Integral Control for Nonlinear Systems Subject to Time-Varying Perturbations with Unknown Magnitudes

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***Abstract*—This paper focuses on the design of linear integral controllers for uncertain nonlinear systems subjected to various types of time-varying perturbations. We propose a method called the feedback domination method to address this problem. The time-varying perturbations can include constant step disturbances, exogenous time-varying disturbances with unknown magnitudes, and modeling uncertainties with unknown system parameters. The approach involves constructing a new linear integral controller with integral dynamics to drive the states of uncertain systems without nonlinear terms to the origin asymptotically Additionally, a high-gain technique is introduced to design a controller that can asymptotically drive the states of uncertain nonlinear systems to the origin. Finally, the paper introduces a new stability criterion and extends the results to a class of uncertain nonlinear systems with more general time-varying perturbations of unknown magnitudes.**

***Index Terms*—nonlinear system, integral controller, time-varying system, unknown magnitude**

I. Introduction

This paper considers the following nonlinear system

$$\left\{\begin{array}{l}\dot{x}_i=x_{i+1}+f_i\left(x_1, \ldots, x_i\right), \ i=1, \ldots, n-1, \\ \dot{x}_n=u+f_n\left(x_1, \ldots, x_n\right)+d\left(t, x\right),\end{array}\right. \tag{1}$$

where $x=(x_1,\ldots,x_n)^T \in \mathbb{R}^n$ and $u \in \mathbb{R}$ are the system states and control input, respectively. In addition, $f_i(\cdot),\ i=1,\ldots,n$ are system nonlinearities and $d\left(t,x\right)$ represents the unknown perturbation. System (1) is widely used to describe dynamics of numerous practical systems in practice [1], [2]. For example, a pendulum system is described by

$$\left\{\begin{array}{l}\dot{x}_1=x_2, \\ \dot{x}_2=\frac{1}{J}u-\frac{MgL}{2J}\sin(x_1)+d(t,x),\end{array}\right.$$

where x_1 is the angle of oscillation, x_2 is the angular velocity, M, g, L and J are system parameters, and $d(t,x)$ represents various uncertainties. In this paper, we are interested in solving the problem of global regulation control for system (1) in the presence of uncertainties. More specifically, we aim to design controllers under which the states of system (1) are globally bounded and converge to the origin [3], [4]. When $d(t,x) = 0$, many important results have been obtained [1], [5], [6], [7], [8]. The nonlinear system (1) with known differentiable non-linearity f_i's can be globally stabilized by the backstepping approach [9]. In the case when f_i's are unknown satisfying a linear growth condition, a domination approach was proposed in [10] to obtain a linear controller. In the case when f_i's only satisfy a Hölder growth condition, a design methodology called adding a power integrator was proposed in work [11] to globally stabilize the nonlinear system (1). In the presence of non-vanishing uncertainties, i.e., $d(t,0) \neq 0$, the aforementioned results cannot guarantee that all states of the nonlinear system (1) converge to the origin. When f_i is linear $i=1,\ldots n$, and $d(t,x)=\theta$ for an unknown constant θ, system (1) can be regulated by the commonly-used PID controller [12]

$$u(t)=-k_0\int_0^t x_1(s)ds-k_1x_1(t)-\cdots-k_nx_n(t), \tag{2}$$

where positive gains $k_0,\ldots,k_n$ are coefficients of Hurwitz polynomial $s^{n+1}+k_ns^n+k_{n-1}s^{n-1}+\cdots+k_1s+k_0$ [13], [14], [15], [16]. The corrective term $\int_0^t x_1(s)ds$ can counteract the effect of a constant disturbance but also will cause a trade-off between stability and convergence rate [12], i.e., a larger k_0 desired for a faster convergence rate may cause instability. Moreover, the controller (2) cannot handle exogenous time-varying disturbances, e.g., $d(t,x)=c(1+0.5\sin(t))$ with an unknown magnitude c, which drive the states $x_i,\ i=1,\ldots,n$ away from origin. The performance of the PID controller will be even worse when $d(t,x)$ is an internal modeling uncertainty such as $d(t,x)=\theta(1+x_2^2)$ with unknown parameter θ.

In this paper, a novel integral controller consisting of an integral dynamic is proposed to solve the global regulation problem of system (1) with various non-vanishing uncertainties[17]. First, an extra integral state in the control law is introduced to tackle $d(t,x)$. By revamping the technique of adding a power integrator [11], a linear control law with a linear corrective term is constructed to handle the various forms of uncertainties. The proposed integral controller will globally regulate the n-dimensional system in the presence of not-precisely-known non-linearities and non-vanishing uncertainties.

