Enumeration of Intersection Graphs of x-Monotone Curves

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Abstract

A curve in the plane is *x-monotone* if every vertical line intersects it at most once. A family of curves are called *pseudo-segments* if every pair of them have at most one point in common. We construct $2^{\Omega(n^{4/3})}$ families, each consisting of n labelled x-monotone pseudo-segments such that their intersection graphs are different. On the other hand, we show that the number of such intersection graphs is at most $2^{O(n^{3/2-\varepsilon})}$, where $\varepsilon > 0$ is a suitable constant. Our proof uses an upper bound on the number of set systems of size m on a ground set of size n, with VC-dimension at most d. Much better upper bounds are obtained if we only count *bipartite* intersection graphs, or, in general, intersection graphs with bounded chromatic number.

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1 Introduction

The intersection graph of a collection $\mathcal{C}$ of sets has vertex set $\mathcal{C}$ and two sets in $\mathcal{C}$ are adjacent if and only if they have nonempty intersection. A curve is a subset of the plane which is homeomorphic to the interval $[0, 1]$. A *string graph* is the intersection graph of a collection of curves. It is straightforward to show the intersection graph of any collection of arcwise connected sets in the plane is a string graph. A collection of curves in the plane is called a collection of *pseudo-segments* if every pair of them have at most one point in common. Finally, we say that a curve in the plane is *x-monotone* if every vertical line intersects it in at most one point.

For a family $\mathcal{F}$ of simple geometric objects (namely those that can be defined by semi-algebraic relations of bounded description complexity), such as segments or disks in the plane, Warren's theorem [24] can be used to show that the number of labelled graphs on n vertices which can be obtained as the intersection graph of a collection of n objects from $\mathcal{F}$ is $2^{O(n \log n)}$ (see [16, 15]). Moreover, for many simple geometric objects, a result of Sauermann [19] shows that these bounds are essentially tight. Unfortunately, for general curves, Warren's theorem cannot be applied. In this paper, we estimate the number of graphs which can be obtained as the intersection graph of curves in the plane under various constraints.

In [17], Pach and Tóth showed that the number of intersection graphs of n labelled pseudo-segments is at most $2^{o(n^2)}$. This bound was later improved by Kynčl [13] to $2^{O(n^{3/2}\log n)}$. It was noted in both papers that the best known lower bound on the number of intersection graphs of n labelled pseudo-segments is $2^{\Omega(n\log n)}$, the number of different labellings of the vertex set. Our first result significantly improves this bound.

▶ **Theorem 1.** *There are at least $2^{\Omega(n^{4/3})}$ labelled n-vertex intersection graphs of x-monotone pseudo-segments.*

In the other direction, we prove the following.

▶ **Theorem 2.** *There is an absolute constant $\varepsilon \in (0,1)$ such that the following holds. There are at most $2^{O(n^{3/2-\varepsilon})}$ labelled n-vertex intersection graphs of x-monotone pseudo-segments in the plane.*

The ε in the theorem above can be taken to be roughly $1/D$, where D is the VC-dimension of intersection graphs of pseudo-segments in the plane. A result of result of Pach and Tóth [17] states that D is at most a tower of 2's of height 8.

In the case of small clique number, we obtain the following.

▶ **Theorem 3.** *There are at most $2^{O(kn\log^2 n)}$ labelled n-vertex intersection graphs of x-monotone pseudo-segments with clique number at most k. Moreover, for $k < n^{1/3}$, this bound is tight up to a polylogarithmic factor in the exponent.*

In [17], Pach and Tóth showed that the number of string graphs on n labelled vertices is $2^{\frac{3}{4}\binom{n}{2}+o(n^2)}$. Moreover, their result holds for x-monotone curves. Our next result shows that there are far fewer *bipartite* intersection graphs of x-monotone curves in the plane.

▶ **Theorem 4.** *There are at most $2^{O(n\log^2 n)}$ labelled n-vertex bipartite intersection graphs of x-monotone curves in the plane.*

Let us remark that the x-monotone condition in the theorem above cannot be removed. An interesting construction due to Keszegh and Pálvölgyi [12] implies that the number of n-vertex *bipartite* string graphs is at least $2^{\Omega(n^{4/3})}$.

For the non-bipartite case, suppose G is an n-vertex intersection of graph of x-monotone curves, such that G has chromatic number $q \geq 3$. Then we can partition $V(G)$ into q parts such that each part is an independent set. By further partitioning each part, arbitrarily, such that the size of each remaining part is at most n/q, we end up with at most $2q$ parts. By applying Theorem 4 to each pair of parts, we obtain the following corollary.

▶ **Corollary 5.** *There are at most $2^{O(qn\log^2 n)}$ labelled n-vertex intersection graphs of x-monotone curves with chromatic number at most q.*

Two drawings of a graph are *isomorphic* if the intersection graphs of their edges (with edges labelled by their endpoints) are the same. A *topological graph* is a graph drawn in the plane with possibly intersecting edges, and it is called *simple* if every pair of edges intersect at most once. A topological graph is *k-quasiplanar* if it has no k pairwise crossing edges with distinct endpoints.

