}$ is a tensor product of a pure (hence admissible) variation with an admissible variation, hence is admissible itself. By the theorem of the fixed part, Theorem 4.2.1, Q carries a canonical real mixed Hodge structure such that the evaluation map

$$Q \otimes \widetilde{\mathbb{L}} \to \mathbb{W}$$

is a map of variations, as desired. The map is non-zero by construction. □

4.3. *Consequences of non-abelian Hodge theory.* The main result from non-abelian Hodge theory that we will use is the following result of Mochizuki, generalizing earlier work of Simpson in the projective case:

THEOREM 4.3.1 ([Moc06, Th. 10.5]). *Let $\overline{X}$ be a smooth projective variety and $D \subset \overline{X}$ a strict normal crossings divisor. Let $X = \overline{X} \setminus D$. Any representation*

$$\rho : \pi_1(X) \to \mathrm{GL}_r(\mathbb{C})$$

with finite determinant admits a deformation with constant determinant to a representation underlying a complex PVHS.

Proof. Aside from the statement about determinants, this is precisely [Moc06, Th. 10.5]; the statement about determinants follows by examining the proof of that theorem. □

We also use the related result that cohomologically rigid representations underlie complex PVHS.

LEMMA 4.3.2. *Let X, D, ρ be as in Theorem 4.3.1. Suppose in addition that ρ is semisimple with finite determinant and $H^1(X, \operatorname{ad}\rho) = 0$. Then ρ underlies a complex PVHS.*

Proof. Let G denote the Zariski-closure of the image of ρ. Note that G is reductive by semisimplicity of ρ. Let $\mathfrak{g}$ be the Lie algebra of G, viewed as a $\pi_1(X)$-representation. Note that $\mathfrak{g}$ is a subalgebra of $\operatorname{ad}\rho$ as ρ has finite determinant, and it is moreover a summand of $\operatorname{ad}\rho$ as $\operatorname{ad}\rho$ is semisimple. Hence ρ is cohomologically rigid as a G-representation, as $H^1(X, \mathfrak{g}) \subset H^1(X, \operatorname{ad}\rho) = 0$.

We may apply [Moc06, Lemma 10.13], which yields that ρ admits a deformation as a $G(\mathbb{C})$-representation to $\rho_0 : \pi_1(X) \to G(\mathbb{C})$, which underlies a PVHS. But ρ is rigid and hence admits no non-trivial deformations. This implies $\rho = \rho_0$, so ρ underlies a PVHS. □

5. The period map associated to a unitary representation

In this section, we give an explicit description of the derivative of the period map on the cohomology of a unitary vector bundle over a punctured curve. In Section 5.1, we set up notation to describe the derivative of the period map, and we state our description in Theorem 5.1.6. In Section 5.2, we study a natural bilinear pairing on vector bundles, which is closely related to the derivative of the period map.

5.1. *The period map.* We now describe the period map associated to the first cohomology of a unitary local system over a punctured curve. We aim to

state Theorem 5.1.6, which is a generalization of the classical statement that, for C a curve, the multiplication map

$$H^0(C,\omega)\otimes H^0(C,\omega)\to H^0(C,\omega^{\otimes 2})$$

can be identified via Serre duality with the derivative of the period map associated to the Hodge structure on $H^1(C,\mathbb{C})$ [Voi07, Lemma 10.22].

Notation 5.1.1. Suppose $\pi:\mathscr{C}\to\mathscr{B}$ is a relative smooth proper curve over a smooth contractible complex-analytic base $\mathscr{B}$. Suppose $s_1,\dots,s_n:\mathscr{B}\to\mathscr{C}$ are sections to π with disjoint images $\mathscr{D}_1,\dots,\mathscr{D}_n$, and let $\mathscr{D}$ be the union of the $\mathscr{D}_i$. Let $\mathscr{C}^\circ=\mathscr{C}\setminus\mathscr{D}$, $j:\mathscr{C}^\circ\to\mathscr{C}$ be the natural inclusion, and let $\pi^\circ:\mathscr{C}^\circ\to\mathscr{B}$ be the composition $\pi^\circ=\pi\circ j$. Let $\mathbb{V}$ be a unitary $\mathbb{C}$-local system on $\mathscr{C}^\circ$, and let $\mathbb{W}:=R^1\pi^\circ_*\mathbb{V}$ denote the higher direct image local system on $\mathscr{B}$. Let $(\mathscr{E},\nabla)$ denote the Deligne canonical extension [Del70, Rems. 5.5(i)] of $(\mathbb{V}\otimes\mathscr{O}_{\mathscr{C}^\circ},\mathrm{id}\otimes d)$ to $\mathscr{C}$; this is a vector bundle with flat connection with regular singularities along $\mathscr{D}$. Note that for each $b\in\mathscr{B}$, $\mathscr{E}|_{\mathscr{C}_b}$ with the parabolic structure induced by ∇ (as in [LL23, Def. 3.3.1]) is the semistable parabolic vector bundle associated to $\mathbb{V}|_{\mathscr{C}_b^\circ}$ by the Mehta-Seshadri correspondence [MS80], [Sim90].

5.1.2. *Constructing the period map.* We next construct the relevant period map in (5.2). Set $\mathscr{H}:=\mathbb{W}\otimes_{\mathbb{C}}\mathscr{O}_{\mathscr{B}}\simeq\mathbb{R}^1\pi_*(\mathscr{E}\xrightarrow{\nabla}\mathscr{E}\otimes\Omega^1_{\mathscr{C}/\mathscr{B}}(\log\mathscr{D}))$, the right-hand side being relative hypercohomology [Del70, Cor. 6.10, Prop. 6.14]. As the vector bundle $\mathscr{H}$ arises from a local system, it carries a natural Gauss-Manin connection ∇_{GM}. Because we are assuming $\mathbb{V}$ is unitary, the Hodge-de Rham spectral sequence degenerates [Tim87, Th. 7.1(a)] (see also [Sai88, Th. 5.3.1] for a much more general result), giving a two-step Hodge filtration of $\mathscr{H}=\mathbb{W}\otimes\mathscr{O}_{\mathscr{B}}$, satisfying

$$\begin{aligned} F^1\mathscr{H}&:=\pi_*(\mathscr{E}\otimes\Omega^1_{\mathscr{C}/\mathscr{B}}(\log\mathscr{D})),\\ \mathscr{H}/F^1\mathscr{H}&\simeq R^1\pi_*\mathscr{E}, \end{aligned}\tag{5.1}$$

as described in Theorem 4.1.1.

Since $\mathscr{B}$ is contractible, we can globally trivialize the local system $\mathbb{W}$, yielding a flat trivialization of $\mathscr{H}$. We therefore obtain a period map

$$P:\mathscr{B}\to\mathrm{Gr}(\mathrm{rk}\,F^1\mathscr{H},\mathrm{rk}\,\mathscr{H}),\tag{5.2}$$

for $\mathrm{Gr}(s,r)$ the Grassmannian of s-dimensional subspaces of an r dimensional vector space, which sends $b\in\mathscr{B}$ to the subspace $F^1\mathscr{H}_b\subset\mathscr{H}_b$.

Remark 5.1.3. In the above Notation 5.1.1, and in what follows, when $\mathscr{C}\to\mathscr{B}$ is a relative curve, we will tend to use $\Omega^1_{\mathscr{C}/\mathscr{B}}(\mathscr{D})$ instead of the isomorphic but arguably more correct $\Omega^1_{\mathscr{C}/\mathscr{B}}(\log\mathscr{D})$ in order to simplify notation.

We will also often identify $\Omega^1_{\mathscr{C}/\mathscr{B}}$ with the relative dualizing sheaf $\omega_{\mathscr{C}/\mathscr{B}}$; we will pass between the two without comment.

5.1.4. *The derivative of the period map.* Having set up notation for the period map, we now describe its derivative. The derivative of the period map is an $\mathscr{O}_{\mathscr{B}}$-linear map

$$dP : T_{\mathscr{B}} \to P^* T_{\mathrm{Gr}(\mathrm{rk}\, F^1\mathscr{H}, \mathrm{rk}\,\mathscr{H})} \simeq (F^1\mathscr{H})^\vee \otimes (\mathscr{H}/F^1\mathscr{H}),$$

where the latter canonical identification follows from [EH16, Th. 3.5] and the fact that the universal sub and quotient bundles on the Grassmannian pull back under P to $F^1\mathscr{H}$ and $\mathscr{H}/F^1\mathscr{H}$. This is dual to a map

$$dP^\vee : (F^1\mathscr{H}) \otimes (\mathscr{H}/F^1\mathscr{H})^\vee \to \Omega^1_{\mathscr{B}}.$$

As $\mathscr{C}, s_1, \ldots, s_n$ is a family of n-pointed curves over $\mathscr{B}$, we also obtain a classifying map $c : \mathscr{B} \to \mathscr{M}_{g,n}$, inducing a pullback map

$$c^* : c^*\Omega^1_{\mathscr{M}_{g,n}} \to \Omega^1_{\mathscr{B}}.$$

5.1.5. *The derivative of the period map at a point.* We now identify the derivative of the period map at a point $b \in \mathscr{B}$. Given a point $b \in \mathscr{B}$, set $C := \mathscr{C}_b$, $D := \mathscr{D}_b$, and $E := \mathscr{E}|_C$. Via Serre duality, we identify $dP_b^\vee$ as a map

$$dP_b^\vee : H^0(C, E \otimes \Omega^1_C(\log D)) \otimes H^0(C, E^\vee \otimes \omega_C) \to \Omega^1_{\mathscr{B},b}$$

and c_b^* as a map

$$c_b^* : H^0(C, \omega_C^{\otimes 2}(D)) \to \Omega^1_{\mathscr{B},b}.$$

We also have a natural $\mathscr{O}_{\mathscr{B}}$-linear map $\overline{\nabla} : F^1\mathscr{H} \to \mathscr{H}/F^1\mathscr{H} \otimes \Omega^1_{\mathscr{B}}$ given as the composition

$$\overline{\nabla} : F^1\mathscr{H} \to \mathscr{H} \xrightarrow{\nabla_{GM}} \mathscr{H} \otimes \Omega^1_{\mathscr{B}} \to \mathscr{H}/F^1\mathscr{H} \otimes \Omega^1_{\mathscr{B}}. \tag{5.3}$$

A computation similar to [Voi07, Lemma 10.19, Th. 10.21, and Lemma 10.22], which we carry out in Appendix A and complete in Section A.2, yields

THEOREM 5.1.6. *Under the identifications above, the map $dP_b^\vee$ factors as*

$$\begin{array}{ccc} H^0(C, E \otimes \Omega^1_C(\log D)) \otimes H^0(C, E^\vee \otimes \omega_C) & \xrightarrow{dP_b^\vee} & \Omega^1_{\mathscr{B},b}, \\ \downarrow \otimes & & c_b^* \uparrow \\ H^0(C, E \otimes E^\vee \otimes \omega_C^{\otimes 2}(D)) & \xrightarrow{\mathrm{tr}} & H^0(C, \omega_C^{\otimes 2}(D)) \end{array} \tag{5.4}$$

where the left vertical map $\otimes$ is the tensor product of global sections, and the bottom map tr *is induced by the trace pairing $E \otimes E^\vee \to \mathscr{O}_C$. Moreover, $dP_b^\vee$ is adjoint to $\overline{\nabla}_b$.*

5.2. *A bilinear pairing.* Let C be a smooth proper connected curve of genus g, $D \subset C$ a reduced divisor, and E a vector bundle on C. In this section, we study the natural perfect pairing

$$B_E : (E \otimes \omega_C(D)) \times (E^\vee \otimes \omega_C) \to \omega_C^{\otimes 2}(D) \tag{5.5}$$

given as the composition

$$B_E : (E \otimes \omega_C(D)) \times (E^\vee \otimes \omega_C) \xrightarrow{\otimes} (E \otimes E^\vee) \otimes \omega_C^{\otimes 2}(D) \xrightarrow{\mathrm{tr} \otimes \mathrm{id}} \omega_C^{\otimes 2}(D),$$

where tr denotes the trace pairing $E \otimes E^\vee \to \mathscr{O}_C$. Our motivation for considering this pairing is the close relationship between (5.5) and Theorem 5.1.6: for E as in Theorem 5.1.6, the pairing on global sections induced by B_E computes the derivative of the period map.

The main result of this section is Proposition 5.2.3, which gives a sufficient criterion for the rank of E to be large in terms of this bilinear pairing. It is this lower bound on the rank of E that ultimately leads to the bound of $\sqrt{g+1}$ in our main result, Theorem 1.2.1. In Sections 6 and 7 we use Proposition 5.2.3 to analyze mapping class group representations appearing in the cohomology of unitary local systems.

We next review background on parabolic bundles, in order to state Proposition 5.2.3. For a more detailed and leisurely introduction to parabolic bundles, we suggest the reader consult [LL23, §2] and references therein.

5.2.1. *A lightning review of parabolic bundles.* Fix a smooth proper curve C, and let $D = x_1 + \cdots + x_n$ be a reduced divisor on C. Recall that a *parabolic bundle* on (C, D) is a vector bundle E on C, a decreasing filtration $E_{x_j} = E_j^1 \supsetneq E_j^2 \supsetneq \cdots \supsetneq E_j^{n_j+1} = 0$ for each $1 \leq j \leq n$, and an increasing sequence of real numbers $0 \leq \alpha_j^1 < \alpha_j^2 < \cdots < \alpha_j^{n_j} < 1$ for each $1 \leq j \leq n$, referred to as *weights*. We use $E_\star = (E, \{E_j^i\}, \{\alpha_j^i\})$ to denote the data of a parabolic bundle.

Given a parabolic bundle $E_\star = (E, \{E_j^i\}, \{\alpha_j^i\})$, let $J \subset \{1, \ldots, n\}$, denote the set of integers $j \in \{1, \ldots, n\}$ for which $\alpha_j^1 = 0$, and define

$$\widehat{E_0} := \ker(E \to \oplus_{j \in J} E_{x_j}/E_j^2).$$

(This is a special case of more general notation used for coparabolic bundles as in [LL23, 2.2.8] or the equivalent [BY96, Def. 2.3], but it is all we will need for this paper.) In particular, $\widehat{E_0} \subset E$ is a subbundle.

Parabolic bundles admit a notion of *parabolic stability*, analogous to the usual notion of stability for vector bundles, which we recall next. First, the *parabolic degree* of a parabolic bundle $E_\star$ is

$$\text{par-deg}(E_\star) := \deg(E) + \sum_{j=1}^{n} \sum_{i=1}^{n_j} \alpha_j^i \dim(E_j^i/E_j^{i+1}).$$

Then, the *parabolic slope* is defined by $\mu_\star(E_\star) := \text{par-deg}(E_\star)/\operatorname{rk}(E_\star)$. Any subbundle $F \subset E$ has an induced parabolic structure $F_\star \subset E_\star$ defined as follows: The filtration over x_j on F is obtained from the filtration

$$F_{x_j} = E_j^1 \cap F_{x_j} \supset E_j^2 \cap F_{x_j} \supset \cdots \supset E_j^{n_j+1} \cap F_{x_j} = 0$$

by removing redundancies. The weight associated to $F_j^i \subset F_{x_j}$ is

$$\max_{k, 1 \le k \le n_j} \{\alpha_j^k : F_j^i = E_j^k \cap F_{x_j}\}.$$

A parabolic bundle $E_\star$ is *parabolically semi-stable* if for every non-zero subbundle $F \subset E$ with induced parabolic structure $F_\star$, we have $\mu_\star(F_\star) \le \mu_\star(E_\star)$.

Mehta and Seshadri [MS80, Th. 4.1, Rem. 4.3], give a correspondence between *parabolically stable parabolic bundles of parabolic degree zero* on (C, D) and irreducible unitary local systems on $C \setminus D$. This bijection sends a local system $\mathbb{V}$ to the Deligne canonical extension of $(\mathbb{V} \otimes \mathscr{O}, \mathrm{id} \otimes d)$ with the parabolic structure induced by the connection (as in [LL23, Def. 3.3.1]).

Remark 5.2.2. It is not entirely trivial to extract from [MS80] that the parabolic vector bundle they construct from a unitary representation is in fact the parabolic vector bundle associated to the Deligne canonical extension of the associated flat vector bundle. What we will use is the fact that the parabolic vector bundle associated to the Deligne canonical extension is parabolically semistable, which follows from [Sim90, Th. 5], for example. A generalization of this fact about unitary local systems (for complex PVHS) is nicely explained in [AHL19, §5].