II. Preliminaries

In order to get our main results, the following lemma is first introduced.

Lemma 1: Consider the time-varying nonlinear system

$$\dot{x} = f(t,x), \tag{3}$$

where $f(t,x) : \mathbb{R}_{\geq 0} \times \mathcal{D} \to \mathbb{R}^n$ is continuous with respect to x in an open neighborhood $\mathcal{D} \subseteq \mathbb{R}^n$ and piecewise continuous with respect to t, $f(t,0) = 0,\ t \in \mathbb{R}_{\geq 0}$ and the initial state is $x(0) = x_0 \in \mathcal{D}$. If there is a continuously differentiable, positive definite function $V(x) : \mathcal{D} \to \mathbb{R}_{\geq 0}$ for system (3) such that

$$\dot{V}(x) \leq -\mu(t)V, \tag{4}$$

where $\mu(t)$ is piecewise continuous (containing removable discontinuities and jump discontinuities) satisfying

$$\lim_{t\to+\infty} \int_0^t \mu(s)ds = +\infty. \tag{5}$$

Then, the trajectories of system (3) are locally convergent to zero. Furthermore, if $\mathcal{D} = \mathbb{R}^n$, the trajectories of system (3) are also globally convergent to zero.

Proof. Firstly, integrating both sides of inequality (4) from 0 to t, we can easily have

$$V(t) \leq V(x_0)e^{-\int_0^t \mu(s)ds}.$$

From the condition (5), we can get that system (3) is globally convergent to zero.

Remark 1: Lemma 1 extends the existing Lyapunov conditions of asymptotically stability since it will degenerate to the results when $\mu(t),\ t \in \mathbb{R}_{\geq 0}$ being a positive constant, i.e., there exists a positive constant c such that $\mu(t) \geq c,\ t \in \mathbb{R}_{\geq 0}$. However, the criterion can allow $\mu(t),\ t \in \mathbb{R}_{\geq 0}$ to be equal to zero and even negative at some time points, which makes the positive constant c not exist. We introduce the following example to further elaborate Lemma 1 and Remark 1.

Example 1: Consider a scalar time-varying nonlinear system

$$\dot{x}(t) = -\mu(t)x(t),\ t \in \mathbb{R}_{\geq 0}, \tag{6}$$

where $x(0) = x_0 > 0$ and the following two cases are analyzed based on system (6).

Case 1. When $\mu(t) = (1 + \cos t)$ is a continuous function with respect to t. Firstly, from the definition of $\mu(t)$, we get that $\mu(t)$ is equal to zero in $T_1 = \{t | t = \pi + 2k\pi,\ t \in \mathbb{R}_{\geq 0},\ k \in \mathbb{Z}_{\geq 0}\}$ and positive in $t \in \mathbb{R}_{\geq 0}/T_1$. Then, we can easily obtain that $\lim_{t\to+\infty} \int_0^t \mu(s)ds = +\infty$. Thus, we can get that the trajectories of system (6) are globally convergent to zero.

Case 2. When $\mu(t) = \left(\frac{1}{2} + \sin t\right)$ is a continuous function with respect to t. Firstly, we get that $\mu(t)$ is a negative function in $T_2 = \left\{t | t \in \left(\frac{7}{6}\pi + 2k\pi, \frac{11}{6}\pi + 2k\pi\right),\ t \in \mathbb{R}_{\geq 0},\ k \in \mathbb{Z}_{\geq 0}\right\}$. Then, integrating both sides of system (6) from 0 to t, we have $\int_0^t \mu(s)ds = \left(\frac{1}{2}t - \cos t + 1\right)$ is equal to zero at $t = 0$, positive when $t > 0$ and converges to $+\infty$ as $t \to +\infty$, which indicates that the trajectories of system (6) are globally convergent to zero.

III. Main Results

In this paper, a new integral controller will be constructed for system (1). Firstly, we will modify the technique of adding an integrator to design a novel regulator for a class of chain systems, and then apply the regulator to regulate system (1).

A. Global Regulation of Chain Systems

Theorem 1: For the following system

$$\begin{cases} \dot{z}_i = z_{i+1},\ i = 1, \ldots, n, \\ \dot{z}_{n+1} = u, \end{cases} \tag{7}$$

there are positive constants k^* and $a_i,\ i = 1, \ldots, n$ such that for any $K(t,\ z) \geq k^*$, the linear controller

$$u = -K(t,z)(a_1 z_1 + \cdots + a_n z_n + z_{n+1}) \tag{8}$$

can globally asymptotically stabilize system (7).