The above results can be used to get upper bounds on the number of non-isomorphic drawings of a graph with certain properties. The next result is an immediate corollary of Theorem 3, combined with the theorem of Valtr [23] stating that the number of edges of a k-quasiplanar simple topological graph on n vertices with x-monotone edges is $O_k(n\log n)$.

▶ **Corollary 6.** *Given any n-vertex graph G, the number of non-isomorphic drawings of G as a k-quasiplanar simple topological graph with x-monotone edges is $2^{O_k(n \log^3 n)}$.*

In Theorems 3 and 4 and Corollaries 5 and 6, we conjecture that one of the logarithmic factors in the exponent can be removed. (In the case of Corollary 6, perhaps a factor $\log^2 n$ in the exponent can removed). We discuss what is known from below at the end of the paper.

Our paper is organized as follows. In the next section, we prove Theorem 1. In Section 3, we establish a bound on the number of set systems of size m on a ground set of size n with VC-dimension d. After completing this work, we learned that Alon, Moran, and Yehudayoff [3] also found a proof of this result. Together with the well-known cutting lemma, we prove Theorem 2 in Section 4. In Section 5, we prove Theorem 4. We conclude the paper with some remarks.

2 Proof of Theorem 1

The proof of Theorem 1 is based on a well-known construction from incidence geometry. We prove the following more general result.

▶ **Theorem 7.** *For $k \leq n^{1/3}$, there are at least $2^{\Omega(kn)}$ n-vertex labelled intersection graphs of x-monotone pseudo-segments with clique number at most k.*

Proof. Let k and n be integers such that $k \leq n^{1/3}$. Take

$$P = \{(a, b) \in \mathbb{N}^2 : a < n^{1/3}, b < n^{2/3}\}$$

and

$$\mathcal{L} = \{a'x + b' = y : a', b' \in \mathbb{N}, a' < k, b' < n^{2/3}/2\}.$$

Then we have $|P| \leq n$ and $|\mathcal{L}| \leq kn^{2/3}/2 \leq n$, and each line in $\mathcal{L}$ is incident to at least $n^{1/3}/4$ points from P. For each point $p = (a, b)$ in P, we replace p with a very short horizontal segment $\overline{p}$ with endpoints (a, b) and $(a + \epsilon, b)$. Let $\mathcal{H}$ be the resulting set of horizontal segments.

For each line $\ell \in \mathcal{L}$, we modify ℓ in a small neighborhood of each point in P that is incident to ℓ as follows. Let $\ell : y = a'x + b'$ and $p \in \ell$. Inside the circle C centered at p with radius $\frac{\epsilon}{2a'}$, we modify ℓ so that it is a half-circle along C that lies either above or below p. After performing this operation at each point p on ℓ, and performing a small perturbation, we obtain an x-monotone curve $\tilde{\ell}$. Moreover, any two resulting x-monotone curves will cross at most once. See Figure 1. Let $\mathcal{L}_x$ be the resulting set of x-monotone curves, and note that $\mathcal{H} \cup \mathcal{L}_x$ is a set of x-monotone pseudo-segments.

We now count the number of intersection graphs between $\mathcal{H}$ and $\mathcal{L}_x$. Since each line $\ell \in \mathcal{L}$ was incident to at least $n^{1/3}/4$ points in P, the number of different neighborhoods that can be generated for $\tilde{\ell}$ is $2^{\Omega(n^{1/3})}$. Moreover, two x-monotone curves $\tilde{\ell}, \tilde{\ell}' \in \mathcal{L}_x$ cross if and only if their original line configuration $\ell, \ell' \in \mathcal{L}$ have distinct slope. Thus, the intersection graph of $\mathcal{H} \cup \mathcal{L}_x$ has clique number at most k, and number of such intersection graphs we can create between $\mathcal{H}$ and $\mathcal{L}_x$ is at least $2^{\Omega(kn)}$. This completes the proof of Theorem 7. ◀

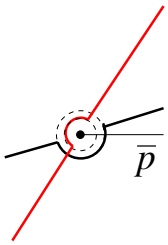

Figure 1 Modifying lines through p.

3 Tools from VC-dimension theory

In this section, we recall and prove results related to the notion of VC-dimension. The *VC-dimension of a set system* $\mathcal{F}$ on a ground set V is the *largest* integer d for which there exists a d-element set $S \subset V$ such that for every subset $B \subset S$, one can find a member $A \in \mathcal{F}$ with $A \cap S = B$. Note that for a multiset system (which allows for the sets to necessarily be distinct), the VC-dimension is the same as for the set system where we include each set that appears once.

