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THE EQUIVARIANT LAZARD RING OF PRIMARY CYCLIC GROUPS

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Abstract. This paper calculates the equivariant Lazard ring for primary cyclic groups, in terms of explicit generators and defining relations. This ring is known to coincide with the coefficient ring of the equivariant stable complex cobordism spectrum, which I compute by the method of isotropy separation, using a "staircase diagram." This calculation provides new tools for constructing equivariant spectra.

Contents

1. Introduction

Equivariant formal group laws for abelian compact Lie groups were introduced by Cole, Greenlees and Kriz in [4, 5], as the structure universally present on the rings of $\mathbb{Z}$-graded coefficients of complex-oriented equivariant spectra. From the beginning, it was conjectured by J. P. C. Greenlees that the stable equivariant complex cobordism ring $(MU_G)_*$ is the ring supporting the universal G-equivariant formal group law. This conjecture was recently proved by Hanke and Wiemeler [7] for $G = \mathbb{Z}/2$, and then in general by Hausmann [8] using global homotopy theory, and by different methods by Kriz and Lu [11]. The term "stable" signifies the fact that this is the ring of homotopy groups of the equivariant complex cobordism Thom spectrum, which, for $G \neq \{e\}$, is known to represent not the actual complex G-cobordism ring of manifolds, but a certain stabilization. For $G = \mathbb{Z}/2$, the equivariant complex cobordism ring of manifolds is represented by a spectrum recently constructed by J. Carlisle [3], which leads to an interesting extension of the concept of equivariant formal group laws.

Equivariant formal group laws turn out to be a powerful tool for constructing and investigating equivariant complex-oriented spectra, similarly as in the non-equivariant case. This has become clear in the work of Strickland [17], and to

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an even greater degree, in the recent paper [9]. However, the algebra is much more complicated, as is the structure of the equivariant Lazard ring L_G, i.e., the equivariant cobordism ring $(MU_G)_*$.

The coefficient ring $(MU_G)_*$, in fact, has long been considered mysterious. The first efforts at calculating it came during the 1990s in the works of Kriz [10] and Sinha [15], with a generalization by Abram and Kriz [1], in the form of certain pullback diagrams and their generalizations. For $G = \mathbb{Z}/2$, a presentation in terms of generators and relations was first obtained by Strickland [16]. While the ring is not polynomial as in the non-equivariant case, it can be understood in terms of the Euler class u of the non-trivial representation of $\mathbb{Z}/2$. Essentially, there is a torsion part which is the same as on the Borel cohomology of $MU_{\mathbb{Z}/2}$, and a non-torsion part, which can be thought of as a coordinate neighborhood of an infinite sequence of blow-ups of a polynomial ring, from a scheme-theoretical point of view. In [17] and also especially in [9], it has become apparent that the algebraic structure of $L_G = (MU_G)_*$ can be used to construct and investigate new equivariant spectra. However, the constructions and computations of [9] were limited by the fact that the appropriate presentation was still only known for $G = \mathbb{Z}/2$.

The purpose of this paper is to give this presentation of $(MU_G)_*$ for a general primary cyclic group $G = \mathbb{Z}/(p^n)$. The computation is qualitatively more difficult than in [16], even in the case of $G = \mathbb{Z}/p$, due to the fact that multiple Euler classes are present. These Euler classes turn out to be multiples of each other, but not unit multiples: a more delicate statement relating them to each other holds, involving "carry-over" in p-adic multiplication. From this, one obtains that divisibility by the different Euler classes is equivalent. The case of $G = \mathbb{Z}/p$ is an important special case in our computation, and is treated separately in Section 2. The main statement is Theorem 2.2.

The general case of $G = \mathbb{Z}/p^n$ is again qualitatively more complex, and new topological calculations are required. This is treated in Section 3. The main statement for this case is Theorem 3.1. The essential topological observation which describes a certain "staircase-shaped" isotropy separation diagram is Proposition 3.4. Substantial work is also needed to show how it applies to the case of complex cobordism, which involves some complicated algebra, filling out most of the remainder of the section. While similar "staircase" diagrams have occurred in the literature before (e.g. [1], [9], [14]), I could not find a convenient reference for the present context, so I state and prove what I need in Proposition 3.4.

In fact, the relationship between the different staircase diagrams from Section 3 of the present paper and from [1] is quite delicate, and leads to certain special effects for $G = \mathbb{Z}/p^2$. This is described in detail in Section 4.

The main results of this paper are rather technical and require a substantial amount of background and specialized notation. For this reason, it is not feasible to state them in the Introduction, and I state them in their respective sections.

One feature of the presentation given in this paper is that it is more explicit than the presentation given in [11], in the sense of giving explicit control over the divisibility by Euler classes. This can be used to construct non-isotropically split examples of complex-oriented $\mathbb{Z}/p^n$-equivariant spectra using spectral algebra. In the case of $n = 1$, we discuss an explicit example in the Remark at the end of Section 2. These spectra are non-trivial in the sense of [2]. No such examples, with the exception of those coming directly from geometry or for $p = 2$ [9], were known prior to the calculations done in the present paper.

I will give a few comments on the setting and notations here. In this paper, we work with G-equivariant spectra, where G is a finite group, indexed over the complete universe, as defined by Lewis-May-Steinberger [13]. For a G-equivariant space (or another G-spectrum) X, and any virtual representation V of G (i.e. elements of the representation ring $RO(G)$), such a spectrum E defines abelian groups $E^V X$ and $E_V X$. By the coefficients of E, we mean the groups $E^n(*) = E_{-n}(*)$, where $n \in \mathbb{Z}$, and $*$ is a point. Recall that E is a commutative associative ring spectrum if we have maps $E \wedge E \to E$ and $S \to E$ satisfying the usual axioms. When E is a commutative associative ring spectrum, then $E_* = \oplus_{n\in\mathbb{Z}} E_n(*)$ forms a graded commutative associative ring with unit.

The example we are interested in is the equivariant complex cobordism spectrum MU_G [18]. Let $\mathcal{U}$ be the complete G-universe, i.e. the sum of infinitely many copies of the regular unitary representation of G. For a finite-dimensional complex representation V in $\mathcal{U}$, let $Gr(|V|,\mathcal{U} \oplus V)$ be the space of all $|V|$-dimensional complex vector subspaces of $\mathcal{U} \oplus V$. Let D_V be the Thom space of the universal G-equivariant $|V|$-dimensional complex bundle on $Gr(|V|,\mathcal{U} \oplus V)$. Then for V, W finite-dimensional orthogonal subrepresentations of $\mathcal{U}$, we have a canonical map

$$\Sigma^W D_V \to D_{V+W}.$$

The G-spaces $\{D_V\}$ form a $\mathcal{U}$-prespectrum, and the Thom spectrum MU_G is the spectrification of $\{D_V\}$. In this case, this means that

$$(MU_G)_V = \operatorname{colim}_{W\perp V} \Omega^W D_{V+W}.$$

We refer to MU_G as the *equivariant complex cobordism spectrum*. The explicit presentation of the ring $L_G = (MU_G)_*$ for $G = \mathbb{Z}/(p^n)$ is the subject of this paper.

2. The $G = \mathbb{Z}/p$ case

In this section, we consider the case where $G = \mathbb{Z}/p$. For this case, our starting point is the work of Kriz [10], who considered the Tate diagram (in the sense of Greenlees and May [6]) for $MU_{\mathbb{Z}/p}$. Our method is very similar to that of Strickland [16], who computed it in the case when $p = 2$.

Recall that the coefficient ring MU_* of the non-equivariant complex cobordism spectrum MU is the Lazard ring. We denote the universal formal group law on MU_* by F, and the formal n-series of a variable x by $[n]_F x = [n]x$. We will write the universal formal group law as

$$F(x, y) = x +_F y = \sum_{i,j\geq 0} a_{i,j} x^j y^i,$$

where $a_{i,j} \in MU_*$. In particular, $a_{i,j} = a_{j,i}$, $a_{0,1} = a_{1,0} = 1$, and $a_{0,k} = a_{k,0} = 0$ for all $k \neq 1$. There are also other relations among the $a_{i,j}$ arising from the associativity of F. Also, recall that the standard grading on MU_* has $|a_{i,j}| = 2(i + j - 1)$, so that if x and y are both given degree -2, then $x +_F y$ is also of degree -2.

For each $\alpha = 1, \ldots, p - 1$, we always use α^{-1} to denote the representative of $\alpha^{-1} \in (\mathbb{Z}/p)^\times$, with $1 \leq \alpha^{-1} \leq p - 1$. Namely, α^{-1} is the smallest positive integer such that

$$\alpha \cdot \alpha^{-1} = 1 + k_\alpha p$$

with $k_\alpha \geq 0$. We also write

$$x +_F [\alpha]y = \sum_{i,j\geq 0} a_{i,j} x^i ([\alpha]y)^j = \sum_{i,j\geq 0} a_{i,j}^{(\alpha)} x^i y^j.$$

It is easy to see that for every α, $a_{1,0}^{(\alpha)} = 1$, $a_{i,0}^{(\alpha)} = 0$ for every $i \neq 1$, and that $|a_{i,j}^{(\alpha)}| = 2(1-i-j)$. Also,

$$[\alpha]y = \sum_{j\geq 0} a_{0,j}^{(\alpha)} y^j,$$

so in particular, $a_{0,1}^{(\alpha)} = \alpha$. We will also write

$$[p]u = \sum_{j\geq 0} c_j u^j.$$

In particular, $c_0 = 0$, $c_1 = p$, and $|c_j| = 2(1-j)$.

For a power series $G(x,y)$ in two variables, and $i \geq 0$, we denote the coefficient of x^i in $G(x,y)$ as

$$\mathrm{Coeff}_{x^i} G(x,y).$$

Clearly,

$$\sum_{i\geq 0} \mathrm{Coeff}_{x^i} G(x,y) \cdot z^i = G(z,y). \tag{2.1}$$

The main result of [10] is the following.

Theorem 2.1 ([10, Theorem 1.1])**.** *There is a pullback square of rings*

$$\begin{array}{ccc} (MU_{\mathbb{Z}/p})_* & \xrightarrow{\kappa} & MU_*[b_i^{(\alpha)}, (b_0^{(\alpha)})^{-1} \mid i \geq 0, \alpha \in (\mathbb{Z}/p)^\times] \\ \downarrow & & \downarrow{\scriptstyle\phi} \\ MU_*[[u]]/([p]u) & \xrightarrow{\iota} & (MU_*[[u]]/[p]u)[u^{-1}]. \end{array} \tag{2.2}$$

Here, the bottom map ι is localization, and

$$\begin{aligned} \phi(b_i^{(\alpha)}) &= \mathrm{Coeff}_{x^i}(x +_F [\alpha]u) = \sum_{j\geq 0} a_{i,j}^{(\alpha)} u^j, \\ \phi((b_0^{(1)})^{-1}) &= u^{-1}. \end{aligned} \tag{2.3}$$

We will write $b_0^{(\alpha)} = u_\alpha$. Our $b_i^{(\alpha)}$ corresponds to the element $b_\alpha^{(i)} u_\alpha$ in the original statement of Theorem 2.1 in [10]. The notation of (2.3) is consistent in that

$$\phi(u_\alpha) = [\alpha]u,$$

so $\phi(b_0^{(1)}) = \phi(u_1) = u$. Hence, we will also write $u_1 = u$. In particular, in the lower right corner of (2.2), we have

$$u = [\alpha^{-1}]([\alpha]u) = \sum_{j\geq 1} a_{j,0}^{(\alpha^{-1})} ([\alpha]u)^j,$$

so there is a series with no negative powers of u

$$f_\alpha(u) = \sum_{j\geq 0} a_{j+1,0}^{(\alpha^{-1})} ([\alpha]u)^j$$

with $f_\alpha(u)\cdot[\alpha]u = u$, so $(u^{-1}f_\alpha(u))\cdot[\alpha]u = 1$ in the lower right corner of (2.2). Hence, $\phi(u_\alpha^{-1}) = ([\alpha]u)^{-1} = u^{-1}f_\alpha(u)$ exists in the target as claimed.

We will write

$$f_\alpha(u) = \frac{u}{[\alpha]u}.$$

However, this notation is slightly misleading, since in the lower left corner of (2.2), we only have

$$\frac{[\alpha]u}{u}\cdot\frac{u}{[\alpha]u}\cdot u = u.$$

It only becomes the inverse of $([\alpha]u)/u$ after we invert u.

Diagram (2.2) is obtained by taking homotopy groups of the fixed points of the right half of the Tate diagram [6] of $MU_{\mathbb{Z}/p}$, in the sense of Greenlees and May [6]. This is the diagram of $\mathbb{Z}/p$-equivariant spectra

$$\begin{array}{ccc} MU_{\mathbb{Z}/p} & \longrightarrow & \widetilde{E\mathbb{Z}/p}\wedge MU_{\mathbb{Z}/p} \\ \downarrow & & \downarrow \\ F(E\mathbb{Z}/p_+, MU_{\mathbb{Z}/p}) & \longrightarrow & \widetilde{E\mathbb{Z}/p}\wedge F(E\mathbb{Z}/p_+, MU_{\mathbb{Z}/p}) \end{array} \tag{2.4}$$

where $E\mathbb{Z}/p$ is the standard free contractible $\mathbb{Z}/p$-equivariant space, and $\widetilde{\ }$ denotes the unreduced suspension. Taking fixed points, the upper right corner becomes the geometric fixed points $\Phi^{\mathbb{Z}/p}MU_{\mathbb{Z}/p}$, and the lower right corner is the Tate cohomology $\widehat{MU_{\mathbb{Z}/p}}$, while the lower left corner is the Borel cohomology.

Our main theorem for this section is the following. For $p = 2$, it was proven by Strickland in [16].

Theorem 2.2. *As an MU_*-algebra, $(MU_{\mathbb{Z}/p})_*$ has the following generators:*

$$\begin{aligned} &u, \\ &b_{i,j}^{(\alpha)}, \quad i\geq 0, j\geq 1, \alpha\in(\mathbb{Z}/p)^\times, \\ &\lambda_\alpha, \quad \alpha\in(\mathbb{Z}/p)^\times, \\ &q_j, \quad j\geq 0 \end{aligned} \tag{2.5}$$

with the relations

$$b_{0,1}^{(1)} = 1, \quad b_{0,j}^{(1)} = 0 \quad \textit{for every } j\geq 2, \tag{2.6}$$

$$b_{i,j}^{(\alpha)} - a_{i,j}^{(\alpha)} = u b_{i,j+1}^{(\alpha)}, \tag{2.7}$$

$$q_0 = 0, \tag{2.8}$$

$$q_j - c_j = u q_{j+1}. \tag{2.9}$$

$$\lambda_1 = 1 \tag{2.10}$$

and

$$\lambda_\alpha b_{0,1}^{(\alpha)} = 1 + k_\alpha q_1 \quad \textit{where } \alpha\alpha^{-1} = k_\alpha p + 1 \textit{ in } \mathbb{Z} \tag{2.11}$$

as well as

$$\lambda_\alpha q_1 = \alpha^{-1}q_1. \tag{2.12}$$

We will write

$$u_\alpha = b_{0,0}^{(\alpha)} = u b_{0,1}^{(\alpha)} \tag{2.13}$$

and

$$b_i^{(\alpha)} = b_{i,0}^{(\alpha)} = \begin{cases} u b_{i,1}^{(\alpha)} + 1 & \text{if } i = 1, \\ u b_{i,1}^{(\alpha)} & \text{if } i \geq 2. \end{cases} \tag{2.14}$$

With this notation, relation (2.7) extends to all $i, j \geq 0$, and $u = u_1 = b_{0,0}^{(1)}$ in particular. Also, note that relation (2.9) for $j = 0$ gives $q_1 u = 0$. Hence, the relation

$$\lambda_\alpha u_\alpha = \lambda_\alpha b_{0,1}^{(\alpha)} u = (1 + k_\alpha q_1) u = u \tag{2.15}$$

follows. In this sense, λ_α should be seen as an "approximate" inverse to $b_{0,1}^{(\alpha)}$. As will be made explicit in Lemma 2.5, relation (2.11) comes from a corresponding relation in $MU_*[[u]]/[p]u$.

There is a grading on $(MU_{\mathbb{Z}/p})_*$, with $|b_{i,j}^{(\alpha)}| = |a_{i,j}^{(\alpha)}| = 2(i + j - 1)$, $|q_j| = |c_j| = 2(j - 1)$, and $|\lambda_\alpha| = 0$. In particular, $|q_1| = 0$, and the Euler classes $b_{0,0}^{(\alpha)} = u_\alpha$ are the only elements with negative degree. It is straightforward to check that this grading is consistent with the relations given in the theorem.

We also note Lemma 2.3.

Lemma 2.3. *In R, we have that*

$$\lambda_\alpha \equiv \alpha^{-1}$$

modulo u.

