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A Fulton–Hansen theorem for almost homogeneous spaces

János Kollár[1] · Aaron Landesman[2]

Dedicated to Fabrizio Catanese on his 70th birthday

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Abstract
We prove a generalization of the Fulton–Hansen connectedness theorem, where $\mathbb{P}^n$ is replaced by a normal variety on which an algebraic group acts with a dense orbit.

The product version of the Fulton–Hansen theorem [6] says that if Y_1, Y_2 are irreducible, projective varieties, $Y_i \to \mathbb{P}^n$ are finite morphisms and $\dim Y_1 + \dim Y_2 > \dim \mathbb{P}^n$, then $Y_1 \times_{\mathbb{P}^n} Y_2$ is connected. As Catanese emphasized both in the panoramic survey [3] and in the shorter version [2], one should look at the topological side of every question in algebraic geometry. Thus one should view the Fulton–Hansen theorem as a generalization of Lefschetz's hyperplane section theorem for the fundamental group. In this form, the quasi-projective version says that if $Y_1, Y_2 \subset \mathbb{P}^n$ are normal, irreducible, quasi-projective varieties, $p_i : Y_i \to \mathbb{P}^n$ are quasi-finite morphisms, and $\dim Y_1 + \dim Y_2 > \dim \mathbb{P}^n$, then

$$\pi_1\big(Y_1 \times_{\mathbb{P}^n, g_1, g_2} Y_2\big) \twoheadrightarrow \pi_1(Y_1) \times \pi_1(Y_2) \quad \text{is surjective}$$

for general $g_1, g_2 \in \mathrm{Aut}(\mathbb{P}^n)$; here our notation means that we first compose p_i with g_i and then take fiber product. See [4,6] and [7, Remark 9.3] for the original proofs and Corollaries 2–3 for further discussion. Also see [18, Theorem 3.3.6] and [12, Theoreme 7.2].

Our main result is Theorem 2, which generalizes the above results. We replace $(\mathrm{PGL}_{n+1}, \mathbb{P}^n)$ by a pair (G, X), where X is a normal quasi-projective variety and G is an algebraic group acting on X with a fixed point and a dense open orbit. We obtain the Fulton-Hansen theorem by applying Theorem 2 to $X = \mathbb{P}^n$ and $G \subset \mathrm{PGL}_{n+1}$ the stabilizer of a point; see Corollary 2. This also implies the original form of the Fulton–Hansen theorem, which begins with a map $Z \to \mathbb{P}^n \times \mathbb{P}^n$ and shows that the preimage of the diagonal is connected, see Corollary 3.

The usual approaches to Fulton–Hansen-type theorems exploit ampleness properties of the tangent bundle T_X. We get the strongest result when T_X is ample, which holds only for $X = \mathbb{P}^n$ [20]. For other homogeneous spaces $X = G/P$, one can develop a theory that

✉ Aaron Landesman
aaronlandesman@stanford.edu

János Kollár
kollar@math.princeton.edu

[1] Princeton University, Princeton, NJ 08544-1000, USA

[2] Stanford University, Stanford, CA 94305, USA

measures the failure of ampleness [5,7,9,21,22]. This leads to results that give π_1-surjectivity if $\dim Y_1 + \dim Y_2$ is large enough. For example, let $0 \leq r \leq n/2$ and let $X = \mathrm{Gr}(\mathbb{P}^r, \mathbb{P}^n)$. By [9, Connectedness Theorem, p. 361] the map $\pi_1(Y_1 \cap Y_2) \to \pi_1(Y_1) \times \pi_1(Y_2)$ is surjective provided

$$\dim Y_1 + \dim Y_2 > \dim X + \dim \mathrm{Gr}(\mathbb{P}^{r-1}, \mathbb{P}^{n-2}). \qquad (*)$$

Here $\dim X$ is what we expect on the right hand side of $(*)$ from the naive dimension count. For $r = 0$, $\dim \mathrm{Gr}(\mathbb{P}^{r-1}, \mathbb{P}^{n-2}) = 0$, and we recover the Fulton-Hansen theorem. However, for $r \gg 0$ the extra term gets quite large. Nonetheless, this bound is optimal, see Example 8.

Our aim is to prove that, for many group actions, the naive dimension estimate $\dim Y_1 + \dim Y_2 > \dim X$ is enough to give π_1-surjectivity, unless the intersection $Y_1 \cap Y_2$ is very degenerate.

Notation 1 We work over an algebraically closed field k unless otherwise specified. Let G be a connected algebraic group acting on a normal k-variety X. We usually assume that the action is faithful.

The action is denoted by $(g, x) \mapsto gx$. If $p : Y \to X$ is a morphism then it is convenient to use the shorthand $gY \to X$ for $g \circ p : Y \to X$.

We consider the case when the G-action has a dense open orbit, denoted by X°, and also a fixed point $x_0 \in X$. For any $n \geq 0$, G acts linearly on the nth order infinitesimal neighborhood $\mathcal{O}_{X,x_0}/\mathfrak{m}_{x_0}^n$, where $(\mathcal{O}_{X,x_0}, \mathfrak{m}_{x_0})$ is the local ring at x_0 and its maximal ideal. If we assume the action of G on X is faithful, this induced action is also faithful for sufficiently large n, and so, in this case, G is a linear algebraic group.

We use $\pi_1(Z, z_0)$ to denote the topological fundamental group of $Z(\mathbb{C})$ if we are over $\mathbb{C}$ and the étale fundamental group of Z otherwise. We will usually include basepoints, but sometimes omit them when it is not important to keep track of them.

The main result is easier to state for normal subvarieties over $\mathbb{C}$. We prove a more general version in Theorem 2, where the characteristic is arbitrary, the stabilizer may be disconnected and the Y_i are not necessarily subvarieties.

Theorem 1 *Let G be a connected algebraic group acting on a normal variety X defined over $\mathbb{C}$. Assume that the action has a fixed point $x_0 \in X$ and a dense open orbit $X^\circ \subset X$ such that the stabilizer $H \subset G$ of any point in X° is connected. Let Y_1, Y_2 be irreducible, normal, locally closed subvarieties of X containing x_0. Assume that*

(1) $\dim Y_1 + \dim Y_2 > \dim X$ *and*
(2) *there is an irreducible component $Z \subset Y_1 \cap Y_2$ of the expected dimension* $\dim Y_1 + \dim Y_2 - \dim X$, *that contains x_0 and is* not *disjoint from X°.*

Then,

$$\pi_1(g_1Y_1 \cap g_2Y_2, x_0) \twoheadrightarrow \pi_1(Y_1, x_0) \times \pi_1(Y_2, x_0) \quad \text{is surjective}$$

for general $(g_1, g_2) \in G \times G$.

Note that the conclusion of Theorem 1 is pretty much the strongest that one can hope for: If $Y_1 \cap Y_2$ is disjoint from X°, then $g_1Y_1 \cap g_2Y_2$ is never in 'general position,' and if an irreducible component does not contain the base point x_0, then its contribution to the fundamental group is hard to control; see also Examples 7–8. For counterexamples with disconnected stabilizers see Examples 10–11.