Given a graph $G = (V, E)$, we define the *VC-dimension of* G to be the VC-dimension of the set system formed by the neighborhoods of the vertices, where the neighborhood of $v \in V$ is $N(v) = \{u \in V : uv \in E\}$. In [2], Alon et al. proved that the number of bipartite graphs with parts of size n and VC-dimenison at most d is at most

$$2^{O(n^{2-1/d}(\log n)^{d+2})}.$$

They further asked if the logarithmic factors can be removed. We make progress on this question, obtaining a better bound for a more general problem. By following their proof but using the Haussler packing lemma [11] (stated below) instead of Lemma 26 in [2], one can obtain a stronger and more general bound. In addition to this, we use a different counting strategy that further removes an additional logarithmic factor.

For the sake of completeness, we include the short proof below. First, we will need some definitions. Given two sets $A, B \in \mathcal{F}$, the *distance* between A and B is $d(A, B) := |A \triangle B|$, where $A \triangle B = (A \cup B) \setminus A \cap B)$ is the symmetric difference of A and B. We say that the set system $\mathcal{F}$ is δ-separated if the distance between any two members in $\mathcal{F}$ is at least δ. The following *packing lemma* was proved by Haussler in [11].

▶ **Lemma 8** ([11]). *Let $\delta > 0$ and $\mathcal{F}$ be a set system on an n-element ground set V such that $\mathcal{F}$ has VC-dimension d. If $\mathcal{F}$ is δ-separated, then $|\mathcal{F}| \leq c_1(n/\delta)^d$ where $c_1 = c_1(d)$.*

Let $h_d(m, n)$ denote the number of multiset systems consisting of m subsets of $[n]$ that have VC-dimension at most d. Let $h'_d(m, n)$ denote the number of set systems of m subsets of $[n]$ that have VC-dimension at most d. Clearly, $h'_d(m, n) \leq h_d(m, n)$. For simplicity, we let $\binom{n}{\leq d} := \sum_{i=0}^{d} \binom{n}{i}$. The Sauer-Shelah lemma [18, 21] says that any set system with ground set $[n]$ and VC-dimension d has size at most $\binom{n}{\leq d}$. It follows that $h'_d(m, n) = 0$ if $m > \binom{n}{\leq d}$. Further, we can relate the two as follows. If we pick a multiset system consisting of m sets that has VC-dimension at most d, then by throwing out repeated sets, we get a set system on the same ground set consisting of $m' \leq m$ sets. We then have to fill out these m' sets to m sets with repeats, including each set at least once. We thus have

$$h_d(m, n) = \sum_{m' \leq m} h'_d(m', n) \binom{m-1}{m'-1}. \tag{1}$$

In what follows, d is fixed and the implicit constant in the big-O depends on d.

▶ **Theorem 9.** *Let $d \geq 2$ be fixed and $n, m \geq 2$. Then the number $h_d(m,n)$ of multiset systems of m subsets of $[n]$ with VC-dimension at most d satisfies*

$$h_d(m,n) = 2^{O(m^{1-1/d} n \log m)}.$$

Furthermore, if $m > n^d$, then

$$h_d(m,n) = 2^{O(n^d \log m)}.$$

Proof. Consider a linear ordering of the subsets of $[n]$. Let $\mathcal{F}$ be a multiset system of m subsets of $[n]$. Let S_1 be the first set in $\mathcal{F}$ by the linear ordering. We will order the sets in $\mathcal{F}$ as $S_1, S_2, \ldots, S_m$ as follows. After picking $S_1, \ldots, S_{i-1}$, let

$$\delta_i = \max_{S \in \mathcal{F} \setminus \{S_1, \ldots, S_{i-1}\}} \min_{1 \leq j \leq i-1} d(S, S_j),$$

and S_i be a set S that obtains the maximum, and j_i be a j that obtains the minimum $d(S_i, S_j)$. By our choice of the sets, we have the minimum of $d(S_a, S_b)$ over all $1 \leq a < b \leq i$ is $d(S_{j_i}, S_i)$. By the Haussler packing lemma, we thus have $i = O((n/\delta_i)^d)$, or equivalently, $\delta_i = O(i^{-1/d} n)$.

We now upper bound the number of choices of $\mathcal{F}$. There are at most 2^n choices of S_1. Each j_i is a positive integer at most $i-1$, so there are at most $(m-1)! \leq m^m$ choices of $j_2, \ldots, j_m$. Having picked out this sequence of j_i's, and having picked $S_1, \ldots, S_{i-1}$, we know S_i must have symmetric difference at most $t_i := O(i^{-1/d} n)$ from S_{j_i}. Thus given this information, the number of choices for S_i is at most $\binom{n}{\leq t_i}$. Thus we get the number of choices of $\mathcal{F}$ is at most

$$\begin{aligned} 2^n m^m \prod_{i=2}^{m} \binom{n}{\leq t_i} &\leq 2^n m^m \prod_{i=2}^{m} (O(i^{1/d}))^{O(i^{-1/d} n)} \\ &= 2^n m^m 2^{n \sum_{i=2}^{m} O(i^{-1/d} \log i)} \\ &= 2^n m^m 2^{O(m^{1-1/d} n \log m)}. \end{aligned}$$

Note that the 2^n factor is at most the last factor. Hence we get that the count is at most $m^m 2^{O(m^{1-1/d} n \log m)}$. If $m \leq n^d$, then the last factor is largest and this gives the desired bound.