Proof. Based on adding a power integrator, we propose a constructive method to design a controller (8) and Lyapunov functions in a recursive manner.

Step 1: First, construct

$$V_1 = \frac{1}{2}z_1^2.$$

The time derivative of V_1 along system (7) is

$$\dot{V}_1|_{(7)} = z_1 z_2 = \xi_1 z_2^* + \xi_1(z_2 - z_2^*) \tag{9}$$

with $\xi_1 := z_1$. For (9), selecting the virtual controller $z_2^* = -\beta_1\xi_1 = -(n+1)\xi_1$ yields

$$\dot{V}_1|_{(7)} = -(n+1)\xi_1^2 + \xi_1(z_2 - z_2^*). \tag{10}$$

Inductive Step: Suppose at step $k-1$, there is a C^1 Lyapunov function $V_{k-1} : \mathbb{R}^{k-1} \to \mathbb{R}$, which is positive definite and proper, and a set of C^1 virtual controllers $z_1^*, \ldots, z_k^*$, defined by

$$\begin{array}{ll} z_1^* = 0, & \xi_1 = z_1 - z_1^*, \\ z_2^* = -\beta_1\xi_1, & \xi_2 = z_2 - z_2^*, \\ \vdots & \vdots \\ z_k^* = -\beta_{k-1}\xi_{k-1}, & \xi_k = z_k - z_k^* \end{array} \tag{11}$$

with positive constants $\beta_1, \ldots, \beta_{k-1}$ such that

$$\begin{aligned}\dot{V}_{k-1}|_{(7)} \leq & -(n-k+3)\left(\xi_1^2+\cdots+\xi_{k-1}^2\right) \\ & +\xi_{k-1}\left(z_k-z_k^*\right).\end{aligned} \tag{12}$$

It is clear that (12) reduces to the inequality (10) when $k=2$ under the definitions of (11). Next, we will prove (12) also holds at step k. To prove this, we choose

$$W_k=\frac{1}{2}(z_k-z_k^*)^2. \tag{13}$$

Therefore, for $V_k=V_{k-1}+W_k$, the time derivative of the Lyapunov function V_k along system (7) is

$$\begin{aligned}\dot{V}_k|_{(7)} \leq & -(n-k+3)\left(\xi_1^2+\cdots+\xi_{k-1}^2\right) \\ & +\xi_{k-1}\left(z_k-z_k^*\right)+\xi_k z_{k+1}^*+\textstyle\sum_{l=1}^{k-1}\frac{\partial W_k}{\partial z_l}\dot{z}_l \\ & +\xi_k\left(z_{k+1}-z_{k+1}^*\right).\end{aligned} \tag{14}$$

Now we give the estimate of the terms $\xi_{k-1}\left(z_k-z_k^*\right)$ and $\sum_{l=1}^{k-1}\frac{\partial W_k}{\partial z_l}\dot{z}_l$ in (14).

Firstly, from (11) and by means of the Young inequality, we have

$$\xi_{k-1}\left(z_k-z_k^*\right)=\xi_{k-1}\xi_k \leq \frac{1}{2}\xi_{k-1}^2+\frac{1}{2}\xi_k^2. \tag{15}$$

Next, from (11), and by means of (13), we have

$$\sum_{l=1}^{k-1}\frac{\partial W_k}{\partial z_l}\dot{z}_l \leq \frac{1}{2}\left(\xi_1^2+\cdots+\xi_{k-1}^2\right)+c_k\xi_k^2, \tag{16}$$

where c_k is a positive constant. Substituting (15) and (16) into (14) yields

$$\begin{aligned}\dot{V}_k|_{(7)} \leq & -(n-k+2)\left(\xi_1^2+\cdots+\xi_{k-1}^2\right) \\ & +\xi_k z_{k+1}^*+\left(c_k+\tfrac{1}{2}\right)\xi_k^2 \\ & +\left(z_k-z_k^*\right)\left(z_{k+1}-z_{k+1}^*\right).\end{aligned} \tag{17}$$

Now construct the virtual controller

$$z_{k+1}^*=-\beta_k\xi_k=-\left(c_k+\tfrac{1}{2}+n-k+2\right)\xi_k \tag{18}$$

and substituting (18) into (17), we have

$$\begin{aligned}\dot{V}_k|_{(7)} \leq & -(n-k+2)\left(\xi_1^2+\cdots+\xi_{k-1}^2+\xi_k^2\right) \\ & +\left(z_k-z_k^*\right)\left(z_{k+1}-z_{k+1}^*\right).\end{aligned}$$

This completes the inductive proof.