The main technical result we will use regarding B_E is

Proposition 5.2.3. *Let $E_\star$ be a semistable parabolic bundle on (C, D) of parabolic degree zero, with underlying vector bundle E. Let $v \in H^0(C, E \otimes \omega_C(D))$ be a non-zero global section, and suppose the map*

$$\begin{aligned} H^0(C, B_E(v, -)) : H^0(C, E^\vee \otimes \omega_C) &\to H^0(C, \omega_C^{\otimes 2}(D)), \\ u &\mapsto B_E(v, u) \end{aligned}$$

has rank r. Then $\operatorname{rk}(E) \ge g - r$.

Before proceeding with the proof, we recall [LL23, Prop. 6.3.6], which the reader may take as a black box. For a less technical variant of this statement, see [LL23, Prop. 6.3.1].

Proposition 5.2.4 ([LL23, Prop. 6.3.6]). *Suppose C is a smooth proper connected genus g curve and $E_\star = (E, \{E_j^i\}, \{\alpha_j^i\})$ is a non-zero parabolic bundle C with respect to $D = x_1 + \cdots + x_n$. Suppose $E_\star$ is parabolically*

semistable. Let $U \subset \widehat{E}_0$ *be a (non-parabolic) subbundle with* $c := \operatorname{rk} E - \operatorname{rk} U$ *and* $\delta := h^0(C, \widehat{E}_0) - h^0(C, U)$.

(I) *If* $\mu_\star(E_\star) > 2g - 2 + n$, *then* $\operatorname{rk} E > gc - \delta$.
(II) *If* $\mu_\star(E_\star) = 2g - 2 + n$, *then* $\operatorname{rk} E \geq gc - \delta$.

Remark 5.2.5. The statement of Proposition 5.2.4 is equivalent to [LL23, Prop. 6.3.6], but it differs slightly in that we write "$E_\star$ is parabolically stable" in place of "$\widehat{E}_\star$ is coparabolically stable" and $\mu_\star(E_\star)$ in place of $\mu_\star(\widehat{E}_\star)$. However, by definition, $E_\star$ is parabolically stable if and only if $\widehat{E}_\star$ is coparabolically stable and $\mu_\star(E_\star) = \mu_\star(\widehat{E}_\star)$ [LL23, Defs. 2.2.9 and 2.4.2].

Proof of Proposition 5.2.3. The idea is to apply Proposition 5.2.4(II) with $c = 1$ and $\delta = r$, and we now set up notation to do so. Set $n = \deg D$. Let $f_v : E^\vee \otimes \omega_C \to \omega_C^{\otimes 2}(D)$ be the map of vector bundles induced by $B_E(v, -)$, and set $U = \ker(f_v)$. As B_E is a perfect pairing and v is non-zero by assumption, U has corank one in $E^\vee \otimes \omega_C$.

We conclude by applying Proposition 5.2.4(II) to the parabolic bundle $(E_\star)^\vee \otimes \omega_C(D)$. This is parabolically stable by definition and has slope $\mu((E_\star)^\vee \otimes \omega_C(D)) = 2g - 2 + n$, since $E_\star$ has parabolic slope 0.

By unwinding the definitions, one may verify $\widehat{(E_\star)^\vee \otimes \omega_C(D)}_0 = E^\vee \otimes \omega_C$. Therefore, the assumption of the proposition implies

$$h^0(C, \widehat{(E_\star)^\vee \otimes \omega_C(D)}_0) - h^0(C, U) = h^0(C, E^\vee \otimes \omega_C) - h^0(C, U) = r.$$

Applying Proposition 5.2.4(II) to the parabolic bundle $(E_\star)^\vee \otimes \omega_C(D)$ with $U = \ker(f_v)$ as defined above so that $c = 1$ and $\delta = r$, we find $\operatorname{rk} E = \operatorname{rk} \widehat{(E_\star)^\vee \otimes \omega_C(D)}_0 \geq g - r$. □

6. **The main cohomological results**

In Section 6.1, we prove our main cohomological result, showing that higher direct images of unitary systems on families of curves contain no low rank local systems. From this we derive a related vanishing result in Section 6.2, which we will use later to establish cohomological rigidity of certain local systems.

Throughout this section, we continue to use Notation 1.10.1. Namely, we use $\pi : \mathscr{C} \to \mathscr{M}$ for a versal family of n-pointed curves of genus g, and $\pi^\circ : \mathscr{C}^\circ \to \mathscr{M}$ for the associated punctured versal family of curves.

6.1. *A rank bound.* We now prove our main result on the cohomology of unitary local systems on versal families of curves, Theorem 1.7.1. That is, for $\mathbb{V}$ a unitary local system on $\mathscr{C}^\circ$, we will show that any non-zero sub-local system of $R^1\pi^\circ_*\mathbb{V}$ has rank at least $2g - 2\operatorname{rk}\mathbb{V}$.

Proof of Theorem 1.7.1 assuming Lemma 6.1.1 (below). Let $m \in \mathscr{M}$, $C = \pi^{-1}(m)$, $C^\circ = (\pi^\circ)^{-1}(m)$ and $D := C - C^\circ$. Let (E, ∇) denote the Deligne canonical extension to C of the vector bundle $\mathbb{V}|_{C^\circ} \otimes \mathscr{O}_{C^\circ}$ with its tautological connection; (E, ∇) is a flat vector bundle on C with regular singularities along D. The residues of ∇ endow E with the structure of a parabolic bundle $E_\star$, as described in [LL23, Def. 3.3.1]. By the Mehta-Seshadri correspondence [MS80], the parabolic bundle $E_\star$ is a sum of parabolically stable bundles, hence parabolically semistable.

The connection ∇ on $\mathscr{H} := R^1\pi^\circ_*\mathbb{V} \otimes_{\mathbb{C}} \mathscr{O}_{\mathscr{M}}$ induces an $\mathscr{O}_{\mathscr{M}}$-linear map

$$F^1\mathscr{H} \xrightarrow{\overline{\nabla}} \mathscr{H}/F^1\mathscr{H} \otimes \Omega_{\mathscr{M}}$$

as in (5.3). Theorem 5.1.6 identifies the fiber of this map at the point $m \in \mathscr{M}$ with the map

$$\begin{aligned} \mu : H^0(C, E \otimes \omega_C(D)) &\to \operatorname{Hom}(H^0(C, E^\vee \otimes \omega_C), H^0(C, \omega_C^{\otimes 2}(D))), \\ v &\mapsto (u \mapsto B_E(v, u)) \end{aligned}$$

defined via B_E from (5.5). For $v \in H^0(C, E \otimes \omega_C(D))$, we denote by $\overline{\nabla}_m(v)$ the induced map

$$\begin{aligned} \overline{\nabla}_m(v) : H^0(C, E^\vee \otimes \omega_C) &\to H^0(C, \omega_C^{\otimes 2}(D)), \\ u &\mapsto B_E(v, u). \end{aligned}$$

Our strategy is to observe that a sub-local system of $R^1\pi^\circ_*\mathbb{V}$ of low rank would provide us with a vector v in $F^1\mathscr{H}_m$ so that the linear transformation $\overline{\nabla}_m(v)$ has low rank. This is in tension with Proposition 5.2.3 above, which will provide us with the required dimension bound.

Now, suppose we have a non-zero irreducible sub-local system $\mathbb{L} \subset R^1\pi^\circ_*\mathbb{V}$. We will show in Lemma 6.1.1 that, after possibly replacing $\mathbb{V}$ with its complex conjugate $\overline{\mathbb{V}}$, there is some non-zero $v \in F^1\mathscr{H}_m \simeq H^0(C, E \otimes \omega_C(D))$ for which $\overline{\nabla}_m(v)$ has rank (as a linear map) at most $\frac{\operatorname{rk}\mathbb{L}}{2}$. Under the identification of Theorem 5.1.6, we obtain that $\mu(v)$ has rank at most $\frac{\operatorname{rk}\mathbb{L}}{2}$. Using Proposition 5.2.3, $\operatorname{rk}\mathbb{V} = \operatorname{rk} E \geq g - \frac{\operatorname{rk}\mathbb{L}}{2}$. Upon rearranging, we find $\operatorname{rk}\mathbb{L} \geq 2g - 2\operatorname{rk}\mathbb{V}$. □

We now complete the proof of Theorem 1.7.1 by verifying the following lemma, which is essentially an application of the theorem of the fixed part.

LEMMA 6.1.1. *Suppose $\pi^\circ : \mathscr{C}^\circ \to \mathscr{M}$ is a punctured versal family of genus g curves and $\mathbb{V}$ is a unitary local system on $\mathscr{C}^\circ$. For $m \in \mathscr{M}$, let $\overline{\nabla}_m$ denote the fiber over m of the map $F^1\mathscr{H} \xrightarrow{\overline{\nabla}} \mathscr{H}/F^1\mathscr{H} \otimes \Omega_{\mathscr{M}}$ from (5.3). Suppose $\mathbb{L} \subset R^1\pi^\circ_*\mathbb{V}$ is a non-zero irreducible sub-local system. After possibly replacing $\mathbb{V}$ and $\mathbb{L}$ with their complex conjugates $\overline{\mathbb{V}}$ and $\overline{\mathbb{L}}$, there is some non-zero $v \in \mathbb{L}_m \cap F^1\mathscr{H}_m$ so that $\overline{\nabla}_m(v)$ has rank at most $\frac{\operatorname{rk}\mathbb{L}}{2}$.*

Proof. For a local system $\mathbb{W}$, let $\widetilde{\mathbb{W}}$ denote the corresponding local system with real structure as in (4.3). Note that

$$\widetilde{R^1\pi_*^\circ \mathbb{V}} = R^1\pi_*^\circ \widetilde{\mathbb{V}}.$$

By Proposition 4.2.2, $\widetilde{\mathbb{L}}$ underlies a real polarizable variation of Hodge structure, and we have a non-zero real mixed Hodge structure $Q_{\mathbb{R}}$ and a non-zero map of real variations of mixed Hodge structures

$$\iota : Q_{\mathbb{R}} \otimes \widetilde{\mathbb{L}} \to \widetilde{R^1\pi_*^\circ \mathbb{V}}.$$

Let $Q := (Q_{\mathbb{R}}) \otimes_{\mathbb{R}} \mathbb{C}$. As the grading on $(\widetilde{R^1\pi_*^\circ \mathbb{V}})_{\mathbb{C}}$ is supported in degrees $(1,0),(0,1),(1,1)$, we may assume (after regrading and possibly replacing Q with a subspace) that either

(1) the bigrading on $\mathbb{L}$ is supported in degree $(0,0)$ and the bigrading on Q has support contained in $\{(1,0),(0,1),(1,1)\}$; or
(2) the bigrading on $\mathbb{L}$ has support equal to $\{(1,0),(0,1)\}$ and the bigrading on Q is supported in degree $(0,0)$.

In the two cases above, we choose v as follows.

Case (1): After possibly replacing $\mathbb{V},\mathbb{L}$ with their complex conjugates, we may assume that

$$\iota_{i,j} : Q^{i,j} \otimes \mathbb{L} \to R^1\pi_*^\circ \mathbb{V}$$

is non-zero for some $(i,j) \in \{(1,0),(1,1)\}$. Choose $\ell \in \mathbb{L}_m$ and $q \in Q^{i,j}$, $(i,j) \in \{(1,0),(1,1)\}$ such that $v := \iota_m(q \otimes \ell)$ is non-zero.

Case (2): After possibly replacing $\mathbb{V}$ and $\mathbb{L}$ with their complex conjugates, we may assume $\dim \mathbb{L}_m^{0,1} \leq \dim \mathbb{L}_m^{1,0}$. Choose arbitrary non-zero $q \in Q$ and $\ell \in \mathbb{L}_m^{1,0}$ such that $v := \iota_m(q \otimes \ell)$ is non-zero.

It is enough to argue that for v as above, $\overline{\nabla}(v)$ has rank at most $\frac{\operatorname{rk}\mathbb{L}}{2}$. Since $\mathbb{L}$ carries a complex PVHS, we may compute the rank of $\overline{\nabla}(v)$ by computing with the analogous period map

$$\overline{\nabla}_{\mathbb{L},m} : F^1\mathbb{L}_m \to \mathbb{L}_m/F^1\mathbb{L}_m \otimes \Omega^1_{\mathscr{M},m}$$

associated to $\mathbb{L}$. In case (1), this map is identically zero, so we are done (i.e., $\operatorname{rk}\overline{\nabla}(v) = 0$). In case (2), from the definition of $\overline{\nabla}_{\mathbb{L},m}$, it is enough to show

$$\dim \mathbb{L}_m/F^1\mathbb{L}_m \leq \frac{\operatorname{rk}\mathbb{L}}{2}.$$

This follows because we have $\dim \mathbb{L}_m = \dim F^1\mathbb{L}_m + \dim \mathbb{L}_m/F^1\mathbb{L}_m$ and we assumed

$$\dim F^1\mathbb{L}_m = \dim \mathbb{L}_m^{1,0} \geq \dim \mathbb{L}_m^{0,1} = \dim \mathbb{L}_m/F^1\mathbb{L}_m. \qquad \square$$

6.2. *A vanishing result.* We apply Theorem 1.7.1 to prove a vanishing result about local systems on $\mathscr{C}^\circ$ that are not necessarily unitary. We assume only that their restriction to each fiber of π° is unitary. The reader may find it useful to consider the case where $A = \mathbb{C}$ below, though we will need the full generality of the result below for our deformation-theoretic applications.

THEOREM 6.2.1. *With notation as in Notation 1.10.1, let $\pi^\circ : \mathscr{C}^\circ \to \mathscr{M}$ be a punctured versal family of genus g curves. Let $m \in \mathscr{M}$ be a point, and set $C^\circ = \mathscr{C}^\circ_m$. Let A be a local Artin $\mathbb{C}$-algebra with residue field $\mathbb{C}$, and let $\mathbb{V}$ be a locally constant sheaf of free A-modules on $\mathscr{C}^\circ$ such that*

(1) *$\mathbb{V}|_{C^\circ}$ is a constant deformation of a unitary local system, i.e., there exists a unitary complex local system $\mathbb{V}_0$ on C° and an isomorphism $\mathbb{V}|_{C^\circ} \simeq \mathbb{V}_0 \otimes A$; and*

(2) *$\operatorname{rk}_A \mathbb{V} < g$.*

Then $H^0(\mathscr{M}, R^1\pi^\circ_*\mathbb{V}) = 0$.

Proof. Given a dominant map $\mathscr{M}' \to \mathscr{M}$, we have an injection

$$H^0(\mathscr{M}, R^1\pi^\circ_*\mathbb{V}) \to H^0(\mathscr{M}', R^1\pi^\circ_*\mathbb{V}|_{\mathscr{M}'}).$$

Hence, it is enough to prove the claim after replacing $\mathscr{M}$ by some $\mathscr{M}'$ with a dominant map to $\mathscr{M}$. Combining this with Lemma 2.4.2, we may assume $\mathbb{V} \simeq \oplus_{i=1}^s \mathbb{U}_i \otimes (\pi^\circ)^*\mathbb{W}_i$, where $\mathbb{W}_i$ are locally constant sheaves of free A-modules on $\mathscr{M}$ and $\mathbb{U}_i$ are unitary complex local systems on $\mathscr{C}^\circ$. It is enough to show that

$$H^0(\mathscr{M}, R^1\pi^\circ_*(\mathbb{U}_i \otimes (\pi^\circ)^*\mathbb{W}_i)) = H^0(\mathscr{M}, (R^1\pi^\circ_*\mathbb{U}_i) \otimes \mathbb{W}_i) = 0$$

for each i. Letting $\mathfrak{m}_A$ be the maximal ideal of A, the $\mathfrak{m}_A$-adic filtration on $\mathbb{W}_i$ has associated-graded pieces isomorphic to direct sums of copies of $\mathbb{W}_i^0 := \mathbb{W}_i \otimes_A \mathbb{C}$. Hence $\mathbb{W}_i \otimes R^1\pi_*\mathbb{U}_i$ has a filtration with associated-graded pieces isomorphic to $\mathbb{W}_i^0 \otimes R^1\pi_*\mathbb{U}_i$. Thus it suffices to prove

$$H^0(\mathscr{M}, R^1\pi^\circ_*(\mathbb{U}_i \otimes (\pi^\circ)^*\mathbb{W}_i^0)) = H^0(\mathscr{M}, (R^1\pi^\circ_*\mathbb{U}_i) \otimes \mathbb{W}_i^0) = 0.$$

Note that $\operatorname{rk} \mathbb{U}_i \otimes (\pi^\circ)^*\mathbb{W}_i^0 < g$.