Proof. Using $b_{0,1}^{(\alpha)} = \alpha + b_{0,2}^{(\alpha)} u$ and $p = q_1 - q_2 u$, we have in R

$$\begin{aligned} \lambda_\alpha b_{0,1}^{(\alpha)} &= 1 + k_\alpha q_1, \\ \lambda_\alpha (\alpha + b_{0,2}^{(\alpha)} u) &= 1 + k_\alpha q_1, \\ \lambda_\alpha (\alpha \alpha^{-1} + \alpha^{-1} b_{0,2}^{(\alpha)} u) &= \alpha^{-1} + \alpha^{-1} k_\alpha q_1, \\ \lambda_\alpha (1 + k_\alpha p + \alpha^{-1} b_{0,2}^{(\alpha)} u) &= \alpha^{-1} + \alpha^{-1} k_\alpha q_1, \\ \lambda_\alpha + k_\alpha p \lambda_\alpha + \alpha^{-1} \lambda_\alpha b_{0,2}^{(\alpha)} u &= \alpha^{-1} + \alpha^{-1} k_\alpha q_1, \\ \lambda_\alpha + k_\alpha (q_1 - q_2 u) \lambda_\alpha + \alpha^{-1} \lambda_\alpha b_{0,2}^{(\alpha)} u &= \alpha^{-1} + \alpha^{-1} k_\alpha q_1, \\ \lambda_\alpha - k_\alpha \lambda_\alpha q_2 u + \alpha^{-1} \lambda_\alpha b_{0,2}^{(\alpha)} u &= \alpha^{-1} + \alpha^{-1} k_\alpha q_1 - k_\alpha \lambda_\alpha q_1, \\ \lambda_\alpha - (k_\alpha \lambda_\alpha q_2 + \alpha^{-1} \lambda_\alpha b_{0,2}^{(\alpha)}) u &= \alpha^{-1} + k_\alpha (\alpha^{-1} q_1 - \lambda_\alpha q_1). \end{aligned}$$

However, the right hand side is α^{-1} by relation (2.12). □

We already have that $b_{i,j}^{(\alpha)} \equiv a_{i,j}^{(\alpha)}$ and $q_j \equiv c_j$ modulo u. These are all the generators of R, so by Lemma 2.3, we do get

$$R/(u) = MU_*.$$

The proof of Theorem 2.2 is analogous to that of [16] for the case of $p = 2$. It makes use of the following standard result of commutative algebra, a proof of which was given in [16].

Proposition 2.4 ([16, Theorem 4]). *Let R be a commutative ring, and x an element of R. Suppose that R has bounded x-torsion, namely that there exists some N such that*

$$\bigcup_{k\geq 0} Ann(d^k) = Ann(d^N).$$

Then R is the pullback of the following square:

$$\begin{array}{ccc} R & \longrightarrow & R[d^{-1}] \\ \downarrow & & \downarrow \\ R^\wedge_d & \longrightarrow & R^\wedge_d[d^{-1}]. \end{array}$$

Let R be the MU_*-algebra defined by the generators and relations in Theorem 2.2. We will show that $R^\wedge_u$ and $R[u^{-1}]$ are respectively isomorphic to the upper right corner and the lower left corner of (2.2), in a compatible way, and that u has bounded torsion in R.

We start by considering the completion of R at u. For any $G(u)$ in $MU_*[[u]]$, denote by $G(u)_{|u|\geq j}$ the terms of $G(u)$ where the power of u is at least j.

Lemma 2.5. *There is a map of MU_*-algebras*

$$\phi : R \to MU_*[[u]]/([p]u),$$

given by

$$\begin{aligned} \phi(u) &= u, \\ \phi(b^{(\alpha)}_{i,j}) &= \mathrm{Coeff}_{x^i}\left(\frac{(x +_F [\alpha]u)_{]u|\geq j}}{u^j}\right) = \sum_{k\geq 0} a^{(\alpha)}_{i,j+k} u^k, \\ \phi(q_j) &= \frac{([p]u)_{|u|\geq j}}{u^j} = \sum_{k\geq 0} c_{j+k} u^k, \\ \phi(\lambda_\alpha) &= f_\alpha(u) = \frac{[1+k_\alpha p]u}{[\alpha]u} = \sum_{j\geq 0} a^{(\alpha^{-1})}_{0,j+1}([\alpha]u)^j \end{aligned}$$

which induces an isomorphism $\widehat{\phi} : R^\wedge_u \to MU_*[[u]]/([p]u)$.

Proof. We need to check that ϕ, as described above, respects the relations in R. For relation (2.6), we have

$$\phi(b^{(1)}_{0,j}) = \frac{(u)_{|u|\geq j}}{u^j}.$$

The numerator is u for $j = 1$, and 0 for $j \geq 2$, which gives (2.6). For relation (2.7), we have

$$\begin{aligned} \phi(b^{(\alpha)}_{i,j}) - a^{(\alpha)}_{i,j} &= \sum_{k\geq 0} a^{(\alpha)}_{i,j+k} u^k - a^{(\alpha)}_{i,j} \\ &= \sum_{k\geq 1} a^{(\alpha)}_{i,j+k} u^k \\ &= u \sum_{l\geq 0} a^{(\alpha)}_{i,j+l+1} u^l \\ &= \phi(u)\phi(b^{(\alpha)}_{i,j+1j}). \end{aligned}$$

From the definition of $\phi(q_j)$, we immediately get $\phi(q_0) = [p]u = 0$, giving relation (2.8). For relation (2.9), we have

$$\begin{aligned}\phi(q_j) - c_j &= \sum_{k\geq 0} c_{j+k}u^k - c_j \\ &= \sum_{k\geq 1} c_{j+k}u^k \\ &= u\sum_{l\geq 0} c_{j+1+l}u^l \\ &= \phi(u)\phi(q_{j+1}).\end{aligned}$$

Relation (2.10) is immediate, since $a^{(1)}_{0,j+1} = 1$ for $j = 0$, and is 0 else. For relation (2.11), note that in $MU_*[[u]]$, $[k_\alpha p + 1]u = [\alpha^{-1}]([\alpha]u)$, i.e.

$$\sum_{j\geq 0} a^{(k_\alpha p+1)}_{0,j} u^j = \sum_{j\geq 0} a^{(\alpha^{-1})}_{0,j}([\alpha]u)^j,$$

and

$$f_\alpha(u) \cdot \frac{[\alpha]u}{u} \cdot u = f_\alpha(u) \cdot [\alpha]u = [k_\alpha p + 1]u.$$

In $MU_*[[u]]$, u is a regular element, so

$$f_\alpha(u) \cdot \frac{[\alpha]u}{u} = \frac{[k_\alpha p + 1]u}{u} = \sum_{j\geq 0} a^{(k_\alpha p+1)}_{0,j+1} u^j \tag{2.16}$$

in $MU_*[[u]]$, and hence in $MU_*[[u]]/([p]u)$. The left hand side of this is $\phi(\lambda_\alpha)\phi(b^{(\alpha)}_{0,1})$. We have that

$$[k_\alpha p + 1]u = [k_\alpha]([p]u) +_F u = k_\alpha[p]u + u + \text{higher terms},$$

where the higher terms either have at least $([p]u)^2$ or both u and $[p]u$, and become 0 in $MU_*[[u]]/([p]u)$ when one reduces the power of u by one. Thus, the right hand side of (2.16) is

$$k_\alpha\left(\frac{[p]u}{u}\right) + 1 = k_\alpha\phi(q_1) + 1.$$

Finally, for relation (2.12), note that $\phi(\lambda_\alpha) = f_\alpha(u)$ has constant term α^{-1}, so in $MU_*[[u]]/([p]u)$,

$$\phi(\lambda_\alpha)\phi(q_1) = f_\alpha(u) \cdot \frac{[p]u}{u} = \alpha^{-1} \cdot \frac{[p]u}{u}.$$

Hence, the map ϕ is well-defined.

Since $MU_*[[u]]$ is complete at u, ϕ induces a map of MU_*-algebras

$$\widehat{\phi} : R^\wedge_u \to MU_*[[u]] \to MU_*[[u]/([p]u).$$

To show that $\widehat{\phi}$ is an isomorphism, we have a map $\widehat{\psi} : MU_*[[u]] \to R^\wedge_u$, induced by the obvious map $MU_*[u] \to R$ that sends u to u. Clearly, $\widehat{\phi}\widehat{\psi} : MU_*[[u]] \to MU_*[[u]]/([p]u)$ is the quotient map.

To show that $\widehat{\psi}$ is onto, note that for any $n \geq 0$, we have by induction on n

$$b^{(\alpha)}_{i,j} = \sum_{k=0}^{n-1} a^{(\alpha)}_{i,j+k}u^k + u^n b^{(\alpha)}_{i,j+n}.$$

The sequence $\{u^n b^{(\alpha)}_{i,j+n}\}$ converges to 0 in $R^\wedge_u$. Hence,

$$b^{(\alpha)}_{i,j} = \sum_{k\geq 0} a^{(\alpha)}_{i,j+k} u^k \tag{2.17}$$

in $R^\wedge_u$. So the same series in $MU_*[[u]]$ maps by $\widehat{\psi}$ to $b^{(\alpha)}_{i,j}$.

Similarly, one can see by induction on n that

$$q_i = q_{i+n}u^n + \sum_{k=0}^{n-1} c_{i+k}u^k.$$

The sequence $\{q_{i+n}u^n\}$ converges to 0 in $R^\wedge_u$, so we have

$$q_i = \sum_{k\geq 0} c_{i+k}u^k \tag{2.18}$$

in $R^\wedge_u$. Hence, the series on the right hand side in $MU_*[[u]]$ maps by $\widehat{\psi}$ to q_i. In particular, for $i = 0$,

$$\widehat{\psi}([p]u) = q_0 = 0.$$

Hence, $\widehat{\psi}$ factors through a map

$$\widehat{\psi} : MU_*[[u]]/([p]u) \to R^\wedge_u.$$

It remains to show that λ_α is in the image of $\widehat{\psi}$. Since $b^{(\alpha)}_{0,1}$ becomes $\frac{[\alpha]u}{u}$ and q_1 becomes $\frac{[p]u}{u}$ in $R^\wedge_u$, relations (2.11) and (2.12) become

$$\lambda_\alpha \cdot \frac{[\alpha]u}{u} = 1 + k_\alpha \cdot \frac{[p]u}{u} \tag{2.19}$$

and

$$\lambda_\alpha \cdot \frac{[p]u}{u} = \alpha^{-1} \cdot \frac{[p]u}{u} \tag{2.20}$$

respectively in $R^\wedge_u$. Multiplying (2.19) by α^{-1} gives

$$\lambda_\alpha \alpha^{-1} \frac{[\alpha]u}{u} = \alpha^{-1} + k_\alpha \alpha^{-1} \frac{[p]u}{u}, \tag{2.21}$$

and multiplying (2.20) by k_α gives

$$\lambda_\alpha k_\alpha \frac{[p]u}{u} = k_\alpha \alpha^{-1} \frac{[p]u}{u}. \tag{2.22}$$

Now subtract (2.22) from (2.21) gives

$$\lambda_\alpha \left(\alpha^{-1} \frac{[\alpha]u}{u} - k_\alpha \frac{[p]u}{u} \right) = \alpha^{-1}. \tag{2.23}$$

Now

$$\alpha^{-1} \frac{[\alpha]u}{u} = \alpha^{-1}\alpha + uA(u)$$

for some $A(u) \in MU_*[[u]]$. Similarly,

$$k_\alpha \frac{[p]u}{u} = k_\alpha p + uB(u)$$

for some $B(u) \in MU_*[[u]]$. Hence, (2.23) becomes

$$\lambda_\alpha \left(\alpha^{-1}\alpha + uA(u) - k_\alpha p - uB(u)\right) = \alpha^{-1},$$
$$\lambda_\alpha \left(1 + u(A(u) - B(u))\right) = \alpha^{-1}.$$

But $1 + u(A(u) - B(u))$ is a unit in $MU_*[[u]]$, so we get that in $R_u^\wedge$,

$$\lambda_\alpha = \alpha^{-1}(1 + u(A(u) - B(u)))^{-1},$$

where the right hand side is in $MU_*[[u]]$. Hence, this power series in $MU_*[[u]]/([p]u)$ maps to λ_α by $\widehat{\psi}$. □

For inverting u, we have Lemma 2.6.

Lemma 2.6. *There is a map of MU_*-algebras*

$$\kappa : R \to MU_*[b_i^{(\alpha)}, u_\alpha^{-1} \mid i \geq 0, \alpha \in (\mathbb{Z}/p)^\times]$$

(in the target, we again have $u_\alpha = b_0^{(\alpha)}$, $u = u_1$), given by

$$\kappa(b_{i,j}^{(\alpha)}) = u^{-j} b_i^{(\alpha)} - \sum_{k=1}^{j} a_{i,j-k}^{(\alpha)} u^{-k}, \tag{2.24}$$

$$\kappa(q_j) = -\sum_{k=1}^{j} c_{j-k} u^{-k}, \tag{2.25}$$

and

$$\kappa(\lambda_\alpha) = u u_\alpha^{-1}. \tag{2.26}$$

The map κ induces an isomorphism

$$\overline{\kappa} : R[u^{-1}] \to MU_*[b_i^{(\alpha)}, u_\alpha^{-1} \mid i \geq 0, \alpha \in (\mathbb{Z}/p)^\times].$$

Proof. Note that for $j = 0$, we get from (2.24), $\kappa(b_i^{(\alpha)}) = b_i^{(\alpha)}$, so the notation makes sense.

For $j = 1$, (2.25) gives $\kappa(q_1) = 0$, so the map is consistent with (2.12). For $i = 0$ and $j = 1$, we have

$$\kappa(b_{0,1}^{(\alpha)}) = u^{-1} b_0^{(\alpha)} - a_{0,0}^{(\alpha)} u^{-1} = u^{-1} u_\alpha,$$

so

$$\kappa(\lambda_\alpha b_{0,1}^{(\alpha)}) = 1 = 1 + k_\alpha \kappa(q_1).$$

This gives that κ is consistent with (2.11).

Relation (2.7) in R implies that in $R[u^{-1}]$,

$$b_{i,j+1}^{(\alpha)} = u^{-1}(b_{i,j}^{(\alpha)} - a_{i,j}^{(\alpha)})$$

which determines $\overline{\kappa}(b_{i,j}^{(\alpha)})$ for all $i \geq 0, j \geq 1$. Namely, by induction on j, one sees that if we replace the left hand side of (2.24) by $b_{i,j}^{(\alpha)}$, it holds as a relation in $R[u^{-1}]$. Hence, $\kappa(b_{i,j}^{(\alpha)}) = \overline{\kappa}(b_{i,j}^{(\alpha)})$ must be as given by (2.24). By construction, relation (2.7) is respected. It also immediately follows that $\kappa(b_{0,1}^{(1)}) = 1$, consistent with the first part of relation (2.6). For $j \geq 2$, $a_{0,j-k} = 1$ for $k = j - 1$, and 0 else, so

$$\kappa(b_{0,j}^{(1)}) = u^{-j} b_0^{(1)} - a_{0,1} u^{-j+1} = 0.$$

Thus, the second part of (2.6) is also preserved.

We have $\kappa(q_0) = \kappa(0) = 0$. By relation (2.9), in $R[u^{-1}]$,

$$q_{i+1} = u^{-1}(q_i - c_i)$$

which determines $\kappa(q_i)) = \overline{\kappa}(q_i)$ for all i. Indeed, by induction on i, one sees that (2.25), with the left hand side replaced by just q_i, holds in $R[u^{-1}]$, so $\kappa(q_i)$ must be as given by (2.25).

There is an evident map $\mu : MU_*[b_i^{(\alpha)}, u_\alpha^{-1}] \to R[u^{-1}]$, with $\mu(b_i^{(\alpha)}) = b_i^{(\alpha)}$, $\mu(u_\alpha^{-1}) = u_\alpha^{-1} = u^{-1}\lambda_\alpha$. Clearly, $\overline{\kappa}\mu$ is the identity. By the versions of (2.24) and (2.25) in $R[u^{-1}]$, $b_{i,j}^{(\alpha)}$ and q_i are in the image of μ, so μ is onto. Hence, μ is the inverse to $\overline{\kappa}$. $\square$

It still remains to check that the element u has bounded torsion in R. We will show that the ideal (q_1) contains all u-power torsion elements in R.

Lemma 2.7. *In R, we have*

$$\bigcup_{k\geq 0} Ann(u^k) = Ann(u) = (q_1).$$

Proof. The proof of this lemma is also similar to that of Strickland in [16], by showing that u is a regular element of $R/(q_1)$. Note first that in $R/(q_1)$, $\lambda_\alpha b_{0,1}^{(\alpha)} = 1$. Fix $k \geq 2$. Let R_k be the $MU_*[u]$-subalgebra of $R/(q_1)$ generated by $b_{i,j}^{(\alpha)}$ and q_j for $j \leq k$, and $\lambda_\alpha = (b_{0,1}^{(\alpha)})^{-1}$. On the other hand, consider

$$A_k = MU_*[u][b_{i,k}^{(\alpha)}, q_k \mid (i,\alpha) \neq (0,1)].$$

For $\alpha \in (\mathbb{Z}/p)^\times$, $0 \leq j \leq k-1$, and $(i,\alpha) \neq (0,1)$, consider the element in A_k

$$h_{i,j}^{(\alpha)} = b_{i,k}^{(\alpha)} u^{k-j} + \sum_{l=0}^{k-j-1} a_{i,j+l}^{(\alpha)} u^l.$$

Define a map of $MU_*[u]$-algebras

$$\eta : A_k[(h_{0,1}^{(\alpha)})^{-1} \mid \alpha \in (\mathbb{Z}/p)^\times \setminus \{1\}] \to R_k$$

given by $\eta(b_{i,k}^{(\alpha)}) = b_{i,k}^{(\alpha)}$, $\eta(q_k) = q_k$, and $\eta((h_{0,1}^{(\alpha)})^{-1}) = \lambda_\alpha$. By induction, it is easy to see that in R (and thus also in R_k),

$$b_{i,j}^{(\alpha)} = b_{i,k}^{(\alpha)} u^{k-j} + \sum_{l=0}^{k-j1} a_{i,j+l}^{(\alpha)} u^l, \tag{2.27}$$

which is the same formula that gives $h_{i,j}^{(\alpha)}$. Hence, $\eta(h_{i,j}^{(\alpha)}) = b_{i,k-j}^{(\alpha)}$. In particular, $\eta(h_{0,1}^{(\alpha)}) = b_{0,1}^{(\alpha)}$, so η is consistent on $(h_{0,1}^{(\alpha)})^{-1}$.