So we may assume we are in the case $m > n^d > \binom{n}{\leq d}$. In this case, by equation (1), the fact that $h'_d(m', n) = 0$ for $m' > n^d$ and $h'_d(m', n) \leq h_d(n^d, n)$, we get

$$h_d(m,n) \leq h_d(n^d, n) \sum_{m' \leq n^d} \binom{m-1}{m'-1} = 2^{O(n^d \log m)}.$$

Notice that in this case, the first bound still holds, as $m^{1-1/d} n \log m \geq n^d \log m$. ◀

4 Intersection graphs of x-monotone pseudo-segments

In this section, we prove Theorem 2. We will need the following lemmas. Recall that a *pseudoline* is a two-way infinite x-monotone curve in the plane. An *arrangement of pseudolines* is a finite collection of pseudolines such that any two members have at most one point in common, at which they cross, and each intersection point has a unique x-coordinate. Given an arrangement $\mathcal{A}$ of n pseudolines, we obtain a sequence of permutations of $1, \ldots, n$

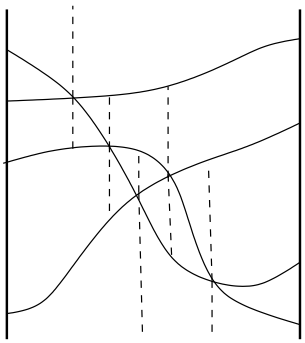

Figure 2 Vertical decomposition of $\mathcal{A}$.

by sweeping a directed vertical line across $\mathcal{A}$. This sequence of permutations is often referred to as an *allowable sequence of permutations*, which starts with the identity permutation $(1, \ldots, n)$, such that i) the move from one permutation to the next consists of swapping two adjacent elements, and ii) each pair of elements switch exactly once. We say that two pseudoline arrangements $\mathcal{A}_1$ and $\mathcal{A}_2$ are *x-isomorphic* if they give rise to the same sequence of permutations, that is, a sweep with a vertical line meets the crossings in the same order.

▶ **Lemma 10** ([22]). *The number of arrangements of m pseudolines, up to x-isomorphism, is at most $2^{\Theta(m^2 \log m)}$.*

We will also need the following result, known as the *zone lemma* for pseudolines.

▶ **Lemma 11** ([8]). *Let $\mathcal{A}$ be an arrangement of m pseudolines. Then for any $\alpha \in \mathcal{A}$, the sum of the numbers of sides in all the cells in the arrangement of $\mathcal{A}$ that are supported by α is at most $O(m)$.*

The next lemma we will need is the following result due to Pach and Tóth.

▶ **Lemma 12** ([17]). *Let G be the intersection graph of a collection of pseudo-segments in the plane. Then the VC-dimension of G is at most an absolute constant d.*

We say that a collection $\mathcal{A}$ of x-monotone pseudo-segments in the plane is *double grounded* if there are vertical lines ℓ_1 and ℓ_2 (called grounds) such that each curve in $\mathcal{A}$ has its left endpoint on ℓ_1 and its right endpoint on ℓ_2. We start by bounding the number of intersection graphs between a family $\mathcal{A}$ of double grounded x-monotone curves and a family $\mathcal{B}$ of x-monotone curves such that $\mathcal{A} \cup \mathcal{B}$ is a collection of pseudo-segments.

Let $\mathcal{A}$ be a collection of double grounded x-monotone pseudo-segments in the plane. The *vertical decomposition* of the arrangement of $\mathcal{A}$ is obtained by drawing a vertical segment from each crossing point and endpoint in the arrangement, in both directions, and extend it until it meets the arrangement of $\mathcal{A}$, else to $\pm\infty$. Since $\mathcal{A}$ is double grounded, the grounds will appear in the vertical decomposition. The vertical decomposition of $\mathcal{A}$ partitions the plane into cells called *generalized trapezoids*, where each generalized trapezoid is bounded by at most two curves from $\mathcal{A}$ from above or below, and at most two vertical segments on the sides. See Figure 2. By applying standard random sampling arguments (e.g., see [6] or Lemma 4.6.1 in [14]), we obtain the following result known as the *weak cutting lemma*.

▶ **Lemma 13** ([6, 14]). *Let $\mathcal{A}$ be a collection of m double grounded x-monotone pseudo-segments in the plane. Then for any parameter r, where $1 \leq r \leq m$, there is a set of at most $s = 6r \log m$ curves in $\mathcal{A}$ whose vertical decomposition partitions the plane $\mathbb{R}^2 = \Delta_1 \cup \cdots \cup \Delta_t$ into t generalized trapezoids, such that $t = O(s^2)$, and the interior of each Δ_i crosses at most m/r members in $\mathcal{A}$.*

Let $f(m,n)$ denote the number of labelled intersection graphs between a collection $\mathcal{A}$ of m double grounded x-monotone curves whose grounds are the vertical lines at $x=0$ and $x=1$, and a collection $\mathcal{B}$ of n x-monotone curves whose endpoints lie inside the strip $S=[0,1]\times\mathbb{R}$ such that $\mathcal{A}\cup\mathcal{B}$ is a collection of pseudo-segments. By Lemma 12, there is an absolute constant d such that the VC-dimension of any intersection graph of pseudo-segments in the plane is at most d. We now prove the following.