It is therefore enough to show that if $\mathbb{U}$ is a unitary complex local system on $\mathscr{C}^\circ$ and $\mathbb{W}$ is an arbitrary complex local system on $\mathscr{M}$ with

$$H^0(\mathscr{M}, R^1\pi^\circ_*(\mathbb{U} \otimes (\pi^\circ)^*\mathbb{W})) = H^0(\mathscr{M}, R^1\pi^\circ_*\mathbb{U} \otimes \mathbb{W}) \neq 0,$$

then $\operatorname{rk} \mathbb{U} \otimes (\pi^\circ)^*\mathbb{W} \geq g$. A non-zero element of $H^0(\mathscr{M}, R^1\pi^\circ_*\mathbb{U} \otimes \mathbb{W})$ yields a non-zero map of local systems

$$\mathbb{W}^\vee \to R^1\pi^\circ_*\mathbb{U}.$$

Since $\mathbb{U}$ is unitary, we have $\operatorname{rk} \mathbb{W} \geq 2g - 2\operatorname{rk} \mathbb{U}$ by Theorem 1.7.1. Hence $\operatorname{rk} \mathbb{W} + 2\operatorname{rk} \mathbb{U} \geq 2g$. As $\operatorname{rk} \mathbb{W}, \operatorname{rk} \mathbb{U}$ are positive integers, we find $\operatorname{rk} \mathbb{W} \cdot \operatorname{rk} \mathbb{U}$ is minimized when $\operatorname{rk} \mathbb{W} = 1$ in which case $\operatorname{rk} \mathbb{W} \cdot \operatorname{rk} \mathbb{U} \geq \lceil (2g-1)/2 \rceil \geq g$. Note

that here we use that $g \geq 2$ in order for condition (2) of the statement to be satisfied as $\mathbb{V} \neq 0$. In general, this implies

$$\operatorname{rk}(\mathbb{U} \otimes (\pi^\circ)^* \mathbb{W}) = \operatorname{rk} \mathbb{W} \cdot \operatorname{rk} \mathbb{U} \geq g, \tag{6.1}$$

as desired. □

7. The asymptotic Putman-Wieland conjecture

7.1. *The statement of the Putman-Wieland conjecture.* We next discuss some applications of our methods to the Putman-Wieland conjecture (Conjecture 7.1.3), a major open problem in geometric topology. Much of the interest in the Putman-Wieland conjecture arises from its close relationship to Ivanov's conjecture that mapping class groups $\mathrm{Mod}_{g,n}$ do not virtually surject onto $\mathbb{Z}$ for $g \gg 0$ [Iva06, §7], as explained in [PW13, Th. 1.3]. In particular, the Putman-Wieland conjecture for $g \gg 0$ is equivalent to Ivanov's conjecture for $g \gg 0$.

Notation 7.1.1. Let Σ_g be an oriented surface of genus $g \geq 0$, and let $\Sigma_{g,b,p}$ be the complement of $b \geq 0$ disjoint open discs and $p \geq 0$ disjoint points in Σ_g, so that $\Sigma_{g,b,p}$ is an oriented genus g surface with b boundary components and p punctures. We let $\mathrm{PMod}_{g,b,p}$ be the subgroup of the mapping class group of $\Sigma_{g,b,p}$ fixing the punctures and boundary components pointwise. Fix a basepoint $v_0 \in \Sigma_{g,b,p}$, which we count as an additional puncture to obtain an action of $\mathrm{PMod}_{g,b,p+1}$ on $\pi_1(\Sigma_{g,b,p}, v_0)$. Let $K \trianglelefteq \pi_1(\Sigma_{g,b,p}, v_0)$ be a finite index normal subgroup corresponding to a finite Galois covering space $\Sigma_{g',b',p'} \to \Sigma_{g,b,p}$, or equivalently a surjection $\phi : \pi_1(\Sigma_{g,b,p}, v_0) \twoheadrightarrow H$, with $H = \pi_1(\Sigma_{g,b,p}, v_0)/K$ a finite group. Let $\Gamma \subset \mathrm{PMod}_{g,b,p+1}$ denote the finite index subgroup preserving ϕ up to conjugacy. Viewing $H_1(\Sigma_{g',b',p'}, \mathbb{C})$ as an H-representation, if ρ is an irreducible H-representation, we let

$$H_1(\Sigma_{g',b',p'}, \mathbb{C})^\rho := \rho \otimes \mathrm{Hom}_H(\rho, H_1(\Sigma_{g',b',p'}, \mathbb{C}))$$

denote the ρ-isotypic component.

We obtain an action of Γ on $K^{\mathrm{ab}} \otimes \mathbb{C} = H_1(\Sigma_{g',b',p'}, \mathbb{C})$ and hence on the characteristic subrepresentation $H^1(\Sigma_{g',b',p'}, \mathbb{C})^\rho$. By filling in the punctures and deleted discs, we also obtain an action of Γ on $V_K := H_1(\Sigma_{g'}, \mathbb{C})$, referred to in [PW13, p. 80-81] as a *higher Prym representation.*

Definition 7.1.2. Fix a finite group H and non-negative integers g, b, and p. Let $\mathrm{PW}^H_{g,b,p}$ be the statement that for any surjection $\phi : \pi_1(\Sigma_{g,b,p}, v_0) \twoheadrightarrow H$, taking $K := \ker \phi$ and $v \in V_K$ any non-zero vector, v has infinite orbit under Γ.

Conjecture 7.1.3 (Putman-Wieland, [PW13, Conj. 1.2]). The statement $\mathrm{PW}^H_{g,b,p}$ holds for every group H with $g \geq 2, b \geq 0, p \geq 0$.

Remark 7.1.4. We note that [PW13, Conj. 1.2] is stated without a group H and with $\mathbb{Q}$ coefficients instead of $\mathbb{C}$ coefficients. We use this equivalent statement to more easily state our results, which imply that $\mathrm{PW}^H_{g,b,p}$ holds whenever $\#H$ is small compared to the genus g.

Remark 7.1.5. There is a counterexample to the Putman-Wieland conjecture when $g = 2$ [Mar22, Th. 1.3].

7.2. *Our results.* We are able to verify $\mathrm{PW}^H_{g,b,p}$ in many new cases. Moreover, we prove the following more general statement, ruling out not only invariant vectors in $H^1(\Sigma_{g'}, \mathbb{C})$, but even ruling out low-dimensional invariant subspaces of isotypic components. The proof is given below in Section 7.3.

THEOREM 7.2.1. *With notation as in Notation 7.1.1, let ρ be an irreducible complex H-representation and $\Gamma' \subset \Gamma$ be a finite-index subgroup. Then $H^1(\Sigma_{g',b',p'}, \mathbb{C})^\rho$ has no non-zero Γ'-invariant subrepresentations of dimension strictly less than $2g - 2\dim\rho$. The same holds for $H_1(\Sigma_{g'}, \mathbb{C})^\rho$ in place of $H^1(\Sigma_{g',b',p'}, \mathbb{C})^\rho$.*

Remark 7.2.2. In Theorem 7.2.1, it is important that we restrict our attention to subrepresentations, as opposed to arbitrary subquotients, as the Γ-representation $H^1(\Sigma_{g',b',p'}, \mathbb{C})$ is not in general semisimple for $b' + p' > 0$.

A perhaps more concrete corollary is the following.

COROLLARY 7.2.3. *The statement $\mathrm{PW}^H_{g,b,p}$ holds for every group H such that every irreducible representation ρ of H has dimension $\dim\rho < g$.*

Proof. Fix an irreducible representation ρ of H. By assumption, $\dim\rho < g$. Then $2g - 2\dim\rho > 1$, and so by Theorem 7.2.1, $H_1(\Sigma_{g'}, \mathbb{C})^\rho$ has no non-zero Γ'-invariant 1-dimensional subrepresentations for any ρ and any finite index $\Gamma' \subset \Gamma$. In particular, $H_1(\Sigma_{g'}, \mathbb{C})$ has no non-zero invariant vectors under any finite index Γ'. But if there was a non-zero vector $v \in V_K$ with finite orbit, its stabilizer would yield a finite index Γ' with a fixed vector. □

Even more concretely, we can give the following bound on $\#H$ independent of its representation theory.

COROLLARY 7.2.4. *For fixed $g, b, p \geq 0$, $\mathrm{PW}^H_{g,b,p}$ holds for every group H with $\#H < g^2$.*

Proof. If H is a finite group of order $\#H < g^2$, then because $\#H$ is the sum of the squares of the dimensions of the irreducible representations of H by [FH91, (2.19)], no irreducible representations of H have dimension $\geq g$. The result follows from Corollary 7.2.3. □

In order to prove Theorem 7.2.1, we first note that it suffices to consider the case of surfaces without boundary.

LEMMA 7.2.5. *With notation as in Notation 7.1.1, for any $g \geq 0, b, p \geq 0$, and ρ an irreducible H-representation, let $\Gamma' \subset \Gamma$ be a finite index subgroup. Then there exist a finite index $\Gamma'' \subset \mathrm{PMod}_{g,0,b+p+1}$ and an isomorphism $H_1(\Sigma_{g',b',p'}, \mathbb{C})^\rho \to H_1(\Sigma_{g,0,b'+p'}, \mathbb{C})^\rho$ inducing a bijection from Γ'-invariant subspaces to Γ''-invariant subspaces.*

Proof. Define the natural inclusion $\iota : \Sigma_{g',b',p'} \hookrightarrow \Sigma_{g,0,b'+p'}$, shrinking boundary components to punctures. There is a deformation retraction from $\Sigma_{g,0,b'+p'}$ onto the subspace $\iota(\Sigma_{g',b',p'})$ and so ι induces an isomorphism

$$H_1(\Sigma_{g',b',p'}, \mathbb{C}) \to H_1(\Sigma_{g,0,b'+p'}, \mathbb{C}).$$

The kernel of the induced surjection $\mathrm{PMod}_{g,b,p} \to \mathrm{PMod}_{g,0,b+p}$ is generated by $\mathbb{Z}^b$ acting on $S_{g,b,p}$ by rotating the boundary components [FM12, Th. 3.18], and hence this $\mathbb{Z}^b$ acts trivially on $H_1(\Sigma_{g',b',p'}, \mathbb{C})$. Let Γ'' be the image of Γ' under this natural map $\mathrm{PMod}_{g,b,p+1} \to \mathrm{PMod}_{g,0,b+p+1}$. The action of Γ' on $H_1(\Sigma_{g',0,b'+p'}, \mathbb{C})$ factors through the natural map $\Gamma' \to \Gamma''$, inducing the claimed bijection. □

7.3. We now prove Theorem 7.2.1, which is more or less an immediate consequence of Theorem 1.7.1 applied to the irreducible representations of H, once one sets up the appropriate local systems. For the proof, we will need the existence of a versal family of ϕ-covers, for $\phi : \pi_1(\Sigma_{g,0,n}, v_0) \twoheadrightarrow H$, which we now define.

Definition 7.3.1. As in Notation 7.1.1, specify a surjection $\phi : \pi_1(\Sigma_{g,0,n}, v_0) \twoheadrightarrow H$. A *versal family of ϕ-covers* is the data of

(1) a dominant étale morphism $\mathscr{M} \to \mathscr{M}_{g,n}$, with $\pi^\circ : \mathscr{C}^\circ \to \mathscr{M}$ the associated family of punctured curves;
(2) a point $c \in \mathscr{C}^\circ$, $m = \pi^\circ(c)$, $C^\circ = \mathscr{C}^\circ_m$, and an identification $i : \pi_1(\Sigma_{g,0,n}, v_0) \simeq \pi_1(C^\circ, c)$; and
(3) a finite étale Galois H-cover $f : \mathscr{X}^\circ \to \mathscr{C}^\circ$ inducing a map

$$\pi_1(C^\circ, c) \to \pi_1(\mathscr{C}^\circ, c)/\pi_1(\mathscr{X}^\circ, x) \simeq H$$

agreeing with the surjection ϕ under the identification of (2).

Proof of Theorem 7.2.1. Using Lemma 7.2.5, it is enough to prove Theorem 7.2.1 when $b = 0$.

By [Wew98, Th. 4], there exists a versal family of ϕ-covers $\mathscr{X}^\circ \xrightarrow{f} \mathscr{C}^\circ \xrightarrow{\pi^\circ} \mathscr{M}$. Technically, [Wew98, Th. 4] constructs $\mathscr{M}$ as a Deligne-Mumford stack, where points have isotropy groups isomorphic to the center of H, but we may replace this stack by a scheme that has a dominant étale map to this stack. Further, after replacing $\mathscr{M}$ with another scheme that has a dominant étale map to $\mathscr{M}$, corresponding generically to an étale multisection of π°, we may

assume π° admits a section, and hence that the map $\mathscr{M} \to \mathscr{M}_{g,n}$ lifts to a map $\mathscr{M} \to \mathscr{M}_{g,n+1}$.

As f is a Galois finite étale H-cover, for each irreducible representation ρ of H, we obtain a local system $\mathbb{U}_\rho$ on $\mathscr{C}^\circ$, with monodromy representation given by ρ, and with $\mathbb{U}_\rho|_{\mathscr{X}^\circ}$ trivial. After replacing $\mathscr{M}$ with a dominant étale cover, we may write

$$f_*\mathbb{C} = \bigoplus_\rho \mathbb{U}_\rho \otimes (\pi^\circ)^* \mathbb{W}_\rho,$$

where $\mathbb{W}_\rho := \pi^\circ_* \operatorname{Hom}(\mathbb{U}_\rho, f_*\mathbb{C})$, by Lemma 2.4.2 (where we take $A = \mathbb{C}$). As $\mathbb{U}_\rho$ has finite monodromy by Lemma 2.4.1(1), and $f_*\mathbb{C}$ has finite monodromy by definition, the same is true for $\mathbb{W}_\rho$. Thus after replacing $\mathscr{M}$ with a finite étale cover we may assume each $\mathbb{W}_\rho$ is a trivial local system.

We next verify the first part of Theorem 7.2.1, about $H^1(\Sigma_{g',0,p'}, \mathbb{Q})^\rho$. Observe that for $m \in \mathscr{M}$, the action of $\pi_1(\mathscr{M}, m)$ on

$$R^1\pi^\circ_*(\mathbb{U}_\rho \otimes (\pi^\circ)^*\mathbb{W}_\rho)_m = (R^1\pi^\circ_*\mathbb{U}_\rho \otimes \mathbb{W}_\rho)_m$$

is identified with its action on $H^1(\mathscr{X}^\circ_m, \mathbb{C})^\rho = H^1(\Sigma_{g',0,p'}, \mathbb{C})^\rho$. Note also that the action of $\pi_1(\mathscr{M})$ factors through Γ, under our given map $\pi_1(\mathscr{M}) \to \pi_1(\mathscr{M}_{g,n+1}) = \mathrm{PMod}_{g,n+1}$. To prove the first part of Theorem 7.2.1, about $H^1(\Sigma_{g',0,p'}, \mathbb{Q})^\rho$, it thus suffices to show that, after replacing $\mathscr{M}$ with an arbitrary finite étale cover, $R^1(\pi^\circ_*\mathbb{U}_\rho) \otimes \mathbb{W}_\rho$ contains no non-trivial sub-local systems of rank less than $2g - 2\dim\rho$. As $\mathbb{W}_\rho$ is a trivial local system, this holds by applying Theorem 1.7.1 to $\mathbb{U}_\rho$.

We have thus far proven a statement for the cohomology of $\Sigma_{g',n'}$, and to conclude we deduce a corresponding statement for the homology of $\Sigma_{g'}$. Using the inclusion $H^1(\Sigma_{g'}, \mathbb{C})^\rho \subset H^1(\Sigma_{g',n'}, \mathbb{C})^\rho$ we deduce that the former can have no non-zero Γ'-invariant subrepresentations (for $\Gamma' \subset \Gamma$ finite index) of dimension less than $2g - 2\dim\rho$, as the latter has no such subrepresentations. Using Poincaré duality and the intersection pairing on $H_1(\Sigma_{g'}, \mathbb{C})$ we obtain a $\mathrm{Mod}_{g,n}$ equivariant isomorphism $H^1(\Sigma_{g'}, \mathbb{C}) \simeq H_1(\Sigma_{g'}, \mathbb{C})$. Hence, $H_1(\Sigma_{g'}, \mathbb{C})^\rho$ also has no subrepresentations of dimension less than $2g - 2\dim\rho$. □

8. **Proof of the main theorem on MCG-finite representations**

We have finally developed the tools to verify our main result, Theorem 1.2.1: that MCG-finite representations of rank $r < \sqrt{g+1}$ have finite image. The proof is fairly involved, so the reader may find it useful to refer to the sketch in Section 1.9. Briefly, we start by reviewing the notion of cohomological rigidity in Section 8.1 and prove the necessary rigidity results in Section 8.2, using our main vanishing result, Theorem 6.2.1. We next deduce the integrality of MCG-finite representations of low rank in Section 8.3 using a result of Klevdal-Patrikis [KP22], which builds on work of Esnault-Groechenig

[EG18] and ultimately relies on input from the Langlands program and known cases of the companion conjectures. We then combine the above ingredients with input from non-abelian Hodge theory to deduce that low rank unitary MCG-finite representations have finite image in Section 8.4. We generalize this to the semisimple but non-unitary case in Section 8.5, again using non-abelian Hodge theory. Finally, in Section 8.6 we bootstrap these results to the general, non-semisimple case, using Theorem 7.2.1 on the Putman-Wieland conjecture.