Similarly, consider the element in A_k

$$g_j = q_k u^{k-j} + \sum_{l=0}^{k-j-1} c_{j+l} u^l.$$

In R, one easily sees by induction that

$$q_j = q_k u^{k-j} + \sum_{l=0}^{k-j-1} c_{j+l} u^l, \tag{2.28}$$

i.e. the same formula that defines g_j, which gives $\eta(g_j) = q_j$. Hence, η is onto. In particular, $\eta(g_1) = q_1 = 0$ in R_k, so we get a surjective map

$$\eta : A_k[(h_{0,1}^{(\alpha)})^{-1}]/(g_1) \to R_k. \tag{2.29}$$

We claim that η is in fact an isomorphism. For its inverse, define

$$\pi : R_k \to A_k[(h_{0,1}^{(\alpha)})^{-1}]/(g_1)$$

by $\pi(b_{i,j}^{(\alpha)}) = h_{i,j}^{(\alpha)}$, $\pi(q_j) = g_j$, and $\pi(\lambda) = (h_{0,1}^{(\alpha)})^{-1}$. To check that π is well-defined, note that the relations in R_k are the relations for R listed in Theorem 2.2, with the changes that the relation (2.11) becomes $b_{0,1}^{(\alpha)}\lambda_\alpha = 1$, the relation (2.12) becomes trivial, and the additional relation $q_1 = 0$. By their definitions, we have that in the target,

$$h_{i,j}^{(\alpha)} - a_{i,j}^{(\alpha)} = uh_{i,j+1}^{(\alpha)},$$
$$g_j - c_j = ug_{j+1}$$

same as in relations (2.7) and (2.9). Also, we have $\pi(q_1) = g_1 = 0$. Therefore, π is well-defined. Also, $\pi(b_{0,0}^{(1)}) = u$ by this same formula, when we take $b_{0,k}^{(1)} = 0$ in the target, so the notation is consistent. It is easy to check that the map π is the inverse of η, so (2.29) is an isomorphism.

The element u is clearly regular in $A_k[(h_\alpha(u))^{-1}]$. Note that $g(u)$ has a constant term $c_1 = p$, which is not a zero divisor in MU_*. A standard argument shows that u is regular in $A_k[(h_\alpha(u))^{-1}]/(g(u))$. Namely, if $uh(u) = 0$ modulo $g(u)$, then $uh(u) = g(u)k(u)$ for some $k(u)$, but from the constant term of $g(u)$, we get that $k(u) = uk_1(u)$ for some $k_1(u)$. Therefore, since u is regular, we have $h(u) = g(u)k_1(u)$ is in the ideal $(g(u))$. Therefore, u is also regular in R_k. As $R/(q_1) = \operatorname{colim}_k R_k$, u is regular in $R/(q_1)$.

In R, suppose $u^n y = 0$ for some y, and $n \geq 1$. Passing to $R/(q_1)$, one sees that $u^{n-1}y = 0$ in this quotient ring. Inductively, we get $y = 0$ in $R/(q_1)$. Hence, all u-power torsion in R are generated by q_1. □

Theorem 2.2 follows from the lemmas above.

Proof of Theorem 2.2. It is straightforward to check that the diagram of MU_*-algebras

$$\begin{array}{ccc} R & \xrightarrow{\kappa} & MU_*[b_i^{(\alpha)}, (b_0^{(\alpha)})^{-1} \mid i \geq 0, \alpha \in (\mathbb{Z}/p)^\times] \\ \downarrow{\scriptstyle \overline{\phi}} & & \downarrow{\scriptstyle \phi} \\ MU_*[[u]]/([p]u) & \xrightarrow{\iota} & (MU_*[[u]]/[p]u)[u^{-1}] \end{array} \tag{2.30}$$

commutes, with ϕ and ι as in Theorem 2.1. By Lemmas 2.5 and 2.6, the three corners other than R are $R[u^{-1}]$, $R_u^\wedge$, and $R_u^\wedge[u^{-1}]$. By Lemma 2.7 and Proposition 2.4, (2.30) is a pullback square. Hence, by Theorem 2.1, $R \cong (MU_{\mathbb{Z}/p})_*$. □

Comments.

(1) In [16], for the case $p = 2$, instead of the generators q_i, the generators t_i were used, satisfying the relations

$$t_i - b_i = ut_{i+1}.$$

These generators are related to our q_i by

$$q_i - t_i = \sum_{k=0}^{i-1} b^{(1)}_{k,i-k}. \tag{2.31}$$

In particular, $q_1 = t_1 + 1$. To see this, by Proposition 6 of [16] and Lemma 2.5, after completion at u, we have

$$q_i = \sum_{k+j\geq i} a_{k,j} u^{k+j-i},$$
$$t_i = \sum_{k\geq i, j\geq 0} a_{k,j} u^{k+j-i},$$

so in $MU_*[[u]]/[p]u$,

$$q_i - t_i = \sum_{k=0}^{i-1}\left(\sum_{j\geq i-k} a_{k,j} u^{k+j-i}\right) = \sum_{k=0}^{i-1} b^{(1)}_{k,i-k}$$

by Lemma 2.5.

On the other hand, for $p = 2$,

$$c_j = \mathrm{Coeff}_{u^j}([2]u) = \sum_{r+s=j} a_{r,s}.$$

By Lemma 2.6 and Proposition 7 of [16], we have in $(MU_{\mathbb{Z}/2})_*[u^{-1}]$,

$$q_i - t_i = \sum_{l=1}^{i} b^{(1)}_{i-l} u^{-l} - \sum_{k=1}^{i} c_{i-k} u^{-k}, \tag{2.32}$$

$$\sum_{k=0}^{i-1} b^{(1)}_{k,i-k} = \sum_{k=0}^{i-1} b^{(1)}_{k} u^{-(i-k)} - \sum_{k=0}^{i-1}\left(\sum_{l=1}^{i-k} a_{i-k-l,k} u^{-l}\right). \tag{2.33}$$

Clearly, the first terms on the right hand sides of (2.32) and (2.33) are the same. The sums at the end of (2.32) and (2.33) are both

$$\sum_{r+s+l=i, l\geq 1} a_{r,s} u^{-l}.$$

Hence, (2.32) and (2.33) are equal in $(MU_{\mathbb{Z}/2})_*[u^{-1}]$ as well, so they are equal in $(MU_{\mathbb{Z}/2})_*$.

(2) In diagram (2.2), there is a natural action by $(\mathbb{Z}/p)^\times$: namely, it acts on the upper right corner $MU_*[b_i^{(\alpha)}, u_\alpha^{-1}]$ by permuting the generators, and on the lower row, it acts by substituting $[\alpha]u$ by u. Hence, it induces an action σ of $(\mathbb{Z}/p)^\times$ on the pullback $(MU_{\mathbb{Z}/p})_*$.

To give an example of how to describe this action, consider the case $p = 5$, and the generator 2 of $(\mathbb{Z}/5)^\times$. We will calculate $\sigma_2(b^{(2)}_{0,1})$. In the completion $MU_*[[u]]/([5]u)$, we have that $b^{(2)}_{0,1}$ is $([2]u)/u$, so it should map to $([4]u)/[2]u$ by σ_2. What element of R becomes this in the completion? We have that $b^{(2)}_{0,1}u = u_2$, so applying σ_2, we get that

$$\sigma_2(b^{(2)}_{0,1})u_2 = u_4.$$

This suggests that we should have $\sigma_2(b_{0,1}^{(2)}) = b_{0,1}^{(4)}\lambda_2$ modulo q_1. In the completion,

$$b_{0,1}^{(4)}\lambda_2 = \frac{[4]u}{u} \cdot \frac{[6]u}{[2]u}$$

has constant term 12, whereas $([4]u)/([2]u)$ has constant term 2. This suggests that we should have

$$\sigma_2(b_{0,1}^{(2)}) = b_{0,1}^{(4)}\lambda_2 - 2q_1. \tag{2.34}$$

In $R_u^\wedge = MU_*[[u]]/([5]u)$, this is

$$\begin{aligned}\frac{[4]u}{u} \cdot \frac{[6]u}{[2]u} - 2 \cdot \frac{[5]u}{u} &= \frac{[4]u}{[2]u} \cdot \frac{[6]u}{u} - 2 \cdot \frac{[5]u}{u} \\ &= \frac{[4]u}{[2]u}\left(1 + \frac{[5]u}{u}\right) - 2 \cdot \frac{[5]u}{u} \\ &= \frac{[4]u}{[2]u} + \frac{[4]u}{[2]u} \cdot \frac{[5]u}{u} - 2 \cdot \frac{[5]u}{u} \\ &= \frac{[4]u}{[2]u}\end{aligned}$$

since in $MU_*[[u]]/([5]u)$, any series times $([5]u)/u$ is equal to its constant term times $([5]u)/u$. On the other hand, in $R[u^{-1}]$, $b_{0,1}^{(2)}$ becomes u_2u^{-1}, and σ_2 takes this to $u_4u_2^{-1}$. However, here we have that $b_{0,1}^{(4)}\lambda_2 - 2q_1$ becomes $(u_4u^{-1})(uu_2^{-1}) - 0 = u_4u_2^{-1}$ as well, so the formula is correct here as well. Hence, (2.34) is the correct formula in $R = (MU_{\mathbb{Z}/5})_*$.

In general, we can find σ_α on the generators of R similarly. In particular, note that $\sigma_{\alpha^{-1}}(b_{0,1}^{(\alpha})) = \lambda_\alpha$. We also have that $\sigma_\alpha(q_j) = q_j$. For an example of the last statement, we calculate $\sigma_2(q_1)$ directly in the case $p = 3$. In the completion $MU_*[[u]]/([3]u)$, we have

$$q_1 = \frac{[3]u}{u} \mapsto \frac{[6]u}{[2]u}$$

by σ_2. However, we also have that $q_1 = (4-3)q_1 = (2\lambda_2 - 3)q_1 = 2q_1\lambda_2 - 3q_1$. In the completion, this is

$$\begin{aligned}2\frac{[3]u}{u} \cdot \frac{[4]u}{[2]u} - 3\frac{[3]u}{u} &= \frac{[6]u}{u} \cdot \frac{[4]u}{[2]u} - 3\frac{[3]u}{u} \\ &= \frac{[6]u}{[2]u} \cdot \frac{[4]u}{u} - 3\frac{[3]u}{u} \\ &= \frac{[6]u}{[2]u}\left(1 + \frac{[3]u}{u}\right) - 3\frac{[3]u}{u} \\ &= \frac{[6]u}{[2]u} + \left(\frac{[6]u}{[2]u} - 3\right)\frac{[3]u}{u} \\ &= \frac{[6]u}{[2]u}.\end{aligned}$$

(3) In the presentation of $(MU_{\mathbb{Z}/p})_*$, we can replace the generator λ_α by $\lambda'_\alpha = \lambda_\alpha + aq_1$ for any $a \in (MU_{\mathbb{Z}/p})_*$. Since $Ann(u_\alpha) = Ann(u) = (q_1)$ by Lemma 2.7,

these are the only choices of λ'_α that satisfy $\lambda'_{\alpha|}u_\alpha = u$. Every $a \in (MU_{\mathbb{Z}/p})_*$ has $a \equiv \overline{a}$ modulo u for some $\overline{a} \in MU_*$. Then relation (2.11) becomes

$$\lambda'_\alpha b_{0,1}^{(\alpha)} = 1 + (k_\alpha + \alpha\overline{a})q_1$$

and relation (2.12) becomes

$$\lambda'_\alpha q_1 = (\alpha^{-1} + \overline{a}p)q_1.$$

All results in this section carry through with α^{-1} replaced by $(\alpha^{-1})' = \alpha^{-1} + \overline{a}p$ and k_α replaced by $k_\alpha^{-1} = k_\alpha + \alpha\overline{a}$, since $\alpha(\alpha^{-1})' = 1 + k'_\alpha p$ in MU_*. In particular, if a is an integer, in the completion, λ_α corresponds to

$$\frac{[1 + k'_\alpha p]u}{[\alpha]u}.$$

Remark. Theorem 2.2 has concrete applications toward constructing interesting complex-oriented $\mathbb{Z}/p$-equivariant spectra. The sequence

$$(x_2, x_3, \ldots)$$

from MU_* is regular in $(MU_{\mathbb{Z}/p})_*$ because it is regular in the associated graded with respect to (u) (which comes from Borel cohomology), and there is no infinite u-divisibility by [8, 11]. So we have a spectrum E with

$$E_* = x_1^{-1}(MU_{\mathbb{Z}/p})_*/(x_2, x_3, \ldots). \tag{2.35}$$

What is interesting about E is the fact that $\Phi^{\mathbb{Z}/p}E_*$ is a localization of E_*, so

$$Spec(\Phi^{\mathbb{Z}/p}E_*) \tag{2.36}$$

is an open set in

$$Spec(E_*). \tag{2.37}$$

What we claim is that (2.36) is not a closed subset of (2.37). Thus, the spectrum E is not isotropically split, so it is a non-trivial example from the point of view of [2].

To see that (2.36) is not a Zariski closed subset of (2.37), we pull back to the spectrum of a simpler ring. For $p = 2$, pull back to $Spec(R)$ where

$$R = E_*/(b_{i,j} \text{ for } i > 1, b_{1,j} \text{ for } j \geq 2, b_{1,0} = 1 + \beta u), \tag{2.38}$$

where β denotes the Bott class. So R is obtained from $\mathbb{Z}[u]$ by making $u(u+2)$ infinitely u-divisible (by keeping the q_js), while the $b_{i,j}$s are assigned to their "obvious" values in Tate cohomology (see for example [10]). Pulling back to $Spec(R)$ amounts to forming

$$\Phi^{\mathbb{Z}/2}E_* \otimes_{E_*} R = \mathbb{Z}[u, u^{-1}]. \tag{2.39}$$

Thus, the map from (2.38) to (2.39) is not onto, because u is not actually inverted in R. This shows that the pullback of $Spec(\Phi^{\mathbb{Z}/2}E_*)$ to $Spec(R)$ is not closed in $Spec(R)$. Therefore, $Spec(\Phi^{\mathbb{Z}/2}E_*)$ is not closed in $Spec(E_*)$.

For $p > 2$, by Theorem 2.2, the story is analogous if we replace R by the quotient of E_* by the relations

$$\begin{aligned} b_{0,j}^{(\alpha)} &= \sum_{k=j}^{\alpha} \binom{\alpha}{k} \beta^{k-1} u^{k-j}, \quad j \geq 1, \\ b_{1,j}^{(\alpha)} &= \sum_{k=i}^{\alpha} \binom{\alpha}{k} \beta^{k} u^{k-j}, \quad j \geq 1, \\ b_{1,0}^{(\alpha)} &= 1 + \beta u, \\ b_{i,j}^{(\alpha)} &= 0, \quad i \geq 2. \end{aligned}$$

3. The $G = \mathbb{Z}/p^n$ case

The goal of this section is to state and prove Theorem 3.1, which is our main theorem for the $G = \mathbb{Z}/p^n$-case. The relations here are more complicated than in the $\mathbb{Z}/p$-case, so we begin with some notations. As in the previous section, we denote by c_j the coefficients of $[p]u \in MU_*[[u]]$. For $1 \leq \alpha \leq p^n - 1$, write $\alpha = \gamma p^t$, with $(\gamma, p) = 1$. For $t + 1 \leq r \leq n$, let $\alpha_{[r]}^{-1}$ be the smallest positive representative of γ^{-1} in $\mathbb{Z}/p^{r-t}$, i.e. it is the smallest positive integer such that in $\mathbb{Z}$,

$$\alpha_{[r]}^{-1} \alpha = p^t + k_\alpha^{[r]} p^r \tag{3.1}$$

for some $k_\alpha^{[r]} \geq 0$.

Next, we need to define certain polynomials that will be needed to state the relations in $(MU_{\mathbb{Z}/p^n})_*$. Consider a sequence of polynomial generators z_k for $k \geq 0$. For $1 \leq l \leq p-1$, the series $(x +_F [l]y)^k$ has terms with $x^i y^j$ with $i + j \geq k$, so there is a series

$$\sum_{k \geq 0} z_k (x +_F [l]y)^k \in MU_*[z_k \mid k \geq 0][[x, y]]. \tag{3.2}$$

We rewrite it as

$$\sum_{i,j \geq 0} p_{i,j}^{(l)}(z_k) x^i y^j, \tag{3.3}$$

where for $i, j \geq 0$, $p_{i,j}^{(l)}(z_k) \in MU_*[z_k \mid k \geq 0]$.

Similarly as in the $\mathbb{Z}/p$-case, there will be elements λ_α and $b_{0,1}^{(\alpha)}$ in $(MU_{\mathbb{Z}/p^n})_*$ for each $\alpha = 1, \ldots, p^n - 1$. There will also be elements $q_{[r],1}$ for each $r = 0, \ldots, n-1$. To state the relations involving λ_α, we need to define certain elements $\underline{b_{0,1}^{(\alpha)}}$ as follows. Let p^r be the highest power of p less than or equal to α, so we can write $\alpha = lp^r + s = \gamma p^t$ for some $1 \leq l \leq p-1$, and $0 \leq s \leq p^r - 1$. If $s = 0$, then $\gamma = l$ and $r = t$. In this case, define

$$\underline{b_{0,1}^{(lp^r)}} = b_{0,1}^{(lp^r)}.$$

If $s \neq 0$, then $r > t$, and we define

$$\underline{b_{0,1}^{(\alpha)}} = b_{0,1}^{(\alpha)} q_{[t],1} \cdots q_{[r-1],1} + \underline{b_{0,1}^{(s)}}. \tag{3.4}$$

For convenience, we also write $b_{1,0}^{(0)} = 1$ and $b_{i,0}^{(0)} = 0$ for all other i.