Last Step: The inductive argument shows that (12) holds for $k=n+1$ with a set of virtual controllers (11). Based on the inductive argument, we can choose the $(n+1)$th Lyapunov function

$$V_{n+1}=V_n+W_{n+1}=V_n+\tfrac{1}{2}\left(z_{n+1}-z_{n+1}^*\right)^2.$$

The time derivative of V_{n+1} along system (7) is

$$\begin{aligned}\dot{V}_{n+1}|_{(7)} \leq & -2\left(\xi_1^2+\cdots+\xi_n^2\right)+\xi_{n+1}u \\ & +\xi_n\xi_{n+1}+\left(z_{n+1}-z_{n+1}^*\right)\left(-\dot{z}_{n+1}^*\right).\end{aligned} \tag{19}$$

Similar to the estimate of the terms (15) and (16), we have

$$\xi_n\xi_{n+1} \leq \frac{1}{2}\xi_n^2+\frac{1}{2}\xi_{n+1}^2 \tag{20}$$

and

$$\begin{aligned}\left(z_{n+1}-z_{n+1}^*\right)\left(-\dot{z}_{n+1}^*\right) \leq & \tfrac{1}{2}\left(\xi_1^2+\cdots+\xi_n^2\right) \\ & +c_{n+1}\xi_{n+1}^2.\end{aligned} \tag{21}$$

Substituting (20) and (21) into (19), we have

$$\begin{aligned}\dot{V}_{n+1}|_{(7)} \leq & -\left(\xi_1^2+\cdots+\xi_n^2\right)+\left(c_{n+1}+\tfrac{1}{2}\right)\xi_{n+1}^2 \\ & +\xi_{n+1}u.\end{aligned} \tag{22}$$

By the adding a power integrator technique, we can simply choose the following controller

$$u=-K(t,z)\xi_{n+1}, \tag{23}$$

where $K(t,z) \geq k^*=c_{n+1}+\frac{3}{2}$. Substituting (23) into (22), we have

$$\dot{V}_{n+1}|_{(7)} \leq -\left(\xi_1^2+\cdots+\xi_n^2+\xi_{n+1}^2\right). \tag{24}$$

Thus, we have achieved that system (7) is globally asymptotically stable under controller (23), that is

$$\begin{aligned}u &= -K(t,z)\xi_{n+1} \\ &= -K(t,z)(z_{n+1}+\beta_n(z_n+\cdots+\beta_2(z_2+\beta_1 z_1)\cdots)),\end{aligned}$$

where $a_i=\beta_i\cdots\beta_n,\ i=1,\ldots,n$. This completes our proof.

Now it is time to present our main results for global regulating system (1). First, we utilize Theorem 1 to solve the regulation problem of chain system (1) when $f_i(\cdot)=0,\ i=1,\ldots,n$, i.e,

$$\left\{\begin{array}{l}\dot{x}_i=x_{i+1},\ i=1,\ldots,n-1, \\ \dot{x}_n=u+d(t,x).\end{array}\right. \tag{25}$$

To solve the problem, we assume that the time-varying perturbations with unknown magnitudes $d(t,x)$ satisfies the following assumption. The assumptions are based on most common unknown disturbance in control system.

Assumption 1: Assume there are an unknown constant θ and a known function $\alpha(t,x) \geq 1$ such that

$$d(t,x)=\theta\alpha(t,x).$$

Remark 2: It should be noted that Assumption 1 comes from our previous work [18]. The uncertain function $d(t,x)$ satisfying Assumption 1 encompasses several types of uncertainties in system (25). First, it includes constant disturbances as its special case when $\alpha(t,x)=1$. For exogenous time-varying disturbance such as $d(t,x)=c(1+0.5\sin t)$ with unknown magnitude c, we can simply choose $\alpha(t,x)=2(1+0.5\sin t) \geq 1$ and $\theta=c/2$. Moreover, $d(t,x)$ can include internal modeling uncertainties such as $d(t,x)=\theta(1+x_2^2)$ with unknown θ.

Theorem 2: Under Assumption 1, there are positive constants k^* and $a_i,\ i=1,\ldots,n$, such that the following integral controller

$$\begin{cases} u=-k^*\alpha(t,x)(a_1x_0+a_2x_1+\cdots \\ \qquad +a_nx_{n-1}+x_n), \\ \dot{x}_0=x_1 \end{cases} \tag{26}$$

globally regulates system (25).