8.1. *Background on Cohomological rigidity.* We now define the notion of cohomological rigidity, which detects whether a representation has any first order deformations.

Definition 8.1.1 (Cohomological rigidity). Let $\overline{X}$ be a smooth projective variety, $Z \subset \overline{X}$ a strict normal crossings divisor, and $X = \overline{X} \setminus Z$. Let G be a reductive group over $\mathbb{C}$ and

$$\rho : \pi_1(X) \to G(\mathbb{C})$$

a homomorphism such that the monodromy at infinity is quasi-unipotent. The representation ρ is said to be *cohomologically rigid* if

$$H^1(\overline{X}, j_{!*} \operatorname{ad} \rho) = 0,$$

where $j : X \hookrightarrow \overline{X}$ is the natural inclusion, $j_{!*}$ is the intermediate extension, and $\operatorname{ad} \rho$ is defined as in Notation 1.10.2.

If $\mathbb{V}$ is the local system on X associated to ρ, we call $\mathbb{V}$ cohomologically rigid if ρ is.

Remark 8.1.2. For the case where ρ is irreducible, see [KP22, Prop. 4.7, Rem. 4.8] for a moduli-theoretic interpretation of cohomological rigidity. This says that ρ is a smooth isolated point of the appropriate character variety, parametrizing representations with fixed local monodromy at infinity. Note that if $\overline{X}$ is a curve, then $j_{!*} = j_*$.

We will typically prove cohomological rigidity by computing $H^1(X, \operatorname{ad} \rho)$, via the next lemma.

Lemma 8.1.3. *In the setting of Definition 8.1.1, assume* $H^1(X, \operatorname{ad} \rho) = 0$. *Then* ρ *is cohomologically rigid.*

Proof. This follows from the identification $H^1(\overline{X}, j_{!*} \operatorname{ad} \rho) \simeq H^1(U, a_* \operatorname{ad} \rho)$ for $U \subset \overline{X}$ a certain dense open subscheme containing X with $a : X \to U$ the inclusion [EG18, Rem. 2.4] (see also [KP22, Rem. 4.8]). The Leray spectral sequence gives an injection $H^1(U, a_* \operatorname{ad} \rho) \hookrightarrow H^1(X, \operatorname{ad} \rho)$, implying the claim. □

Definition 8.1.4. If $H^1(X, \operatorname{ad}\rho) = 0$, we say that ρ is *strongly cohomologically rigid*.

8.2. *Cohomological rigidity and unitary MCG-finite representations.* Let $\rho : \pi_1(\Sigma_{g,n}) \to \mathrm{GL}_r(\mathbb{C})$ be a unitary MCG-finite representation of rank $< \sqrt{g+1}$. We will study ρ by associating to ρ a certain unitary local system on a well-chosen versal family of curves. Crucially, this unitary local system will be cohomologically rigid. This will follow from our main vanishing result Theorem 6.2.1.

Throughout this section, we use notation as in Notation 1.10.1. In particular, $\pi : \mathscr{C} \to \mathscr{M}$ is a versal family of n-pointed curves of genus g, $\pi^\circ : \mathscr{C}^\circ \to \mathscr{M}$ is the associated family of punctured curves, $m \in \mathscr{M}$ is a basepoint, and $C^\circ = \mathscr{C}^\circ_m$.

Proposition 8.2.1. *Let $\mathbb{V}$ be a GL_r (respectively, PGL_r)-local system on $\mathscr{C}^\circ$ with $r < \sqrt{g+1}$. Suppose that for $m \in \mathscr{M}$, $\mathbb{V}|_{C^\circ}$ is (respectively, is the projectivization of) an irreducible, unitary local system. Then $\mathbb{V}$ is strongly cohomologically rigid. In particular, $\mathbb{V}$ is cohomologically rigid.*

Proof. By Lemma 8.1.3, it is enough to show $\mathbb{V}$ is strongly cohomologically rigid. We let $\operatorname{ad}\mathbb{V}$ denote the adjoint local system as in Notation 1.10.2. We will check $H^1(\mathscr{C}^\circ, \operatorname{ad}\mathbb{V}) = 0$. Using the Leray spectral sequence associated to the map π°, it suffices to show that

$$H^0(\mathscr{M}, R^1\pi^\circ_* \operatorname{ad}\mathbb{V}) = H^1(\mathscr{M}, \pi^\circ_* \operatorname{ad}\mathbb{V}) = 0.$$

We have $H^0(\mathscr{M}, R^1\pi^\circ_* \operatorname{ad}\mathbb{V}) = 0$ by Theorem 6.2.1 (taking $A = \mathbb{C}$), as $\operatorname{rk}\operatorname{ad}\mathbb{V} = r^2 - 1 < g$ and $\operatorname{ad}\mathbb{V}|_{C^\circ}$ is unitary by assumption.

To conclude, it is enough to show $\pi^\circ_* \operatorname{ad}\mathbb{V} = 0$. This is follows from Schur's lemma because $\mathbb{V}|_{C^\circ}$ is (the projectivization of) an irreducible local system, and so $\operatorname{ad}\mathbb{V}|_{C^\circ}$ has no $\pi_1(C^\circ)$-invariants. □

8.3. *Verifying integrality.* The next step of the proof is to use the cohomological rigidity provided by Proposition 8.2.1 to deduce integrality of MCG-finite unitary representations.

Definition 8.3.1. Let G be a group scheme over $\mathbb{Z}$. We say a representation $\pi_1(X, x) \to G(\mathbb{C})$ is *integral* if it is conjugate to a representation that factors through $G(\mathscr{O}_K)$ for some number field K. A local system on a connected space X is integral if the corresponding monodromy representation is.

We will primarily be concerned with integrality with respect to the group schemes $G = \mathrm{PGL}_r$ and $G = \mathrm{GL}_r$. The basic idea to show our local system is integral is to apply the main result of [KP22], which builds on [EG18] and ultimately relies on Lafforgue's work on the Langlands program for function fields

and consequent work on Deligne's companion conjectures. To apply this result, we need to know the monodromy of our cohomologically rigid local system $\mathbb{V}$ around boundary components of a strict normal crossings compactification of $\mathscr{C}^\circ$ is quasi-unipotent, which we will verify using the following result.

PROPOSITION 8.3.2 ([AS16, Prop. 2.4]). *Suppose that $g \geq 3$, $\gamma \subset \Sigma_{g,n}$ is a simple closed curve, and $\Gamma \subset \mathrm{Mod}_{g,n}$ is a finite index subgroup. Let $\delta_\gamma \in \mathrm{Mod}_{g,n}$ be the Dehn twist about γ. Let $\rho : \Gamma \to \mathrm{GL}_r(\mathbb{C})$ be a representation. Then for m such that $\delta_\gamma^m \in \Gamma$, $\rho(\delta_\gamma^m)$ is quasi-unipotent.*

We now verify that the projectivization of a unitary MCG-finite representation is integral.

LEMMA 8.3.3. *Let $g \geq 3$ be an integer, and let $\rho : \pi_1(\Sigma_{g,n}) \to \mathrm{GL}_r(\mathbb{C})$ be an irreducible, unitary, MCG-finite representation, with $r < \sqrt{g+1}$. Then the composition*

$$\mathbb{P}\rho : \pi_1(\Sigma_{g,n}) \xrightarrow{\rho} \mathrm{GL}_r(\mathbb{C}) \to \mathrm{PGL}_r(\mathbb{C})$$

is integral.

Proof. By Lemma 2.2.3, there exist a scheme $\mathscr{M}$ with a finite étale map $\mathscr{M} \to \mathscr{M}_{g,n}$, with associated family of punctured curves $\pi^\circ : \mathscr{C}^\circ \to \mathscr{M}$, and a representation

$$\widetilde{\rho} : \pi_1(\mathscr{C}^\circ) \to \mathrm{PGL}_r(\mathbb{C})$$

agreeing with $\mathbb{P}\rho$ on restriction to a fiber of π°. It suffices to show that $\widetilde{\rho}$ is integral. By Proposition 8.2.1, $\widetilde{\rho}$ is cohomologically rigid. Because the abelianization of $\mathrm{PGL}_r(\mathbb{C})$ is trivial, it follows from [KP22, Th. 1.2] that $\widetilde{\rho}$ is integral once we verify that $\widetilde{\rho}$ has quasi-unipotent local monodromy around the boundary components of a good (i.e., strict normal crossing) compactification of $\mathscr{C}^\circ$.

One may construct a strict normal crossing compactification as follows. The scheme $\mathscr{C}^\circ$ is a finite étale cover of $\mathscr{M}_{g,n+1}$ by construction; let $\overline{\mathscr{M}_{g,n+1}}'$ be a strict normal crossing compactification of $\mathscr{M}_{g,n+1}$ obtained by blowing up boundary strata of the Deligne-Mumford compactification $\overline{\mathscr{M}_{g,n+1}}$ (which is only a normal crossing compactification, and is not in general strict). Let $\overline{\mathscr{C}}$ be the normalization of $\overline{\mathscr{M}_{g,n+1}}'$ in the function field of $\mathscr{C}^\circ$. By Proposition 8.3.2, it suffices to check that the local monodromy about the boundary components of $\overline{\mathscr{C}}$ correspond to products of commuting Dehn twists about simple closed curves, under the identification of $\pi_1(\mathscr{C}^\circ)$ with a subgroup of $\mathrm{PMod}_{g,n+1} = \pi_1(\mathscr{M}_{g,n+1})$. This is because a product of commuting quasi-unipotent matrices is quasi-unipotent. To check the claim about local monodromy about boundary components, it suffices to check the corresponding claim for $\overline{\mathscr{M}_{g,n+1}}'$. But this is [LLSS23, Lemma 2.1.1]. □

The next result shows we can lift integrality from the projectivization of a unitary irreducible MCG-finite representation to the original representation.

LEMMA 8.3.4. *Let $g \geq 3$ be an integer, and let $\rho : \pi_1(\Sigma_{g,n}) \to \mathrm{GL}_r(\mathbb{C})$ be an irreducible, unitary, MCG-finite representation, with $r < \sqrt{g+1}$. Then ρ is integral.*

Proof. By Proposition 2.3.1, $\det \rho$ has finite image, and hence ρ factors through $G(\mathbb{C})$, where $G \subset \mathrm{GL}_r$ is a flat affine $\mathbb{Z}$-group scheme containing SL_r with finite index. (For example, we can take $G = \det^{-1}(\mu_m)$, where $\det \rho$ factors through μ_m.) Note that the natural morphism $G \to \mathrm{PGL}_r$ is finite. After conjugating ρ by some matrix we may assume $\mathbb{P}\rho$ factors through $\mathrm{PGL}_r(\mathscr{O}_K)$ for some number field K, by Lemma 8.3.3.

As $\pi_1(\Sigma_{g,n})$ is finitely generated, it suffices to show that for a generator $\alpha \in \pi_1(\Sigma_{g,n})$, there exists some finite extension K'/K such that the matrix $\rho(\alpha)$ lies in $G(\mathscr{O}_{K'})$. The obstruction to lifting any given $\mathscr{O}_K$ point of PGL_r to G is a $\ker(G \to \mathrm{PGL}_r)$ torsor, which becomes trivial on some finite flat extension, and hence over $\mathscr{O}_{K'}$ for some finite K'/K. □

8.4. *Verifying finiteness.* We now show that a unitary MCG-finite representation ρ of low rank has finite image. We have already shown that, in the irreducible setting, such representations are defined over $\mathscr{O}_K$. We will use the techniques of [LL23] to deduce that they have finite monodromy. To do so, we will argue that for each embedding $\tau : \mathscr{O}_K \hookrightarrow \mathbb{C}$, $\rho \otimes_{\mathscr{O}_K,\tau} \mathbb{C}$ has unitary monodromy. We know this for one such τ but not for the rest. To prove unitarity for all such τ, we will use [LL23, Th. 1.2.12], and to verify its hypotheses we will need to use non-abelian Hodge theory to construct certain complex PVHS.

PROPOSITION 8.4.1. *Let*

$$\rho : \pi_1(\Sigma_{g,n}) \to \mathrm{GL}_r(\mathbb{C})$$

be a unitary MCG-finite representation with $r < \sqrt{g+1}$. Then ρ has finite image.

Proof. As ρ is unitary, it is semisimple. As an irreducible sub-representation of an MCG-finite representation is MCG-finite by Proposition 2.1.1, we may without loss of generality assume ρ is irreducible.

If $g < 3$, then $r \leq 1$, and so we may conclude by Proposition 2.3.1. Thus we may assume $g \geq 3$. By Lemma 8.3.4, we may assume ρ factors through $\mathrm{GL}_r(\mathscr{O}_K)$ for some number field K and some embedding $\iota : \mathscr{O}_K \hookrightarrow \mathbb{C}$. We now adjust notation so that ρ denotes this $\mathscr{O}_K$-representation and $\rho_\iota = \rho \otimes_{\mathscr{O}_K,\iota} \mathbb{C}$ denotes our original MCG-finite representation. By Corollary 2.3.5, there exists a punctured versal family $\mathscr{C}^\circ \to \mathscr{M}$ and a representation $\rho'_\iota : \pi_1(\mathscr{C}^\circ) \to \mathrm{GL}_r(\mathbb{C})$ with finite determinant, restricting to ρ_ι on the fundamental group of a fiber of π°, i.e., for C° a fiber of π°, $\rho'_\iota|_{\pi_1(C^\circ)} = \rho_\iota$. Note that

ρ'_ι is irreducible (as ρ_ι is irreducible), and strongly cohomologically rigid, by Proposition 8.2.1.

Choose any embedding $\tau : \mathscr{O}_K \hookrightarrow \mathbb{C}$, and let $\rho_\tau := \rho \otimes_{\mathscr{O}_K,\tau} \mathbb{C}$. Note that ρ_τ is MCG-finite, as the same is true for ρ_ι. We aim to show ρ_τ is unitary.

Choose $\sigma : \mathbb{C} \xrightarrow{\sim} \mathbb{C}$ so that $\sigma \circ \iota = \tau$, and set $\rho'_\tau = \rho'_\iota \otimes_{\mathbb{C},\sigma} \mathbb{C}$. Note that $\rho'_\tau|_{\pi_1(C^\circ)} = \rho_\tau$. Now observe that ρ'_τ is strongly cohomologically rigid, as the same is true for ρ'_ι, and strong cohomological rigidity (being a cohomological condition) is preserved by automorphisms of $\mathbb{C}$. Moreover, ρ'_τ is irreducible with finite determinant, as the same is true for ρ'_ι.

We next show ρ_τ is unitary. By [LL23, Th. 1.2.12], in order to show ρ_τ is unitary, it is enough to show ρ'_τ underlies a complex PVHS on $\mathscr{C}^\circ$. Since ρ'_τ is strongly cohomologically rigid and irreducible with finite determinant, this follows from Lemma 4.3.2.

We are thus in the following situation: $\rho : \pi_1(\Sigma_{g,n}) \to \mathrm{GL}_r(\mathscr{O}_K)$ is a representation such that for each $\tau : \mathscr{O}_K \hookrightarrow \mathbb{C}$, $\rho \otimes_{\mathscr{O}_K,\tau} \mathbb{C}$ is unitary. Such a representation has finite image by [LL23, Lemma 7.2.1]. □

8.5. *Reduction from the semisimple case to the unitary case.* In this section we will prove Theorem 1.2.1 for semisimple representations. The idea is to first deform our representation to a complex PVHS via non-abelian Hodge theory. Then we will argue that this complex PVHS is in fact unitary, using the results of [LL23], so that we can apply Proposition 8.4.1. Finally we will use the rigidity properties of unitary MCG-finite representations to argue that our deformation was in fact trivial—the deformed representation was in fact the one we started with.