Theorem 3.1. *As an MU_*-algebra, $(MU_{\mathbb{Z}/p^n})_*$ has the following generators:*

$$\begin{aligned} &u_1, u_p, \dots, u_{p^{n-1}}, \\ b^{(\alpha)}_{i,j}, \quad & i, i \geq 0, \ \alpha = 1, 2, \dots, p^n - 1, \\ \lambda_\alpha, \quad & \alpha = 1, 2, \dots, p^n - 1, \\ q_{[r],j}, \quad & r = 0, \dots, n-1, \ j \geq 0. \end{aligned} \tag{3.5}$$

For each $\alpha = 1, \dots, p^n - 1$, we write $\alpha = lp^r + s$, where $1 \leq l \leq p-1$ and $0 \leq s \leq p^r - 1$.

The relations are as follows:

$$b^{(p^r)}_{0,1} = 1, \quad b^{(p^r)}_{0,j} = 0 \qquad \text{for every } j \geq 2, \tag{3.6}$$

$$b^{(\alpha)}_{i,j} - p^{(l)}_{i,j}(b^{(s)}_{k,0}) = u_{p^r} b^{(\alpha)}_{i,j+1}, \tag{3.7}$$

$$q_{[n-1],0} = 0, \qquad q_{[r],0} = u_{p^{r+1}} \qquad \text{for every } r \leq n-2, \tag{3.8}$$

$$q_{[r],j} - c_j = u_{p^r} q_{[r],j+1}. \tag{3.9}$$

The relations involving λ_α are:

$$\lambda_{p^r} = 1 \quad \text{for } 0 \leq r \leq n-1, \tag{3.10}$$

$$\lambda_\alpha \underline{b^{(\alpha)}_{0,1}} = 1 + k^{[n]}_\alpha q_{[t],1} \cdots q_{[n-1],1} \tag{3.11}$$

and

$$\lambda_\alpha q_{[t],1} \cdots q_{[n-1],1} = \alpha^{-1}_{[n]} q_{[t],1} \cdots q_{[n-1],1}. \tag{3.12}$$

For simplicity, we will write

$$u_\alpha = b^{(\alpha)}_{0,0}, \quad b^{(\alpha)}_i = b^{(\alpha)}_{i,0},$$

$$q_{[t,r]} = q_{[t],1} \cdots q_{[r-1],1} \tag{3.13}$$

for all $t < r \leq n$, so

$$u_{p^t}\ q_{[t,r]} = u_{p^r} \tag{3.14}$$

by relations (3.9) and (3.8). (Here, take $u_{p^n} = 0$.) Thus, in Theorem 3.1, instead of all the $u_1, \dots, u_{p^{n-1}}$, there is actually only the generator $u = u_1$. However, it is more convenient to write all u_{p^r} for $r = 0, \dots, n-1$.

Let R be the ring described in Theorem 3.1. Before proving the theorem, we note some relations in R that follow. First, consider the case $\alpha = lp^r$, where $1 \leq l \leq p-1$. Here, $s = 0$, so the series (3.2), with $z_k = b^{(0)}_k$, reduces to

$$\sum_{k \geq 0} b^{(0)}_k (x +_F [l]y)^k = x +_F [l]y = \sum_{i,j \geq 0} a^{(l)}_{i,j} x^i y^j.$$

Hence, we get that $p^{(l)}_{i,j}(b^{(0)}_k) = a^{(l)}_{i,j}$, and relation (3.7) becomes

$$b^{(lp^r)}_{i,j} - a^{(l)}_{i,j} = u_{p^r} b^{(lp^r)}_{i,j+1}. \tag{3.15}$$

For general s, the series (3.2), with $z_k = b^{(s)}_k$, has

$$\mathrm{Coeff}_{y^0}\left(\sum_{k \geq 0} b^{(s)}_k (x +_F [l]y)^k\right) = \sum_{k \geq 0} b^{(s)}_k x^k.$$

Hence, $p_{i,0}^{(\alpha)} = b_i^{(s)}$, and for $j = 0$, (3.7) becomes

$$b_i^{(\alpha)} - b_i^{(s)} = u_{p^r} b_{i,1}^{(\alpha)}. \tag{3.16}$$

In particular, for $i = 0$,

$$u_\alpha - u_s = u_{p^r} b_{0,1}^{(\alpha)}. \tag{3.17}$$

Write

$$\alpha = l_r p^r + l_{r-1} p^{r-1} + \cdots + l_t p^t.$$

Using induction on $r - t$, while combining (3.16) and (3.15), we get that

$$b_i^{(\alpha)} \equiv b_i^{(l_t p^t)} \equiv a_{i,0}^{(l_t)} \tag{3.18}$$

modulo u_{p^t}. If $i = 1$, this is 1, and if $i \geq 2$, this is 0.

We also record the following lemmas.

Lemma 3.2. *Let $\alpha = lp^r + s = \gamma p^t$, with $1 \leq l \leq p-1$, $0 \leq s \leq p^r - 1$, and $(\gamma, p) = 1$ as before. Then*

$$u_{p^t} \underline{b_{0,1}^{(\alpha)}} = u_\alpha.$$

Proof. By induction on $r - t$. If $r - t = 0$, then $\alpha = lp^r$, and relation (3.15) gives

$$u_\alpha = b_{0,0}^{(\alpha)} = b_{0,0}^{(\alpha)} - a_{0,0}^{(\alpha)} = u_{p^r} b_{0,1}^{(\alpha)} = u_{p^t} b_{0,1}^{(\alpha)}.$$

But in this case, $\underline{b_{0,1}^{(\alpha)}} = b_{0,1}^{(\alpha)}$.

As above, write

$$\alpha = l_r p^r + l_{r-1} p^{r-1} + \cdots + l_t p^t$$

(i.e. in base p, α has $r - t$ digits, followed by t zeros). For general $\alpha = lp^r + s$, with $s > 0$, we have

$$\begin{aligned} u_{p^t} \underline{b_{0,1}^{(\alpha)}} &= u_{p^t} (b_{0,1}^{(\alpha)} q_{[t],1} \cdots q_{[r-1],1} + \underline{b_{0,1}^{(s)}}) \\ &= u_{p^r} b_{0,1}^{(\alpha)} + u_{p^t} \underline{b_{0,1}^{(s)}}. \end{aligned}$$

But $s = \delta p^t$ for some $(\delta, p) = 1$, and also has one fewer non-zero digit than α in base p, so by the induction hypothesis,

$$u_{p^t} \underline{b_{0,1}^{(s)}} = u_s.$$

So the statement follows by (3.17). □

Write $\alpha = \gamma p^t = s + lp^r$ for $(\gamma, p) = 1$, $\leq l \leq p-1$, $0 \leq s \leq p^r - 1$ as before. If $s \neq 0$, it is straightforward to check that p^{r-t} divides $\alpha_{[n]}^{-1} - s_{[n]}^{-1}$. Define the integer

$$\epsilon_\alpha = \frac{\alpha_{[n]}^{-1} - s_{[n]}^{-1}}{p^{r-t}}.$$

Lemma 3.3. *In the notation above, we have:*

(1) *Modulo u_{p^t},*

$$\underline{b_{0,1}^{(\alpha)}} \equiv \gamma.$$

(2) *Modulo u_{p^t},*

$$\lambda_\alpha \equiv \alpha^{-1}.$$

(3) *For* $\alpha = s + lp^r$, $s \neq 0$,

$$\lambda_\alpha \equiv \lambda_s + \epsilon_\alpha q_{[t,r]}$$

modulo u_{p^r}.

Proof. The first statement is again by induction on $r - t$. If $r - t = 0$, $\alpha = lp^r$ for $1 \leq l \leq p - 1$. By (3.15),

$$\underline{b_{0,1}^{(\alpha)}} = b_{0,1}^{(\alpha)} \equiv a_{0,1}^{(l)} = l = \gamma$$

modulo u_{p^t}.

For general α, if $\alpha = lp^r + s = \gamma p^t$, and $s = \gamma' p^t$, then $\gamma = lp^{r-t} + \gamma'$. By the induction hypothesis, we have that $\underline{b_{0,1}^{(s)}} \equiv \gamma'$ modulo u_{p^t}. By definition,

$$\underline{b_{0,1}^{(\alpha)}} = b_{0,1}^{(\alpha)} q_{[t],1} \cdots q_{[r-1],1} + \underline{b_{0,1}^{(s)}}.$$

We have

$$b_{0,1}^{(\alpha)} \equiv p_{0,1}^{(l)}(b_k^{(s)}) = lb_1^{(s)}$$

modulo u_{p^r}. This is because $p_{0,1}^{(l)}(b_k^{(s)})$ is the constant term of

$$\frac{\sum_{k\geq 1} b_k^{(s)}([l]u_{p^r})^k}{u_{p^r}}.$$

But by (3.18), $b_1^{(s)} \equiv 1$ modulo u_{p^t}. Also, for each $k \geq t$, $q_{[k],1} \equiv p$ modulo u_{p^k}, and so modulo u_{p^t} as well. So modulo u_{p^t},

$$\underline{b_{0,1}^{(\alpha)}} \equiv lp^{r-t} + \gamma' = \gamma.$$

The second statement about λ_α is an argument similar to the $\mathbb{Z}/p$ case. For short, write $\alpha^{-1} = \alpha_{[n]}^{-1}$ and $k_\alpha = k_\alpha^{[n]}$ here. We have $\gamma\alpha^{-1} = 1 + k_\alpha p^{n-t}$. From relation (3.11) and the first statement of this lemma, we have that modulo u_{p^t},

$$\begin{aligned}
\lambda_\alpha \underline{b_{0,1}^{(\alpha)}} \alpha^{-1} &= \alpha^{-1} + k_\alpha \alpha^{-1} q_{[t,n]}, \\
\lambda_\alpha \gamma \alpha^{-1} &\equiv \alpha^{-1} + k_\alpha \alpha^{-1} q_{[t,n]}, \\
\lambda_\alpha (1 + k_\alpha p^{n-t}) &\equiv \alpha^{-1} + k_\alpha \alpha^{-1} q_{[t,n]}, \\
\lambda_\alpha &\equiv \alpha^{-1}.
\end{aligned}$$

The last congruence follows from relation (3.12) and the fact that $q_{[r],1} \equiv p$ modulo u_{p^t} for all $r \geq t$, so $q_{[n,t]} \equiv p^{n-t}$ modulo u_{p^t}.

For Statement (3), start with relation (3.11) and multiply both sides by $\lambda_s + \epsilon_\alpha q_{[t,r]}$. Using the definition of $\underline{b_{0,1}^{(\alpha)}}$, we have

$$\lambda_\alpha(\underline{b_{0,1}^{(s)}} + b_{0,1}^{(s)} q_{[t,r]})(\lambda_s + \epsilon_\alpha q_{[t,r]}) = (1 + k_\alpha q_{[t,n]})(\lambda_s + \epsilon_\alpha q_{[t,r]}).$$

The right hand side is

$$(\lambda_s + \epsilon_\alpha q_{[t,r]}) + k_\alpha q_{[t,n]}(\lambda_s + \epsilon_\alpha q_{[t,r]})$$

with an extra term

$$k_\alpha q_{[t,n]}(\lambda_s + \epsilon_\alpha q_{[t,r]}) = k_\alpha q_{[r,n]}(\lambda_s + \epsilon_\alpha q_{[t,r]}) q_{[t,r]}.$$

The left hand side is λ_α plus an extra term

$$\lambda_\alpha \left[k_s q_{[r,n]} + b_{0,1}^{(\alpha)} \lambda_s + \epsilon_\alpha \underline{b_{0,1}^{(s)}} + b_{0,1}^{(\alpha)} \epsilon_\alpha q_{[t,r]}\right] q_{[t,r]}.$$

To show that the two extra terms on the left and right are congruent modulo u_{p^r}, since $u_{p^t} q_{[t,r]} = u_{p^r}$, it suffices to show that these two terms, without the factor $q_{[t,r]}$ at their ends, are congruent modulo u_{p^t}. From the right hand side, we have that modulo u_{p^t},

$$k_\alpha q_{[r,n]}(\lambda_s + \epsilon_\alpha q_{[t,r]}) \equiv k_\alpha p^{n-r}(s^{-1} + \epsilon_\alpha p^{r-t}) = k_\alpha \alpha^{-1} p^{n-r} \tag{3.19}$$

by Part (2) of this lemma.

For the left hand side, we have that modulo u_{p^r}, $b_{0,1}^{(\alpha)}$ is congruent to the polynomial

$$p_{0,1}^{(l)}(b_j^{(s)}) = \mathrm{Coeff}_{x^0 u_{p^r}^1} \left(\sum_{j \geq 0} b_j^{(s)} (x +_F [l] u_{p^r})^j \right) = l b_1^{(s)}.$$

Inductively, $b_1^{(s)} \equiv 1$ modulo u_{p^t}, so $b_{0,1}^{(\alpha)} \equiv l$ modulo u_{p^t}. So on the left hand side, modulo u_{p^t},

$$\begin{aligned} &\lambda_\alpha \left[k_s q_{[r,n]} + b_{0,1}^{(\alpha)} \lambda_s + \epsilon_\alpha \underline{b_{0,1}^{(s)}} + b_{0,1}^{(\alpha)} \epsilon_\alpha q_{[t,r]}\right] \\ &\equiv \alpha^{-1} \left(k_s p^{n-r} + l s^{-1} + \epsilon_\alpha \frac{s}{p^t} + l \epsilon_\alpha p^{r-t} \right). \end{aligned} \tag{3.20}$$

It is straightforward to check that the integers (3.20) and (3.19) match. $\square$

From Lemma 3.2, it follows that

$$\lambda_\alpha u_\alpha = \lambda_\alpha \underline{b_{0,1}^{(\alpha)}} u_{p^t} = (1 + k_\alpha^{[n]} q_{[t,n]}) u_{p^t} = u_{p^t}. \tag{3.21}$$

In this sense, $\underline{b_{0,1}^{(\alpha)}}$ and λ_α are "approximate" inverses to each other.

Comment. The elements $\underline{b_{0,1}^{(\alpha)}}$ are examples of certain sequences of elements $\underline{b_{i,j}^{(\alpha)}}$ in R. For $1 \leq \alpha \leq p^n - 1$, again write $\alpha = \gamma p^t$ with $(\gamma, p) = 1$. For all α, we have

$$\underline{b_{i,0}^{(\alpha)}} = b_{i,0}^{(\alpha)}.$$

The $\underline{b_{i,j}^{(\alpha)}}$ satisfy the divisibility conditions

$$\underline{b_{i,j}^{(\alpha)}} - a_{i,j}^{(\gamma)} = u_{p^t} \underline{b_{i,j+1}^{(\alpha)}}. \tag{3.22}$$

Just like $\underline{b_{0,1}^{(\alpha)}}$, the formulas for these elements in terms of $b_{i,j}^{(\alpha)}$ can be written down inductively. In particular, if $\alpha = lp^r$ for some $1 \leq l \leq p-1$, then

$$\underline{b_{i,j}^{(lp^r)}} = b_{i,j}^{(lp^r)}$$

for all i, j, and (3.22) is just (3.15).

As an example, we write down $\underline{b_{i,1}^{(\alpha)}}$ explicitly. By the above, $\underline{b_{i,1}^{(1)}} = b_{i,1}^{(1)}$. Suppose that $\underline{b_{i,1}^{(s)}}$ is defined for all $s < \alpha$. If $\alpha = lp^r + s$ for some $1 \leq l \leq p-1$ and $1 \leq s \leq p^r - 1$, define

$$\underline{b_{i,1}^{(\alpha)}} = \underline{b_{i,1}^{(s)}} + q_{[t,r]} b_{i,1}^{(\alpha)}.$$

One can check inductively that the divisibility relation (3.22) is satisfied.

The proof of the theorem will be by induction on n. For $n = 1$, this is Theorem 2.2 of the previous section. From now on, we will assume that Theorem 3.1, as well as all claims in the rest of this section, holds for $(MU_{\mathbb{Z}/p^r})_*$, for $r = 1, \ldots, n-1$.

For our main tool, we will use the pullback diagram

$$(3.23)\qquad \begin{array}{ccc} MU_{\mathbb{Z}/p^n} & \longrightarrow & \widetilde{E\mathbb{Z}/p} \wedge MU_{\mathbb{Z}/p^n} \\ \downarrow & & \downarrow \\ F(E\mathbb{Z}/p_+, MU_{\mathbb{Z}/p^n}) & \longrightarrow & \widetilde{E\mathbb{Z}/p} \wedge F(E\mathbb{Z}/p_+, MU_{\mathbb{Z}/p^n}) \end{array}$$

which is the right half of the Tate diagram of [6].

Before calculating the three other corners of (3.23) and showing that it is also a pullback after taking coefficients, we will fit it into a staircase diagram, which is an approach similar to that used by Abram and Kriz [1] to describe $(MU_G)_*$ for a general abelian group G. Although this diagram is bigger (the square (3.23) is a part of it), it gives further understanding of $(MU_{\mathbb{Z}/p^n})_*$, and also has other applications, being for example used in [9].

Recall that for a finite abelian group G and a subgroup H, $\mathcal{F}[H]$ is the family of subgroups K of G with $H \not\subseteq K$. The classifying space $E\mathcal{F}$ has the property that $E\mathcal{F}^K$ is $*$ for $K \in \mathcal{F}$, and $\emptyset$ for $K \notin \mathcal{F}$. We also have the family $\mathcal{F}(H)$ of subgroups that are contained in H. For $G = \mathbb{Z}/p^n$, we have $E\mathcal{F}(\mathbb{Z}/p^r) = E\mathcal{F}[\mathbb{Z}/p^{r+1}]$, which we write as $E\mathbb{Z}/p^{n-r}$ (where $\mathbb{Z}/p^{n-r}$ is the quotient group).

Proposition 3.4 is analogous to a special case of Theorem 2 of Abram-Kriz [1]. The diagram set-up, however, is different, and we will clarify the relationship between this diagram and the one from [1] in Section 4.