We next state and prove Theorem 2, from which Theorem 1 follows. We then derive generalizations of the Fulton–Hansen theorem and further applications. We conclude with various examples and counterexamples.

Proof of the main theorem

We start with some preliminary lemmas.

Lemma 1 *Let $f : U \to V$ be a dominant morphism of irreducible varieties and $\sigma : V \to U$ a section. Then U_v is connected for general $v \in V$.*

Proof It is enough to show that the geometric generic fiber of f is connected. By irreducibility of U, the generic fiber of f is irreducible. The generic fiber may fail to be geometrically irreducible, but all geometric components must pass through the image of σ, and hence the geometric generic fiber is connected. □

We also require the following slight variant of [14, Theorem 2], which follows from [14, Theorem 2] and upper semicontinuity of dimension of fibers.

Lemma 2 *Let G be a connected algebraic group acting transitively on a normal k-variety X. Let Y_1 and Y_2 be irreducible, normal varieties and $p_i : Y_i \to X$ quasi-finite morphisms. Then there is a dense open subset $U \subset G \times G$ such for all $(g_1, g_2) \in U$, $g_1Y_1 \times_X g_2Y_2$ is*

(1) *either empty*
(2) *or of pure dimension* $\dim Y_1 + \dim Y_2 - \dim X$

Moreover, if $g_1Y_1 \times_X g_2Y_2$ is non-empty and of dimension $\leq \dim Y_1 + \dim Y_2 - \dim X$ for some $(g_1, g_2) \in G \times G$, then (2) holds. □

Notation 2 Let X be a k-variety, $x_0 \in X$ a closed point and $X^\circ \subset X$ a dense open subset. Let $p : Z \to X$ be a quasi-finite morphism. We let Z° denote $p^{-1}(X^\circ)$. For $z_0 \in p^{-1}(x_0)$, let $Z(z_0) \subset Z$ denote the union of those irreducible components that contain z_0 and whose images are *not* disjoint from X°. We say that z_0 is a *good limit point* of Z if $Z(z_0) \neq \emptyset$.

Lemma 3 *Let G be a connected algebraic group acting on a normal variety X with a fixed point $x_0 \in X$ and a dense open orbit X° with stabilizer $H \subset G$. Let $H_0 \subset H$ denote the identity component of H.*

Let Y_1 and Y_2 be irreducible, normal varieties and $p_i : Y_i \to X$ quasi-finite morphisms such that $Y_i^\circ := p_i^{-1}(X^\circ)$ are nonempty.

Let $I \subset G \times G \times Y_1 \times Y_2$ parametrize quadruples (g_1, g_2, y_1', y_2') such that $g_1^{-1}p_1(y_1') = g_2^{-1}p_2(y_2')$. Let $I^\circ \subset I$ be the preimage of $Y_1^\circ \times Y_2^\circ$ and $\bar{I}^\circ \subset I$ its closure. We have

$$G \times G \xleftarrow{\pi} \bar{I}^\circ \xrightarrow{\tau} Y_1 \times Y_2. \tag{6.1}$$

Then

(2) *$I^\circ \to Y_1^\circ \times Y_2^\circ$ is a locally trivial $G \times H$-bundle.*
(3) *I° has at most $|H/H_0|$ irreducible components.*
(4) *Every good limit point $z_0 = (y_1, y_2)$ lying on a component of $Y_1 \times_X Y_2$ of dimension $\dim Y_1 + \dim Y_2 - \dim X$ gives a rational section $\pi_{z_0} : (g_1, g_2) \mapsto (g_1, g_2, y_1, y_2)$.*

Let j index those irreducible components $\bar{I}^\circ(z_0, j) \subset \bar{I}^\circ$ that contain the image of π_{z_0}. Then the following hold.

(5) *For any j as above, the image of $\pi_1\big(\bar{I}^\circ(z_0, j)\big) \to \pi_1(Y_1 \times Y_2)$ has index $\leq |H/H_0|$.*
(6) *$\bar{I}^\circ(z_0, j) \cap \pi^{-1}(g_1, g_2)$ is a nonempty union of irreducible components of $(g_1Y_1 \times_X g_2Y_2)(z_0)$ for general $(g_1, g_2) \in G \times G$.*

(7) *If H is connected then $\pi^{-1}(g_1, g_2)$ is connected for general $(g_1, g_2) \in G \times G$.*

Remark 3.8. It can happen that $g_1Y_1 \times_X g_2Y_2$ is always reducible, though this seems to be rare; see Example 4.

Proof Claims (2) and (4) are clear and (2) implies (3). There is nothing further to prove if there are no good limit points.

Otherwise let $z_0 = (y_1, y_2)$ be a good limit point. Set $K = k(G \times G)$. We next prove (6). By Lemma 2, the generic fiber $\bar{I}^\circ(z_0, j)_K$ of the projection $\bar{I}^\circ(z_0, j) \to G \times G$ is nonempty, hence irreducible over K, and it has a K-point, given by z_0. Note there is some such component $\bar{I}^\circ(z_0, j)$ because z_0 is a good limit point. The union of those geometric irreducible components of $\bar{I}^\circ(z_0, j)_K$ that contain this K-point is defined over an extension of K. Thus, every geometric irreducible component of $\bar{I}^\circ(z_0, j)_K$ contains this K-point. Hence, for general $(g_1, g_2) \in G \times G$, every irreducible component of $\bar{I}^\circ(z_0, j) \cap \pi^{-1}(g_1, g_2)$ contains (g_1, g_2, z_0), proving (6).

As for (5), $I^\circ \to Y_1^\circ \times Y_2^\circ$ factors as

$$I^\circ \xrightarrow{\tau_1} I^\circ/(G \times H_0) \xrightarrow{\tau_2} Y_1^\circ \times Y_2^\circ.$$

Here τ_1 is a principal $G \times H_0$-bundle with connected fibers and τ_2 is finite, étale of degree $|H/H_0|$ (but possibly disconnected). Thus the image of

$$\pi_1(I^\circ) \to \pi_1(Y_1^\circ \times Y_2^\circ)$$

is a subgroup of index $\leq |H/H_0|$. Observe that $\pi_1(Y_1^\circ \times Y_2^\circ) \to \pi_1(Y_1 \times Y_2)$ is surjective as Y_1 and Y_2 are normal; see [7, 0.7.B]. We can thus factor $\tau : \bar{I}^\circ(z_0, j) \to Y_1 \times Y_2$ as

$$\bar{I}^\circ(z_0, j) \xrightarrow{\tau_0} (Y_1 \times Y_2)^\sim \xrightarrow{\tilde{\tau}} Y_1 \times Y_2, \tag{6.9}$$

where $\tilde{\tau}$ is finite, étale of degree $\leq |H/H_0|$ and

$$\pi_1\big(\bar{I}^\circ(z_0, j)\big) \to \pi_1\big((Y_1 \times Y_2)^\sim\big) \quad \text{is surective.} \tag{6.10}$$

This proves (5) and (7) follows from Lemma 1. □

The proof of Theorem 2 will follow from the above, together with the following important fact from topology.