▶ **Lemma 14.** *For $m,n\geq 1$, we have*

$$f(m,n)\leq 2^{O(n^{d/(2d-1)}m^{(2d-2)/(2d-1)}\log^2 m)}+2^{O(n^{3/2-1/d}\log n)}+2^{O(m\log^3 m)}.$$

Proof. We can assume that m,n are sufficiently large. Let $\mathcal{A}\cup\mathcal{B}$ be a collection of pseudo-segments where $\mathcal{A}$ and $\mathcal{B}$ are as above.

Suppose $n>m^2$. Then by the first part of Theorem 9, the number of intersection graphs between $\mathcal{A}$ and $\mathcal{B}$ is at most

$$h_d(n,m)\leq 2^{O(mn^{1-1/d}\log n)}\leq 2^{O(n^{3/2-1/d}\log n)}. \tag{2}$$

If $n<m^{1/d}\log^2 m$, then by the second part of Theorem 9, the number of intersection graphs between $\mathcal{A}$ and $\mathcal{B}$ is at most

$$h_d(m,n)\leq 2^{O(m^{1-1/d}n\log m)}\leq 2^{O(m\log^3 m)}. \tag{3}$$

Let us assume that $m^{1/d}<n<m^2$. Set $r=\frac{n^{d/(2d-1)}}{(m\log^{2d}m)^{1/(2d-1)}}$ and $s=6r\log m$. Since m and n are sufficiently large, we have $1\leq r<s\leq m$. For a set of m double grounded x-monotone curves whose grounds are on the vertical lines $x=0$ and $x=1$, there are $(m!)^2$ ways to order the left and right endpoints. Let us fix such an ordering.

Let $\mathcal{A}'\subset\mathcal{A}$ be a set of at most $s=6r\log m$ curves from $\mathcal{A}$ whose arrangement gives rise to a vertical decomposition satisfying Lemma 13 with parameter r. Note that there are at most m^s choices for $\mathcal{A}'$, and by Lemma 10, there are at most $2^{O(s^2\log s)}$ ways to fix the arrangement, up to x-isomorphism, for $\mathcal{A}'$. Once the arrangement of $\mathcal{A}'$ is fixed, the vertical decomposition of $\mathcal{A}'$ is determined.

Let $\mathbb{R}^2=\Delta_1\cup\cdots\cup\Delta_t$ be the vertical decomposition corresponding to $\mathcal{A}'$, where $t=O(s^2)$. Let $\mathcal{A}_i\subset\mathcal{A}$ be the curves in $\mathcal{A}$ that cross the cell Δ_i. For each curve $\alpha\in\mathcal{A}'$, by Lemma 11, at most $O(s)$ vertical segments from the vertical decomposition have an endpoint on α. Moreover, at most m curves from $\mathcal{A}$ cross α. Among these $O(s+m)$ points along α, let us fix the order in which they appear along α, from left to right. Since there are at most s^2 vertical segments, there are at most $(s^2+m)^{O(s+m)}=m^{O(m)}$ ways to fix this ordering, and therefore, there are at most $m^{O(sm)}$ ways to fix such an ordering for each curve $\alpha\in\mathcal{A}'$.

Let $\beta\in\mathcal{B}$. Then there are $O(s^4)$ choices for the cells Δ_i for which the endpoints of β lie in. Suppose that the left endpoint of β lies in cell Δ_i and the right endpoint lies in Δ_j, and consider the vertical lines ℓ_1 and ℓ_2 that goes through the left and right endpoint of β respectively. Then for each $\alpha'\in\mathcal{A}\setminus(\mathcal{A}_i\cup\mathcal{A}_j)$, we have already determined if α' crosses β. Indeed, let us consider the left endpoint of β and the cell Δ_i. By the vertical decomposition, Δ_i is bounded either above or below by some curve $\alpha\in\mathcal{A}'$. Without loss of generality, let us assume that Δ_i is bounded from above by α. Let p be the point on α that intersects the left side of Δ_i. Then for any $\alpha'\in\mathcal{A}\setminus(\mathcal{A}_i\cup\mathcal{A}_j)$, we have already determined if the left endpoint of α' is above or below the left endpoint of α along the ground $x=0$. Moreover, we have already determined if α' crosses α to the left of point p. Since α' does not cross Δ_i, we have determined if α' crosses ℓ_1 above or below β. See Figure 3. By the same argument, we have determined if α' crosses ℓ_2 above or below the right endpoint of β. Therefore, by the pseudo-segment condition, we have determined if α' crosses β.

