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THE EXPECTED EULER CHARACTERISTIC APPROXIMATION TO EXCURSION PROBABILITIES OF GAUSSIAN VECTOR FIELDS

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Let $\{(X(t), Y(s)) : t \in T, s \in S\}$ be an $\mathbb{R}^2$-valued, centered, unit-variance smooth Gaussian vector field, where T and S are compact rectangles in the Euclidean space. It is shown that, as $u \to \infty$, the joint excursion probability $\mathbb{P}\{\sup_{t\in T} X(t) \ge u, \sup_{s\in S} Y(s) \ge u\}$ can be approximated by $\mathbb{E}\{\chi(A_u)\}$, the expected Euler characteristic of the excursion set $A_u = \{(t,s) \in T \times S : X(t) \ge u, Y(s) \ge u\}$, such that the error is super-exponentially small. This verifies the expected Euler characteristic heuristic (cf. Taylor, Takemura and Alder (2005), Alder and Taylor (2007)) for a large class of smooth Gaussian vector fields.

1. Introduction. For a real-valued Gaussian random field $\{Z(t), t \in \mathbb{R}^N\}$ and a compact rectangle $T \subset \mathbb{R}^N$, the study of excursion probability $\mathbb{P}\{\sup_{t\in T} Z(t) \ge u\}$ is a classical and very important problem in both probability and statistics due to its vast applications in many areas such as p-value computations, risk control, and extreme event analysis, etc. Various methods for precise approximations of $\mathbb{P}\{\sup_{t\in T} Z(t) \ge u\}$ have been developed. These include the double sum method, the tube method, the Euler characteristic method, and the Rice method. We refer to the monographs Piterbarg [10], Adler [1], Adler and Taylor [2], Azaïs and Wschebor [4] and the references therein for comprehensive accounts. However, the extreme value theory of multivariate random fields (or random vector fields) is still under-developed and only a few authors have studied the joint excursion probability of multivariate random fields. Piterbarg and Stamatovic [12] and Debicki et al. [7] established large deviation results for the excursion probability in the multivariate case. Anshin [3] obtained precise asymptotics for a special class of nonstationary bivariate Gaussian processes under quite restrictive conditions. Hashorva and Ji [8] and Debicki et al. [6] derived precise asymptotics for the excursion probability of certain multivariate Gaussian processes defined on the real line $\mathbb{R}$ with specific cross dependence structures. Zhou and Xiao [16] studied the excursion probability of a class of nonsmooth bivariate Gaussian random fields by applying the double sum method. Their main results show explicitly that the excursion probabilities of bivariate Gaussian random fields depend not only on the smoothness parameters of the coordinate fields but also on their maximum cross-correlation.

In statistical applications, such joint excursion probabilities are critical tools for constructing simultaneous confidence regions in a continuous-domain approach [13]. In particular, motivated by the expected Euler characteristic (EEC) approximation to excursion probabilities of real-valued Gaussian random fields [2, 14], we prove in this work that the EEC approximation holds in general for the joint excursion probability of Gaussian vector fields.

Let $\{(X(t), Y(s)) : t \in T, s \in S\}$ be an $\mathbb{R}^2$-valued, centered, unit-variance smooth Gaussian vector field, where T and S are compact rectangles in $\mathbb{R}^N$ and $\mathbb{R}^{N'}$ respectively. Let $A_u = \{(t,s) \in T \times S : X(t) \ge u, Y(s) \ge u\}$ be the excursion set where both components X

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and Y exceed the level u. Our main objective is to show that, as $u \to \infty$, the joint excursion probability $\mathbb{P}\{\sup_{t\in T} X(t) \geq u, \sup_{s\in S} Y(s) \geq u\}$ can be approximated by $\mathbb{E}\{\chi(A_u)\}$, the EEC of A_u, such that the error is super-exponentially small; see Theorem 3.1 below for a precise description. This approximation result shows that the maximum correlation between $X(t)$ and $Y(s)$, denoted by R (see (1) below), plays an important role in both $\mathbb{E}\{\chi(A_u)\}$ and the super-exponentially small error. Moreover, as we will see in the proof of Theorem 3.1 (cf. $\mathcal{M}_0$ in (44), $\mathcal{M}_1$ in (56), and $\mathcal{M}_2$ in (63)), the points where R is attained make the major contribution to $\mathbb{E}\{\chi(A_u)\}$. Based on this observation, we also establish two simpler approximations in Corollary 3.1 under the boundary condition (7) on nonzero derivatives of the correlation function over boundary points where R is attained, and in Theorem 3.2 under the condition that there is a unique point attaining R, respectively.

In general, the EEC approximation $\mathbb{E}\{\chi(A_u)\}$ can be expressed by the Kac–Rice formula as an integral; see (6) in Theorem 3.1. In [2, 14], the authors derived a nice expression for $\mathbb{E}\{\chi(A_u)\}$ called the Gaussian kinematic formula, since they assumed that the real-valued Gaussian field has unit variance, which is an important condition to simplify the integration formula of $\mathbb{E}\{\chi(A_u)\}$. However, in our case here, the integration formula of $\mathbb{E}\{\chi(A_u)\}$ (see (6)) mainly depends on the conditional correlation of $X(t)$ and $Y(s)$, which varies over $T \times S$. It turns to be very difficult to get an explicit expression for $\mathbb{E}\{\chi(A_u)\}$. Instead, one can apply the Laplace method to extract the term with the largest order of u from the integral such that the remaining error is $o(1/u)\mathbb{E}\{\chi(A_u)\}$. To explain this, we show several examples of specific calculations in Section 4. For an intuitive understanding of the EEC approximation, we may roughly treat the main term $\mathbb{E}\{\chi(A_u)\}$ as $g(u)e^{-u^2/(1+R)}$ where $g(u)$ is a polynomial in u (by approximating the integral in (6)), and the error term $o(e^{-u^2/(1+R)-\alpha u^2})$ is super-exponentially small compared with $\mathbb{E}\{\chi(A_u)\}$.

The paper is organized as follows. We first introduce the notation and assumptions in Section 2, and then state our main results Theorem 3.1, Corollary 3.1, and Theorem 3.2 in Section 3. We provide in Section 4 several examples on evaluating the EEC and hence approximating the joint excursion probability explicitly. The rest of the paper focuses on the proofs of the main results, which are demonstrated in three steps: (i) sketching the main ideas in Section 5; (ii) studying the super-exponentially small errors between the joint excursion probability and EEC in Sections 6 and 7; and (iii) providing final proofs for the main results in Section 8.

2. Notation and assumptions. Let $\{(X(t), Y(s)) : t \in T, s \in S\}$ be an $\mathbb{R}^2$-valued, centered, unit-variance Gaussian vector field, where T and S are compact rectangles in $\mathbb{R}^N$ and $\mathbb{R}^{N'}$ respectively. Let

$$r(t,s) = \mathbb{E}\{X(t)Y(s)\}, \qquad R = \sup_{t\in T, s\in S} r(t,s). \tag{1}$$

For a function $f(\cdot) \in C^2(\mathbb{R}^n)$ and $t \in \mathbb{R}^n$ with $n \geq 1$, let

$$\begin{aligned} f_i(t) &= \frac{\partial f(t)}{\partial t_i}, \qquad f_{ij}(t) = \frac{\partial^2 f(t)}{\partial t_i \partial t_j}, \quad \forall i, j = 1, \dots, n, \\ \nabla f(t) &= (f_1(t), \dots, f_n(t)), \qquad \nabla^2 f(t) = (f_{ij}(t))_{i,j=1,\dots,n}, \\ \mathrm{vech}(\nabla^2 f(t)) &= (f_{ij}(t), 1 \leq i \leq j \leq n), \\ (f, \nabla f, \mathrm{vech}(\nabla^2 f))(t) &= (f(t), \nabla f(t), \mathrm{vech}(\nabla^2 f(t))), \end{aligned} \tag{2}$$

where "vech" denotes the half-vectorization operator. For a symmetric matrix B, denote by $B \prec 0$ and $B \preceq 0$ if all the eigenvalues of B are negative (i.e., B is negative definite) and nonpositive (i.e., B is negative semidefinite), respectively. For two functions $f(x)$ and $g(x)$, we write $f(x) \sim g(x)$ as $x \to \infty$ if $\lim_{x\to\infty} f(x)/g(x) = 1$.

Denote by $T = \prod_{i=1}^{N}[a_i, b_i]$ and $S = \prod_{i=1}^{N'}[a_i', b_i']$, where $-\infty < a_i < b_i < \infty$ and $-\infty < a_i' < b_i' < \infty$. Following the notation in Adler and Taylor [2], p. 134, we show below that T and S can be decomposed into unions of their interiors and the lower dimension faces. Based on these decompositions, the Euler characteristic of the excursion set A_u can be represented (see Section 3).

A face K of dimension k is defined by fixing a subset $\sigma(K) \subset \{1, \dots, N\}$ of size k and a subset $\varepsilon(K) = \{\varepsilon_j, j \notin \sigma(K)\} \subset \{0, 1\}^{N-k}$ of size $N - k$ so that

$$K = \{t = (t_1, \dots, t_N) \in T : a_j < t_j < b_j \text{if } j \in \sigma(K),$$
$$t_j = (1 - \varepsilon_j)a_j + \varepsilon_j b_j \text{if } j \notin \sigma(K)\}.$$

Denote by $\partial_k T$ the collection of all k-dimensional faces in T. Then the interior of T is denoted by $\overset{\circ}{T} = \partial_N T$ and the boundary of T is given by $\partial T = \bigcup_{k=0}^{N-1} \bigcup_{K\in\partial_k T} K$. For each $t \in K \in \partial_k T$ and $s \in L \in \partial_l S$, let

$$\nabla X_{|K}(t) = (X_{i_1}(t), \dots, X_{i_k}(t))_{i_1,\dots,i_k\in\sigma(K)}, \qquad \nabla^2 X_{|K}(t) = (X_{mn}(t))_{m,n\in\sigma(K)},$$
$$\nabla Y_{|L}(s) = (Y_{i_1}(s), \dots, Y_{i_l}(s))_{i_1,\dots,i_l\in\sigma(L)}, \qquad \nabla^2 Y_{|L}(s) = (Y_{mn}(s))_{m,n\in\sigma(L)}.$$

We can decompose T and S into

$$T = \bigcup_{k=0}^{N} \bigcup_{K\in\partial_k T} K, \qquad S = \bigcup_{l=0}^{N'} \bigcup_{L\in\partial_l S} L.$$

For each $K \in \partial_k T$ and $L \in \partial_l S$, define the *number of extended outward maxima above u* as

$$M_u^E(X, K) := \#\{t \in K : X(t) \ge u, \nabla X_{|K}(t) = 0, \nabla^2 X_{|K}(t) \prec 0, \varepsilon_j^* X_j(t) \ge 0, \forall j \notin \sigma(K)\},$$
$$M_u^E(Y, L) := \#\{s \in L : Y(s) \ge u, \nabla Y_{|L}(s) = 0, \nabla^2 Y_{|L}(s) \prec 0, \varepsilon_j^* Y_j(s) \ge 0, \forall j \notin \sigma(L)\},$$

where $\varepsilon_j^* = 2\varepsilon_j - 1$, and define the *number of local maxima above u* as

$$M_u(X, K) := \#\{t \in K : X(t) \ge u, \nabla X_{|K}(t) = 0, \nabla^2 X_{|K}(t) \prec 0\},$$
$$M_u(Y, L) := \#\{s \in L : Y(s) \ge u, \nabla Y_{|L}(s) = 0, \nabla^2 Y_{|L}(s) \prec 0\}.$$

Clearly, $M_u^E(X, K) \le M_u(X, K)$ and $M_u^E(Y, L) \le M_u(Y, L)$.

