Edge limits of truncated circular beta ensembles[*]

Yun Li[†]   Benedek Valkó[‡]

Abstract

We study the scaling limit of the rank-one truncation of various beta ensemble generalizations of classical unitary/orthogonal random matrices: the circular beta ensemble, the real orthogonal beta ensemble, and the circular Jacobi beta ensemble. We derive the scaling limit of the normalized characteristic polynomials and the point process limit of the eigenvalues near the point 1. We also treat multiplicative rank one perturbations of our models. Our approach relies on a representation of truncated beta ensembles given by Killip-Kozhan [24], together with the random operator framework developed in [42, 43, 44] to study scaling limits of beta ensembles.

1 Introduction

For the classical unitary and orthogonal random matrix ensembles the point process scaling limit of the eigenvalues is well understood. The eigenvalues are on the unit circle, and if one scales the eigenangles appropriately, one obtains a point process limit on the real line. More recently the scaling limit of the (normalized) characteristic polynomials of these classical ensembles has been derived and characterized as well, these limits lead to random entire functions where the zero set is given by the point process limit of the eigenvalues.

[†]Yau Mathematical Sciences Center, Tsinghua University, Beijing, China.
 E-mail: `liyun1723@mail.tsinghua.edu.cn`
[‡]Department of Mathematics, University of Wisconsin – Madison, Wisconsin, USA.
 E-mail: `valko@math.wisc.edu`

If we remove the first row and first column of a unitary (or orthogonal) matrix then the eigenvalues of the resulting matrix are within the unit disk. It is natural to ask what one can say about the limits of the eigenvalues and the characteristic polynomial if one studies the truncated random matrices, and what connections can be shown between the limit objects of the original and the truncated models. Our main goal is to investigate these questions for beta-generalizations of classical orthogonal and unitary ensembles introduced by [24]. We also consider similar questions for multiplicative rank one perturbations of these models. We demonstrate how the random operator framework developed in [42, 43, 44] to study circular ensembles can be used to derive scaling limits for truncated and the perturbed versions of circular and orthogonal beta ensembles. We also introduce and study a new truncated beta ensemble model constructed from the circular Jacobi beta ensemble. Non-normal perturbations of classical ensembles have a rich history, see e.g. the surveys [19] and [12] and the references within. Our results provide the first examples of point process limits of non-normal perturbations of beta ensembles.

1.1 Haar unitary matrices and their truncations

To start with a concrete example, we first consider the case of Haar unitary matrices. Let M_n be an $n \times n$ uniformly chosen unitary matrix. With probability one M_n has n distinct eigenvalues $e^{i\theta_k}, 1 \le k \le n$, all on the unit circle. The joint eigenvalue density is given by

$$\frac{1}{Z_n} \prod_{1 \le j < k \le n} |e^{i\theta_j} - e^{i\theta_k}|^2, \qquad \theta_j \in [-\pi, \pi), \tag{1.1}$$

where Z_n is an explicit normalizing constant (see e.g. [11]). The distribution given by (1.1) is called the size n *circular unitary ensemble*. Because of the appearance of the squared Vandermonde determinant in the probability density, this ensemble is *determinantal* ([2, 21]), all finite dimensional marginal densities can be expressed via determinants built from a fixed kernel function. (We will provide more detail on the results discussed within this section in the Appendix.) The point process scaling limits of finite determinantal ensembles can be derived by studying the corresponding scaling limits of the determinantal kernel. It is a classical result due to Gaudin, Mehta, Dyson [2, 32] that if we scale the eigenangles of M_n by n then we get a translation invariant determinantal point process in the limit. We call this point process the Sine_2 process.

In a more recent result, Chhaibi, Najnudel and Nikeghbali [7] studied the scaling limit of the (normalized) characteristic polynomial

$$p_n(z) := \frac{\det(I_n - zM_n^{-1})}{\det(I_n - M_n^{-1})} = \prod_{j=1}^{n} \frac{1 - ze^{-i\theta_j}}{1 - e^{-i\theta_j}}$$

of the circular unitary ensemble. They showed that under the scaling of the Gaudin-Mehta-Dyson theorem one obtains a random entire function ζ (named the *stochastic zeta function*) with zero set given by the Sine_2 process.

For a square matrix M we denote by $\mathrm{Trunc}(M)$ the matrix obtained by removing the first row and column from M. Note that we can write $\mathrm{Trunc}(M)$ as $\Pi^\dagger M \Pi$ where Π is the appropriate projection matrix, and † denotes the transpose.

Now consider the truncated version of a uniformly chosen $(n+1) \times (n+1)$ unitary matrix, i.e. $\mathrm{Trunc}(M_{n+1})$. With probability one this matrix has eigenvalues in the open unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. The obtained random matrix has been studied in the physics literature because of its connection to chaotic scattering problems (see [17, 23] for further discussion and references). In [47] Życzkowski and Sommers proved that the

joint eigenvalue density of $\text{Trunc}(M_{n+1})$ with respect to the Lebesgue measure in the unit disk is given by

$$\frac{1}{\pi^n} \prod_{1 \le j < k \le n} |z_j - z_k|^2, \qquad z_j \in \mathbb{D}. \tag{1.2}$$

We call this distribution the *truncated circular unitary ensemble*. (Note that the [47] provides a description for the eigenvalue distribution for general rank-k truncation as well.) The squared Vandermonde term in (1.2) indicates that this is also a determinantal point process. By studying the determinantal kernel one can show that the point process limit of the eigenvalues of $\text{Trunc}(M_{n+1})$ *without any additional scaling* leads to a determinantal point process limit in $\mathbb{D}$. We may call the resulting process the *bulk scaling limit* of the truncated circular unitary ensemble.

It is reasonable to ask if this point process can be connected to the zeroes of a 'nice' random analytic function, since it is the scaling limit of the zeros of the characteristic polynomial of $\text{Trunc}(M_{n+1})$. In Peres-Virág [34] it was shown that this is indeed the case, the bulk scaling limit of the eigenvalues of $\text{Trunc}(M_{n+1})$ has the same distribution as the zero set of the so-called Gaussian analytic function.

One can treat the eigenvalues of $\text{Trunc}(M_{n+1})$ as perturbations of the original eigenvalues of M_{n+1}. This leads to the study of the behavior of the eigenvalues of the truncated matrix under the scaling

$$z \mapsto -ni \log z, \tag{1.3}$$

since this corresponds to the scaling $e^{i\theta} \mapsto n\theta$ that takes the original (unit length) eigenvalues to the Sine_2 process.[1] See Figure 1 for an illustration. It was shown in [1] that under this scaling the kernel of the truncated circular unitary ensemble (and hence the ensemble itself) indeed has a limit. The limiting point process is determinantal, and it is supported in the open upper half plane $\mathbb{H} = \{z \in \mathbb{C} : \Im z > 0\}$. We call this the *(hard) edge scaling limit* of the truncated model, since we zoom in near $z = 1$.

The point process obtained as the edge limit of the truncated circular ensembles limit process in [1] appeared before in [15] and [18] as the point process limit of the rank-one additive anti-Hermitian perturbation for the Gaussian unitary ensemble under the appropriate scaling. See also [16, 8, 9] for recent results on this model.
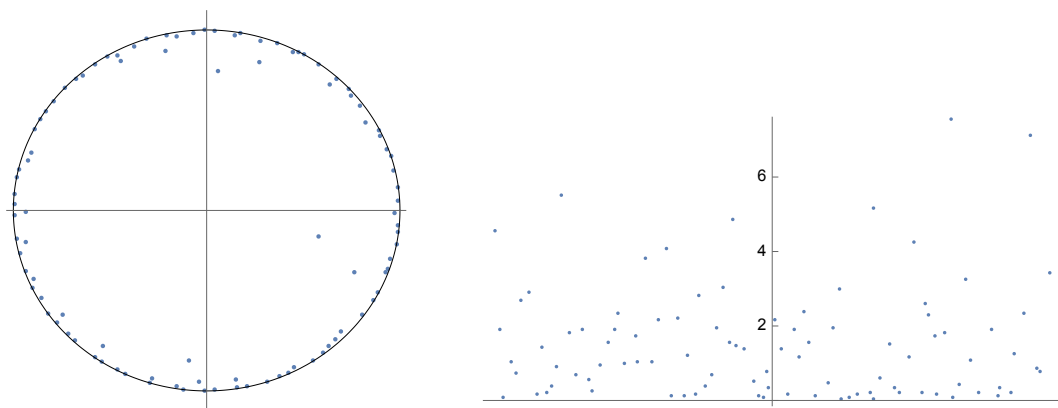

Figure 1: The picture on the left shows the eigenvalues of a truncated uniformly chosen 100×100 unitary matrix. The picture on the right shows the same eigenvalues under the edge scaling (1.3).

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[1]Throughout the paper we are considering the branch of logarithm that is defined on $\mathbb{C} \setminus (-\infty, 0]$ and satisfies $\log(1) = 0$.

As in the bulk case, we can ask if one can connect the edge limit of the truncated circular ensemble to the zero set of a random analytic function, and whether one can characterize this random function in a natural way. We answer this question in the affirmative in our main result, Theorem 1.2 below. We provide a scaling limit for the normalized characteristic polynomial of the truncated model under the edge scaling, and describe the limiting random entire function. In fact, our goal is to study this and related questions in a more general setting: for the beta-generalizations of the circular unitary and other random unitary and orthogonal ensembles.

1.2 CMV matrices, beta ensembles, and their truncations

The size n circular beta ensemble with $\beta > 0$ is the distribution of n points $\{e^{i\theta_j}, 1 \le j \le n\}$ on the unit circle with joint probability density given by

$$\frac{1}{Z_{n,\beta}} \prod_{j < k \le n} |e^{i\theta_j} - e^{i\theta_k}|^\beta, \qquad \theta_j \in [-\pi, \pi). \tag{1.4}$$

Here $Z_{n,\beta}$ is an explicit normalizing constant, see [11]. When $\beta = 2$ we get the circular unitary ensemble. The cases when $\beta = 1$ and 4 correspond to symmetric/self-dual random unitary matrices, but for general $\beta > 0$ there is no known invariant random matrix ensemble with the appropriate joint eigenvalue distribution. Note however that (1.4) has a natural interpretation as the Gibbs measure corresponding to a log-gas of n-points restricted to the unit circle and interacting via a logarithmic potential.

In [25] Killip and Nenciu (motivated by the results of [10]) constructed a family of sparse random unitary matrix models $\{\mathsf{Circ}_{n,\beta}, n \ge 1\}$ with joint eigenvalue distribution given by (1.4). Their construction is based on the theory of orthogonal polynomials on the unit circle. We provide here a quick overview of their approach, the precise statements will be reviewed in Section 3.

Suppose that μ is a discrete probability measure on the unit circle $\partial \mathbb{D}$ with a finite support of n points. The probability measure μ can be encoded with its system of monic orthogonal polynomials. These polynomials satisfy the so-called Szegő recursion, which can be parameterized with a finite collection of complex numbers $\alpha_0, \dots, \alpha_{n-1}$, called the Verblunsky coefficients. In [5] Cantero, Moral, and Velasquez provided a construction for a 'canonical' sparse (five-diagonal) $n \times n$ unitary matrix (called the *CMV matrix*)

$$\mathcal{C} = \mathcal{C}(\alpha_0, \dots, \alpha_{n-1})$$

in terms of the Verblunski coefficients, so that the spectral measure of $\mathcal{C}$ with respect to the unit vector $\mathbf{e}_1 = (1, 0, \dots, 0)^\dagger$ is exactly μ. Moreover, if the probability measure μ is the spectral measure of an $n \times n$ unitary matrix U with respect to $\mathbf{e}_1$ then the CMV matrix $\mathcal{C}$ corresponding to μ is unitary equivalent to U. Note that the CMV matrix is the analogue of the tridiagonal (Jacobi) matrix constructed from the coefficients of the three-term recursion of the orthogonal polynomials of a finitely supported probability measure on $\mathbb{R}$.

Let M_n be an $n \times n$ Haar unitary matrix, and consider its spectral measure μ_n with respect to $\mathbf{e}_1$. This is a (random) probability measure with support given by the circular unitary ensemble (1.1). Using unitary invariance one can show that the joint distribution of the weights of μ_n is given by a particular Dirichlet distribution, and that the weights are independent of the support of μ_n. Moreover, the Verblunsky coefficients of μ_n are independent random variables, and their distributions can be computed explicitly. This motivated Killip and Nenciu in [25] to study the random probability measure $\mu_{n,\beta}^{\mathrm{KN}}$ with support given by the circular beta ensemble (1.4) and weights chosen independently from a particular (β-dependent) Dirichlet distribution.

[25] showed that the Verblunsky coefficients of $\mu_{n,\beta}^{\mathrm{KN}}$ are still independent, with explicitly given distributions. The corresponding CMV matrix $\mathrm{Circ}_{n,\beta} := \mathcal{C}$ provides a natural sparse random unitary matrix with spectrum given by the circular beta ensemble (1.4). For $\beta = 2$ this matrix is unitary equivalent to the Haar unitary matrix M_n, and their spectral measures with respect to $\mathbf{e}_1$ have the same distribution.

In [24] Killip and Kozhan studied how removing the first row and column changes the spectrum of classical random unitary and orthogonal matrices. An important observation of [24] (which is crucial for our paper as well) is the following: if U is an $n \times n$ unitary matrix then the truncated matrix $\mathrm{Trunc}(U)$ is unitary equivalent to the truncated version of the CMV matrix $\mathcal{C}$ corresponding to U, which in turn is unitary equivalent to an $(n-1) \times (n-1)$ CMV matrix built from a simple transformation of the Verblunsky coefficients of U. This means that if we know the Verblunsky coefficients of U then we can construct a sparse matrix whose spectrum is the same as that of $\mathrm{Trunc}(U)$.

This observation allowed [24] to provide a sparse matrix model with spectrum distributed as (1.2). Their approach also allowed them to study the matrix models $\mathrm{Circ}_{n,\beta}$ of [25] with the first row and column removed. They proved that the joint eigenvalue density of the truncated matrix $\mathrm{Trunc}(\mathrm{Circ}_{n+1,\beta})$ is given by

$$\frac{\beta^n}{(2\pi)^n} \prod_{1 \le j,k \le n} (1 - z_j \bar{z}_k)^{\frac{\beta}{2}-1} \prod_{j<k \le n} |z_j - z_k|^2, \qquad z_j \in \mathbb{D}. \tag{1.5}$$

We call the resulting distribution the size n *truncated circular beta ensemble*. Note that for $\beta = 2$ we recover (1.2). (We remark that [24] also provided a log-gas interpretation for (1.5).) Our goal is to study this ensemble (together with some other related models) under the edge scaling (1.3).

The approach of Killip and Nenciu [25] can be extended to provide random matrix representations of beta-generalizations of other random unitary and orthogonal ensembles where the joint distribution of the Verblunsky coefficients can be described explicitly. The results of Killip and Kozhan [24] then provide a natural random matrix representation of the *truncated* version of these beta ensembles. Our main results provide descriptions of the edge scaling limits of these truncated ensembles.

1.3 Scaling limits of circular beta ensembles and their truncations

Using the Killip-Nenciu representation Killip and Stoiciu in [26] showed that under the scaling (1.3) the circular beta ensemble has a point process limit. They characterized the limiting point process via its counting function using a system of stochastic differential equations. This limit process was later shown to be the same as the Sine_β process, the bulk scaling limit of the Gaussian beta ensemble ([33, 42]). Note that Sine_β is not determinantal for general β, in fact there is no known description for its joint intensity functions in the general case.

In a series of papers [42, 43, 44] Valkó and Virág developed a framework to study the scaling limits of beta ensembles using Dirac-type differential operators (see Section 2 for a more detailed discussion). [42] showed that the spectra of unitary CMV matrices and some of their point process limits (including the Sine_β process) can be represented as the eigenvalues of random Dirac-type differential operators. A Dirac-type differential operator can be parametrized by a path in the upper half plane $\mathbb{H} := \{z : \Im z > 0\}$ together with two boundary points in $\partial \mathbb{H} = \mathbb{R} \cup \{\infty\}$. In the case of a unitary CMV matrix these parameters can be built from the Verblunsky coefficients. [43] showed how this representation can be used to prove operator level convergence of the circular beta ensemble to the Sine_β process. The path parameter of the random differential operator corresponding to $\mathrm{Circ}_{n,\beta}$ is a random walk, which under the appropriate scaling

converges to a time-changed hyperbolic Brownian motion. This process is the path parameter of the random Dirac operator corresponding to the Sine_β process.

[44] developed a framework to study scaling limits of normalized characteristic polynomials of beta ensembles. In particular, [44] proved that the normalized and scaled characteristic polynomials of the $\mathrm{Circ}_{n,\beta}$ converge to a random entire function ζ_β with zero set given by Sine_β. (For $\beta = 2$ this random entire function is the stochastic zeta function introduced in [7].) The random function ζ_β is characterized via various equivalent ways, in particular as the solution of the following random shooting problem.

Theorem 1.1 ([44]). *Let b_1, b_2 be independent two-sided standard Brownian motion, and q an independent standard Cauchy random variable. Consider the unique strong solution $\mathcal{H}_\beta : (-\infty, 0] \times \mathbb{C} \to \mathbb{C}^2$ of the stochastic differential equation*

$$d\mathcal{H}_\beta = \begin{pmatrix} 0 & -db_1 \\ 0 & db_2 \end{pmatrix} \mathcal{H}_\beta - z\frac{\beta}{8}e^{\frac{\beta}{4}u}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \mathcal{H}_\beta du, \quad u \leq 0, \tag{1.6}$$

subject to the initial condition $\lim_{u \to -\infty} \sup_{|z| < 1} |\mathcal{H}_\beta(u, z) - \binom{1}{0}| = 0$. Then ζ_β has the same distribution as the random function $\mathcal{H}_\beta(0, z)^\dagger \binom{1}{-q}$.

Our main result gives the edge scaling limit of the truncated circular beta ensemble (1.5) together with the scaling limit of its normalized characteristic polynomial. This result also provides a connection to the limit objects of the original circular beta ensemble.

Theorem 1.2. *Under the edge-scaling (1.3) the truncated circular beta ensemble converges in distribution to a point process $\mathcal{X}_\beta$, which has the same distribution as the zero set of the random entire function $\mathcal{E}_\beta = \mathcal{H}_\beta(0, \cdot)^\dagger \binom{1}{-i}$ defined via (1.6). Moreover, $\mathcal{E}_\beta$ is the scaling limit of the normalized characteristic polynomials of the truncated circular beta ensemble.*

Theorem 1.2 is proved in Section 6.1. In fact, we will show that there is a coupling of the finite ensembles and the limiting object so that the stated limits hold with probability one with effective (random) error bounds (see Proposition 6.4). The proof of the theorem uses the random operator framework to analyze the scaling limit of the normalized characteristic polynomial of truncated CMV matrices.

Theorem 1.2 provides a connection between the scaling limits of the full and the truncated circular beta ensemble that is new even in the classical $\beta = 2$ case. It shows that the scaling limit of the characteristic polynomials of the circular beta ensemble can be obtained from the corresponding limit of the truncated model and an independent Cauchy random variable. Both ζ_β and $\mathcal{E}_\beta$ are random entire functions, and hence they are determined by their restriction to $\mathbb{R}$. By Theorems 1.1 and 1.2 we have the following equality in distribution:

$$\{\zeta_\beta(s) : s \in \mathbb{R}\} \stackrel{d}{=} \{\Re\mathcal{E}_\beta(s) + q\,\Im\mathcal{E}_\beta(s) : s \in \mathbb{R}\},$$

where q is a Cauchy random variable independent of $\mathcal{E}_\beta$.

If M is a square matrix then the spectrum of $\mathrm{Trunc}(M)$ can also be studied by considering the spectrum of the rank one multiplicative perturbation

$$M \cdot \mathrm{diag}(0, 1, 1, \ldots, 1)$$

instead. (Of course, this also adds an additional zero eigenvalue.) This motivates the study of general rank one multiplicative perturbations. For $r \in \mathbb{R}$ define

$$M^{[r]} := M \cdot \mathrm{diag}(r, 1, 1, \ldots, 1). \tag{1.7}$$

If M is a Haar unitary matrix then the eigenvalue distribution of $M^{[r]}$ has been studied in [14]. See also [13], where this problem was studied in the case when r decays as $n^{-1/2}$ as $n \to \infty$. (See also [15] and [18] for related results on rank-one additive anti-Hermitian perturbations for the Gaussian unitary ensemble.) Note that because of the various symmetries of the model, we may assume $r \in [0, 1]$.

Following the definition of the truncated circular beta ensemble, it is natural to define the appropriate rank-one multiplicative perturbation of the circular beta ensemble as the spectrum of $\mathrm{Circ}^{[r]}_{n,\beta}$. In [24] Killip and Kozhan derived the joint eigenvalue distribution of $\mathrm{Circ}^{[r]}_{n,\beta}$. Moreover, they showed that if $\mathcal{C}$ is a unitary CMV matrix then the spectrum of $\mathcal{C}^{[r]}$ is the same as a certain explicitly determined CMV matrix. Using their results we are able to extend the results of Theorem 1.2.

Theorem 1.3. *Fix $r \in [0, 1]$. Consider the random function $\mathcal{H}_\beta$ defined via (1.6), and let q be a standard Cauchy random variable independent of b_1, b_2 appearing in (1.6). Under the edge-scaling (1.3) the eigenvalues of $\mathrm{Circ}^{[r]}_{n,\beta}$ converge in distribution to a point process $\mathcal{X}_{r,\beta}$, which has the same distribution as the zero set of the random entire function $\mathcal{E}_{r,\beta} = \mathcal{H}_\beta(0, \cdot)^\dagger \binom{1}{-c_r}$ with*

$$c_r = \frac{q + i\frac{1-r}{1+r}}{1 - iq\frac{1-r}{1+r}}. \tag{1.8}$$

Moreover, $\mathcal{E}_{r,\beta}$ is the scaling limit of the normalized characteristic polynomials of $\mathrm{Circ}^{[r]}_{n,\beta}$ under the same scaling.

Note that $c_r = q$ for $r = 1$ and $c_r = i$ for $r = 0$, so this result gives an interpolation between the scaling limits of the unperturbed and the truncated circular beta ensemble.

Our approach extends to other matrix models as well. We provide versions of Theorems 1.2 and 1.3 for the *real orthogonal beta ensemble* and the *circular Jacobi beta ensemble*.

The real orthogonal beta ensemble was introduced in [25] (see also [24]) as a generalization of the joint eigenvalue distributions of a certain class of the classical compact random matrix models. (See Section 4.1 for more detail.) The operator level limit of the real orthogonal beta ensemble in the hard-edge limit was derived in [30]. The real orthogonal beta ensemble can be naturally transformed into another classical model, the (real) Jacobi beta ensemble, whose edge scaling limits were studied in [20]. The truncated version of the real orthogonal beta ensemble was introduced in [24], where the authors constructed a sparse matrix model and derived the joint eigenvalue distribution. In Theorem 6.8 and Corollary 6.9 of Section 6.2, we will establish the scaling limit of the truncated (and perturbed) real orthogonal beta ensemble, together with the scaling limit of its normalized characteristic polynomial.

The circular Jacobi beta ensemble is a one-parameter extension of the circular beta ensemble. For a complex parameter δ with $\Re\delta > -1/2$ it is given by the joint density function

$$\frac{1}{Z^{\mathrm{CJ}}_{n,\beta,\delta}} \prod_{j<k\leq n} \left| e^{i\theta_j} - e^{i\theta_k} \right|^\beta \prod_{k=1}^n (1 - e^{-i\theta_k})^\delta (1 - e^{i\theta_k})^{\bar{\delta}} \tag{1.9}$$

with respect to the uniform measure on the unit circle. For $\delta = 0$ this is just the circular beta ensemble. When $\delta = \beta\frac{k}{2}$ with a positive integer k then this model can be viewed as the circular beta ensemble conditioned to have k particles at $e^{i\theta} = 1$. (See Section 4.1 for additional details.) In [4] the authors constructed a family of unitary matrix models whose eigenvalues are distributed according to (1.9). Following the Killip-Nenciu approach they studied a random probability measure where the support is given by the circular Jacobi

beta ensemble, and the weights are given by an independently chosen beta-dependent Dirichlet distribution. [4] showed that although the Verblunsky coefficients for this measure are usually not independent, a modified version of these coefficients are in fact independent, and their distributions can be described explicitly. [30] used this representation to derive the point process and operator level scaling limit of the circular Jacobi beta ensemble. We build on these results, together with the ideas of [24], to define the truncated (and the perturbed) version of the circular Jacobi beta ensemble (see Section 7.1). We show that the joint density of the truncated circular beta ensemble is given by a constant multiple of

$$\prod_{j,k=1}^{n} (1 - z_j \bar{z}_k)^{\frac{\beta}{2}-1} \prod_{j<k} |z_k - z_j|^2 \prod_{j=1}^{n} \left((1 - z_j)^{\bar{\delta}} (1 - \bar{z}_j)^{\delta} \right), \tag{1.10}$$

generalizing (1.5). We then derive the point process scaling limit of these new models, together with the scaling limit of their normalized characteristic polynomial. See Theorem 7.13 and Corollary 7.14 of Section 7.2 for the precise statements.

1.4 Outline of the paper

Sections 2–4 provide the necessary background on Dirac-type operators and beta ensembles. Our goal was to present this material in a self-contained manner. In Section 2 we review the required background for the considered Dirac-type differential operators. In Section 3 we give an overview of how finitely supported probability measures on the unit circle can be represented with CMV matrices and Dirac-type differential operators. Section 4 describes the considered finite beta ensembles and their operator level limits, and summarizes the known results on scaling limits of characteristic polynomials of these models.

Sections 5–7 include the proofs of our new results. In Section 5 we provide general results regarding the convergence of the eigenvalues of truncated and perturbed CMV matrices. Section 6 provides the proofs for Theorems 1.2 and 1.3 on the scaling limits of the truncated and the perturbed circular beta ensemble, and proves the corresponding results for the real orthogonal beta ensemble. Section 7 proves our results on the truncated and the perturbed circular Jacobi beta ensemble.

Section 8 is an appendix that contains a more detailed discussion of the determinantal point processes mentioned in the Introduction, together with a few open problems.

2 Dirac-type differential operators

This section provides a brief overview of Dirac-type operators. A more detailed discussion can be found in [42], [44], and [46].

2.1 Basics of Dirac operators

Let $\mathcal{I}$ be $[0, 1)$ or $(0, 1]$. Suppose $x + iy : \mathcal{I} \mapsto \mathbb{H} = \{z \in \mathbb{C} : \Im z > 0\}$ is a locally bounded measurable function, we define

$$R = \frac{X^\dagger X}{2 \det X}, \qquad X = \begin{pmatrix} 1 & -x \\ 0 & y \end{pmatrix} \tag{2.1}$$

to be the positive definite matrix-valued function that encodes $x + iy$. We consider differential operators of the form

$$\tau : f \to R^{-1}(t) J f', \qquad f : \mathcal{I} \mapsto \mathbb{R}^2, \qquad J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{2.2}$$

We call τ a *Dirac-type operator*, $x + iy$ the *generating path* of τ, and $R(t)$ the *weight function* of τ.

The boundary conditions for τ at $t = 0, 1$ are given by two non-zero, non-parallel vectors $\mathfrak{u}_0, \mathfrak{u}_1 \in \mathbb{R}^2$. We assume that these vectors are normalized and satisfy certain integrability conditions with respect to the weight function.