Remark 1 Suppose we are given a map $f : X \to Y$ of varieties over $\mathbb{C}$. We claim that there is a Zariski open subset $U \subset Y$ such that $X \times_Y U \to U$ is locally a trivial bundle in the Euclidean topology. More precisely, for any point $p \in U^{\mathrm{an}}$, the analytic space associated to U, there is an analytic open set $V \subset U^{\mathrm{an}}$ with $p \in V$ and a homeomorphism of topological spaces spaces $(f^{\mathrm{an}})^{-1}(V) \simeq (f^{\mathrm{an}})^{-1}(p) \times V$ over V.

The claim above follows from [8, Part I, Theorem 1.7, p. 43] and its proof, although it is not explicitly stated there, so we comment on the details. First, [11, §4, Theorem 1] shows that the pieces in the stratification of [8, Part I, Theorem 1.7, p. 43] can be taken to be complex analytic. (Note that the pieces of the stratification are not necessarily real analytic when the map is real analytic, see [8, Caution, p. 43].) To prove the existence of the desired Zariski open set $U \subset Y$, one may work with proper compactifications $\overline{X}, \overline{Y}$ of X and Y, which exist by Nagata's compactification theorem. We may then use the last sentence of [8, Part I, Theorem 1.7, p. 43] to ensure the boundaries of these closures are also unions of strata of the stratifications. Hence, we also obtain stratifications of the open parts X and Y. Then, [8, Part I, §1.6, p. 43] explains why the resulting map is topologically trivial over pieces of

the stratification. Finally, to show that one of the pieces of the stratification of Y is Zariski open, we note that there is a piece whose complement is a complex analytic proper closed subvariety, hence algebraic by Chow's theorem.

8(Fundamental groups of fibers) Let S, W, Z be irreducible k-varieties, Z is normal. Assume that we have dominant morphisms

$$S \xleftarrow{q} W \xrightarrow{p} Z$$

and a section $\sigma : S \to W$ of q such that $p \circ \sigma : S \to Z$ is a constant morphism with image $z_0 \in Z$. Assume that $\pi_1(W) \to \pi_1(Z)$ is surjective and $W \to Z$ is a flat map with irreducible fibers. We aim to understand whether the induced homomorphism $\pi_1\big(W_s, \sigma(s)\big) \to \pi_1(Z, z_0)$ is surjective for general $s \in S$.

Case .1. $k = \mathbb{C}$. By Remark 1, there is an open, dense subset $S^\circ \subset S$ such that q is a topological fiber bundle over S°. Set $W^\circ := q^{-1}(S^\circ)$. Note that $W^\circ \to S^\circ$ has a section that gets contracted by p. Thus

$$\pi_1\big(W_s, \sigma(s)\big) \twoheadrightarrow \pi_1(Z, z_0) \quad \text{is surjective for every } s \in S^\circ. \tag{(8.1.a)}$$

Case .2. $\operatorname{char} k = 0$. By the Lefschetz principle we get that (.1.a) holds for the étale fundamental group.

Case .3. $\operatorname{char} k > 0$. Let $\pi_Z : (Z', z_0') \to (Z, z_0)$ be an irreducible, finite, étale cover corresponding to a finite quotient $\pi_1(Z, z_0) \twoheadrightarrow H$. By pullback we obtain a connected, finite, étale cover $\pi_W : W' \to W$. Here, W' is irreducible because $W' \to Z'$ is flat with irreducible fibers. Further, σ lifts (non-uniquely) to a section $\sigma' : S \to W'$ such that $p' \circ \sigma'$ maps S to z_0'. Thus, by Lemma 1, there is an open, dense subset $S^\circ_H \subset S$ such that W'_s is connected for $s \in S^\circ_H$. That is,

$$\pi_1\big(W_s, \sigma(s)\big) \to \pi_1(Z, z_0) \to H \quad \text{is surjective for every } s \in S^\circ_H. \tag{8.3.a}$$

In this case we say that $\pi_1\big(W_s, \sigma(s)\big) \to \pi_1(Z, z_0)$ is *surjective on finite quotients* for general $s \in S$.

Remark .4 The complication in .3 is that S°_H depends on H, and, if $\operatorname{char} k > 0$,then $\cap_H S^\circ_H$ may be empty. That is, there may not be any closed point $s \in S$ for which $\pi_1\big(W_s, \sigma(s)\big) \to \pi_1(Z, z_0)$ is surjective; see Example 9.

Combining Lemma 3 with Paragraph , can now prove our main theorem.

Theorem 2 *Let G be a connected algebraic group acting on a normal k-variety X with a fixed point $x_0 \in X$ and a dense open orbit X° with stabilizer $H \subset G$. Let $(y_1, Y_1), (y_2, Y_2)$ be irreducible, normal, pointed varieties and $p_i : Y_i \to X$ quasi-finite morphisms such that $p_i(y_i) = x_0$. Assume that*

(1) $1 \le \dim(Y_1^\circ \times_X Y_2^\circ) \le \dim Y_1 + \dim Y_2 - \dim X$, *and*
(2) $Y_1 \times_X Y_2$ *has a good limit point* $z_0 = (y_1, y_2)$, *as in Notation 2. Then, for general* $(g_1, g_2) \in G \times G$,
(3) *the closure of* $g_1 Y_1^\circ \times_X g_2 Y_2^\circ$ *in* $g_1 Y_1 \times_X g_2 Y_2$ *has at most* $|H/H_0|$ *connected components, and*
(4) *there is a subgroup* $\Gamma \subset \pi_1(Y_1, y_1) \times \pi_1(Y_2, y_2)$ *of index at most* $|H/H_0|$ *such that the natural map,*

$$\pi_1\big(g_1 Y_1 \times_X g_2 Y_2, z_0\big) \to \Gamma$$

is surjective if $\operatorname{char} k = 0$, *and surjective on finite quotients if* $\operatorname{char} k > 0$.

Proof. Fix an index j as in Lemma 3 and consider the diagram

$$G \times G \xleftarrow{\pi} \bar{I}^{\circ}(z_0, j) \xrightarrow{\tau_0} (Y_1 \times Y_2)^{\sim} \xrightarrow{\tilde{\tau}} Y_1 \times Y_2,$$

whose right hand side is defined in (3.9). Set

$$\Gamma := \pi_1\big((Y_1 \times Y_2)^{\sim}\big) \subset \pi_1(Y_1, y_1) \times \pi_1(Y_2, y_2).$$

Next apply the discussions in Paragraph to conclude that

$$\pi_1\big(\bar{I}^{\circ}(z_0, j) \cap \pi^{-1}(g_1, g_2)\big) \to \pi_1\big((Y_1 \times Y_2)^{\sim}\big)$$

is surjective if $\operatorname{char} k = 0$, and surjective on finite quotients if $\operatorname{char} k > 0$. □

Fulton–Hansen-type theorems

We get the following version of the Fulton–Hansen theorem.

Corollary 1 (Fulton–Hansen theorem I) *Let $\mathcal{G}$ be a connected algebraic group acting 2-transitively on a quasi-projective variety X. Let Z be a normal, irreducible variety with* $\dim Z > \dim X$ *and $p : Z \to X \times X$ quasi-finite. Then, for general $(g_1, g_2) \in \mathcal{G} \times \mathcal{G}$, and $\Delta_X : X \to X \times X$ the diagonal map,*

(1) $Z \times_{(g_1,g_2)\circ p, X\times X, \Delta_X} X$ *is connected and*
(2) *the natural map,*

$$\pi_1\big(Z \times_{(g_1,g_2)\circ p, X\times X, \Delta_X} X, z_0\big) \twoheadrightarrow \pi_1(Z, z) \times \pi_1(X, x),$$

is surjective if $\operatorname{char} k = 0$ *and surjective on finite quotients if* $\operatorname{char} k > 0$.

