Immersions and Albertson's Conjecture

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Abstract

A graph is said to contain K_k (a clique of size k) as a *weak immersion* if it has k vertices, pairwise connected by edge-disjoint paths. In 1989, Lescure and Meyniel made the following conjecture related to Hadwiger's conjecture: Every graph of chromatic number k contains K_k as a weak immersion. We prove this conjecture for graphs with at most $1.4(k-1)$ vertices. As an application, we make some progress on Albertson's conjecture on crossing numbers of graphs, according to which every graph G with chromatic number k satisfies $\text{cr}(G) \geq \text{cr}(K_k)$. In particular, we show that the conjecture is true for all graphs of chromatic number k, provided that they have at most $1.4(k-1)$ vertices and k is sufficiently large.

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1 Introduction

There are several famous problems in graph theory which state that over all graphs of a given chromatic number, some graph parameter is minimized by a complete graph. Obviously, the chromatic number of a graph is at least its clique number. The converse is false, but partial converses have been of central interest in graph theory. Hadwiger's conjecture states that every graph of chromatic number k contains a K_k-minor. Wagner proved in 1937 that the case $k = 5$ is equivalent to the four color theorem. Hadwiger's conjecture was verified for $k \leq 6$ by Robertson, Seymour, and Thomas [22], and is open for $k \geq 7$. In the 1980's, Kostochka [17] and Thomason [24] proved that every graph of chromatic number k contains a K_t-minor, where $t = \Omega(k/\sqrt{\log k})$, which was improved to $t = \Omega(k/(\log k)^{1/4+\epsilon})$ by Norin, Postle, and Song [20], and very recently, to $t = \Omega(k/\log\log k)$ by Delcourt and Postle [9].

In 1961, Hajós conjectured the following strengthening of Hadwiger's conjecture: Every graph of chromatic number k contains a *subdivision* of the complete graph K_k, i.e., it has k so-called "branch vertices" connected by $\binom{k}{2}$ internally *vertex-disjoint* paths. Hajós' conjecture is true for $k \leq 4$, but for $k \geq 7$ it was disproved by Catlin [8]. In fact, Erdős and Fajtlowicz [13] showed that almost all graphs are counterexamples (see also [14]). The conjecture remains open for $k = 5, 6$.

Lescure and Meyniel [19] suggested a conjecture weaker than Hajós', which may still be true for every k. Instead of requiring that G contains a subdivision of K_k, they wanted to prove the existence of k branch vertices connected by $\binom{k}{2}$ *edge-disjoint* paths. Moreover,

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these paths may pass through some branch vertices other than their endpoints. They called such a subgraph of G a *weak immersion* of K_k.

More precisely, a graph G contains H as a weak immersion if there is a mapping ϕ from $V(H) \cup E(H)$, which maps each vertex of H to a vertex in G and each edge of H to a path in G such that

1. $\phi(u) \neq \phi(v)$, for distinct vertices $u, v \in V(H)$;
2. for distinct edges $e, f \in E(H)$, the paths $\phi(e)$ and $\phi(f)$ are edge-disjoint; and
3. for each edge $e = uv \in E(H)$, $\phi(e)$ is a path in G with endpoints $\phi(u)$ and $\phi(v)$.

If the following condition is also satisfied, we say that G contains H as a *strong immersion*.

4. For each edge $e \in E(H)$, the path $\phi(e)$ intersects the set of branch vertices, $\phi(V(H))$, only at its endpoints.

In 1989, Lescure and Meyniel conjectured the following.

▶ **Conjecture 1** ([19]). *Every graph with chromatic number k contains a weak immersion of the complete graph K_k.*

For $k \geq 3$, the Lescure-Meyniel conjecture is an immediate corollary of Hajós' conjecture. DeVos, Kawarabayashi, Mohar, and Okamura [11] verified Conjecture 1 for $4 \leq k \leq 6$. Conjecture 1 remains open for $k \geq 7$. According to a result of Gauthier, Le, and Wollan [16], every graph with chromatic number k contains a weak immersion of K_t, where $t = (k-4)/3.54$ (see also [10, 12] for earlier bounds).