Proposition 3.4. *A $\mathbb{Z}/p^n$-equivariant spectrum E is the pullback of the homotopy groups of the fixed points of the following maps of $\mathbb{Z}/p^n$-equivariant spectra:*

$$(3.24)\qquad \begin{array}{cccccccc} & & & & & & & E_n \\ & & & & & & & \downarrow{\scriptstyle \phi_n} \\ & & & & & E_{n-1} & \xrightarrow{\iota_{n-1}} & F_{n-1} \\ & & & & & \downarrow{\scriptstyle \phi_{n-1}} & & \\ & & & \cdots & \longrightarrow & F_{n-2} & & \\ & & & \downarrow & & & & \\ & E_1 & \xrightarrow{\iota_1} & F_1 & & & & \\ & \downarrow{\scriptstyle \phi_1} & & & & & & \\ E_0 \xrightarrow{\iota_0} & F_0 & & & & & & \end{array}$$

where

$$
\begin{aligned}
E_r &= \widetilde{E\mathcal{F}[\mathbb{Z}/p^r]} \wedge F(E\mathcal{F}(\mathbb{Z}/p^r)_+, E)\\
&= \widetilde{E\mathbb{Z}/p^{n-r+1}} \wedge F((E\mathbb{Z}/p^{n-r})_+, E),\\
F_r &= \widetilde{E\mathcal{F}[\mathbb{Z}/p^{r+1}]} \wedge F(E\mathcal{F}(\mathbb{Z}/p^r)_+, E)\\
&= \widetilde{E\mathbb{Z}/p^{n-r}} \wedge F((E\mathbb{Z}/p^{n-r})_+, E).
\end{aligned}
$$

In our particular case with $E = MU_{\mathbb{Z}/p^n}$,

$$E_n = \widetilde{E\mathbb{Z}/p} \wedge MU_{\mathbb{Z}/p^n},$$

the geometric fixed points spectrum $\Phi^{\mathbb{Z}/p^n} MU_{\mathbb{Z}/p^n}$, and the upper right corner of (3.23).

Proof of Proposition 3.4. For $0 \le r \le n$, let X_r be the pullback of the diagram

$$
\begin{array}{ccccccc}
 & & & & & & E_n\\
 & & & & & & \downarrow{\scriptstyle \phi_n}\\
 & & & & E_{n-1} & \xrightarrow{\iota_{n-1}} & F_{n-1}\\
 & & & & \downarrow{\scriptstyle \phi_{n-1}} & & \\
 & & \cdots & \longrightarrow & F_{n-2} & & \\
 & & \downarrow & & & & \\
E_r & \xrightarrow{\iota_r} & F_r & & & &
\end{array}
\tag{3.25}
$$

.

We claim that

$$X_r \simeq \widetilde{E\mathbb{Z}/p^{n-r+1}} \wedge E \tag{3.26}$$

as a $\mathbb{Z}/p^n$-spectrum.

To show claim (3.26), we use induction on $n - r$. For $r = n$, we just have

$$X_n = E_n = \widetilde{E\mathbb{Z}/p} \wedge E.$$

Suppose the claim holds for $r + 1$. Then X_r is the pullback of the diagram

$$
\begin{array}{ccc}
X_r & \longrightarrow & \widetilde{E\mathbb{Z}/p^{n-r}} \wedge E\\
\downarrow & & \downarrow\\
 & & \widetilde{E\mathbb{Z}/p^{n-r}} \wedge F(E\mathbb{Z}/p_+^{n-r-1}, E)\\
 & & \downarrow\\
\widetilde{E\mathbb{Z}/p^{n-r+1}} \wedge F(E\mathbb{Z}/p_+^{n-r}, E) & \longrightarrow & \widetilde{E\mathbb{Z}/p^{n-r}} \wedge F(E\mathbb{Z}/p_+^{n-r}, E)
\end{array}
\tag{3.27}
$$

where all maps are the obvious ones. Let the space M_r be the fiber of the map

$$\widetilde{E\mathbb{Z}/p^{n-r+1}} \to \widetilde{E\mathbb{Z}/p^{n-r}}.$$

Consider the diagram

$$\begin{array}{ccccc} M_r \wedge E & \longrightarrow & \widetilde{E\mathbb{Z}/p^{n-r+1}} \wedge E & \longrightarrow & \widetilde{E\mathbb{Z}/p^{n-r}} \wedge E \\ \downarrow & & \downarrow & & \downarrow \\ M_r \wedge F(E\mathbb{Z}/p^{n-r}_+, E) & \longrightarrow & \widetilde{E\mathbb{Z}/p^{n-r+1}} \wedge F(E\mathbb{Z}/p^{n-r}_+, E) & \longrightarrow & \widetilde{E\mathbb{Z}/p^{n-r}} \wedge F(E\mathbb{Z}/p^{n-r}_+, E) \end{array}$$

where the rows are cofiber sequences. However, recall that for $0 \le i \le n$, we have

$$(\widetilde{E\mathbb{Z}/p^{n-r}})^{\mathbb{Z}/p^i} \simeq \begin{cases} S^0 & \text{if } i > r \\ * & \text{if } i \le r \end{cases}.$$

Thus, the $\mathbb{Z}/p^i$-fixed points of M_r are S^0 if $i = r$, and $*$ else. However, the Z/p^i-fixed points of $E\mathbb{Z}/p^{n-r}_+$ are S^0 for all $i \le r$. Hence after smashing with M_r, $F(E\mathbb{Z}/p^{n-r}_+, E)$ becomes indistinguishable from $F(S^0, E) = E$. In other words, the left vertical map is an equivalence:

$$M_r \wedge E \simeq M_r \wedge F(E\mathbb{Z}/p^{n-r}_+, E).$$

Also by induction, the right square of the diagram has the property that the two maps into the lower right corner are jointly onto. Hence, it is a pullback. However, it is also the same as diagram (3.27), so we get claim (3.26).

For $r = 0$, this gives that X_0, the pullback of the entire diagram, is just E itself. □

We again specialize to the case $E = MU_{\mathbb{Z}/p^n}$. For each r, let Y_r be the pullback of the partial diagram

$$\begin{array}{cccc} & & & E_r \\ & & & \downarrow \\ & & \cdots \longrightarrow & \\ & & \downarrow & \\ & E_1 \longrightarrow & F_1 & \\ & \downarrow & & \\ E_0 \longrightarrow & F_0 & & \end{array}$$

Then

$$Y_r \simeq F(E\mathbb{Z}/p^{n-r}_+, MU_{\mathbb{Z}/p^n}). \tag{3.28}$$

Again, this is shown by induction on r. For $f = 0$, $Y_0 = E_0 = F(E\mathbb{Z}/p^n_+, MU_{\mathbb{Z}/p^n})$. Suppose (3.28) holds for $r - 1$. Let N_r be the cofiber of the map $E\mathbb{Z}/p^{n-r+1}_+ \to E\mathbb{Z}/p^{n-r}$. Recall that for $0 \le i \le r$, $(E\mathbb{Z}/p^{n-r})^{\mathbb{Z}/p^i}_+$ is S^0 if $i \le r$ and $*$ if $i > r$.

Hence, $(N_r)^{\mathbb{Z}/p^i}$ is S^0 only for $i = r$, and $*$ for all other i. Consider the diagram

(3.29)

$$\begin{array}{ccc}
F(N_r, MU_{\mathbb{Z}/p^n}) & \longrightarrow & \widetilde{E\mathbb{Z}/p^{n-r+1}} \wedge F(N_r, MU_{\mathbb{Z}/p^n}) \\
\downarrow & & \downarrow \\
F(E\mathbb{Z}/p_+^{n-r}, MU_{\mathbb{Z}/p^n}) & \longrightarrow & E_r = \widetilde{E\mathbb{Z}/p^{n-r+1}} \wedge F(E\mathbb{Z}/p_+^{n-r}, MU_{\mathbb{Z}/p^n}) \\
\downarrow & & \downarrow \\
F(E\mathbb{Z}/p_+^{n-r+1}, MU_{\mathbb{Z}/p^n}) & \longrightarrow & F_r = \widetilde{E\mathbb{Z}/p^{n-r+1}} \wedge F(E\mathbb{Z}/p_+^{n-r+1}, MU_{\mathbb{Z}/p^n})
\end{array}$$

for all $i \geq r$, where the vertical sides are fibration sequences. However, we have that

$$(\widetilde{E\mathbb{Z}/p^{n-r+1}})^{\mathbb{Z}/p^i} = S^0 \text{ for all } i \geq r,$$

so on $F(N_r, MU_{\mathbb{Z}/p^n})$, smashing with $\widetilde{E\mathbb{Z}/p^{n-r+1}}$ is indistinguishable from smashing with S^0. Hence, the top row of diagram (3.29) is an equivalence, giving that the bottom square is a pullback. But the pullback of the bottom square is Y_r by definition, giving (3.28).

It is helpful to put everything together and write down a fuller version of diagram (3.24):

(3.30)

$$\begin{array}{ccccccccc}
MU_{\mathbb{Z}/p^n} & \longrightarrow & X_1 & \longrightarrow & \cdots & \longrightarrow & X_{n-1} & \longrightarrow & E_n \\
\downarrow & & & & & & & & \downarrow{\scriptstyle \phi_n} \\
Y_{n-1} & & & & & & E_{n-1} & \xrightarrow{\iota_{n-1}} & F_{n-1} \\
\downarrow & & & & & & \downarrow{\scriptstyle \phi_{n-1}} & & \\
\vdots & & & & \cdots & \longrightarrow & F_{n-2} & & \\
\downarrow & & & & \downarrow & & & & \\
Y_1 & & E_1 & \xrightarrow{\iota_1} & F_1 & & & & \\
\downarrow & & \downarrow{\scriptstyle \phi_1} & & & & & & \\
E_0 & \xrightarrow{\iota_0} & F_0 & & & & & &
\end{array}$$

Diagram (3.23) is now just the topmost "wide" rectangle of (3.30).

Now we come to calculating the three other corners of (3.23) algebraically. For the upper right corner, recall that in general, for a G-equivariant spectrum E, the geometric fixed points $\Phi^G E$ is $(\widetilde{E\mathcal{F}} \wedge E)^G$, where $\mathcal{F}$ is the family of proper subgroups of G. Hence, the upper right corner is the geometric fixed points $\Phi^{\mathbb{Z}/p^n} MU_{\mathbb{Z}/p^n}$. This can be calculated by methods similar to that of [10], using the methods of tom Dieck [18].

Lemma 3.5. *We have*

$$(\widetilde{E\mathbb{Z}/p} \wedge MU_{\mathbb{Z}/p^n})_* = MU_*[b_i^{(\alpha)}, u_\alpha^{-1}],$$

where $1 \leq \alpha \leq p^n - 1$, $i \geq 0$, *and* $u_\alpha = b_0^{(\alpha)}$.

Proof. We begin with some notation. For $0 \leq \alpha \leq p^n - 1$, we will denote the irreducible complex representation of $\mathbb{Z}/p^n$, where the generator of $\mathbb{Z}/p^n$ acts by $e^{2\pi\alpha i/p^n}$,by V_α. The complete $\mathbb{Z}/p^n$-universe is

$$\mathcal{U} = \bigoplus_{0\leq\alpha\leq p^n-1} V_\alpha^\infty$$

and $\mathcal{U}^{\mathbb{Z}/p^n} = \mathbb{R}^\infty$ is the trivial universe. For each $\alpha \in (\mathbb{Z}/p^n)^\times$, let u_α be the Euler class of the irreducible representation V_α.

Recall also that the prespectrum D giving $MU_{\mathbb{Z}/p^n}$ is as follows. For a finite-dimensional representation V of $\mathbb{Z}/p^n$, let $Gr_{|V|}(\mathcal{U} \oplus V)$ be the Grassmannian of all $|V|$-dimensional complex representations of $\mathcal{U} \oplus V$ (note that this is the same as $Gr_{|V|}(\mathcal{U})$), and let $\gamma_{|V|}$ be the $|V|$-dimensional canonical bundle on it. The $\mathcal{U}$-prespectrum D is given by

$$D_V = Gr_{|V|}(\mathcal{U} \oplus V)^{\gamma_{|V|}},$$

where the superscript denotes Thom space. The point of the geometric fixed point spectrum is that it can be calculated on the prespectrum level. Namely,

$$\Phi^{\mathbb{Z}/p^n} MU_{\mathbb{Z}/p^n} = \operatorname{colim} \Sigma^{-V^{\mathbb{Z}/p^n}} (D_V)^{\mathbb{Z}/p^n} \tag{3.31}$$

over all finite-dimensional representations $V \subset \mathcal{U}$.

Given a finite-dimensional representation

$$V = \bigoplus_{\alpha=0}^{p^n-1} n_\alpha V_\alpha,$$

the space $(D_V)^{\mathbb{Z}/p^n}$ can be calculated by the methods of tom Dieck [18]. Namely, it is

$$\bigvee \left(Gr(k_0, \mathcal{U}^{\mathbb{Z}/p^n})^{\gamma_{k_0}} \wedge (\bigwedge_{1\leq\alpha\leq p^n-1} Gr(k_\alpha, V_\alpha^\infty)_+) \right),$$

where the wedge sum is taken over all partitions $|V| = \sum k_\alpha$, where $0 \leq \alpha \leq p^n - 1$. Meanwhile, $V^{\mathbb{Z}/p^n} = \mathbb{R}^{2n_0}$ (the multiple of 2 arises from the fact that $V_0 = \mathbb{C}$). Also note that

$$|V| = \sum_{\alpha=0}^{p^n-1} n_\alpha = \sum_{\alpha=0}^{p^n-1} k_\alpha,$$

so

$$\begin{aligned}
&\Sigma^{-V^{\mathbb{Z}/p^n}} (D_V)^{\mathbb{Z}/p^n} \\
&= \bigvee \Sigma^{-2n_0} \left(Gr(k_0, \mathcal{U}^{\mathbb{Z}/p^n})^{\gamma_{k_0}} \wedge (\bigwedge_{1\leq\alpha\leq p^n-1} Gr(k_\alpha, V_\alpha^\infty)_+) \right) \\
&= \bigvee \left(\Sigma^{-2k_0} Gr(k_0, \mathcal{U}^{\mathbb{Z}/p^n})^{\gamma_{k_0}} \wedge (\bigwedge_{1\leq\alpha\leq p^n-1} \Sigma^{2(k_\alpha - n_\alpha)} Gr(k_\alpha, V_\alpha^\infty)_+) \right).
\end{aligned}$$

Write $m_\alpha = k_\alpha - n_\alpha$. Then this is

$$\bigvee_{(m_\alpha)} \left(\Sigma^{2(n_0 - \sum m_\alpha)} Gr(n_0 - \Sigma m_\alpha, \mathcal{U}^{\mathbb{Z}/p^n})^{\gamma_{n_0 - \Sigma m_\alpha}} \wedge (\bigwedge \Sigma^{2m_\alpha} Gr(m_\alpha + n_\alpha, V_\alpha^\infty)_+) \right),$$

where the wedge sum is over all $(p^n - 1)$-tuples of integers $(m_\alpha)_{1\leq\alpha\leq p^n-1}$.

To obtain (3.31), we pass to the colimit where all n_α go to infinity. This gives

$$\bigvee_{(m_\alpha)} MU \wedge \left(\bigwedge_{1\leq\alpha\leq p^n-1} \Sigma^{2m_\alpha} BU_+ \right).$$

Let the element $b_i^{(\alpha)}$ correspond to the element b_i in

$$MU_* Gr(1, V_\alpha^\infty) = MU_*\{b_0, b_1, \ldots\}.$$

(In particular, $b_0^{(\alpha)} = u_\alpha$.) Then the coefficients of

$$MU \wedge \left(\bigwedge_{1\leq\alpha\leq p^n-1} BU_+ \right)$$

have generators $b_i^{(\alpha)}$. For the wedge summand suspended by $(2m_\alpha)$, we need to attach the power $u_\alpha^{m_\alpha}$ for each α. This gives the calculation of $\Phi^{\mathbb{Z}/p^n} MU_{\mathbb{Z}/p^n}$. $\square$

The terms $(Y_r)_*$ on the left hand side of diagram (3.30) can be computed by the standard methods, namely the Borel cohomology spectral sequence. This spectral sequence gives an associated graded object of $(Y_r)_*$ as

$$(MU_{\mathbb{Z}/p^r})_*[[u_{p^r}]]/[p^{n-r}]u_{p^r}.$$

By comparison with the case $r = 0$, which is the lower left corner of (3.30), we see that there should also be an extension $u_{p^{r-1}} q_{[r-1],1} = u_{p^r}$. We can describe $(Y_r)_*$ as follows. For $1 \leq \alpha \leq p^r - 1$, take $\alpha_{[n]}^{-1}\alpha = p^t + k_\alpha^{[n]} p^n = p^t + (k_\alpha^{[n]} p^{n-r})p^r$. Instead of λ_α, choose alternative generator

$$\widetilde{\lambda}_\alpha = \lambda_\alpha + m_\alpha q_{[t,r]}$$

in $(MU_{\mathbb{Z}/p^r})_*$, where

$$m_\alpha = \frac{\alpha_{[n]}^{-1} - \alpha_{[r]}^{-1}}{p^{r-t}}.$$

In this presentation of $(MU_{\mathbb{Z}/p^r})_*$, relation (3.11) is now

$$\widetilde{\lambda}_\alpha \underline{b_{0,1}^{(\alpha)}} = 1 + k_\alpha^{[n]} p^{n-r} q_{[t,r]} \tag{3.32}$$

and relation (3.12)) is now

$$\widetilde{\lambda}_\alpha q_{[t,r]} = \alpha_{[n]}^{-1} q_{[t,r]}. \tag{3.33}$$

Let $\overline{(MU_{\mathbb{Z}/p^r})_*}$ be a ring with the same generators as $(MU_{\mathbb{Z}/p^r})_*$, and all relations except $q_{[r-1],0} = 0$ and relations (3.32), (3.33). Write $u_{p^r} = q_{[r-1],0} = q_{[r-1],1} u_{p^{r-1}}$. Complete $\overline{(MU_{\mathbb{Z}/p^r})_*}$ with respect to u_{p^r}. Write

$$q_{[r,n]} = \frac{[p^{n-r}]u_{p^r}}{u_{p^r}}$$

and $q_{[t,n]} = q_{[t,r]} q_{[r,n]}$. Relation (3.32) becomes

$$\widetilde{\lambda}_\alpha \underline{b_{0,1}^{(\alpha)}} = 1 + k_\alpha^{[n]} q_{[t,n]}. \tag{3.34}$$

(This arises from the fact that it needs to map to

$$\frac{[1+k_\alpha^{[n]}p^{n-t}]u_{p^t}}{u_{p^t}}$$

if we complete at u_{p^t}.) Relation (3.33) becomes

$$\widetilde{\lambda}_\alpha q_{[t,n]} = \alpha_{[n]}^{-1} q_{[t,n]}. \tag{3.35}$$

We have

$$(Y_r)_* = (\overline{(MU_{\mathbb{Z}/p^r})_*})^{\wedge}_{u_{p^r}}/[p^{n-r}]u_{p^r}, \sim, \tag{3.36}$$

where $\sim$ are the relations (3.34), (3.35). Lemma 3.3 carries through modulo u_{p^r}, so we have $\widetilde{\lambda}_\alpha \equiv \alpha_{[n]}^{-1}$ modulo u_{p^t}. We also have

$$\widetilde{\lambda}_\alpha u_\alpha = u_{p^t}. \tag{3.37}$$

It is clear that (3.36), modulo u_{p^r}, becomes $(MU_{\mathbb{Z}/p^r})_*$ as it should.