Assumption 2.1. *We assume that*

$$\mathfrak{u}_0^\dagger J \mathfrak{u}_1 = 1, \tag{2.3}$$

$$\int_0^1 \int_0^t \mathfrak{u}_0^\dagger R(s) \mathfrak{u}_0 \, \mathfrak{u}_1^\dagger R(t) \mathfrak{u}_1 \, ds dt < \infty, \tag{2.4}$$

and

$$\int_0^1 \|R(s)\mathfrak{u}_1\| ds < \infty \text{ if } \mathcal{I} = [0, 1), \quad \text{and} \quad \int_0^1 \|\mathfrak{u}_0^\dagger R(s)\| ds < \infty \text{ if } \mathcal{I} = (0, 1].$$

Let L_R^2 denote the L^2 space of functions $f : \mathcal{I} \mapsto \mathbb{R}^2$ with the L^2-norm

$$\|f\|_R^2 := \int_{\mathcal{I}} f(s)^\dagger R(s) f(s) ds.$$

Under Assumption 2.1, the operator τ defined according to (2.1) and (2.2) is self-adjoint on the following domain:

$$\mathrm{dom}(\tau) = \{v \in L_R^2 \cap \mathrm{AC} : \tau v \in L_R^2, \lim_{s \to 0} v(s)^\dagger J \mathfrak{u}_0 = 0, \lim_{s \to 1} v(s)^\dagger J \mathfrak{u}_1 = 0\}. \tag{2.5}$$

Here $\mathrm{AC}(\mathcal{I})$ is the set of absolutely continuous real functions on $\mathcal{I}$.

Throughout the paper, we use the notations $\mathrm{Dir}(R, \mathfrak{u}_0, \mathfrak{u}_1)$ or $\mathrm{Dir}(x + iy, \mathfrak{u}_0, \mathfrak{u}_1)$ interchangeably for the operator τ defined via (2.2) and (2.1) on the domain (2.5). We can identify a nonzero vector in $\mathbb{R}^2$ with a boundary point in $\partial\mathbb{H} = \mathbb{R} \cup \{\infty\}$ using the projection operator

$$\mathcal{P}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{cases} x_1/x_2 & \text{if } x_2 \neq 0, \\ \infty & \text{if } x_2 = 0. \end{cases} \tag{2.6}$$

Using this identification the boundary conditions can be parametrized by two points in $\partial\mathbb{H}$.

Under Assumption 2.1, the operator τ is invertible, and τ^{-1} is a Hilbert-Schmidt integral operator.

Proposition 2.2. *Suppose that $\tau = \mathrm{Dir}(R, \mathfrak{u}_0, \mathfrak{u}_1)$ satisfies Assumption 2.1. Then*

$$\|\tau^{-1}\|_{\mathrm{HS}}^2 = 2 \int_0^1 \int_0^t \mathfrak{u}_0^\dagger R(s) \mathfrak{u}_0 \, \mathfrak{u}_1^\dagger R(t) \mathfrak{u}_1 \, ds dt < \infty,$$

and τ^{-1} is a Hilbert-Schmidt integral operator on L_R^2 given by

$$\tau^{-1} f(s) = \int_{\mathcal{I}} K_{\tau^{-1}}(s, t) f(t) dt, \quad K_{\tau^{-1}}(s, t) = (\mathfrak{u}_0 \mathfrak{u}_1^\dagger \mathbf{1}_{s < t} + \mathfrak{u}_1 \mathfrak{u}_0^\dagger \mathbf{1}_{s \geq t}) R(t). \tag{2.7}$$

Consider the conjugated operator $X\tau X^{-1}$, where X is defined as in (2.6). We denote the inverse of this operator by $\mathbf{r}\,\tau := (X\tau X^{-1})^{-1}$, this is an integral operator on $L^2(\mathcal{I})$ with kernel

$$K_{\mathbf{r}\,\tau}(s, t) = \tfrac{1}{2}(\mathfrak{a}_0(s)\mathfrak{a}_1(t)^\dagger \mathbf{1}_{s < t} + \mathfrak{a}_1(s)\mathfrak{a}_0(t)^\dagger \mathbf{1}_{s \geq t}), \qquad \mathfrak{a}_j(s) = \frac{X(s)\mathfrak{u}_j}{\sqrt{\det X}}, \quad j = 0, 1. \tag{2.8}$$

The conjugated operator $X\tau X^{-1}$ has the same spectrum as the operator τ, and $\|\mathbf{r}\,\tau\|_{\mathrm{HS}} = \|\tau^{-1}\|_{\mathrm{HS}}$.

Proposition 2.2 and the fact that τ is self-adjoint implies that τ has a discrete spectrum, with countably many eigenvalues that are all nonzero real numbers, and can only accumulate near $\pm\infty$. We label the eigenvalues of τ in increasing order by $\lambda_k, k \in \mathbb{Z}$ so that $\lambda_{-1} < 0 < \lambda_0$.

Let us remark that if $\tau = \mathtt{Dir}(R, \mathfrak{u}_0, \mathfrak{u}_1)$ satisfies Assumptions 2.1, then for any $\sigma \in (0, 1)$, the operator τ restricted in $\mathcal{I} \cap [0, \sigma]$ is well-defined with boundary conditions $\mathfrak{u}_0, \mathfrak{u}_1$ at endpoints $t = 0, \sigma$ and the restricted weight function $R|_{t \in \mathcal{I} \cap [0,\sigma]}$. We denote the restricted operator and its resolvent as $\tau_\sigma, \mathtt{r}\,\tau_\sigma$ respectively.

2.2 Canonical systems and the secular functions

In order to study scaling limits of characteristic polynomials, [44] introduced the secular function and structure function of a Dirac-type operator. We briefly review the required definitions and results.

Suppose that $\mathcal{I} = (0, 1]$ and consider the eigenvalue equation of a Dirac operator $\tau H = zH$ as a canonical system, i.e.,

$$J\frac{d}{dt}H(t, z) = zR(t)H(t, z).$$

[44] showed that this system has a unique solution under our assumptions if we set the initial condition to be $\mathfrak{u}_0$.

Proposition 2.3. *Suppose that $\mathcal{I} = (0, 1]$ and $\tau = \mathtt{Dir}(R, \mathfrak{u}_0, \mathfrak{u}_1)$ satisfies Assumptions 2.1. Then there exists a unique vector valued function $H : \mathcal{I} \times \mathbb{C} \mapsto \mathbb{C}^2$ so that for every $z \in \mathbb{C}$, the function $H(\cdot, z)$ solves the following ordinary differential equation*

$$J\frac{d}{dt}H(t, z) = zR(t)H(t, z), \qquad t \in \mathcal{I}, \qquad \lim_{t \to 0} H(t, z) = \mathfrak{u}_0. \tag{2.9}$$

Write $H = \binom{A}{B}$. For any $t \in \mathcal{I}$, the function $H(t, z)$ satisfies $\|H(t, z)\| \geq 0$, and its components $A(t, \cdot), B(t, \cdot)$ are analytic functions such that $A(t, x), B(t, x) \in \mathbb{R}$ for $x \in \mathbb{R}$.

Definition 2.4. *Under the assumptions of Proposition 2.3, we define the secular function of τ as*

$$\zeta_\tau(\cdot) = H(1, \cdot)^\dagger J\mathfrak{u}_1, \tag{2.10}$$

and the structure function of τ as

$$\mathcal{E}_\tau(\cdot) = H(1, \cdot)^\dagger \begin{pmatrix} 1 \\ -i \end{pmatrix} = A(1, \cdot) - iB(1, \cdot). \tag{2.11}$$

We define the integral trace of $\mathtt{r}\,\tau$ as the integral of the trace of the kernel $K_{\mathtt{r}\,\tau}$, and denote it by $\mathfrak{t}_\tau$:

$$\mathfrak{t}_\tau = \int_0^1 \operatorname{tr} K_{\mathfrak{t}_\tau}(s, s)ds = \frac{1}{2}\int_0^1 \mathfrak{a}_0(s)^\dagger \mathfrak{a}_1(s)ds = \int_0^1 \mathfrak{u}_0^\dagger R(s)\mathfrak{u}_1 ds. \tag{2.12}$$

It was proved in [44] that the secular function ζ_τ can be represented as

$$\zeta_\tau(z) = e^{-z\mathfrak{t}_\tau}\det{}_2(I - z\,\mathtt{r}\,\tau) = e^{-\frac{z}{2}\int_0^1 \mathfrak{a}_0(s)^\dagger \mathfrak{a}_1(s)ds}\prod_k (1 - z/\lambda_k)e^{z/\lambda_k}, \tag{2.13}$$

where $\det_2$ is the second regularized determinant, see [39]. Note that the integral trace is finite under Assumption 2.1, and the secular function ζ_τ is an entire function with zero set given by $\mathrm{spec}(\tau)$, the spectrum of τ. We refer to [44] for additional details. The secular function of τ can be viewed as a generalization of the normalized characteristic polynomial of a matrix.

The next proposition provides comparisons for the solutions of two canonical systems of the form (2.9). It also shows that $H(t, \cdot)$ is continuous on compacts in $z \in \mathbb{C}$ at $t = 0$.

Proposition 2.5 (Proposition 12 of [45])**.** *Suppose that* $\mathcal{I} = (0, 1]$ *and* $\tau = \mathtt{Dir}(R, \mathfrak{u}_0, \mathfrak{u}_1)$, $\tilde{\tau} = \mathtt{Dir}(\tilde{R}, \mathfrak{u}_0, \tilde{\mathfrak{u}}_1)$ *satisfy Assumption 2.1. Let* $H, \tilde{H}$ *be the solutions of the corresponding canonical systems (2.9), and define* $\mathfrak{a}_0, \tilde{\mathfrak{a}}_0$ *according to (2.8). Recall that* τ_t *is the operator* τ *restricted to* $(0, t]$. *Then there is an absolute constant* $c > 1$ *depending only on* $\mathfrak{u}_0$ *so that for all* $t \in \mathcal{I}$, $z \in \mathbb{C}$ *we have*

$$|H(t, z) - \tilde{H}(t, z)| \leq \left(c^{|z| \left(|\mathfrak{t}_{\tau_t} - \tilde{\mathfrak{t}}_{\tau_t}| + \|\mathfrak{r}\,\tau_t - \mathfrak{r}\,\tilde{\tau}_t\| + \sqrt{\int_0^t |\mathfrak{a}_0(s) - \tilde{\mathfrak{a}}_0(s)|^2 ds \int_0^t (|\mathfrak{a}_0(s)|^2 + |\tilde{\mathfrak{a}}_0(s)|^2) ds} \right)} - 1 \right)$$
$$\times\, c^{\left(|z|(|\mathfrak{t}_{\tau_t}| + |\mathfrak{t}_{\tilde{\tau}_t}| + \|\mathfrak{r}\,\tau_t\| + \|\mathfrak{r}\,\tilde{\tau}_t\| + \int_0^t (|\mathfrak{a}_0(s)|^2 + |\tilde{\mathfrak{a}}_0(s)|^2) ds) + 1 \right)^2}.$$
(2.14)

Moreover, we have for all $t \in \mathcal{I}$, $z \in \mathbb{C}$

$$|H(t, z) - \mathfrak{u}_0| \leq \left(c^{|z|(|\mathfrak{t}_{\tau_t}| + \|\mathfrak{r}\,\tau_t\| + \int_0^t |\mathfrak{a}_0(s)|^2 ds)} - 1 \right) c^{\left(|z|(|\mathfrak{t}_{\tau_t}| + \|\mathfrak{r}\,\tau_t\| + \int_0^t |\mathfrak{a}_0(s)|^2 ds) + 1 \right)^2}. \quad (2.15)$$

Proposition 2.5 together with the Hoffman-Wielandt inequality in infinite dimensions (see e.g. [3]) provide similar comparisons for the secular functions and the spectra of two Dirac operators.

Remark 2.6. Let τ_1, τ_2 be two Dirac operators on $\mathcal{I}$ satisfying Assumption 2.1. Denote by $\lambda_{k,i}, \zeta_i, \mathfrak{r}\,\tau_i, \mathfrak{t}_{\tau_i}$ the eigenvalues, secular function, resolvent and integral trace of τ_i. Then we have

$$\sum_k \left| \lambda_{k,1}^{-1} - \lambda_{k,2}^{-1} \right|^2 \leq \|\mathfrak{r}\,\tau_1 - \mathfrak{r}\,\tau_2\|_{\mathrm{HS}}^2, \quad (2.16)$$

and there is an absolute constant $c > 1$ so that for all $z \in \mathbb{C}$

$$|\zeta_1(z) - \zeta_2(z)| \leq \left(c^{|z||\mathfrak{t}_{\tau_1} - \mathfrak{t}_{\tau_2}|} - 1 + |z| \|\mathfrak{r}\,\tau_1 - \mathfrak{r}\,\tau_2\| \right) c^{|z|^2(\|\mathfrak{r}\,\tau_1\|^2 + \|\mathfrak{r}\,\tau_2\|_{\mathrm{HS}}^2) + |z|(|\mathfrak{t}_{\tau_1}| + |\mathfrak{t}_{\tau_2}|) + 1}. \quad (2.17)$$

Transformations of Dirac-type operators

We finish this section with a short discussion on simple transformations of Dirac-type operators. First note that the two cases $\mathcal{I} = (0, 1]$ and $\mathcal{I} = [0, 1)$ can be connected by a time reversal transformation ρ on functions defined on $(0, 1]$ or $[0, 1)$ by $\rho f(t) = f(1 - t)$. Let $\mathfrak{r} : \mathbb{H} \to \mathbb{H}$ be the reflection with respect to the imaginary axis defined by $\mathfrak{r}(x + iy) = -x + iy$. The next statement provides a description of the effect of the composition of the time reversal ρ and the reflection $\mathfrak{r}$, see [44] or [45].

Lemma 2.7 ([44])**.** *Suppose that the Dirac operator* $\tau = \mathtt{Dir}(R, \mathfrak{u}_0, \mathfrak{u}_1)$ *satisfies Assumption 2.1. Set* $S = \mathrm{diag}(1, -1)$, *then the operator* $\rho^{-1} S \tau S \rho$ *also satisfies Assumption 2.1 with boundary conditions* $S\mathfrak{u}_1, S\mathfrak{u}_0$, *weight function* ρSRS, *and generating path* $\mathfrak{r}\rho z$. *The operators* τ *and* $\rho^{-1} S \tau S \rho$ *are orthogonally equivalent in the respective* L^2 *spaces, and they have the same integral traces and secular functions.*

Consider the projection operator $\mathcal{P}$ defined in (2.6). It can be naturally generalized to the projection operator on non-zero two-dimensional complex vectors with $\mathcal{P}\binom{z_1}{z_2} = z_1/z_2$ given $z_2 \neq 0$. In this way, a 2×2 non-singular matrix A can be identified with a linear fractional transformation via $z \to \mathcal{P}A\binom{z}{1}$. When A is real, the corresponding linear fractional transformation is an isometry of $\mathbb{H}$. The next lemma describes the effect of a hyperbolic isometry on a Dirac operator.

Lemma 2.8 ([44])**.** *Let* Q *be a* 2×2 *orthogonal matrix with determinant 1. Let* $\mathcal{Q} : \overline{\mathbb{H}} \to \overline{\mathbb{H}}$ *be the corresponding linear isometry of* $\overline{\mathbb{H}}$ *mapping* $z \in \overline{\mathbb{H}}$ *to the ratio of the entries of*

$Q\binom{z}{1}$. *Suppose that the Dirac operator* $\tau = \mathrm{Dir}(R, \mathfrak{u}_0, \mathfrak{u}_1)$ *satisfies Assumption 2.1. Then the operator* $Q\tau Q^{-1}$ *also satisfies Assumption 2.1, with boundary conditions* $\mathcal{Q}\mathfrak{u}_0, \mathcal{Q}\mathfrak{u}_1$ *and generating path* $\mathcal{Q}(x + iy)$. *The two operators are orthogonally equivalent, they have the same integral traces and secular functions.*

3 Finitely supported measures on the unit circle

In this section we provide an overview of the CMV construction for finitely supported probability measures on the unit circle, and review how Dirac-type operators can be used to study them.

3.1 CMV matrices

We briefly review some of the needed facts from the theory of orthogonal polynomials on the unit circle, together with some properties of CMV matrices. See [38] for a comprehensive treatment of the subject, or [37] for a shorter summary.

Let ν be a probability measure on $\partial\mathbb{D}$ with a support of n distinct points $e^{i\lambda_j}, 1 \leq j \leq n$. Let $\Phi_k, k = 0, \ldots, n$ be the Gram-Schmidt orthogonalization of the polynomials $1, z, \cdots, z^n$ with respect to ν. Then $\Phi_k, k = 0, \ldots, n-1$ are the monic orthogonal polynomials with respect to ν, and $\Phi_n(z) := \prod_{j=1}^{n}(z - e^{i\lambda_j})$. Together with the reversed polynomials

$$\Phi_k^*(z) := z^k \overline{\Phi_k(1/\bar{z})},$$

these polynomials satisfy the famous Szegő recursion (see e.g. Section 1.5, vol. 1 of [38]):

$$\begin{pmatrix} \Phi_{k+1} \\ \Phi_{k+1}^* \end{pmatrix} = \begin{pmatrix} 1 & -\bar{\alpha}_k \\ -\alpha_k & 1 \end{pmatrix} \begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \Phi_k \\ \Phi_k^* \end{pmatrix}, \qquad \begin{pmatrix} \Phi_0 \\ \Phi_0^* \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \qquad 0 \leq k \leq n-1. \quad (3.1)$$

The constants $\alpha_k, 0 \leq k \leq n-1$ are called the Verblunsky coefficients, they satisfy $|\alpha_k| < 1$ for $0 \leq k \leq n-2$ and $|\alpha_{n-1}| = 1$. The map between the probability measures supported on n points on $\partial\mathbb{D}$ and the possible Verblunsky coefficients $(\alpha_0, \ldots, \alpha_{n-1}) \in \mathbb{D}^{n-1} \times \partial\mathbb{D}$ is invertible, and both the map and its inverse are continuous. (See e.g. Theorem 1.7.11 of [38].)

The next definition introduces the CMV matrix.

Definition 3.1. *For fixed* $n \geq 1$, *let* $\{\alpha_k, 0 \leq k \leq n-1\}$ *be a sequence of complex coefficients with* $|\alpha_k| \leq 1$. *Define*

$$\Xi_k = \begin{pmatrix} \bar{\alpha}_k & \rho_k \\ \rho_k & -\alpha_k \end{pmatrix}, \qquad \rho_k = \sqrt{1 - |\alpha_k|^2}, \qquad 0 \leq k \leq n-2$$

and set $\Xi_{-1} = (1)$ *and* $\Xi_{n-1} = (\bar{\alpha}_{n-1})$ *to be* 1×1 *matrices.*

For $n \geq 2$ *we define the* CMV matrix *corresponding to* $\{\alpha_k, 0 \leq k \leq n-1\}$ *as*

$$\mathcal{C}(\alpha_0, \cdots, \alpha_{n-1}) := \mathcal{L}\mathcal{M}, \qquad (3.2)$$

where $\mathcal{L}, \mathcal{M}$ *are* $n \times n$ *block-diagonal matrices*

$$\mathcal{L} = \mathrm{diag}\left(\Xi_0, \Xi_2 \cdots, \Xi_{2\lfloor \frac{n-1}{2} \rfloor}\right), \qquad \mathcal{M} = \mathrm{diag}\left(\Xi_{-1}, \Xi_1 \cdots, \Xi_{2\lfloor \frac{n}{2} \rfloor - 1}\right).$$

For $n = 1$ *we define the* 1×1 *CMV matrix as* $\mathcal{C}(\alpha_0) = (\bar{\alpha}_0)$.

Note that $\mathcal{C}(\alpha_0, \ldots, \alpha_{n-1})$ is unitary if and only if $|\alpha_{n-1}| = 1$. The following proposition provides a crucial link between CMV matrices and orthogonal polynomials of a discrete probability measure on $\partial\mathbb{D}$.

Proposition 3.2 ([24, 38]). *Suppose that ν is a probability measure with a support of n distinct points on $\partial\mathbb{D}$, and let $\alpha_k, 0 \leq k \leq n-1$ be its sequence of Verblunsky coefficients. Then for any $1 \leq k \leq n$ we have*

$$\det(zI_k - \mathcal{C}(\alpha_0, \ldots, \alpha_{k-1})) = \Phi_k(z). \tag{3.3}$$

Suppose that U is an $n \times n$ unitary matrix for which $\mathbf{e}_1 = (1, 0, \ldots, 0)^\dagger$ is cyclic. The spectral measure of U with respect to $\mathbf{e}_1$ is given by the following discrete probability measure:

$$\mu = \sum_{k=1}^{n} \delta_{\lambda_k} \cdot |\langle \mathbf{e}_1, \mathbf{v}_k \rangle|^2. \tag{3.4}$$

Here $\lambda_k, 1 \leq k \leq n$ are the eigenvalues of U, and $\mathbf{v}_k, 1 \leq k \leq n$ are the corresponding unit length (right) eigenvectors. The following proposition summarizes how a unitary matrix and the CMV matrix of its spectral measure are connected.

Proposition 3.3 ([5]). *Suppose that U is an $n \times n$ unitary matrix for which $\mathbf{e}_1$ is cyclic. Let the spectral measure of U with respect to $\mathbf{e}_1$ be ν, and denote the Verblunsky coefficients of ν by $\alpha_k, 0 \leq k \leq n-1$. Then there is a unitary matrix V with $V\mathbf{e}_1 = \mathbf{e}_1$ and $U = V\mathcal{C}(\alpha_0, \ldots, \alpha_{n-1})V^{-1}$. In particular, U and $\mathcal{C}$ are unitary equivalent, and they have the same spectral measure with respect to $\mathbf{e}_1$.*

The next definition introduces a simple transformation on a collection of Verblunsky coefficients and on the corresponding CMV matrix.

Definition 3.4. *Suppose that $(\alpha_0, \ldots, \alpha_{n-1}) \in \mathbb{D}^{n-1} \times \partial\mathbb{D}$ is the sequence of Verblunsky coefficients of the discrete probability measure ν on $\partial\mathbb{D}$. Let $\mathcal{C}$ denote the CMV matrix corresponding to these Verblunsky coefficients. We define the 'reversed' version of the Verblunsky coefficients as*

$$(\widetilde{\alpha}_0, \widetilde{\alpha}_1, \ldots, \widetilde{\alpha}_{n-1}) := (-\alpha_{n-1}\bar{\alpha}_{n-2}, -\alpha_{n-1}\bar{\alpha}_{n-3}, \ldots, -\alpha_{n-1}\bar{\alpha}_0, \alpha_{n-1}). \tag{3.5}$$

We denote by $\widetilde{\nu}$ and $\widetilde{\mathcal{C}}$ the probability measure and CMV matrix corresponding to the sequence $\widetilde{\alpha}_k, 0 \leq k \leq n-1$, and call these the reversed version of μ and $\mathcal{C}$, respectively.

Note that since $|\alpha_{n-1}| = 1$, the reversal operation is an involution. This operation appeared in [25] where it was shown that $\widetilde{\mathcal{C}}$ is unitary equivalent to $\mathcal{C}$ (if n is even), and to $\mathcal{C}^\dagger$ (if n is odd). This implies that $\mathcal{C}$ and $\widetilde{\mathcal{C}}$ have the same eigenvalues, or equivalently, μ and $\widetilde{\mu}$ have the same support. In fact, the arguments of Proposition B.2 of [25] also imply that $\widetilde{\mu}$ is the spectral measure of $\mathcal{C}$ with respect to the unit vector $\mathbf{e}_n = (0, 0, \ldots, 0, 1)^\dagger$.

The following results were proved in [24]. They provide key ingredients for our study of truncated and perturbed unitary matrices.

Proposition 3.5 ([24]). *Consider the same setup as in Proposition 3.3. Then if $n \geq 2$ then the truncated matrix $\mathrm{Trunc}(U)$ is unitary equivalent to $\mathrm{Trunc}(\mathcal{C})$, which in turn is unitary equivalent to*

$$\mathcal{C}(\widetilde{\alpha}_0, \ldots, \widetilde{\alpha}_{n-2}), \quad \textit{if } n \textit{ is even, and} \quad \mathcal{C}(\widetilde{\alpha}_0, \ldots, \widetilde{\alpha}_{n-2})^\dagger, \quad \textit{if } n \textit{ is odd.} \tag{3.6}$$

For $r \in [0, 1]$ the perturbed matrix $U^{[r]}$ (as defined in (1.7)) is unitary equivalent to

$$\mathcal{C}(\widetilde{\alpha}_0, \ldots, \widetilde{\alpha}_{n-2}, r\widetilde{\alpha}_{n-1}), \quad \textit{if } n \textit{ is even, and} \quad \mathcal{C}(\widetilde{\alpha}_0, \ldots, \widetilde{\alpha}_{n-2}, r\widetilde{\alpha}_{n-1})^\dagger, \quad \textit{if } n \textit{ is odd.} \tag{3.7}$$

3.2 Connection to Dirac-type operators

In this section, we show how the Dirac operator framework introduced in [42, 43, 44] can be used to study finitely supported probability measures on $\partial\mathbb{D}$.

The following definition constructs a Dirac-type differential operator corresponding to a finitely supported probability measure on $\partial\mathbb{D}$.

Definition 3.6. *Suppose that μ is a probability measure on $\partial\mathbb{D}$ supported on n distinct points $\{e^{i\lambda_j}, 1 \le j \le n\}$, and let $\alpha_k, 0 \le k \le n-1$ be the corresponding Verblunsky coefficients. For $0 \le k \le n$ we define*

$$b_0 = 0, \quad b_k = \mathcal{P} \begin{pmatrix} 1 & \bar{\alpha}_0 \\ \alpha_0 & 1 \end{pmatrix} \cdots \begin{pmatrix} 1 & \bar{\alpha}_{k-1} \\ \alpha_{k-1} & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \text{for } 1 \le k \le n. \tag{3.8}$$

Set $z_k = \mathcal{U}^{-1}(b_k)$, where $\mathcal{U}$ is the Cayley transform mapping $\overline{\mathbb{H}}$ to $\overline{\mathbb{D}}$ defined via

$$\mathcal{U}(z) = \mathcal{P}U \begin{pmatrix} z \\ 1 \end{pmatrix} = \frac{z-i}{z+i}, \qquad U = \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}. \tag{3.9}$$

We call $b_k, 0 \le k \le n$ and $z_k, 0 \le k \le n$ the (discrete) path parameters *of μ in $\mathbb{D}$ and in $\mathbb{H}$, respectively. We say that $z : [0,1) \to \mathbb{H}$ defined via $z(t) = (x+iy)(t) := z_{\lfloor nt \rfloor}$ is the* generating path *associated to μ. We set $\mathfrak{u}_0 = \binom{1}{0}$, $\mathfrak{u}_1 = \binom{-z_n}{-1}$, and we call $\tau = \mathrm{Dir}(z(\cdot), \mathfrak{u}_0, \mathfrak{u}_1)$ the Dirac-type operator corresponding to μ.*

We may call τ the Dirac-type operator corresponding to the Verblunsky coefficients $\alpha_k, 0 \le k \le n-1$, or the path $b_0, \ldots, b_n$. We may use the notation $\tau = \mathrm{Dir}(b_{-1}, b_0, \ldots, b_n)$ with $b_{-1} = \mathcal{P}U^{-1}\mathfrak{u}_0 = 1$ to emphasize the path dependence. We call b_{-1} and b_n the *left* and *right boundary point* of τ, respectively.