We shall make use of the following smoothness condition (**H**1) and regularity conditions (**H**2) and (**H**3).

(**H**1) $X, Y \in C^2(\mathbb{R}^N)$ almost surely and their second derivatives satisfy the *uniform mean-square Hölder condition*: there exist constants $C, \delta > 0$ such that

$$\mathbb{E}(X_{ij}(t) - X_{ij}(t'))^2 \le C\|t - t'\|^{2\delta}, \quad \forall t, t' \in T, i, j = 1, \dots, N,$$
$$\mathbb{E}(Y_{mn}(s) - Y_{mn}(s'))^2 \le C\|s - s'\|^{2\delta}, \quad \forall s, s' \in S, m, n = 1, \dots, N'.$$

(**H**2) For every $(t, t', s) \in T^2 \times S$ with $t \ne t'$, the Gaussian vector

$$((X, \nabla X, \mathrm{vech}(\nabla^2 X))(t), (X, \nabla X, \mathrm{vech}(\nabla^2 X))(t'), (Y, \nabla Y, \mathrm{vech}(\nabla^2 Y))(s))$$

is nondegenerate; and for every $(s, s', t) \in S^2 \times T$ with $s \ne s'$, the Gaussian vector

$$((Y, \nabla Y, \mathrm{vech}(\nabla^2 Y))(s), (Y, \nabla Y, \mathrm{vech}(\nabla^2 Y))(s'), (X, \nabla X, \mathrm{vech}(\nabla^2 X))(t))$$

is nondegenerate.

(**H**3) For every $(t,s) \in \partial_k T \times S$, $0 \le k \le N-2$, such that $r(t,s) = R$ and the index set $\mathcal{I}_X^R(t,s) = \{\ell : \frac{\partial r}{\partial t_\ell}(t,s) = 0\}$ contains at least two indices, we have

$$\left(\frac{\partial^2 r}{\partial t_i \partial t_j}(t,s)\right)_{i,j\in\mathcal{I}_X^R(t,s)} \preceq 0. \tag{3}$$

For every $(t,s) \in T \times \partial_l S$, $0 \le l \le N'-2$, such that $r(t,s) = R$ and the index set $\mathcal{I}_Y^R(t,s) = \{\ell : \frac{\partial r}{\partial s_\ell}(t,s) = 0\}$ contains at least two indices, we have

$$\left(\frac{\partial^2 r}{\partial s_m \partial s_n}(t,s)\right)_{m,n\in\mathcal{I}_Y^R(t,s)} \preceq 0.$$

The smoothness condition (**H**1) and regularity condition (**H**2) imply the validity of Corollary 11.3.2 in [2], showing that X and Y are almost surely Morse functions on T and S respectively. Additionally, the conditions required for Kac–Rice formulas in Theorems 11.2.1 and 11.5.1 in [2] are satisfied, so that we can apply them to compute moments of the number of critical points.

Although (**H**3) looks technical, it is in fact a mild condition imposed only on the lower-dimension boundary points (t,s) with $r(t,s) = R$. Roughly speaking, it shows that the correlation function should have a negative semidefinite Hessian matrix on boundary critical points where the maximum correlation R is attained. Since $r(t,s) = R$ implies $\frac{\partial r}{\partial t_\ell}(t,s) = 0$ for all $\ell \in \sigma(K)$, we have $\mathcal{I}_X^R(t,s) \supset \sigma(K)$. Similarly, $\mathcal{I}_Y^R(t,s) \supset \sigma(L)$. We show below that, for $k = N-1$ or $k = N$, the property (3) is always satisfied.

(i) If $k = N$, then t becomes a maximum point of r (as a function of t) in the interior of T and $\mathcal{I}_X^R(t,s) = \sigma(K) = \{1,\dots,N\}$, implying (3).

(ii) For $k = N-1$, we distinguish two cases. If $\mathcal{I}_X^R(t,s) = \sigma(K)$, then t becomes a maximum point of r restricted on K, hence (3) holds. If $\mathcal{I}_X^R(t,s) = \{1,\dots,N\}$, let s be fixed, it follows from Taylor's formula that

$$r(t',s) = r(t,s) + (t'-t)\nabla^2 r(t,s)(t'-t)^T + o(\|t'-t\|^2), \quad t' \in T,$$

where $\nabla^2 r(t,s)$ is the Hessian with respect to t. Notice that $\{(t'-t)/\|t'-t\| : t' \in T\}$ contains all directions in $\mathbb{R}^N$ since $t \in K \in \partial_{N-1}T$, together with the fact $r(t,s) = R$, we see that $\nabla^2 r(t,s)$ cannot have any positive eigenvalue and hence (3) holds.

It is also evident from the 1D Taylor's formula that (3) holds if $\mathcal{I}_X^R(t,s)$ contains only one index. Combining these facts, together with the observations that

$$\frac{\partial r}{\partial t_i}(t,s) = \mathbb{E}\{X_i(t)Y(s)\}, \qquad \frac{\partial^2 r}{\partial t_i \partial t_j}(t,s) = \mathbb{E}\{X_{ij}(t)Y(s)\},$$
$$\frac{\partial r}{\partial s_i}(t,s) = \mathbb{E}\{X(t)Y_i(s)\}, \qquad \frac{\partial^2 r}{\partial s_i \partial s_j}(t,s) = \mathbb{E}\{X(t)Y_{ij}(s)\},$$

we obtain the following result.

PROPOSITION 2.1. *Under the condition* (**H**3), *we have that, for every* $(t,s) \in T \times S$ *such that* $r(t,s) = R$, *the matrices*

$$(\mathbb{E}\{X_{ij}(t)Y(s)\})_{i,j\in\mathcal{I}_X^R(t,s)} \preceq 0 \quad \text{and} \quad (\mathbb{E}\{X(t)Y_{kl}(s)\})_{k,l\in\mathcal{I}_Y^R(t,s)} \preceq 0,$$

where the index sets $\mathcal{I}_X^R(t,s)$ *and* $\mathcal{I}_Y^R(t,s)$ *are defined respectively as*

$$\mathcal{I}_X^R(t,s) = \{\ell : \mathbb{E}\{X_\ell(t)Y(s)\} = 0\} \quad \text{and} \quad \mathcal{I}_Y^R(t,s) = \{\ell : \mathbb{E}\{X(t)Y_\ell(s)\} = 0\}.$$

3. Main results. Here, we shall state our main results Theorem 3.1, Corollary 3.1, and Theorem 3.2, whose proofs will be given in Section 8. Define respectively the excursion sets of X, Y, and (X, Y) above level u by

$$A_u(X, T) = \{t \in T : X(t) \geq u\},$$
$$A_u(Y, S) = \{s \in S : Y(s) \geq u\}, \quad \text{and}$$
$$A_u := A_u(X, T) \times A_u(Y, S) = \{(t, s) \in T \times S : X(t) \geq u, Y(s) \geq u\}.$$

Let the *number of extended outward critical points of index i above level u* be

$$\mu_i(X, K) := \#\{t \in K : X(t) \geq u, \nabla X_{|K}(t) = 0, \text{index}(\nabla^2 X_{|K}(t)) = i, \varepsilon_j^* X_j(t) \geq 0 \text{ for all } j \notin \sigma(K)\},$$
$$\mu_i(Y, L) := \#\{s \in L : Y(s) \geq u, \nabla Y_{|L}(s) = 0, \text{index}(\nabla^2 Y_{|L}(s)) = i, \varepsilon_j^* Y_j(s) \geq 0 \text{ for all } j \notin \sigma(L)\}.$$

Recall that $\varepsilon_j^* = 2\varepsilon_j - 1$ and the index of a matrix is defined as the number of its negative eigenvalues. It follows from (**H**1), (**H**2), and the Morse theorem (see Corollary 9.3.5 or pages 211–212 in Adler and Taylor [2]) that the Euler characteristic of the excursion set can be represented as

$$\chi(A_u(X, T)) = \sum_{k=0}^{N} \sum_{K \in \partial_k T} (-1)^k \sum_{i=0}^{k} (-1)^i \mu_i(X, K),$$
(4)
$$\chi(A_u(Y, S)) = \sum_{l=0}^{N'} \sum_{L \in \partial_l S} (-1)^l \sum_{i=0}^{l} (-1)^i \mu_i(Y, L).$$

Since for two sets D_1 and D_2, $\chi(D_1 \times D_2) = \chi(D_1)\chi(D_2)$, we have

$$\chi(A_u) = \chi(A_u(X, T) \times A_u(Y, S)) = \chi(A_u(X, T)) \times \chi(A_u(Y, S))$$
(5)
$$= \sum_{k,l} \sum_{K \in \partial_k T, L \in \partial_l S} (-1)^{k+l} \left(\sum_{i=0}^{k} (-1)^i \mu_i(X, K)\right)\left(\sum_{j=0}^{l} (-1)^j \mu_j(Y, L)\right).$$

Here and in the sequel, to simplify the notation, we denote by $\sum_{k,l}$ the sum taken over all $k = 0, \ldots, N$ and $l = 0, \ldots, N'$.

Now we state the following general result on the EEC approximation of the joint excursion probability.