Our first result shows that the Lescure-Meyniel conjecture is true for graphs whose number of vertices is not much larger than its clique number.

▶ **Theorem 2.** *If G is a graph with chromatic number k and at most $1.4(k-1)$ vertices, then G contains K_k as a weak immersion.*

As an application, we use Theorem 2 to obtain new bounds on an old conjecture of Albertson.

The *crossing number* of a graph G, $\mathrm{cr}(G)$, is the smallest number of edge crossings in any drawing of G in the plane. In 2007, Albertson conjectured the following.

▶ **Conjecture 3.** *Every graph G with chromatic number k satisfies $\mathrm{cr}(G) \geq \mathrm{cr}(K_k)$.*

Clearly, Albertson's conjecture is weaker than Hajós' conjecture. Moreover, Conjecture 3 vacuously holds for $k \leq 4$, since $\mathrm{cr}(K_4) = 0$ and, for $k = 5$, Conjecture 3 is equivalent to the four color theorem. After a sequence of results [3, 6, 1], it is now known that Albertson's conjecture holds for $k \leq 18$, but it is open for $k \geq 19$.

A graph G is *k-critical* if $\chi(G) = k$, and every proper subgraph of G has chromatic number less than k. A 1-critical graph is just a graph consisting of a single vertex. As $\mathrm{cr}(G) \geq \mathrm{cr}(H)$ holds for all subgraphs $H \subset G$, it suffices to prove Albertson's conjecture for k-critical graphs. In [6], Barát and Tóth verified Conjecture 3 for all k-critical graphs on at most $k + 4$ vertices, and Ackerman [1] proved the conjecture for all k-critical graphs with at least $3.03k$ vertices. Our next result is the following.

▶ **Theorem 4.** *There is a constant $\delta > 0$ such that the following holds. If k is sufficiently large, then every graph G of chromatic number k on $n \leq (1.4+\delta)k$ vertices satisfies $\mathrm{cr}(G) \geq \mathrm{cr}(K_k)$.*

In the last section, we discuss some concluding remarks on how to improve on the constant factor 1.4 in Theorem 2, which in turn implies a larger range for which Theorem 4 holds.

2 Weak immersion

In this section, we prove Theorem 2. First, let us recall the following lemma. A classic result due to Gallai states that if G is a k-critical graph on n vertices, where $n \leq 2k-2$, then the complement of G is disconnected. This implies the following.

▶ **Lemma 5** ([15]). *Let k, n be positive integers with $n \leq 2k-2$. If G is a k-critical graph on n vertices, then there is a vertex partition*

$$V(G) = V_1 \cup V_2 \cup \cdots \cup V_t,$$

where $t \geq 2$, such that V_i is complete to V_j, for $i \neq j$, $|V_i| = n_i$, and the induced subgraph $G[V_i]$ is k_i-critical with $n_i \geq 2k_i - 1$.

The *chromatic index* $\chi'(H)$ of a multigraph H without loops is the minimum number of colors needed to properly color the edges of H, i.e., to color them in such a way that no two edges that share a vertex receive the same color. A classical theorem of Shannon [23] states that every multigraph H with maximum degree Δ satisfies $\chi'(H) \leq 3\Delta/2$.