As the next proposition shows, the Dirac-type operator corresponding to μ encodes the support of μ in the spectrum.

Proposition 3.7 ([44, 45])**.** *Suppose that μ is a probability measure on $\partial\mathbb{D}$ supported on n distinct points $e^{i\lambda_j}, 1 \le j \le n$, with $\mu(\{1\}) = 0$. Then the Dirac-type operator τ corresponding to μ satisfies Assumption 2.1 with $\mathcal{I} = [0,1)$, and its spectrum is given by*

$$\mathrm{spec}(\tau) = n\Lambda_n + 2\pi n\mathbb{Z}, \qquad \Lambda_n = \{\lambda_1, \ldots, \lambda_n\}.$$

In [42, 45] it was observed that the Dirac operator representation for μ can be simplified using the modified Verblunsky coefficients. These are defined in terms of the Verblunsky coefficients via the recursion

$$\gamma_k = \bar{\alpha}_k \prod_{j=0}^{k-1} \frac{1 - \bar{\gamma}_j}{1 - \gamma_j}, \qquad 0 \le k \le n-1. \tag{3.10}$$

Note that the modified Verblunsky coefficients satisfy $|\gamma_k| = |\alpha_k|$. Denote by $\mathcal{T}$ the mapping from the Verblunsky coefficients to the modified ones. This mapping is invertible, in fact for any $k \ge 1$ it provides a one-to-one correspondence between the first k Verblunsky coefficients and the first k modified Verblunsky coefficients. We will use the notation $\mathcal{T}_k$ for this restricted map.

The modified Verblunsky coefficients of a discrete probability measure μ are connected to its normalized orthogonal polynomials (with normalization at 1). Let Φ_k be the monic orthogonal polynomials of μ, with Φ_k^* being the reversed polynomials. We set

$$\Psi_k(z) = \frac{\Phi_k(z)}{\Phi_k(1)}, \qquad \Psi_k^*(z) = \frac{\Phi_k^*(z)}{\Phi_k^*(1)}. \tag{3.11}$$

These are always well defined for $0 \le k \le n-1$, and Ψ_n is defined as long as $\mu(\{1\}) = 0$. The polynomials Ψ, Ψ^* satisfy the following modified version of the Szegő recursion (3.1):

$$\begin{pmatrix} \Psi_{k+1} \\ \Psi_{k+1}^* \end{pmatrix} = \begin{pmatrix} \frac{1}{1-\gamma_k} & -\frac{\gamma_k}{1-\gamma_k} \\ -\frac{\bar{\gamma}_k}{1-\bar{\gamma}_k} & \frac{1}{1-\bar{\gamma}_k} \end{pmatrix} \begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \Psi_k \\ \Psi_k^* \end{pmatrix}, \quad \begin{pmatrix} \Psi_0 \\ \Psi_0^* \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad 0 \le k \le n-1. \tag{3.12}$$

The matrices appearing in this recursion correspond to affine transformations.

Affine transformations of $\mathbb{H}$ can be parametrized by the elements of $\mathbb{H}$ as follows. For $z = x + iy \in \mathbb{H}$ we define the matrix A_z and the corresponding linear fractional transformation $\mathcal{A}_{z,\mathbb{H}} : \overline{\mathbb{H}} \mapsto \overline{\mathbb{H}}$ as follows:

$$A_{x+iy,\mathbb{H}} = \begin{pmatrix} 1 & -x \\ 0 & y \end{pmatrix}, \qquad \mathcal{A}_{x+iy,\mathbb{H}}(w) = \mathcal{P} A_{x+iy,\mathbb{H}} \begin{pmatrix} w \\ 1 \end{pmatrix}. \tag{3.13}$$

Note that $x + iy$ is the pre-image of i under $\mathcal{A}_{x+iy,\mathbb{H}}$.

The transformations $\mathcal{A}_{z,\mathbb{H}}$ are isometries of the half-plane model of the hyperbolic plane. The corresponding transformations on the unit-disk model of the hyperbolic plane can be obtained by conjugating with the Cayley transform. For $\gamma \in \mathbb{D}$ we set

$$A_{\gamma,\mathbb{D}} = U A_{\mathcal{U}^{-1}(\gamma),\mathbb{H}} U^{-1}, \qquad \mathcal{A}_{\gamma,\mathbb{D}} = \mathcal{U} \circ \mathcal{A}_{\mathcal{U}^{-1}(\gamma),\mathbb{H}} \circ \mathcal{U}^{-1}, \tag{3.14}$$

which leads to

$$A_{\gamma,\mathbb{D}} = \begin{pmatrix} \frac{1}{1-\bar{\gamma}} & \frac{\gamma}{\bar{\gamma}-1} \\ \frac{\bar{\gamma}}{\bar{\gamma}-1} & \frac{1}{1-\bar{\gamma}} \end{pmatrix}, \qquad \mathcal{A}_{\gamma,\mathbb{D}}(z) = \mathcal{P} A_{\gamma,\mathbb{D}} \begin{pmatrix} z \\ 1 \end{pmatrix}. \tag{3.15}$$

Note that the matrix coefficient in (3.12) is exactly $A_{\gamma_k,\mathbb{D}}$, and $\mathcal{A}_{\gamma,\mathbb{D}}$ maps γ to 0.

The following proposition shows that the generating path of a discrete probability measure μ can be expressed via a simple (affine) recursion using the modified Verblunsky coefficients. Define $w_k, v_k \in \mathbb{R}$ from the modified Verblunsky coefficients as

$$v_k + i w_k := \frac{2i\gamma_k}{1 - \gamma_k} = \mathcal{U}^{-1}(\gamma_k) - i. \tag{3.16}$$

Proposition 3.8 ([44, 45]). *Suppose that μ is a probability measure on $\partial\mathbb{D}$ supported on n distinct points $e^{i\lambda_j}, 1 \le j \le n$, and $\mu(\{1\}) = 0$. Let $\gamma_k, 0 \le k \le n-1$ be the modified Verblunsky coefficients of μ. Let $b_k, z_k, 0 \le k \le n$ be the path parameters of μ defined as in Definition 3.6. Then the following identities hold for $0 \le k \le n-1$:*

$$z_{k+1} = z_k + (v_k + i w_k)\Im z_k, \tag{3.17}$$

with v_k, w_k defined in (3.16), and

$$b_{k+1} = \frac{b_k + \gamma_k \frac{1-b_k}{1-\bar{b}_k}}{1 + \bar{b}_k \gamma_k \frac{1-b_k}{1-\bar{b}_k}}. \tag{3.18}$$

We also have

$$b_k = \mathcal{A}_{\gamma_0,\mathbb{D}}^{-1} \circ \cdots \circ \mathcal{A}_{\gamma_{k-1},\mathbb{D}}^{-1}(0). \tag{3.19}$$

The normalized orthogonal polynomials of μ can be expressed using the canonical system (2.9) of the Dirac operator τ corresponding to μ.

Proposition 3.9 ([44, 45]). *Under the same setup as in Proposition 3.7, consider the solution to the canonical system (2.9)*

$$\tau H = \lambda H, \qquad H(0, \lambda) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad (t, \lambda) \in [0, 1] \times \mathbb{C}.$$

Then the normalized orthogonal polynomials Ψ_k, Ψ_k^* of μ (defined via (3.11)) satisfy

$$\begin{pmatrix} \Psi_k(e^{i\lambda/n}) \\ \Psi_k^*(e^{i\lambda/n}) \end{pmatrix} = e^{i\lambda k/(2n)} \begin{pmatrix} 1 & -z_k \\ 1 & -\bar{z}_k \end{pmatrix} H(k/n, \lambda), \qquad 0 \le k \le n. \tag{3.20}$$

Note that Proposition 3.9 applied with $k = n$ gives

$$e^{-i\lambda/2}\Psi_n(e^{i\lambda/n}) = \begin{pmatrix} 1 \\ -z_n \end{pmatrix}^\dagger H(1,\lambda) = H(1,\lambda)^\dagger J \begin{pmatrix} 1 \\ z_n \end{pmatrix}. \tag{3.21}$$

Recall that $\Psi_n(\cdot)$ is just the normalized characteristic polynomial corresponding to the support of μ:

$$\Psi_n(z) = \frac{\Phi_n(z)}{\Phi_n(1)} = \prod_{j=1}^{n} \frac{z - e^{i\lambda_j}}{1 - e^{i\lambda_j}}$$

Hence (2.10) and (3.21) imply that the scaled and normalized characteristic polynomial of μ is the same as the secular function of the Dirac-type operator corresponding to μ.

4 Beta ensembles from classical unitary and orthogonal random matrices

This section collects the matrix models for various beta-generalizations of unitary and orthogonal random matrices, along with their Dirac operator representations and operator limits.

4.1 Finite ensembles and their Dirac operator representations

Circular beta ensemble

Recall the definition of the size n circular beta ensemble with density function (1.4) and the spectral measure with respect to $\mathbf{e}_1$ (3.4). Recall also that for $a_1, \ldots, a_n > 0$ the Dirichlet $(a_1, \ldots, a_n)$ distribution is a probability measure on the set

$$\{(x_1, x_2, \ldots, x_n) \in [0,1]^n : \sum_{i=1}^{n} x_i = 1\}$$

with joint probability density function

$$\frac{\Gamma(\sum_{i=1}^{n} a_i)}{\prod_{i=1}^{n} \Gamma(a_i)} \prod_{i=1}^{n} x_i^{a_i - 1}.$$

We define the size n Killip-Nenciu measure as a random probability measure on $\partial\mathbb{D}$ as

$$\mu_{n,\beta}^{\mathrm{KN}} = \sum_{j=1}^{n} \pi_j \delta_{e^{i\lambda_j}}, \tag{4.1}$$

where the support is distributed as the size n circular beta ensemble, and the weights $\pi_j, 1 \leq j \leq n$ are chosen according to Dirichlet $(\beta/2, \ldots, \beta/2)$ distribution, independently of the support. The joint distribution of the Verblunsky coefficients with respect to $\mu_{n,\beta}^{\mathrm{KN}}$ was computed in [25].

Definition 4.1. *For $a > 0$ we denote by $\Theta(a+1)$ the distribution on $\mathbb{D}$ that has probability density function*

$$\frac{a}{2\pi}(1 - |z|^2)^{a/2 - 1}. \tag{4.2}$$

The definition is extended for $a = 0$ as follows: $\Theta(1)$ is the uniform distribution on $\partial\mathbb{D}$.

Proposition 4.2 ([25])**.** *For a fixed n and $\beta > 0$, the sequence of Verblunsky coefficients $\alpha_k, 0 \leq k \leq n-1$ of $\mu_{n,\beta}^{\mathrm{KN}}$ are independent, and $\alpha_k \sim \Theta(\beta(n-k-1)+1)$ for $0 \leq k \leq n-1$.*

Definition 4.3. *For fixed n and $\beta > 0$, we denote by $\mathrm{Circ}_{n,\beta}$ and $\mathtt{Circ}_{n,\beta}$ the CMV matrix and Dirac-type operator corresponding to $\mu_{n,\beta}^{\mathrm{KN}}$.*

Real orthogonal beta ensemble

The *real orthogonal beta ensemble* is a family of distributions supported on conjugated pairs of points on the unit circle indexed by real parameters $\beta > 0, a > -1, b > -1$. If we parametrize the size $2n$ real orthogonal beta ensemble as $\{\pm e^{i\theta_1}, \ldots, \pm e^{i\theta_n}\}$ with $\theta_j \in (0, \pi)$ then the joint density for $(\theta_1, \ldots, \theta_n)$ is proportional to

$$\prod_{j < k \leq n} |\cos(\theta_j) - \cos(\theta_k)|^\beta \times \prod_{k=1}^n |1 - \cos(\theta_k)|^{\frac{\beta}{2}(a+1)-1/2} |1 + \cos(\theta_k)|^{\frac{\beta}{2}(b+1)-1/2}. \quad (4.3)$$

The ensemble was introduced in [25], it can be viewed as a generalization of joint eigenvalue distributions of some classical orthogonal ensembles. For example when $\beta = 2$, $a = b = \frac{1}{\beta} - 1$, (4.3) describes the joint eigenvalue distribution of a $2n \times 2n$ special orthogonal matrix chosen uniformly with respect to the Haar measure on $\mathbb{SO}(2n)$.

Consider the probability measure

$$\mu^{\mathrm{RO}}_{2n,\beta,a,b} = \sum_{j=1}^n \frac{1}{2}\pi_j(\delta_{e^{i\theta_j}} + \delta_{e^{-i\theta_j}}) \quad (4.4)$$

on the unit circle, where the support is distributed as a $2n$ real orthogonal beta ensemble, and the weights $(\pi_1, \ldots, \pi_n)$ is an independent random vector that has a Dirichlet $(\beta/2, \ldots, \beta/2)$ distribution. The joint distribution of the Verblunsky coefficients of $\mu^{\mathrm{RO}}_{2n,\beta,a,b}$ was derived in [24].

Definition 4.4. *For $s, t > 0$ let $\tilde{\mathrm{B}}(s, t)$ denote the scaled (and flipped) beta distribution on $(-1, 1)$ that has probability density function*

$$\frac{2^{1-s-t}\Gamma(s+t)}{\Gamma(s)\Gamma(t)}(1-x)^{s-1}(1+x)^{t-1}.$$

Proposition 4.5 (Theorem 2 of [25], Proposition 4.5 in [24])**.** *For given $\beta > 0$, $a, b > -1$ and fixed n, the Verblunsky coefficients $\alpha_k, 0 \leq k \leq 2n - 1$ of the random probability measure $\mu^{\mathrm{RO}}_{2n,\beta,a,b}$ are independent with $\alpha_{2n-1} = -1$, and for $0 \leq k \leq 2n - 2$*

$$\alpha_k \sim \begin{cases} \tilde{\mathrm{B}}\big(\frac{\beta}{4}(2n - k + 2a), \frac{\beta}{4}(2n - k + 2b)\big), & \text{if } k \text{ is even,} \\ \tilde{\mathrm{B}}\big(\frac{\beta}{4}(2n - k + 2a + 2b + 1), \frac{\beta}{4}(2n - k - 1)\big), & \text{if } k \text{ is odd.} \end{cases}$$

Definition 4.6. *For fixed $n \geq 1$, $a, b > -1$ and $\beta > 0$, we define $\mathrm{RO}_{2n,\beta,a,b}$ and $\mathtt{RO}_{2n,\beta,a,b}$ as the CMV matrix model and the Dirac-type operator corresponding to $\mu^{\mathrm{RO}}_{2n,\beta,a,b}$, respectively.*

Circular Jacobi beta ensemble

The circular Jacobi beta ensemble can be viewed as a one-parameter generalization of the circular beta ensemble. Let δ be a complex parameter such that $\Re\delta > -1/2$. The size n circular Jacobi beta ensemble is defined as the probability measure on the n distinct points $\{e^{i\theta_1}, \ldots, e^{i\theta_n}\}$ with $\theta_j \in [-\pi, \pi)$, where the joint density function of the angles θ_j is given by (1.9). For $\beta = 2$ the distribution was studied by Hua [22] and Pickrell [35], this special case is sometimes called the Hua-Pickrell measure.

Consider the random probability measure

$$\mu^{\mathrm{CJ}}_{n,\beta,\delta} = \sum_{j=1}^n \pi_j \delta_{e^{i\theta_j}} \quad (4.5)$$

on the unit circle, where the support is distributed as the size n circular Jacobi beta ensemble, the weights are Dirichlet $(\beta/2, \ldots, \beta/2)$ distributed and are independent

of the support. In [4] the authors showed that the modified Verblunsky coefficients $\gamma_k, 0 \le k \le n-1$ of $\mu_{n,\beta,\delta}^{\mathrm{CJ}}$ are independent and described explicitly their joint distribution.

We first introduce a generalization of the $\Theta(a+1)$ distribution defined in Definition 4.1.

Definition 4.7. *For $a > 0$ and $\Re\delta > -1/2$ let $\Theta(a+1,\delta)$ be the distribution on $\mathbb{D}$ that has probability density function*

$$c_{a,\delta}(1-|z|^2)^{a/2-1}(1-z)^{\bar{\delta}}(1-\bar{z})^{\delta}, \qquad c_{a,\delta} = \frac{\Gamma(a/2+1+\delta)\Gamma(a/2+1+\bar{\delta})}{\pi\Gamma(a/2)\Gamma(a/2+1+\delta+\bar{\delta})}. \tag{4.6}$$

For the $a = 0$, $\Re\delta > -1/2$ case we define $\Theta(1,\delta)$ to be the distribution on $\partial\mathbb{D} = \{|z| = 1\}$ with probability density function

$$\frac{\Gamma(1+\delta)\Gamma(1+\bar{\delta})}{\Gamma(1+\delta+\bar{\delta})}(1-z)^{\bar{\delta}}(1-\bar{z})^{\delta}. \tag{4.7}$$

Proposition 4.8 ([4]). *For given $n \ge 1, \beta > 0, \Re\delta > -1/2$, the sequence of modified Verblunsky coefficients $\gamma_k, 0 \le k \le n-1$ of $\mu_{n,\beta,\delta}^{\mathrm{CJ}}$ are independent with $\gamma_k \sim \Theta(\beta(n-k-1)+1,\delta)$ for $0 \le k \le n-1$.*

Note that for $\delta \ne 0$ the Verblunsky coefficients $\alpha_k, 0 \le k \le n-1$ are not independent.

Definition 4.9. *For given $n \ge 1, \beta > 0, \Re\delta > -1/2$, we define $\mathrm{CJ}_{n,\beta,\delta}$ and $\mathsf{CJ}_{n,\beta,\delta}$ as the CMV matrix model and the Dirac-type operator corresponding to $\mu_{n,\beta,\delta}^{\mathrm{CJ}}$, respectively.*

4.2 Random operator limits from beta ensembles

This section discusses the operator limits of the finite ensembles in Section 4.1. The main idea is that under the appropriate scaling, the piece-wise constant generating paths of the $\mathrm{Circ}_{n,\beta}, \mathrm{RO}_{2n,\beta,a,b}$ and $\mathsf{CJ}_{n,\beta,\delta}$ operators converge to certain diffusions in the hyperbolic plane. Then one can construct random differential operators in terms of these diffusions. These limiting operators will be denoted as $\mathrm{Sine}_{\beta}, \mathrm{Bess}_{\beta,a}$ and $\mathrm{HP}_{\beta,\delta}$, respectively. For convenience, we will define the limiting operators with generating paths that lie in $\mathbb{H}$. We also set

$$v(t) = v_\beta(t) = -\frac{4}{\beta}\log(1-t)$$

be the logarithmic time change function.

The Sine_{β} operator was introduced in [42] as the $n \to \infty$ limit of the $\mathrm{Circ}_{n,\beta}$ operator. The operator level convergence was proved in [43] with an explicit rate of convergence, see Proposition 6.3 below.

Definition 4.10. *Fix $\beta > 0$. Let B_1, B_2 be independent standard Brownian motion, and let $\mathsf{x}_v + i\mathsf{y}_v, v \ge 0$ be the strong solution of the SDE*

$$d\mathsf{y} = \mathsf{y}dB_1, \quad d\mathsf{x} = \mathsf{y}dB_2, \quad \mathsf{y}(0) = 1, \mathsf{x}(0) = 0. \tag{4.8}$$

For $t \in [0,1)$ define $z(t) = x(t) + iy(t) = \mathsf{x}_v + i\mathsf{y}_v$ where $v = v(t)$. Let $\mathfrak{u}_0 = \binom{1}{0}$, $\mathfrak{u}_1 = \binom{-q}{-1}$, where $q = \lim_{t\to\infty} \mathsf{x}(t)$. Set $\mathrm{Sine}_{\beta} = \mathrm{Dir}(z(\cdot), \mathfrak{u}_0, \mathfrak{u}_1)$.

Note that $\mathsf{x}_v + i\mathsf{y}_v, v \ge 0$ is just a hyperbolic Brownian motion in $\mathbb{H}$, started from i.

The hard-edge Dirac operator $\mathrm{Bess}_{\beta,a}$ and the Hua-Pickrell operator $\mathrm{HP}_{\beta,\delta}$ were also introduced in [42]. It has been shown in [30] that they are indeed the operator level limits of the $\mathrm{RO}_{2n,\beta,a,b}$ and $\mathsf{CJ}_{n,\beta,\delta}$ operators, respectively.

Definition 4.11. *Fix $\beta > 0, a > -1$, and let B be a standard Brownian motion. Set $\mathsf{y}(t) = e^{-\frac{\beta}{4}(2a+1)t - B(2t)}$, $y(t) = \mathsf{y}(v(t))$, $\mathfrak{u}_0 = \binom{1}{0}$, and $\mathfrak{u}_1 = \binom{0}{-1}$. Define the hard-edge operator as $\mathrm{Bess}_{\beta,a} := \mathrm{Dir}(iy, \mathfrak{u}_0, \mathfrak{u}_1)$.*

Definition 4.12. *Fix $\beta > 0$ and $\delta \in \mathbb{C}$ with $\Re\delta > -1/2$. Let B_1, B_2 be independent standard Brownian motion, and let $\mathsf{x}_v + i\mathsf{y}_v, v \geq 0$ be the strong solution of the SDE*

$$d\mathsf{y} = (-\Re\delta dt + dB_1)\,\mathsf{y}, \quad d\mathsf{x} = (\Im\delta dt + dB_2)\,\mathsf{y}, \quad \mathsf{y}(0) = 1, \mathsf{x}(0) = 0. \tag{4.9}$$

For $t \in [0,1)$ define $z(t) = x(t) + iy(t) = \mathsf{x}_v + i\mathsf{y}_v$ where $v = v(t)$. Let $\mathfrak{u}_0 = \binom{1}{0}$, $\mathfrak{u}_1 = \binom{-q}{-1}$, where $q = \lim_{v \to \infty} \mathsf{x}(v)$. Set $\mathrm{HP}_{\beta,\delta} = \mathrm{Dir}(z(\cdot), \mathfrak{u}_0, \mathfrak{u}_1)$.

Note that the SDE (4.9) can be solved explicitly and the solution is given by

$$\mathsf{y}(v) = e^{B_1(v) - (\Re\delta + \frac{1}{2})v}, \quad \mathsf{x}(v) = \int_0^v \mathsf{y}(s)dB_2(s) + \Im\delta \int_0^v \mathsf{y}(s)ds. \tag{4.10}$$

In the case when $\delta = 0$ the equation reduces to (4.8). In particular, the $\mathrm{HP}_{\beta,\delta}$ operator can be viewed as the δ-generalization of Sine_β, and we have $\mathrm{HP}_{\beta,0} = \mathrm{Sine}_\beta$.

4.3 Random analytic functions from beta ensembles

As explained in Section 2.2, the Hilbert–Schmidt convergence of the resolvents of Dirac-type operators and the convergence of the integral traces imply the uniform on compacts convergence of the secular functions. In Section 3.2 we saw that we can express the characteristic polynomials of finite ensembles on the unit circle via the secular function of an associated Dirac-type operator. The operator level limits discussed in the previous section lead then lead to convergence statements regarding the scaled and normalized characteristic polynomials of the respective finite ensembles.

In this section, we briefly review the secular functions and structure functions arising from the limits of the considered beta ensembles. The constructions rely on certain time-reversed and transformed versions of the limiting operators introduced in Section 4.2. Recall the transformations introduced in Section 2.2.

In [44] Valkó and Virág constructed the following Dirac-type operator on $(0, 1]$ and showed that it is orthogonally equivalent to the Sine_β operator. Consider the time change

$$u(t) = u_\beta(t) = \frac{4}{\beta} \log t, \qquad t \in (0, 1]. \tag{4.11}$$

Definition 4.13. *Let b_1, b_2 be independent two-sided standard Brownian motion, and set*

$$\mathsf{y}_u = e^{b_2(u) - u/2}, \qquad \mathsf{x}_u = -\int_u^0 e^{b_2(s) - \frac{s}{2}} db_1, \tag{4.12}$$

For $t \in (0, 1]$ set $\hat{z}(t) = (\hat{x} + i\hat{y})(t) := \mathsf{x}_u + i\mathsf{y}_u$, $\mathfrak{u}_0 = \binom{1}{0}$, and $\mathfrak{u}_1 = \binom{-q}{-1}$, where q is a Cauchy distributed random variable independent of b_1, b_2. Define $\tau_\beta = \tau_\beta^{\mathrm{Sine}} = \mathrm{Dir}(\hat{z}(\cdot), \mathfrak{u}_0, \mathfrak{u}_1)$, and denote by $\zeta_\beta := \zeta_{\tau_\beta}$ and $\mathcal{E}_\beta := \mathcal{E}_{\tau_\beta}$ the corresponding secular and structure function according to Definition (2.4).

Proposition 4.14 ([44])**.** *Let q and τ_β be defined as in Definition 4.13. Then the τ_β operator satisfies Assumption 2.1, and the orthogonally equivalent operator*

$$\rho^{-1} S Q \tau_\beta Q^{-1} S \rho, \qquad Q = \frac{1}{\sqrt{1 + q^2}} \begin{pmatrix} q & 1 \\ -1 & q \end{pmatrix},$$

has the same distribution as Sine_β. Let $H_\beta(t, z)$ be the solution to the canonical system (2.9) of the τ_β operator, and let x_u and y_u be defined according to (4.12). Set

$$\mathcal{H}_\beta(u, z) = \begin{pmatrix} 1 & -\mathsf{x}_u \\ 0 & \mathsf{y}_u \end{pmatrix} H_\beta(t(u), z), \qquad t = t(u) = e^{\beta u/4}, \tag{4.13}$$

then $\mathcal{H}$ satisfies the stochastic differential equation (1.6). Since $\mathcal{H}_\beta(0, z) = H_\beta(1, z)$, the structure-function $\mathcal{E}_\beta(z)$ can be represented as $\mathcal{E}_\beta(z) = \mathcal{H}_\beta(0, z)^\dagger \binom{1}{-i}$.

Using the results of Proposition 4.14 [44] showed the convergence of the scaled and normalized characteristic polynomials of the circular beta ensemble to the stochastic zeta function.

Proposition 4.15 ([44]). *Consider the size n (unperturbed) circular beta ensemble, i.e., the eigenvalues of $\mathsf{Circ}_{n,\beta}$, and its normalized characteristic polynomial $p_{n,\beta}(z) := \frac{\det(zI - \mathsf{Circ}_{n,\beta})}{\det(I - \mathsf{Circ}_{n,\beta})}$. There exists a coupling of $p_{n,\beta}(z), n \geq 1$, the secular function $\zeta_\beta(z) = \mathcal{H}_\beta(0,z)^\dagger \binom{1}{-q}$, and an a.s. finite C so that for all $z \in \mathbb{C}$ we have*

$$|p_{n,\beta}(e^{iz/n})e^{-iz/2} - \zeta_\beta(z)| \leq C^{|z|^2+1}\left(e^{|z|\frac{\log^3 n}{\sqrt{n}}} - 1\right).$$

The time-reversed and transformed version of the $\mathsf{Bess}_{\beta,a}$ and $\mathsf{HP}_{\beta,\delta}$ operators were constructed in [30] in a similar spirit.