For $1 \leq \alpha \leq p^r - 1$, and x a multiple of u_{p^r}, write

$$u_\alpha \widetilde{+}_F x = \sum_{i\geq 0} b_i^{(\alpha)} x^i = u_\alpha + \sum_{i \geq 1} b_i^{(\alpha)} x^i.$$

For $x = [n]u_{p^r}$, this is a version of $u_\alpha +_F [n]u_{p^r}$ that exists in $(\overline{(MU_{\mathbb{Z}/p^r})_*})^{\wedge}_{u_{p^r}}$. In $(Y_r)_*$, the λ_α from the original presentation of $(MU_{\mathbb{Z}/p^r})_*$ can be written in terms of $\widetilde{\lambda}_\alpha$ as

$$\lambda_\alpha = \frac{u_{p^t} \widetilde{+}_F [k_\alpha^{[r]}]u_{p^r}}{u_\alpha} + \delta_\alpha q_{[t,n]}.$$

Here, the division is done in the obvious manner: we use $\widetilde{\lambda}_\alpha$ for the first term u_{p^t}/u_α, and each higher term of the numerator is a multiple of u_{p^t}. There is a correction term by an appropriate integer δ_α multiple of $q_{[t,n]}$, so that $\lambda_\alpha \equiv \alpha_{[r]}^{-1}$ modulo u_{p^t}. This also gives

$$\lambda_\alpha \equiv \overline{\lambda}_\alpha - m_\alpha q_{[t,r]}$$

modulo u_{p^r}. We have

$$\lambda_\alpha \underline{b_{0,1}^{(\alpha)}} = \frac{u_{p^t} \widetilde{+}_F [k_\alpha^{[r]}]u_{p^r}}{u_{p^t}}$$

(with the obvious way to divide). Note that the right hand side of this is congruent to $1 + k_\alpha^{[r]} q_{[t,r]}$ modulo u_{p^r}.

For simplicity of notation, we write

$$S_r = (Y_r)_*.$$

Using the induction hypothesis and carrying out an argument similar to that of Lemma 2.5 of the previous section, we have

$$(S_r)^{\wedge}_u = MU_*[[u]]/([p^n]u). \tag{3.38}$$

By Greenlees and May [6], S_r is the completion of $(MU_{\mathbb{Z}/p^n})_*$ at u_{p^r}. Also, by Kriz and Lu [11], no element of $(MU_{\mathbb{Z}/p^n})_*$ can be infinitely divisible by u. Hence, all vertical maps $(MU_{\mathbb{Z}/p^n})_* \to S^r$ on the left hand side of (3.30) are injective.

On coefficients, diagram (3.23) becomes the following pullback diagram:

$$
\begin{array}{ccc}
(MU_{\mathbb{Z}/p^n})_* & \longrightarrow & MU_*[b_i^{(\alpha)}, u_\alpha^{-1}] \\
\Big\downarrow & & \Big\downarrow \phi \\
S_{n-1} & \xrightarrow[\iota]{} & S_{n-1}[u_{p^{n-1}}^{-1}],
\end{array}
\tag{3.39}
$$

where α, i are as in Lemma 3.5. The bottom horizontal map is localization.

Before describing the right vertical map ϕ, we record Lemma 3.6.

Lemma 3.6. *For $p^r < \alpha \le p^n - 1$, $p^r \nmid \alpha$, write $\alpha = s + lp^r$ with $1 \le s \le p^r - 1$, $1 \le l \le p^{n-r} - 1$. Then $u_s \widetilde{+}_F [l] u_{p^r}$ divides u_{p^t} in S_r.*

Proof. Write $q = q_{[t,r]}$ in this proof. Note that u_{p^r} multiplied by a sufficiently high power of q is divisible by $q^2_{p^r}$. Namely, let m be the smallest positive integer such that $m(r-t) \ge n - r$. Since $q \equiv p^{r-t}$ modulo u_{p^t}, we get $q^m \equiv p^{m(r-t)}$ modulo u_{p^t}, hence $q^{m+1} \equiv p^{m(r-t)} q$ modulo u_{p^r}. This gives

$$q^{m+1} u_{p^r} \equiv p^{m(r-t)} q u_{p^r} \quad \text{modulo } u^2_{p^r}.$$

However, $p^{m(r-t)} u_{p^r} \equiv [p^{m(r-t)}] u_{p^r} = 0$ modulo $u^2_{p^r}$.

Modulo $u^2_{p^r}$,

$$
\begin{aligned}
u_s \widetilde{+}_F [l] u_{p^r} &= u_s + b_1^{(s)}([l]u_{p^r}) + b_2^{(s)}([l]u_{p^r})^2 + \cdots \\
&\equiv u_s + b_1^{(s)}(l u_{p^r}) \\
&\equiv u_s + (1 + u_s \widetilde{\lambda_s} \underline{b_{1,1}^{(s)}}) l u_{p^r} \\
&\equiv u_s + l u_{p^r} + (u_s u_{p^r}\text{-term}) \\
&\equiv u_s (1 + l \widetilde{\lambda_s} q) + (u_s u_{p^r}\text{-term}).
\end{aligned}
$$

Write $y = -l\widetilde{\lambda_s} q$. Then $u_s(1 + l\widetilde{\lambda_s} q) = u_s(1-y)$ divides u_{p^r} modulo $u^2_{p^r}$. Namely, again take a positive integer m such that $m(r-t) \ge n - r$. Then

$$u_s(1-y)\widetilde{\lambda_s} q (1 + y + \cdots + y^m) = u_{p^r}(1 - y^{m+1}) = u_{p^r} - u_{p^r} y^{m+1}.$$

But $u_{p^r} y^{m+1}$ is a multiple of $q^{a+1} u_{p^r}$, which is congruent to 0 modulo $u^2_{p^r}$.

Write the quotient as

$$z = \widetilde{\lambda_s} q (1 + y + \cdots + y^m).$$

Then $z u_s u_{p^r} = (1 + y + \cdots + y^m) u^2_{p^r}$, so modulo $u^2_{p^r}$,

$$z(u_s \widetilde{+}_F [l] u_{p^r}) \equiv x\,(u_s(1-y) + u_s u_{p^r}\text{-term}) \equiv u_{p^r}.$$

This is enough to make u_{p^r} divisible by $u_s \widetilde{+}_F [l] u_{p^r}$ in S_r. Now we can do long division in S_r to divide u_{p^t} by $u_s \widetilde{+}_F [l] u_{p^r}$: start with $u_s \widetilde{\lambda_s} = u_{p^t}$, and after the first step, the remainder is a multiple of u_{p^r}. □

Comment. If x is any representative of the quotient from Lemma 3.6, then $x + y q_{[t,n]}$ is another representative of the quotient for any y. Examining the proof of the lemma for $r = n-1$, for $\alpha = s + lp^{n-1} = \gamma p^t$, we see there is an x satisfying $x(u_s \widetilde{+}_F [l] u_{p^r}) = x_{p^t}$ with x congruent to an integer modulo u_{p^t}. This integer must be a representative of γ^{-1} in $\mathbb{Z}/p^{n-t}$. Adjusting by an appropriate multiple of $q_{[t,n]}$, we get another representative x_α of the quotient, which is congruent to $\alpha_{[n]}^{-1}$ modulo u_{p^t}.

Now consider the completion map

$$\phi_u : S_{n-1} \to (S_{n-1})^\wedge_u = MU_*[[u]]/[p^n]u.$$

In both cases of α, $\phi_u(x_\alpha)$ satisfies

$$([\alpha]u)\phi_u(x_\alpha) = u_{p^t}$$

and $\phi_u(x_\alpha) \equiv \alpha_{[n]}^{-1}$ modulo u. The only element of $MU_*[[u]]/([p^n]u)$ satisfying this is

$$\frac{[\alpha_{[n]}^{-1}]([\alpha]u)}{[\alpha]u} = \frac{[p^t + k_\alpha^{[n]}p^n]u}{[\alpha]u}$$

(where the quotient is taken in the obvious manner), so this must be $\phi_u(x_\alpha)$. Also, the completion map ϕ_u is injective as S_{n-1} contains no element that is infinitely divisible by u, so x_α is unique in S_{n-1}.

Now we can describe the right vertical map ϕ of (3.39). Write $\alpha = lp^{n-1} + s$, with $0 \le l \le p-1$ and $0 \le s \le p^{n-1} - 1$. In the case of $G = \mathbb{Z}/p$, $\phi(b_i^{(\alpha)})$ is calculated in Lemma 2.14 of [10]. Examining the proof of that lemma, one sees that it carries through for $G = \mathbb{Z}/p^n$ as well. Namely, for the map going all the way down the left hand side of (3.30)

$$(MU_{\mathbb{Z}/p^n})_* \to F(E(\mathbb{Z}/p^n)_+, MU_{\mathbb{Z}/p^n})_* = MU_*[[u]]/[p^n]u,$$

$b_i^{(\alpha)}$ maps to $\mathrm{Coeff}_{x^i}(x +_F [\alpha]u)$. So we have

$$\phi(b_i^{(\alpha)}) = \mathrm{Coeff}_{x^i}\left(\sum_{k\ge 0} b_k^{(s)}(x +_F [l]u_{p^{n-1}})^k\right), \tag{3.40}$$

as this is the only element of S_{n-1} that maps to $\mathrm{Coeff}_{x^i}(x +_F [\alpha]u)$. In particular, for $1 \le \alpha \le p^{n-1} - 1$, we get $\phi(b_i^{(\alpha)}) = b_{i,0}^{(\alpha)} \in \overline{(MU_{\mathbb{Z}/p^{n-1}})_*}$.

It remains to describe ϕ on the u_α^{-1}, which is forced by the map on u_α. For $\alpha = lp^{n-1}$ where $1 \le l \le p-1$, (3.40) gives

$$\phi(u_{lp^{n-1}}) = [l]u_{p^{n-1}}.$$

Write l^{-1} to be the smallest positive representation of the inverse of l in $\mathbb{Z}/p$. Then $[l^{-1}]([l]u_{p^{n-1}}) = u_{p^{n-1}}$ in S_{n-1}, so we must have

$$\begin{aligned}\phi(u_\alpha^{-1}) &= u_{p^{n-1}}^{-1} \cdot \frac{[l^{-1}]([l]u_{p^{n-1}})}{[l]u_{p^{n-1}}}\\ &= u_{p^{n-1}}^{-1}\sum_{j\ge 0} a_{0,j+1}^{(l^{-1})}([l]u_{p^{n-1}})^j.\end{aligned} \tag{3.41}$$

For $p^{n-1} \nmid \alpha$ (with again $\alpha = \gamma p^t$), since u_{p^t} is a factor of $u_{p^{n-1}}$, it also becomes a unit once we invert $u_{p^{n-1}}$. If $1 \le \$u_\alpha \le p^{n-1} - 1$, u_α is also a unit in $S_{n-1}[u_{p^{n-1}}^{-1}]$ since it divides u_{p^t}. Since we have $\phi(u_\alpha) = u_\alpha$, the inverse must map to u_α^{-1} in $S_{n-1}[u_{p^{n-1}}^{-1}]$, which is $xq_{[t,n-1]}u_{p^{n-1}}^{-1}$ for any choice of x in S_{n-1} such that $xu_\alpha = u_{p^t}$.

Similarly, if $p^{n-1} \le \alpha$, write $\alpha = lp^{n-1} + s$, $1 \le s \le p^{n-1} - 1$. We have

$$\phi(u_\alpha) = u_s \widetilde{+}_F [l]u_{p^{n-1}}.$$

By Lemma 3.6, this is a factor of u_{p^t}, so it is a unit in $S_{n-1}[u_{p^{n-1}}^{-1}]$. Then $\phi(u_\alpha^{-1})$ must map to $((u_s \widetilde{+}_F [l] u_{p^{n-1}})^{-1}$ in $S_{n-1}[u_{p^{n-1}}^{-1}]$.

We now show that the algebraic diagram (3.39) is also a pullback. Since it is the coefficient of the right half of a Tate diagram, by a standard result in homological algebra (see e.g. [10, Lemma 2.1]), it being a pullback is equivalent to the maps ι and ϕ being jointly onto as maps of abelian groups. Consider a monomial $x = u_{p^{n-1}}^{-m} f(u_{p^{n-1}})$, where $m > 0$ and $f(u_{p^{n-1}}) \in S_{n-1}$ is a series in $u_{p^{n-1}}$. Then x can be written as $\overline{x} + u_{p^{n-1}}^{-m} \overline{f}(u_{p^{n-1}})$, where $\overline{x} \in Im(\iota)$, and $\overline{f}(u_{p^{n-1}})$ has only finitely many terms:

$$\overline{f}(u_{p^{n-1}}) = \sum_{l=0}^{m-1} z_l u_{p^{n-1}}^j,$$

where each z_l is a polynomial in $b_{i,j}^{(\alpha)}$, $q_{[r],j}$ and $\widetilde{\lambda_\alpha}$ (with $1 \le \alpha \le p^{n-1} - 1$). However, in $S_{n-1}[u_{p^{n-1}}^{-1}]$, all u_α are inverted, so $\widetilde{\lambda_\alpha} = u_\alpha^{-1} u_{p^t}$,

$$b_{i,j}^{(\alpha)} = u_{p^r}^{-1} \left(b_{i,j-1}^{(\alpha)} - p_{i,j-1}^{(l)}(b_k^{(s)}) \right)$$

and

$$q_{[r],j} = u_{p^r}^{-1} \left(q_{[r],j-1} - c_j \right).$$

So inductively, all z_l can be written as a polynomial in $MU_*[b_i^{(\alpha)}, u_\alpha^{-1}]$, so $u_{p^{n-1}}^{-m} \overline{f} \in Im(\phi)$. Thus, $Im(\iota) + Im(\phi) = S_{n-1}[u_{p^{n-1}}^{-1}]$.

In the staircase diagram (3.30), every horizontal rectangle is the right half of a Tate diagram, so a similar argument can be applied to show that it is also a pullback on coefficients. So on coefficients, (3.30) is a pullback as well.

The proof of Theorem 3.1 follows from the next lemmas. Let R be the ring given in the theorem.

Lemma 3.7. *We have*

$$R[u_{p^{n-1}}^{-1}] \cong MU_*[b_i^{(\alpha)}, u_\alpha^{-1}], \tag{3.42}$$

where $i \ge 0$, $1 \le \alpha \le p^{n-1}$, and $u_\alpha = b_0^{(\alpha)}$.

Lemma 3.8. *We have*

$$R^\wedge_{u_{p^{n-1}}} \cong S_{n-1}. \tag{3.43}$$

In particular, Lemma 3.8 also gives that

$$R^\wedge_{u_{p^r}} \cong (S_{n-1})^\wedge_r \cong S_r.$$

Lemma 3.9 gives that the element u_{p^r} has bounded torsion in R.

Lemma 3.9. *For $m = 0, \ldots, n-1$,*

$$\bigcup_{k \ge 0} Ann(u_{p^m}^k) = Ann(u_{p^m}) = (q_{[m,n]})$$

in R.

Given Lemmas 3.7 and 3.8, the upper right, lower left, and lower right corners of diagram 3.9 become

$$\begin{array}{ccc} & & R[u_{p^{n-1}}^{-1}] \\ & & \downarrow \\ R^{\wedge}_{u_{p^{n-1}}} & \longrightarrow & R^{\wedge}_{u_{p^{n-1}}}[u_{p^{n-1}}^{-1}] \end{array}$$

By Lemma 3.9 and Proposition 2.4, the pullback of this is R, which gives Theorem 3.1.

The remainder of this section is devoted to the proofs of Lemmas 3.7, 3.8, and 3.9.

Proof of Lemma 3.7. By relations (3.8) and (3.9) of Theorem 3.1, as well as (3.21), for all $1 \le \alpha \le p^n - 1$, u_α is a factor of $u_{p^{n-1}}$, so u_α is invertible in $R[u_{p^{n-1}}^{-1}]$.