Proof (assuming Corollary 5). For general $(g_1, g_2) \in \mathcal{G} \times \mathcal{G}$ choose $z_0 = (z, x) \in Z \times_{(g_1,g_2)\circ p, X\times X, \Delta_X} X$ and define $x_0 \in X$ as the point such that $(x_0, x_0) = ((g_1, g_2) \circ p)(z)$. Let $G \subset \mathcal{G}$ denote the identity component of the stabilizer of x_0. Note that G acts transitively on $X \setminus \{x_0\}$. (By Remark 2, the stabilizer in $\mathcal{G}$ of x_0 is already connected, though we will not need this fact.)

We aim to apply Theorem 2 to $G \times G$ acting on $X \times X$ with fixed point (x_0, x_0) and dense open orbit $(X \setminus \{x_0\}) \times (X \setminus \{x_0\})$. We now check the hypotheses. As an itinitial reduction, the statement for general translates of both Z and X is equivalent to the analogous one for only translates of Z because $Z \times_{(g_1,g_2)\circ p, X\times X, (h_1,h_2)\circ\Delta_X} X \simeq Z \times_{(h_1^{-1}g_1, h_2^{-1}g_2)\circ p, X\times X, \Delta_X} X$. Now, note that, $\dim_{z_0}(Z \times_{(g_1,g_2)\circ p, X\times X, \Delta_X} X) = \dim Z + \dim X - \dim X \times X > 0$ by Lemma 2. Since $\Delta_X(X) \setminus \{(x_0, x_0)\}$ is contained in the dense orbit $(X \setminus \{x_0\}) \times (X \setminus \{x_0\})$, z_0 is a good limit point of $Z \times_{(g_1,g_2)\circ p, X\times X, \Delta_X} X$. Additionally, it follows from Corollary 5 that the stabilizer $H \subset G$ of the G action on $X \setminus \{x_0\}$ is connected. Therefore, the stabilizer in $G \times G$ of a point of $X \setminus \{x_0\} \times X \setminus \{x_0\}$ is also connected. □

We could have proven Corollary 1 using Remark 2 in place of Corollary 5. We opted to use the latter as it leads to a more self-contained proof.

An important special case of Corollary 1 is the the case where $Z = Y_1 \times Y_2$ and $p = p_1 \times p_2$ for $p_i : Y_i \to X$ quasi-finite morphisms. Because this is typically how Corollary 1 is applied, we now restate it in this case.

Corollary 2 (Fulton–Hansen theorem II) *Let $\mathcal{G}$ be a connected algebraic group acting 2-transitively on a quasi-projective variety X. Let Y_1, Y_2 be normal, irreducible varieties and*

$p_i : Y_i \to X$ quasi-finite morphisms. Assume that $\dim Y_1 + \dim Y_2 > \dim X$. *Then, for general* $(g_1, g_2) \in \mathcal{G} \times \mathcal{G}$,

(1) $g_1 Y_1 \times_X g_2 Y_2$ *is connected and*
(2) *the natural map,*

$$\pi_1\big(g_1 Y_1 \times_X g_2 Y_2, z_0\big) \twoheadrightarrow \pi_1(Y_1, y_1) \times \pi_1(Y_2, y_2),$$

is surjective if $\operatorname{char} k = 0$ *and surjective on finite quotients if* $\operatorname{char} k > 0$.

If X and Z as in Corollary 1 are proper, then connectedness of $Z \times_{(g_1,g_2)\circ p, X\times X, \Delta_X} X$ for general $(g_1, g_2) \in \mathcal{G} \times \mathcal{G}$ implies connectedness for every (g_1, g_2). This is sometimes called the Enriques-Severi-Zariski connectedness principle, proved by combining Stein factorization with Zariski's main theorem.

Thus we recover the original setting of the Fulton–Hansen therem [6].

Corollary 3 (Fulton–Hansen theorem III) *Let $\mathcal{G}$ be a connected algebraic group acting 2-transitively on a projective variety X. Let Z be a normal irreducible proper variety with* $\dim Z > \dim X$ *and $p : Z \to X \times X$ a finite morphism. Then $Z \times_{p, X\times X, \Delta_X} X$ is connected. In particular, if $Z = Y_1 \times Y_2$ and $p = p_1 \times p_2$ for $p_i : Y_i \to X$ finite morphisms, then $Y_1 \times_{p_1, X, p_2} Y_2$ is connected.* □

Remark 2 The most important example of a 2-transitive action is $(\mathrm{PGL}_{n+1}, \mathbb{P}^n)$,

The 2-transitive case seems much more general, but in fact there are very few such pairs $(\mathcal{G}, X)$. By [15] $(\mathrm{PGL}_{n+1}, \mathbb{P}^n)$ is the only pair with X projective. The pairs with X quasi-projective are all of the form $(G \ltimes \mathbb{G}_a^n, \mathbb{A}^n)$ where $G \subset \mathrm{GL}_n$ is a product $C \cdot \overline{G}$, for C a subgroup of the central $\mathbb{G}_m \subset \mathrm{GL}_n$ and $\overline{G}$ is one of the following:

(1) $n = 1$, $\overline{G} = \mathrm{GL}_1$,
(2) $n \geq 2$, $\overline{G} = \mathrm{SL}_n$,
(3) $n = 2m$ is even, $\overline{G} = \mathrm{Sp}_m$,
(4) $n = 6$, the characteristic is 2, and $\overline{G} = \mathrm{G}_2$.

(Note that G_2 does not have a nontrivial 6-dimensional representation in characteristics $\neq 2$.)

There is, however, a very long list of pairs (G, X) such that $G(\mathbb{R})$ acts 2-transitively on $X(\mathbb{R})$; see [16,25]. So the following variant applies in many more cases.

Corollary 4 *Let X be a variety defined over $\mathbb{R}$ and G a connected algebraic group acting on it such that the $G(\mathbb{R})$ action on $X(\mathbb{R})$ is 2-transitive. Assume that for $x_0 \neq x_1 \in X(\mathbb{R})$, the stabilizer of the ordered pair (x_0, x_1) is connected (over $\mathbb{C}$). Let Z_1, Z_2 be irreducible, normal varieties and $p_i : Z_i \to X$ quasi-finite morphisms. Assume that* $\dim Z_1 + \dim Z_2 > \dim X$ *and the Z_i have smooth real points. Then,*

$$\pi_1(g_1 Z_1 \times_X g_2 Z_2) \to \pi_1(Z_1) \times \pi_1(Z_2) \quad \textit{is surjective}$$

for general $(g_1, g_2) \in G(\mathbb{R}) \times G(\mathbb{R})$ *and for general* $(g_1, g_2) \in G(\mathbb{C}) \times G(\mathbb{C})$. □

There are also some non-transitive group actions for which we get a Fulton–Hansen-type result, with obvious exceptions.