THEOREM 3.1. *Let $\{(X(t), Y(s)) : t \in T, s \in S\}$ be an $\mathbb{R}^2$-valued, centered, unit-variance Gaussian vector field satisfying* (**H**1), (**H**2), *and* (**H**3). *Then there exists a constant $\alpha > 0$ such that as $u \to \infty$,*

$$\mathbb{P}\Big\{\sup_{t \in T} X(t) \geq u, \sup_{s \in S} Y(s) \geq u\Big\} = \mathbb{E}\{\chi(A_u)\} + o\left(\exp\left\{-\frac{u^2}{1 + R} - \alpha u^2\right\}\right),$$

where the EEC has the expression

$$\mathbb{E}\{\chi(A_u)\} = \sum_{k,l} \sum_{K \in \partial_k T, L \in \partial_l S} (-1)^{k+l} \int_K \int_L dt\, ds\, p_{\nabla X_{|K}(t), \nabla Y_{|L}(s)}(0, 0)$$
(6)
$$\times \mathbb{E}\{\det \nabla^2 X_{|K}(t) \det \nabla^2 Y_{|L}(s) \mathbb{1}_{\{X(t) \geq u, \varepsilon_\ell^* X_\ell(t) \geq 0 \text{ for all } \ell \notin \sigma(K)\}}$$
$$\times \mathbb{1}_{\{Y(s) \geq u, \varepsilon_\ell^* Y_\ell(s) \geq 0 \text{ for all } \ell \notin \sigma(L)\}} | \nabla X_{|K}(t) = \nabla Y_{|L}(s) = 0\}.$$

Note that condition (**H**2) implies $R > -1$. In general, the EEC approximation $\mathbb{E}\{\chi(A_u)\}$ is hard to compute, since the conditional expectation in (6) involves the joint tail estimate and hence the conditional correlation on $X(t)$ and $Y(s)$, which varies over $T \times S$. However, one can apply the Laplace method to extract the term with the largest order of u from $\mathbb{E}\{\chi(A_u)\}$ such that the remaining error is $o(1/u)\mathbb{E}\{\chi(A_u)\}$; see Section 4 for examples on this.

Note that, in (6), if $k = 0$, then all terms involving $\nabla X_{|K}(t)$ and $\nabla^2 X_{|K}(t)$ vanish. In particular, if $k = l = 0$, then the integral in (6) becomes a joint tail probability. We adopt such notation in the results in Corollary 3.1 and Theorem 3.2 below as well.

It can be seen from the proof of Theorem 3.1 that those points attaining the maximal correlation R make the major contribution to $\mathbb{E}\{\chi(A_u)\}$. Therefore, in many cases, the general EEC approximation $\mathbb{E}\{\chi(A_u)\}$ can be simplified. To address this, we introduce the following *boundary condition*: for all faces $K_1 \subset T$ and $L_1 \subset S$,

$$\left\{(t,s) \in K_1 \times L_1 : r(t,s) = R, \prod_{i \notin \sigma(K_1)} \frac{\partial r}{\partial t_i}(t,s) \prod_{j \notin \sigma(L_1)} \frac{\partial r}{\partial s_j}(t,s) = 0\right\} = \varnothing. \tag{7}$$

The boundary condition (7) is on nonzero derivatives of the correlation function over boundary points where R is attained. In other words, (7) indicates that, for any point $(t,s) \in K_1 \times L_1$ attaining the maximal correlation R, there must be $\frac{\partial r}{\partial t_i}(t,s) \neq 0$ for all $i \notin \sigma(K_1)$ and $\frac{\partial r}{\partial s_j}(t,s) \neq 0$ for all $j \notin \sigma(L_1)$. In particular, as an important fact, we see that the boundary condition (7) implies condition (**H**3). Based on this boundary condition, we obtain the following refined approximation in Corollary 3.1, which eliminates the partial derivatives in the indicator functions in (6).

COROLLARY 3.1. *Let* $\{(X(t), Y(s)) : t \in T, s \in S\}$ *be an* $\mathbb{R}^2$*-valued, centered, unit-variance Gaussian vector field satisfying* (**H**1), (**H**2), *and the boundary condition* (7). *Then there exists a constant* $\alpha > 0$ *such that as* $u \to \infty$,

$$\begin{aligned}
&\mathbb{P}\Big\{\sup_{t\in T} X(t) \geq u, \sup_{s\in S} Y(s) \geq u\Big\} \\
&= \sum_{k,l} \sum_{K\in\partial_k T, L\in\partial_l S} (-1)^{k+l} \int_K \int_L dt\, ds\, p_{\nabla X_{|K}(t), \nabla Y_{|L}(s)}(0,0) \mathbb{E}\{\det\nabla^2 X_{|K}(t) \det\nabla^2 Y_{|L}(s) \\
&\times \mathbb{1}_{\{X(t)\geq u, Y(s)\geq u\}} | \nabla X_{|K}(t) = \nabla Y_{|L}(s) = 0\} + o\left(\exp\left\{-\frac{u^2}{1+R} - \alpha u^2\right\}\right).
\end{aligned}$$

The following Theorem 3.2 provides an asymptotic approximation for the case when the correlation attains its maximum R at a unique point. Such case is quite common in applications, especially when the correlation function has certain monotone property. We will show refined explicit approximation results and examples in Section 4 below by employing the Laplace method.

THEOREM 3.2. *Let* $\{(X(t), Y(s)) : t \in T, s \in S\}$ *be an* $\mathbb{R}^2$*-valued, centered, unit-variance Gaussian vector field satisfying* (**H**1), (**H**2), *and* (**H**3). *Suppose that the correlation attains its maximum* R *at a single point* $(t^*, s^*) \in K \times L$, *where* $K \in \partial_k T$ *and* $L \in \partial_l S$ *with* $k, l \geq 0$. *Then there exists a constant* $\alpha > 0$ *such that as* $u \to \infty$,

$$\begin{aligned}
&\mathbb{P}\Big\{\sup_{t\in T} X(t) \geq u, \sup_{s\in S} Y(s) \geq u\Big\} \\
&= \sum_J \sum_F (-1)^{\dim(J)+\dim(F)} \int_J \int_F dt\, ds\, p_{\nabla X_{|J}(t), \nabla Y_{|F}(s)}(0,0)
\end{aligned}$$

$$\times \mathbb{E}\{\det\nabla^2 X_{|J}(t)\det\nabla^2 Y_{|F}(s)\mathbb{1}_{\{X(t)\geq u,\varepsilon^*_\ell X_\ell(t)\geq 0 \text{ for all } \ell\in\mathcal{I}^R_X(t^*,s^*)\setminus\sigma(J)\}}$$
$$\times \mathbb{1}_{\{Y(s)\geq u,\varepsilon^*_\ell Y_\ell(s)\geq 0 \text{ for all } \ell\in\mathcal{I}^R_Y(t^*,s^*)\setminus\sigma(F)\}}|\nabla X_{|J}(t)=\nabla Y_{|F}(s)=0\}$$
$$+o\left(\exp\left\{-\frac{u^2}{1+R}-\alpha u^2\right\}\right),$$

where the sums are taken over all faces J of T such that $t^*\in\bar{J}$ and $\sigma(J)\subset\mathcal{I}^R_X(t^*,s^*)$, and all faces F of S such that $s^*\in\bar{F}$ and $\sigma(F)\subset\mathcal{I}^R_Y(t^*,s^*)$.

4. Examples. Throughout this section, we assume that $\{(X(t),Y(s)):t\in T,s\in S\}$, where $T=S=[0,1]$, is an $\mathbb{R}^2$-valued, centered, unit-variance Gaussian vector field satisfying (**H**1), (**H**2), and (**H**3).

4.1. *Example with correlation attaining the maximum at a unique point.* Suppose $r(t,s)$ attains the maximum R only at a point (t^*,s^*), that is, $r(t^*,s^*)=R$. Let

$$\lambda_1(t)=\mathrm{Var}(X'(t)),\qquad \lambda_2(s)=\mathrm{Var}(Y'(s)),\qquad \lambda_1=\lambda_1(t^*),\qquad \lambda_2=\lambda_2(s^*),$$
$$r_{11}(t,s)=\mathbb{E}\{X''(t)Y(s)\},\qquad r_{22}(t,s)=\mathbb{E}\{X(t)Y''(s)\},\qquad r_{12}(t,s)=\mathbb{E}\{X'(t)Y'(s)\},$$
$$R_{11}=r_{11}(t^*,s^*),\qquad R_{22}=r_{22}(t^*,s^*),\qquad R_{12}=r_{12}(t^*,s^*),$$
$$r_1(t,s)=\mathbb{E}\{X'(t)Y(s)\},\qquad r_2(t,s)=\mathbb{E}\{X(t)Y'(s)\}.$$

Case 1: $(t^*,s^*)=(0,0)$ *and* $r_1(0,0)r_2(0,0)\neq 0$. By Corollary 3.1,

$$\mathbb{P}\Big\{\sup_{t\in T}X(t)\geq u,\sup_{s\in S}Y(s)\geq u\Big\}=\mathbb{P}\{X(0)\geq u,Y(0)\geq u\}+o\left(\exp\left\{-\frac{u^2}{1+R}-\alpha u^2\right\}\right)$$
$$=\frac{(1+R)^2}{2\pi\sqrt{1-R^2}}\frac{1}{u^2}e^{-\frac{u^2}{1+R}}(1+o(1)),$$

where the last line is due to a well-know asymptotics for $\mathbb{P}\{X(0)\geq u,Y(0)\geq u\}$, see [9].

As a concrete example, consider cosine fields (cf. [2], p. 382) $X(t)=\xi_1\cos(1-t)+\xi_2\sin(1-t)$ and $Y(s)=\xi_1\cos(2+s)+\xi_2\sin(2+s)$, where ξ_1 and ξ_2 are independent standard Gaussian random variables and $t,s\in[0,1]$. Then $(X(t),Y(s))$ is an $\mathbb{R}^2$-valued, centered, unit-variance Gaussian vector field satisfying (**H**1), (**H**2), and (**H**3). In particular, the correlation $r(t,s)=\cos(1+t+s)$ attains the maximum $R=\cos(1)$ only at $(t^*,s^*)=(0,0)$ with $r_1(0,0)=r_2(0,0)=-\sin(1)\neq 0$. Thus we can apply the derived result above with $R=\cos(1)$ to approximate the joint excursion probability.