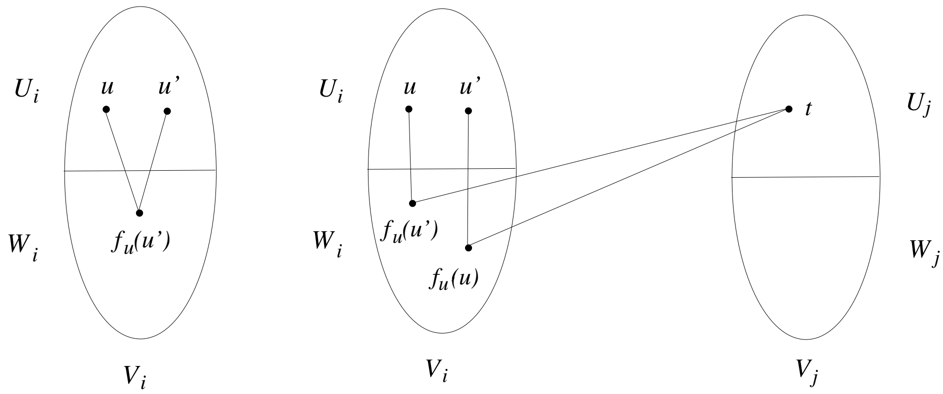
Proof of Theorem 2. We may assume that $k \geq 7$, since otherwise we obtain a weak immersion of K_k by [16]. By possibly deleting vertices and edges, we may assume without loss of generality that G is k-critical. Let $n = |V(G)|$ so $n \leq 1.4(k-1) \leq 2k-2$. By Lemma 5, there is a vertex partition $V(G) = V_1 \cup \cdots \cup V_t$, where $t \geq 2$, such that V_i is complete to V_j for each $i \neq j$, and the induced subgraph $G[V_i]$ is k_i-critical for each i with $n_i \geq 2k_i - 1$ vertices. Hence, $n = \sum_{i=1}^{t} n_i$ and $k = \sum_{i=1}^{t} k_i$. For each i, arbitrarily partition $V_i = U_i \cup W_i$ with $|U_i| = k_i$, so $|W_i| = n_i - k_i$. Let $U = \bigcup_i U_i$.

In what follows, we will construct a weak immersion of K_k with U being the set of branch vertices. Moreover, we will use all edges in U as paths of length one in the weak immersion. By the Gallai decomposition, each nonadjacent pair of vertices in U has both of its vertices in U_i for some i. For each i and vertex $u \in U_i$, let f_u be a one-to-one function from the set of non-neighbors of u in $U_i \setminus \{u\}$ to the set of neighbors of u in W_i. Such a function f_u exists as the degree of u in $G[V_i]$ is at least $k_i - 1$, as $G[V_i]$ is k_i-critical. If a nonadjacent pair (u, u') of vertices in U_i satisfies $f_u(u') = f_{u'}(u)$, then we connect u and u' in the weak immersion by the path of length two with middle vertex $f_u(u')$. Moreover, these paths will be edge-disjoint as f_u is one-to-one. See Figure 1a.

So far we constructed edge-disjoint paths (which are of length one or two) connecting some pairs of branch vertices. We next describe how we connect the remaining pairs of vertices in U by paths, which will each be of length four. For a pair (u, u') of nonadjacent vertices in U_i with $f_u(u') \neq f_{u'}(u)$, we will pick a vertex $t \in U \setminus U_i$ and the path of length four connecting u to u' will have the vertices in order as $u, f_u(u'), t, f_{u'}(u), u'$. We next describe how to pick the vertex $t = t(u, u')$ for each such pair u, u' of nonadjacent vertices in the same U_i with $f_u(u') \neq f_{u'}(u)$. See Figure 1b.

Make an auxiliary multigraph H_i on W_i as follows. For each nonadjacent pair (u, u') with $u, u' \in U_i$ and $f_u(u') \neq f_{u'}(u)$, we add an edge between $f_u(u')$ and $f_{u'}(u)$ in H_i. Clearly, H_i does not contain loops as we require $f_u(u') \neq f_{u'}(u)$. Since f_u is one-to-one, the maximum degree in H_i is at most $|U_i| = k_i$. Being able to pick the desired vertex $t = t(u, u') \in U \setminus U_i$ for each nonadjacent pair $u, u' \in U_i$ with $f_u(u') \neq f_{u'}(u)$, in order to obtain the desired paths of length four for the immersion, is equivalent to being able to properly color the edges of H_i with color set $U \setminus U_i$. As $n_i \geq 2k_i - 1$, we have

$$k_i \leq n_i - k_i + 1 = |W_i| + 1 \leq n - k + 1 \leq 0.4(k-1),$$



(a) $f_u(u') = f_{u'}(u)$ **(b)** $f_u(u') \neq f_{u'}(u)$

Figure 1 Constructing a path from u to u'.