Proof. Define $z_1=x_0-\frac{\theta}{k^*a_1}$, $z_i=x_{i-1}, i=2,\ldots,n+1$. Under the new coordinates, it is clear that the closed-loop system (25) and (26) can be rewritten as

$$\begin{aligned} \dot{z} &= \begin{bmatrix} z_2 \\ \vdots \\ z_{n+1} \\ -k^*\alpha(t,x)(a_1z_1+\cdots+a_nz_n+z_{n+1}) \end{bmatrix} \\ &= F(t,z). \end{aligned} \tag{27}$$

By the proof of Theorem 1, we can find positive constants k^* and $a_i,\ i=1,\ldots,n$ such that for $K(t,z)=k^*\alpha(t,x)\geq k^*$, system (27) is globally regulated. In the case when $d(t,x)=\theta$ is an unknown constant, Theorem 2 holds for a controller with constant gains.

Corollary 1: The system (25) with $d(t,x)=\theta$ being an unknown constant θ can be globally regulated by the integral controller

$$\begin{cases} u=-k^*(a_1x_0+a_2x_1+\cdots+a_nx_{n-1}+x_n), \\ \dot{x}_0=x_1 \end{cases}$$

for appropriate positive constants k^* and $a_i,\ i=1,\ldots,n$.

B. Global Regulation of Nonlinear Systems

This section considers the global regulation problem of system (1) when system nonlinearities satisfy the following assumption.

Assumption 2: For $i=1,\ldots,n$, there is a known constant c such that

$$|f_i(x_1,\ldots,x_i)|\leq c(|x_1|+\cdots+|x_i|).$$

Theorem 3: Under Assumptions 1 and 2, there are positive constants k^* and $a_1,\ldots,a_n$ such that for a large enough constant $L\geq 1$, the following integral controller

$$\begin{cases} u=-L^nk^*\alpha(t,x)\left(a_1x_0+a_2x_1+a_3\frac{x_2}{L}\right. \\ \qquad \left.+\cdots+a_n\frac{x_{n-1}}{L^{n-2}}+\frac{x_n}{L^{n-1}}\right), \\ \dot{x}_0=Lx_1 \end{cases} \tag{28}$$

with $\alpha(t,x)$ defined in Assumption 1 solves the global regulation problem of system (1).

Proof. Define $z_1=x_0-\frac{\theta}{k^*a_1L^n}$, $z_i=\frac{x_{i-1}}{L^{i-2}},\ i=2,\ldots,n+1$. By choosing the same constants k^* and $a_i,\ i=1,\ldots,n$ as in Theorem 2, the closed-loop system (1) and (28) under the new coordinates can be rewritten as

$$\begin{cases} \dot{z}_1=Lz_2, \\ \dot{z}_2=Lz_3+f_1(x_1), \\ \dot{z}_3=Lz_4+\frac{f_2(x_1,x_2)}{L}, \\ \quad\vdots \\ \dot{z}_n=Lz_{n+1}+\frac{f_{n-1}(x_1,\ldots,x_{n-1})}{L^{n-2}}, \\ \dot{z}_{n+1}=-Lk^*\alpha(t,x)(a_1z_1+\cdots+a_nz_n+z_{n+1}) \\ \qquad +\frac{f_n(x_1,\ldots,x_n)}{L^{n-1}}, \end{cases}$$

which can be further rewritten as the following matrix form

$$\dot{z}=LF(t,z)+\varPhi, \tag{29}$$

where $F(t,z)$ is the same as the one in (27) and $\varPhi=\left(0,f_1(x_1),\frac{f_2(x_1,x_2)}{L},\ldots,\frac{f_n(x_1,\ldots,x_n)}{L^{n-1}}\right)^T$.

By using the same Lyapunov function V_{n+1} constructed in Theorem 1, we can see that the time derivative of V_{n+1} along system (29) is

$$\begin{aligned} &\dot{V}_{n+1}|_{(29)} \\ &=\frac{\partial V_{n+1}}{\partial z}(LF(t,z)+\varPhi) \\ &\leq -L\textstyle\sum_{l=1}^{n+1}\xi_l^2+\sum_{i=2}^{n+1}\sum_{l=i}^{n+1}\frac{\partial V_l}{\partial z_i}\frac{f_{i-1}(x_1,\ldots,x_{i-1})}{L^{i-2}}. \end{aligned} \tag{30}$$