Case 2: $(t^*,s^*)=(0,0)$*,* $r_1(0,0)=0$*, and* $r_2(0,0)\neq 0$. By Theorem 3.2,

$$\mathbb{P}\Big\{\sup_{t\in T}X(t)\geq u,\sup_{s\in S}Y(s)\geq u\Big\}$$
$$=\mathbb{P}\{X(0)\geq u,Y(0)\geq u,X'(0)<0\}+I(u)+o\left(\exp\left\{-\frac{u^2}{1+R}-\alpha u^2\right\}\right),\tag{8}$$

where

$$I(u)=-\int_0^1 p_{X'(t)}(0)\,dt\int_u^\infty\int_u^\infty p_{X(t),Y(0)}(x,y|X'(t)=0)$$
$$\times\mathbb{E}\{X''(t)|X(t)=x,Y(0)=y,X'(t)=0\}\,dx\,dy.$$

Since $X'(0)$ is independent of both $X(0)$ and $Y(0)$, we have

$$\mathbb{P}\{X(0)\geq u,Y(0)\geq u,X'(0)<0\}=\frac{(1+R)^2}{4\pi\sqrt{1-R^2}}\frac{1}{u^2}e^{-\frac{u^2}{1+R}}(1+o(1)).\tag{9}$$

Let $\Sigma(t) = (\Sigma_{ij}(t))_{i,j=1,2} = \mathrm{Cov}((X(t), Y(0))|X'(t) = 0)$, implying $\Sigma_{11}(t) = 1$, $\Sigma_{22}(t) = 1 - r_1^2(t,0)/\lambda_1(t)$ and $\Sigma_{12}(t) = \Sigma_{21}(t) = r(t,0)$. Then

$$I(u) = -\int_0^1 \frac{1}{\sqrt{2\pi\lambda(t)}}\,dt \int_0^\infty \int_0^\infty \frac{e^{-\frac{1}{2}(x+u,y+u)\Sigma(t)^{-1}(x+u,y+u)^T}}{2\pi\sqrt{\det(\Sigma(t))}} \times \mathbb{E}\{X''(t)|X(t) = x+u, Y(0) = y+u, X'(t) = 0\}\,dx\,dy,$$

where the expectation can be written as $f(t)u + g(t)$ with f and g not depending on u and

$$f(0) = (-\lambda_1, R_{11})\begin{pmatrix} 1 & R \\ R & 1 \end{pmatrix}^{-1}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{R_{11} - \lambda_1}{1+R}.$$

By Theorem 7.5.3 in Tong [15], as $u \to \infty$, the Mills ratio

$$\int_0^\infty \int_0^\infty e^{-\frac{1}{2}(x,y)\Sigma(t)^{-1}(x,y)^T - (u,u)\Sigma(t)^{-1}(x,y)^T}\,dx\,dy \sim \frac{1}{u^2[(\Sigma(t)^{-1})_{11} + (\Sigma(t)^{-1})_{21}][(\Sigma(t)^{-1})_{12} + (\Sigma(t)^{-1})_{22}]}. \tag{10}$$

Therefore,

$$I(u) \sim -\int_0^1 \frac{1}{\sqrt{2\pi\lambda_1(t)}} \frac{1}{2\pi\sqrt{\det(\Sigma(t))}} f(t) \frac{1}{u} e^{-\frac{1}{2}u^2(1,1)\Sigma(t)^{-1}(1,1)^T} \times \frac{1}{[(\Sigma(t)^{-1})_{11} + (\Sigma(t)^{-1})_{21}][(\Sigma(t)^{-1})_{12} + (\Sigma(t)^{-1})_{22}]}\,dt.$$

It can be checked that the function

$$h(t) := \frac{1}{2}(1,1)\Sigma(t)^{-1}(1,1)^T = \frac{2 - r_1(t,0)^2/\lambda_1(t) - 2r(t,0)}{2[1 - r_1(t,0)^2/\lambda_1(t) - r(t,0)^2]}$$

attains its minimum only at 0 with $h(0) = 1/(1+R)$ and $h''(0) = R_{11}(R_{11} - \lambda_1)/[\lambda_1(1+R)^2]$. Applying the Laplace method (see, e.g., Lemma A.3 in Cheng and Xiao [5]), we obtain

$$\begin{aligned} I(u) &\sim \frac{1}{2}\frac{1}{\sqrt{2\pi\lambda_1}}\frac{1}{2\pi\sqrt{1-R^2}}\frac{\lambda_1 - R_{11}}{1+R}(1+R)^2\left(\frac{2\pi}{u^2}\frac{\lambda_1(1+R)^2}{R_{11}(R_{11}-\lambda_1)}\right)^{1/2}\frac{1}{u}e^{-\frac{u^2}{1+R}} \\ &= \frac{\sqrt{\lambda_1 - R_{11}}}{2\sqrt{-R_{11}}}\frac{(1+R)^2}{2\pi\sqrt{1-R^2}}\frac{1}{u^2}e^{-\frac{u^2}{1+R}}. \end{aligned} \tag{11}$$

Combining (8) with (9) and (11), we obtain

$$\mathbb{P}\Big\{\sup_{t\in T} X(t) \ge u, \sup_{s\in S} Y(s) \ge u\Big\} = \left(\frac{1}{2} + \frac{\sqrt{\lambda_1 - R_{11}}}{2\sqrt{-R_{11}}}\right)\frac{(1+R)^2}{2\pi\sqrt{1-R^2}}\frac{1}{u^2}e^{-\frac{u^2}{1+R}}\big(1+o(1)\big).$$

Case 3: $(t^*, s^*) = (0,0)$ *and* $r_1(0,0) = r_2(0,0) = 0$. By Theorem 3.2,

$$\mathbb{P}\Big\{\sup_{t\in T} X(t) \ge u, \sup_{s\in S} Y(s) \ge u\Big\} = I_1(u) + I_2(u) + I_3(u) + I_4(u) + o\left(\exp\left\{-\frac{u^2}{1+R} - \alpha u^2\right\}\right), \tag{12}$$

where

$$\begin{aligned} I_1(u) &= \mathbb{P}\{X(0) \ge u, Y(0) \ge u, X'(0) < 0, Y'(0) < 0\} \\ &= \mathbb{P}\{X'(0) < 0, Y'(0) < 0\}\frac{(1+R)^2}{2\pi\sqrt{1-R^2}}\frac{1}{u^2}e^{-\frac{u^2}{1+R}}\big(1+o(1)\big), \end{aligned} \tag{13}$$

since $(X'(0), Y'(0))$, which has covariance matrix $\mathrm{Var}(X'(0)) = \lambda_1$, $\mathrm{Var}(Y'(0)) = \lambda_2$ and $\mathbb{E}\{X'(0)Y'(0)\} = R_{12}$, is independent of $(X(0), Y(0))$; and

$$\begin{aligned}
I_2(u) &= -\int_0^1 p_{X'(t)}(0)\,dt \int_u^\infty \int_u^\infty \int_{-\infty}^0 p_{X(t),Y(0),Y'(0)}\big(x, y, z|X'(t)=0\big) \\
&\quad\times \mathbb{E}\big\{X''(t)|X(t)=x, Y(0)=y, Y'(0)=z, X'(t)=0\big\}\,dx\,dy\,dz, \\
I_3(u) &= -\int_0^1 p_{Y'(s)}(0)\,ds \int_u^\infty \int_u^\infty \int_{-\infty}^0 p_{X(0),Y(s),X'(0)}\big(x, y, z|Y'(s)=0\big) \\
&\quad\times \mathbb{E}\big\{Y''(s)|X(0)=x, Y(s)=y, X'(0)=z, Y'(s)=0\big\}\,dx\,dy\,dz, \\
I_4(u) &= \int_0^1 \int_0^1 p_{X'(t),Y'(s)}(0,0)\,dt\,ds \int_u^\infty \int_u^\infty p_{X(t),Y(s)}\big(x, y|X'(t)=Y'(s)=0\big) \\
&\quad\times \mathbb{E}\big\{X''(t)Y''(s)|X(t)=x, Y(s)=y, X'(t)=Y'(s)=0\big\}\,dx\,dy.
\end{aligned}$$

Similar to (11), we have

$$\begin{aligned}
I_2(u) &\sim \frac{\sqrt{\lambda_1 - R_{11}}}{2\sqrt{-R_{11}}} \frac{(1+R)^2}{2\pi\sqrt{1-R^2}} \frac{1}{u^2} e^{-\frac{u^2}{1+R}}\big(1+o(1)\big), \\
I_3(u) &\sim \frac{\sqrt{\lambda_2 - R_{22}}}{2\sqrt{-R_{22}}} \frac{(1+R)^2}{2\pi\sqrt{1-R^2}} \frac{1}{u^2} e^{-\frac{u^2}{1+R}}\big(1+o(1)\big).
\end{aligned} \tag{14}$$

Let us compute $I_4(u)$. Let $\Sigma(t,s) = (\Sigma_{ij}(t,s))_{i,j=1,2} = \mathrm{Cov}((X(t), Y(s))|X'(t) = Y'(s) = 0)$, implying

$$\Sigma_{11}(t,s) = 1 - \frac{\lambda_1(t) r_2^2(t,s)}{\lambda_1(t)\lambda_2(s) - r_{12}^2(t,s)}, \qquad \Sigma_{22}(t,s) = 1 - \frac{\lambda_2(s) r_1^2(t,s)}{\lambda_1(t)\lambda_2(s) - r_{12}^2(t,s)},$$

$$\Sigma_{12}(t,s) = \Sigma_{21}(t,s) = r(t,s) + \frac{r_{12}(t,s) r_1(t,s) r_2(t,s)}{\lambda_1(t)\lambda_2(s) - r_{12}^2(t,s)}.$$

Then

$$\begin{aligned}
I_4(u) &= \int_0^1 \int_0^1 \frac{1}{2\pi\sqrt{\lambda_1(t)\lambda_2(s) - r_{12}^2(t,s)}}\,dt\,ds \\
&\quad\times \int_0^\infty \int_0^\infty \frac{e^{-\frac{1}{2}(x+u,y+u)\Sigma(t,s)^{-1}(x+u,y+u)^T}}{2\pi\sqrt{\det(\Sigma(t,s))}} \\
&\quad\times \mathbb{E}\big\{X''(t)Y''(s)|X(t)=x+u, Y(s)=y+u, X'(t)=Y'(s)=0\big\}\,dx\,dy,
\end{aligned} \tag{15}$$

where the expectation is on the product of two noncentered (conditional) Gaussian variables and hence its highest-order term in u can be derived from the product of the means of Gaussian variables. We can write $\mathbb{E}\{X''(t)|X(t) = x + u, Y(s) = y + u, X'(t) = Y'(s) = 0\} = f(t,s)u + f_0(t,s,x,y)$ such that

$$f(0,0) = (-\lambda_1, R_{11}) \begin{pmatrix} 1 & R \\ R & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{R_{11} - \lambda_1}{1+R};$$

and write $\mathbb{E}\{Y''(s)|X(t) = x+u, Y(s) = y+u, X'(t) = Y'(s) = 0\} = g(t,s)u + g_0(t,s,x,y)$ such that

$$g(0,0) = (R_{22}, -\lambda_2) \begin{pmatrix} 1 & R \\ R & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{R_{22} - \lambda_2}{1+R}.$$

Therefore, in the expectation in (15), the highest-order term in u evaluated at $(0,0)$ is given by $[(\lambda_1 - R_{11})(\lambda_2 - R_{22})/(1+R)^2]u^2$. Note that the Mills ratio in (10) with $\Sigma(t)$ replaced by $\Sigma(t,s)$ is asymptotically $(1+R)^2/u^2$ at $(t,s)=(0,0)$. Plugging these into (15) yields

$$I_4(u) \sim \int_0^1 \int_0^1 \frac{1}{2\pi\sqrt{\lambda_1(t)\lambda_2(s) - r_{12}^2(t,s)}} \frac{1}{2\pi\sqrt{\det(\Sigma(t,s))}} f(t,s)g(t,s)u^2 \times \frac{1}{u^2[(\Sigma(t,s)^{-1})_{11} + (\Sigma(t,s)^{-1})_{21}]^2} e^{-\frac{1}{2}u^2(1,1)\Sigma(t,s)^{-1}(1,1)^T}\, dt\, ds.$$