where the last inequality follows from the fact that $n \leq 1.4(k-1)$. This implies that

$$|U \setminus U_i| = k - k_i \geq 3k/5.$$

On the other hand, by Shannon's theorem, we have $\chi'(H_i) \leq 3k_i/2 \leq 3(k-1)/5$. By combining the last two inequalities above, we have $\chi'(H_i) \leq |U \setminus U_i|$, and therefore, we are able to find such a proper edge-coloring, completing the proof. ◀

3 Albertson's conjecture

In this section, we prove Theorem 4.

We recall an old conjecture of Hill, according to which the crossing number of the complete graph on k vertices satisfies $cr(K_k) = H(k)$, where

$$H(k) := \frac{1}{4}\left\lfloor\frac{k}{2}\right\rfloor\left\lfloor\frac{k-1}{2}\right\rfloor\left\lfloor\frac{k-2}{2}\right\rfloor\left\lfloor\frac{k-3}{2}\right\rfloor.$$

It is known that $cr(K_k) \leq H(k)$ by a particular drawing of K_k. In the other direction, Balogh, Lidický, and Salazar [5] proved that $cr(K_k)$ is at least $0.9855H(k)$, for large enough k. In particular, we have the following lemma.

▶ **Lemma 6** ([5]). *If k is sufficiently large, then $\mathrm{cr}(K_k) > k^4/65$.*

A related old conjecture of Zarankiewicz, is that $\mathrm{cr}(K_{a,b}) = \lfloor\frac{a}{2}\rfloor\lfloor\frac{a-1}{2}\rfloor\lfloor\frac{b}{2}\rfloor\lfloor\frac{b-1}{2}\rfloor$. Towards this conjecture, Balogh et al. [4] recently proved the following result.

▶ **Lemma 7** ([4]). *If a, b are sufficiently large, then $\mathrm{cr}(K_{a,b}) > .9118a^2b^2/16$.*

Before turning to the proof of Theorem 4, as a warm-up, we establish the following useful asymptotic result.

▶ **Lemma 8.** *Let G be a graph on n vertices with $\chi(G) = k$ such that $n \leq 1.4(k-1)$. Then*

$$\mathrm{cr}(G) > \mathrm{cr}(K_k) - k^3/3.$$

In particular, $\mathrm{cr}(G) \geq (1 - o(1))\mathrm{cr}(K_k)$, as $k \to \infty$.

Proof. Let G be drawn in the plane with $\mathrm{cr}(G)$ crossings. For a vertex v of G, let $d(v)$ denote the degree of v in G. By Theorem 2, G contains K_k as a weak immersion. Let $v_1, \ldots, v_k$ be the branch vertices of the immersion, and let P_{ij} be the path in G used in the weak immersion with endpoints v_i and v_j.

Consider a drawing of K_k in the plane, with vertices $v_1, \ldots, v_k$, where the edge between v_i and v_j is drawn along the path P_{ij} such that it goes around every branch vertex that is an internal vertex of P_{ij}. By going either clockwise or counterclockwise around a branch vertex v, we can achieve that in the neighborhood of v, the drawing of the edge between v_i and v_j participates in at most $(d(v) - 2)/2$ crossings. Apart from small neighborhoods of the branch vertices along the path P_{ij}, the drawing of the edge connecting v_i to v_j coincides with the drawing of P_{ij}. In particular, the drawing of the edge between v_i and v_j passes through every non-branch vertex that is an internal vertex of P_{ij}.

There are two types of crossings in this drawing of K_k. All crossings that are already crossings in the original drawing of G are of *type 1*, so there are at most $\mathrm{cr}(G)$ of them. The remaining crossings are of *type 2*. They occur in small neighborhoods of vertices of G. The latter crossings fall into two categories depending on whether they occur in a small neighborhood of a non-branch vertex or a branch vertex.