By Assumption 2 and the fact $L\geq 1$, we have

$$\left|\frac{f_{i-1}(\cdot)}{L^{i-2}}\right|\leq\frac{c(|x_1|+\cdots+|x_{i-1}|)}{L^{i-2}}\leq\tilde{c}(|\xi_1|+\cdots+|\xi_i|) \tag{31}$$

for a positive constant $\tilde{c}$. In addition, from the definition of system (15), we have

$$\textstyle\sum_{l=i}^{n+1}\frac{\partial V_l}{\partial z_i}\leq\bar{c}\sum_{l=i}^{n+1}|\xi_i| \tag{32}$$

for a positive constant $\bar{c}$. Then, by substituting (31) and (32) into (30), we have

$$\dot{V}_{n+1}|_{(29)}\leq -L\textstyle\sum_{l=1}^{n+1}\xi_l^2+\hat{c}\sum_{l=1}^{n+1}\xi_l^2$$

for a positive constant $\hat{c}$.

Selecting a large enough $L\geq\hat{c}+1$, we can get a relation same as (24). As a result, the integral controller (28) can regulate system (1) under Assumptions 1 and 2.

C. Extension

We reconsider system (1) where system vanishing uncertainties $f_i(x_1,\ldots,x_i),\ i=1,2,\ldots,n$ satisfy Assumption 2 and the non-vanishing uncertainty $d(t,x)$ satisfies the following assumption.

Assumption 3: Assume the non-vanishing uncertainty $d(t,x)$ satisfies

$$d(t,x)=\theta\alpha(t,x),$$

where θ is an unknown constant and $\alpha(t,x)$ is a known continuous function. Moreover, $\alpha(t,x)$ also satisfies $\alpha(t,x)\geq\bar{\alpha}(t)\geq 0$ where the continuous function $\bar{\alpha}(t)$ is a periodic function with $T>0$ being the period.

Remark 3: Assumption 3 is significantly different from Assumption 1, and brings great difficulties to the stability analysis of system (1). Specifically speaking, due to the condition $\alpha(t,x) \geq 1$ in Assumption 1, there must be a constant k^* in Theorem 2 such that $k^*\alpha(t,x) > c_{n+1} + \frac{1}{2}$ for all $(t,x) \in \mathbb{R}_{\geq 0} \times \mathbb{R}^n$. However, since $\alpha(t,x) \geq \bar{\alpha}(t) \geq 0$ holds in Assumption 3 and for all $(t,x) \in \mathbb{R}_{\geq 0} \times \mathbb{R}^n$, we cannot choose the constant k^*, such that $k^*\alpha(t,x) > c_{n+1} + \frac{1}{2}$, which brings great challenges to the stability analysis of the closed-loop system.

Based on Assumptions 2 and 3, we can have the following result.

Theorem 4: Under Assumptions 2 and 3, there are positive constants k^* and $a_1, \ldots, a_n$ such that for a large enough constant $L \geq 1$, the following integral controller

$$\begin{cases} u = -L^n k^* \alpha(t,x)\big(a_1 x_0 + a_2 x_1 + \cdots + a_n \frac{x_{n-1}}{L^{n-2}} \\ \qquad + \frac{x_n}{L^{n-1}}\big), \\ \dot{x}_0 = L x_1 \end{cases} \tag{33}$$

with $\alpha(t,x)$ defined in Assumption 3 solves the global regulation problem of system (1).

Proof. Following the proof of Theorems 1, 2 and 3, we can easily obtain

$$\dot{V}_{n+1}|_{(29)} \leq -L\sum_{l=1}^{n} \xi_l^2 + C\xi_{n+l}^2 + \xi_{n+1} u, \tag{34}$$

where C is a positive constant independent of L. From (33), we construct the controller

$$u = -L^2 k^* \alpha(t,x)\xi_{n+1} \tag{35}$$

and substituting (35) into (34), we have

$$\dot{V}_{n+1}|_{(29)} \leq -L\sum_{l=1}^{n} \xi_l^2 + C\xi_{n+1}^2 - L^2 k^* \alpha(t,x)\xi_{n+1}^2, \tag{36}$$

and by means of Assumption 3 and $L \geq 1$, we further obtain

$$\dot{V}_{n+1}|_{(29)} \leq -L\sum_{l=1}^{n} \xi_l^2 - (Lk^*\bar{\alpha}(t) - C)\xi_{n+l}^2. \tag{37}$$

Then, from Assumption 3, we have $\bar{\alpha}(t) \geq 0$. If $\bar{\alpha}(t) > 0$ for $t \in [0,T]$, and following the continuous of $\bar{\alpha}(t)$ and the proof of Theorem 3, we can easily prove Theorem 4.