Definition 4.16. *Let B be a standard two-sided Brownian motion. Let $\mathsf{y}_u = e^{-\frac{\beta}{4}(2a+1)u+B(2u)}$ and $\hat{y}(t) = \mathsf{y}(u(t))$ for $t \in (0,1]$, where u is defined in (4.11). Set $\mathfrak{u}_0 = \binom{1}{0}, \mathfrak{u}_1 = \binom{0}{-1}$ and define $\tau_{\beta,a} = \tau_{\beta,a}^{\mathsf{Bess}} = \mathtt{Dir}(i\hat{y}(t), \mathfrak{u}_0, \mathfrak{u}_1)$. We also define $\zeta_{\beta,a} = \zeta_{\tau_{\beta,a}}$ and $\mathcal{E}_{\beta,a} = \mathcal{E}_{\tau_{\beta,a}}$ as the secular and structure function of the $\tau_{\beta,a}$ operator via Definition 2.4.*

Proposition 4.17 ([30]). *The operator $\tau_{\beta,a}$ satisfies Assumption 2.1, and we have*

$$\rho^{-1}J\tau_{\beta,a}J\rho \overset{d}{=} \mathsf{Bess}_{\beta,a}, \qquad J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Let $H_{\beta,a}(t,z)$ be the solution to the canonical system (2.9) of the $\tau_{\beta,a}$ operator and set $\mathcal{H}_{\beta,a}(u,z) = \mathrm{diag}(1, \mathsf{y}_u)H_{\beta,a}(e^{\frac{\beta}{4}u}, z)$ for $u \leq 0, z \in \mathbb{C}$, where y_u is defined as in Definition 4.16. Then $\mathcal{H}_{\beta,a}$ is the unique strong solution to the SDE

$$d\mathcal{H} = \begin{pmatrix} 0 & 0 \\ 0 & \sqrt{2}dB + (1 - \frac{\beta}{4}(2a+1))du \end{pmatrix}\mathcal{H} - z\frac{\beta}{8}e^{\beta u/4}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\mathcal{H}du, \qquad (4.14)$$

with boundary conditions $\lim_{u \to -\infty} \sup_{|z|<1}\left|\mathcal{H}(u,z) - \binom{1}{0}\right| = 0$. Since $\mathcal{H}_{\beta,a}(0,z) = H_{\beta,a}(1,z)$, we have $\mathcal{E}_{\beta,a}(z) = \mathcal{H}_{\beta,a}(0,z)^\dagger \binom{1}{-i}$ and $\zeta_{\beta,a} = \mathcal{H}_{\beta,a}(1,z)^\dagger \binom{1}{0}$.

[30] showed that the secular function $\zeta_{\beta,a}$ of the $\mathsf{Bess}_{\beta,a}$ operator is the limit in distribution of the normalized characteristic polynomial of the real orthogonal beta ensemble under the edge scaling (1.3).

Definition 4.18. *Let B_1, B_2 be independent two-sided standard Brownian motion, and for $u \leq 0$ define*

$$\mathsf{y}_u = e^{B_2(u)-(\Re\delta+\frac{1}{2})u}, \quad \mathsf{x}_u = -\int_u^0 e^{B_2(s)-(\Re\delta+\frac{1}{2})s}dB_1 - \Im\delta\int_u^0 e^{B_2(s)-(\Re\delta+\frac{1}{2})s}ds, \quad (4.15)$$

For $t \in (0,1]$ let $\hat{z}(t) = (\hat{x} + i\hat{y})(t) = \mathsf{x}_u + i\mathsf{y}_u$ with $u = u(t)$ defined in (4.11). Set $\mathfrak{u}_0 = \binom{1}{0}$, and $\mathfrak{u}_1 = \binom{-q}{-1}$, where $q \sim \Theta(1,\delta)$ is independent of B_1, B_2. Define $\tau_{\beta,\delta} = \tau_{\beta,\delta}^{\mathsf{HP}} = \mathtt{Dir}(\hat{z}(\cdot), \mathfrak{u}_0, \mathfrak{u}_1)$, and denote by $\zeta_{\beta,\delta} = \zeta_{\tau_{\beta,\delta}}$ and $\mathcal{E}_{\beta,\delta} = \mathcal{E}_{\tau_{\beta,\delta}}$ the secular and structure function of $\tau_{\beta,\delta}$, respectively.[2]

Proposition 4.19 ([30]). *Let q and $\tau_{\beta,\delta}$ be defined as in Definition 4.18. Then the $\tau_{\beta,\delta}$ operator satisfies Assumption 2.1, and the orthogonally equivalent operator*

$$\rho^{-1}SQ\tau_{\beta,\delta}Q^{-1}S\rho, \qquad Q = \frac{1}{\sqrt{1+q^2}}\begin{pmatrix} q & 1 \\ -1 & q \end{pmatrix},$$

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[2]From the context it will always be clear if $\zeta_{\beta,\cdot}, \mathcal{E}_{\beta,\cdot}$ refer to the objects related to the $\mathsf{Bess}_{\beta,a}$ or the $\mathsf{HP}_{\beta,\delta}$ operators.

has the same distribution as $\mathtt{HP}_{\beta,\delta}$. Let $H_{\beta,\delta}$ be the solution of the canonical system (2.9) of the $\tau_{\beta,\delta}$ operator, and let $\mathsf{x}_u, \mathsf{y}_u$ be defined as in (4.15). For $u \leq 0, z \in \mathbb{C}$ set

$$\mathcal{H}_{\beta,\delta}(u,z) = \begin{pmatrix} 1 & -\mathsf{x}_u \\ 0 & \mathsf{y}_u \end{pmatrix} H_{\beta,\delta}(e^{\beta u/4}, z),$$

then $\mathcal{H}_{\beta,\delta}$ is the unique solution of the SDE

$$d\mathcal{H} = \begin{pmatrix} 0 & -dB_1 \\ 0 & dB_2 \end{pmatrix}\mathcal{H} + \begin{pmatrix} 0 & -\Im\delta du \\ 0 & -\Re\delta du \end{pmatrix}\mathcal{H} - z\frac{\beta}{8}e^{\beta u/4}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\mathcal{H}du, \qquad (4.16)$$

with the boundary condition $\lim\limits_{u \to -\infty} \sup_{|z|<1}\left|\mathcal{H}(u,z) - \binom{1}{0}\right| = 0$, and we have $\mathcal{E}_{\beta,\delta}(z) = \mathcal{H}_{\beta,\delta}(0,z)^\dagger\binom{1}{-i}$.

[30] also showed that the secular function $\zeta_{\beta,\delta} = \mathcal{H}_{\beta,\delta}(0,z)^\dagger\binom{1}{-q}$ of the $\mathtt{HP}_{\beta,\delta}$ operator is the limit in distribution of the normalized characteristic polynomials of the circular Jacobi beta ensemble under the edge scaling (1.3).

5 Convergence of the truncated models

This section establishes general convergence results for the rank-one truncations and multiplicative perturbations of the finite ensembles when the generating paths satisfy certain path bounds. The statements in this section hold in the deterministic setting.

Throughout the section, we assume μ_n is a probability measure on $\partial\mathbb{D}$ supported on n distinct points, with Verblunsky coefficients $\alpha_0, \ldots, \alpha_{n-1}$. We denote by τ_n the Dirac operator corresponding to μ_n.

Recall the reversed (discrete) probability measures defined in Definition 3.4. We first introduce the reversed Dirac operator and its pulled-back version, which are closely connected to the truncated ensembles.

Definition 5.1. Let $\widetilde{\mu}_n$ be the reversed version of μ_n, i.e., the probability measure corresponding to the reversed Verblunsky coefficients $(\widetilde{\alpha}_0, \widetilde{\alpha}_1, \ldots, \widetilde{\alpha}_{n-1})$. We define $\widetilde{\tau}_n$ as the Dirac operator corresponding to $\widetilde{\mu}_n$ and call it the reversed version of τ_n. We denote by $\widetilde{b}_k, 0 \leq k \leq n$ the path parameters of $\widetilde{\tau}_n$ in $\mathbb{D}$.

Recall the affine transformation $\mathcal{A}_{\gamma,\mathbb{D}}$ defined in (3.15).

Definition 5.2. Set $\overleftarrow{b}_k := \mathcal{A}_{\widetilde{b}_{n-1},\mathbb{D}}(\widetilde{b}_k)$ for $0 \leq k \leq n$ with the extension that $\overleftarrow{b}_{-1} = \mathcal{A}_{\widetilde{b}_{n-1},\mathbb{D}}(\widetilde{b}_{-1}) = 1$. We define $\overleftarrow{\tau}_n := \mathcal{A}_{\widetilde{b}_{n-1},\mathbb{D}}(\widetilde{\tau}_n) := \mathtt{Dir}(1, \overleftarrow{b}_0, \ldots, \overleftarrow{b}_{n-1}, \overleftarrow{b}_n)$ as the Dirac operator with path parameters $\overleftarrow{b}_k, 0 \leq k \leq n$ in $\mathbb{D}$. We call $\overleftarrow{\tau}_n$ the pulled-back version of $\widetilde{\tau}_n$.

To summarize, we are considering three sets of path parameters corresponding to μ_n, and each one has a Dirac-type operator associated to it. $b_k, 0 \leq k \leq n$ are the path parameters constructed from the Verblunsky coefficients $\alpha_k, 0 \leq k \leq n-1$ according to Definition 3.6. The reversed path $\widetilde{b}_k, 0 \leq k \leq n$ is constructed via the same definition, but from the reversed Verblunsky coefficients (3.5). Finally, the path $\overleftarrow{b}_k, 0 \leq k \leq n$ can be obtained from the reversed path by applying an affine transformation that maps $\widetilde{b}_{n-1}$ to 0 in $\mathbb{D}$. See Table 1 for a summary of the defined objects.

Note that by the definition of $\mathcal{A}_{\gamma,\mathbb{D}}$ and the recursion (3.18), we have $\overleftarrow{b}_{n-1} = 0$ and $\overleftarrow{b}_n = \widetilde{\gamma}_{n-1}$, where $\widetilde{\gamma}_{n-1}$ is the last modified Verblunsky coefficient corresponding to $\widetilde{\mu}_n$.

The next statement represents the orthogonal polynomial of degree $n-1$ of $\widetilde{\mu}_n$ using the operator $\overleftarrow{\tau}_n$.

Table 1: Table of objects associated to a probability measure μ supported on n points on $\partial \mathbb{D}$.

original objects	reversed objects	pulled-back objects
μ discrete prob. measure on $\partial\mathbb{D}$	$\widetilde{\mu}$ reversed measure	
$\updownarrow$ (3.1) $\qquad$ (3.5)	$\uparrow$ (3.1)	
$\alpha_k, 0 \le k \le n-1$ Verblunsky coefficients	$\longleftrightarrow \quad \widetilde{\alpha}_k, 0 \le k \le n-1$ reversed Verbl. coeffs	
$\updownarrow$ (3.10)		
$\gamma_k, 0 \le k \le n-1$ modified Verblunsky coeffs		
$\updownarrow$ Def. 3.6	$\Big\vert$ Def. 3.6	Def. 5.2
$b_k, 0 \le k \le n$ $z_k, 0 \le k \le n$ path parameters	$\widetilde{b}_k, 0 \le k \le n$ $\widetilde{z}_k, 0 \le k \le n$ reversed path	$\longrightarrow \quad \overleftarrow{b}_k, 0 \le k \le n$ $\overleftarrow{z}_k, 0 \le k \le n$ pulled-back path
$\downarrow$ Def. 3.6	$\downarrow$ Def. 3.6	$\downarrow$ Def. 3.6, 5.2
$\tau = \texttt{Dir}(b_{-1}, \ldots, b_n)$ Dirac-type operator	$\widetilde{\tau} = \texttt{Dir}(\widetilde{b}_{-1}, \ldots, \widetilde{b}_n)$ reversed operator	$\overleftarrow{\tau} = \texttt{Dir}(\overleftarrow{b}_{-1}, \ldots, \overleftarrow{b}_n)$ pulled-back operator

Proposition 5.3. *Let $\widetilde{\mu}_n$ be the reversed version of μ_n, and $\widetilde{\tau}_n, \overleftarrow{\tau}_n$ be defined as in Definitions 5.1 and 5.2. For $0 \le k \le n-1$, let $\widetilde{\Phi}_k(\cdot)$ be the monic orthogonal polynomials of degree k associated to $\widetilde{\mu}_n$, and set $\widetilde{\Psi}_k(\cdot) = \widetilde{\Phi}_k(\cdot)/\widetilde{\Phi}_k(1)$. Then we have*

$$\widetilde{\Psi}_{n-1}(e^{iz/n}) = e^{iz(n-1)/(2n)} \overleftarrow{H}_n\big(\tfrac{n-1}{n}, z\big)^\dagger \begin{pmatrix} 1 \\ -i \end{pmatrix},$$

where $\overleftarrow{H}_n(t,z)$ is the solution of the canonical system (2.9) corresponding to the operator $\overleftarrow{\tau}_n$.

Proof. Let $\widetilde{b}_k, 0 \le k \le n$ be the path parameters of $\widetilde{\tau}_n$ and set $\widetilde{z}_k = \mathcal{U}^{-1}(\widetilde{b}_k), 0 \le k \le n$ to be the corresponding path parameters in $\mathbb{H}$. Here $\mathcal{U}$ is the Cayley transform defined in (3.9). Consider the linear fractional transformation $\mathcal{A}_{\widetilde{z}_{n-1}, \mathbb{H}}$ (the upper half plane representation of $\mathcal{A}_{\widetilde{b}_{n-1}, \mathbb{D}}$) such that

$$\mathcal{A}_{\widetilde{z}_{n-1}, \mathbb{H}}(w) = \mathcal{U}^{-1} \circ \mathcal{A}_{\widetilde{b}_{n-1}, \mathbb{D}} \circ \mathcal{U}(w) = \frac{w - \Re\widetilde{z}_{n-1}}{\Im\widetilde{z}_{n-1}}, \qquad w \in \mathbb{H}.$$

Define $\overleftarrow{z}_k = \overleftarrow{x}_k + i\overleftarrow{y}_k = \mathcal{A}_{\widetilde{z}_{n-1}, \mathbb{H}}(\widetilde{z}_k)$ for $0 \le k \le n$. Then by (3.14) we have $\overleftarrow{z}_k = \mathcal{U}^{-1}(\overleftarrow{b}_k), 0 \le k \le n$ with the extension that $\overleftarrow{z}_{-1} = \mathcal{U}^{-1}(\overleftarrow{b}_{-1}) = \infty$.

Introduce the temporary notation

$$\widetilde{X}_k = A_{\widetilde{z}_k, \mathbb{H}} = \begin{pmatrix} 1 & -\widetilde{x}_k \\ 0 & \widetilde{y}_k \end{pmatrix}, \quad \overleftarrow{X}_k = A_{\overleftarrow{z}_k, \mathbb{H}} = \begin{pmatrix} 1 & -\overleftarrow{x}_k \\ 0 & \overleftarrow{y}_k \end{pmatrix}, \qquad 0 \le k \le n,$$

then we have

$$\overleftarrow{X}_k = \widetilde{X}_k(\widetilde{X}_{n-1})^{-1}, \qquad 0 \le k \le n. \tag{5.1}$$

For a matrix of the form

$$X = \begin{pmatrix} 1 & -x \\ 0 & y \end{pmatrix}$$

with $y > 0$ we have the identity

$$y^{-1} X^\dagger J = J X^{-1}.$$

Using this together with (2.1) and (2.2) we obtain

$$\overleftarrow{\tau}_n = \widetilde{X}_{n-1} \widetilde{\tau}_n (\widetilde{X}_{n-1})^{-1}. \tag{5.2}$$

Let $\overleftarrow{\widetilde{H}}_n(t,z)$ and $\widetilde{H}_n(t,z)$ be the solutions to the canonical systems (2.9) of $\overleftarrow{\tau}_n$ and $\widetilde{\tau}_n$ respectively, then (5.2) leads to

$$\widetilde{H}_n(t,z) = (\widetilde{X}_{n-1})^{-1} \overleftarrow{H}_n(t,z). \tag{5.3}$$

Let $\widetilde{\Psi}_k^*(z) = z^k \overline{\widetilde{\Psi}_k(1/\bar{z})}$ be the reversed polynomial of $\widetilde{\Psi}_k(z)$. Using (3.9), (3.20), (5.1), and (5.3) we get

$$\begin{pmatrix} \widetilde{\Psi}_k(e^{iz/n}) \\ \widetilde{\Psi}_k^*(e^{iz/n}) \end{pmatrix} = e^{izk/(2n)} U \widetilde{X}_k \widetilde{H}_n(k/n, z) = e^{izk/(2n)} U \overleftarrow{X}_k \overleftarrow{H}_n(k/n, z).$$

Since $\overleftarrow{z}_{n-1} = \mathcal{A}_{\widetilde{z}_{n-1}, \mathbb{H}}(\widetilde{z}_{n-1}) = i$, and $\overleftarrow{X}_{n-1} = I$, we have

$$\widetilde{\Psi}_{n-1}(e^{iz/n}) = e^{iz(n-1)/(2n)} \overleftarrow{H}_n((n-1)/n, z)^\dagger \begin{pmatrix} 1 \\ -i \end{pmatrix},$$

finishing the proof. $\square$

Suppose now that the probability measure μ_n is the spectral measure of an $n \times n$ unitary or orthogonal matrix U with respect to e_1. By Propositions 3.2 and 3.5, the eigenvalues of the truncated matrix $\mathrm{Trunc}(U)$ are exactly the zeros of the normalized orthogonal polynomial $\widetilde{\Psi}_{n-1}(\cdot)$, hence they can be expressed from the vector-valued function $\overleftarrow{H}_n$.

Next we turn to the discussion of the scaling limit of the eigenvalues of the truncated matrices. If we can prove uniform on compacts convergence of the normalized orthogonal polynomials, then this leads to the convergence of the eigenvalues of the truncated models. By Proposition 5.3 it is sufficient to show the convergence the convergence of $\overleftarrow{H}_n$. Using Proposition 2.5, this can be done if we have sufficient control of the integral traces, resolvent norms, and the proper integrals of the kernels of $\mathrm{r} \overleftarrow{\tau}_n$. We summarize this idea in the following (deterministic) statement.

Proposition 5.4. *Suppose that $\mu_n, n \geq 1$ is a sequence of probability measures on $\partial\mathbb{D}$, with μ_n supported on n distinct points. Consider the setup of Proposition 5.3 and denote by $\overleftarrow{z}_n(\cdot)$ the generating path of $\overleftarrow{\tau}_n$ for $n \geq 1$. Suppose that there exists a Dirac operator $\tau_\infty = \mathrm{Dir}(z_\infty(\cdot), \mathfrak{u}_0, \mathfrak{u}_1)$ with $\mathfrak{u}_0 = \binom{1}{0}$ and $\mathcal{I} = (0,1]$ so that as $n \to \infty$*

$$\|\mathrm{r}\overleftarrow{\tau}_n - \mathrm{r}\tau_\infty\|_{\mathrm{HS}} \to 0, \qquad \mathfrak{t}_{\overleftarrow{\tau}_n} - \mathfrak{t}_{\tau_\infty} \to 0, \qquad \int_0^1 |\overleftarrow{\mathfrak{a}}_{0,n}(s) - \mathfrak{a}_{0,\infty}(s)|^2 ds \to 0, \tag{5.4}$$

where $\overleftarrow{\mathfrak{a}}_{0,n}(\cdot) = \binom{\Im \overleftarrow{z}_n(\cdot)^{-\frac{1}{2}}}{0}, n \geq 1$ and $\mathfrak{a}_{0,\infty} = \binom{\Im z_\infty(\cdot)^{-\frac{1}{2}}}{0}$ are defined according to (2.8). Let $\widetilde{\Psi}_{n-1,n}$ be the normalized orthogonal polynomial of degree $n-1$ of $\widetilde{\mu}_n$, and let $\mathcal{E}(z) := H(1,z)^\dagger \binom{1}{-i}$ be the structure function of τ_∞. Then we have

$$|e^{-iz/2} \widetilde{\Psi}_{n-1,n}(e^{iz/n}) - \mathcal{E}(z)| \to 0 \quad \text{uniformly on compacts in } \mathbb{C} \text{ as } n \to \infty. \tag{5.5}$$

Note that (5.5) implies that the zeros of $\widetilde{\Psi}_{n-1,n}$ converge to the zeros of $\mathcal{E}$ under the edge scaling (1.3).

Proof of Proposition 5.4. Let $\overleftarrow{H}_n, n \geq 1$ be the solution of the canonical system (2.9) of $\overleftarrow{\tau}_n$. By Proposition 5.3 and the triangle inequality, we have

$$
\begin{aligned}
|\widetilde{\Psi}_{n-1,n}(e^{iz/n})e^{-iz/2} - \mathcal{E}(z)| \leq\ & |e^{-iz/(2n)}| \left| \left(\overleftarrow{H}_n(\tfrac{n-1}{n}, z)^\dagger - H(\tfrac{n-1}{n}, z)^\dagger \right) \cdot \begin{pmatrix} 1 \\ -i \end{pmatrix} \right| \\
& + |e^{-iz/(2n)}| \left| \left(H(\tfrac{n-1}{n}, z)^\dagger - H(1, z)^\dagger \right) \cdot \begin{pmatrix} 1 \\ -i \end{pmatrix} \right| \\
& + \left| e^{-iz/(2n)} - 1 \right| \left| H(1, z)^\dagger \cdot \begin{pmatrix} 1 \\ -i \end{pmatrix} \right|.
\end{aligned}
\tag{5.6}
$$

For z in a compact set of $\mathbb{C}$ and large enough n, the sum of the second and the third term of the right-hand side of (5.6) is upper bounded by an error term of the order $O(n^{-1})$. Hence it remains to provide an upper bound for the difference $|\overleftarrow{H}_n(\tfrac{n-1}{n}, z) - H(\tfrac{n-1}{n}, z)|$. By Proposition 2.5, it suffices to control the terms

$$
|\mathfrak{t}_{\tau_\infty} - \mathfrak{t}_{\overleftarrow{\tau}_n}|, \qquad \|\mathbf{r}\,\tau_\infty - \mathbf{r}\,\overleftarrow{\tau}_n\|_{\mathrm{HS}}, \qquad \int_0^1 |\mathfrak{a}_{0,\infty}(s) - \overleftarrow{\mathfrak{a}}_{0,n}(s)|^2 ds.
$$

Applying the assumptions (5.4) finishes the proof. $\qquad\square$

Next, we turn to the discussion of the rank-one multiplicative perturbation of the matrix models. Suppose U is an $n \times n$ unitary or orthogonal matrix with CMV matrix form $\mathcal{C}(\alpha_0, \ldots, \alpha_{n-1})$. Let $r \in [0, 1]$ and $U^{[r]} = U \cdot \mathrm{diag}(r, 1, 1, \ldots, 1)$ be its rank-one multiplicative perturbation. Recall from Proposition 3.5 that the perturbed matrix $U^{[r]}$ has the same eigenvalues as the CMV matrix

$$
\mathcal{C}_n^{[r]} := \mathcal{C}(\widetilde{\alpha}_0, \ldots, \widetilde{\alpha}_{n-2}, r\widetilde{\alpha}_{n-1}).
\tag{5.7}
$$

Note that this matrix is a function of the spectral measure of U. In particular, for any probability measure μ_n on $\partial\mathbb{D}$ (supported on n points) we can define the matrix $\mathcal{C}_n^{[r]}$ from its Verblunsky coefficients.

Suppose now that we have a sequence of probability measures $\mu_n, n \geq 1$, just as in Proposition 5.4. Let $\widetilde{\gamma}_{n-1}$ be the last modified Verblunsky coefficient corresponding to the Verblunsky coefficients $\widetilde{\alpha}_0, \ldots, \widetilde{\alpha}_{n-1}$ of $\widetilde{\mu}_n$. (Note the abuse of notation: we should write $\widetilde{\alpha}_{k,n}, 0 \leq k \leq n-1$ here, but we drop the extra n from the notation.) The next result shows that if $\widetilde{\gamma}_{n-1}$ converges and the assumptions of Proposition 5.4 hold, then the eigenvalues of $\mathcal{C}_n^{[r]}$ converge as well.

Proposition 5.5. *Consider the same setup as in Proposition 5.4. Fix $r \in [0, 1]$, let $\Psi_n^{[r]}(z) = \prod_{i=1}^n \frac{z - \lambda_i}{1 - \lambda_i}$ be the normalized characteristic polynomial of $\mathcal{C}_n^{[r]}$. Assume that the convergence (5.5) holds, and assume further that $\lim_{n\to\infty} \widetilde{\gamma}_{n-1} = \gamma \neq 1$. Then we have*

$$
|\Psi_n^{[r]}(e^{iz/n})e^{-iz/2} - \mathcal{E}^{[r]}(z)| \to 0, \qquad \mathcal{E}^{[r]}(z) = H(1, z)^\dagger \begin{pmatrix} 1 \\ -i\frac{1+r\gamma}{1-r\gamma} \end{pmatrix}
\tag{5.8}
$$

uniformly on compacts in $\mathbb{C}$ as $n \to \infty$.

Note that (5.8) implies the convergence of the eigenvalues of $\Psi_n^{[r]}$ to the zeros of $\mathcal{E}^{[r]}$ under the edge scaling (1.3).

Proof of Proposition 5.5. By the modified Szegő recursion (3.12), we have

$$
\Psi_n^{[r]}(z) = \frac{1}{1 - r\widetilde{\gamma}_{n-1}} z \Psi_{n-1}^{[r]}(z) - \frac{r\widetilde{\gamma}_{n-1}}{1 - r\widetilde{\gamma}_{n-1}} \Psi_{n-1}^{[r],*}(z),
\tag{5.9}
$$

where $\Psi_{n-1}^{[r],*}(\cdot)$ is the reversed polynomial of $\Psi_{n-1}^{[r]}(\cdot)$. Note that the polynomials $\Psi_{n-1}^{[r]}$ and $\Psi_{n-1}^{[r],*}$ do not depend on $\widetilde{\gamma}_{n-1}$.

Introduce $\mathcal{E}^*(z) = \overline{\mathcal{E}(\bar{z})}$, by the definition of the reversed polynomials, (3.20) and (5.5), we have

$$\left| \Psi_{n-1}^{[r],*}(e^{iz/n})e^{-iz/2} - \mathcal{E}^*(z) \right| \to 0 \quad \text{uniformly on compacts as } n \to \infty. \tag{5.10}$$

Together with (5.9) and the convergence of $\widetilde{\gamma}_{n-1} \to \gamma$ as $n \to \infty$, we get

$$\lim_{n\to\infty} \Psi_n^{[r]}(e^{iz/n})e^{-iz/2} = \tfrac{1}{1-r\gamma}\mathcal{E}(z) - \tfrac{r\gamma}{1-r\gamma}\mathcal{E}^*(z) = H(1,z)^\dagger \begin{pmatrix} 1 \\ -i\frac{1+r\gamma}{1-r\gamma} \end{pmatrix},$$

uniformly on compacts in z. $\qquad\square$

In order to work with our probabilistic models, we present below a sufficient condition for the convergence results in (5.4). The statement and its proof are similar to Corollary 30 of [30], where the convergence of the resolvents and integral traces of Dirac-type operators was discussed in a similar context.