Since

$$h(t) := \frac{1}{2}(1,1)\Sigma(t,s)^{-1}(1,1)^T = \frac{1}{2}\frac{\Sigma_{11}(t,s) + \Sigma_{22}(t,s) - 2\Sigma_{12}(t,s)}{\Sigma_{11}(t,s)\Sigma_{22}(t,s) - \Sigma_{12}^2(t,s)}$$

attains its minimum only at $(t,s)=(0,0)$ with $h(0) = 1/(1+R)$ and

$$\nabla^2 h(0,0) = \frac{1}{(1+R)^2(\lambda_1\lambda_2 - R_{12}^2)} \times \begin{pmatrix} (\lambda_1 - R_{11})(R_{12}^2 - \lambda_2 R_{11}) & R_{12}(\lambda_1 - R_{11})(R_{22} - \lambda_2) \\ R_{12}(\lambda_1 - R_{11})(R_{22} - \lambda_2) & (\lambda_2 - R_{22})(R_{12}^2 - \lambda_1 R_{22}) \end{pmatrix},$$

$$\det(\nabla^2 h(0,0)) = \frac{(\lambda_1 - R_{11})(\lambda_2 - R_{22})(R_{11}R_{22} - R_{12}^2)}{(1+R)^4(\lambda_1\lambda_2 - R_{12}^2)}.$$

Applying the Laplace method (see Lemma A.3 in [5]) yields

$$\begin{aligned} I_4(u) &\sim \mathbb{P}(Z_1 > 0, Z_2 > 0)\frac{1}{2\pi\sqrt{\lambda_1\lambda_2 - R_{12}^2}} \frac{1}{2\pi\sqrt{1-R^2}} \frac{(\lambda_1 - R_{11})(\lambda_2 - R_{22})}{(1+R)^2} u^2 \\ &\quad \times \frac{(1+R)^2}{u^2}\frac{2\pi}{u^2}\left(\frac{(1+R)^4(\lambda_1\lambda_2 - R_{12}^2)}{(\lambda_1 - R_{11})(\lambda_2 - R_{22})(R_{11}R_{22} - R_{12}^2)}\right)^{1/2} e^{-\frac{u^2}{1+R}} \\ &= \mathbb{P}(Z_1 > 0, Z_2 > 0)\frac{\sqrt{(\lambda_1 - R_{11})(\lambda_2 - R_{22})}}{\sqrt{R_{11}R_{22} - R_{12}^2}} \frac{(1+R)^2}{2\pi\sqrt{1-R^2}} \frac{1}{u^2} e^{-\frac{u^2}{1+R}}, \end{aligned} \tag{16}$$

where (Z_1, Z_2) is a centered bivariate Gaussian variable with covariance $\nabla^2 h(0,0)$. Plugging (13), (14), and (16) into (12), we obtain

$$\begin{aligned} &\mathbb{P}\Big\{\sup_{t\in T} X(t) \ge u, \sup_{s\in S} Y(s) \ge u\Big\} \\ &= \bigg[\mathbb{P}(X'(0) < 0, Y'(0) < 0) + \frac{\sqrt{\lambda_1 - R_{11}}}{2\sqrt{-R_{11}}} + \frac{\sqrt{\lambda_2 - R_{22}}}{2\sqrt{-R_{22}}} \\ &\quad + \mathbb{P}(Z_1 > 0, Z_2 > 0)\frac{\sqrt{(\lambda_1 - R_{11})(\lambda_2 - R_{22})}}{\sqrt{R_{11}R_{22} - R_{12}^2}}\bigg] \frac{(1+R)^2}{2\pi\sqrt{1-R^2}} \frac{1}{u^2} e^{-\frac{u^2}{1+R}}(1 + o(1)), \end{aligned}$$

where the two probabilities on the right side become $1/4$ when $R_{12} = 0$.

Case 4: $(t^*, s^*) = (t^*, 0)$, *where* $t^* \in (0,1)$ *and* $r_2(t^*, 0) \neq 0$. By Theorem 3.2 and similar arguments in Case 2, we obtain

$$\mathbb{P}\Big\{\sup_{t\in T} X(t) \ge u, \sup_{s\in S} Y(s) \ge u\Big\} = \frac{\sqrt{\lambda_1 - R_{11}}}{\sqrt{-R_{11}}} \frac{(1+R)^2}{2\pi\sqrt{1-R^2}} \frac{1}{u^2} e^{-\frac{u^2}{1+R}}(1 + o(1)).$$

Case 5: $(t^*, s^*) = (t^*, 0)$, *where* $t^* \in (0, 1)$ *and* $r_2(t^*, 0) = 0$. By Theorem 3.2 and similar arguments in Case 3, we obtain

$$\mathbb{P}\Big\{\sup_{t\in T} X(t) \geq u, \sup_{s\in S} Y(s) \geq u\Big\} = \bigg[\frac{\sqrt{\lambda_1 - R_{11}}}{\sqrt{-R_{11}}} + \frac{\sqrt{(\lambda_1 - R_{11})(\lambda_2 - R_{22})}}{2\sqrt{R_{11}R_{22} - R_{12}^2}}\bigg] \times \frac{(1+R)^2}{2\pi\sqrt{1-R^2}} \frac{1}{u^2} e^{-\frac{u^2}{1+R}}\big(1+o(1)\big).$$

Case 6: $(t^*, s^*) \in (0, 1)^2$. By Theorem 3.2 and similar arguments in Case 3, we obtain

$$\mathbb{P}\Big\{\sup_{t\in T} X(t) \geq u, \sup_{s\in S} Y(s) \geq u\Big\} = \frac{(1+R)^2\sqrt{(\lambda_1 - R_{11})(\lambda_2 - R_{22})}}{2\pi\sqrt{1-R^2}\sqrt{R_{11}R_{22} - R_{12}^2}} \frac{e^{-\frac{u^2}{1+R}}}{u^2}\big(1+o(1)\big).$$

4.2. *Examples with correlation attaining the maximum on a line*. Here we consider the bivariate Gaussian random fields in Zhou and Xiao [16], where the smooth case was not studied since the double sum method therein is not applicable. Let $X(t)$ and $Y(s)$ be smooth stationary Gaussian processes with covariances satisfying

$$\mathbb{E}\{X(0)X(t)\} = 1 - \frac{\lambda_1}{2}|t|^2\big(1+o(1)\big) \quad \text{as } |t| \to 0,$$
$$\mathbb{E}\{Y(0)Y(s)\} = 1 - \frac{\lambda_2}{2}|s|^2\big(1+o(1)\big) \quad \text{as } |s| \to 0,$$

which implies $\mathrm{Var}(X'(t)) = -\mathbb{E}\{X(t)X''(t)\} = \lambda_1$ and $\mathrm{Var}(Y'(s)) = -\mathbb{E}\{Y(s)Y''(s)\} = \lambda_2$. Assume that the correlation of X and Y satisfies

$$r(t,s) = \mathbb{E}\{X(t)Y(s)\} = \rho\big(|t-s|\big), \quad \forall t, s \in [0,1],$$

where ρ is a real function. Suppose ρ attains its maximum R only at 0 with $\rho'(0) = 0$ and $\rho''(0) < 0$. This indicates that the maximal correlation R is only achieved on the diagonal line $\{t = s : 0 \leq t, s \leq 1\}$. By Theorem 3.1, we have

$$\begin{aligned}
&\mathbb{P}\Big\{\sup_{t\in[0,1]} X(t) \geq u, \sup_{s\in[0,1]} Y(s) \geq u\Big\} \\
&= \int_0^1\int_0^1 p_{X'(t),Y'(s)}(0,0)\,dt\,ds \int_u^\infty\int_u^\infty p_{X(t),Y(s)}\big(x,y|X'(t) = Y'(s) = 0\big) \\
&\quad \times \mathbb{E}\big\{X''(t)Y''(s)|X(t) = x, Y(s) = y, X'(t) = Y'(s) = 0\big\}\,dx\,dy \\
&\quad + \mathbb{P}\big\{X(0) \geq u, Y(0) \geq u, X'(0) < 0, Y'(0) < 0\big\} \\
&\quad + \mathbb{P}\big\{X(1) \geq u, Y(1) \geq u, X'(1) > 0, Y'(1) > 0\big\} + o\bigg(\exp\bigg\{-\frac{u^2}{1+R} - \alpha u^2\bigg\}\bigg) \\
&= I(u)\big(1+o(1)\big),
\end{aligned}$$

where $I(u)$ denotes the integral term in the second and third lines. We shall derive below the asymptotics of $I(u)$ which gives the highest-order term in u. By the stationarity and change of variables (using $z = s$ and $w = t - s$ for $0 < s < t < 1$ and the symmetry property),

$$
\begin{aligned}
I(u) = &\int_0^1 \int_0^1 p_{X'(0),Y'(|t-s|)}(0,0)\,dt\,ds \\
&\times \int_u^\infty \int_u^\infty p_{X(0),Y(|t-s|)}\big(x,y|X'(0)=Y'(|t-s|)=0\big) \\
&\times \mathbb{E}\big\{X''(0)Y''(|t-s|)|X(0)=x, Y(|t-s|)=y, X'(0)=Y'(|t-s|)=0\big\}\,dx\,dy \\
= &\,2\int_0^1 (1-t)p_{X'(0),Y'(t)}(0,0)\,dt \int_u^\infty \int_u^\infty p_{X(0),Y(t)}\big(x,y|X'(0)=Y'(t)=0\big) \\
&\times \mathbb{E}\big\{X''(0)Y''(t)|X(0)=x, Y(t)=y, X'(0)=Y'(t)=0\big\}\,dx\,dy \\
:= &\,2I_0(u).
\end{aligned}
$$

Let $\Sigma(t) = (\Sigma_{ij}(t))_{i,j=1,2} = \mathrm{Cov}((X(0), Y(t))|X'(0) = Y'(t) = 0)$, implying

$$
\Sigma_{11}(t) = 1 - \frac{\lambda_1\rho'(t)^2}{\lambda_1\lambda_2 - \rho''(t)^2}, \qquad \Sigma_{22}(t) = 1 - \frac{\lambda_2\rho'(t)^2}{\lambda_1\lambda_2 - \rho''(t)^2},
$$

$$
\Sigma_{12}(t) = \Sigma_{21}(t) = \rho(t) + \frac{\rho''(t)\rho'(t)^2}{\lambda_1\lambda_2 - \rho''(t)^2}.
$$