For non-branch vertices v, the number of crossings in the drawing of K_k at v is at most $\binom{f(v)}{2}$, where $f(v)$ denotes the number of paths P_{ij} in the K_k immersion, in which v is an internal vertex of the path P_{ij}. Note that $f(v) \leq d(v)/2$, as v is an internal vertex of at most $d(v)/2$ edge-disjoint paths. Obviously, $d(v) \leq n - 1$ and there are at most $n - k$ non-branch vertices. Therefore, at non-branch vertices, the total number of crossings of type 2 is at most

$$(n-k)\binom{(n-1)/2}{2} \leq (n-k)n^2/8.$$

For each of the k branch vertices v_i, there are $k - 1$ paths P_{ij} ending at v_i. Thus, v_i is an internal vertex of at most $(d(v_i) - (k-1))/2 \leq (n-k)/2$ paths $P_{\ell j}$. In the drawing of K_k, the edge from v_ℓ to v_j participates in at most $(d(v_i) - 2)/2 < n/2$ crossings in the neighborhood of v_i, by going either clockwise or counterclockwise around v_i. Thus, in the neighborhoods of branch vertices, altogether there are at most

$$k\frac{n-k}{2}\frac{n}{2}$$

crossings of type 2.

Adding up the above two bounds and using our assumption that $n \leq 1.4(k-1)$, we conclude that the total number of crossings in the drawing of K_k, which occur in small neighborhoods of the vertices of G is at most $21k^3/64 < k^3/3$. Thus, we have produced a drawing of K_k with fewer than $\mathrm{cr}(G) + k^3/3$ crossings. Consequently, we have

$$\mathrm{cr}(G) > \mathrm{cr}(K_k) - k^3/3 = (1 - o(1))\mathrm{cr}(K_k),$$

as desired. ◀

The well-known crossing lemma discovered by Ajtai, Chvátal, Newborn, Szemerédi [2] and independently, by Leighton [18], states that every graph G with n vertices and $m \geq 4n$ edges satisfies $\mathrm{cr}(G) \geq cm^3/n^2$, where $c > 0$ is an absolute constant. The constant has been improved by several authors. The currently best constant is due to Büngener and Kaufmann.

▶ **Lemma 9** ([7]). *Let G be a graph on n vertices with m edges. If $m \geq 6.95n$, then* $\mathrm{cr}(G) \geq \frac{1}{27.48}\frac{m^3}{n^2}$.

We also need the following two simple lemmas.

▶ **Lemma 10** ([21]). *Let G be a graph with m edges, and let x and y be two nonadjacent vertices. Let $G+xy$ denote the graph obtained by adding the edge (x,y). Then we have*

$$\mathrm{cr}(G+xy) \le \mathrm{cr}(G) + m.$$

▶ **Lemma 11.** *Let G be a graph on n vertices with m edges and $1 \le a \le n$ be an integer. Then G has an induced subgraph on a vertices with at least $m\binom{a}{2}/\binom{n}{2}$ edges.*

Proof. If we take a uniform random subset A of a vertices, the expected number of edges in A is $m\binom{a}{2}/\binom{n}{2}$, and hence, there is an induced subgraph with at least that many edges. ◀

We are now ready to prove Theorem 4. We first prove a variant for k-critical graphs with at most $1.4(k-1)$ vertices.

▶ **Theorem 12.** *There is a constant $c > 0$ such that for k sufficiently large, every k-critical graph G on $n \le 1.4(k-1)$ vertices satisfies $\mathrm{cr}(G) \ge \mathrm{cr}(K_k) + c(n-k)k^3$.*

Proof. The proof is by induction on $\ell := n-k$. The base case $\ell = n-k = 0$ is trivial, as in this case $G = K_k$. Let ℓ be a positive integer and suppose we have established the desired result for smaller nonnegative integer values of ℓ. Let k be a large constant that will be specified later, and let G be a k-critical graph on $n \le 1.4(k-1)$ vertices with $n-k=\ell$. By Lemma 5 (Gallai's theorem), G has a vertex partition

$$\mathcal{P} : V(G) = V_1 \cup \ldots \cup V_t$$

into $t \ge 2$ nonempty parts such that each $G[V_i]$ is k_i-critical with n_i vertices with $n_i \ge 2k_i - 1$, and every pair of vertices in different parts are adjacent. In particular, we have $\sum_{i=1}^t k_i = k$ and $\sum_{i=1}^t n_i = n$. Set $\epsilon = 1/8$, $\epsilon' = 2^{-8}$, and $c = 2^{-44}$, say. We distinguish three cases.