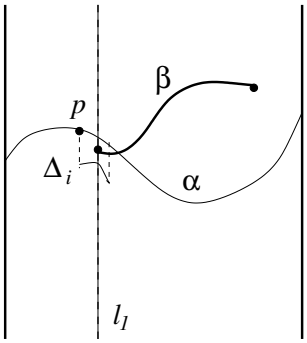

Figure 3 Cell Δ_i bounded above by α and contains the left endpoint of β.

It remains to determine how many ways β can cross the curves in $\mathcal{A}_i$ and $\mathcal{A}_j$. By Lemma 13, $|\mathcal{A}_i| \leq m/r$. Let $\mathcal{B}_i$ denote the curves in $\mathcal{B}$ that has at least one endpoint in the cell Δ_i. Set $n_i = |\mathcal{B}_i|$. By Theorem 9, there are at most

$$h_d(|\mathcal{A}_i|, |\mathcal{B}_i|) \quad \leq \quad 2^{O((m/r)^{1-1/d} n_i \log m)}$$

ways the curves in $\mathcal{A}_i$ cross the curves in $\mathcal{B}_i$. Putting everything together, the number of ways the curves in $\mathcal{A}$ cross the curves in $\mathcal{B}$ is at most

$$(m!)^2 m^s 2^{O(s^2 \log s)} m^{O(sm)} \left(s^4\right)^n \prod_{i=1}^{t} 2^{O((m/r)^{1-1/d} n_i \log m)}.$$

Since $t = O(s^2)$, $r = \frac{n^{d/(2d-1)}}{(m \log^{2d} m)^{1/(2d-1)}}$, and $s = 6r \log m \leq m$, this quantity is at most

$$2^{O((m/r)^{1-1/d} n \log m + s^2 (m/r) \log m)} \leq 2^{O(n^{d/(2d-1)} m^{(2d-2)/(2d-1)} \log^2 m)}. \tag{4}$$

Combining (2), (3), and (4), we have

$$f(m,n) \leq 2^{O(n^{d/(2d-1)} m^{(2d-2)/(2d-1)} \log^2 m)} + 2^{O(n^{3/2-1/d} \log n)} + 2^{O(m \log^3 m)}. \quad \blacktriangleleft$$

Hence, we have $f(n,n) \leq 2^{O\left(n^{3/2-1/(4d-2)} \log^2 n\right)}$, where d is the absolute constant from Lemma 12.

Proof of Theorem 2. Let d be the absolute constant from Lemma 12. Let $g(n;p)$ be the number of labeled intersection graphs of at most n x-monotone pseudo-segments in the vertical strip $[0,1] \times \mathbb{R}$, such that there are at most p endpoints with x-coordinate in $(0,1)$. Note that some pseudo-segments may contribute two endpoints to p. Then we have the following recurrence.

▷ Claim 15. We have

$$g(n;p) \leq 2^{O\left(n^{3/2-1/(4d-2)} \log^2 n\right)} g^2(\lceil p/2 \rceil; \lceil p/2 \rceil).$$

Proof. For n x-monotone curves in the strip $S = [0,1] \times \mathbb{R}$, with p endpoints in the interior of S, we can assume that these p endpoints have distinct x-coordinates. We partition the interval $[0,1]$ into two parts I_1, I_2, so that the interior of each strip $S_i = I_i \times \mathbb{R}$ has at most $\lceil p/2 \rceil$ endpoints. Next, we upper bound the number of labeled intersection graphs of the curves restricted to the strip S_i. Note that there are $n!$ ways to label the curves.

Among the curves restricted to the strip S_i, let $\mathcal{A}_i$ denote the set of curves that go entirely through S_i, and let $\mathcal{B}_i$ be the curve what at least one endpoint in the interior of S_i. There are at most $n!$ ways to determine the intersection graph among the curves in $\mathcal{A}_i$. By Lemma 14, there are at most

$$f(|\mathcal{A}_i|, |\mathcal{B}_i|) \leq f(n,n) \leq 2^{O(n^{3/2-1/(4d-2)} \log^2 n)}$$

ways to determine the intersection graph between $\mathcal{A}_i$ and $\mathcal{B}_i$. Finally, there are at most $g(\lceil p/2 \rceil; \lceil p/2 \rceil)$ ways to determine the intersection graph among the curves in $\mathcal{B}_i$. Putting everything above together gives the desired recurrence. ◁

Since $p \leq 2n$, the recurrence above gives

$$g(n;2n) \leq 2^{\sum_{i=1}^{\log n} 2^i O\left((n/2^i)^{3/2-1/(4d-2)} \log^2(n/2^i)\right)} g(1;1) \leq 2^{O(n^{3/2-1/(4d-2)} \log^2 n)}.$$ ◀

Intersection graphs with small clique number

In this subsection, we prove Theorem 3.