LEMMA 8.5.1. *Let $\pi^\circ : \mathscr{C}^\circ \to \mathscr{M}$ be a punctured versal family of curves of genus g. Let $c \in \mathscr{C}^\circ$ be a point, set $m = \pi^\circ(c)$, and let $C^\circ = \mathscr{C}^\circ_m$. Let*

$$\rho_\infty : \pi_1(\mathscr{C}^\circ, c) \to \mathrm{GL}_r(\mathbb{C}[[t]])$$

be a representation with $r < \sqrt{g+1}$ and constant determinant, and let $\rho_0 = \rho_\infty \otimes \mathbb{C}$. Suppose $\rho_0|_{\pi_1(C^\circ,c)}$ is unitary. Then $\rho_\infty|_{\pi_1(C^\circ,c)}$ is conjugate to a constant representation.

Proof. We apply Proposition 3.2.1 to the short exact sequence

$$1 \to \pi_1(C^\circ, c) \to \pi_1(\mathscr{C}^\circ, c) \to \pi_1(\mathscr{M}, m) \to 1.$$

We must verify that for any n and any deformation

$$\gamma : \pi_1(\mathscr{C}^\circ, c) \to \mathrm{GL}_r(\mathbb{C}[t]/t^{n+1})$$

of ρ_0 with $\gamma|_{\pi_1(C^\circ,c)}$ constant, we have

$$H^0(\mathscr{M}, R^1\pi_* \mathrm{ad}(\gamma)) = 0.$$

But this follows from Theorem 6.2.1. Indeed,

$$\mathrm{rk}_{\mathbb{C}[t]/t^{n+1}} \mathrm{ad}(\gamma) = r^2 - 1 < g$$

as $r < \sqrt{g+1}$ by assumption. □

LEMMA 8.5.2. *Let $\pi^\circ : \mathscr{C}^\circ \to \mathscr{M}$ be a punctured versal family of curves of genus g. Let $c \in \mathscr{C}^\circ$ be a point, set $m = \pi^\circ(c)$, and let $C^\circ = \mathscr{C}^\circ_m$. Let X be a finite-type connected $\mathbb{C}$-scheme, and let*

$$\rho : \pi_1(\mathscr{C}^\circ, c) \to \mathrm{GL}_r(\mathscr{O}_X(X))$$

be a representation with $r < \sqrt{g+1}$ and constant determinant. For a closed point $x \in X$ with residue field $\kappa(x) = \mathbb{C}$, let $\rho_x := \rho \otimes_{\mathscr{O}_X(X)} \kappa(x)$. Suppose there exists a closed point $y \in X$ such that $\rho_y|_{\pi_1(C^\circ,c)}$ is unitary. Then for each closed point $x \in X$, $\rho_x|_{\pi_1(C^\circ,c)}$ is conjugate to $\rho_y|_{\pi_1(C^\circ,c)}$.

Proof. Without loss of generality X is a reduced (not necessarily irreducible) connected curve, as any closed point $x \in X$ is connected to y by such a curve. We claim it suffices to prove the statement if the parameter space X is smooth and irreducible.

To reduce to this case, suppose X is neither smooth nor irreducible, and let $\tilde{Y}$ be the normalization of the component Y of X containing y, and $\iota : \tilde{Y} \to X$ the natural map. By the case of smooth irreducible curves, $\iota^*\rho|_{\pi_1(C^\circ,c)} \otimes \kappa(y')$ is independent of $y' \in \tilde{Y}(\mathbb{C})$, and thus $\rho|_{\pi_1(C^\circ,c)} \otimes \kappa(y'')$ is independent of $y'' \in Y$ as well and, in particular, unitary for all y''. Let Z be a connected component of the closure of $X - \mathrm{im}(\iota)$. As X is connected, Z intersects Y non-trivially, say at some point $z \in Y$. But as z is in Y, $\rho_z|_{\pi_1(C^\circ,c)}$ is unitary, and hence we are done by induction on the number of components of X.

So we now assume X is a smooth connected curve. Let

$$\mathrm{Hom}(\pi_1(C^\circ, c), \mathrm{GL}_r(\mathbb{C}))$$

be the representation variety parametrizing r-dimensional complex representations of $\pi_1(C^\circ, c)$. There is a natural $\mathrm{GL}_r(\mathbb{C})$ action on $\mathrm{Hom}(\pi_1(C^\circ, c), \mathrm{GL}_r(\mathbb{C}))$ induced by conjugation. Let

$$f : X \to \mathrm{Hom}(\pi_1(C^\circ, c), \mathrm{GL}_r(\mathbb{C}))$$

be the map sending a point x to $\rho_x|_{\pi_1(C^\circ,c)}$. We wish to show that the image of f lies in a single $\mathrm{GL}_r(\mathbb{C})$-orbit, namely that of $\rho_y|_{\pi_1(C^\circ,c)}$.

Choose a local parameter t of X at y, so that $\widehat{\mathscr{O}_{X,y}} \simeq \mathbb{C}[[t]]$. Let

$$\rho_\infty : \pi_1(\mathscr{C}^\circ, c) \to \mathrm{GL}_r(\mathbb{C}[[t]])$$

be the corresponding representation. By Lemma 8.5.1, $\rho_\infty|_{\pi_1(C^\circ,c)}$ is conjugate to a constant representation. It follows that there exists a dense open subset $U \subset X$ containing y such that $f(U)$ is contained in the $\mathrm{GL}_r(\mathbb{C})$-orbit of $\rho_y|_{\pi_1(C^\circ,c)}$.

As $\rho_y|_{\pi_1(C^\circ,c)}$ is unitary, hence semisimple, its $\mathrm{GL}_r(\mathbb{C})$-orbit in

$$\mathrm{Hom}(\pi_1(C^\circ, c), \mathrm{GL}_r(\mathbb{C}))$$

is closed [Sik12, Th. 30]. Hence the closure $\overline{U}$ of U maps into the $\mathrm{GL}_r(\mathbb{C})$-orbit of $\rho_y|_{\pi_1(C^\circ,c)}$. But as X is irreducible, $\overline{U} = X$, completing the proof. □

THEOREM 8.5.3. *Let*

$$\rho : \pi_1(\Sigma_{g,n}) \to \mathrm{GL}_r(\mathbb{C})$$

be a semisimple MCG-finite representation, and suppose $r < \sqrt{g+1}$. *Then* ρ *has finite image.*

Proof. Observe that if ρ is a sum of irreducible representations, each of which have finite monodromy, then ρ has finite monodromy as well. As a summand of an MCG-finite representation is MCG-finite by Proposition 2.1.1, it suffices to treat the case that ρ is irreducible.

By Corollary 2.3.5, there exist a dominant étale map $\mathscr{M} \to \mathscr{M}_{g,n}$ and a local system $\mathbb{V}$ with finite determinant on the total space $\mathscr{C}^\circ$ of the associated family of n-times punctured curves of genus g, such that for C° a fiber of π°, $\mathbb{V}|_{C^\circ}$ has monodromy representation given by ρ. By Theorem 4.3.1, $\mathbb{V}$ admits a deformation with constant determinant to a local system $\mathbb{V}_0$ underlying a complex PVHS. Since $r < 2\sqrt{g+1}$, it follows from [LL23, Th. 1.2.12] that the local system $\mathbb{V}_0|_{C^\circ}$ is unitary. Moreover, $\mathbb{V}_0|_{C^\circ}$ is MCG-finite by Proposition 2.1.3.

It follows from Proposition 8.4.1 that $\mathbb{V}_0|_{C^\circ}$ has finite monodromy. To conclude, it is enough to show that $\mathbb{V}|_{C^\circ} = \mathbb{V}_0|_{C^\circ}$. But this is immediate from Lemma 8.5.2. □

8.6. *Completion of the non-semisimple case.* We come to the final step, where we verify the non-semisimple case of Theorem 1.2.1. The idea is to show that if we do have a non-semisimple MCG-finite representation of $\pi_1(\Sigma_{g,n})$ of low rank, we can produce a certain finite cover $\Sigma_{g',n'}$ of $\Sigma_{g,n}$, which violates our results toward the Putman-Wieland conjecture.

If ρ is a G-representation and $\rho' \subset \rho$ is a subrepresentation, we say that ρ' is *characteristic* if it is stable under all G-automorphisms of ρ (for example, the socle of ρ, or an isotypic component thereof).

LEMMA 8.6.1. *Let* ρ *be an MCG-finite representation of* $\pi_1(\Sigma_{g,n})$, *and let* $\rho_1 \subset \rho$ *be a semisimple, characteristic subrepresentation of* ρ. *Let* $\rho_2 = \rho/\rho_1$, *and suppose* ρ_2 *is semisimple as well. If* ρ_1, ρ_2 *have finite image and* $\mathrm{rk}\,\rho < \sqrt{g+1}$, *then* ρ *has finite image as well, i.e., the extension of* ρ_2 *by* ρ_1 *splits.*

Proof. If $g = 0$, ρ is the zero representation, so without loss of generality we may assume g is positive.

Decompose $\rho_2^\vee \otimes \rho_1$ as $\rho_2^\vee \otimes \rho_1 \simeq \oplus_i \sigma_i^{\oplus n_i}$, with the σ_i irreducible and pairwise non-isomorphic, and the n_i positive integers. Since $\dim \rho_1 + \dim \rho_2 = \dim \rho < \sqrt{g+1}$, we obtain from the AM-GM inequality that $\dim \rho_1 \cdot \dim \rho_2 < (\sqrt{g+1}/2)^2 = (g+1)/4$. Therefore, for every i,

$$n_i \cdot \dim \sigma_i < (g+1)/4$$

and hence $n_i, \dim \sigma_i < (g+1)/4$. To put ourselves in the setting of Theorem 7.2.1, we choose an additional basepoint $v \in \Sigma_{g,n}$ and let $\Sigma_{g,n+1} := \Sigma_{g,n} \setminus \{v\}$. Let $\Gamma \subset \mathrm{Mod}_{g,n+1}$ be a finite index subgroup stabilizing the conjugacy class of each σ_i.

Let $\Sigma_{g',n'} \to \Sigma_{g,n}$ be a finite Galois cover, with Galois group H, upon which the local systems corresponding to both ρ_1 and ρ_2 trivialize (for example, the cover defined by $\ker(\rho_1 \oplus \rho_2)$). It suffices to show that $\rho|_{\pi_1(\Sigma_{g',n'})}$ is trivial. As $\rho|_{\pi_1(\Sigma_{g',n'})}$ has abelian image, it factors through an H-equivariant map

$$H_1(\Sigma_{g',n'}) \to \rho_2^\vee \otimes \rho_1.$$

Equivalently, this is the extension class corresponding to ρ in

$$\begin{aligned}\mathrm{Ext}^1_{\pi_1(\Sigma_{g',n'})}(\rho_2, \rho_1) &= H^1\left(\pi_1(\Sigma_{g',n'}), \mathrm{Hom}(\rho_2, \rho_1)\right) \\ &= \mathrm{Hom}\left(\pi_1(\Sigma_{g',n'}), \mathrm{Hom}(\rho_2, \rho_1)\right).\end{aligned}$$

Projecting onto the σ_i-isotypic piece of $\rho_2^\vee \otimes \rho_1$ yields a Γ-stable quotient of $H_1(\Sigma_{g',n'})^{\sigma_i}$, or equivalently a Γ-stable subspace of $H^1(\Sigma_{g',n'})^{\sigma_i}$, of rank at most $(g+1)/4$. Call this subspace Q_i. But by Theorem 7.2.1, we have that any non-zero Γ-stable subspace of $H^1(\Sigma_{g',n'})^{\sigma_i}$ has rank at least $2g - 2\dim \sigma_i$, which satisfies the inequality

$$2g - 2\dim \sigma_i > 2g - (g+1)/2 = \frac{3g}{2} - \frac{1}{2} > (g+1)/4 > n_i \dim \sigma_i.$$

Hence Q_i equals zero. As this holds for all i, $\rho|_{\pi_1(\Sigma_{g',n'})}$ is trivial as desired. □

8.7. We finally complete the proof of our main theorem, that MCG-finite representations

$$\rho : \pi_1(\Sigma_{g,n}) \to \mathrm{GL}_r(\mathbb{C})$$

with $r < \sqrt{g+1}$ have finite image.

Proof of Theorem 1.2.1. The proof is by induction on the length of the socle filtration. The base case is Theorem 8.5.3. Now let ρ_1 be the socle of ρ. Because the socle of a representation is characteristic, both ρ_1 and ρ/ρ_1 are MCG-finite. Therefore, both ρ_1 and ρ/ρ_1 have finite image by induction. We conclude by Lemma 8.6.1. □

9. Consequences for arithmetic representations

The main arithmetic consequence of Theorem 1.2.1 is Theorem 9.1.2 below, which verifies a prediction of the Fontaine-Mazur conjecture, as we now explain. Throughout this section, we will no longer be working over $\mathbb{C}$.

9.1. *Application to relative Fontaine-Mazur.*

Definition 9.1.1. Let X be a variety over a finitely-generated field K with algebraic closure $\overline{K}$, and let

$$\rho : \pi_1^{\text{ét}}(X_{\overline{K}}) \to \mathrm{GL}_r(\overline{\mathbb{Q}}_\ell)$$

be a continuous representation. We say that ρ is *arithmetic* if its conjugacy class has finite orbit under the action of $\mathrm{Gal}(\overline{K}/K)$ on the set of isomorphism classes of $\pi_1^{\text{ét}}(X_{\overline{K}})$-representations. Here the action is induced by the natural outer action of $\mathrm{Gal}(\overline{K}/K)$ on $\pi_1^{\text{ét}}(X_{\overline{K}})$.

The relative form of the Fontaine-Mazur conjecture predicts that all semisimple arithmetic representations are of geometric origin [Pet23, Conj. 1 bis]. Our main arithmetic result, a straightforward application of Theorem 1.2.1, is that this is true for representations of low rank on the *generic* n-pointed curve of genus g. In fact, we show such representations are not only of geometric origin: they even have finite image.

THEOREM 9.1.2. *Let K be a finitely-generated field of characteristic zero and $(C, x_1, \dots, x_n)$ a smooth n-pointed geometrically connected curve of genus g over K, such that the corresponding map $\mathrm{Spec}(K) \to \mathscr{M}_{g,n,\mathbb{Q}}$ factors through the generic point. If $r < \sqrt{g+1}$, then any continuous arithmetic representation*

$$\rho : \pi_1^{\text{ét}}(C_{\overline{K}} \setminus \{x_1, \dots, x_n\}) \to \mathrm{GL}_r(\overline{\mathbb{Q}}_\ell)$$

has finite image.

We prove this at the end of Section 9.1 after spelling out some consequences.

Remark 9.1.3. Let $K, (C, x_1, \dots, x_n)$ be as in the theorem, and choose any embedding $K \hookrightarrow \mathbb{C}$. Let $C^{\text{an}} \setminus \{x_1, \dots, x_n\}$ be the associated Riemann surface. By [LL23, Lemma 7.5.1], Theorem 9.1.2 implies that arithmetic representations are not Zariski-dense in the $\overline{\mathbb{Q}}_\ell$-points of the character variety parametrizing semisimple representations of $\pi_1(C^{\text{an}} \setminus \{x_1, \dots, x_n\})$ of rank r for $1 < r < \sqrt{g+1}$. This answers negatively a question of Esnault and Kerz [EK20, Question 9.1(1)].

Recall that a local system $\mathbb{V}$ on a smooth complex variety X is of geometric origin if there is some Zariski open $U \subset X$ and a smooth proper morphism $f : Y \to U$ so that $\mathbb{V}|_U$ is a subquotient of $R^i f_* \mathbb{C}$ for some i.