Example 1 (Orthogonal group) Let $\mathrm{GO}_q := \mathbb{G}_m \cdot \mathrm{O}_q$ be the group of orthogonal similitudes acting on the n-dimensional vector space V^n, where q is a nondegenerate quadratic form. There are 3 orbits, $\{0\}$, $(q = 0) \setminus \{0\}$, and the dense open orbit is $V^n \setminus (q = 0)$.

Claim 1.1. Let $0 \in Y_i \subset \mathbb{A}^n$ be irreducible, normal, locally closed subvarieties. Assume that $Y_i \not\subset (q = 0)$ and $\dim Y_1 + \dim Y_2 > n$. Then

$$\pi_1(g_1 Y_1 \cap g_2 Y_2, 0) \twoheadrightarrow \pi_1(Y_1, 0) \times \pi_1(Y_2, 0) \quad \text{is surjective}$$

for general $(g_1, g_2) \in \mathrm{GO}_q \times \mathrm{GO}_q$.

Proof Since $\mathbb{A}^n$ is smooth,

$$\dim_0(g_1 Y_1 \cap g_2 Y_2) \geq \dim Y_1 + \dim Y_2 - n.$$

Thus we have a good limit point if

$$\dim_0\big(g_1 Y_1 \cap g_2 Y_2 \cap (q = 0)\big) < \dim Y_1 + \dim Y_2 - n.$$

Since $Y_i \not\subset (q = 0)$, we see that $\dim_0\big(g_i Y_i \cap (q = 0)\big) \leq \dim Y_i - 1$. Since $(q = 0) \setminus \{0\}$ is homogeneous, using Lemma 2 we see that

$$\dim_0\big(g_1 Y_1 \cap g_2 Y_2 \cap (q = 0)\big) \leq (\dim Y_1 - 1) + (\dim Y_2 - 1) - (n - 1),$$

as needed. □

The above arguments show that our approach gives the best results if the orbits of an action are fully understood. In the most extreme case, we have the following classification. The proof relies on some results of [15], that we recall afterwards.

Proposition 1 *Let X be an irreducible, normal variety of dimension ≥ 2 over a field k and G a connected linear algebraic group acting on X. Assume that all orbits have dimension either* 0 *or* $\dim X$.

(1) *There is at most 1 orbit of dimension* 0.
(2) *If* $\operatorname{char} k = 0$ *and there is a* 0*-dimensional orbit, then X is isomorphic to either an affine or a projective cone over a projective, homogeneous G-variety Y.*

Proof. For (1) we may assume that k is algebraically closed. We may then replace G by its reduction to assume G is smooth. Let $P = \{p_i\}$ be the union of the 0-dimensional orbits and assume P is nonempty. By Proposition 2 there is a projective G-variety Y and a G-equivariant, affine, surjective morphism $f : X \setminus P \to Y$, whose general fiber is 1-dimensional by Lemma 4.

By [24, Theorem 3], there is a normal, G-equivariant compactification $\bar{X} \supset X$. Let Z be the normalization of the closure of the graph of f with projections π_X and π_Y.

Since G acts transitively on $X \setminus P$, it also acts transitively on Y, hence $E := Z \setminus (X \setminus P)$ is a union of G-orbits. Thus every fiber of $\pi_Y : Z \to Y$ is a geometrically rational curve and E is a disjoint union of (possibly multiple) sections.

For any $p_i \in P$ there is an irreducible component $E_i \subset E$ that is contracted by π_X to p_i. Let $C \subset Y$ be a general curve, X_C the normalization of $\pi_Y^{-1}(C)$ and $F_i \subset X_C$ the preimage of E_i. Then $X_C \to C$ is a $\mathbb{P}^1$-bundle. Note that a $\mathbb{P}^1$-bundle over a smooth, projective curve contains at most 1 curve with negative self-intersection, and this curve is a section. Thus P has at most 1 point.

In order to prove (2), we assume from now on that $\operatorname{char} k = 0$. Then $\pi_Y : Z \to Y$ is a $\mathbb{P}^1$-bundle and E consists of 1 or 2 sections.

If E consists of 2 sections, then $Z = \mathbb{P}_Y(\mathcal{O}_Y + L)$ for some anti-ample line bundle L on Y. Thus X is the affine cone over (Y, L^{-1}).

If E consists of 1 section, then $Z = \mathbb{P}_Y(E)$ where E is obtained as an extension

$$0 \to \mathcal{O}_Y \to E \to L \to 0,$$

for some anti-ample line bundle L on Y. Now we again use that $\operatorname{char} k = 0$, hence $-K_Y$ is ample and Kodaira's vanishing theorem implies that the extension splits. Thus X is the projective cone over (Y, L^{-1}). □

Remark 1.3. We believe that (2) is not true if $\operatorname{char} k \neq 0$, but it should be possible to get a complete description of all cases.

From this we deduce a useful corollary, used in the proof of Corollary 2 above.

Corollary 5 *Let X be an irreducible, smooth variety over a field k and G a connected linear algebraic group acting on X with 2 orbits, one of which is a point $x \in X$. Then $X \setminus \{x\} \cong G/H$ where H is connected.*

Proof This is clear if $\dim X = 1$, so assume that $\dim X \geq 2$.

Let $H_0 \subset H$ be the identity component. If $H_0 \neq H$ then $G/H_0 \to G/H$ is an étale cover, which extends to an étale cover $\pi : \tilde{X} \to X$ by purity. The G-action on $\tilde{X}$ has an open orbit $\tilde{X} \setminus \pi^{-1}(x)$ and $\pi^{-1}(x)$ is a union of 0-dimensional orbits. Thus $\deg \pi = 1$ by Propositon 1. □

Lemma 4 (cf. [15, p. 443]) *Let X be a normal irreducible variety, $P \subset X$ a 0-dimensional subset and $f : X \setminus P \to Y$ an affine morphism. Then* $\dim X \leq \dim Y + 1$.

Proof Assume that $\dim X \geq \dim Y + 1$. Choose normal compactifications $\bar{X} \supset X$, $\bar{Y} \supset Y$ and let Z be the normalization of the closure of the graph of f with projections π_X and π_Y. Note that π_X cannot contract a whole fiber of π_Y. Thus there is point $\bar{y} \in \bar{Y}$ and an irreducible component $Z_z \subset \pi_Y^{-1}(\bar{y})$ such that $Z_z \cap \pi_X^{-1}(P)$ and $Z_z \setminus \pi_X^{-1}(P)$ are both nonempty. Set $W = \pi_X(Z_z)$. Note that $P \cap W \neq \emptyset$, so $W^\circ := W \cap X$ is dense in W. Thus $W^\circ \cap (X \setminus P)$ is the fiber of f, hence affine. By Hartogs's theorem, this implies $\dim W = 1$. So the general fiber dimension of f is ≤ 1. □

The following group theoretic result is proved, but not stated, on [15, p. 443]. It does not seem to be well known, so we now state it and give a proof in Paragraph , following suggestions of Brian Conrad and Zhiwei Yun.