Corollary 5.6. *Suppose that $\overleftrightarrow{\tau}_n = \mathrm{Dir}(\overleftrightarrow{z}_n, \mathfrak{u}_0, \overleftarrow{\mathfrak{u}}_{1,n})$ and $\tau_\infty = \mathrm{Dir}(z_\infty, \mathfrak{u}_0, \mathfrak{u}_1)$ are random Dirac-type operators with $\mathcal{I} = (0,1]$, $\mathfrak{u}_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\overleftarrow{\mathfrak{u}}_{1,n} = \begin{pmatrix} -\overleftarrow{q}_n \\ -1 \end{pmatrix}$, and $\mathfrak{u}_1 = \begin{pmatrix} -q \\ -1 \end{pmatrix}$. Assume that there is a coupling of these operators so that all of the following limits hold:*

$$\| \mathbf{r}\,\overleftrightarrow{\tau}_n - \mathbf{r}\,\tau_\infty \|_{\mathrm{HS}} \to 0, \quad \mathfrak{t}_{\overleftrightarrow{\tau}_n} - \mathfrak{t}_{\tau_\infty} \to 0, \quad \overleftarrow{z}_n \to z_\infty \text{ pointwise on } (0,1]. \tag{5.11}$$

Assume also that there exists a constant $c < 1$ and a sequence of tight positive random variables $\kappa_n, n \geq 1$ so that for $0 < t \leq 1$

$$\kappa_n^{-1}(\lfloor nt \rfloor/n)^c \leq \Im\overleftarrow{z}_n(t), \qquad \kappa^{-1}t^c \leq \Im z_\infty(t). \tag{5.12}$$

Then there is a coupling of $\overleftrightarrow{\tau}_n, n \geq 1$ and τ_∞ so that the conditions in (5.4) hold almost surely as $n \to \infty$.

Proof. Set

$$s_n := \int_0^1 |\overleftarrow{\mathfrak{a}}_{0,n}(s) - \mathfrak{a}_{0,\infty}(s)|^2 ds = \int_0^1 \left| \left(\Im\overleftarrow{z}_n(s)\right)^{-1/2} - (\Im z_\infty(s))^{-1/2} \right|^2 ds.$$

It is sufficient to prove that in the appropriate product space

$$(\overleftarrow{z}_n, \mathbf{r}\,\overleftrightarrow{\tau}_n, \mathfrak{t}_{\overleftrightarrow{\tau}_n}, s_n) \to (z_\infty, \mathbf{r}\,\tau_\infty, \mathfrak{t}_{\tau_\infty}, 0) \tag{5.13}$$

jointly in distribution, the Skorokhod representation theorem then implies the statement. To show the convergence (5.13) in distribution, it is enough to show that any subsequence $n_j, j \geq 1$ has a further subsequence $n_{j(m)}, m \geq 1$ along which the appropriate limit holds. Consider the coupling in the assumption and choose the second subsequence $n_{j(m)}$ so that $\kappa_{n_{j(m)}} \to \kappa_\infty$ in distribution with an a.s. finite κ_∞. Then $(\overleftarrow{z}_n, \overleftarrow{q}_n, \mathbf{r}\,\overleftrightarrow{\tau}_n, \mathfrak{t}_{\overleftrightarrow{\tau}_n}, \kappa_n) \to (z_\infty, q_\infty, \mathbf{r}\,\tau_\infty, \mathfrak{t}_{\tau_\infty}, \kappa_\infty)$ jointly in distribution (along our subsequence), so by Skorokhod's representation theorem there is a coupling where this limit holds in the a.s. sense as well. The dominated convergence theorem together with the bounds (5.12) imply that we also have $s_n \to 0$ a.s. along our subsequence, which means that (5.13) holds a.s. as well. This implies the required joint convergence in distribution and completes the proof. $\qquad\square$

6 Edge limits of the truncated circular and real orthogonal beta ensembles

Our goal is to prove Theorems 1.2 and 1.3, the edge scaling limits of the rank-one truncation and multiplicative perturbation of the circular beta ensemble using the framework developed in Section 5. We will also prove the analogue results for the real orthogonal beta ensemble.

In both cases we will check that the random measures associated to the respective beta ensembles (introduced in Section 4.1) satisfy the conditions in Propositions 5.4 and 5.5.

6.1 The truncated circular beta ensemble

Recall the definition of the random probability measure $\mu_{n,\beta}^{\mathrm{KN}}$ introduced in Section 4.1 and the corresponding CMV matrix $\mathrm{Circ}_{n,\beta}$. Proposition 4.2 gives the distribution of the Verblunsky coefficients $\alpha_k, 0 \leq k \leq n-1$ corresponding to $\mu_{n,\beta}^{\mathrm{KN}}$. The truncated circular beta ensemble is the joint distribution of the eigenvalues of the truncated CMV matrix $\mathrm{Trunc}(\mathrm{Circ}_{n,\beta})$. The rank-one multiplicative perturbation of the circular beta ensemble (corresponding to a parameter $r \in [0,1]$) is defined as the distribution of the eigenvalues of $\mathrm{Circ}_{n,\beta}^{[r]}$.

Note that by Proposition 4.2 the Verblunsky coefficients $\alpha_k, 0 \leq k \leq n-1$ are independent and rotationally invariant. Hence from (3.5) we get the following distributional identity:

$$(\widetilde{\alpha}_0, \widetilde{\alpha}_1, \ldots, \widetilde{\alpha}_{n-2}, \widetilde{\alpha}_{n-1}) \overset{d}{=} (\alpha_{n-2}, \alpha_{n-3}, \ldots, \alpha_0, \alpha_{n-1}). \tag{6.1}$$

Using this together with Proposition 3.5, we recover the following result of Killip and Kozhan [24].

Proposition 6.1 ([24]). *For fixed n and $\beta > 0$, let $\alpha_k, 0 \leq k \leq n-1$ be distributed according to Proposition 4.2. Then the joint eigenvalue distribution of the CMV matrix $\mathcal{C}(\alpha_{n-2}, \alpha_{n-3}, \ldots, \alpha_0)$ is the same as that of $\mathrm{Trunc}(\mathrm{Circ}_{n,\beta})$, which is given by the truncated circular beta ensemble. For fixed $r \in [0,1]$, the matrix $\mathcal{C}(\alpha_{n-2}, \alpha_{n-3}, \ldots, \alpha_0, r\alpha_{n-1})$ has the same distribution as $\mathrm{Circ}_{n,\beta}^{[r]}$, in particular its joint eigenvalue distribution is given by the rank-one multiplicative perturbation of the circular beta ensemble.*

Note that [24] also proved that the joint density of the truncated circular beta ensemble is given by (1.5), and described explicitly the joint eigenvalue distribution of the perturbed matrix $\mathrm{Circ}_{n,\beta}^{[r]}$ (see Proposition 7.2 of [24]).

Recall the Dirac operator $\mathrm{Circ}_{n,\beta}$ in Definition 4.3. We denote by $\widetilde{\tau}_{n,\beta}$ the reversed version of $\mathrm{Circ}_{n,\beta}$, and by $\overleftarrow{\tau}_{n,\beta}$ the pulled-back version of $\widetilde{\tau}_{n,\beta}$ (see Definitions 5.1 and 5.2). By studying the generating paths of the operators $\overleftarrow{\tau}_{n,\beta}$ and $\mathrm{Circ}_{n,\beta}$, we claim that these operators are orthogonally equivalent. Indeed, this observation follows from Proposition 52 of [45] using the rotational invariance of the models and a conditioning argument. We will provide a different proof that holds for the circular Jacobi beta ensemble ($\delta = 0$ corresponding to the circular beta ensemble). To avoid repetition, we will state the result without proof here.

Proposition 6.2. *Recall the definitions of the operators ρ and S from Lemma 2.7. Denote by $\overleftarrow{b}_k, 0 \leq k \leq n$ the path parameters of $\overleftarrow{\tau}_{n,\beta}$, and let*

$$Q = \frac{1}{\sqrt{1+q^2}} \begin{pmatrix} q & 1 \\ -1 & q \end{pmatrix}, \qquad q = \mathcal{U}^{-1}(\overleftarrow{b}_n).$$

Then the operator $\rho^{-1}(SQ)\overleftarrow{\tau}_{n,\beta}(SQ)^{-1}\rho$ has the same distribution as $\mathrm{Circ}_{n,\beta}$.

Recall the definition of the $\mathtt{Sine}_\beta$ operator from Definition 4.10, and its reversed and transformed version τ_β defined in Definition 4.13. Proposition 6.2 shows that one can obtain the operators $\overleftarrow{\tau}_{n,\beta}$ and τ_β from $\mathtt{Circ}_{n,\beta}, n \geq 1$ and $\mathtt{Sine}_\beta$ under the same orthogonal transformations. The final ingredient of proving Theorems 1.2 and 1.3 is the following strong operator level convergence of $\mathtt{Circ}_{n,\beta}$ to the $\mathtt{Sine}_\beta$ operator. Let $d_{\mathbb{H}}(z_1, z_2) = \operatorname{arccosh}(1 + \frac{|z_1 - z_2|^2}{2\Im z_1 \Im z_2})$ denote the hyperbolic distance in $\mathbb{H}$.

Proposition 6.3 ([43, 44])**.** *There exists an explicit coupling of the operators* $\mathtt{Circ}_{n,\beta} = \mathtt{Dir}(z_n(\cdot), \mathfrak{u}_0, \mathfrak{u}_1^{(n)}), n \geq 1$ *and* $\mathtt{Sine}_\beta = \mathtt{Dir}(z(\cdot), \mathfrak{u}_0, \mathfrak{u}_1)$ *such that* $\mathfrak{u}_1^{(n)} = \mathfrak{u}_1$*, and for large enough* n*,*

$$d_{\mathbb{H}}(z_n(t), z(t)) \leq \frac{\log^{3-1/8} n}{\sqrt{(1-t)n}}, \qquad 0 \leq t \leq t_n := 1 - \frac{\log^6 n}{n},$$

$$d_{\mathbb{H}}(z_n(t_n), z(t)) \leq (\log \log n)^4, \qquad t_n \leq t < 1. \tag{6.2}$$

Under this coupling, we have

$$\| \mathtt{r}\,\mathtt{Circ}_{n,\beta} - \mathtt{r}\,\mathtt{Sine}_\beta \|_{\mathrm{HS}}^2 \leq \frac{\log^6 n}{n}, \qquad | \mathtt{t}_{\mathtt{Circ}_{n,\beta}} - \mathtt{t}_{\mathtt{Sine}_\beta} | \leq \frac{\log^3 n}{\sqrt{n}}. \tag{6.3}$$

Here the first inequality of (6.3) was proved in [43] and the second one follows from the estimates used in the proof of Theorem 49 of [44] (in particular equations (107)-(109)). 'Large enough n' means that there is a finite random variable N_0 so that the statements hold for $n \geq N_0$.

Proposition 6.3 provides us with the necessary ingredients so that we can apply Proposition 5.4 to prove the convergence of the normalized characteristic polynomials of the truncated circular beta ensemble.

Proposition 6.4. *Let* $\lambda_i, 1 \leq i \leq n-1$ *be the size* $n-1$ *truncated circular beta ensemble and set* $p_{n-1,\beta}(z) = \prod_{i=1}^{n-1} \frac{z - \lambda_i}{1 - \lambda_i}$ *be the normalized characteristic polynomial. Let* $\mathcal{E}_\beta$ *be the structure function of* τ_β *defined via (2.11). Then there is a coupling of* $p_{n-1,\beta}, n \geq 2$ *and* $\mathcal{E}_\beta$ *such that*

$$| p_{n-1,\beta}(e^{iz/n}) e^{-iz/2} - \mathcal{E}_\beta(z) | \leq \left(e^{|z| \frac{\log^3 n}{\sqrt{n}}} - 1 \right) C^{1+|z|^2} \tag{6.4}$$

for all $z \in \mathbb{C}$ *and* $n \geq 1$*, where* C *is an a.s. finite constant.*

Proof. Let $\widetilde{\mu}_n$ be the reversed version of the Killip-Nenciu probablity measures $\mu_n := \mu_{n,\beta}^{\mathrm{KN}}$, and let $\overleftarrow{\tau}_{n,\beta}$ be the pulled-back operator. By Propositions 3.2 and 6.1, we can represent $p_{n-1,\beta}$ as the monic orthogonal polynomial of degree $n-1$ associated to $\widetilde{\mu}_n$. Then by Proposition 5.3, we have

$$p_{n-1,\beta}(e^{iz/n}) e^{-iz/2} = e^{-iz/(2n)} \overleftarrow{H}_{n,\beta}((n-1)/n, z)^\dagger \binom{1}{-i},$$

where $\overleftarrow{H}_{n,\beta}$ solves the ODE (2.9) of $\overleftarrow{\tau}_{n,\beta}$. Recall that $\mathcal{E}_\beta(z) = H_\beta(1, z)^\dagger \binom{1}{-i}$.

Consider the coupling of Proposition 6.3. Using the triangle inequality as in (5.6) within the proof of Proposition 5.4, we have

$$| p_{n-1,\beta}(e^{iz/n}) e^{-iz/2} - \mathcal{E}_\beta(z) | \leq |e^{-iz/(2n)}| \left| \left(\overleftarrow{H}_{n,\beta}(\tfrac{n-1}{n}, z)^\dagger - H_\beta(\tfrac{n-1}{n}, z)^\dagger \right) \cdot \binom{1}{-i} \right|$$

$$+ |e^{-iz/(2n)}| \left| \left(H_\beta(\tfrac{n-1}{n}, z)^\dagger - H_\beta(1, z)^\dagger \right) \cdot \binom{1}{-i} \right| \tag{6.5}$$

$$+ \left| e^{-iz/(2n)} - 1 \right| \left| H_\beta(1, z)^\dagger \cdot \binom{1}{-i} \right|.$$

Note that for any compact subset of $(0, 1]$ the operator norm of the weight function $R(s)$ of τ_β is bounded by a finite random constant. Hence by the standard theory of ordinary differential equations, for z in a compact set of $\mathbb{C}$, the second term on the right hand side of (6.5) can be upper bounded by Cn^{-1} where C is an a.s. finite constant. The third term on the right side of (6.5) can also be bounded similarly, since $z \mapsto H_\beta(1, z)^\dagger \cdot \left(\begin{smallmatrix} 1 \\ -i \end{smallmatrix} \right)$ is a random entire function.

It remains to estimate the first term on the right hand side of (6.5). By Propositions 4.14 and 6.2, the operators $\overleftarrow{\tau}_{n,\beta}, n \geq 1$ and τ_β can be obtained from $\mathtt{Circ}_{n,\beta}, n \geq 1$ and $\mathtt{Sine}_\beta$ under the same orthogonal transformations. Hence in the coupling of Proposition 6.3, for large enough n we have

$$\| \mathbf{r}\, \tau_\beta - \mathbf{r}\, \overleftarrow{\tau}_{n,\beta} \|_{\mathrm{HS}} \leq \frac{\log^3 n}{\sqrt{n}}, \qquad |\mathbf{t}_{\tau_\beta} - \mathbf{t}_{\overleftarrow{\tau}_{n,\beta}}| \leq \frac{\log^3 n}{\sqrt{n}}, \tag{6.6}$$

Let $\overleftarrow{z}_n(\cdot) \equiv \overleftarrow{z}_{n,\beta}(\cdot)$ and $z(\cdot) \equiv z_\beta(\cdot)$ be the driving paths of the $\overleftarrow{\tau}_{n,\beta}$ and τ_β operators, respectively. It has also been shown in the proof of Proposition 3 of [45] (see equations (69)-(72) therein) that

$$\int_0^1 |\overleftarrow{\mathfrak{a}}_{0,n}(s) - \mathfrak{a}_0(s)|^2 ds = \int_0^2 \left| (\Im \overleftarrow{z}_n(s))^{-\frac{1}{2}} - (\Im z(s))^{-\frac{1}{2}} \right|^2 ds \leq \frac{\log^6 n}{n}. \tag{6.7}$$

The estimates (6.6) and (6.7) allow us to use Proposition 2.5 to bound $|\overleftarrow{H}_{n,\beta}(\frac{n-1}{n}, z) - H_\beta(\frac{n-1}{n}, z)|$ for n large enough. From this it follows that (6.5) can be bounded from above according to (6.4), which proves the proposition. $\qquad \square$

We are now ready to prove Theorems 1.2 and 1.3.

Proof of Theorem 1.2. First note that by (4.13) we have $\mathcal{H}_\beta(0, \cdot) = H_\beta(1, \cdot)$, hence $\mathcal{E}_\beta(\cdot) = \mathcal{H}_\beta(0, \cdot)^\dagger \left(\begin{smallmatrix} 1 \\ -q \end{smallmatrix} \right)$. Then by Proposition 6.4, we obtain the convergence of the normalized characteristic polynomials of the truncated circular beta ensemble to the random analytic function $\mathcal{E}_\beta$ with an explicit error bound. Under the edge scaling (1.3), the size n truncated circular beta ensemble has the same distribution as the zeros of the function $p_{n,\beta}(e^{iz/n})$ defined in Proposition 6.4. The weak convergence of the truncated circular beta ensemble to $\mathcal{X}_\beta := \{z \in \mathbb{H} : \mathcal{E}_\beta(z) = 0\}$ follows directly from Proposition 6.4 and Hurwitz's theorem. $\qquad \square$

Proof of Theorem 1.3. Consider the coupling of $\overleftarrow{\tau}_{n,\beta}$ and τ_β described in Proposition 6.4 again, the right boundary condition of these operators are coupled together. By Propositions 5.5 and 6.4 we get the uniform-on-compacts convergence of the normalized characteristic polynomials of $\mathtt{Circ}_{n,\beta}^{[r]}$ to $\mathcal{E}_{r,\beta}$ with a similar error bound. This in turn gives the weak convergence of the eigenvalues of $\mathtt{Circ}_{n,\beta}^{[r]}$ under the edge scaling and completes the proof. $\qquad \square$

6.2 Edge limits of the truncated real orthogonal beta ensemble

This section discusses the edge limits of the rank-one truncation and multiplicative perturbation of the real orthogonal beta ensemble. We will follow the same approach as in Section 6.1.

Recall the size $2n$ real orthogonal ensemble with joint density (4.3) and the random probability measure $\mu_{2n,\beta,a,b}^{\mathrm{RO}}$ in (4.4). Proposition 4.5 describes the distribution of the corresponding Verblunsky coefficients $\alpha_k, 0 \leq k \leq 2n-1$. Since α_k's are all real and $\alpha_{2n-1} = -1$, from (3.5) we get that the reversed Verblunsky coefficients $\widetilde{\alpha}_k, 0 \leq k \leq 2n-1$ satisfy

$$(\widetilde{\alpha}_0, \widetilde{\alpha}_1, \cdots, \widetilde{\alpha}_{2n-2}, \widetilde{\alpha}_{2n-1}) = (\alpha_{2n-2}, \alpha_{2n-3}, \cdots, \alpha_0, \alpha_{2n-1}). \tag{6.8}$$

The rank-one truncation and multiplicative perturbation of the real orthogonal beta ensemble are defined as the joint eigenvalue distributions of the truncated CMV matrix $\text{Trunc}(\text{RO}_{2n,\beta,a,b})$ and the perturbed CMV matrix $\text{RO}_{2n,\beta,a,b}^{[r]}$ (indexed by $r \in [0,1]$), respectively. Using (6.8) with Proposition 3.5 yields the following result of Killip and Kozhan [24].

Proposition 6.5 ([24]). *For given $\beta > 0$, $a, b > -1$ and $n \geq 1$, let $\alpha_k, 0 \leq k \leq 2n-1$ be distributed according to Proposition 4.5. Then the CMV matrix $\mathcal{C}(\alpha_{2n-2}, \alpha_{2n-3} \ldots, \alpha_0)$ has the same joint eigenvalue distribution as $\text{Trunc}(\text{RO}_{2n,\beta,a,b})$. For fixed $r \in [0,1]$, the CMV matrix $\mathcal{C}(\alpha_{2n-2}, \ldots, \alpha_0, -r)$ has the same joint eigenvalue distribution as $\text{RO}_{2n,\beta,a,b}^{[r]}$.*

Note that the joint distributions of the rank-one truncation and multiplicative perturbation of the real orthogonal beta ensemble were described explicitly in Theorem 6.4 and Proposition 7.2 (b) of [24].

Recall the Dirac operator $\text{RO}_{2n,\beta,a,b}$ in Definition 4.6. Denote by $\widetilde{\tau}_{2n,\beta,a,b}$ the reversed version of $\text{RO}_{2n,\beta,a,b}$, and by $\overleftarrow{\widetilde{\tau}}_{2n,\beta,a,b}$ the pulled-back version of $\widetilde{\tau}_{2n,\beta,a,b}$. Recall also the limiting $\text{Bess}_{\beta,a}$ operator in Definition 4.11 and its reversed version $\tau_{\beta,a}$ in Definition 4.16. By Proposition 4.17 we get that $\rho^{-1} J \tau_{\beta,a} J \rho \overset{d}{=} \text{Bess}_{\beta,a}$. The next result shows that under the same transformations, the operators $\overleftarrow{\widetilde{\tau}}_{2n,\beta,a,b}$ and $\text{RO}_{2n,\beta,a,b}$ are orthogonally equivalent.

Proposition 6.6.
$$\rho^{-1} J \overleftarrow{\widetilde{\tau}}_{2n,\beta,a,b} J \rho \overset{d}{=} \text{RO}_{2n,\beta,a,b}.$$

Proof. Let $\alpha_k, 0 \leq k \leq 2n-1$ be distributed as in Proposition 4.5, and let $\gamma_k, 0 \leq k \leq 2n-1$ be the corresponding modified Verblunsky coefficients. Since α_k's are all real, from (3.10) we have $\gamma_k = \alpha_k$ for all $0 \leq k \leq 2n-1$.

Let $z_k, 0 \leq k \leq 2n$ and $\widetilde{z}_k, 0 \leq k \leq 2n$ be the path parameters of $\text{RO}_{2n,\beta,a,b}$ and $\widetilde{\tau}_{2n,\beta,a,b}$ in $\mathbb{H}$, respectively. Using (3.17) and (6.8) we get for $0 \leq k \leq 2n$,

$$z_k = i \prod_{j=0}^{k-1} \frac{1+\gamma_j}{1-\gamma_j}, \qquad \widetilde{z}_k = i \prod_{j=0}^{k-1} \frac{1+\gamma_{2n-2-j}}{1-\gamma_{2n-2-j}}.$$

Recall the affine transformation $\mathcal{A}_{z,\mathbb{H}}$ in (3.13) and the corresponding transformation on the uni-disk model $\mathcal{A}_{\gamma,\mathbb{D}}$ in (3.15). By Definition 5.2 and (3.14), the path parameters of $\overleftarrow{\widetilde{\tau}}_{2n,\beta,a,b}$ in $\mathbb{H}$, denoted by $\overleftarrow{\widetilde{z}}_k, 0 \leq k \leq 2n$, are obtained from $\widetilde{z}_k, 0 \leq k \leq 2n$ via the affine transformation $\mathcal{A}_{\widetilde{z}_{2n-1},\mathbb{H}}$. More precisely, we have

$$\overleftarrow{\widetilde{z}}_k = \mathcal{P} \begin{pmatrix} 1 & 0 \\ 0 & \Im \widetilde{z}_{2n-1} \end{pmatrix} \begin{pmatrix} \widetilde{z}_k \\ 1 \end{pmatrix} = i \prod_{j=0}^{2n-2-k} \frac{1-\gamma_j}{1+\gamma_j}, \qquad 0 \leq k \leq 2n-1,$$

with $\overleftarrow{\widetilde{z}}_{2n} = \mathcal{A}_{\widetilde{z}_{2n-1},\mathbb{H}}(\widetilde{z}_{2n}) = 0$.

Note that conjugating with the permutation matrix J maps $z \mapsto -1/z$ for $z \in i\mathbb{R}$. Together with the time-reversal, we see that the path parameters of the operator $\rho J \overleftarrow{\widetilde{\tau}}_{2n,\beta,a,b} J \rho$ are the same as the path parameters of $\text{RO}_{2n,\beta,a,b}$. By Lemmas 2.7 and 2.8, the left and right boundary points of $\rho^{-1} J \overleftarrow{\widetilde{\tau}}_{2n,\beta,a,b} J \rho$ are given by 1 and -1, which corresponds to the vectors $\mathfrak{u}_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\mathfrak{u}_1 = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$ as desired. This completes the proof. $\qquad\square$

In the rest of the section, we aim to prove the convergence of the normalized characteristic polynomials of the truncated real orthogonal beta ensemble using Proposition 5.4 and Corollary 5.6. The main ingredient will be the operator level convergence of $\text{RO}_{2n,\beta,a,b}$ to its limit $\text{Bess}_{\beta,a}$ and the corresponding path bounds proved in [30].

Proposition 6.7 ([30]). *Let* $\mathfrak{u}_0 = \binom{1}{0}, \mathfrak{u}_1 = \binom{0}{-1}$. *Consider the random Dirac-type operators* $\mathrm{RO}_{2n,\beta,a,b} = \mathrm{Dir}(z_n(\cdot), \mathfrak{u}_0, \mathfrak{u}_1), n \geq 1$ *and* $\mathrm{Bess}_{\beta,a} = \mathrm{Dir}(z(\cdot), \mathfrak{u}_0, \mathfrak{u}_1)$. *There exists a coupling of the operators* $\mathrm{RO}_{2n,\beta,a,b}, n \geq 1$ *and* $\mathrm{Bess}_{\beta,a}$ *such that as* $n \to \infty$ *we have almost surely* $z_n \to z$ *pointwise on* $[0,1)$, *and almost surely* $\|\mathbf{r}\,\mathrm{RO}_{2n,\beta,a,b} - \mathbf{r}\,\mathrm{Bess}_{\beta,a}\|_{\mathrm{HS}} \to 0$.

Moreover, for $\varepsilon > 0$ *small there exists a sequence of tight random variables* κ_n *and an a.s. finite random variable* $\kappa > 0$ *such that for* $0 \leq t < 1$

$$\Im z_n(t) \leq \kappa_n (1 - \lfloor nt \rfloor /n)^{2a+1-\varepsilon}, \qquad \Im z(t) \leq \kappa (1-t)^{2a+1-\varepsilon}. \tag{6.9}$$

Now we are ready to prove the main result of this section.

Theorem 6.8. *For fixed* $\beta > 0, a, b > -1$ *and* $n \geq 1$, *let* $\lambda_i, 1 \leq i \leq 2n-1$ *be the size* $(2n-1)$ *truncated real orthogonal beta ensemble and set* $p_{2n-1,\beta,a,b}(z) = \prod_{i=1}^{2n-1} \frac{z-\lambda_i}{1-\lambda_i}$ *be the normalized characteristic polynomial. Let* $\mathcal{E}_{\beta,a}$ *be the structure function of* $\tau_{\beta,a}$ *defined via* (2.11). *Then there is a coupling of* $p_{2n-1,\beta,a,b}, n \geq 1$ *and* $\mathcal{E}_{\beta,a}$ *such that almost surely*

$$\left| p_{2n-1,\beta,a,b}(e^{iz/(2n)}) e^{-iz/2} - \mathcal{E}_{\beta,a}(z) \right| \to 0 \quad \text{uniformly on compacts as } n \to \infty. \tag{6.10}$$

Consequently, the truncated real orthogonal beta ensembles converge weakly to the zeros of $\mathcal{E}_{\beta,a}(\cdot)$ *under the edge scaling* (1.3) *as* $n \to \infty$.