COROLLARY 9.1.4. *Let K and $(C, x_1, \dots, x_n)$ be as in Theorem 9.1.2, and let ρ be a continuous representation*

$$\rho : \pi_1^{\text{ét}}(C_{\overline{K}} \setminus \{x_1, \dots, x_n\}) \to \mathrm{GL}_r(\overline{\mathbb{Q}}_\ell)$$

with $\dim \rho < \sqrt{g+1}$. *The following are equivalent*:

(1) *ρ has finite image*;
(2) *ρ is arithmetic*;
(3) *for any embedding $K \hookrightarrow \mathbb{C}$ and any isomorphism $\mathbb{C} \simeq \overline{\mathbb{Q}}_\ell$, the local system corresponding to $\rho_{\mathbb{C}}|_{C^{\mathrm{an}} \setminus \{x_1, \dots, x_n\}}$ on $C^{\mathrm{an}} \setminus \{x_1, \dots, x_n\}$ is of geometric origin*;
(4) *for any embedding $K \hookrightarrow \mathbb{C}$ and any isomorphism $\mathbb{C} \simeq \overline{\mathbb{Q}}_\ell$, the local system corresponding to $\rho_{\mathbb{C}}|_{C^{\mathrm{an}} \setminus \{x_1, \dots, x_n\}}$ on $C^{\mathrm{an}} \setminus \{x_1, \dots, x_n\}$ underlies an integral PVHS.*

Moreover, (1), (3), *and* (4) *are equivalent whenever* $\dim \rho < 2\sqrt{g+1}$.

Proof. The equivalence of (1) and (2) when $\dim \rho < \sqrt{g+1}$ holds by Theorem 9.1.2, and (1) certainly implies (3) and (4) for any value of $\dim \rho$. Next, (3) implies (4) for any value of $\dim \rho$ since any local system of geometric origin underlies an integral PVHS. This is well known, but see, for example, the proof of [LL23, Cor. 1.2.7].

To conclude, it remains to show that (4) implies (1) when $\dim \rho < 2\sqrt{g+1}$. This follows from [LL23, Cor. 1.2.7], by choosing an embedding $K \hookrightarrow \mathbb{C}$ corresponding to an analytically very general complex point of $\mathscr{M}_{g,n}$. By [LL23, Cor. 1.2.7], a local system on an analytically very general curve underlying an integral PVHS has finite monodromy whenever its dimension is $< 2\sqrt{g+1}$. Therefore, ρ also has finite monodromy. □

Remark 9.1.5. The relative Fontaine-Mazur conjecture predicts that all semisimple arithmetic local systems are of geometric origin. Hence on a generic curve, all semisimple arithmetic local systems ρ with $\dim \rho < 2\sqrt{g+1}$ should have finite monodromy by Corollary 9.1.4. In particular, this suggests one should be able to improve the bound of $\sqrt{g+1}$ in Theorem 9.1.2 to $2\sqrt{g+1}$ if one restricts to semisimple representations.

Note that arithmeticity of ρ on C over K is a stronger condition than having finite orbit under the action of the mapping class group after base change to $\mathbb{C}$, and so this does not mean the bound of Theorem 8.5.3 can be improved to $2\sqrt{g+1}$. Indeed, it cannot be improved, even in the semisimple case, when $g = 1$; see Remark 10.2.1.

Proof of Theorem 9.1.2. Let $C^\circ_{\overline{K}} := C_{\overline{K}} \setminus \{x_1, \dots, x_n\}$. For R a ring, let $\mathscr{M}_{g,n,R}$ denote the moduli stack of genus g, n-pointed curves over $\operatorname{Spec} R$ and let $\pi^\circ_R : \mathscr{C}^\circ_R \to \mathscr{M}_{g,n,R}$ denote the universal n-punctured genus g curve over R. We use C°_R to denote a fiber of π°_R.

We claim (suppressing basepoints) that there is a map of exact sequences of étale fundamental groups

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \pi_1^{\text{ét}}(C^\circ_{\overline{K}}) & \longrightarrow & \pi_1^{\text{ét}}(C^\circ_K) & \longrightarrow & \pi_1^{\text{ét}}(\operatorname{Spec} K) & \longrightarrow & 0 \\ & & \downarrow \simeq & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \pi_1^{\text{ét}}(C^\circ_{\overline{K(M_{g,n,\mathbb{Q}})}}) & \longrightarrow & \pi_1^{\text{ét}}(\mathscr{C}^\circ_{\mathbb{Q}}) & \longrightarrow & \pi_1^{\text{ét}}(\mathscr{M}_{g,n,\mathbb{Q}}) & \longrightarrow & 0. \end{array} \tag{9.1}$$

Indeed, the first sequence is the standard homotopy exact sequence; see [Stacks, Tag 0BTX]. The second one is less standard but, using [Stacks, Tag 0BTX] again, its exactness can be reduced to verifying exactness of

$$0 \longrightarrow \pi_1^{\text{ét}}(C^\circ_{\overline{K(M_{g,n,\mathbb{Q}})}}) \longrightarrow \pi_1^{\text{ét}}(\mathscr{C}^\circ_{\mathbb{Q}}) \longrightarrow \pi_1^{\text{ét}}(\mathscr{M}_{g,n,\overline{\mathbb{Q}}}) \longrightarrow 0. \tag{9.2}$$

By comparison of fundamental groups of algebraically closed fields of characteristic 0, it is enough to verify exactness of the sequence with all fields in subscripts replaced by the complex numbers. The version for topological fundamental groups is given in Lemma 2.0.1. Since the étale fundamental group is the profinite completion of the topological fundamental group [GR71, Exp. XII, Cor. 5.2], and profinite completions of exact sequences where the left term has trivial center remain exact [And74, Prop. 3], we obtain exactness of (9.2).

We next use (9.1) to deduce that ρ has finite orbit under the natural action of $\pi_1(\mathscr{M}_{g,n,\mathbb{Q}})$, to be constructed below. The two exact sequences in (9.1) induce compatible outer actions of $\pi_1(\operatorname{Spec} K)$ and $\pi_1(\mathscr{M}_{g,n,\mathbb{Q}})$ on $\pi_1(C^\circ_{\overline{K(M_{g,n,\mathbb{Q}})}})$. Hence, the outer action of $\pi_1(\operatorname{Spec} K)$ factors through that of $\pi_1(\mathscr{M}_{g,n,\mathbb{Q}})$. Because $\operatorname{Spec} K \to \mathscr{M}_{g,n,\mathbb{Q}}$ is dominant, the induced map $\pi_1(\operatorname{Spec} K) \to \pi_1(\mathscr{M}_{g,n,\mathbb{Q}})$ has image of finite index. In particular, the orbit of the conjugacy class of ρ under $\pi_1^{\text{ét}}(\mathscr{M}_{g,n,\mathbb{Q}}, \overline{\eta})$, and hence under $\pi_1^{\text{ét}}(\mathscr{M}_{g,n,\overline{\mathbb{Q}}}, \overline{\eta})$, is finite.

Choose an embedding $\overline{K} \hookrightarrow \mathbb{C}$, and let $(\mathscr{C}^\circ)^{\text{an}}$ be the corresponding punctured Riemann surface, so that, by residual finiteness of surface groups, there is a natural inclusion $\pi_1((\mathscr{C}^\circ)^{\text{an}}) \hookrightarrow \pi_1^{\text{ét}}(\mathscr{C}^\circ_{\overline{K}})$. As the conjugacy class of ρ has finite orbit under $\pi_1^{\text{ét}}(\mathscr{M}_{g,n,\overline{\mathbb{Q}}}, \overline{\eta}) \simeq \pi_1^{\text{ét}}(\mathscr{M}_{g,n,\mathbb{C}}, \overline{\eta}_{\mathbb{C}})$, the standard comparison theorems between étale and topological π_1 [GR71, Exp. XII, Cor. 5.2] imply that $\rho|_{\pi_1((\mathscr{C}^\circ)^{\text{an}})}$ has finite orbit under the topological fundamental group $\pi_1(\mathscr{M}^{\text{an}}_{g,n,\mathbb{C}}, \overline{\eta}_{\mathbb{C}})$. Hence $\rho|_{\pi_1((\mathscr{C}^\circ)^{\text{an}})}$ is MCG-finite, and hence has finite image by Theorem 1.2.1, which may be applied after choosing an isomorphism between $\overline{\mathbb{Q}}_\ell$ and $\mathbb{C}$. But $\pi_1((\mathscr{C}^\circ)^{\text{an}})$ is dense in $\pi_1^{\text{ét}}(\mathscr{C}^\circ_{\overline{K}})$, so ρ also has finite image. □

9.2. *Application to lifting residual representations.* Let $K, (C, x_1, \dots, x_n)$ be as in Theorem 9.1.2. We next explain why there are residual representations of $\pi_1^{\text{ét}}(C_{\overline{K}} \setminus \{x_1, \dots, x_n\})$ that do not lift to arithmetic representations in characteristic zero. In particular, in Example 9.2.3 we construct residual

representations of geometric fundamental groups that are not "of geometric origin," as we now define. To our knowledge these are the first such examples.

Definition 9.2.1. Let K be an algebraically closed field, and let X/K be a variety. Let $\mathbb{F}$ be a finite field of characteristic different from that of K. Let L be the fraction field of the Witt vectors $W(\mathbb{F})$. We say that a continuous representation

$$\rho : \pi_1^{\text{ét}}(X_{\overline{K}}) \to \mathrm{GL}_r(\mathbb{F})$$

is *of geometric origin* if there exists a continuous representation

$$\xi : \pi_1^{\text{ét}}(X_{\overline{K}}) \to \mathrm{GL}_r(\overline{L})$$

such that

(1) there exists a ξ-stable $\mathscr{O}_{\overline{L}}$-lattice $W \subset \overline{L}^r$ with $W \otimes \overline{\mathbb{F}} \simeq \rho \otimes \overline{\mathbb{F}}$;
(2) there exist a dense open subscheme $U \subset X$ and a smooth proper morphism $\pi : Y \to U$, such that $\xi|_{\pi_1^{\text{ét}}(U_{\overline{K}})}$ arises as a subquotient of the monodromy representation of $R^i\pi_*\overline{L}$ for some $i \geq 0$.

In other words, we say a residual representation is of geometric origin if it arises as the reduction of a characteristic zero representation of geometric origin.

Corollary 9.2.2. *Let $K, (C, x_1, \ldots, x_n)$ be as in Theorem 9.1.2. Then for $1 < r < \sqrt{g+1}$ and $p \gg_r 0$, no surjective representation*

$$\rho : \pi_1^{\text{ét}}(C_{\overline{K}} \setminus \{x_1, \ldots, x_n\}) \to \mathrm{GL}_r(\mathbb{F}_p)$$

admits an arithmetic lift to characteristic zero. In particular, no such representation is of geometric origin.

Proof. By definition, a mod p representation of geometric origin lifts to a representation of geometric origin over $\overline{\mathbb{Q}}_p$. As representations of geometric origin are arithmetic, the second statement follows from the first.

Thus it suffices to show that for $p \gg 0$ and ρ as in the theorem, ρ does not admit an arithmetic lift. Suppose to the contrary that it did admit an arithmetic lift ξ; by Theorem 9.1.2, ξ would have finite image. Thus it suffices to show that for Q a finite, totally ramified extension of $\mathbb{Q}_p$, there do not exist any finite subgroups of $\mathrm{GL}_r(\mathscr{O}_Q)$ surjecting onto $\mathrm{GL}_r(\mathbb{F}_p)$ if $p \gg_r 0$.

This follows from Jordan's theorem on finite subgroups of $\mathrm{GL}_r(Q)$, where Q is any field of characteristic zero ([Jor78, p. 91] or [CR06, Th. 36.13]). Recall that Jordan's theorem says that there exists some constant $n(r)$ such that if $G \subset \mathrm{GL}_r(Q)$ is a finite subgroup, G contains an abelian normal subgroup of index dividing $n(r)$. In particular, for any $g_1, g_2 \in G$, we have

$$[g_1^{n(r)}, g_2^{n(r)}] = I_r,$$

where I_r is the $r \times r$ identity matrix.

Now define $\overline{g_1}, \overline{g_2} \in \mathrm{GL}_r(\mathbb{F}_p)$ by

$$\overline{g_1} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \oplus I_{r-2}, \qquad \overline{g_2} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \oplus I_{r-2}.$$

Direct computation shows that $[\overline{g_1}^{n(r)}, \overline{g_2}^{n(r)}] \neq I_r$ as long as p does not divide $n(r)$. But if $\mathrm{GL}_r(\mathbb{F}_p)$ was the image of a finite subgroup of $\mathrm{GL}_r(\mathscr{O}_Q)$, this equality would hold by Jordan's theorem, giving the claim. Indeed, we have shown the result for all p not dividing the constant $n(r)$ from Jordan's theorem. □

Using Corollary 9.2.2, we now construct examples of residual representations with no arithmetic lifts.

Example 9.2.3. Fix non-negative integers g, n so that $n \geq 1$ if $g = 1$ and $n \geq 3$ if $g = 0$. Fix r with $1 < r < \sqrt{g+1}$. We claim that for any n-pointed genus g curve $(C, x_1, \ldots, x_n)$, and any p, there exist surjective representations

$$\rho : \pi_1^{\text{ét}}(C_{\overline{K}} \setminus \{x_1, \ldots, x_n\}) \to \mathrm{GL}_r(\mathbb{F}_p)$$

as in Corollary 9.2.2. In particular, these representations do not admit arithmetic lifts to characteristic 0.

If $n > 0$, maintaining the assumption $n \geq 3$ when $g = 0$, we have that $\pi_1^{\text{ét}}(C_{\overline{K}} \setminus \{x_1, \ldots, x_n\})$ is profinite free on at least two generators. The claim in this case follows from the fact that $\mathrm{GL}_r(\mathbb{F}_p)$ is generated by two elements [Wat89].

We now deal with the case $n = 0$ and $g \geq 2$. Indeed, this follows from the fact that $\pi_1^{\text{ét}}(C_{\overline{K}})$ surjects onto a free profinite group on g generators by writing it as the profinite completion of $\langle a_1, \ldots, a_g, b_1, \ldots, b_g \rangle / \prod_{i=1}^{g} [a_i, b_i]$ and considering the map to the free profinite group generated by $c_1, \ldots, c_g$ sending $a_i \mapsto c_i, b_i \mapsto \mathrm{id}$.

We conclude the section with several remarks describing consequences of the above non-liftable residual representations.

Remark 9.2.4. Corollary 9.2.2 provides a counterexample to a particularly optimistic extension of de Jong's conjecture [dJ01, Conj. 2.3, Th. 3.5]. De Jong's conjecture, proven by Gaitsgory [Gai07], implies that an absolutely irreducible residual $\mathbb{F}_q$-representations of geometric fundamental groups of smooth curves over finite fields of characteristic not dividing q always lift to arithmetic representations over a field of characteristic zero. One might naturally ask if the same is true for curves over arbitrary finitely-generated fields; Corollary 9.2.2 shows that this is not the case.

Remark 9.2.5. The solution to de Jong's conjecture, as described in Remark 9.2.4, implies that deformation rings of absolutely irreducible $\mathbb{F}_p$-representations of arithmetic fundamental groups of curves over finite fields (of characteristic different from p) are always complete intersections over $\mathbb{Z}_p$. Our results

show that the analogous statement is not true for the arithmetic fundamental group $\pi_1^{\text{ét}}(C_K \setminus \{x_1, \dots, x_n\})$ as in Theorem 9.1.2 because they are not flat over $\mathbb{Z}_p$ by Example 9.2.3.

Remark 9.2.6. Flach asks [CO05, p. 7] if deformation rings of absolutely irreducible residual representations of profinite groups are always complete intersections. By now it is well known that the answer to this question is in general "no." (See, e.g., [EM16] for a more or less complete answer to this question, and the references therein.) Our result Corollary 9.2.2 shows that this question has a negative answer even for arithmetic fundamental groups of generic smooth curves, as explained in Remark 9.2.5.

10. **Questions and examples**

10.1. *Rigidity questions for mapping class groups.* One of the key ingredients of our arguments is the statement that certain representations of (finite index subgroups of) $\text{Mod}_{g,n}$ are rigid.

Question 10.1.1. Let $g \gg 0$. Is every irreducible complex representation of every finite index subgroup of $\text{Mod}_{g,n}$ rigid?

The answer is, perhaps, plausibly "yes" if one accepts the well-known analogy between $\text{Mod}_{g,n}$ and lattices in simple Lie groups of rank > 1, as all complex representations of such lattices are rigid. For g small, there are examples of irreducible non-rigid representations of $\text{Mod}_{g,n}$ (for example, $\pi_1(\mathscr{M}_{0,4})$ is free on two generators).

Remark 10.1.2. A positive answer to Question 10.1.1 would imply Ivanov's conjecture that $\text{Mod}_{g,n}$ does not virtually surject onto $\mathbb{Z}$ as stated in [Iva06, §7] and [Kir97, Prob. 2.11.A]. Indeed, suppose some finite index subgroup $\Gamma \subset \text{Mod}_{g,n}$ admitted a surjection $f : \Gamma \twoheadrightarrow \mathbb{Z}$. For $t \in \mathbb{C}^\times$, the representation

$$\begin{aligned} \rho_t : \mathbb{Z} &\to \mathbb{C}^\times, \\ n &\mapsto t^n \end{aligned}$$

is a non-trivial family of one-dimensional representations of $\mathbb{Z}$. Then the representation $\rho_t \circ f$ is a non-constant family of irreducible representations of Γ.