Then

$$
\begin{aligned}
I_0(u) = &\int_0^1 \frac{1-t}{2\pi\sqrt{\lambda_1\lambda_2 - \rho''(t)^2}}\,dt \\
&\times \int_0^\infty \int_0^\infty \frac{1}{2\pi\sqrt{\det(\Sigma(t))}} e^{-\frac{1}{2}(x+u,y+u)\Sigma(t)^{-1}(x+u,y+u)^T} \\
&\times \mathbb{E}\big\{X''(0)Y''(t)|X(0)=x+u, Y(t)=y+u, X'(0)=Y'(t)=0\big\}\,dx\,dy.
\end{aligned}
\tag{17}
$$

We have $\mathbb{E}\{X''(0)|X(0) = x + u, Y(t) = y + u, X'(0) = Y'(t) = 0\} = f(t)u + f_0(t, x, y)$ with

$$
f(0) = \big(-\lambda_1, \rho''(0)\big)\begin{pmatrix} 1 & R \\ R & 1 \end{pmatrix}^{-1}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{\rho''(0) - \lambda_1}{1+R};
$$

and $\mathbb{E}\{Y''(t)|X(0) = x + u, Y(t) = y + u, X'(0) = Y'(t) = 0\} = g(t)u + g_0(t, x, y)$ with

$$
g(0) = \big(\rho''(0), -\lambda_2\big)\begin{pmatrix} 1 & R \\ R & 1 \end{pmatrix}^{-1}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{\rho''(0) - \lambda_2}{1+R}.
$$

Therefore, in the expectation in (17), the highest-order term in u evaluated at $t = 0$ is given by $[(\lambda_1 - \rho''(0))(\lambda_2 - \rho''(0))/(1 + R)^2]u^2$. Note that the Mills ratio in (10) is asymptotically $(1 + R)^2/u^2$ at $t = 0$. Plugging these into (17) yields

$$
\begin{aligned}
I_0(u) \sim &\int_0^1 \frac{1-t}{2\pi\sqrt{\lambda_1\lambda_2 - \rho''(t)^2}} \frac{1}{2\pi\sqrt{\det(\Sigma(t))}} f(t)g(t) \\
&\times \frac{1}{[(\Sigma(t)^{-1})_{11} + (\Sigma(t)^{-1})_{12}]^2} e^{-\frac{1}{2}u^2(1,1)\Sigma(t)^{-1}(1,1)^T}\,dt.
\end{aligned}
$$

Since

$$
h(t) := \frac{1}{2}(1,1)\Sigma(t)^{-1}(1,1)^T = \frac{1}{2}\frac{\Sigma_{11}(t) + \Sigma_{22}(t) - 2\Sigma_{12}(t)}{\Sigma_{11}(t)\Sigma_{22}(t) - \Sigma_{12}(t)^2}
$$

attains its minimum only at $t = 0$ with $h(0) = 1/(1+R)$ and

$$h''(0) = \frac{-\rho''(0)(\lambda_1 - \rho''(0))(\lambda_2 - \rho''(0))}{(1+R)^2[\lambda_1\lambda_2 - \rho''(0)^2]}.$$

Applying the Laplace method (see Lemma A.3 in [5]) yields

$$\begin{aligned} I_0(u) &\sim \frac{1}{2}\frac{1}{2\pi\sqrt{\lambda_1\lambda_2 - \rho''(0)^2}}\frac{1}{2\pi\sqrt{1-R^2}}\frac{[\lambda_1 - \rho''(0)][\lambda_2 - \rho''(0)]}{(1+R)^2}u^2 \\ &\quad\times \frac{(1+R)^2}{u^2}\left(-\frac{2\pi}{u^2}\frac{(1+R)^2[\lambda_1\lambda_2 - \rho''(0)^2]}{\rho''(0)(\lambda_1 - \rho''(0))(\lambda_2 - \rho''(0))}\right)^{1/2} e^{-\frac{u^2}{1+R}} \\ &= \frac{1}{2}\frac{\sqrt{(\lambda_1 - \rho''(0))(\lambda_2 - \rho''(0))}(1+R)}{(2\pi)^{3/2}\sqrt{1-R^2}\sqrt{-\rho''(0)}}\frac{1}{u}e^{-\frac{u^2}{1+R}}. \end{aligned}$$

Thus we obtain

$$\begin{aligned} &\mathbb{P}\Big\{\sup_{t\in[0,1]} X(t) \geq u, \sup_{s\in[0,1]} Y(s) \geq u\Big\} \\ &\quad= \frac{1}{(2\pi)^{3/2}}\sqrt{\frac{(\lambda_1 - \rho''(0))(\lambda_2 - \rho''(0))(1+R)}{-\rho''(0)(1-R)}}\frac{1}{u}e^{-\frac{u^2}{1+R}}\big(1+o(1)\big). \end{aligned} \tag{18}$$

As a concrete example, consider the modified cosine fields $X(t) = (\xi_1\cos(t) + \xi_2\sin(t) + \xi_3)/\sqrt{2}$ and $Y(s) = (\xi_1\cos(s) + \xi_2\sin(s) + \xi_4)/\sqrt{2}$, where ξ_1, ξ_2, ξ_3, and ξ_4 are independent standard Gaussian random variables and $t, s \in [0,1]$. Then $(X(t), Y(s))$ is an $\mathbb{R}^2$-valued, centered, unit-variance Gaussian vector field satisfying (**H**1), (**H**2), and (**H**3). In particular, we have $\lambda_1 = \lambda_2 = 1/2$, and the correlation $r(t,s) = [\cos(t-s)]/2$ attains the maximum $R = 1/2$ on the line $\{t = s : 0 \leq t, s \leq 1\}$. Let $\rho(x) = [\cos(x)]/2$. Then $\rho'(0) = 0$ and $\rho''(0) = -1/2$. Thus we can apply the derived result (18) to approximate the joint excursion probability as

$$\mathbb{P}\Big\{\sup_{t\in[0,1]} X(t) \geq u, \sup_{s\in[0,1]} Y(s) \geq u\Big\} = \frac{\sqrt{6}}{(2\pi)^{3/2}}\frac{1}{u}e^{-\frac{2u^2}{3}}\big(1+o(1)\big).$$

5. Sketch of the proofs of main results. Note that, for a smooth real-valued function f, $\sup_{t\in T} f(t) \geq u$ if and only if there exists at least one extended outward local maximum above u on some face of T. Thus, under conditions (**H**1) and (**H**2), the following relation holds for each $u \in \mathbb{R}$:

$$\begin{aligned} &\Big\{\sup_{t\in T} X(t) \geq u, \sup_{s\in S} Y(s) \geq u\Big\} \\ &\quad= \bigcup_{k=0}^{N}\bigcup_{l=0}^{N'}\bigcup_{K\in\partial_k T, L\in\partial_l S}\{M_u^E(X,K) \geq 1, M_u^E(Y,L) \geq 1\} \quad \text{a.s.} \end{aligned} \tag{19}$$

Therefore, we obtain the following upper bound for the joint excursion probability:

$$\begin{aligned} &\mathbb{P}\Big\{\sup_{t\in T} X(t) \geq u, \sup_{s\in S} Y(s) \geq u\Big\} \\ &\quad\leq \sum_{k,l}\sum_{K\in\partial_k T, L\in\partial_l S}\mathbb{P}\{M_u^E(X,K) \geq 1, M_u^E(Y,L) \geq 1\} \\ &\quad\leq \sum_{k,l}\sum_{K\in\partial_k T, L\in\partial_l S}\mathbb{E}\{M_u^E(X,K)M_u^E(Y,L)\}. \end{aligned} \tag{20}$$

On the other hand, notice that

$$\begin{aligned}
&\mathbb{E}\{M_u^E(X,K)M_u^E(Y,L)\} - \mathbb{P}\{M_u^E(X,K)\ge 1, M_u^E(Y,L)\ge 1\}\\
&\quad = \sum_{i,j=1}^{\infty}(ij-1)\mathbb{P}\{M_u^E(X,K)=i, M_u^E(Y,L)=j\}\\
&\quad \le \sum_{i,j=1}^{\infty}\big[i(i-1)j + j(j-1)i\big]\mathbb{P}\{M_u^E(X,K)=i, M_u^E(Y,L)=j\}\\
&\quad = \mathbb{E}\{M_u^E(X,K)\big[M_u^E(X,K)-1\big]M_u^E(Y,L)\} + \mathbb{E}\{M_u^E(Y,L)\big[M_u^E(Y,L)-1\big]M_u^E(X,K)\}
\end{aligned}$$

and

$$\begin{aligned}
&\mathbb{P}\{M_u^E(X,K)\ge 1, M_u^E(Y,L)\ge 1, M_u^E(X,K')\ge 1, M_u^E(Y,L')\ge 1\}\\
&\quad \le \mathbb{P}\{M_u^E(X,K)\ge 1, M_u^E(Y,L)\ge 1, M_u^E(Y,L')\ge 1\}\\
&\quad \le \mathbb{E}\{M_u^E(X,K)M_u^E(Y,L)M_u^E(Y,L')\}.
\end{aligned}$$

Combining these two inequalities with (19) and applying the Bonferroni inequality, we obtain the following lower bound for the joint excursion probability:

$$\begin{aligned}
&\mathbb{P}\Big\{\sup_{t\in T} X(t)\ge u, \sup_{s\in S} Y(s)\ge u\Big\}\\
&\quad \ge \sum_{k,l}\sum_{K\in\partial_k T, L\in\partial_l S}\big\{\mathbb{E}\{M_u^E(X,K)M_u^E(Y,L)\}\\
&\qquad - \mathbb{E}\{M_u^E(X,K)\big[M_u^E(X,K)-1\big]M_u^E(Y,L)\}\\
&\qquad - \mathbb{E}\{M_u^E(Y,L)\big[M_u^E(Y,L)-1\big]M_u^E(X,K)\}\big\}\\
&\qquad - \sum_{k,k',l}\sum_{\substack{K\in\partial_k T, L\in\partial_l S\\ K'\in\partial_{k'}T, K\neq K'}}\mathbb{E}\{M_u^E(X,K)M_u^E(X,K')M_u^E(Y,L)\}\\
&\qquad - C_N\sum_{k,l,l'}\sum_{\substack{K\in\partial_k T, L\in\partial_l S\\ L'\in\partial_{l'}S, L\neq L'}}\mathbb{E}\{M_u^E(X,K)M_u^E(Y,L)M_u^E(Y,L')\},
\end{aligned}\tag{21}$$

where C_N is a constant depending only on N.

REMARK 5.1. Note that, following the same arguments above, we have that the expectations on the number of extended outward maxima $M_u^E(\cdot)$ in both (20) and (21) can be replaced by the expectations on the number of local maxima $M_u(\cdot)$.