Proof. Let $\widetilde{\mu}_{2n}$ be the reversed version of the random probability measures $\mu_{2n} := \mu_{2n,\beta,a,b}^{\mathrm{RO}}$, and let $\overleftarrow{\tau}_{2n,\beta,a,b}$ be the pulled-back operator. By Propositions 3.2 and 6.5, one can represent $p_{2n-1,\beta,a,b}$ as the monic orthogonal polynomial of degree $2n-1$ associated to $\widetilde{\mu}_{2n}$. By Proposition 5.3, we have

$$p_{2n-1,\beta,a,b}(e^{iz/(2n)}) e^{-iz/2} = e^{-iz/(4n)} \overleftarrow{H}_{2n,\beta,a,b}((2n-1)/(2n), z)^\dagger \binom{1}{-i},$$

where $\overleftarrow{H}_{2n,\beta,a,b}$ solves the ODE (2.9) of $\overleftarrow{\tau}_{2n,\beta,a,b}$. Recall that $\mathcal{E}_{\beta,a} = H_{\beta,a}(1,\cdot)^\dagger \binom{1}{-i}$. It suffices to provide a coupling of $\overleftarrow{\tau}_{2n,\beta,a,b}$ and $\tau_{\beta,a}$ under which the uniform-on-compacts convergence of $\overleftarrow{H}_{2n,\beta,a,b}(1,z)$ to $H_{\beta,a}(1,z)$ holds.

By Propositions 4.17 and 6.6, the generating paths $\overleftarrow{z}_n(\cdot)$ of the $\overleftarrow{\tau}_{2n,\beta,a,b}$ operator and $z_n(\cdot)$ of the $\mathrm{RO}_{2n,\beta,a,b}$ operator satisfy that $\overleftarrow{z}_n(t) \overset{d}{=} -1/z_n(1-t)$ for $t \in (0,1]$. (Note that an analogous relation also holds for the generating paths of the $\tau_{\beta,a}$ and $\mathrm{Bess}_{\beta,a}$ operators.) Hence the path bounds (6.9) can be translated into

$$\kappa_n^{-1}(\lfloor nt \rfloor /n)^{-2a-1+\varepsilon} \leq \Im \overleftarrow{z}_n(t), \quad \kappa^{-1} t^{-2a-1+\varepsilon} \leq (\Im z(1-t))^{-1}$$

where $\kappa_n = \kappa_n(\varepsilon)$ is a sequence of tight random variables. Here $\varepsilon < 1$ small is chosen so that the condition $-2a - 1 + \varepsilon < 1$ in (5.12) holds. Note the driving paths for $\overleftarrow{\tau}_{2n,\beta,a,b}$ and $\tau_{\beta,a}$ are purely imaginary, and the integral trace $\mathfrak{t}$ is zero. Applying Corollary 5.6 yields the a.s. convergence of the conditions in (5.4). Using Propositions 5.4 and 6.7 completes the proof of the statement. $\qquad\square$

Note that by Proposition 4.19 the limiting random analytic function $\mathcal{E}_{\beta,a}$ can also be characterized as $\mathcal{E}_{\beta,a} = \mathcal{H}_{\beta,a}(0,z)^\dagger \binom{1}{-i}$, where $\mathcal{H}_{\beta,a}$ solves the SDE (4.16).

Using Proposition 5.5 and Theorem 6.8, one obtains the following result on the convergence of normalized characteristic polynomial of the perturbed matrix $\mathrm{RO}_{2n,\beta,a,b}^{[r]}$.

Corollary 6.9. *For fixed* $r \in [0,1]$ *let* $\Lambda_{2n} = \{\lambda_1, \ldots, \lambda_{2n}\}$ *be the size* $2n$ *rank-one multiplicative perturbed real orthogonal ensemble and set* $p_{2n,\beta,a,b}^{[r]}(z) = \prod_{i=1}^{2n} \frac{z-\lambda_i}{1-\lambda_i}$ *to be*

the normalized characteristic polynomial. Then under the coupling of Proposition 6.7, we have almost surely

$$\left| \Psi_{2n,\beta,a,b}^{[r]}(e^{iz/n})e^{-iz/(2n)} - \mathcal{E}_{\beta,a}^{[r]}(z) \right| \to 0, \quad \text{uniformly on compacts as } n \to \infty,$$

where $\mathcal{E}_{\beta,a}^{[r]}(z) = H_{\beta,a}(1,z)^\dagger \left(\begin{smallmatrix} 1 \\ -i\frac{1-r}{1+r} \end{smallmatrix} \right)$. Consequently, the rank-one multiplicative perturbed real orthogonal beta ensemble Λ_{2n} converges weakly to the zero set of the random analytic function $\mathcal{E}_{\beta,a}^{[r]}$ under the edge scaling (1.3) as $n \to \infty$.

Recall that the secular function $\zeta_{\beta,a}$ of the $\mathtt{Bess}_{\beta,a}$ operator is given by $\zeta_{\beta,a} = H_{\beta,a}(1,z)^\dagger \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix} \right)$. The above result gives an interpolation between the scaling limits of the normalized characteristic polynomials of the unperturbed and the truncated real orthogonal beta ensemble.

7 The truncated circular Jacobi beta ensemble

In Section 7.1 we construct the truncated and the multiplicative perturbed circular Jacobi beta ensemble, and we compute the joint eigenvalue density of the truncated model. In Section 7.2, we derive the edge scaling limits of the truncated and perturbed models using Propositions 5.4 and 5.5.

7.1 Matrix model and joint eigenvalue density for truncated circular Jacobi beta ensemble

Consider the random probability measure $\mu_{n,\beta,\delta}^{\mathrm{CJ}}$ in (4.5), with support given by the circular Jacobi beta ensemble. Recall also the matrix model $\mathsf{CJ}_{n,\beta,\delta}$ defined in Proposition 4.8 via the sequence of regular Verblunsky coefficients $\alpha_k, 0 \le k \le n-1$ of $\mu_{n,\beta,\delta}^{\mathrm{CJ}}$.

Definition 7.1. *For fixed $n \ge 1$, $\beta > 0$, $\Re\delta > -1/2$ we define the* truncated circular Jacobi beta ensemble *as the joint eigenvalue distribution of the truncated matrix* $\mathrm{Trunc}(\mathsf{CJ}_{n,\beta,\delta})$. *For a fixed $r \in [0,1]$ we define the perturbed circular Jacobi beta ensemble as the joint eigenvalue distribution of the perturbed matrix $\mathsf{CJ}_{n,\beta,\delta}^{[r]} := \mathsf{CJ}_{n,\beta,\delta} \cdot \mathrm{diag}(r,1,1,\dots,1)$.*

The main challenge to study these ensembles is that the Verblunsky coefficients of $\mu_{n,\beta,\delta}^{\mathrm{CJ}}$ are not independent, hence one cannot expect a nice description of the CMV matrices appearing in Proposition 3.5. However, as the next proposition shows, we can still preserve the independence by expressing the appearing CMV matrices in terms of the *modified* Verblunsky coefficients.

Recall the definition of modified Verblunsky coefficients given by the recursion (3.10). The recursion provides a one-to-one map between the first $k \le n-1$ Verblunsky coefficients and the first k modified Verblunsky coefficients, we denoted this map by $\mathcal{T}_k$.

Proposition 7.2. *Fix $\beta > 0, \delta \in \mathbb{C}$ with $\Re\delta > -1/2$, and let $\gamma_k, 0 \le k \le n-1$ be the sequence of modified Verblunsky coefficients of $\mu_{n,\beta,\delta}^{\mathrm{CJ}}$. Then the sub-unitary CMV matrix $\mathcal{C}(\mathcal{T}_{n-1}^{-1}(\gamma_{n-2}, \gamma_{n-3}\cdots, \gamma_0))$ has the same joint eigenvalue distribution as the truncated model $\mathrm{Trunc}(\mathsf{CJ}_{n,\beta,\delta})$. For fixed $r \in [0,1]$, the matrix $\mathcal{C}(\mathcal{T}_n^{-1}(\gamma_{n-2}, \gamma_{n-3}\dots, \gamma_0, r\gamma_{n-1}))$ has the same joint eigenvalue distribution as the perturbed model $\mathsf{CJ}_{n,\beta,\delta}^{[r]}$.*

Before proving the proposition, we introduce a simple mapping on $\mathbb{D}$. For $\gamma \in \mathbb{D}$ recall the linear fractional transformation $\mathcal{A}_{\gamma,\mathbb{D}}$ from (3.15). This is an isometry of the Poincaré disk that corresponds to an affine transformation in the Poincaré half-plane $\mathbb{H}$. The inverse of $\mathcal{A}_{\gamma,\mathbb{D}}$ is also an isometry, and it also corresponds to an affine transformation in

$\mathbb{H}$, we denote the corresponding element in $\mathbb{D}$ by γ^ι:

$$\mathcal{A}_{\gamma,\mathbb{D}}^{-1} = \mathcal{A}_{\gamma^\iota,\mathbb{D}}, \qquad \gamma^\iota = -\gamma \frac{1-\bar\gamma}{1-\gamma}. \tag{7.1}$$

Note that the $\iota : \gamma \mapsto \gamma^\iota$ mapping is an involution such that $\mathcal{A}_{\gamma,\mathbb{D}}(0) = \gamma^\iota$.

We also need the following distributional identity for $\Theta(a+1,\delta)$ random variables (see Definition 4.7).

Claim 7.3. *Fix* $\delta \in \mathbb{C}$ *with* $\Re\delta > -1/2$. *Let* $\zeta_0, \zeta_1, \ldots, \zeta_{n-1}$ *be a sequence of independent random variables such that* $\zeta_i \sim \Theta(a_i+1,\delta)$, *with* $a_0 = 0$ *and* $a_i \geq 0$ *for* $i \geq 1$. *Then*

$$\left(\frac{\zeta_1^\iota}{\zeta_0}, \frac{\zeta_2^\iota}{\zeta_1}, \ldots \frac{\zeta_{n-1}^\iota}{\zeta_{n-2}}, \frac{1}{\zeta_{n-1}} \right) \stackrel{d}{=} \left(\bar\zeta_1, \frac{\bar\zeta_2}{\bar\zeta_1^\iota}, \ldots, \frac{\bar\zeta_{n-1}}{\bar\zeta_{n-2}^\iota}, \frac{\bar\zeta_0}{\bar\zeta_{n-1}^\iota} \right). \tag{7.2}$$

Proof. Our statement will follow from the following (simpler) distributional identity. Let $\zeta \sim \Theta(a+1,\delta)$ and $\eta \sim \Theta(1,\delta)$ be independent. Then

$$(\eta\bar\zeta^\iota, \zeta) \stackrel{d}{=} (\zeta, \eta\bar\zeta^\iota). \tag{7.3}$$

Since $|\zeta^\iota| = |\zeta|$, it is sufficient to prove that the unit length random variables $\eta\bar\zeta^\iota/|\zeta|$ and $\zeta/|\zeta|$ are conditionally exchangeable given $|\zeta| = r$. Let $z = \zeta/|\zeta|$, then under the condition $|\zeta| = r$ we have

$$\eta\bar\zeta^\iota/|\zeta| = -\eta\frac{1-rz}{z-r}.$$

By the independence of η and ζ, the conditional joint density of (η,z) (given $|\zeta| = r$) is proportional

$$(1-\eta)^{\bar\delta}(1-\eta^{-1})^\delta(1-rz)^{\bar\delta}(1-rz^{-1})^\delta.$$

Since the Jacobian of the mapping $\left(-\eta\frac{1-rz}{z-r}, z \right) \mapsto (\eta,z)$ is equal to 1, the conditional joint density of $(\xi_1, \xi_2) := \left(-\eta\frac{1-rz}{z-r}, z \right)$ given $|\zeta| = r$) is proportional to

$$(1 - r(\xi_1 + \xi_2) + \xi_1\xi_2)^{\bar\delta}(1 - r(\xi_1^{-1} + \xi_2^{-1}) + \xi_1^{-1}\xi_2^{-1})^\delta.$$

This shows the (conditional) exchangeability $(\xi_1, \xi_2) = (\eta\bar\zeta^\iota/|\zeta|, \zeta/|\zeta|)$ and proves (7.3).

Now we turn to the proof of the statement. Note that (7.3) implies that

$$(\zeta_1^\iota/\zeta_0, \bar\zeta_1) \stackrel{d}{=} (\bar\zeta_1, \zeta_1^\iota/\zeta_0).$$

Starting from the random vector on the left-hand side of (7.2), we apply (7.3) repeatedly and get

$$\left(\frac{\zeta_1^\iota}{\zeta_0}, \frac{\zeta_2^\iota}{\zeta_1}, \ldots \frac{\zeta_{n-1}^\iota}{\zeta_{n-2}}, \frac{1}{\zeta_{n-1}} \right) \stackrel{d}{=} \left(\bar\zeta_1, \frac{\bar\zeta_0\zeta_2^\iota}{\bar\zeta_1^\iota}, \ldots \frac{\zeta_{n-1}^\iota}{\zeta_{n-2}}, \frac{1}{\zeta_{n-1}} \right) \stackrel{d}{=} \ldots$$

$$\stackrel{d}{=} \left(\bar\zeta_1, \frac{\bar\zeta_2}{\bar\zeta_1^\iota}, \ldots, \frac{\bar\zeta_0\zeta_{n-1}^\iota}{\bar\zeta_{n-2}^\iota}, \frac{1}{\zeta_{n-1}} \right) \stackrel{d}{=} \left(\bar\zeta_1, \frac{\bar\zeta_2}{\bar\zeta_1^\iota}, \ldots, \frac{\bar\zeta_{n-1}}{\bar\zeta_{n-2}^\iota}, \frac{\bar\zeta_0}{\bar\zeta_{n-1}^\iota} \right),$$

proving (7.2). $\qquad\square$

Now we return to the proof of Proposition 7.2.

Proof of Proposition 7.2. Let $\alpha_k, 0 \leq k \leq n-1$ be the (regular) Verblunsky coefficients of $\mu_{n,\beta,\delta}^{\mathrm{CJ}}$ and set $\tilde\alpha_k, 0 \leq k \leq n-1$ to be the reversed version of $\alpha_k, 0 \leq k \leq n-1$ defined via (3.5). By Proposition 3.5, the truncated and perturbed models $\mathrm{Trunc}(\mathsf{CJ}_{n,\beta,\delta})$ and $\mathsf{CJ}_{n,\beta,\delta}^{[r]}$ have the same eigenvalues as the CMV matrices $\mathcal{C}(\tilde\alpha_0, \tilde\alpha_1, \cdots, \tilde\alpha_{n-2})$ and

$\mathcal{C}(\widetilde{\alpha}_0, \widetilde{\alpha}_1, \cdots, \widetilde{\alpha}_{n-2}, r\widetilde{\alpha}_{n-1})$, respectively. Hence, the statement of the proposition follows if we can show

$$(\widetilde{\alpha}_0, \widetilde{\alpha}_1, \ldots, \widetilde{\alpha}_{n-2}, \widetilde{\alpha}_{n-1}) \overset{d}{=} \mathcal{T}_n^{-1}(\gamma_{n-2}, \gamma_{n-3}, \cdots, \gamma_0, \gamma_{n-1}). \tag{7.4}$$

Introduce the temporary notation

$$(\hat{\alpha}_0, \hat{\alpha}_1, \ldots, \hat{\alpha}_{n-1}) = \mathcal{T}_n^{-1}(\gamma_{n-2}, \gamma_{n-3}, \ldots, \gamma_0, \gamma_{n-1}).$$

We need to show $(\widetilde{\alpha}_0, \widetilde{\alpha}_1, \ldots, \widetilde{\alpha}_{n-2}, \widetilde{\alpha}_{n-1}) \overset{d}{=} (\hat{\alpha}_0, \hat{\alpha}_1, \ldots, \hat{\alpha}_{n-1})$, which will follow from

$$\left(\widetilde{\alpha}_0, -\frac{\widetilde{\alpha}_1}{\widetilde{\alpha}_0} \ldots, -\frac{\widetilde{\alpha}_{n-1}}{\widetilde{\alpha}_{n-2}} \right) \overset{d}{=} \left(\hat{\alpha}_0, -\frac{\hat{\alpha}_1}{\hat{\alpha}_0} \ldots, -\frac{\hat{\alpha}_{n-1}}{\hat{\alpha}_{n-2}} \right).$$

From (3.10) and (7.1) it follows that

$$-\frac{\alpha_k}{\alpha_{k-1}} = \frac{\bar{\gamma}_k}{\bar{\gamma}_{k-1}^\iota}, \quad k \geq 1, \qquad \alpha_0 = \bar{\gamma}_0. \tag{7.5}$$

Hence

$$\left(\hat{\alpha}_0, -\frac{\hat{\alpha}_1}{\hat{\alpha}_0} \ldots, -\frac{\hat{\alpha}_{n-1}}{\hat{\alpha}_{n-2}} \right) = \left(\bar{\gamma}_{n-2}, \frac{\bar{\gamma}_{n-3}}{\bar{\gamma}_{n-2}^\iota}, \ldots, \frac{\bar{\gamma}_0}{\bar{\gamma}_1^\iota}, \frac{\bar{\gamma}_{n-1}}{\bar{\gamma}_0^\iota} \right).$$

On the other hand, from (3.5) and (7.5) we have

$$\left(\widetilde{\alpha}_0, -\frac{\widetilde{\alpha}_1}{\widetilde{\alpha}_0} \ldots, -\frac{\widetilde{\alpha}_{n-1}}{\widetilde{\alpha}_{n-2}} \right) = \left(-\frac{\bar{\alpha}_{n-2}}{\bar{\alpha}_{n-1}}, -\frac{\bar{\alpha}_{n-3}}{\bar{\alpha}_{n-2}}, \ldots, -\frac{\bar{\alpha}_0}{\bar{\alpha}_1}, \frac{1}{\bar{\alpha}_0} \right) = \left(\frac{\gamma_{n-2}^\iota}{\gamma_{n-1}}, \frac{\gamma_{n-3}^\iota}{\gamma_{n-2}}, \ldots, \frac{\gamma_0^\iota}{\gamma_1}, \frac{1}{\gamma_0} \right)$$

From Proposition 4.8 we know that $\gamma_k, 0 \leq k \leq n-1$ are independent with $\gamma_k \sim \Theta(\beta(n-k-1)+1), \delta)$. Using Claim 7.3 with $\zeta_j = \gamma_{n-1-j}, 0 \leq j \leq n-1$ we get

$$\left(\frac{\gamma_{n-2}^\iota}{\gamma_{n-1}}, \frac{\gamma_{n-3}^\iota}{\gamma_{n-2}}, \ldots, \frac{\gamma_0^\iota}{\gamma_1}, \frac{1}{\gamma_0} \right) \overset{d}{=} \left(\bar{\gamma}_{n-2}, \frac{\bar{\gamma}_{n-3}}{\bar{\gamma}_{n-2}^\iota}, \ldots, \frac{\bar{\gamma}_0}{\bar{\gamma}_1^\iota}, \frac{\bar{\gamma}_{n-1}}{\bar{\gamma}_0^\iota} \right),$$

and the statement of the proposition follows. $\qquad\qquad\square$

Proposition 7.2 allows us to derive the joint eigenvalue distribution of the finite truncated circular Jacobi beta ensemble.

Theorem 7.4. *Fix $\beta > 0, \delta \in \mathbb{C}$ with $\Re\delta > -1/2$. Then the eigenvalues of $\mathrm{Trunc}(\mathsf{CJ}_{n+1,\beta,\delta})$ are distributed in $\mathbb{D}^n$ according to the density*

$$c_{n,\beta,\delta} \prod_{j,k=1}^n (1-z_j\bar{z}_k)^{\frac{\beta}{2}-1} \prod_{j<k} |z_k - z_j|^2 \prod_{j=1}^n \left((1-z_j)^{\bar{\delta}}(1-\bar{z}_j)^\delta \right) \tag{7.6}$$

with respect to the Lebesgue measure on $\mathbb{D}^n$. Here $c_{n,\beta,\delta} = \frac{1}{\pi^n n!} \prod_{j=1}^n \frac{\Gamma(\frac{\beta}{2}j+1+\delta)\Gamma(\frac{\beta}{2}j+1+\bar{\delta})}{\Gamma(\frac{\beta}{2}j)\Gamma(\frac{\beta}{2}j+1+\delta+\bar{\delta})}$ is the normalizing constant.

Proof. The proof of the statement relies on the following computations of Jacobian determinants. We refer to Section 6 and Appendix B of [24] for more details.

Let $\gamma_k, 0 \leq k \leq n-1$ be distributed as in Proposition 7.2, and let $(\alpha_0, \alpha_1, \cdots, \alpha_{n-1}) = \mathcal{T}_n^{-1}(\gamma_0, \gamma_1, \cdots, \gamma_{n-1})$. From (3.10) we have

$$\left| \frac{\partial(\alpha_0, \ldots, \alpha_{n-1})}{\partial(\gamma_0, \ldots, \gamma_{n-1})} \right| = 1. \tag{7.7}$$

Denote by $z_k, 1 \leq k \leq n$ the eigenvalues of $\mathrm{Trunc}(\mathsf{CJ}_{n+1,\beta,\delta})$. It has been shown in [24] that

$$\left| \frac{\partial(\alpha_0, \ldots, \alpha_{n-1})}{\partial(z_1, \ldots, z_n)} \right| = |\Delta(z_1, \ldots, z_n)|^2 \prod_{j=0}^{n-1} (1 - |\alpha_j|^2)^{-j}, \qquad (7.8)$$

where $\Delta(z_1, \ldots, z_n) = \prod_{1 \leq j < k \leq n} (z_j - z_k)$ denotes the Vandermonde determinant of $z_k, 1 \leq k \leq n$. Using $|\alpha_k| = |\gamma_k|$, (7.7), and (7.8), we obtain

$$\left| \frac{\partial(\gamma_0, \ldots, \gamma_{n-1})}{\partial(z_1, \ldots, z_n)} \right| = |\Delta(z_1, \ldots, z_n)|^2 \prod_{j=0}^{n-1} (1 - |\gamma_j|^2)^{-j}. \qquad (7.9)$$

Proposition 2.5 of [4] shows that we have

$$\prod_{j=0}^{n-1} (1 - \gamma_j) = \Phi_n(1) = \prod_{j=1}^{n} (1 - z_j), \quad \prod_{j=0}^{n-1} (1 - \bar{\gamma}_j) = \bar{\Phi}_n(1) = \prod_{j=1}^{n} (1 - \bar{z}_j). \qquad (7.10)$$

Since $|\alpha_k| = |\gamma_k|$, by Lemma B.1 (vi) of [24], we have

$$\prod_{j=0}^{n-1} (1 - |\gamma_j|^2)^{(\frac{\beta}{2}-1)(j+1)} = \prod_{j=0}^{n-1} (1 - |\alpha_j|^2)^{(\frac{\beta}{2}-1)(j+1)} = \prod_{j,k=1}^{n} (1 - z_j \bar{z}_k)^{\frac{\beta}{2}-1}. \qquad (7.11)$$

By (7.9)-(7.11) and the explicit joint distribution of $\gamma_k, 0 \leq k \leq n - 1$, the joint density of $z_k, 1 \leq k \leq n$ is given by

$$c_{n,\beta,\delta} \prod_{j=0}^{n-1} (1 - |\gamma_j|^2)^{(\frac{\beta}{2}-1)(j+1)} (1 - \gamma_j)^{\bar{\delta}} (1 - \bar{\gamma}_j)^{\delta} |\Delta(z_1, \ldots, z_n)|^2$$

$$= c_{n,\beta,\delta} \prod_{j,k=1}^{n} (1 - z_j \bar{z}_k)^{\frac{\beta}{2}-1} \prod_{j=1}^{n} \left((1 - z_j)^{\bar{\delta}} (1 - \bar{z}_j)^{\delta} \right) |\Delta(z_1, \ldots, z_n)|^2.$$

Collecting the normalizing constants of the joint distribution of $\gamma_k, 0 \leq k \leq n - 1$, we get

$$c_{n,\beta,\delta} = \frac{1}{\pi^n n!} \prod_{j=0}^{n-1} \frac{\Gamma(\frac{\beta}{2}(j+1) + 1 + \delta)\Gamma(\frac{\beta}{2}(j+1) + 1 + \bar{\delta})}{\Gamma(\frac{\beta}{2}(j+1))\Gamma(\frac{\beta}{2}(j+1) + 1 + \delta + \bar{\delta})}.$$

This finishes the proof. $\qquad \square$

Note that using the explicit description of the joint distribution of the modified Verblunsky coefficients of the random probability measure $\mu_{n,\beta,\delta}^{\mathrm{CJ}}$, and the method developed in Section 7 of [24], one can also obtain the joint density of the perturbed circular Jacobi beta ensemble. We omit the computation to shorten our representation.

7.2 Edge limit of the truncated circular Jacobi beta ensemble

In this section, we prove the edge limits of the rank-one truncation and multiplicative perturbation of the circular Jacobi beta ensemble.

Recall the Dirac operator representation $\mathsf{CJ}_{n,\beta,\delta}$ defined in Definition 4.9. We denote by $\widetilde{\tau}_{n,\beta,\delta}$ the reversed version of $\mathsf{CJ}_{n,\beta,\delta}$ via Definition 5.1, and by $\overleftarrow{\tau}_{n,\beta,\delta}$ to be the pulled-back version of $\widetilde{\tau}_{n,\beta,\delta}$ via Definition 5.2.

Our approach will be similar to the one used for the circular beta ensemble case (which correspond to $\delta = 0$). We will show that under appropriate transformations, the operators $\overleftarrow{\tau}_{n,\beta,\delta}$ and $\mathsf{CJ}_{n,\beta,\delta}$ are orthogonally equivalent. Note however, that in the $\delta \neq 0$ the measure $\mu_{n,\beta,\delta}^{\mathrm{CJ}}$ is no longer invariant under rotations, which requires us to develop a new method to prove the orthogonal equivalence. The key ingredient is the following proposition, providing equivalent descriptions of the conditioned path parameters of $\mathsf{CJ}_{n,\beta,\delta}$.

Proposition 7.5. *Let $\mu = \mu_{n,\beta,\delta}^{\mathrm{CJ}}$ and let $\gamma_k, 0 \le k \le n-1$ be its modified Verblunsky coefficients. Let $b_k, 0 \le k \le n$ be the path parameters of $\mathrm{CJ}_{n,\beta,\delta}$ in $\mathbb{D}$. Then the following sequences have the same joint distribution.*

(1). The path parameters $b_k, 0 \le k \le n$ conditioned on $b_n = 1$.

(2). The path parameters corresponding to the sequence of modified Verblunsky coefficients $\bar{\gamma}_0^\iota, \bar{\gamma}_1^\iota, \ldots, \bar{\gamma}_{n-2}^\iota, 1$.

(3). The 'pulled-back and time-reversed' path parameters $b_k' := \mathcal{A}_{\hat{b}_{n-1}, \mathbb{D}}(\hat{b}_{n-k-1}), 0 \le k \le n$, where $\hat{b}_{-1} = 1$ and $\hat{b}_k, 0 \le k \le n-1$ are the first n elements of the path parameters produced by the sequence of modified Verblunsky coefficients $\bar{\gamma}_{n-2}, \bar{\gamma}_{n-3}, \ldots, \bar{\gamma}_0$.