A positive answer to Question 10.1.1 would also imply a positive answer to the following question:

Question 10.1.3. Let $g \gg 0$. Let $\Gamma \subset \text{Mod}_{g,n}$ be a finite index subgroup, and let $X_r(\pi_1(\Sigma_{g,n}))$ be the character variety parametrizing r-dimensional semisimple representations of $\pi_1(\Sigma_{g,n})$. Is the fixed locus $X_r(\pi_1(\Sigma_{g,n}))^\Gamma$ finite?

Remark 10.1.4. One reason to believe Question 10.1.3 has a positive answer is that the profinite analogue does. Namely, let X be any normal connected algebraic variety over $\mathbb{C}$, and let $\pi_1^{\text{ét}}(X)$ be its profinite fundamental group. Let Γ be a finite index subgroup of $\mathrm{Out}(\pi_1^{\text{ét}}(X))$. Then Γ acts on the set of conjugacy classes of continuous representations

$$\pi_1^{\text{ét}}(X) \to \mathrm{GL}_r(\overline{\mathbb{Q}}_\ell).$$

It follows from [Lit21, Rem. 1.1.4] that the Γ-fixed points are *discrete* in the ℓ-adic topology. (The result there is stated for curves, but one may reduce to this case by the Lefschetz hyperplane theorem.)

If one accepts in addition Simpson's motivicity conjecture for rigid local systems [Sim92, Conj. on p. 9], a positive answer to Question 10.1.1 would imply that every semisimple local system on $\mathscr{M}_{g,n}$ is of geometric origin for $g \gg 0$. This would be of particular interest for those local systems not obviously of geometric origin, e.g., the local systems arising from TQFT constructions. Thus one might ask

Question 10.1.5. For $g \gg 0$, is every semisimple local system on $\mathscr{M}_{g,n}$ of geometric origin? Let C be a Riemann surface of genus $g \gg 0$. Is every semisimple MCG-finite representation of $\pi_1(C)$ of geometric origin?

Example 10.1.6. The analogy between the representation theory of mapping class groups and the representation theory of lattices in simple Lie groups of rank greater than one only goes so far. Indeed, mapping class groups typically admit non-rigid reducible representations, as we now explain.

(1) Let $H = H_1(\Sigma_g, \mathbb{Q})$. Morita [Mor93] produces a non-zero class

$$\sigma \in H^1(\mathrm{Mod}_g, (\wedge^3 H)/H),$$

closely related to the Johnson homomorphism, yielding a non-split extension of Mod_g-representations

$$1 \to \wedge^3 H/H \to W \to \mathbb{Q} \to 1.$$

The representation W is evidently a non-trivial deformation of $\mathbb{Q} \oplus \wedge^3 H/H$, and hence this latter representation is non-rigid.

(2) An essentially identical but perhaps slightly simpler argument gives examples for punctured surfaces. For $n > 0$, the group $\mathrm{PMod}_{g,n}$ acts on $\pi_1(\Sigma_{g,n-1}, x)$, and hence on the group algebra $\mathbb{Q}[\pi_1(\Sigma_{g,n-1}, x)]$. Let $\mathscr{I} \subset \mathbb{Q}[\pi_1(\Sigma_{g,n-1}, x)]$ be the augmentation ideal. Then direct computation shows that the short exact sequence of $\mathrm{PMod}_{g,n}$-representations

$$0 \to \mathscr{I}^2/\mathscr{I}^3 \to \mathscr{I}/\mathscr{I}^3 \to \mathscr{I}/\mathscr{I}^2 \to 0$$

does not split, and hence the representation $\mathscr{I}/\mathscr{I}^2 \oplus \mathscr{I}^2/\mathscr{I}^3$ is not rigid.

(3) For another example, one may consider the non-torsion class in

$$H^1(\mathrm{Mod}_{g,1}, H^1(\Sigma_g, \mathbb{Z}))$$

constructed in [Mor89, Prop. 6.4], which yields a non-split extension of $\mathrm{Mod}_{g,1}$-representations

$$1 \to H^1(\Sigma_g, \mathbb{Q}) \to W \to \mathbb{Q} \to 1.$$

One may make the restriction to the point-pushing subgroup $W|_{\pi_1(\Sigma_g)}$ explicit as follows. Let

$$a : \pi_1(\Sigma_g) \to \pi_1(\Sigma_g)^{\mathrm{ab}} \otimes \mathbb{Q} \simeq H_1(\Sigma_g, \mathbb{Q})$$

be the map induced by the Hurewicz isomorphism $H_1(\Sigma_g, \mathbb{Z}) = \pi_1(\Sigma_g)^{\mathrm{ab}}$. Choose a splitting $W = H_1(\Sigma_g, \mathbb{Q}) \oplus \mathbb{Q}e$, where we view $H_1(\Sigma_g, \mathbb{Q})$ as isomorphic to $H^1(\Sigma_g, \mathbb{Q})$ via the universal coefficient theorem and Poincaré duality. Then $\pi_1(\Sigma_g)$ acts trivially on $H_1(\Sigma_g, \mathbb{Q})$, and an element $\gamma \in \pi_1(\Sigma_g)$ acts on e via the map

$$e \mapsto e + a(\gamma).$$

Remark 10.1.7. For a related discussion of ways mapping class groups *do not* behave like lattices in simple Lie groups of rank greater than one; see [FLM01]. Interestingly, [FLM01, Th. 1.6] shows there are no faithful linear representations of finite index subgroups of $\mathrm{Mod}_{g,0}$ of dimension $< 2\sqrt{g-1}$. A related result follows from our work here, as we now explain.

Let $\Gamma \subset \mathrm{PMod}_{g,n+1}$ be a finite-index subgroup containing the point-pushing subgroup. It follows from Theorem 1.2.1, Lemma 2.4.1(1), and Proposition 2.1.2 that if ρ is a representation of Γ of rank $< \sqrt{g+1}$

(1) whose restriction to the point-pushing subgroup is irreducible, and
(2) with finite determinant,

then ρ has finite image. Although the two results are not comparable, it is interesting that the bound in [FLM01] is also asymptotic to $\sqrt{g}$.

10.2. *Bounds and examples.* It is natural to ask how sharp the bound of $\sqrt{g+1}$ in Theorem 1.2.1 is. We have no reason to believe it is sharp. That said, one cannot expect a bound that is much stronger, as the extension W constructed in Examples 10.1.6(3) yields a non-trivial unipotent representation of $\pi_1(\Sigma_g)$ of rank $2g+1$, fixed by the action of Mod_g, namely $W|_{\pi_1(\Sigma_g)}$.

Remark 10.2.1. The bounds of many of our main results are sharp when $g = 1$. In $g = 1$, there are semisimple 2-dimensional "special dihedral" MCG-finite representations of $\pi_1(\Sigma_{1,n})$ for $n > 0$ [BGMW22, Th. B], with image contained in the infinite dihedral group. This shows sharpness of Theorem 1.2.1 and also the semisimple case, Theorem 8.5.3. Moreover, there

exist 2-dimensional non-semisimple MCG-finite representations of $\pi_1(\Sigma_{1,n})$ for $n > 0$; see [CH21, Th. B]. This implies sharpness of the bound in Lemma 8.6.1.

Note that the "special dihedral" MCG-finite representations of [BGMW22, Th. B] are not in general arithmetic in the sense of Definition 9.1.1, as in general their local monodromy at infinity is not quasi-unipotent.

Remark 10.2.2. On the other hand, the bound $r < \sqrt{g+1}$ in Theorem 1.2.1 is not sharp when $g = 2$ and $g = 3$. Indeed, to verify this, we only need show there are no 2-dimensional MCG-finite representations with infinite image. There are no non-semisimple such representations by [CH21, Th. B], and so it only remains to show there are no irreducible 2-dimensional representations. Any such representation $\rho : \pi_1(\Sigma_{g,n}) \to \mathrm{GL}_2(\mathbb{C})$ has finite determinant by Lemma 2.3.2. Replacing ρ by $\rho \otimes \det \rho^{-1/2}$, we would obtain another MCG-finite 2-dimensional representation with infinite image factoring through $\mathrm{SL}_2(\mathbb{C})$. No such representations exist by [BGMW22, Th. A].

Question 10.2.3. Let $g, n \geq 0$. What is the minimal rank of an MCG-finite representation of $\pi_1(\Sigma_{g,n})$ with infinite image?

It is plausible that much stronger bounds than $\sqrt{g+1}$ hold in Theorem 1.2.1 if one assumes the representation is semisimple (see Figure 1).

Example 10.2.4. We know of examples of MCG-finite semisimple representations of $\pi_1(\Sigma_g)$ with infinite image, of rank exponential in g. For example, one may via TQFT techniques construct representations of $\mathrm{Mod}_{g,n}$ of rank exponential in g [KS16, Cor. 4.3], [BKMS18, Th. 5.1]; these representations are non-trivial when restricted to the point-pushing subgroup, and hence by Proposition 2.1.2 yield MCG-finite representations. Similarly, representations of $\mathrm{Mod}_{g,n}$ constructed via variants of the Kodaira-Parshin trick [LL22, Exam. 3.3.1] (see also [LL23, Prop. 5.1.1 and Rem. 5.1.2]) are semisimple of rank exponential in g and restrict to MCG-finite, semisimple representations of the point-pushing subgroup, with infinite image.

Example 10.2.5. In genus zero, there is a huge collection of MCG-finite representations, namely the *rigid local systems* studied by Katz [Kat96]. The use of rigidity in our proof of Theorem 1.2.1, and Question 10.1.1, suggest that the study of MCG-finite representations is a natural generalization of the study of rigid local systems to the higher genus setting.

Question 10.2.6. Let $g, n \geq 0$. What is the minimal rank of a semisimple MCG-finite representation of $\pi_1(\Sigma_{g,n})$ with infinite image?

In our view it would be extremely interesting to give an improved bound for semisimple MCG-finite representations, and to produce fundamentally new examples of semisimple MCG-finite representations.

Remark 10.2.7 (Bounds for free groups). A variant of the construction in Examples 10.1.6(3) gives an example of a representation of the free group F_N on N generators of rank $N+1$, whose conjugacy class is fixed by the action of $\mathrm{Out}(F_N)$; see [PR00, p. 1444]. For example, when $N = 2$, this representation is given by the matrices

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

Thus again the bound of $\sqrt{g+1}$ in Corollary 1.6.1 cannot be improved too much further if one allows non-semisimple representations. However it is in principle possible that there is *no* semisimple representation of F_N with infinite image whose conjugacy class has finite orbit under $\mathrm{Out}(F_N)$ when $N \geq 3$. Indeed, this would follow from a conjecture of Grunewald and Lubotzky (see [GL09, §9.2]) using the main result of [FH17].

There are interesting semisimple representations of F_2 whose conjugacy class has finite orbit under $\mathrm{Out}(F_2)$. This follows, for example, from the linearity of $\mathrm{Aut}(F_2)$ [Kra00], e.g., by choosing a faithful representation of $\mathrm{Aut}(F_2)$ and restricting to the inner automorphisms, and then semisimplifying. Alternately, one may construct examples by applying the Kodaira-Parshin trick to construct representations of $\mathrm{Mod}_{1,2}$, which is commensurable with $\mathrm{Aut}(F_2)$.

As remarked in the introduction, we conjecture the opposite of Grunewald and Lubotzky:

Conjecture 10.2.8. For all $N > 0$, there exist semisimple representations of F_N with infinite image, whose conjugacy class has finite orbit under $\mathrm{Out}(F_N)$.

Our main impetus for this conjecture is the existence of many interesting MCG-finite representations of $\pi_1(\Sigma_{g,n})$; see Example 10.2.4. In our view this conjecture is of great importance, both because of the intrinsic interest of "canonical" representations of F_N (i.e., those with finite orbit under $\mathrm{Out}(F_N)$), and because of its relationship to the representation theory of $\mathrm{Aut}(F_N)$.

Remark 10.2.9. We briefly spell out the the relationship between representations of F_N whose conjugacy class has finite orbit under $\mathrm{Out}(F_N)$, and the representation theory of $\mathrm{Aut}(F_N)$. If $\Gamma \subset \mathrm{Aut}(F_N)$ is a finite index subgroup containing the inner automorphisms, and

$$\rho : \Gamma \to \mathrm{GL}_r(\mathbb{C})$$

is a representation, then the conjugacy class of $\rho|_{F_N}$ has finite orbit under $\mathrm{Out}(F_N)$, via a proof analogous to that of Proposition 2.1.2. Conversely, as in to the proof of Lemma 2.2.2, the projectivization of any irreducible F_N-representation whose conjugacy class has finite orbit under $\mathrm{Out}(F_N)$ arises

from a projective representation of a finite index subgroup of $\mathrm{Aut}(F_N)$. See [BKP19, Th. 7.12] for a related result about low-rank projective representations of certain finite index subgroups of $\mathrm{Aut}(F_n)$.

10.3. *Classification.* The following is evidently quite difficult, but in our view is of great interest:

Question 10.3.1. Can one classify MCG-finite representations of $\pi_1(\Sigma_{g,n})$ of rank r? Equivalently, can one classify algebraic solutions to the rank r isomonodromy differential equations over $\mathscr{M}_{g,n}$? Can one classify representations of the free group F_N on N generators whose conjugacy class has finite orbit under $\mathrm{Out}(F_N)$?

Remark 10.3.2. It would also be extremely interesting to learn of any new sources of *examples* of MCG-finite representations. For example, are there MCG-finite representations $\pi_1(\Sigma_{g,n}) \to \mathrm{SL}_r(\mathbb{C})$ with (Zariski-)dense image for r arbitrarily large?

Appendix A. Proof of Section 5.1.6

We now explain the proof of Theorem 5.1.6, which loosely follows [Voi07, §9 and §10]. We retain notation as in Section 5.

A.1. *Preparatory Lemmas.* Recall that the derivative of the period map dP was a map

$$dP : T_{\mathscr{B}} \to P^* T_{\mathrm{Gr}(\mathrm{rk}\, F^1\mathscr{H}, \mathrm{rk}\,\mathscr{H})} \simeq (F^1\mathscr{H})^\vee \otimes (\mathscr{H}/F^1\mathscr{H}).$$

By adjointness, we obtain from dP a map

$$dP' : F^1\mathscr{H} \to (\mathscr{H}/F^1\mathscr{H}) \otimes T^\vee_{\mathscr{B}}.$$

Recall that in Section 5.1 we defined a map $\overline{\nabla}$ with the same source and target.

Lemma A.1.1. *We have an equality* $\overline{\nabla} = dP'$.

Proof. Let W be a vector space and $K \subset W$ a subspace of rank r. It is shown in the course of the proof of [Voi07, Lemma 10.7] that for $[K] \in \mathrm{Gr}(r, W)$, the identification $T_{\mathrm{Gr}(r,W),[K]} \simeq \mathrm{Hom}(K, W/K)$ is given as follows. Let $\mathscr{S} \subset \mathscr{O}_{\mathrm{Gr}(r,W)} \otimes W$ denote the universal subbundle. Given $\sigma \in K$, let $\widetilde{\sigma}$ denote a choice of holomorphic section of $\mathscr{S}$ defined on a neighborhood of $[K] \in \mathrm{Gr}(r, W)$ with $\widetilde{\sigma}([K]) = \sigma$. Then, the map

$$T_{\mathrm{Gr}(r,W),[K]} \to \mathrm{Hom}(K, W/K),$$
$$u \mapsto \left(\sigma \mapsto \frac{\partial}{\partial u}(\widetilde{\sigma}) \bmod K\right)$$

is well defined and independent of the choice of lift $\widetilde{\sigma}$.

Choosing $b \in \mathscr{B}$, the derivative of the period map dP'_b is then given as follows: send $\sigma \in F^1\mathscr{H}_b$ to the function that sends $v \in T_{\mathscr{B},b}$ to $\frac{\partial}{\partial v}\widetilde{\sigma} \bmod F^1\mathscr{H}_b$, for $\widetilde{\sigma}$ a local holomorphic lift of σ to $F^1\mathscr{H}$. Here $\frac{\partial}{\partial v}$ makes sense as we have chosen a flat trivialization of $\mathscr{H}$, but we may equivalently write this map as

$$\sigma \mapsto (v \mapsto \nabla_{GM}(\widetilde{\sigma})(v) \bmod F^1\mathscr{H}_b),$$

or equivalently

$$\sigma \mapsto (v \mapsto \overline{\nabla}(\widetilde{\sigma})(v)). \qquad \square$$

After introducing some notation, we next give an explicit computation of the Gauss-Manin connection ∇_{GM} on $\mathscr{H}$.