Case 1. There is a part V_i with $1 < n_i \le \epsilon k$.

Add missing edges to V_i, one at a time, until V_i is complete. Each time we add an edge, we upper bound the increase of the crossing number by applying Lemma 10. As $G[V_i]$ is k_i-critical, each vertex in $G[V_i]$ has degree at least $k_i - 1$. Hence, the number of nonadjacent pairs in $G[V_i]$ is at most $n_i(n_i-k_i)/2$. In total, by making $G[V_i]$ complete, we increase the crossing number by at most $n^2 n_i(n_i-k_i)/4$. The resulting graph G', obtained by completing part V_i, has n vertices and is k'-critical with $k' = k + n_i - k_i$. Thus,

$$\begin{aligned}\mathrm{cr}(G') &\le \mathrm{cr}(G) + n^2 n_i(n_i-k_i)/4 = \mathrm{cr}(G) + n^2 n_i(k'-k)/4 \\ &\le \mathrm{cr}(G) + \left(\frac{4}{65} - c\right) k^3(k'-k), \end{aligned} \tag{1}$$

where in the last inequality we used that $n \le 1.4(k-1)$, $n_i \le k/8$, and $c = 2^{-44}$.

Applying the induction hypothesis to G', we obtain

$$\mathrm{cr}(G') \ge \mathrm{cr}(K_{k'}) + c(n-k')k'^3. \tag{2}$$

Note that by averaging, we have

$$\begin{aligned}\mathrm{cr}(K_{k'}) &\ge \mathrm{cr}(K_k)\binom{k'}{4}/\binom{k}{4} \ge (k'/k)^4 \mathrm{cr}(K_k) \ge \left(1 + 4\left(\frac{k'}{k} - 1\right)\right)\mathrm{cr}(K_k) \\ &= \mathrm{cr}(K_k) + 4\mathrm{cr}(K_k)(k'-k)/k \ge \mathrm{cr}(K_k) + \frac{4}{65}k^3(k'-k), \end{aligned} \tag{3}$$

where in the last inequality we used Lemma 6.

Putting (1)–(3) together, we obtain

$$\begin{aligned} \mathrm{cr}(G) &\geq \mathrm{cr}(K_{k'}) + c(n-k')k'^3 - \left(\frac{4}{65} - c\right)k^3(k'-k) \\ &\geq \mathrm{cr}(K_k) + ck^3(k'-k) + c(n-k')k'^3 \\ &\geq \mathrm{cr}(K_k) + ck^3(n-k). \end{aligned}$$

This completes the proof in this case.

Case 2. There is a part V_i with $k_i \geq \epsilon' k$.

Applying Theorem 2 to G, we obtain K_k as a weak immersion in G. By the proof of Lemma 8, the number of crossings between the edges used in this weak immersion is at least

$$\mathrm{cr}(K_k) - k^3/3. \tag{4}$$

Furthermore, the proof of Theorem 2 shows that no matter how we partition part V_i into $V_i = U_i \cup W_i$ with $|U_i| = k_i$, the edges in $G[W_i]$ are not used in the weak immersion. Note that $G[V_i]$ has minimum degree at least $k_i - 1$, and hence, has at least $n_i(k_i-1)/2$ edges. By Lemma 11, we can pick this partition $V_i = W_i \cup U_i$ so that the number of edges in $G[W_i]$ is at least

$$m_i := \frac{1}{2}n_i(k_i-1)\binom{n_i-k_i}{2}\Big/\binom{n_i}{2} = \frac{1}{2}(k_i-1)(n_i-k_i)(n_i-k_i-1)/(n_i-1).$$