We call a function $h(u)$ *super-exponentially small* [when compared with the joint excursion probability $\mathbb{P}\{\sup_{t\in T} X(t)\ge u, \sup_{s\in S} Y(s)\ge u\}$], if there exists a constant $\alpha>0$ such that $h(u)=o(e^{-\alpha u^2-u^2/(1+R)})$ as $u\to\infty$. The main idea for proving the EEC approximation Theorem 3.1 consists of the following two steps: (i) showing that, except for the upper bound in (20), all terms in the lower bound in (21) are super-exponentially small; and (ii) proving that the difference between the upper bound in (20) and $\mathbb{E}\{\chi(A_u)\}$ is also super-exponentially small. The ideas for proving Corollary 3.1 and Theorem 3.2 are similar.

6. Estimation of super-exponentially small terms in the lower bound.

6.1. *Auxiliary results on multivariate Gaussian tails.*

LEMMA 6.1. *Let* $\{(\xi_1(x_1), \xi_2(x_2), \xi_3(x_3)) : (x_1, x_2, x_3) \in D_1 \times D_2 \times D_3\}$ *be an* $\mathbb{R}^3$-*valued*, C^2, *centered, unit-variance, nondegenerate Gaussian vector field, where* D_i, $i = 1, 2, 3$, *are compact sets in* $\mathbb{R}^N$. *Let* $R_{ij} = \sup_{x_i \in D_i, x_j \in D_j} \mathbb{E}\{\xi_i(x_i)\xi_j(x_j)\}$, *where* $i, j = 1, 2, 3$, *and* $i < j$. *If* $R_{12} \le \min\{R_{13}, R_{23}\}$, *then there exists a constant* $\alpha > 0$ *such that for every integer* $m \ge 0$, *as* $u \to \infty$,

$$\begin{aligned}&\sup_{x_1\in D_1,x_2\in D_2,x_3\in D_3} \mathbb{E}\{|\xi_1(x_1)\xi_2(x_2)\xi_3(x_3)|^m \mathbb{1}_{\{\xi_1(x_1)\ge u,\xi_2(x_2)\ge u,\xi_3(x_3)\ge u\}}\}\\ &\quad = o\left(\exp\left\{-\alpha u^2 - \frac{u^2}{1+R_{12}}\right\}\right).\end{aligned} \tag{22}$$

PROOF. Due to the exponential decay of Gaussian tails, it suffices to prove that there exists $\alpha' > 0$ such that as $u \to \infty$,

$$\begin{aligned}&\sup_{x_1\in D_1,x_2\in D_2,x_3\in D_3} \mathbb{P}\{\xi_1(x_1)\ge u, \xi_2(x_2)\ge u, \xi_3(x_3)\ge u\}\\ &\quad = o\left(\exp\left\{-\alpha' u^2 - \frac{u^2}{1+R_{12}}\right\}\right).\end{aligned} \tag{23}$$

Note that,

$$\mathbb{P}\{\xi_1(x_1)\ge u, \xi_2(x_2)\ge u, \xi_3(x_3)\ge u\} \le \mathbb{P}\{(\xi_1(x_1)+\xi_2(x_2))/2\ge u, \xi_3(x_3)\ge u\},$$

where $(\xi_1(x_1)+\xi_2(x_2))/2$ is a centered Gaussian variable with variance bounded by

$$\sup_{x_1\in D_1,x_2\in D_2} \operatorname{Var}((\xi_1(x_1)+\xi_2(x_2))/2) = \frac{1+R_{12}}{2}.$$

It is known that (see, e.g., Tong [15]), for a centered nondegenerate bivariate Gaussian vector (Z_1, Z_2) with $\operatorname{Var}(Z_1) = \sigma^2$, there exists $\alpha' > 0$ such that as $u \to \infty$,

$$\mathbb{P}\{Z_1 \ge u, Z_2 \ge u\} = o\left(\exp\left\{-\alpha' u^2 - \frac{u^2}{2\sigma^2}\right\}\right).$$

Combining these yields (23) and hence (22). □

LEMMA 6.2. *Let* $\{(\xi_1(x_1), \dots, \xi_n(x_n) : x_i \in D_i, i = 1, \dots, n\}$ *be an* $\mathbb{R}^n$-*valued*, C^2, *centered, unit-variance, nondegenerate Gaussian vector field, where* $D_1, \dots, D_n$ $(n \ge 3)$ *are compact sets in* $\mathbb{R}^N$. *Let* $R_{12} = \sup_{x_1\in D_1,x_2\in D_2} \mathbb{E}\{\xi_1(x_1)\xi_2(x_2)\}$. *If*

$$\begin{aligned}&\{(x_1,\dots,x_n)\in D_1\times\cdots\times D_n :\\ &\quad \mathbb{E}\{\xi_1(x_1)\xi_2(x_2)\} = R_{12}, \mathbb{E}\{(\xi_1(x_1)+\xi_2(x_2))\xi_i(x_i)\} = 0, \forall i = 3,\dots,n\} = \varnothing,\end{aligned} \tag{24}$$

then there exists $\alpha > 0$ *such that as* $u \to \infty$,

$$\begin{aligned}&\sup_{x_i\in D_i, i=1,\dots,n} \mathbb{E}\{|\xi_1(x_1)\xi_2(x_2)|^m \mathbb{1}_{\{\xi_1(x_1)\ge u,\xi_2(x_2)\ge u\}} | \xi_3(x_3) = \cdots = \xi_n(x_n) = 0\}\\ &\quad = o\left(\exp\left\{-\alpha u^2 - \frac{u^2}{1+R_{12}}\right\}\right),\end{aligned}$$

where $m \ge 0$ *is any fixed integer.*

PROOF. Let $\overline{\xi}(x_1, x_2) = [\xi_1(x_1) + \xi_2(x_2)]/2$. Then

$$\begin{aligned}&\mathbb{E}\{|\xi_1(x_1)\xi_2(x_2)|^m \mathbb{1}_{\{\xi_1(x_1)\geq u, \xi_2(x_2)\geq u\}} | \xi_3(x_3) = \cdots = \xi_n(x_n) = 0\}\\ &\quad\leq \mathbb{E}\{\overline{\xi}(x_1, x_2)^{2m} \mathbb{1}_{\{\overline{\xi}(x_1,x_2)\geq u\}} | \xi_3(x_3) = \cdots = \xi_n(x_n) = 0\}.\end{aligned}$$

Note that $(\overline{\xi}(x_1, x_2)|\xi_3(x_3) = \cdots = \xi_n(x_n) = 0)$ is a centered Gaussian variable with variance

$$\begin{aligned}\mathrm{Var}(\overline{\xi}(x_1, x_2)|\xi_3(x_3) = \cdots = \xi_n(x_n) = 0) \leq \mathrm{Var}(\overline{\xi}(x_1, x_2)) &= \frac{1 + \mathbb{E}\{\xi_1(x_1)\xi_2(x_2)\}}{2}\\ &\leq \frac{1 + R_{12}}{2},\end{aligned}$$

where the first inequality becomes equality if and only if $\overline{\xi}(x_1, x_2)$ is independent of each $\xi_i(x_i)$, $i \geq 3$. The desired result follows from the continuity of the conditional variance in x_i and the compactness of D_i, $i = 1, \ldots, n$. □

6.2. *Nonadjacent faces*. For two sets $D, D' \subset \mathbb{R}^N$, let $d(D, D') = \inf\{\|t - t'\| : t \in D, t' \in D'\}$ denote their distance. The following result shows that the last two sums involving the joint moment of two nonadjacent faces in (21) are super-exponentially small.

LEMMA 6.3. *Let $\{(X(t), Y(s)) : t \in T, s \in S\}$ be an $\mathbb{R}^2$-valued, centered, unit-variance Gaussian vector field satisfying* (**H**1) *and* (**H**2). *Then there exists $\alpha > 0$ such that as $u \to \infty$,*

$$\begin{aligned}&\mathbb{E}\{M_u(X, K)M_u(X, K')M_u(Y, L)\} = o\left(\exp\left\{-\frac{u^2}{1+R} - \alpha u^2\right\}\right),\\ &\mathbb{E}\{M_u(X, K)M_u(Y, L)M_u(Y, L')\} = o\left(\exp\left\{-\frac{u^2}{1+R} - \alpha u^2\right\}\right),\end{aligned} \tag{25}$$

where K and K' are different faces of T with $d(K, K') > 0$, L and L' are different faces of S with $d(L, L') > 0$.

PROOF. We only prove the first line in (25), since the proof for the second line is similar. Consider first the case when $\dim(K) = k \geq 1$, $\dim(K') = k' \geq 1$, and $\dim(L) = l \geq 1$. By the Kac–Rice metatheorem ([2], Theorem 11.2.1),

$$\begin{aligned}&\mathbb{E}\{M_u(X, K)M_u(X, K')M_u(Y, L)\}\\ &= \int_K dt \int_{K'} dt' \int_L ds \mathbb{E}\{|\det\nabla^2 X_{|K}(t)||\det\nabla^2 X_{|K'}(t')||\det\nabla^2 Y_{|L}(s)|\\ &\quad\times \mathbb{1}_{\{X(t)\geq u, X(t')\geq u, Y(s)\geq u\}} \mathbb{1}_{\{\nabla^2 X_{|K}(t)\prec 0, \nabla^2 X_{|K'}(t')\prec 0, \nabla^2 Y_{|L}(s)\prec 0\}}|\\ &\quad\nabla X_{|K}(t) = 0, \nabla X_{|K'}(t') = 0, \nabla Y_{|L}(s) = 0\} p_{\nabla X_{|K}(t), \nabla X_{|K'}(t'), \nabla Y_{|L}(s)}(0, 0, 0)\\ &\leq \int_K dt \int_{K'} dt' \int_L ds \int_u^\infty dx \int_u^\infty dx' \int_u^\infty dy p_{X(t), X(t'), Y(s)}(x, x', y)\\ &\quad\times \mathbb{E}\{|\det\nabla^2 X_{|K}(t)||\det\nabla^2 X_{|K'}(t')||\det\nabla^2 Y_{|L}(s)||\\ &\quad X(t) = x, X(t') = x', Y(s) = y, \nabla X_{|K}(t) = 0, \nabla X_{|K'}(t') = 0, \nabla Y_{|L}(s) = 0\}\\ &\quad\times p_{\nabla X_{|K}(t), \nabla X_{|K'}(t'), \nabla Y_{|L}(s)}(0, 0, 0|X(t) = x, X(t') = x', Y(s) = y).\end{aligned} \tag{26}$$