Proposition 2 *Suppose $X = G/H$ is a homogeneous space for a smooth connected linear algebraic group G over an algebraically closed field k. Then, there exists a parabolic subgroup $H \subset P \subset G$ with P/H affine. In particular, there exists a projective variety Y and a surjective and affine map $X \to Y$.*

Remark 3 Knop's proof of Proposition 2 in [15] is slightly different from ours in that he produces a specific choice of P_H associated to H, whereas our proof merely takes P to be an arbitrary minimal parabolic containing H. It is not clear to us what internal property distinguishes it from the other choices.

An interesting aspect is that P is usually not unique and the set of such parabolics has neither a smallest nor a largest element. For example, for

$$H := \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix} \subset \mathrm{SL}_2$$

the maximal choice of P is SL_2; the minimal choices are either the upper or the lower triangular matrices. For

$$H := \begin{pmatrix} 1 & 0 & * \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \subset \mathrm{SL}_3$$

the minimal choice is $B :=$ (upper triangular matrices). The maximal choices are the 2 maximal proper parabolics that contain B.

In what follows, for G an algebraic group, we use $R_u(G)$ to denote its unipotent radical, the maximal smooth normal connected unipotent subgroup of G.

Lemma 5 *Let H be a subgroup of a smooth connected reductive group G over a perfect field k. Then H is either reductive or contained in a k-parabolic subgroup of G.*

Proof The key input in this proof is the fact that any smooth connected unipotent subgroup U of a connected linear algebraic group G over a perfect field k is contained in the unipotent radical of a parabolic k-subgroup $P \subset G$. This follows from a theorem of Bruhat–Tits, see the "Refined Theorem" in [19].

Applying this to our situation, suppose H is not reductive. We wish to show H is contained in a proper parabolic subgroup. By the above fact, there is some $P \subsetneq G$ with $R_u(H) \subset R_u(P)$. It follows that $R_u(H) \subset H \subset N_G(R_u(H)) \subset N_G(R_u(P)) \subset N_G(P) = P$. Therefore, $H \subset P \subsetneq G$ for P parabolic. □

88(Proof of Knop's Proposition 2.) We can write $X = G/H$ for $H \subset G$ a subgroup. Let P denote a minimal parabolic containing H. We wish to show P/H is affine. Let $L := P/R_u(P)$ and let K denote the image of H in L. Because P was chosen to be a minimal parabolic containing H, K is not contained in any proper parabolic subgroup of L. By Lemma 5, K is reductive. Let $U := \ker(H \to K)$. Then, $U \subset R_u(P)$ so U is unipotent. The quotient $R_u(P)/U$ is an affine group scheme acting on P/H with quotient L/K. It follows that P/H is a principal $R_u(P)/U$-bundle over L/K, and so to show P/H is affine, it suffices to show L/K is. This follows from the general claim that a quotient of a connected reductive group by a connected reductive subgroup is affine, see [1, Theorem 1.5]. □

Other applications

Example 2 (Projective homogeneous spaces) These are of the form $X = G/P$ where G is a semisimple algebraic group and $P \subset G$ a parabolic subgroup. We get a Schubert cell decomposition with a single fixed point x_0 and an open cell $X^* \subset X$. Note that X^* is a homogeneous space under the unipotent radical $U \subset P$. The stabilizer of the U-action on X^* is trivial, hence connected. Thus we get the following.

Claim 2.1. Let Y_1, Y_2 be irreducible, normal varieties and $p_i : Y_i \to X$ quasi-finite morphisms. Assume that there is an irreducible component

$$Z^* \subset Y_1 \times_X Y_2 \times_X X^* \quad \text{and a point} \quad z_0 = (y_1, y_2) \in Z \subset Y_1 \times_X Y_2,$$

such that $\dim Z^* = \dim Y_1 + \dim Y_2 - \dim X$ and $p_i(y_i) = x_0$, where Z denotes the closure of Z^*. Then, for general $(g_1, g_2) \in U \times U$, (hece also for general $(g_1, g_2) \in G \times G$), the natural map

$$\pi_1\big(g_1 Y_1 \times_X g_2 Y_2, z_0\big) \to \pi_1(Y_1, y_1) \times \pi_1(Y_2, y_2) \quad \text{is}$$

(a) surjective if $\operatorname{char} k = 0$ and
(b) surjective on finite quotients if $\operatorname{char} k > 0$.

Note that we could consider instead the P-action, which has a usually larger open orbit $X^\circ \supset X^*$. This gives the following variant.

Claim 2.2. Using the above notation, assume that

$$\dim Y_1 + \dim Y_2 > \dim X + \dim(X \setminus X^\circ). \qquad (23.2.a)$$

Then the natural map

$$\pi_1\bigl(g_1Y_1 \times_X g_2Y_2, z_0\bigr) \twoheadrightarrow \pi_1(Y_1, y_1) \times \pi_1(Y_2, y_2) \quad \text{is}$$

(b) surjective if $\operatorname{char} k = 0$, and
(c) surjective on finite quotients if $\operatorname{char} k > 0$,

for general $(g_1, g_2) \in P \times P$.

Proof For dimension reasons there is an irreducible component $Z^\circ \subset Y_1 \times_X Y_2 \times_X X^\circ$ that contains a good limit point z_0. Next we use the P-action to see that z_0 is also a good limit point in $gZ^\circ \times_X X^*$ for general $g \in P$. Now we can apply (1). □

Results of this type have been considered in [5,9,10,22]. Our bound (2.2.a) is optimal in some cases, but weaker in several of them. Using the full G-action, as in the above articles, leads to further improvements, but we did not find a natural way to recover the bounds of [5,9] in all cases.

Example 3 (Prehomogeneous vector spaces) A *prehomogeneous vector space* is a pair (G, V^n) where V^n is a k-vector space of dimension n and $G \subset \mathrm{GL}_n$ is a connected subgroup that has a dense orbit $W \subset V^n$. See [13] for an introduction and detailed classification.

The infinite series of irreducible ones all have connected generic stabilizers. Using the original Sato-Kimura numbering as in [13], the basic examples are built from

(1) (SL_n, V^n),
(2) $(\mathrm{SL}_n, \operatorname{Sym}^2 V^n)$,
(3) $(\mathrm{SL}_n, \wedge^2 V^n)$,
(13) (Sp_n, V^{2n}),
(15) (O_n, V^n),

These lead to further examples by enlarging the group to contain the scalars or replacing (G, V^n) with $(G \times \mathrm{SL}_m, V^n \otimes V^m)$ for certain values of m.

Most of the sporadic examples either have disconnected generic stabilizer or the connectedness is not known. A nice example is $(\mathrm{E}_6 \cdot \mathbb{G}_m, V^{27})$, which is no. 27[1] on the list. The connected component of the generic stabilizer is F_4. Since F_4 has no outer automorphisms, the stabilizer is F_4, hence connected. See also [17,23,26,27] for several other examples.

Counterexamples

Example 4 Start with $(\mathrm{GL}_3, \mathbb{A}^3_{xyz})$ and let $Y_1, Y_2 \subset \mathbb{A}^3$ be cones with vertex 0. Then $g_1Y_1 \cap g_2Y_2$ conists of $\deg Y_1 \cdot \deg Y_2$ lines for general $g_1, g_2 \in \mathrm{GL}_3$. Thus, so long as Y_1 and Y_2 are not both planes, $(g_1Y_1 \cap g_2Y_2)(0) = g_1Y_1 \cap g_2Y_2$ is reducible and the origin is a good limit point.