As all three numbers k, $k_i \geq \epsilon' k$, and $n_i - k_i \geq k_i - 1$ are sufficiently large, we have $m_i \geq 6.95(n_i - k_i)$. Thus, we can apply Lemma 9 to obtain that

$$cr(G[W_i]) \geq \frac{1}{27.48}\frac{m_i^3}{(n_i-k_i)^2} \geq 2^{-11}k_i^3(n_i-k_i).$$

We obtain that

$$\begin{aligned} \mathrm{cr}(G) &\geq \mathrm{cr}(K_k) - k^3/3 + \mathrm{cr}(G[W_i]) \\ &\geq \mathrm{cr}(K_k) - k^3/3 + 2^{-11}k_i^3(n_i-k_i) \\ &\geq \mathrm{cr}(K_k) + c(n-k)k^3. \end{aligned}$$

In the last inequality, we used that k is sufficiently large, $k_i \geq \epsilon' k$, $n_i - k_i \geq k_i - 1$, $n \leq 1.4(k-1)$, and $c \leq 2^{-44}$. This completes the proof in Case 2.

Case 3. Each part V_i is either a singleton or satisfies $n_i > \epsilon k$ and $k_i < \epsilon' k$.

In this case, as $\ell = n - k > 0$, we must have at least one part that is not a singleton. Recall that $n - k < 0.4k$ and $n - k = \sum_i (n_i - k_i)$. For every i for which V_i is not a singleton, we have $n_i - k_i > \epsilon k - \epsilon' k = (\epsilon - \epsilon')k$. Thus, the number of parts that are not singletons is smaller than $0.4k/(\epsilon - \epsilon')k < 4$, which implies that there are at most *three* non-singleton parts.

Let A be the union of the singleton parts and $B = V(G) \setminus A \neq \emptyset$. Since B is the union of non-singleton parts V_i, each of which is larger than ϵk, we have $|B| > \epsilon k$. The chromatic number of G is k. The chromatic number of $G[B]$, the subgraph of G induced by B, is smaller than $3\epsilon' k$. Using that $A \cup B$ is a vertex partition of G, we obtain that the chromatic number of $G[A]$ is larger than $k - 3\epsilon' k$. As $G[A]$ is a clique, we have $|A| > k - 3\epsilon' k = (1 - 3\epsilon')k$.

It follows by averaging over all cliques of size k in K_{k+1}, just like in (3), that $\mathrm{cr}(K_k)/\binom{k}{4}$ is a monotonically increasing function. Since it is bounded from above, it must converge. As $|A| > (1-3\epsilon')k$, we obtain that

$$\mathrm{cr}(K_{|A|}) \geq (1-12\epsilon')\mathrm{cr}(K_k),$$

provided that k is sufficiently large.

Notice that the clique $G[A]$ and the complete bipartite graph between A and B are disjoint subgraphs of G. Therefore, we get

$$\begin{aligned}\mathrm{cr}(G) &\geq \mathrm{cr}(K_{|A|}) + \mathrm{cr}(K_{|A|,|B|}) \geq (1-12\epsilon')\mathrm{cr}(K_k) + .9118|A|^2|B|^2/16 \\ &\geq \mathrm{cr}(K_k) + \left(.9118(1-6\epsilon')\epsilon^2/16 - 12\epsilon'/64\right)k^4 \\ &\geq \mathrm{cr}(K_k) + 2^{-7}k^4 > \mathrm{cr}(K_k) + c(n-k)k^3.\end{aligned}$$

Here the second inequality follows by substituting in the bound from Lemma 7 on the crossing number of complete bipartite graphs and using the bound $\mathrm{cr}(K_k) \leq k^4/64$. The last inequality holds with $c = 2^{-44}$, say, because $n - k < .4k$. ◀

Proof of Theorem 4. It suffices to prove the statement for k-critical graphs as every graph of chromatic number k has a k-critical subgraph. So let G be a k-critical graph with chromatic number k with $n \leq (1.4+\delta)k$ vertices. If $n \leq 1.4(k-1)$, then the statement already follows from Theorem 12. So we may assume $1.4(k-1) \leq n \leq (1.4+\delta)k$.