If there exists a time $t^* \in [0,T]$ such that $\bar{\alpha}(t^*) = 0$ and without loss of generality, we assume that only $t^* \in [0,T]$ exists making $\bar{\alpha}(t^*) = 0$. Firstly, define $T_1 = \{\bar{\alpha}(t) \leq \varepsilon |\ t \in [0,T]\}$, $T_2 = \{\varepsilon < \bar{\alpha}(t) \leq L |\ t \in [0,T]\}$ and $T_3 = \{\bar{\alpha}(t) \geq L |\ t \in [0,T]\}$, where ε is a small positive constant and T_3 can be an empty set. We can get that $t^* \in T_1$ and from (37), we have

$$\dot{V}_{n+1}|_{(29)} \leq \begin{cases} C\sum_{l=1}^{n+1} \xi_l^2, & t \in T_1, \\ -(Lk^*\bar{\alpha}(t) - C)\sum_{l=1}^{n+1} \xi_l^2, & t \in T_2, \\ -L\sum_{l=1}^{n+1} \xi_l^2, & t \in T_3. \end{cases} \tag{38}$$

By means of $\varepsilon < \bar{\alpha}(t) \leq L$ for $t \in T_2$ and from (11), (38) can be further rewritten as

$$\dot{V}_{n+1}|_{(29)} \leq \begin{cases} 2CV_{n+1}, & t \in T_1, \\ -2(Lk^*\varepsilon - C)V_{n+1}, & t \in T_2, \\ -2LV_{n+1}, & t \in T_3. \end{cases} \tag{39}$$

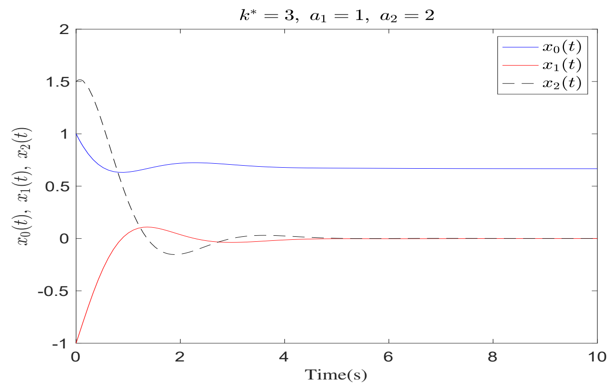

Fig. 1. Trajectories of (25) and (41) with $d(t,x) = 2$

Integrating both sides of inequality (39) from 0 to T, we have

$$V(T) \leq V(0)e^{2\left(\int_{T_1} Cds - \int_{T_2}(Lk^*\varepsilon - C)ds - \int_{T_3} Lds\right)}. \tag{40}$$

From (40), we can get a large $L \geq 1$, such that $Lk^*\varepsilon - C > 0$ and $\int_{T_1} Cds < \int_{T_2}(Lk^*\varepsilon - C)ds$ hold, which indicates $V(T) \leq V(0)$. On the other hand, since $\bar{\alpha}(t)$ is a periodic function with $T > 0$ being the period, we can eventually get $\lim_{t\to+\infty} V(t) = 0$. By Lemma 1, we can achieve the proof of Theorem 4.

IV. EXAMPLES

Example 2: To show the feasibility of the proposed strategy, we first consider the 2-dimension case when the disturbance is a constant in system (25), i.e., $d(t,x) = \theta = 2$. By Corollary 1, an integral controller is constructed as

$$\begin{cases} u = -k^*(a_1 x_0 + a_2 x_1 + x_2), \\ \dot{x}_0 = x_1, \end{cases} \tag{41}$$

where $k^* = 3$, $a_1 = 1$, $a_2 = 2$ and the initial condition is $(x_0(0), x_1(0), x_2(0)) = (1, -1, 1.5)$. From the simulation result shown in Figure 1, we can see that the states x_1 and x_2 will converge to zero asymptotically and x_0 will converge to the constant $\frac{\theta}{a_1 k^*} = \frac{2}{3}$.

When $d(t,x)$ is a time-varying function with an unknown magnitude, for example $d(t,x) = \theta(1 + 0.5\sin(t))$, the controller (41) with constant gain will not be sufficient to drive the output to zero. As a matter of fact, as shown in the simulation in Figure 2 under $\theta = -2$ and the same initial condition, there are oscillations even for a large k^*.

By Theorem 2, we construct an integral controller

$$\begin{cases} u = -k(t,x)(a_1 x_0 + a_2 x_1 + x_2), \\ \dot{x}_0 = x_1 \end{cases} \tag{42}$$

with a time-varying gain $k(t,x) = 6(1 + 0.5\sin(t)) \geq k^* = 3$.