As before, we write $\alpha = lp^r + s$ with $1 \le l \le p-1$ and $0 \le s \le p^r - 1$. We also write $\alpha = \gamma p^t$ with $(\gamma, \alpha) = 1$. Define a map

$$\kappa : R \to MU_*[b_i^{(\alpha)}, u_\alpha^{-1}]$$

as follows:

$$\begin{aligned} b_{i,j}^{(\alpha)} &\mapsto u_{p^r}^{-j} b_i^{(\alpha)} - \sum_{k=1}^{j} p_{i,j-k}^{(l)} u_{p^r}^{-k}, \\ \lambda_\alpha &\mapsto u_\alpha^{-1} u_{p^t}, \\ q_{[r],j} &\mapsto u_{p^r}^{-j} u_{p^{r+1}} - \sum_{k=0}^{j-1} c_k u_{p^r}^{k-j}. \end{aligned}$$

Recall that $p_{i,j-k}^{(l)}$ are the polynomials defined before Theorem 3.1, whose variables are $b_m^{(s)}$ with $m \ge 0$.

For $j = 0$, we have

$$b_{i,0}^{(\alpha)} \mapsto b_i^{(\alpha)},$$

so the notation is consistent. It is routine to check that κ is consistent with relations (3.7), (3.8), (3.9), and (3.10) in R. For relation (3.6), note that for $\alpha = p^r$, we have $l = 1$ and $s = 0$, and $b_m^{(s)} = b_m^{(0)} = 1$ for $m = 1$, and 0 for all other m. Hence, the series (3.2) becomes

$$\sum_{m \ge 0} b_m^{(0)} (x +_F y)^m = x +_F y = \sum_{i,j \ge 0} a_{i,j} x^i y^j,$$

so the polynomials $p_{i,j}^{(0)} = a_{i,j}$. In particular, $p_{0,j}^{(0)} = 1$ for $j = 1$, and 0 for all other values of j. So

$$b_{0,1}^{(p^r)} \mapsto u_{p^r}^{-1} u_{p^r} - p_{0,0}^{(0)} = 1 - 0 = 1.$$

For $j \ge 2$, we get

$$b_{0,j}^{(p^r)} \mapsto u_{p^r}^{-j} u_{p^r} - p_{0,1}^{(0)} u_{p^r}^{1-j} = 0.$$

Hence, κ is consistent with relation (3.6).

For relation (3.11), we claim that for $\alpha = \gamma t$, $(\gamma, p) = 1$, we have

$$\kappa(\underline{b_{0,1}^{(\alpha)}}) = u_{p^t}^{-1} u_\alpha. \qquad (3.44)$$

Note that by (3.2), we have

$$p^{(l)}_{0,0}(b^{(s)}_m) = b^{(s)}_0 = u_s$$

for all s. This gives

$$\kappa(b^{(\alpha)}_{0,1}) = u^{-1}_{p^r} u_\alpha - u^{-1}_{p^r} u_s \tag{3.45}$$

for all α. Hence, for the case $\alpha = lp^r$, $u_s = 0$, so (3.44) holds.

For general α, we prove (3.44) by induction. Suppose that (3.44) holds for all numbers less than α. We write $\alpha = lp^r + s$ with $s \neq 0$. Note that the highest power of p that divides α and s is the same, and we denote this power by p^t. By the induction hypothesis, $\kappa(\underline{b^{(s)}_{0,1}}) = u^{-1}_{p^t} u_s$. Also, we have

$$\kappa(q_{[r],1}) = u^{-1}_{p^r} u_{p^{r+1}}.$$

So we get

$$\begin{aligned} \underline{b^{(\alpha)}_{0,1}} &= b^{(\alpha)}_{0,1} q_{[t],1} \cdots q_{[r-1],1} + \underline{b^{(s)}_{0,1}} \\ &\mapsto (u^{-1}_{p^r} u_\alpha - u^{-1}_{p^r} u_s)(u^{-1}_{p^t} u_{p^{t+1}}) \cdots (u^{-1}_{p^{r-1}} u_{p^r}) + \kappa(\underline{b^{(s)}_{0,1}}) \\ &= u^{-1}_{p^t} u_\alpha - u^{-1}_{p^t} u_s + u^{-1}_{p^t} u_s \\ &= u^{-1}_{p^t} u_\alpha. \end{aligned}$$

Hence, (3.44) is proven. This gives that

$$\kappa(\lambda_\alpha \underline{b^{(\alpha)}_{0,1}}) = 1.$$

However, $\kappa(q_{[n-1],1}) = 0$, so the right hand side of relation (3.11) also has

$$1 + k_\alpha q_{[r],1} \cdots q_{[n-1],1} \mapsto 1.$$

Thus, relation (3.11) is respected by κ. Thus, κ is well-defined, and induces a map

$$\overline{\kappa} : R[u^{-1}_{p^{n-1}}] \to (MU_{\mathbb{Z}/p^{n-1}})_*[b^{(\alpha)}_i, u^{-1}_\alpha].$$

We have an obvious map

$$\mu : MU_*[b^{(\alpha)}_i, u^{-1}_\alpha] \to R[u^{-1}_{p^{n-1}}].$$

Namely,

$$\mu(b^{(\alpha)}_i) = b^{(\alpha)}_{i,0}$$

and

$$\mu(u^{-1}_\alpha) = \lambda_\alpha q_{[t],1} \cdots q_{[n-2],1} \cdot u^{-1}_{p^{n-1}}.$$

It is straightforward to check that μ is the inverse to $\overline{\kappa}$. □

Proof of Lemma 3.8. We define a map

$$\phi : R^\wedge_{u_{p^{n-1}}} \to S_{n-1}$$

similarly as the right vertical map ϕ of (3.39). We start by describing ϕ on the elements $b^{(\alpha)}_{i,j}$. For $1 \leq \alpha \leq p^{n-1} - 1$, let

$$\phi(b^{(\alpha)}_{i,j}) = b^{(\alpha)}_{i,j}$$

in $\overline{(MU_{\mathbb{Z}/p^{n-1}})_*}$. For $p^{n-1} \le \alpha \le p^n - 1$, write $\alpha = s + lp^{n-1}$, where $1 \le l \le p-1$ and $0 \le s \le p^{n-1} - 1$. Then

$$\phi(b_{i,j}^{(\alpha)}) = \mathrm{Coeff}_{x^i}\left(\frac{(\sum_{k\ge 0} b_k^{(s)}(x +_F [l]u_{p^{n-1}})^k)_{|u_{p^{n-1}}|\ge j}}{(u_{p^{n-1}})^j}\right).$$

For the elements $q_{[r],j}$, let

$$\phi(q_{[r],j}) = q_{[r],j}$$

for $0 \le j \le n-2$, and

$$\phi(q_{[n-1],j}) = \frac{([p]u_{p^{n-1}})_{|u_{p^{n-1}}|\ge j}}{(u_{p^{n-1}})^j}.$$

It remains to define $\phi(\lambda_\alpha)$. If $\alpha = lp^{n-1}$, let

$$\phi(\lambda_\alpha) = \frac{[1 + k_\alpha^{[n]}p]u_{p^{n-1}}}{[l]u_{p^{n-1}}} = \frac{[\alpha_{[n]}^{-1}]([l]u_{p^{n-1}})}{[l]u_{p^{n-1}}},$$

where the division is done in the obvious manner.

For $p^{n-1} \nmid \alpha$, if $1 \le \alpha \le p^{n-1}$, define

$$\phi(\lambda_\alpha) = \widetilde{\lambda_\alpha}.$$

For $p^{n-1} + 1 \le \alpha \le p^n - 1$, define

$$\phi(\lambda_\alpha) = x_\alpha,$$

where x_α is the element of S_{n-1} defined in the comment after Lemma 3.6.

For all $b_{i,j}^{(\alpha)}$, $q_{[r],j}$, $\lambda_{lp^{n-1}}$ and λ_α for $1 \le \alpha \le p^{n-1}-1$, it is immediate to see that their images under ϕ satisfy the necessary relations. For $\alpha > p^{n-1}$ with $p^{n-1} \nmid \alpha$, $x_\alpha \equiv \alpha_{[n]}^{-1}$ modulo u_{p^t}, so it satisfies relation (3.12). For relation (3.11), we have

$$(x_\alpha \underline{b_{0,1}^{(\alpha)}})u_{p^t} = x_\alpha u_\alpha = u_{p^t},$$

so $x_\alpha \underline{b_{0,1}^{(\alpha)}} - 1$ is in the annihilator of u_{p^t}. By construction, the annihilator of $u_{p^{n-1}}$ in S_{n-1} is generated by $q_{[n-1,n]} = ([p]u_{p^{n-1}})/u_{p^{n-1}}$. In $S_{n-1}/(u_{p^{n-1}}) \cong (MU_{\mathbb{Z}/p^{n-1}})_*$, the annihilator of u is $(q_{[t,n-1]})$ by the induction hypothesis, so by a standard argument, the annihilator of u in S_{n-1} is $(q_{[t,n]})$. So

$$x_\alpha \underline{b_{0,1}^{(\alpha)}} = z q_{[t,n]} \tag{3.46}$$

for some z. Also, $S_{n-1}/(u) \cong MU_*$. Write $\overline{z} \in MU_*$ as the class of z modulo u_{p^t}. Modulo u_{p^t}, (3.46) becomes

$$\alpha_{[n]}^{-1}\gamma - 1 = \overline{z}p^{n-t}.$$

Hence $\overline{z} = k_\alpha^{[n]}$ since MU_* is an integral domain. So in S_{n-1},

$$x_\alpha \underline{b_{0,1}^{(\alpha)}} = 1 + z q_{[t,n]} = 1 + k_\alpha^{[n]} q_{[t,n]}.$$

This shows that ϕ is a consistent map.

For the inverse map, define

$$\psi : \overline{(MU_{\mathbb{Z}/p^{n-1}})_*}^{\wedge}_{u_{p^{n-1}}} \to R^{\wedge}_{u_{p^{n-1}}}$$

which maps $u_{p^{n-1}}$ to $u_{p^{n-1}}$. For the generators of $\overline{(MU_{\mathbb{Z}/p^{n-1}})_*}$, $\widehat{\psi}$ takes $b_{i,j}^{(\alpha)}$ and $q_{[r],j}$ to the same elements in $R^{\wedge}_{p^{n-1}}$, and $\widetilde{\lambda_\alpha}$ to λ_α. All relations in $\overline{(MU_{\mathbb{Z}/p^{n-1}})_*}^{\wedge}_{u_{p^{n-1}}}$, as well as (3.34) and (3.35), are straightforward to check.

Clearly, $b_{i,j}^{(\alpha)}$ for $1 \le \alpha \le p^{n-1}-1$ and $q_{[r],j}$ for $0 \le r \le n-2$ are in the image of ψ. Similarly as in the $\mathbb{Z}/p$-case, we have that

$$q_{[n-1],j} = \sum_{k\ge 0} c_{j+k} u^k_{p^{n-1}}$$

in $R^{\wedge}_{u_{p^{n-1}}}$, so ψ of this series goes to $q_{[n-1],j}$. In particular, $\psi([p]u_{p^{n-1}}) = q_{[n-1],0} = 0$. So we can replace the source of ψ by S_{n-1}, and $\phi \circ \psi$ is the identity map.

Next, check that $b_{i,j}^{(\alpha)}$ with $p^{n-1} \le \alpha \le p^n - 1$ is in the image of ψ. We write $\alpha = s + lp^{n-1}$ as usual, where $0 \le s \le p^{n-1}-1$, $1 \le l \le p-1$. Then by relation (3.7) of Theorem 3.1, we again have that in $R^{\wedge}_{u_{p^{n-1}}}$,

$$b_{i,j}^{(\alpha)} = \mathrm{Coeff}_{x^i}\left(\frac{(\sum b_k^{(s)}(x +_F [l]u_{p^{n-1}})^k)_{|u_{p^{n-1}}|\ge j}}{(u_{p^{n-1}})^j}\right),$$

so this series maps to $b_{i,j}^{(\alpha)}$.

It only remains to check that λ_α is in the image of ψ for $p^{n-1} \le \alpha \le p^n - 1$. If $\alpha = lp^{n-1}$, this is by an argument similar to that of the $\mathbb{Z}/p$-case. For $p^{n-1} \nmid \alpha$, by Part (3) of Lemma 3.3, λ_α is congruent in R to an element $a_0^{(\alpha)}$ that is in the image of ψ modulo $u_{p^{n-1}}$. So by relations (3.7) and (3.9), every generator of R, and hence every element of R, is congruent modulo $u_{p^{n-1}}$ in R to an element that is in the image of ψ. Write $\lambda_\alpha = a_0^{(\alpha)} + e_1^{(\alpha)} u_{p^{n-1}}$ for some $e_1^{(\alpha)} \in R$. Then $e_1^{(\alpha)} \cong a_1^{(\alpha)}$ modulo $u_{p^{n-1}}$ for some $a_1^{(\alpha)} \in Im(\psi)$, and we can write

$$\lambda_\alpha = a_0^{(\alpha)} + a_1^{(\alpha)} u_{p^{n-1}} + e_2^{(\alpha)} u_{p^{n-1}}$$

for some $e_2^{(\alpha)} \in R$. Continuing this process, we get that in the completion,

$$\lambda_\alpha = \sum_{j\ge 0} a_j^{(\alpha)} u^j_{p^{n-1}},$$

where $a_j^{(\alpha)}$ are all in $Im(\psi)$, so the corresponding series in S_{n-1} map to λ_α. This concludes the proof that ψ is onto. □

Note that by the induction hypothesis, we have that the vertical maps on the left edge of diagram (3.39) give isomorphism

$$(S_{n-1})^{\wedge}_{u_{p^r}} \simeq S_r$$

for all $0 \le r \le n-2$. Thus, we also have that

$$R^{\wedge}_{u_{p^r}} \simeq S_r.$$

It remains to show that the element $u_{p^{n-1}}$ has bounded torsion in R.

Proof of Lemma 3.9. We have

$$u_{p^m} q_{[m,n]} = 0.$$

We will show that u_{p^m} is regular in $R/(q_{[m,n]})$. In $R/(q_{[r,n]})$, we have $\lambda_\alpha \underline{b_{0,1}^{(\alpha)}} = 1$ for every α with $p^{m+1} \nmid \alpha$. Let R_k be the subring of $R/(q_{[m,n]})$ generated by λ_α, $b_{i,j}^{(\alpha)}$ and $q_{[s],j}$ for $j \leq k$. We define for $k \geq 2$

$$A_k = MU_*[b_{i,k}^{(\beta)}, q_{[s],k}, \lambda_\alpha \mid (i,\beta) \neq (0, p^s), \alpha = p^{m+1}\gamma][u].$$

We define two sets of polynomials $g_t(u)$, $\overline{g}_t(u)$ in A_k for $t = 0, \dots, n-1$ (which will correspond respectively to u_{p^t} and $q_{[t],1}$). Namely,

$$\begin{aligned} g_0(u) &= u, \\ \overline{g}_t(u) &= (g_t(u))^{k-1} q_{[t],k} + \sum_{l=0}^{k-2} c_{l+1}(g_t(u))^l, \\ g_{t+1}(u) &= \overline{g}_t(u) g_t(u). \end{aligned}$$

Now for $1 \leq \beta \leq p^n - 1$, β not a power of p, we have $\beta = \gamma p^t$ for some t, and $\gamma \in (\mathbb{Z}/p)^\times \setminus \{1\}$. Define the element

$$h_\beta(u) = (h_t(u))^{k-1} b_{0,k}^{(\beta)} + \sum_{l=0}^{k-2} a_{0,l+1}^{\gamma} (\overline{g}_t(u))^l.$$

These are the elements that correspond to $\underline{b_{0,1}^{(\beta)}}$. We do need to impose the relations that for $\alpha = p^t\gamma$, $s \geq m+1$ and $(\gamma, p) = 1$,

$$\lambda_\alpha h_\alpha(u) = 1 + k_\alpha g_t(u) \cdots g_{n-1}(u). \tag{3.47}$$

Let B_k be the quotient of A_k by the relations (3.47). Define a map of MU_*-algebras

$$\eta_k : B_k[(h_\beta(u))^{-1}] \to R_k$$

given by $\eta_k(b_{i,k}^{(\beta)}) = b_{i,k}^{(\beta)}$, $\eta_k(q_{[t],k}) = q_{[t],k}$, $\eta_k(u) = u$. For $p^{m+1} \nmid \beta$, we have $\eta_k((h_\beta(u))^{-1}) = \lambda_\beta = (\underline{b_{0,1}^{(\beta)}})^{-1}$. For $p^{m+1} | \beta$, $\eta_k(\lambda_\beta) = \lambda_\beta$. By induction on j (similarly as in Lemma 2.7, we have for all $j \leq k$

$$q_{[t],j} = u_{p^t}^{k-j} q_{[t],k} + \sum_{l=0}^{k-j-1} c_{j+l} u_{p^t}^l$$

in R. Now apply induction on t to get

$$\begin{aligned} \eta_k(\overline{g}_t(u)) &= q_{[t],1}, \\ \eta_k(g_t(u)) &= u_{p^t} \end{aligned}$$

and

$$\eta_k\left(g_t(u)^{k-j} q_{[t],k} + \sum_{l=0}^{k-1} c_{j+l}(g_t(u))^l \right) = q_{[t],j}. \tag{3.48}$$

For all $(i,\beta) \neq (0, p^s)$, we write $\beta = s + lp^r$ as usual, with $0 \leq s \leq p^r - 1$, and $1 \leq l \leq p-1$. Then we also have by induction on $j \leq k$

$$b_{i,j}^{(\beta)} = u_{p^r}^{k-j} b_{i,k}^{(\beta)} + \sum_{d=0}^{k-j-1} p_{i,j+d}^{(l)} u_{p^r}^d$$

in R, where $p^{(l)}_{i,j+d}$ are polynomials defined in (3.3) before the statement of Theorem 3.1. Hence,

$$\eta_k\left((g_r(u))^{k-j}b^{(\beta)}_{i,k}+\sum_{d=0}^{k-j-1}p^{(l)}_{i,j+d}\cdot(g_s(u))^d\right)=b^{(\beta)}_{i,j}. \tag{3.49}$$

By the above, η_k is onto. We define

$$\overline{g}(u)=\overline{g}_r(u)\cdots\overline{g}_{n-1}(u).$$

Then $\eta_k(\overline{g}(u))=q_{[r],1}\cdots q_{[n-1],1}=0$ in R_k, so we get a surjective map

$$\eta_k: B_k[(h_\beta(u))^{-1}]/(\overline{g}(u))\to R_k. \tag{3.50}$$

We can define a map

$$\pi_k: R_k\to B_k[(h_\beta(u))^{-1}]/(g(u))$$

similarly as in Lemma 2.7. Namely, π_k takes $q_{[t],j}$ to the left hand side of (3.48), u_{p^t} to $g_t(u)$, and $b^{(\beta)}_{i,j}$ to the left hand side of (3.49). It is straightforward to show that π_k is inverse to η_k, so η_k is an isomorphism.