The proof relies on a special decomposition property of the $\Theta(a+1,\delta)$ distribution and an application of Doob's h-transform. Before presenting the proof, we first introduce a Pearson-type distribution, and a couple of facts about the $\Theta(a+1,\delta)$ distribution.

Definition 7.6. *For $m > 1/2$ and $\mu \in \mathbb{R}$ we denote by $P_{IV}(m,\mu)$ the distribution of the (unscaled) Pearson type IV distribution on $\mathbb{R}$ that has density function*

$$\frac{2^{2m-2}|\Gamma(m+\frac{\mu}{2}i)|^2}{\pi\,\Gamma(2m-1)}(1+x^2)^{-m}e^{-\mu\arctan x}. \tag{7.12}$$

Note that the random variable $\Theta(1,\delta)$ can be connected to the Pearson random variable $P_{IV}(\Re\delta + 1, -2\Im\delta)$ via the mapping $e^{i\theta} \mapsto -\cot(\theta/2)$.

Fact 7.7 ([30]). *Suppose that $\gamma \sim \Theta(a+1,\delta)$ with $a \ge 0$ and $\Re\delta > -1/2$. Define $w, v \in \mathbb{R}$ with $\frac{2\gamma}{1-\gamma} = w - iv$. Then the joint density of $(v, 1+w) \in \mathbb{R} \times \mathbb{R}_+$ is given by*

$$f_{a,\delta}(x,y) = c_{a,\delta}\, y^{\frac{a}{2}-1}(x^2 + (1+y)^2)^{-(\frac{a}{2}+\Re\delta+1)}e^{2\Im\delta\arctan\frac{x}{1+y}}, \tag{7.13}$$

with $c_{a,\delta} = 2^{a+2\Re\delta}\frac{\Gamma(a/2+1+\delta)\Gamma(a/2+1+\bar{\delta})}{\pi\Gamma(a/2)\Gamma(a/2+1+2\Re\delta)}$.

Moreover, the random variables w and $\frac{v}{2+w}$ are independent. The distribution of the random variable $\frac{v}{2+w}$ is given by $P_{IV}(\frac{a}{2}+\Re\delta+1, -2\Im\delta)$. The distribution of $1+w$ is the same as the distribution of G_1/G_2, where G_1, G_2 are the independent (standard) Gamma random variables with parameters $\frac{a}{2}$ and $\frac{a}{2}+2\Re\delta+1$, respectively.

Using Fact 7.7 and the definition of γ^ι in (7.1), one obtains the following property.

Fact 7.8. *Suppose that $\gamma \sim \Theta(a+1,\delta)$ with $a \ge 0$ and $\Re\delta > -1/2$. Then $\gamma^\iota \sim \Theta(\tilde{a}+1,\tilde{\delta})$ with $\tilde{a} = a + 4\Re\delta + 2$, $\tilde{\delta} = -(1+\delta)$.*

Proof. With a bit of abuse of notation, define $w, v, w^\iota, v^\iota \in \mathbb{R}$ with

$$w - iv = \frac{2\gamma}{1-\gamma}, \qquad w^\iota - iv^\iota = \frac{2\gamma^\iota}{1-\gamma^\iota}. \tag{7.14}$$

They satisfy the identities $\mathcal{U}^{-1}(\gamma) = v + i(1+w)$ and $\mathcal{U}^{-1}(\gamma^\iota) = v^\iota + i(1+w^\iota)$. By (7.14) and (7.1) we have

$$1 + w^\iota = \frac{1}{1+w}, \qquad \frac{v^\iota}{2+w^\iota} = -\frac{v}{2+w}.$$

From Fact 7.7, we obtain that the joint density of $(v^\iota, 1+w^\iota)$ is proportional to

$$y^{\frac{a}{2}+2\Re\delta}(x^2 + (1+y)^2)^{-(\frac{a}{2}+\Re\delta+1)}e^{-2\Im\delta\arctan\frac{x}{1+y}}.$$

This shows that if $\gamma \sim \Theta(a+1,\delta)$, then $\gamma^\iota \sim \Theta(\tilde{a}+1,\tilde{\delta})$, with $\tilde{a} := a + 4\Re\delta + 2$ and $\tilde{\delta} := -(1+\delta)$. Note that the random variable γ^ι is well-defined since the conditions $\tilde{a} > 0$ and $\tilde{a}/2 + \Re\tilde{\delta} = a/2 + \Re\delta > -1/2$ are still satisfied. $\square$

The next result shows that the right boundary point of the $\mathtt{CJ}_{n,\beta,\delta}$ operator has the same distribution as a $\Theta(1,\delta)$ random variable.

Proposition 7.9. *Let b_n be the right boundary point of the Dirac operator $\mathtt{CJ}_{n,\beta,\delta}$ in $\mathbb{D}$. Then $b_n \stackrel{d}{=} \Theta(1,\delta)$.*

Proof. Fix n, recall from Proposition 4.8 that the modified Verblunsky coefficients $\gamma_k = \gamma_k^{(n)}, 0 \le k \le n-1$ of $\mathtt{CJ}_{n,\beta,\delta}$ are independent, with $\gamma_k \sim \Theta(\beta(n-k-1)+1,\delta)$. Let $b_k := b_k^{(n)}, 0 \le k \le n-1$ be the path parameters of $\mathtt{CJ}_{n,\beta,\delta}$ in $\mathbb{D}$, these solve the recursion (3.18) with $b_0 = 0$. Let $b_{k,\gamma}, 0 \le k \le n$ be the solution of this recursion for any initial value $b_{0,\gamma} = \gamma \in \mathbb{D}$, and denote by $P_n(\gamma,\eta)$ the probability density function of $b_{n,\gamma}$.

Recall the isometries $\mathcal{A}_{\gamma,\mathbb{D}}$ defined according to (3.13). By the recursions (3.18), (3.19) and the fact that $\mathcal{A}_{\gamma,\mathbb{D}}^{-1}(0) = \gamma$, we have $b_{n,\gamma} = \mathcal{A}_{\gamma,\mathbb{D}}^{-1}(b_n)$. This can also be checked by noticing that the corresponding path parameters in $\mathbb{H}$ are invariant in law with respect to the affine transformations $\mathcal{A}_{x+iy,\mathbb{H}}$ defined as in (3.13).

In the case when $n = 1$, we get $b_n = b_1 = \gamma_0$ from the recursion (3.18), so the statement follows. More precisely, we have

$$P_1(0,\eta) = c_\delta (1-\eta)^{\bar{\delta}} (1-\bar{\eta})^{\delta}, \quad c_\delta = \frac{\Gamma(1+\delta)\Gamma(1+\bar{\delta})}{\Gamma(1+\delta+\bar{\delta})},$$

and

$$P_1(\gamma,\eta) = c_\delta \left(\frac{(1-|\gamma|^2)(1-\eta)}{(1-\bar{\gamma}\eta)(1-\gamma)} \right)^{\bar{\delta}} \left(\frac{(1-|\gamma|^2)(1-\bar{\eta})}{(1-\gamma\bar{\eta})(1-\bar{\gamma})} \right)^{\delta} \frac{1-|\gamma|^2}{|1-\bar{\gamma}\eta|^2}. \tag{7.15}$$

For $n \ge 2$ we will proceed by induction. Assume that $P_{n-1}(0,\eta) = P_1(0,\eta)$ for any $\eta \in \partial\mathbb{D}$. This assumption implies that $P_{n-1}(\gamma,\eta) = P_1(\gamma,\eta)$ since $b_{n-1,\gamma}^{(n-1)} = \mathcal{A}_{\gamma,\mathbb{D}}^{-1}(b_{n-1}^{(n-1)})$. The proof will be completed if we can show that $P_n(0,\eta) = P_1(0,\eta)$. Let

$$f_{a,\delta} = c_{a,\delta}(1-|z|^2)^{\frac{a}{2}-1}(1-z)^{\bar{\delta}}(1-\bar{z})^{\delta}, \quad a = \beta(n-1)$$

be the density function of $\gamma_0^{(n)}$ as in Definition 4.7. Note that the random variable $\gamma_0^{(n)}$ is independent of $\gamma_k^{(n)}, 1 \le k \le n-1$, and the latter sequence has the same joint distribution as $\gamma_k^{(n-1)}, 0 \le k \le n-2$. Hence from (3.18) and (7.15) we get

$$P_n(0,\eta) = \int_{z \in \mathbb{D}} P_{n-1}(z,\eta) f_{a,\delta}(z) dz = \int_{z \in \mathbb{D}} P_1(z,\eta) f_{a,\delta}(z) dz$$

$$= c_\delta (1-\eta)^{\bar{\delta}}(1-\bar{\eta})^{\delta} \int_{z \in \mathbb{D}} c_{a,\delta} \frac{(1-|z|^2)^{\frac{a}{2}+\delta+\bar{\delta}}}{(1-\bar{z}\eta)^{\delta+1}(1-z\bar{\eta})^{\bar{\delta}+1}} dz.$$

Using the change of variable $z \mapsto z\bar{\eta}$ we see that the integral does not depend on η. Hence $P_n(0,\eta)$ is a constant multiple of $P_1(0,\eta)$ which means that it must be equal to it. This completes the induction step and finishes the proof. $\qquad\square$

Now we turn to the proof of Proposition 7.5.

Proof of Proposition 7.5. We first prove the equivalence between (2) and (3). By the recursion (3.19) and the fact $\mathcal{A}_{\gamma,\mathbb{D}}^{-1}(0) = \gamma$, we have

$$b_k' = \mathcal{A}_{\bar{\gamma}_0,\mathbb{D}} \circ \mathcal{A}_{\bar{\gamma}_1,\mathbb{D}} \circ \cdots \circ \mathcal{A}_{\bar{\gamma}_{n-2},\mathbb{D}} \left(\mathcal{A}_{\bar{\gamma}_{n-2},\mathbb{D}}^{-1} \circ \cdots \circ \mathcal{A}_{\bar{\gamma}_k,\mathbb{D}}^{-1}(0) \right)$$

$$= \mathcal{A}_{\bar{\gamma}_0^\iota,\mathbb{D}}^{-1} \circ \cdots \circ \mathcal{A}_{\bar{\gamma}_{k-1}^\iota,\mathbb{D}}^{-1}(0).$$

This shows that the sequence $b_k', 0 \le k \le n-1$ are the first n elements of the path produced by $\bar{\gamma}_0^\iota, \ldots, \bar{\gamma}_{n-2}^\iota$. (This fact was also observed in Lemma 50 of [45].) By setting

$\tilde{b}_{-1} = 1$, we have $b'_n = \mathcal{A}_{b_{n-1}, \mathbb{D}}(1) = 1$. Note also, that $b'_n = 1$ if and only if the corresponding last modified Verblunsky coefficient is equal to 1. This proves that the path parameters described in (2) and (3) are equal in law.

We now prove that the paths described in (1) and (2) have the same distribution. Let $z_k = \mathcal{U}^{-1}(b_k), 0 \le k \le n$ be the path parameters of $\mathtt{CJ}_{n,\beta,\delta}$ in $\mathbb{H}$. By (3.17) this is a (time-inhomogeneous) Markov chain, with transition densities that are invariant under affine transformations. By Proposition 7.9 it follows that $z_n \overset{d}{=} \mathcal{U}^{-1}(\gamma_{n-1}) \sim P_{IV}(\Re\delta + 1, -2\Im\delta)$ with density

$$g(q) = \frac{4^{\Re\delta} |\Gamma(1+\delta)|^2}{\pi\, \Gamma(2\Re\delta + 1)} (1+q^2)^{-\Re\delta - 1} e^{2\Im\delta \arctan q}.$$

Moreover, the proof of the same proposition also implies that the conditional distribution of z_n given $z_k = z = x + iy$ for a fixed $0 \le k \le n-2$ is the same as that of $y\mathcal{U}^{-1}(\gamma_{n-1}) + x$. This random variable has density

$$g_{x,y}(q) := y^{-1} g\left(\frac{q-x}{y}\right).$$

For any fixed $z = x + iy, z' = x' + iy' \in \mathbb{H}$ we get

$$\lim_{q \to \infty} \frac{g_{x',y'}(q)}{g_{x,y}(q)} = \frac{h(z')}{h(z)}, \qquad h(z) := (\Im z)^{2\Re\delta + 1}. \tag{7.16}$$

Let $f_{a_k,\delta}(x,y)$ be the density function of $\mathcal{U}^{-1}(\gamma_k)$ defined via (7.13) with parameters $a_k = \beta(n-k-1)$ and δ. Then the transition density of the Markov chain $z_k, 0 \le k \le n$ is given by

$$P\Big((z,k),(z',k+1)\Big) = f_{a_k,\delta}\left(\frac{\Re(z'-z)}{\Im z}, \frac{\Im z'}{\Im z}\right), \qquad 0 \le k \le n-1.$$

Now consider the distribution of $z_k, 0 \le k \le n$ conditioned on $z_n = q$ with $q \to \infty$. By Doob's h-transform, the conditioned transition density is given by

$$Q\Big((z,k),(z',k+1)\Big) = P\Big((z,k),(z',k+1)\Big) \frac{h(z')}{h(z)}$$

$$= c_{a_k,\delta}\, v^{\frac{\beta}{2}(n-k-1)+2\Re\delta} (u^2 + (1+v^2))^{-\frac{\beta}{2}(n-k-1)-\Re\delta-1} e^{2\Im\delta \arctan(u/(1+v))}$$

where $u = (\Re z' - \Re z)/\Im z$ and $v = \Im z'/\Im z$. By Proposition 7.8, $Q((z,k),(z',k+1))$ is exactly the density function of the random variable $\mathcal{U}^{-1}(\bar{\gamma}_k^\iota)$. This proves the equivalence between the statements (1) and (2), and hence completes the proof. $\qquad\square$

By Proposition 7.5 and Fact 7.8 we see that the effect of conditioning the path parameters in $\mathbb{D}$ to hit 1 is equivalent to changing the parameter $\delta \mapsto -(1+\bar{\delta})$. This coincides with a similar factorization lemma for the generating path of the $\tau_{\beta,\delta}$ operator, see Theorem 43 of [30].

Corollary 7.10. *Consider the same setup as in Proposition 7.5, in particular the path $b'_k, 0 \le k \le n$ defined in (3). Let $\eta \sim \Theta(1,\delta)$ be independent of $\gamma_k, 0 \le k \le n-2$. Then the rotated path parameters $\breve{b}_k := \eta b'_k, 0 \le k \le n$ have the same joint distribution as the path parameters of the $\mathtt{CJ}_{n,\beta,\delta}$ operator.*

Proof. By Proposition 7.5, the path parameters $b'_k, 0 \le k \le n$ can be produced by the sequence of modified Verblunsky coefficients $\bar{\gamma}_0^\iota, \ldots, \bar{\gamma}_{n-2}^\iota, 1$. Let $(\alpha'_0, \ldots, \alpha'_{n-1}) := \mathcal{T}_n^{-1}(\bar{\gamma}_0^\iota, \ldots, \bar{\gamma}_{n-2}^\iota, 1)$ be the corresponding sequence of Verblunsky coefficients. By (3.10) and (7.5) we get

$$\alpha'_k = (-1)^k \frac{\gamma_k^\iota \cdots \gamma_0^\iota}{\gamma_{k-1} \cdots \gamma_0}, 0 \le k \le n-2, \qquad \alpha'_{n-1} = (-1)^{n-1} \frac{\gamma_{n-2}^\iota \cdots \gamma_0^\iota}{\gamma_{n-2} \cdots \gamma_0}.$$

Recall also the (equivalent) description of $b'_k, 0 \leq k \leq n$ using $\alpha'_k, 0 \leq k \leq n-1$ in (3.8). Observe that for $Z = \mathrm{diag}(z, 1)$, we have

$$\mathcal{P}Z^{-1} \begin{pmatrix} 1 & \bar{\alpha}_0 \\ \alpha_0 & 1 \end{pmatrix} \cdots \begin{pmatrix} 1 & \bar{\alpha}_{k-1} \\ \alpha_{k-1} & 1 \end{pmatrix} Z \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \bar{z} b_k. \tag{7.17}$$

This shows that the Verblunsky coefficients $\check{\alpha}_k, 0 \leq k \leq n-1$ corresponding to $\check{b}_k, 0 \leq k \leq n$ are given by $\check{\alpha}_k = \bar{\eta} \alpha'_k$, in particular

$$\check{\alpha}_k = (-1)^k \bar{\eta} \frac{\gamma_k^\iota \cdots \gamma_0^\iota}{\gamma_{k-1} \cdots \gamma_0}, 0 \leq k \leq n-2, \qquad \check{\alpha}_{n-1} = (-1)^{n-1} \bar{\eta} \frac{\gamma_{n-2}^\iota \cdots \gamma_0^\iota}{\gamma_{n-2} \cdots \gamma_0}.$$

(The relation between rotating the sequences of path parameters and the corresponding Verblunsky coefficients is related to the so-called Aleksandrov measure. We refer to [38] for more background.)

Finally, using Claim 7.3 for $\zeta_0 = \eta$, $\zeta_j = \gamma_{j-1}, 1 \leq j \leq n$ we get the following distributional identity:

$$\left(\gamma_0^\iota \bar{\eta}, \frac{\gamma_1^\iota}{\gamma_0}, \ldots, \frac{\gamma_{n-2}^\iota}{\gamma_{n-3}}, \frac{1}{\gamma_{n-2}} \right) \stackrel{d}{=} \left(\bar{\gamma}_0, \frac{\bar{\gamma}_1}{\bar{\gamma}_0^\iota}, \ldots, \frac{\bar{\gamma}_{n-2}}{\bar{\gamma}_{n-3}^\iota}, \frac{\bar{\eta}}{\bar{\gamma}_{n-2}^\iota} \right).$$

This means that $\check{\alpha}_k, 0 \leq k \leq n-1$ have the same joint distribution as the random variables

$$\dot{\alpha}_k = (-1)^k \frac{\bar{\gamma}_k \cdots \bar{\gamma}_0}{\bar{\gamma}_{k-1}^\iota \cdots \bar{\gamma}_0^\iota}, 0 \leq k \leq n-2, \qquad \dot{\alpha}_{n-1} = (-1)^{n-1} \frac{\bar{\eta} \bar{\gamma}_{n-2} \cdots \bar{\gamma}_0}{\bar{\gamma}_{n-2}^\iota \cdots \bar{\gamma}_0^\iota}.$$

Since the joint distribution of $\gamma_0, \ldots, \gamma_{n-2}, \eta$ is the same as the joint distribution of $\gamma_0, \ldots, \gamma_{n-1}$, (3.10) and (7.5) now shows that

$$(\check{\alpha}_0, \check{\alpha}_1, \ldots, \check{\alpha}_{n-1}) \stackrel{d}{=} (\dot{\alpha}_0, \ldots, \dot{\alpha}_{n-1}) \stackrel{d}{=} \mathcal{T}_n^{-1}(\gamma_0, \ldots, \gamma_{n-1}),$$

which implies that $\check{b}_k, 0 \leq k \leq n$ have the same joint distribution as the path parameters $b_k, 0 \leq k \leq n$ for $\mathtt{CJ}_{n,\beta,\delta}$. $\qquad \square$

Recall the path $\overleftarrow{b}_k, 0 \leq k \leq n$, the pulled back version of the reversed path $\widetilde{b}_k, 0 \leq k \leq n$ corresponding to the random measure $\mu_{n,\beta,\delta}^{\mathrm{CJ}}$. By Proposition 7.2 $\widetilde{b}_k, 0 \leq k \leq n-1$ has the same distribution as the path built from the modified Verblunsky coefficients $\gamma_{n-2}, \ldots, \gamma_0$. This path is the complex conjugate of the path $\hat{b}_k, 0 \leq k \leq n-1$ in (3) of Proposition 7.5. This also means that $\overleftarrow{b}_k, 0 \leq k \leq n$ is just the time-reversed and complex conjugated version of the path $b'_k, 0 \leq k \leq n$ in (3) of Proposition 7.5. Together with Corollary 7.10 this implies that applying an independent random rotation, a complex conjugation, and a time reversal to $\overleftarrow{b}_k, 0 \leq k \leq n$ produces a path that has the same distribution as the driving path of the operator $\mathtt{CJ}_{n,\beta,\delta}$. This statement allows us to show that $\overleftarrow{\tau}_{n,\beta,\delta}$ and $\mathtt{CJ}_{n,\beta,\delta}$ are orthogonally equivalent.

Proposition 7.11. *Let* $\overleftarrow{b}_k, 0 \leq k \leq n$ *be the path parameters defined in Corollary 7.10,* $\overleftarrow{\tau}_{n,\beta,\delta}$ *the corresponding Dirac-type operator, and let*

$$Q = \frac{1}{\sqrt{1+q^2}} \begin{pmatrix} q & 1 \\ -1 & q \end{pmatrix}, \qquad q = \mathcal{U}^{-1}(\overleftarrow{b}_n).$$

Then the operator $\rho^{-1}(SQ)\overleftarrow{\tau}_{n,\beta,\delta}(SQ)^{-1}\rho$ *has the same distribution as* $\mathtt{CJ}_{n,\beta,\delta}$.

Proof. Recall the transformation $\mathcal{Q}$ defined in Section 2.2. Observe that in the unit-disk model, the transformation $\mathcal{Q}$ behaves exactly as a rotation, i.e. for $z \in \mathbb{D}$ we have $\mathcal{Q}(z) = \mathcal{U} \circ Q \circ \mathcal{U}^{-1}(z) = \bar{\eta} z$, where we denote $\eta := \overleftarrow{b}_n$.

Together with Propositions 7.2 and 7.5, this observation shows that the path parameters of $\rho^{-1}(SQ)\overleftarrow{\tau}_{n,\beta,\delta}(SQ)^{-1}\rho$ have the same joint distribution as $\check{b}_k = \eta b'_k, 0 \le k \le n$ where $b'_k, 0 \le k \le n$ are distributed as in Proposition 7.5. Moreover, by the same argument one can check that the left boundary point of $\rho^{-1}(SQ)\overleftarrow{\tau}_{n,\beta,\delta}(SQ)^{-1}\rho$ is equal to 1 as desired. The proof is now completed by Corollary 7.10. $\qquad\square$

Recall the limiting operator $\mathtt{HP}_{\beta,\delta}$ defined in Definition 4.12, and its reversed and transformed version $\tau_{\beta,\delta}$ defined in Definition 4.18. By Propositions 7.9 and 7.11 the transformation connecting $\mathtt{HP}_{\beta,\delta}$ and $\tau_{\beta,\delta}$ has the same distribution as the one connecting $\mathtt{CJ}_{n,\beta,\delta}$ and $\overleftarrow{\tau}_{n,\beta,\delta}$.

The operator level convergence of the $\mathtt{CJ}_{n,\beta,\delta}$ to the limiting $\mathtt{HP}_{\beta,\delta}$ operator was proved in [30] and can be summarized as follows. This is also the final ingredient to prove the convergence of the normalized characteristic polynomial of the truncated circular Jacobi beta ensemble.

Proposition 7.12 ([30]). *Fix $\beta > 0$ and $\Re\delta > -1/2$. Consider the Dirac-type operators $\mathtt{CJ}_{n,\beta,\delta} = \mathtt{Dir}(z_n(\cdot), \mathfrak{u}_0, \mathfrak{u}_1^{(n)}), n \ge 1$ with $\mathfrak{u}_0 = \binom{1}{0}, \mathfrak{u}_1^{(n)} = \binom{-z_n(1)}{-1}$, and $\mathtt{HP}_{\beta,\delta} = \mathtt{Dir}(z(\cdot), \mathfrak{u}_0, \mathfrak{u}_1)$ with $\mathfrak{u}_1 = \binom{-z(1)}{-1}$. There exists a coupling of the operators $\mathtt{CJ}_{n,\beta,\delta}, n \ge 1$, and $\mathtt{HP}_{\beta,\delta} = \mathtt{Dir}(z(\cdot), \mathfrak{u}_0, \mathfrak{u}_1)$ such that as $n \to \infty$ we have almost surely $z_n \to z$ pointwise on $[0, 1)$, and almost surely*

$$\|\mathbf{r}\,\mathtt{CJ}_{n,\beta,\delta} - \mathbf{r}\,\mathtt{HP}_{\beta,\delta}\|_{\mathrm{HS}} \to 0, \qquad \mathfrak{t}_{\mathtt{CJ}_{n,\beta,\delta}} - \mathfrak{t}_{\mathtt{HP}_{\beta,\delta}} \to 0.$$

Set $c_\delta = \frac{4}{\beta}(\Re\delta + \frac{1}{2}) > 0$, and for $\varepsilon > 0$ small define $c_1 = c_\delta - \varepsilon, c_2 = c_\delta + \varepsilon$. Then there exists a sequence of tight random variables $\kappa_n, n \ge 1$ and an a.s. finite random variable $\kappa > 0$ such that for $0 \le t < 1$

$$\kappa_n^{-1}\left(1 - \frac{\lfloor nt \rfloor}{n}\right)^{c_2} \le \Im z_n(t) \le \kappa_n\left(1 - \frac{\lfloor nt \rfloor}{n}\right)^{c_1}, \quad |z_n(1) - \Re z_n(t)| \le \kappa_n\left(1 - \frac{\lfloor nt \rfloor}{n}\right)^{c_1}$$
$$\tag{7.18}$$

and similarly,

$$\kappa^{-1}(1 - t)^{c_2} \le \Im z(t) \le \kappa(1 - t)^{c_1}, \quad |z(1) - \Re z(t)| \le \kappa(1 - t)^{c_1}. \tag{7.19}$$

We now have all the ingredients to prove the edge scaling limit of the truncated circular Jacobi beta ensemble.

Theorem 7.13. *For fixed $n \ge 1$, $\beta > 0$ and $\Re\delta > -1/2$, let $\lambda_i, 1 \le i \le n - 1$ be size n truncated circular Jacobi beta ensemble nd set $p_{n-1,\beta,\delta}(z) = \prod_{i=1}^{n-1} \frac{z - \lambda_i}{1 - \lambda_i}$ be the normalized characteristic polynomial. Let $\mathcal{E}_{\beta,\delta}$ be the structure function of $\tau_{\beta,\delta}$ defined via (2.11). Then there is a coupling of $p_{n,\beta,\delta}, n \ge 1$ and $\mathcal{E}_{\beta,\delta}$ such that*

$$|p_{n-1,\beta}(e^{iz/n})e^{-iz/2} - \mathcal{E}_{\beta,\delta}(z)| \to 0 \quad \text{a.s. uniformly on compacts as } n \to \infty$$

Consequently, under the edge scaling (1.3), the truncated circular Jacobi beta ensembles converge weakly to the zeros of the random analytic function $\mathcal{E}_{\beta,\delta}(\cdot)$.