Proof of Theorem 3. Next, we prove Theorem 3. Let $g_k(n;p)$ be the number of labeled intersection graphs of at most n x-monotone pseudosegments with clique number at most k in the vertical strip $[0,1] \times \mathbb{R}$, such that there are at most p endpoints with x-coordinate in $(0,1)$. Similar to above, we will show

$$g_k(n;p) \leq n^{6n+2kp} g_k^2(\lceil p/2 \rceil; \lceil p/2 \rceil).$$

Indeed, for n x-monotone pseudosegments in the strip $S = [0,1] \times \mathbb{R}$, with p endpoints in the interior of S, we can assume that these p endpoints have distinct x-coordinate. We partition the interval $[0,1]$ into two parts I_1, I_2, so that the interior of each strip $S_i = I_i \times \mathbb{R}$ has at most $\lceil p/2 \rceil$ endpoints. We now bound the number of labeled intersection graphs of the curves restricted to S_1.

There are at most $n!$ ways to label the curves in S_1. There are at most 2^n ways to choose the set $\mathcal{A}$ of pseudo-segments that goes entirely through S_1. Let $G_{\mathcal{A}}$ denote its intersection graph of $\mathcal{A}$, restricted to S_1. Then $G_{\mathcal{A}}$ depends entirely on the permutation of the endpoints of $\mathcal{A}$. Hence, there are at most $n!$ ways to determine $G_{\mathcal{A}}$. Since $G_{\mathcal{A}}$ has clique number at most k, by Dilworth's theorem [7], $G_{\mathcal{A}}$ has has chromatic number at most k. Thus, there are at most k^n ways to properly color the vertices of $G_{\mathcal{A}}$. After fixing such a coloring, let $\mathcal{A}_1, \ldots, \mathcal{A}_k$ denote the color classes. Since the curves in $\mathcal{A}_i$ are pairwise disjoint and goes through S_1, for each curve γ with an endpoint in the interior of S_1, there are at most n^2 ways γ can intersect the curves in $\mathcal{A}_i$. Therefore, there are at most $(n^2)^k$ ways γ can intersect the curves in $\mathcal{A}$. Since $k \leq n$, there are at most

$$n! 2^n n! k^n (n^2)^{kp/2} g_k(\lceil p/2; p/2 \rceil) \leq n^{4n+kp} g_k(\lceil p/2 \rceil; \lceil p/2 \rceil)$$

labeled intersection graphs among the curves restricted to S_1. A similar argument holds for the curves restricted to S_2. Hence,

$$g_k(n;p) \leq n^{8n+2kp} g_k^2(\lceil p/2 \rceil; \lceil p/2 \rceil).$$

Iterating the inequality above t times gives

$$g_k(n;p) \leq n^{8n+2kp} \left(\frac{p}{2}\right)^{8p+2kp} \left(\frac{p}{2^2}\right)^{8p+2kp} \cdots \left(\frac{p}{2^{t-1}}\right)^{8p+2kp} g^{2^t}(\lceil p/2^t \rceil; \lceil p/2^t \rceil).$$

Hence for $t = \lceil \log_2 n \rceil$, we have

$$g_k(n;p) \leq n^{8n+2kp} p^{(8p+2kp)t}.$$

By setting $p = 2n$, we have

$$g_k(n;2n) \leq 2^{O(kn \log^2 n)},$$

and Theorem 3 follows. ◀

5 Bipartite intersection graphs of x-monotone curves

In this section, we prove Theorem 4. The proof is very similar to the proof of Theorem 3 above. Let $w(n;p)$ be the number of labeled bipartite intersection graphs of at most n x-monotone curves in the vertical strip $[0,1] \times \mathbb{R}$, such that there are at most p endpoints with x-coordinate in $(0,1)$. We establish the following recurrence.

▶ **Lemma 16.** *We have*

$$w(n;p) \leq n^{6n} w^2(\lceil p/2 \rceil; \lceil p/2 \rceil).$$

Proof. For n x-monotone curves in the strip $S = [0,1] \times \mathbb{R}$, with p endpoints in the interior of S, we can assume that these p endpoints have distinct x-coordinate. We partition the interval $[0,1]$ into two parts I_1, I_2, so that the interior of each strip $S_i = I_i \times \mathbb{R}$ has at most $\lceil p/2 \rceil$ endpoints. Next, we upper bound the number of labeled intersection graphs of the curves restricted to the strip S_i. Note that there are $n!$ ways to label the curves.

For each curve γ, as the graph is bipartite, let us count the number of ways γ intersects the set of pairwise disjoint curves that go entirely through S_i. By ordering these pairwise disjoint curves vertically, this intersection set is an interval with respect to this vertical ordering. Hence, γ has at most n^2 ways to intersect the family of curves that goes entirely through S_i. This gives a total of at most $n!(n^2)^n < n^{3n}$ ways of determining the intersection graph in S_i, apart from the induced subgraph on the curves with at least one endpoint in the interior of S_i. Since there are $p/2$ such endpoints, there are at most $p/2$ such curves. Thus we have at most $w(\lceil p/2 \rceil; \lceil p/2 \rceil)$ possible such intersection graphs of the curves with one end point in S_i. Thus we have at most $n^{3n} w(\lceil p/2 \rceil, \lceil p/2 \rceil)$ possible intersection graphs restricted to S_i. Since the intersection graph of all n curves is the union of the intersection graphs on S_1 and S_2, we get in total at most $(n^{3n} w(\lceil p/2 \rceil; \lceil p/2 \rceil))^2$ such choices. ◀