[1] 27=27 is a coincidence.

Example 5 Consider $(\mathbb{G}_m^3, \mathbb{A}^3_{xyz})$ and let let $Y_1, Y_2 \subset \mathbb{A}^3$ be surfaces that contain the z-axis. Usually $g_1Y_1 \cap g_2Y_2$ is reducible, having both moving and fixed irreducible components for $g_1, g_2 \in \mathbb{G}_m^3$.

If Y_1, Y_2 intersect transversally at the origin then the origin is not a good limit point, even though $g_1Y_1 \cap g_2Y_2$ may have non-empty intersection with the dense open orbit. For example, this happens for $Y_1 = (y = x^2),\ Y_2 = (x = y^2)$.

In any case, the origin is a good limit point if $Y_1 \cap Y_2$ has an irreducible component Z that passes through the origin but is not contained in a coordinate hyperplane.

Example 6 Suppose k has characteristic 0 (or at least has characteristic not equal to 2). Let $X \subset \mathbb{P}^4$ be the projective cone over a smooth quadric surface with vertex x_0. Let $G \subset \mathrm{Aut}(X)$ be the identity component. Then, G acts with 2 orbits: $\{x_0\}$ and $X \setminus \{x_0\}$. Let Y_1 be a 2-plane contained in X and containing x_0. We claim that there is a divisor Y_2 with $x_0 \in Y_2 \subset X$ such that $g_1Y_1^n \times_X g_2Y_2^n$ is the union of a curve and a point for general $(g_1, g_2) \in G \times G$. Thus one preimage of x_0 is a good limit point, the other is not.

The computation is local at x_0, thus we choose affine coordinates such that $X = (xy - uv = 0)$. We can then choose $Y_1 = (x = u = 0)$. We choose Y_2 to be the complete intersection $(xy - uv = 4u - (x - y)^2 = 0)$. We can eliminate u to get

$$Y_2 \cong (4xy - (x - y)^2 v = 0) \subset \mathbb{A}^3.$$

This is an irreducible hypersurface, but

$$4xy - (x - y)^2 v = \bigl(x + y + \sqrt{1 + v}(x - y)\bigr)\bigl(x + y - \sqrt{1 + v}(x - y)\bigr)$$

shows that it is non-normal along $(x = y = 0)$ and Y_2^n has 2 points over the origin. We will next show that only one of them is a good limit point.

A typical translate of Y_1 by G is $(x - c^{-1}v = y - cu = 0)$. Add these to the equations

$$x + y \pm \sqrt{1 + v}(x - y) = 4u - (x - y)^2 = 0$$

and eliminate u and v to get

$$x + y \pm \sqrt{1 + cx}(x - y) = 0 \quad \text{and} \qquad (6.1)4c^{-1}y - (x - y)^2 = 0. \tag{6.1}$$

Here (6.2) defines a curve whose tangent line at the origin is $y = 0$. With $+$ sign, (6.1) defines a curve whose tangent line at the origin is $x = 0$. So we get an isolated intersection point at the origin. Finally,

$$y = x\frac{\sqrt{1 + cx} - 1}{\sqrt{1 + cx} + 1}$$

satisfies both (6.1) with a $+$ sign and (6.2).

Example 7 Suppose k has characteristic 0. For $X = \mathbb{P}^n \times \mathbb{P}^n$, consider $G = \mathrm{PGL}_{n+1} \times \mathrm{PGL}_{n+1}$, $Y_1 = \mathbb{P}^n \times C$, and $Y_2 = \mathbb{P}^n \times H$, where $C \subset \mathbb{P}^n$ is a smooth projective curve of positive genus and $H \subset \mathbb{P}^n$ a hypersurface. Then $\dim Y_1 + \dim Y_2 = \dim X + n$.

For general $(g_1, g_2) \in G \times G$,

$$g_1Y_1 \cap g_1Y_2 = \mathbb{P}^n \times (\deg C \cdot \deg H \text{ points}).$$

Thus, the intersection is disconnected and its conected components are simply connected. So, they do not contribute to the fundamental group of C and hence $\pi_1(g_1Y_1 \cap g_2Y_2, z_0) \to \pi_1(Y_1, y_1)$ has infinite index for any basepoint $z_0 \in g_1Y_1 \cap g_2Y_2$.

Example 8 (Variant of [10, Example, p. 634]) Suppose k has characteristic 0. In $X = \mathrm{Gr}(\mathbb{P}^1, \mathbb{P}^n)$, take $G = \mathrm{PGL}_{n+1}$ and consider

$$\begin{aligned} Y_1 &= (\text{lines through a point}) \cong \mathbb{P}^{n-1}, \\ Z_L &= (\text{lines in a hyperplane } L \subset \mathbb{P}^n) \cong \mathrm{Gr}(\mathbb{P}^1, \mathbb{P}^{n-1}), \quad \text{and} \\ Y_2 &= \cup_{c \in C} Z_{L_c}, \end{aligned}$$

where $C \subset \check{\mathbb{P}}^n$ is a smooth projective curve of positive genus parametrizing hyperplanes $\{L_c : c \in C\}$. Then

$$\dim Y_1 + \dim Y_2 = \dim X + n - 2.$$

The key property is that $g_1 Y_1 \cap g_2 Z_L$ is either empty or is isomorphic to $\mathbb{P}^{n-2}$. Thus, for general $(g_1, g_2) \in G \times G$, $g_1 Y_1 \cap g_2 Y_2$ is reducible and its connected components are isomorphic to $\mathbb{P}^{n-2}$. So again they do not contribute to the fundamental group of C.

Example 9 Let k be a field of positive characteristic, take $X = \mathbb{P}^2$ with the standard action of $G = \mathrm{PGL}_3$. Let $Y_1 \subset \mathbb{P}^2$ be a line and let $Y_2 \subset \mathbb{P}^2$ be the complement of a line. A general translate of an intersection of Y_1 with Y_2 is isomorphic to $\mathbb{A}^1$, and the map $g_1 Y_1 \cap g_2 Y_2 \to Y_2$ is identified with the inclusion of a line $\mathbb{A}^1 \to \mathbb{A}^2$. Hence, after applying an automorphism, we may assume it is given by $\mathrm{Spec}\, k[x, y]/(y) \to \mathrm{Spec}\, k[x, y]$. This is not surjective on fundamental groups because the pullback of the connected finite étale Artin-Schreier cover $W := \mathrm{Spec}\, k[x, y, t]/(t^p - t - y) \to \mathrm{Spec}\, k[x, y]$ is not connected.

Examples with disconnected stabilizers

We conclude by giving examples showing that $\pi_1(g_1 Y_1 \times_X g_2 Y_2, z_0) \to \pi_1(Y_1, y_1) \times \pi_1(Y_2, y_2)$ may be non-surjective when H is disconnected in the setting of Theorem 2. For the remainder of the paper, we assume k has characteristic 0.