Consider a proper k-coloring of G. Let $k' = (1-3\delta)k$. Let A be the union of the $3\delta k$ largest color classes of this proper k-coloring. Either each of these color classes has size at least two, or the remaining color classes forms a clique on k' vertices.

In the first case, $|A| \geq 6\delta k$, and the remaining induced subgraph after deleting A has chromatic number k'. Let B be the vertex set of a k'-critical subgraph of the induced subgraph on $V(G) \setminus A$. Observe that

$$|B| \leq n - 6\delta k \leq (1.4-5\delta)k = 1.4k' - .8\delta k \leq 1.4(k'-1),$$

provided that $\delta k \geq 7/4$, which holds as k is sufficiently large. By Theorem 12, the induced subgraph on B has crossing number at least $\mathrm{cr}(K_{k'}) + c(|B|-k')k'^3$, where $c > 0$ is an absolute constant (We have seen that $c = 2^{-44}$ will do.) If $|B| \geq 1.2k$, this induced subgraph already has crossing number at least

$$\mathrm{cr}(K_{k'}) + c(|B|-k')k'^3 \geq (1-12\delta)\mathrm{cr}(K_k) + .1ck^4 \geq \mathrm{cr}(K_k) - 12\delta k^4/64 + .1ck^4 \geq \mathrm{cr}(K_k),$$

where we used $\delta = 2^{-45}$. Otherwise, as G is k-critical, each vertex of G has degree at least $k-1$, and there at least $(n-|B|)(k-1)/2 \geq k^2/16$ edges not in $G[B]$. Let G_0 denote the subgraph of G consisting of the edges of G not in $G[B]$. Applying the crossing number bound (Lemma 9) to G_0 and using $n < 2k$, we obtain that

$$\mathrm{cr}(G_0) \geq \frac{1}{27.48}\frac{(k^2/16)^3}{(2k)^2} \geq 2^{-19}k^4.$$

Then

$$\mathrm{cr}(G) \geq \mathrm{cr}(G[B]) + \mathrm{cr}(G_0) \geq \mathrm{cr}(K_{k'}) + 2^{-19}k^4 \geq \mathrm{cr}(K_k) - 12\delta k^4/64 + 2^{-19}k^4 \geq \mathrm{cr}(K_k),$$

where we used $\delta = 2^{-45}$.

In the second case, $|A| = n - k'$ and the edges between A and B form a complete bipartite graph in G with $|A| \geq 2k/5$ and $|B| \geq 4k/5$. This complete bipartite graph has crossing number at least $k^4/200$ by Lemma 7. Together with the $\mathrm{cr}(K_{k'})$ crossings between the edges of the clique of size k', we obtain

$$\mathrm{cr}(G) \geq \mathrm{cr}(K_{k'}) + k^4/200 \geq \mathrm{cr}(K_k) - 12\delta k^4/64 + k^4/200 \geq \mathrm{cr}(K_k).$$ ◀

4 Concluding remarks

One can improve the constant factor 1.4 in Theorem 2, when k is sufficiently large. This can be achieved by being more careful how we pick U and the functions f_u (recall in the proof of Theorem 2, these choices were made arbitrarily). Instead, it is better to make sure that the largest degree vertices in $G[V_i]$ are in U_i, while the remaining vertices in U_i are chosen at random. If there is a pair $u, u' \in U$ for which $f_u(u')$ and $f_{u'}(u)$ haven't already been chosen, then we choose $f_u(u')$, $f_{u'}(u)$ to be equal. Once there is no such pair u, u', we make the remaining choices for the functions f_u uniformly at random. Vizing's theorem [25] states that any multigraph H with maximum degree Δ and maximum edge multiplicity μ satisfies $\chi'(H) \leq \Delta + \mu$. A careful analysis of this procedure produces a considerably smaller bound on the maximum degree of H_i than k_i, and also gives that the maximum edge multiplicity of H_i is $o(k_i)$. This can be proved through optimizing expected values and using standard concentration inequalities.

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