Notation A.1.2. Recall from Notation 5.1.1 that $(\mathscr{E}, \nabla)$ denotes the Deligne canonical extension of $(\mathbb{V} \otimes \mathscr{O}_{\mathscr{C}^\circ}, \mathrm{id} \otimes d)$ to $\mathscr{C}$. Let $\mathcal{A}^{i,j}_{\log \mathscr{D}}(\mathscr{E})$ be the sheaf of C^∞ logarithmic (i,j)-forms valued in $\mathscr{E}$, and let $\mathcal{A}^n_{\log \mathscr{D}}(\mathscr{E}) = \oplus_{i+j=n}\mathcal{A}^{i,j}_{\log \mathscr{D}}(\mathscr{E})$. Then, $(\mathcal{A}^{i,\bullet}_{\log \mathscr{D}}(\mathscr{E}), \overline{\partial})$ is the Dolbeault complex of $\mathscr{E} \otimes \Omega^i_{\mathscr{C}}(\log \mathscr{D})$. The complex $(\mathcal{A}^\bullet_{\log \mathscr{D}}(\mathscr{E}), \nabla + \overline{\partial})$ is the de Rham resolution of the unitary local system $\mathbb{V}$. We similarly define the relative variants $\mathcal{A}^{i,j}_{\log \mathscr{D}, \mathscr{C}/\mathscr{B}}(\mathscr{E})$, $\mathcal{A}^n_{\log \mathscr{D}, \mathscr{C}/\mathscr{B}}(\mathscr{E})$.

Notation A.1.3. For v a C^∞ vector field on an open $U \subset \mathscr{C}$, tangent to $\mathscr{D}$, and w a C^∞ section of $\mathcal{A}^i_{\log \mathscr{D}}(\mathscr{E})$ on U, the interior product $\mathrm{int}(v)(w) \in H^0(U, \mathcal{A}^{i-1}_{\log \mathscr{D}}(\mathscr{E}))$ is defined by

$$\mathrm{int}(v)(w)(X_1, \ldots, X_{i+j-1}) = w(v, X_1, \ldots, X_{i+j-1})$$

for any C^∞ vector fields $X_1, \ldots, X_{i+j-1}$ on U tangent to $\mathscr{D}$.

Lemma A.1.4. *Let $\sigma \in \mathcal{A}^1_{\log \mathscr{D}}(\mathscr{E})$ be a differential form such that for each $b' \in \mathscr{B}$, $\sigma|_{\mathscr{C}_{b'}}$ is closed, so that σ represents a section $[\sigma]$ of $\mathscr{H}$. Fix $b \in \mathscr{B}$ and a tangent vector $u \in T_{\mathscr{B},b}$, and let v be a C^∞ section of $T_{\mathscr{C}}|_{\mathscr{C}_b}$, tangent to $\mathscr{D}$ with $u = \pi_*(v)$. We have*

$$\nabla_{GM}([\sigma])|_b(u) = [(\mathrm{int}(v)((\nabla + \overline{\partial})(\sigma)|_{\mathscr{C}_b})] \in \mathscr{H}_b.$$

Proof. This proof essentially follows [Voi07, Prop. 9.14]. After possibly shrinking $\mathscr{B}$, which will not alter the statement of this lemma, we may write $\mathscr{C} \simeq \mathscr{C}_b \times \mathscr{B}$ (as C^∞ manifolds), by Ehresmann's theorem. Let $t_1, \ldots, t_m$ be C^∞ functions on $C_b \times \mathscr{B}$ pulled back from a system of local coordinates for $\mathscr{B}$. After possibly shrinking $\mathscr{B}$, we can write σ as

$$\sigma = \Phi + \sum_{i=1}^m dt_i \otimes \psi_i,$$

where Φ is a section of $\mathscr{A}^1_{\log \mathscr{D}}(\mathscr{E})$ independent of dt_i, each ψ_i is a C^∞ section of $\mathscr{E}$, and $\Phi|_{\mathscr{C}_b} = \sigma|_{\mathscr{C}_b}$. Since σ is fiber-wise closed, applying $\nabla + \overline{\partial}$ gives

$$(\nabla + \overline{\partial})\sigma = \sum_i dt_i \operatorname{int}(\partial/\partial t_i)((\nabla + \overline{\partial})\Phi) - \sum_i dt_i \wedge (\nabla + \overline{\partial})\psi_i$$

and hence

$$\operatorname{int}(\partial/\partial t_i)((\nabla + \overline{\partial})\sigma)|_{\mathscr{C}_b} = \operatorname{int}(\partial/\partial t_i)((\nabla + \overline{\partial})\Phi|_{\mathscr{C}_b}) - (\nabla + \overline{\partial})\psi_i|_{\mathscr{C}_b}.$$

Since $\Phi|_{\mathscr{C}_b} = \sigma|_{\mathscr{C}_b}$,

$$\operatorname{int}(\partial/\partial t_i)((\nabla + \overline{\partial})\Phi|_{\mathscr{C}_b}) = \nabla_{GM}([\sigma])|_b(\pi_*(\partial/\partial t_i)),$$

by definition of the Gauss-Manin connection. But as $(\nabla + \overline{\partial})\psi_i|_{\mathscr{C}_b}$ is exact, $\operatorname{int}(\partial/\partial t_i)((\nabla + \overline{\partial})\sigma)|_{\mathscr{C}_b}$ and $\operatorname{int}(\partial/\partial t_i)((\nabla + \overline{\partial})\Phi|_{\mathscr{C}_b})$ represent the same cohomology class. Thus

$$\begin{aligned}\mathscr{H}_b \ni [(\operatorname{int}(\partial/\partial t_i)((\nabla + \overline{\partial})(\sigma)|_{\mathscr{C}_b})] &= [(\operatorname{int}(\partial/\partial t_i)((\nabla + \overline{\partial})(\Phi)|_{\mathscr{C}_b})] \\ &= \nabla_{GM}([\sigma])|_b(\pi_*(\partial/\partial t_i)).\end{aligned}$$

Since $\sigma|_{\mathscr{C}_b}$ is closed, the same result holds after replacing $\partial/\partial t_i$ with any vector field v satisfying $\pi_* v = \pi_*(\partial/\partial t_i)$; as the $\pi_*(\partial/\partial t_i)$ form a basis for $T_{\mathscr{B},b}$, the proof is complete. □

For the next result, we need to recall the Kodaira-Spencer map.

Definition A.1.5. With notation as in Notation 5.1.1, we have a short exact sequence

$$0 \longrightarrow T_{\mathscr{C}_b}(-\log \mathscr{D}) \longrightarrow T_{\mathscr{C}}(-\log \mathscr{D})|_{\mathscr{C}_b} \longrightarrow \pi^* T_{\mathscr{B}}|_{\mathscr{C}_b} \longrightarrow 0, \tag{A.1}$$

where $T_{\mathscr{C}}(-\log \mathscr{D})$ is the dual of $\Omega^1_{\mathscr{C}}(\log \mathscr{D})$. The *Kodaira-Spencer* map

$$\kappa : T_{\mathscr{B},b} \to H^1(T_{\mathscr{C}_b}(-\log \mathscr{D}))$$

sends a vector $v \in T_{\mathscr{B},b} \simeq H^0(\mathscr{C}_b, \pi^* T_{\mathscr{B}}|_{\mathscr{C}_b})$ to its image in $H^1(T_{\mathscr{C}_b}(-\log \mathscr{D}))$ under the connecting homomorphism in the long exact sequence in cohomology arising from (A.1).

Remark A.1.6. Geometrically, the Kodaira-Spencer map sends a deformation of $(\mathscr{C}_b, \mathscr{D}_b)$ to the corresponding cohomology class in $H^1(T_{\mathscr{C}_b}(-\mathscr{D}))$ (identifying $T_{\mathscr{C}_b}(-\mathscr{D})$ with $T_{\mathscr{C}_b}(-\log \mathscr{D})$). For an explanation of why the boundary map coincides with this description in the case $\mathscr{D} = \emptyset$, see, [Voi07, §9.1.2]. In the case $\mathscr{D} \neq \emptyset$, an analogous argument goes through.

PROPOSITION A.1.7. *Keep notation as in Notation 5.1.1. For $b \in \mathscr{B}$, set $C = \mathscr{C}_b$, $D = \mathscr{D}_b$, and $E = \mathscr{E}|_C$. We have a commuting diagram*
(A.2)

$$\begin{array}{ccc} H^0(C, E \otimes \Omega^1_C(D)) \otimes T_{\mathscr{B},b} & \xrightarrow{\quad dP''_b \quad} & H^1(C, E), \\ \downarrow{\scriptstyle 1\otimes\kappa} & & \uparrow{\scriptstyle \alpha} \\ H^0(C, E \otimes \Omega^1_C(D)) \otimes H^1(C, T_C(-D)) & \xrightarrow{\ \cup\ } & H^1(C, E \otimes \Omega^1_C(D) \otimes T_C(-D)) \end{array}$$

where dP''_b is adjoint to the derivative of the period map dP_b, κ is the Kodaira-Spencer map, α is the map induced by the pairing $\Omega^1_C(D) \otimes T_C(-D) \to \mathscr{O}_C$ and $\cup$ is the cup product.

Proof. Let $\sigma \otimes u$ be an element of $H^0(C, E \otimes \Omega^1_C(D)) \otimes T_{\mathscr{B},b}$. By, e.g., [Voi07, Prop. 9.22] (applied to the $\overline{\partial}$-Laplacian—here unitarity is used to define the adjoint of $\overline{\partial}$ and consequently the Laplacian), we may choose a lift $\widetilde{\sigma}$ of σ to a C^∞ section of $\mathscr{E} \otimes \Omega^1_{\mathscr{C}}(\log \mathscr{D})$, which is holomorphic on fibers. Combining Lemmas A.1.1 and A.1.4, we find that $dP''(\sigma \otimes u)$ can be expressed as $[((\mathrm{int}(v)(\nabla + \overline{\partial})\widetilde{\sigma})|_{\mathscr{C}_b})^{0,1}]$ for v a C^∞ vector field of type $(1,0)$ tangent to $\mathscr{D}$ and lifting u, as in Lemma A.1.4. As $\mathscr{C}$ is a relative curve, for degree reasons we have

$$[((\mathrm{int}(v)(\nabla + \overline{\partial})\widetilde{\sigma}|_{\mathscr{C}_b}))^{0,1}] = [\mathrm{int}(v)\overline{\partial}\widetilde{\sigma}|_{\mathscr{C}_b}].$$

Direct computation gives

$$\overline{\partial}(\mathrm{int}(v)(\widetilde{\sigma}|_{\mathscr{C}_b})) = -\mathrm{int}(v)\overline{\partial}\widetilde{\sigma}|_{\mathscr{C}_b} + \mathrm{int}(\overline{\partial}v)(\widetilde{\sigma}|_{\mathscr{C}_b}).$$

Using the vanishing of the Dolbeault cohomology class $[\overline{\partial}(\mathrm{int}(v)(\widetilde{\sigma}))]$ we obtain

$$[\mathrm{int}(v)(\overline{\partial}\widetilde{\sigma}|_{\mathscr{C}_b})] = [\mathrm{int}(\overline{\partial}v)(\widetilde{\sigma}|_{\mathscr{C}_b})].$$

Combining the above yields that $dP''_b(\sigma \otimes u) = [\mathrm{int}(\overline{\partial}v)(\widetilde{\sigma}|_{\mathscr{C}_b})]$. By definition of the Kodaira Spencer map, $\overline{\partial}v$ is identified with $\kappa(u)$. So

$$dP''_b(\sigma \otimes u) = [\mathrm{int}(\overline{\partial}v)(\widetilde{\sigma}|_{\mathscr{C}_b})] = [\alpha\left(\overline{\partial}v \cup \widetilde{\sigma}|_{\mathscr{C}_b}\right)] = [\alpha\left(\overline{\partial}v \cup \sigma\right)] = [\alpha\left(\kappa(u) \cup \sigma\right)],$$

verifying the commutativity of (A.2) as desired. □

LEMMA A.1.8. *Suppose E is a vector bundle on a smooth curve C, and suppose $D \subset C$ is a divisor. Let $H^1(C, T_C(-D)) \otimes H^0(C, E \otimes \omega_C(D)) \to H^1(C, E)$ denote the map corresponding to the Yoneda cup product composition $\alpha \circ \cup$ from Proposition A.1.7, induced by the natural pairing $E \otimes \omega_C(D) \otimes T_C(-D) \to E$. This is adjoint to a map $v\colon H^0(C, E \otimes \omega_C(D)) \otimes H^1(C, E)^\vee \to$*

$H^1(C, T_C(-D))^\vee$. *We have a commutative diagram*

$$\begin{array}{ccc} H^0(C, E \otimes \omega_C(D)) \otimes H^0(C, E^\vee \otimes \omega_C) & \xrightarrow{\mu} & H^0(C, \omega_C^{\otimes 2}(D)) \\ \downarrow \eta & & \downarrow \zeta \\ H^0(C, E \otimes \omega_C(D)) \otimes H^1(C, E)^\vee & \xrightarrow{\upsilon} & H^1(C, T_C(-D))^\vee, \end{array} \tag{A.3}$$

where the two vertical maps are induced by Serre duality, and the map μ is the map on global sections induced by

$$(E \otimes \omega_C(D)) \times (E^\vee \otimes \omega_C) \to (E \otimes E^\vee) \otimes (\omega_C(D) \otimes \omega_C) \to \omega_C^{\otimes 2}(D),$$

where the first map is the tensor product and the second is obtained from the trace pairing $E \otimes E^\vee \to \mathscr{O}_C$.

Proof. Recall that for F a vector bundle on C, the Serre duality pairing between $H^0(C, F)$ and $H^1(C, F^\vee \otimes \omega_C)$ is obtained as the composition $H^0(C, F) \otimes H^1(C, F^\vee \otimes \omega_C) \to H^1(C, F^\vee \otimes F \otimes \omega_C) \to H^1(C, \omega_C) \to \mathbb{C}$, where the first map is induced by cup product, the second is induced by the pairing $F \otimes F^\vee \to \mathscr{O}$, and the third map is the trace map $\mathrm{tr} : H^1(C, \omega_C) \simeq \mathbb{C}$ [Vak17, 30.3.15].

Using the above description, we now check commutativity of (A.3). We use $\cup$ for the usual cup product and $\overline{\cup}$ to denote the composition of the cup product and the pairing $E \otimes E^\vee \to \mathscr{O}$. Choose $\alpha \in H^0(C, E \otimes \omega_C(D)), \beta \in H^0(C, E^\vee \otimes \omega_C)$; we will denote by s a general element of $H^1(C, E)$ and by t a general element of $H^1(C, T_C(-D))$. On the one hand, $\mu(\alpha \otimes \beta) = \alpha \overline{\cup} \beta$, and so $\zeta(\mu(\alpha \otimes \beta))$ is given by

$$t \mapsto \mathrm{tr}((\alpha \overline{\cup} \beta) \cup t).$$

On the other hand, using the above description of Serre duality,

$$\begin{aligned} \upsilon(\eta(\alpha \otimes \beta)) &= \upsilon(\alpha \otimes (s \mapsto \mathrm{tr}(\beta \overline{\cup} s))) \\ &= (t \mapsto \mathrm{tr}(\beta \overline{\cup} (\alpha \cup t))) \\ &= (t \mapsto \mathrm{tr}((\beta \overline{\cup} \alpha) \cup t)), \end{aligned}$$

and so we obtain the desired commutativity (using that $\alpha \overline{\cup} \beta = \beta \overline{\cup} \alpha$). □

A.2. *Proof of Theorem 5.1.6.*

Proof. The final statement claiming adjointness follows from Lemma A.1.1. The map μ in Lemma A.1.8 is another name for the composition $\mathrm{tr} \circ \otimes$ in (5.4), and by Lemma A.1.8, μ is identified with the map labeled υ there, upon applying Serre duality. Moreover, υ is by definition adjoint to the composition $\alpha \circ \cup$ in Proposition A.1.7. The result now follows by identifying the Kodaira-Spencer map κ of Proposition A.1.7 as Serre dual to the pullback map c_b^* in the statement of Theorem 5.1.6; this identification follows from Remark A.1.6. □

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