Based on the same initial condition and $\theta = -2$, the state trajectories of closed-loop system (25) and (42) are shown in Figure 3. Clearly the states x_1 and x_2 of the closed-loop system converge to zero asymptotically.

Next, we consider a system with both a vanishing uncertainty and a non-vanishing uncertainty.

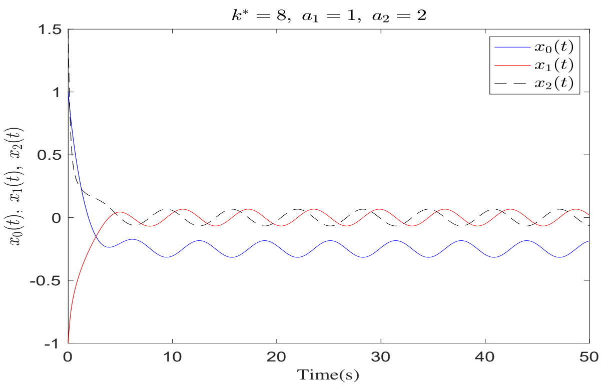

Fig. 2. Trajectories of (25) and (41) with time-varying $d(t, x)$

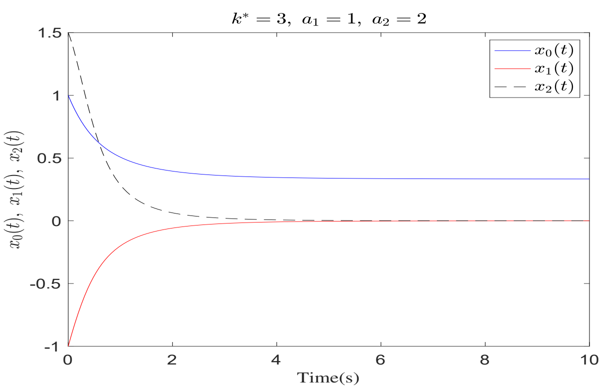

Fig. 3. Trajectories of (25) and (42) with time varying $d(t, x)$

Example 3: Consider the following system inspired by the pendulum dynamic

$$\begin{cases} \dot{x}_1 = x_2, \\ \dot{x}_2 = u + \theta(1 + x_2^2) + \sin(x_1)\delta(t), \end{cases} \tag{43}$$

where θ is an unknown constant and $\delta(t)$ is an unknown disturbance satisfying $|\delta(t)| \leq 1$.
Firstly, we can verify that

$$|\sin(x_1)\delta(t)| \leq |x_1|,$$

which satisfies Assumption 2. By Theorem 3, we can construct the following integral controller

$$\begin{cases} u = -L^2 k^*(1 + x_2^2)(a_1 x_0 + \frac{x_2}{L} + a_2 x_1), \\ \dot{x}_0 = L x_1. \end{cases} \tag{44}$$

In the simulation shown in Figure 4, we chose $\delta(t) = \sin(t)$, $\theta = 3$, $L = 2$, $k^* = 3$, $a_1 = 1$, $a_2 = 2$ and the initial conditions $(x_0(0), x_1(0), x_2(0)) = (1, -1, 1.5)$. Clearly the states x_1 and x_2 of the closed-loop system converge to zero asymptotically.

V. Conclusion

This paper has presented a new method to design an integral controller to regulate the states of a class of uncertain nonlinear systems. Compared to the traditional PID controller, our proposed controller can handle more general uncertainties beyond constant step disturbance, such as external time-varying disturbances with unknown magnitudes and internal modeling uncertainties due to unknown parameters. Moreover, the system states will converge to the origin and the unknown magnitude/parameter can be recovered from the integral state.

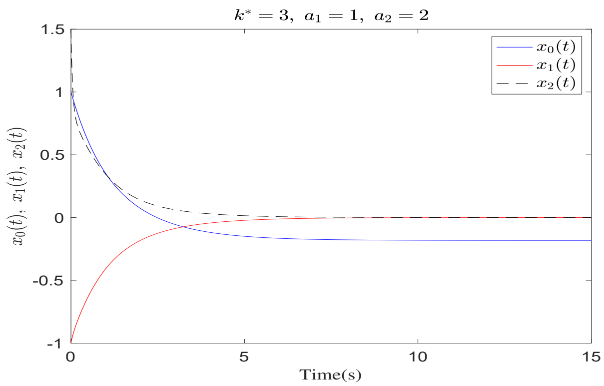

Fig. 4. Trajectories of (43) and (44) with time varying $d(t, x)$

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