Example 10 If X as in Theorem 2 has nontrivial fundamental group, we can take $Y_1 = Y_2 = X$, and the resulting map $\pi_1(X, x_0) \to \pi_1(X, x_0) \times \pi_1(X, x_0)$ will fail to be surjective. By applying Theorem 2, we conclude that H must be disconnected, and have at least $\pi_1(X, x_0)$ components.

For a concrete example of such a variety, take X to be the moduli space parametrizing unordered pairs of distinct points in $\mathbb{P}^2$. Then X is homogeneous under PGL_3. The open orbit in $X \times X$ is formed by those $(\{p_1, p_2\}, \{p_1', p_2'\})$ for which no three points are on a line. The stabilizer is $\mathbb{Z}/2 \times \mathbb{Z}/2$. The space parametrizing ordered pairs of distinct points in $\mathbb{P}^2$ is the universal cover of X. Thus $\pi_1(X) \cong \mathbb{Z}/2$.

We conclude by giving a somewhat more involved example where H is disconnected, but nevertheless $\pi_1(X, x_0) = 1$.

Example 11 (Constructing X and G) Let X be the moduli space of smooth plane conics in $\mathbb{P}^4$ over $\mathbb{C}$. The group GL_5 acts transitively on X via its action on $\mathbb{P}^4$.

Choose coordinates $x_0, \ldots, x_4$ and a reference conic $C_0 = (x_0^2 + x_1^2 + x_3^2 = x_3 = x_4 = 0)$. The stablizer of C_0 is the set of matrices

$$\tilde{G} := \left\{ \begin{pmatrix} \mathrm{GO}_3 & * \\ 0 & \mathrm{GL}_2 \end{pmatrix} \right\} \subset \mathrm{GL}_5.$$

Thus $\tilde{G}$ is connected and so is $\tilde{G} \cap \mathrm{SL}_5$ as shown by the retraction

$$\begin{pmatrix} A_3 & B \\ 0 & A_2 \end{pmatrix} \mapsto \begin{pmatrix} d^{-1}A_3 & B \\ 0 & dA_2 \end{pmatrix} \quad \text{where} \quad d = \det A_3 \cdot \det A_2.$$

Thus $X \cong \mathrm{SL}_5/(\tilde{G} \cap \mathrm{SL}_5)$ is simply connected. Since we prefer faithful actions, our group $G \subset \mathrm{PGL}_5$ is the image of $\tilde{G}$.

For $Z \subset \mathbb{P}^4$, let $\langle Z \rangle$ denote the linear span of Z in $\mathbb{P}^4$. Observe that G has a dense orbit $X^\circ \subset X$, consisting of those conics C_1 so that $\langle C_0 \rangle \cap \langle C_1 \rangle$ is a point p and neither of the C_i contains p.

If $C_1 = (x_0 = x_1 = x_2^2 + x_3^2 + x_4^2 = 0)$ then the stabilizer of the ordered pair (C_0, C_1) is the set of matrices

$$H := \left\{ \begin{pmatrix} \mathrm{O}_2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \mathrm{O}_2 \end{pmatrix} \right\} \subset \mathrm{PGL}_5.$$

Thus H has 4 connected components, which can be geometrically described as follows. Set $p := \langle C_0 \rangle \cap \langle C_1 \rangle$. From p one can draw two distinct tangent lines to each C_i. Let these tangent lines be $\{T_0^0, T_0^1\}$ and $\{T_1^0, T_1^1\}$. The H action permutes these lines, giving a surjection $H \twoheadrightarrow \mathbb{Z}/2 \times \mathbb{Z}/2$.

Example 12 (Constructing Y_1 and Y_2) Continuing Example 11, we next construct the subvarieties Y_1 and $Y_2 \subset X$.

Choose a point $q \in \langle C_0 \rangle \setminus C_0$ and let $Y_1 := Y_1(q) \subset X$ be the set of those conics C for which $q \in \langle C \rangle \setminus C$. Then Y_1 is a smooth, locally closed subvariety of dimension 9 in X.

To construct Y_2, fix 2 distinct points $x_1, x_2 \in C_0$ and general 2-planes S_i such that $x_i \in S_i$. Let $Y_2 := Y_2(S_1, S_2) \subset X$ be the set of conics C with the following two properties.

(1) We have that $c_i := C \cap S_i$ is a single point and $c_1 \neq c_2$.
(2) Let $\tau(C) \in \langle C \rangle \setminus C$ denote the intersection point of the lines tangent to C at the c_i. Then $\tau(C) \in \langle C_0 \rangle \setminus C_0$.

Given distinct $c_i \in S_i$ and $p \in \langle C_0 \rangle \setminus C_0$, the set of such conics for $c_i = C \cap S_i$ and $\tau(C) = p$ is a principal $\mathbb{G}_m$-bundle whenever c_1, c_2, and p are not collinear. Indeed, the conics in $\mathbb{P}^2$ that pass through $(1:0:0)$, $(0:1:0)$ which have tangent lines at those points intersecting at $(0:0:1)$ are precisely the hyperbolas $x_1x_2 = \lambda x_0^2$ for $\lambda \in \mathbb{G}_m(\mathbb{C})$. Thus Y_2 is a smooth, locally closed subvariety of dimension 7 in X.

The intersection $Z := Y_1 \cap Y_2$ consists of those conics in Y_2 for which $\tau(C) = q$. Thus Z is a smooth, locally closed subvariety of dimension 5 in X.

Proposition 3 *We use the notation of Examples 11–12. For general $(g_1, g_2) \in G \times G$, $g_1Y_1 \cap g_2Y_2$ is irreducible and the map*

$$\pi_1\big(g_1Y_1 \cap g_2Y_2, [C_0]\big) \to \pi_1\big(Y_1, [C_0]\big) \quad \textit{has index 4.}$$

Proof Note first that the group action sends $Y_1(q)$ to $Y_1(g_1q)$ and $Y_2(S_1, S_2)$ to $Y_2(g_2S_1, g_2S_2)$. Thus, letting $Z = Y_1 \cap Y_2$, it is enough to show that

$$\pi_1\big(Z, [C_0]\big) \to \pi_1\big(Y_1, [C_0]\big) \times \pi_1\big(Y_2, [C_0]\big) \quad \text{has index 4.}$$

The index is at most 4 by Theorem 2 because H has 4 connected components, as shown in Example 11. We now show the index is at least 4. Note that Y_1 has a connected degree 2 finite étale cover $\widetilde{Y}_1 \to Y_1$ parametrizing pairs (C, c) where $C \in Y_1$ and $c \in C$ is one of the

2 points of C whose tangent line passes through q. Similarly, Y_2 has a connected degree 2 finite étale cover parametrizing pairs (C, d) where d is one of the two points of C_0 whose tangent line passes through $\tau(C)$. Let q_1 and q_2 denote the two points of C_0 whose tangent lines pass through q. Then, the restriction of the cover $\widetilde{Y}_1 \times \widetilde{Y}_2$ to Z splits into the 4 connected components

$$\widetilde{Z}_{i,j} := \{(C, C \cap S_i, q_j)\} \subset Z \times_{Y_1 \times Y_2} (\widetilde{Y}_1 \times \widetilde{Y}_2) \quad \text{for} \quad 1 \leq i, j \leq 2.$$

□

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