Proof. Let $\widetilde{\mu}_n$ be the reversed version of the measure $\mu_n := \mu_{n,\beta,\delta}^{\mathrm{CJ}}$, and $\overleftarrow{\tau}_{n,\beta,\delta}$ be the pulled-back operator. By Propositions 3.2 and 7.2, we may regard $p_{n-1,\beta,\delta}$ as the monic orthogonal polynomial of degree $n - 1$ associated to $\widetilde{\mu}_n$. By Proposition 5.3, we have

$$p_{n-1,\beta,\delta}(e^{iz/n})e^{-iz/2} = e^{-iz/(2n)}\overleftarrow{H}_n((n-1)/n, z)^\dagger \binom{1}{-i},$$

where $\overleftarrow{H}_n$ solves the ODE (2.9) of $\overleftarrow{\tau}_{n,\beta,\delta}$. Recall $\mathcal{E}_{\beta,\delta} = H_{\beta,\delta}(1,\cdot)^\dagger \cdot \binom{1}{-i}$, where $H_{\beta,\delta}(t,z)$ solves (2.9) of $\tau_{\beta,\delta}$. It is enough to show the uniform-on-compacts convergence of $\overleftarrow{H}_n(1,z)$ to $H_{\beta,\delta}(1,z)$.

By Propositions 4.19 and 7.11, one can obtain the operators $\overleftarrow{\tau}_{n,\beta,\delta}, n \geq 1$ and $\tau_{\beta,\delta}$ from $\mathrm{CJ}_{n,\beta,\delta}, n \geq 1$ and $\mathrm{HP}_{\beta,\delta}$ under the same orthogonal transformations. Therefore, under the coupling of the operators $\mathrm{CJ}_{n,\beta,\delta}$ and $\mathrm{HP}_{\beta,\delta}$ in Proposition 7.12 gives

$$\|\mathbf{r}\,\overleftarrow{\tau}_{n,\beta,\delta} - \mathbf{r}\,\tau_{\beta,\delta}\|_{\mathrm{HS}} \to 0, \qquad |\mathbf{t}_{\overleftarrow{\tau}_{n,\beta,\delta}} - \mathbf{t}_{\tau_{\beta,\delta}}| \to 0 \quad \text{a.s. as } n \to \infty. \tag{7.20}$$

Let $\{\overleftarrow{z}_k, 0 \leq k \leq n\}$, $\{\widetilde{z}_k, 0 \leq k \leq n\}$, and $\{z_k, 0 \leq k \leq n\}$ be the path parameters of $\overleftarrow{\tau}_{n,\beta,\delta}$, $\widetilde{\tau}_{n,\beta,\delta}$ and $\mathrm{CJ}_{n,\beta,\delta}$ in $\mathbb{H}$, respectively. By Definitions 5.1 and 5.2, we have $\Im\overleftarrow{z}_k = \Im\widetilde{z}_k/\Im\widetilde{z}_{n-1}$. From (3.16) and (3.17) we have $\Im z_k = \prod_{j=0}^{k-1} \frac{1-|\gamma_j|^2}{|1-\gamma_j|^2}$. By (7.4) we have

$$(\Im\widetilde{z}_k, 0 \leq k \leq n-1) \overset{d}{=} \Big(\prod_{j=n-k-1}^{n-2} \frac{1-|\gamma_j|^2}{|1-\gamma_j|^2}, 0 \leq k \leq n-1 \Big)$$

which implies $(\Im\overleftarrow{z}_k, 0 \leq k \leq n-1) \overset{d}{=} ((\Im z_{n-k-1})^{-1}, 0 \leq k \leq n-1)$. (Note an analogous relation also holds between the imaginary part of the generating paths of the $\tau_{\beta,\delta}$ and $\mathrm{HP}_{\beta,\delta}$ operators.) Recall $\overleftarrow{z}_n(\cdot) := \overleftarrow{z}_{\lfloor nt \rfloor}$ for $t \in (0,1]$. For ε small, the first bound in (7.18) yields that

$$\kappa_n^{-1} \left(\frac{\lfloor nt \rfloor}{n} \right)^{-c_\delta+\varepsilon} \leq \Im\overleftarrow{z}_n(t) \leq \kappa_n \left(\frac{\lfloor nt \rfloor}{n} \right)^{-c_\delta-\varepsilon}, \quad c_\delta = \frac{4}{\beta}(\Re\delta + \frac{1}{2}), \tag{7.21}$$

with a sequence of tight random variables κ_n. By choosing $\varepsilon < \min\{c_\delta, \frac{1}{2}\}$ one can check that the path bound (7.21) satisfies the condition (5.12). We can now combine Proposition 5.4, Corollary 5.6, and Proposition 7.12 to obtain the theorem. $\qquad\square$

Note that by Proposition 4.19, the structure function $\mathcal{E}_{\beta,\delta}$ can also be characterized as $\mathcal{E}_{\beta,\delta} = \mathcal{H}_{\beta,\delta}(0,z)^\dagger \binom{1}{-i}$, where $\mathcal{H}_{\beta,\delta}(u,z)$ solves the SDE (4.16).

Note that in the coupling of Proposition 7.12, the path bounds (7.18), (7.19) and the point-wise convergence of the generating paths $z_n \to z$ imply that $\mathfrak{u}_1^{(n)} \to \mathfrak{u}_1$ a.s. (see also Proposition 7.9). From Theorem 7.13 and Proposition 5.5 one obtains the following result on the (edge) scaling limit of the multiplicative perturbed circular Jacobi beta ensemble.

Corollary 7.14. *For fixed $r \in [0,1]$ let $\Lambda_n^{[r]} = \{\lambda_1, \ldots, \lambda_n\}$ be the set of eigenvalues of the perturbed matrix $\mathrm{CJ}_{n,\beta,\delta}^{[r]}$. Set $p_{n,\beta,\delta}^{[r]}(z) = \prod_{i=1}^n \frac{z-\lambda_i}{1-\lambda_i}$ to be the normalized characteristic polynomial, and let $H_{\beta,\delta}$ be the solution to the ODE (2.9) of $\tau_{\beta,\delta}$. Let $q \sim \Theta(1,\delta)$ be independent of $H_{\beta,\delta}$. Then under the coupling of Proposition 7.12, we have*

$$\left| p_{n,\beta,\delta}^{[r]}(e^{iz/n})e^{-iz/(2n)} - \mathcal{E}_{\beta,\delta}^{[r]}(z) \right| \to 0, \quad \text{a.s. uniformly on compacts as } n \to \infty,$$

where $\mathcal{E}_{\beta,\delta}^{[r]}(z) = H_{\beta,\delta}(1,z) \cdot \binom{1}{-c_r}$ with $c_r = \frac{q+i\frac{1-r}{1+r}}{1-iq\frac{1-r}{1+r}}$. In particular, this implies the weak convergence of $\Lambda_n^{[r]}$ under the edge scaling (1.3) to the zero set of the random analytic function $\mathcal{E}_{\beta,\delta}^{[r]}$ as $n \to \infty$.

Similar to the statement of Theorem 1.3, we have $c_r = q$ when $r = 1$ and $c_r = i$ when $r = 0$, hence this result shows the connection between the scaling limits of the truncated and the unperturbed circular Jacobi beta ensemble.

8 Appendix

8.1 Overview of some results for $\beta = 2$

Let Λ be a locally compact Polish space (in this section it will always be $\mathbb{C}$). A simple point process on Λ is called *determinantal* with respect to a reference measure μ with kernel function $K : \Lambda \times \Lambda \to \mathbb{R}$ if for any $k \geq 1$ the kth joint intensity function of the process with respect to μ is given by

$$\rho_k(x_1, \ldots, x_k) = \det(K(x_i, x_j))_{1 \leq i,j \leq k}. \tag{8.1}$$

(See [21] for more on determinantal point processes.) Determinantal processes appear naturally from joint probability densities containing the square of the Vandermonde. In this section we will always assume that all absolute moments of μ are finite.

Proposition 8.1 ([32, 27]). *Suppose that the $X_1, X_2, \ldots, X_n$ are complex valued random variables with joint density given by*

$$\frac{1}{Z_{n,\mu}} \prod_{1 \leq i < j \leq n} |z_i - z_j|^2 \tag{8.2}$$

with respect to a product measure $\mu^{\otimes n}$ (on $\mathbb{R}^n$ of $\mathbb{C}^n$). (Here $Z_{n,\mu}$ is a finite constant.) Let $\varphi_k(z)$ be the degree k orthonormal polynomials with respect to μ. Then $\sum_{k=1}^n \delta_{X_k}$ is a determinantal point process with respect to μ with kernel given by

$$K(z, w) = \sum_{k=0}^{n-1} \varphi_k(z) \bar{\varphi}_k(w). \tag{8.3}$$

Proposition 8.1 together with (1.1) and (1.2) immediately imply that both the circular unitary ensemble and the truncated circular unitary ensemble are determinantal. The size n circular unitary ensemble has kernel function

$$K_{\mathsf{Circ}_{n,2}}(z, w) = \sum_{k=0}^{n-1} z^k \bar{w}^k, \tag{8.4}$$

with respect to the uniform measure on the unit circle. The size n truncated circular unitary ensemble has a similar kernel function

$$K_{\mathrm{Trunc}(\mathsf{Circ}_{n,2})}(z, w) = \sum_{k=0}^{n-1} (k+1) z^k \bar{w}^k, \tag{8.5}$$

with respect to the uniform measure on the unit disk. Note that [47] also treats more general truncations of Haar unitary matrices. They showed that if we delete the first m rows and columns of a size $n + m$ Haar unitary matrix then the eigenvalues of the resulting submatrix have joint eigenvalue density given by

$$\frac{1}{Z_{n,m}} \prod_{1 \leq j < k \leq n} |z_j - z_k|^2 \prod_{k=1}^n (1 - |z_k|^2)^{m-1}, \qquad z_j \in \mathbb{D}, \tag{8.6}$$

with respect to the Lebesgue measure on the unit disk. By Proposition 8.1 this is also a determinantal point process, with respect to the measure with density $(1 - |z|^2)$ on the unit disk.

Determinantal processes have a number of nice analytic features. The following proposition shows that if we understand the scaling limit of the kernels of a sequence of determinantal processes then we can derive the scaling limit and the limit is also determinantal.

Proposition 8.2 ([40, 36]). *Suppose that $\mathcal{X}, \mathcal{X}_1, \mathcal{X}_2, \ldots$ are determinantal processes on $\mathbb{C}$ with respect to the common reference measure μ, with kernel functions $K, K_1, K_2, \ldots$. Assume that*

$$K_n(z, w) \to K(z, w) \tag{8.7}$$

uniformly on compacts in $\mathbb{C}^2$. Then $\mathcal{X}_n$ converges in distribution to $\mathcal{X}$.

Using some simple transformations one can rewrite the determinantal kernel of the circular unitary ensemble as

$$\tilde{K}_{\mathsf{Circ}_{n,2}}(e^{it}, e^{is}) = D_n(t - s), \qquad D_n(u) = \frac{\sin(nu/2)}{\sin(u/2)}.$$

By taking the limit of this kernel as $n \to \infty$ we get the celebrated result of Gaudin, Mehta, Dyson regarding the the scaling limit of the circular ensemble.

Theorem 8.3 ([2, 32]). *Let Λ_n be the angles in the size n circular unitary ensemble parametrized in $(-\pi, \pi]$. Then $n\Lambda_n \Rightarrow \Lambda$ where Λ is a determinantal point process on $\mathbb{R}$ with kernel*

$$K_{\mathrm{Sine}_2}(s, t) = \frac{\sin((s - t)/2)}{\pi(s - t)}$$

with respect to the Lebesgue measure.

Taking the limit of the kernel (8.5) (without any additional scaling) we obtain the point process studied by Peres and Virág in [34].

Theorem 8.4 ([34]). *Let Λ_n be the eigenvalues of $\mathrm{Trunc}(\mathsf{Circ}_{n+1,2})$. Then Λ_n converges to a determinantal point process on the unit disk with kernel*

$$K_{\mathrm{Bergman}}(z, w) = \frac{1}{(1 - z\bar{w})^2} \tag{8.8}$$

with respect to the uniform measure on the unit disk. The resulting point process has the same distribution as the zero set of the Gaussian analytic function

$$f_{GAF}(z) = \sum_{n=0}^{\infty} \xi_n z^n, \tag{8.9}$$

where $\xi_n, n \geq 0$ are i.i.d. standard complex normals.

Note that [29] provides a generalization of this result by connecting the limit of the eigenvalues of rank m truncation of Haar unitary matrices with the singular points of a matrix valued Gaussian analytic function that generalizes (8.9).

The circular Jacobi beta ensemble (1.9) for $\beta = 2$ is also determinantal, this is also called the Hua-Pickrell distribution. The kernel function can be expressed in terms of the orthogonal polynomials with respect to the probability measure μ_δ that has probability density function proportional to

$$(1 - \bar{z})^\delta (1 - z)^{\bar{\delta}}$$

on the unit circle. The Hua-Pickrell distribution can be realized as the joint eigenvalue distribution of the random unitary matrices that have density proportional to $|\det(1 - U)^\delta|^2$ with respect to the Haar measure.

In [31] the authors studied the truncations of these random matrices. They showed that the eigenvalues of the rank-m truncated matrices form a determinantal point process on the unit disk, and derived their kernel function. Moreover, they showed that the scaling limit of the joint eigenvalue distribution (without any additional scaling) leads to the same limits as in the Haar unitary case (as described in [34] and [29]).

If we are interested in the (hard) edge scaling limit of the eigenvalues of the truncated matrix $\text{Trunc}(\text{Circ}_{n+1,2})$ then one needs to transform the kernel (8.5) according to (1.3), and take the limit. This was considered in [1] where the following result was shown.

Theorem 8.5 ([1]). *Let Λ_n be the eigenvalues of $\text{Trunc}(\text{Circ}_{n+1,2})$. Then the sequence $-ni\log\Lambda_n$ converges to a determinantal point process $\mathcal{X}_2$ supported on the upper half plane $\mathbb{H}$ with kernel function*

$$K_{\text{edge}}(z,w) = f(z-\bar{w}), \quad f(u) = \frac{1}{\pi}\int_0^1 te^{itu}dt, \quad z,w\in\mathbb{H}, \tag{8.10}$$

with respect to the Lebesgue measure on $\mathbb{H}$.

In fact, this is a special case of a more general problem that [1] considers: the product of independent copies of rank-m truncations of Haar unitary matrices. The point process $\mathcal{X}_2$ also appears in [15] as the scaling limit of the rank-one additive anti-Hermitian perturbation for the Gaussian unitary ensemble.

Note that our Theorem 1.2 for $\beta=2$ provides a new characterization for the determinantal point process $\mathcal{X}_2$. It would be interesting to see whether the determinantal structure could be proved directly from that result.

8.2 Open problems

We end with a couple of open problems.

Problem 1. (Bulk scaling limits) Our results for general $\beta > 0$ consider the models with the edge scaling (1.3). It would be interesting to explore the limiting behavior under the bulk scaling, i.e. when we do not scale at all. In other words, it would be interesting to see whether one could extend the results of [34] (see Theorem 8.4) and [31] for general β, to characterize the point process scaling limit of the truncated circular and circular Jacobi beta ensembles under no additional scaling. Moreover, it would be interesting to find the scaling limit of the characteristic polynomial in this scaling regime.

For $\beta = 2$, Krishnapur proved a result in [29] that connects the characteristic polynomials of the truncated circular unitary ensemble with the Gaussian analytic function of Theorem 8.4. Let $\phi_{n-1}(z)$ be the characteristic polynomial of the rank one truncated Haar unitary matrix (the truncated circular unitary ensemble), and $\phi_{n-1}^*(z) = z^{n-1}\overline{\phi_{n-1}(1/\bar{z})}$ the reversed version of this polynomial. Theorem 5 of [29] proves that

$$\sqrt{n}\,\frac{\phi_{n-1}(z)}{\phi_{n-1}^*(z)} \to f_{GAF}(z), \qquad z\in\mathbb{D} \tag{8.11}$$

in distribution, uniformly on compacts in $\mathbb{D}$. (See also the comments at the end of Section 5 of [29].) This of course implies that the truncated circular unitary ensemble as a point process converges to the zero set of the Gaussian analytic function f_{GAF}. It is not known whether a similar limit holds for general β, and what the limit should be. It is interesting to note however that for general β the sequence $\phi_{n-1}^*(z), n\geq 2$ converges uniformly on compacts in $\mathbb{D}$, this was proved in [6]. The limiting random function can be written as $\exp\left(\sqrt{\frac{2}{\beta}}g(z)\right)$ where $g(z) = \sum_{k=1}^\infty \frac{z^k}{\sqrt{k}}\xi_n$, and $\xi_n, n\geq 1$ are i.i.d. complex standard normals.

Problem 2. (Edge to bulk transition) A simple calculation shows that under the scaling $z\mapsto c^{-1}z, c\to\infty$ the kernel (8.10) converges to a transformed version of the kernel (8.8), where the transformation is the Cayley transform mapping the unit disk $\mathbb{D}$ to the upper half plane $\mathbb{H}$. This implies that as $c\to\infty$ the scaled edge limit process $c^{-1}\mathcal{X}_2$ converges to the image of the bulk limit process of Peres-Virág under the Cayley

transform. Note that similar 'edge-to-bulk' limits are known for other random matrix ensembles, see e.g. [41] for a similar transition involving the point process limits of the Gaussian beta ensemble. It would be interesting to see if a similar limit exists for $\mathcal{X}_\beta$ for general $\beta > 0$. Presumably, this could also provide a way to prove the 'edge-to-bulk' transition in the $\beta = 2$ case without using the determinantal process framework.

Problem 3. (Purely imaginary additive rank one perturbations of beta ensembles) In [28] Kozhan constructs tridiagonal matrix models for non-Hermitian (purely imaginary) rank-one perturbations of the Gaussian and Laguerre beta ensembles. It is known from [14] that in the $\beta = 2$ case the eigenvalues of these perturbed ensembles converge (under the appropriate edge scaling) to the point process limits of the rank-one multiplicative perturbations of the circular and real orthogonal beta ensembles. One would expect that the result should hold in the general β case as well. This would require the extension of the random operator framework to limits of tridiagonal (Jacobi) matrices.

Problem 4. (Multi-rank truncation of circular ensembles) In the $\beta = 2$ case, Życzkowski and Sommers [47] also derived the joint eigenvalue distribution for a general rank-k truncated circular unitary ensemble. In fact several of the $\beta = 2$ limit scaling results extend to truncated models where $k \geq 1$ rows and columns are removed from a Haar unitary matrix. It would be interesting to see if any of these results could be extended for general $\beta > 0$. The first step would be to characterize the rank-k truncated circular beta ensemble. As noted in [24], this problem could potentially be attacked using the theory of matrix-valued Verblunsky coefficients.

References

[1] Gernot Akemann, Zdzislaw Burda, Mario Kieburg, and Taro Nagao. Universal microscopic correlation functions for products of truncated unitary matrices. *Journal of Physics A: Mathematical and Theoretical*, 47(25):255202, 2014. MR3224113

[2] Greg W. Anderson, Alice Guionnet, and Ofer Zeitouni. *An introduction to random matrices*. Number 118. Cambridge University Press, 2010. MR2760897

[3] Rajendra Bhatia and Ludwig Elsner. The Hoffman-Wielandt inequality in infinite dimensions. In *Proceedings of the Indian Academy of Sciences-Mathematical Sciences*, volume 104, pages 483–494. Springer, 1994. MR1314392

[4] Paul Bourgade, Ashkan Nikeghbali, and Alain Rouault. Circular Jacobi ensembles and deformed Verblunsky coefficients. *International Mathematics Research Notices*, 2009(23):4357–4394, 2009. MR2558335

[5] Maria J. Cantero, Leandro Moral, and Luis Velázquez. Five-diagonal matrices and zeros of orthogonal polynomials on the unit circle. *Linear Algebra and its Applications*, 362:29–56, 2003. MR1955452

[6] Reda Chhaibi and Joseph Najnudel. On the circle, $GMC^\gamma = \varprojlim C\beta E_n$ for $\gamma = \sqrt{\frac{2}{\beta}}, (\gamma \leq 1)$. *arXiv preprint arXiv:1904.00578*, 2019.

[7] Reda Chhaibi, Joseph Najnudel, and Ashkan Nikeghbali. The Circular Unitary Ensemble and the Riemann zeta function: the microscopic landscape and a new approach to ratios. *Inventiones mathematicae*, pages 1–91, 2015. MR3592756

[8] Guillaume Dubach and László Erdős. Dynamics of a rank-one perturbation of a Hermitian matrix. *Electronic Communications in Probability*, 28:1–13, 2023. MR4555404

[9] Guillaume Dubach and Jana Reker. Dynamics of a rank-one multiplicative perturbation of a unitary matrix. *arXiv preprint arXiv:2212.14638*, 2022. MR4775373

[10] Ioana Dumitriu and Alan Edelman. Matrix models for beta ensembles. *J. Math. Phys.*, 43(11):5830–5847, 2002. MR1936554

[11] Peter J. Forrester. *Log-gases and random matrices (LMS-34)*. Princeton University Press, 2010. MR2641363

[12] Peter J. Forrester. Rank 1 perturbations in random matrix theory — A review of exact results. *Random Matrices: Theory and Applications*, aug 2023. MR4672877

[13] Peter J. Forrester and Jesper R Ipsen. A generalisation of the relation between zeros of the complex Kac polynomial and eigenvalues of truncated unitary matrices. *Probability Theory and Related Fields*, 175(3):833–847, 2019. MR4026606

[14] Yan V. Fyodorov. Spectra of random matrices close to unitary and scattering theory for discrete-time systems. In *AIP Conference Proceedings*, volume 553, pages 191–196. American Institute of Physics, 2001. MR1870891

[15] Yan V. Fyodorov and Boris A Khoruzhenko. Systematic analytical approach to correlation functions of resonances in quantum chaotic scattering. *Physical review letters*, 83(1):65, 1999.

[16] Yan V Fyodorov, Boris A Khoruzhenko, and Mihail Poplavskyi. Extreme eigenvalues and the emerging outlier in rank-one non-Hermitian deformations of the Gaussian unitary ensemble. *Entropy*, 25(1):74, 2022. MR4539184

[17] Yan V. Fyodorov and Dmitry Savin. Resonance scattering of waves in chaotic systems. In *The Oxford Handbook of Random Matrix Theory*. Oxford University Press, 09 2015.

[18] Yan V. Fyodorov and Hans-Jürgen Sommers. Spectra of random contractions and scattering theory for discrete-time systems. *Journal of Experimental and Theoretical Physics Letters*, 72(8):422–426, 2000.

[19] Yan V. Fyodorov and Hans-Jürgen Sommers. Random matrices close to Hermitian or unitary: overview of methods and results. *Journal of Physics A: Mathematical and General*, 36(12):3303, 2003. MR1986421

[20] Diane Holcomb and Gregorio R. Moreno Flores. Edge Scaling of the β-Jacobi Ensemble. *Journal of Statistical Physics*, 149(6):1136–1160, November 2012. MR3006323

[21] John Ben Hough, Manjunath Krishnapur, Yuval Peres, and Bálint Virág. *Zeros of Gaussian analytic functions and determinantal point processes*, volume 51. American Mathematical Soc., 2009. MR2552864

[22] Loo-Keng Hua. *Harmonic analysis of functions of several complex variables in the classical domains*. Translated from the Russian by Leo Ebner and Adam Korányi. American Mathematical Society, Providence, R.I., 1963. MR0171936

[23] Boris Khoruzhenko and Hans-Jürgen Sommers. Non-Hermitian ensembles. In *The Oxford Handbook of Random Matrix Theory*. Oxford University Press, 09 2015.

[24] Rowan Killip and Rostyslav Kozhan. Matrix models and eigenvalue statistics for truncations of classical ensembles of random unitary matrices. *Communications in Mathematical Physics*, pages 1–37, 2015. MR3602821

[25] Rowan Killip and Irina Nenciu. Matrix models for circular ensembles. *International Mathematics Research Notices*, 2004(50):2665–2701, 2004. MR2127367

[26] Rowan Killip and Mihai Stoiciu. Eigenvalue statistics for CMV matrices: from Poisson to clock via random matrix ensembles. *Duke Math. J.*, 146(3):361–399, 2009. MR2484278

[27] Wolfgang König. Orthogonal polynomial ensembles in probability theory. *Probability Surveys*, 2:385 – 447, 2005. MR2203677

[28] Rostyslav Kozhan. Rank one non-hermitian perturbations of hermitian β-ensembles of random matrices. *Journal of Statistical Physics*, 168(1):92–108, 2017. MR3659979

[29] Manjunath Krishnapur. From random matrices to random analytic functions. *Ann. Probab.*, 37:314–346, 2009. MR2489167

[30] Yun Li and Benedek Valkó. Operator level limit of the circular Jacobi β-ensemble. *Random Matrices: Theory and Applications*, 11(04):2250043, 2022. MR4480838

[31] Zhaofeng Lin, Yanqi Qiu, and Kai Wang. Truncations of random unitary matrices drawn from Hua-Pickrell distribution. *Complex Analysis and Operator Theory*, 17(1):6, 2023. MR4519787

[32] Madan Lal Mehta. *Random matrices*, volume 142 of *Pure and Applied Mathematics (Amsterdam)*. Elsevier/Academic Press, Amsterdam, third edition, 2004. MR2129906

[33] Fumihiko Nakano. Level statistics for one-dimensional Schrödinger operators and Gaussian beta ensemble. *Journal of Statistical Physics*, 156(1):66–93, 2014. MR3215116

[34] Yuval Peres and Bálint Virág. Zeros of the iid Gaussian power series: a conformally invariant determinantal process. *Acta Mathematica*, 194(1):1–35, 2005. MR2231337

[35] Doug Pickrell. Measures on infinite-dimensional Grassmann manifolds. *J. Funct. Anal.*, 70(2):323–356, 1987. MR0874060

[36] Tomoyuki Shirai and Yoichiro Takahashi. Random point fields associated with certain Fredholm determinants I: fermion, Poisson and boson point processes. *Journal of Functional Analysis*, 205(2):414–463, 2003. MR2018415

[37] Barry Simon. OPUC on one foot. *Bulletin of the American Mathematical Society*, 42(4):431–460, 2005. MR2163705

[38] Barry Simon. *Orthogonal polynomials on the unit circle. Part 1*, volume 54 of *American Mathematical Society Colloquium Publications*. American Mathematical Society, Providence, RI, 2005. Classical theory. MR2105088

[39] Barry Simon. *Trace ideals and their applications*, volume 120 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, second edition, 2005. MR2154153

[40] Alexander Soshnikov. Determinantal random point fields. *Russian Mathematical Surveys*, 55(5):923, 2000. MR1799012

[41] Benedek Valkó and Bálint Virág. Continuum limits of random matrices and the Brownian carousel. *Inventiones Math.*, 177:463–508, 2009. MR2534097

[42] Benedek Valkó and Bálint Virág. The Sine$_\beta$ operator. *Inventiones mathematicae*, 209(1):275–327, Dec 2016. MR3660310

[43] Benedek Valkó and Bálint Virág. Operator limit of the circular beta ensemble. *The Annals of Probability*, 48(3):1286–1316, 2020. MR4112715

[44] Benedek Valkó and Bálint Virág. The many faces of the stochastic zeta function. *Geometric and Functional Analysis*, 32(5):1160–1231, 2022. MR4498842

[45] Benedek Valkó and Bálint Virág. Palm measures for Dirac operators and the Sine$_\beta$ process. *Stochastic Processes and their Applications*, 2023. MR4605452

[46] Joachim Weidmann. *Spectral theory of ordinary differential operators*, volume 1258 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1987. MR0923320

[47] Karol Życzkowski and Hans-Jürgen Sommers. Truncations of random unitary matrices. *Journal of Physics A: Mathematical and General*, 33(10):2045, 2000. MR1748745