Proof of Theorem 4. It suffices to bound $w(n;2n)$ as the original n curves have $2n$ endpoints. Iterating the recurrence in Lemma 16 t times gives

$$w(n;p) \leq n^{6n} \left(\frac{p}{2}\right)^{6p} \left(\frac{p}{2^2}\right)^{6p} \cdots \left(\frac{p}{2^{t-1}}\right)^{6p} w^{2^t}(\lceil p/2^t \rceil; \lceil p/2^t \rceil).$$

Thus for $t = \lceil \log_2 n \rceil$, we get

$$w(n;p) \leq n^{6n} p^{6pt}.$$

Hence,

$$w(n;2n) \leq 2^{O(n \log^2 n)}.$$ ◀

Let us remark that in [9], the first two authors showed that there is an absolute constant $c > 0$ such that every n-vertex string graph with clique number k has chromatic number at most $(C\frac{\log n}{\log k})^{c \log k}$. Together with Corollary 5, we obtain the following.

▶ **Corollary 17.** *For every $\epsilon > 0$, there is $\delta > 0$ such that the number of intersection graphs of n x-monotone curves with clique number at most n^δ is at most $2^{n^{1+\epsilon}}$.*

6 Concluding remarks

An important motivation for enumerating intersection graphs of curves of various kinds came from a question in graph drawing [17]: How many ways can one draw a graph? The number of different (non-isomorphic) drawings of K_n, a complete graph of n vertices, can be upper-bounded by the number intersection graphs of $\binom{n}{2}$ curves. By [17], this is at most $2^{(3/2^5+o(1))n^4}$.

The number of non-isomorphic *straight-line drawings* of K_n cannot exceed the the number of different intersection graphs of $\binom{n}{2}$ segments in the plane, which is $2^{(4+o(1))n^2 \log n}$; see [19, 16]. However, the true order of magnitude of the number of straight-line drawings of K_n is much smaller. As was pointed out in [17], this quantity is equal to the number of *order types* of n points in general position in the plane. The latter quantity is $2^{(4+o(1))n \log n}$, according to seminal results of Goodman–Pollack [10] and Alon [1], based on Warren's theorem in real algebraic geometry [24].

Recall that Theorem 9 in Section 3 shows that for $d \geq 2$ fixed and $m, n \geq 2$, the number $h'_d(m, n)$ of set systems of m subsets of $[n]$ that have VC-dimension at most d is at most $2^{O(nm^{1-1/d} \log m)}$. It would be interesting to remove the logarithmic factor in the exponent, which would answer the question of Alon et al. [2] mentioned in the beginning of Section 3. A natural approach, which has worked for similar enumerative problems, is to recast the problem as counting independent sets in an auxiliary hypergraph and use the hypergraph container method. Consider the 2^{d+1}-uniform hypergraph H with vertex set $2^{[n]}$ (so the vertices are just the subsets of $[n]$) and a 2^{d+1}-tuple of vertices forms an edge if they shatter a subset of the ground set of size $d + 1$. The function $h'_d(m, n)$ then just counts the number of independent sets of size m in H. The hypergraph container method (introduced in [4, 20], see also [5]) is a powerful tool that is useful for counting independent sets in similar settings. It would be interesting if one could adapt these techniques to give better bounds on $h'_d(m, n)$.

The last five results in the introduction give upper bounds on the number of intersection graphs or the number of non-isomorphic drawings of graphs under various constraints. It would be interesting to close the gap between these upper bounds and lower bounds.

The following simple construction shows that there are $2^{\Omega(n \log n)}$ unlabelled bipartite graphs on n vertices that are intersection graphs of segments. One can fix the first $k = n/\log n$ segments to be vertical and cross the x-axis, and then have the freedom to choose the remaining $n - k$ segments to be horizontal, deciding which interval of vertical segments (ordered by x-axis intersection point) to intersect. By having, for $i \in [k]$, a horizontal segment that intersects precisely the first i of the vertical segments, we can fix the underlying ordering of the vertical segments, up to reversing the order, and use the remaining $n - 2k$ horizontal segments to pick any interval of the vertical segments to intersect. One gets $2^{(2-o(1))n \log_2 n}$ labelled bipartite intersection graphs (and hence at least $2^{(1-o(1))n \log_2 n}$ unlabelled bipartite intersection graphs). This shows that Theorem 4 is tight up to a single logarithmic factor in the exponent.

Viewing the same construction as a drawing of a matching (with the endpoints of segments as vertices of the matching), gives $2^{\Omega(n \log n)}$ non-isomorphic straight-line drawings of a matching on n vertices whose edge-intersection graph is bipartite, providing a lower bound for Corollary 6.

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