We will show that $g_m(u)$ is regular in $B_k[(h_\beta(u))^{-1}]/(\overline{g}(u))$. The polynomial $g_m(u)$ is regular in $A_k[(h_\beta(u))^{-1}]$. The relation (3.47) does not affect this, so $g_m(u)$ is regular in $B_k[(h_\beta(u))^{-1}]$. Note that $\overline{g}_m(u)$ is a polynomial in $g_m(u)$, with the constant term p. Thus, $g_{m+1}(u)=\overline{g}_m(u)g_m(u)$ is a polynomial in $g_m(u)$ with no constant term. By induction, it is easy to see that for every $t\geq m$, $\overline{g}_t(u)$ is a polynomial in $g_m(u)$, with the constant term p. Hence, $\overline{g}_n(u)=f(g_m(u))$, where $f(x)$ is a polynomial with the constant term p^{n-m}, which is not a zero divisor in MU_*. Suppose that $g_m(u)$ is a zero-divisor modulo $f(g_m(u))$, then for some polynomials $k(u)$, $m(u)$

$$g_m(u)k(u)=f(g_m(u))m(u) \tag{3.51}$$

in $B_k[(h_\beta(u))^{-1}]$. By the constant term of f, we get that $g_m(u)$ divides $p^{n-m}m(u)$, hence it divides $m(u)$. Substituting $m(u)=g_m(u)\overline{m}(u)$ in (3.51) and canceling $g_m(u)$, we get that $k(u)$ is a multiple of $\overline{g}_n(u)$, so it is 0 in the quotient ring.

Therefore, u_{p^m} is regular in R_k for each k. As $R/(q_{[m,n]})=\operatorname{colim}_k R_k$, we get u_{p^m} is regular in $R/(q_{[m,n]})$. The statement of the lemma follows. □

We can now give explicit descriptions of the terms in the staircase diagram (3.30).

Proposition 3.10. *We have*

$$(E_r)_*=S_r[u^{-1}_{p^{r-1}}] \tag{3.52}$$

and

$$(F_r)_*=S_r[u^{-1}_{p^r}]=(E_r)_*[u^{-1}_{p^r}]. \tag{3.53}$$

Note that for all $1\leq\beta\leq p^n-1$. $p^r\nmid\beta$, $\widetilde{u}_\beta$ is also inverted in (3.52), since as we showed above, $\widetilde{u}_\beta$ divides $u_{p^{r-1}}$. Also, since S_{r-1} is complete in u_{p^r}, the map $\phi: R\to S_{r-1}$ induces a map $R^\wedge_{u_{p^r}}\simeq S_r\to S_{r-1}$. The vertical maps $(E_r)_*\to(F_{r-1})_*$ are these maps with $u_{p^{r-1}}$ inverted.

4. Special calculations

In this section, we show that at the bottom of the staircase diagram (3.24), there are in fact different algebraic characterizations of the terms F_0 and E_1. This allows one to compare (3.24) with the staircase diagram for $MU_{\mathbb{Z}/p^n}$ described in [1]. In particular, for $MU_{\mathbb{Z}/p^2}$, the two staircase diagrams turn out to be actually the same. However, this is not true for general $MU_{\mathbb{Z}/p^n}$. For the rest of this section, we assume $n \geq 2$.

We begin by briefly recalling the staircase diagram of [1], where it was shown that $(MU_{\mathbb{Z}/p^n})_*$ is the pullback of the coefficients of the diagram of spectra

$$\begin{array}{ccccccccc}
 & & & & & & & & C_n \\
 & & & & & & & & \downarrow{\scriptstyle \phi_n} \\
 & & & & & & C_{n-1} & \xrightarrow{\iota_{n-1}} & D_{n-1} \\
 & & & & & & \downarrow{\scriptstyle \phi_{n-1}} & & \\
 & & & & \cdots & \longrightarrow & D_{n-2} & & \\
 & & & & \downarrow & & & & \\
 & & C_1 & \xrightarrow{\iota_1} & D_1 & & & & \\
 & & \downarrow{\scriptstyle \phi_1} & & & & & & \\
C_0 & \xrightarrow{\iota_0} & D_0 & & & & & &
\end{array} \tag{4.1}$$

where

$$C_r = F(E\mathbb{Z}/p^{n-r}_+, \widetilde{E\mathcal{F}[\mathbb{Z}/p^r]} \wedge MU_{\mathbb{Z}/p^n})$$

and

$$D_r = F(E\mathbb{Z}/p^{n-r-1}_+, \widetilde{E\mathcal{F}[\mathbb{Z}/p^{r+1}]} \wedge F(E\mathbb{Z}/p^{n-r}_+, \widetilde{E\mathcal{F}[\mathbb{Z}/p^r]} \wedge MU_{\mathbb{Z}/p^n}).$$

From [1] and also [9], we have

$$(C_r)_* = MU_*[b_i^{(\alpha)}, u_\alpha^{-1} \mid 1 \leq \alpha \leq p^r - 1, i \geq 0][[u_{p^r}]]/[p^{n-r}]u_{p^r}$$

(where $u_\alpha = b_0^{(\alpha)}$ as usual) and

$$(D_r)_* = ((C_r)_*[u_{p^r}^{-1}])^\wedge_{[p]u_{p^r}}.$$

It is easy to see that there is a map from (3.24) to (4.1). In general, this map is not an isomorphism, except for $E_0 \to C_0$, $F_0 \to D_0$, $E_1 \to C_1$, and $E_n \to C_n$. For E_0 and E_n, one immediately sees that the corresponding spectra in the two staircase diagram are the same. We prove the cases of E_1 and F_0 here, which happen due to connectivity reasons.

Lemma 4.1. *We have*

$$\begin{aligned}
(E_1)_* &= \left((\overline{(MU_{\mathbb{Z}/p})_*})^\wedge_{u_p}/[p^{n-1}]u_p \right)[u_1^{-1}] \\
&\cong \left(MU_*[b_i^{(\alpha)}, u_\alpha^{-1} \mid 1 \leq \alpha \leq p-1] \right)^\wedge_{u_p}/[p^{n-1}]u_p = (C_1)_*.
\end{aligned} \tag{4.2}$$

Proof. It is easy to see that

$$\overline{(MU_{\mathbb{Z}/p})_*}[u_1^{-1}] = MU_*[u_\alpha^{-1}, b_i^{(\alpha)}]$$

with $1 \leq \alpha \leq p-1$, $i \geq 0$ (with $u_\alpha = b_0^{(\alpha)}$, so the right hand side of (4.2) is

$$\left(\overline{(MU_{\mathbb{Z}/p})_*}[u_1^{-1}]\right)^\wedge_{u_p} /[p^{n-1}]u_p. \tag{4.3}$$

It is straightforward to see that on coefficients, the map of spectra $E_1 \to C_1$ is the obvious map

$$\left((\overline{(MU_{\mathbb{Z}/p})_*})^\wedge_{u_p}/[p^{n-1}]u_p\right)[u_1^{-1}] \to \left(\overline{(MU_{\mathbb{Z}/p})_*}[u_1^{-1}]\right)^\wedge_{u_p}/[p^{n-1}]u_p. \tag{4.4}$$

We need to show that (4.4) is an isomorphism.

By Lemma 2.7, the annihilator of u_1 is generated by q_1 in $(MU_{\mathbb{Z}/p})_*$. In $\overline{(MU_{\mathbb{Z}/p})_*}$, we have $u_1 q_1 = u_p$ instead of 0, so u_1 is a regular element of $\overline{(MU_{\mathbb{Z}/p})_*}$, so the map $\overline{(MU_{\mathbb{Z}/p})_*} \to \overline{(MU_{\mathbb{Z}/p})_*}[u_1^{-1}]$ is injective. It is straightforward to check that for every m, the induced map $\overline{(MU_{\mathbb{Z}/p})_*}/(u_p^m) \to \overline{(MU_{\mathbb{Z}/p})_*}[u_1^{-1}]/(u_p^m)$ is injective, passing to an injective map on the completions $(\overline{(MU_{\mathbb{Z}/p})_*})^\wedge_{u_p} \to (\overline{(MU_{\mathbb{Z}/p})_*}[u_1^{-1}])^\wedge_{u_p}$. Now it is again straightforward to check that the preimage of the ideal generated by $[p^{n-1}]u_p$ is again $([p^{n-1}]u_p$, so we get that

$$\left((\overline{(MU_{\mathbb{Z}/p})_*})^\wedge_{u_p}\right)/[p^{n-1}]u_p \to \left((\overline{(MU_{\mathbb{Z}/p})_*}[u_1^{-1}])^\wedge_{u_p}\right)/[p^{n-1}]u_p$$

is injective. This shows that (4.4) is injective.

To show that (4.4) is onto, we will show that every homogeneous series in (4.3) can be rewritten with only a finite negative power of u_1, putting it in the left hand side of (4.2). Recall that in the grading on $\overline{(MU_{\mathbb{Z}/p})_*}$, we have $|b_{i,j}^{(\alpha)}| = 2(i+j-1)$, $|q_j| = |q_{[0],j}| = 2(j-1)$, and $|\lambda_\alpha| = 0$. Hence, the only elements with negative degree are u_α and $u_p = q_0$. For any element in (4.3), rewrite all occurrences of u_α as $b_{0,1}^{(\alpha)} u_1$. So the element can be written as

$$\sum_{j \geq 0} x_j u_1^{-k(j)} u_p^j \tag{4.5}$$

in (4.3), where $x_j \in \overline{(MU_{\mathbb{Z}/p})_*}$ has non-negative degree. Assume without loss of generality that $k(j) > 0$, then

$$|x_j u_1^{-k(j)} u_p^j| = |x_j| + 2k(j) - 2j|$$

is equal to the degree of the homogeneous series. Hence, $j - k(j)$ is bounded below. If $j - k(j) \geq -N$ for some natural number N, by factoring out u_1^{-N}, we may assume without loss of generality that $j - k(j) \geq 0$. In $\overline{(MU_{\mathbb{Z}/p})_*}[u_1^{-1}]$, we also have $q_1 = u_1^{-1} u_p$. Thus, (4.5) can be rewritten as

$$\sum_{j \geq 0} x_j q_1^{k(j)} u_p^{j-k(j)},$$

where $j - k(j) \geq 0$. In turn, rewrite this as

$$\sum_{s,t \geq 0} x_{s,t} q_1^s u_p^t, \tag{4.6}$$

where $x_{s,t} \in \overline{(MU_{\mathbb{Z}/p})_*}$ with non-negative degree. We will show that this can be rewritten so that it exists in $\left((\overline{(MU_{\mathbb{Z}/p})_*})^\wedge_{u_p}/[p^{n-1}]u_p\right)[u_1^{-1}]$.

Note that from $[p^{n-1}]u_p = 0$, we get $[p^{n-1}]u_p \equiv 0$ modulo u_p^2 in $\left(\overline{(MU_{\mathbb{Z}/p})_*}\right)^\wedge_{u_p}/[p^{n-1}]u_p$. Dividing once by u_1, we get $p^{n-1}q_1 \equiv 0$ modulo u_pq_1 and hence also modulo u_p. In fact, $p^{n-1}q_1 = u_py$ for some y in $\left(\overline{(MU_{\mathbb{Z}/p})_*}\right)^\wedge_{u_p}/[p^{n-1}]u_p$. Also, $q_1 \equiv 0$ modulo u_1, so $q_1^2 \equiv pq_1$ modulo u_p in $\overline{(MU_{\mathbb{Z}/p})_*}$. Hence, we get

$$q_1^{2n-2} \equiv p^{n-1}q_1^{n-1} \equiv 0 \quad \text{modulo } u_p.$$

Hence,

$$q_1^{(2n-2)m} \equiv 0 \quad \text{modulo } u_p^m.$$

From the above,

$$q_1^{(2n-2)m} = y_{(2n-2)m}u_p^m + \text{terms with } u_p^{m+1},$$

where $y_{(2n-2)m} \in \overline{(MU_{\mathbb{Z}/p})_*}$, and the higher terms are in

$$\left(\overline{(MU_{\mathbb{Z}/p})_*}\right)^\wedge_{u_p}/[p^{n-1}]u_p.$$

Thus, we get

$$x_{s,t}q_1^su_p^t = x_{s,t}y_{(2n-2)m}u_p^{t+\lfloor s/(2n-2)\rfloor} + \text{terms with } u_p^{t+\lfloor s/(2n-2)\rfloor+1}.$$

Again, the higher terms are in $\left(\overline{(MU_{\mathbb{Z}/p})_*}\right)^\wedge_{u_p}/[p^{n-1}]u_p$. From this, one sees that (4.6) exists in $\left((\overline{(MU_{\mathbb{Z}/p})_*})^\wedge_{u_p}/[p^{n-1}]u_p\right)[u_1^{-1}]$. Hence, (4.4) is onto. □

For the "Tate" terms of the staircases, we also have that in general, $(F_r)_* \not\cong (D_r)_*$, except for the case $r = 0$. This is shown using connectivity, similarly as for $(E_1)_*$.

Lemma 4.2. *We have*

$$(4.7)\quad (F_0)_* = (MU_*[[u_1]]/[p^n]u_1)[u_1^{-1}] \cong \left((MU_*[[u_1]]/[p^n]u_1)[u_1^{-1}]\right)^\wedge_{[p]u_1} = (D_0)_*.$$

Proof. On coefficients, the map $F_0 \to D_0$ is just the completion map

$$(4.8)\qquad (MU_*[[u_1]]/[p^n]u_1)[u_1^{-1}] \to \left((MU_*[[u_1]]/[p^n]u_1)[u_1^{-1}]\right)^\wedge_{[p]u_1}.$$

Similarly as in Lemma 4.1, it is straightforward to check that (4.8) is injective. To show that it is onto, a generic homogeneous series from the right hand side is of the form

$$\sum_{j\geq 0} f_j(u_1)([p]u_1)^j,$$

where $f_j(u_1)$ is a Laurent series in u_1. If the $f_j(u_1)$ have no negative powers of u_1, this can clearly be summed, so without loss of generality, we can assume each $f_j(u_1)$ is in fact a polynomial in u_1^{-1}. Rearranging terms, the homogeneous series can be written as

$$(4.9)\qquad \sum_{s,t\geq 0} x_{s,t}u_1^{-s}([p]u_1)^t,$$

where $x_{s,t} \in MU_*$ (and hence has non-negative degree). Again, the degree of each non-zero term is $|x_{s,t}|+2s-2t$, which is bounded above, giving that $t-s$ is bounded below. By factoring out some u_1^{-N}, we can assume without loss of generality that $t-s \geq 0$.

We still have the element

$$q_1 = \sum c_{j+1} u_1^j = ([p]u_1)/u_1 \in MU_*[[u_1]]/[p^n]u_1$$

with $q_1 \equiv p$ modulo u_1 in this ring, and $q_1 u_1 = [p]u_1$. This allows us to rewrite (4.9) as

$$\sum_{s,t-s\geq 0} x_{s,t} q_1^s ([p]u_1)^{t-s}. \tag{4.10}$$

By the same arguments as in Lemma 4.1, we still have that

$$q_1^{(2n-2)m} \equiv 0 \quad \text{modulo } ([p]u)^m$$

in $(MU_*[[u_1]]/[p^n]u_1)[u_1^{-1}]$. In fact, we have

$$q_1^{(2n-2)m} = y_{(2n-2)m}([p]u_1)^m + \text{terms with } ([p]u_1)^{m+1},$$

where $y_{(2n-2)m} \in MU_*$, and the higher terms are in $MU_*[[u_1]]/[p^n]u_1$. From this, we can sum (4.10) in $(MU_*[[u_1]]/[p^n]u_1)[u_1^{-1}]$, giving that (4.8) is onto. □

Now consider the case of $MU_{\mathbb{Z}/p^2}$. Here, we have shown that in the two five term staircases, the terms in the positions of E_0, F_0, E_1 and E_2 all coincide. We have

$$(F_1)_* = (E_1)_*[u_p^{-1}] = \left(MU_*[b_i^{(\alpha)}, u_\alpha^{-1}][[u_p]]/[p]u_p\right)[u_p^{-1}]$$

and

$$(D_1)_* = ((C_1)_*[u_p^{-1}])^\wedge_{[p]u_p} = \left(\left(MU_*[b_i^{(\alpha)}, u_\alpha^{-1}][[u_p]]/[p]u_p\right)[u_p^{-1}]\right)^\wedge_{[p]u_p}.$$

Clearly, the last completion does nothing as $[p]u_p = 0$, so $(F_1)_*$ and $(D_1)_*$ also coincide. This shows that for $MU_{\mathbb{Z}/p^2}$, diagram (3.24) coincides with that from [1]. However, this turns out to be false for general $MU_{\mathbb{Z}/p^n}$. Further explanation of the relationship between the staircase diagram from this paper and from [1] is given in [9].

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