Notice that the following two inequalities hold: for constants a_{i_1}, b_{i_2}, and c_{i_3},

$$\prod_{i_1=1}^k |a_{i_1}| \prod_{i_2=1}^{k'} |b_{i_2}| \prod_{i_3=1}^l |c_{i_3}| \leq \frac{\sum_{i_1=1}^k |a_{i_1}|^{k+k'+l} + \sum_{i_2=1}^{k'} |b_{i_2}|^{k+k'+l} + \sum_{i_3=1}^l |c_{i_3}|^{k+k'+l}}{k + k' + l};$$

and for any Gaussian variable ξ and positive integer m, by Jensen's inequality,

$$\mathbb{E}|\xi|^m \le \mathbb{E}\big(|\mathbb{E}\xi| + |\xi - \mathbb{E}\xi|\big)^m$$
$$\le 2^{m-1}\big(|\mathbb{E}\xi|^m + \mathbb{E}|\xi - \mathbb{E}\xi|^m\big) = 2^{m-1}\big(|\mathbb{E}\xi|^m + B_m\big(\mathrm{Var}(\xi)\big)^{m/2}\big),$$

where B_m is some constant depending only on m. Combining these two inequalities with the well-known conditional formula for Gaussian variables, we obtain that there exist positive constants C_1 and N_1 such that for large x, x', and y,

$$\begin{aligned}\sup_{t\in K,t'\in K',s\in L} &\mathbb{E}\{|\det\nabla^2 X_{|K}(t)||\det\nabla^2 X_{|K'}(t')||\det\nabla^2 Y_{|L}(s)||X(t)=x, X(t')=x',\\ &Y(s)=y, \nabla X_{|K}(t)=0, \nabla X_{|K'}(t')=0, \nabla Y_{|L}(s)=0\} \le C_1 + (xx'y)^{N_1}.\end{aligned} \tag{27}$$

Further, there exists $C_2 > 0$ such that

$$\begin{aligned}&\sup_{t\in K,t'\in K',s\in L} p_{\nabla X_{|K}(t),\nabla X_{|K'}(t'),\nabla Y_{|L}(s)}(0,0,0|X(t)=x, X(t')=x', Y(s)=y)\\ &\quad\le \sup_{t\in K,t'\in K',s\in L} (2\pi)^{-(k+k'+l)/2}\big[\det\mathrm{Cov}\big(\nabla X_{|K}(t), \nabla X_{|K'}(t'), \nabla Y_{|L}(s)|\\ &\qquad X(t)=x, X(t')=x', Y(s)=y\big)\big]^{-1/2} \le C_2.\end{aligned} \tag{28}$$

Plugging (27) and (28) into (26), we obtain that there exists $C_3 = \mathrm{Vol}(K)\mathrm{Vol}(K')\mathrm{Vol}(L)$ such that

$$\begin{aligned}&\mathbb{E}\{M_u(X,K)M_u(X,K')M_u(Y,L)\}\\ &\quad\le C_3C_2 \sup_{t\in K,t'\in K',s\in L} \mathbb{E}\{(C_1 + |X(t)X(t')Y(s)|^{N_1})\mathbb{1}_{\{X(t)\ge u, X(t')\ge u, Y(s)\ge u\}}\}.\end{aligned} \tag{29}$$

The desired result then follows from Lemma 6.1. The case when one of the dimensions of K, K', and L is zero can be proved similarly. □

6.3. *Factorial moments.* The following result shows that the factorial moments in (21) are super-exponentially small.

LEMMA 6.4. *Let $\{(X(t), Y(s)) : t \in T, s \in S\}$ be an $\mathbb{R}^2$-valued, centered, unit-variance Gaussian vector field satisfying* (**H**1), (**H**2), *and* (**H**3). *Then there exists a constant $\alpha > 0$ such that for all $K \in \partial_k T$ and $L \in \partial_l T$ with $k, l \ge 0$, as $u \to \infty$,*

$$\begin{aligned}\mathbb{E}\{M_u(X,K)[M_u(X,K)-1]M_u(Y,L)\} &= o\bigg(\exp\bigg\{-\frac{u^2}{1+R} - \alpha u^2\bigg\}\bigg),\\ \mathbb{E}\{M_u(X,K)M_u(Y,L)[M_u(Y,L)-1]\} &= o\bigg(\exp\bigg\{-\frac{u^2}{1+R} - \alpha u^2\bigg\}\bigg).\end{aligned} \tag{30}$$

PROOF. We only prove the first line in (30), since the proof for the second line is similar. Note that, if $k = 0$, then $M_u(X,K)[M_u(X,K)-1] \equiv 0$ and hence the desired result holds. Without loss of generality, we assume $k \ge 1$ or even $k = N$ for simplifying notation. We first focus on the estimation when K is replaced by a small N-dimensional subset $J \subset K$.

Case (i): $l = 0$. The face L becomes a single point, say $L = \{s\}$. Applying the Kac–Rice metatheorem for high moments [2], we have the following upper bounds (removing one restriction on u and another restriction on the negative definiteness of Hessian matrices),

$$\begin{aligned}
&\mathbb{E}\{M_u(X,J)[M_u(X,J)-1]M_u(Y,L)\}\\
&\quad\le \int_J dt \int_J dt' \mathbb{E}\{|\det\nabla^2 X(t)||\det\nabla^2 X(t')|\mathbb{1}_{\{X(t)\ge u, Y(s)\ge u\}}|\nabla X(t)=\nabla X(t')=0\}\\
&\qquad\times p_{\nabla X(t),\nabla X(t')}(0,0)\\
&\quad\le \int_J dt \int_J dt' \int_u^\infty dx p_{\frac{X(t)+Y(s)}{2}}(x|\nabla X(t)=\nabla X(t')=0) p_{\nabla X(t),\nabla X(t')}(0,0)\\
&\qquad\times \mathbb{E}\{|\det\nabla^2 X(t)||\det\nabla^2 X(t')||(X(t)+Y(s))/2=x, \nabla X(t)=\nabla X(t')=0\},
\end{aligned}\tag{31}$$

where the last inequality is due to the fact $\mathbb{1}_{\{X(t)\ge u, Y(s)\ge u\}} \le \mathbb{1}_{\{[X(t)+Y(s)]/2\ge u\}}$. Following the same arguments for proving Lemma 3 in Piterbarg [11], we obtain from (31) that, for any $\varepsilon>0$, there exists $\delta>0$ such that for J with $\operatorname{diam}(J)=\sup_{t,t'\in J}\|t-t'\|\le\delta$ and u large enough,

$$\mathbb{E}\{M_u(X,J)[M_u(X,J)-1]M_u(Y,L)\}\le \exp\left\{-\frac{u^2}{2\beta(J,L)+\varepsilon}\right\},\tag{32}$$

where

$$\beta(J,L)=\sup_{t\in J, s\in L, e\in\mathbb{S}^{N-1}} \operatorname{Var}((X(t)+Y(s))/2|\nabla X(t)=0, \nabla^2 X(t)e=0),\tag{33}$$

and $\mathbb{S}^{N-1}$ the $(N-1)$-dimensional unit sphere in $\mathbb{R}^N$.

Case (ii): $l\ge 1$. To simplify the notation, without loss of generality, we assume $l=N$. Applying again the Kac–Rice metatheorem for high moments, we have the following upper bounds:

$$\begin{aligned}
&\mathbb{E}\{M_u(X,J)[M_u(X,J)-1]M_u(Y,L)\}\\
&\quad\le \int_J dt \int_J dt' \int_L ds \mathbb{E}\{|\det\nabla^2 X(t)||\det\nabla^2 X(t')||\det\nabla^2 Y(s)|\mathbb{1}_{\{X(t)\ge u, Y(s)\ge u\}}|\\
&\qquad \nabla X(t)=0, \nabla X(t')=0, \nabla Y(s)=0\} p_{\nabla X(t),\nabla X(t'),\nabla Y(s)}(0,0,0)\\
&\quad\le \int_J dt \int_J dt' \int_L ds \int_u^\infty dx p_{\frac{X(t)+Y(s)}{2}}(x|\nabla X(t)=0, \nabla X(t')=0, \nabla Y(s)=0)\\
&\qquad\times \mathbb{E}\{|\det\nabla^2 X(t)||\det\nabla^2 X(t')||\det\nabla^2 Y(s)||[X(t)+Y(s)]/2=x,\\
&\qquad \nabla X(t)=0, \nabla X(t')=0, \nabla Y(s)=0\} p_{\nabla X(t),\nabla X(t'),\nabla Y(s)}(0,0,0).
\end{aligned}\tag{34}$$

Comparing (34) with (31), the only essential difference is the additional effect of $\nabla Y(s)=0$, which however will not affect the desired super-exponentially small estimation since (X,Y) is nondegenerate under the condition (**H**2). Therefore, similar to (32), we have that, for any $\varepsilon>0$, there exists $\delta>0$ such that for J with $\operatorname{diam}(J)\le\delta$ and u large enough,

$$\begin{aligned}
\mathbb{E}\{M_u(X,J)[M_u(X,J)-1]M_u(Y,L)\}&\le \exp\left\{-\frac{u^2}{2\gamma(J,L)+\varepsilon}\right\}\\
&\le \exp\left\{-\frac{u^2}{2\beta(J,L)+\varepsilon}\right\},
\end{aligned}\tag{35}$$

where

$$\gamma(J,L)=\sup_{t\in J, s\in L, e\in\mathbb{S}^{N-1}} \operatorname{Var}((X(t)+Y(s))/2|\nabla X(t)=\nabla Y(s)=\nabla^2 X(t)e=0)\le\beta(J,L).$$

The set K may be covered by congruent cubes J_i with disjoint interiors, edges parallel to coordinate axes and sizes small enough such that $\operatorname{diam}(J_i \cup J_j) \le \delta$ for any two neighboring cubes J_i and J_j (i.e., $d(J_i, J_j) = 0$). Then

$$
\begin{aligned}
&\mathbb{E}\{M_u(X,K)[M_u(X,K)-1]M_u(Y,L)\}\\
&\quad\le \mathbb{E}\Big\{\Big(\sum_i M_u(X,J_i)\Big)\Big[\sum_j M_u(X,J_j)-1\Big]M_u(Y,L)\Big\}\\
&\quad= \mathbb{E}\Big\{\Big(\sum_i M_u(X,J_i)\sum_j M_u(X,J_j)-\sum_i M_u(X,J_i)\Big)M_u(Y,L)\Big\}\\
&\quad= \sum_i \mathbb{E}\{M_u(X,J_i)^2 M_u(Y,L)\} + \sum_{i\neq j}\mathbb{E}\{M_u(X,J_i)M_u(X,J_j)M_u(Y,L)\}\\
&\qquad - \sum_i \mathbb{E}\{M_u(X,J_i)M_u(Y,L)\}\\
&\quad= \sum_i \mathbb{E}\{M_u(X,J_i)[M_u(X,J_i)-1]M_u(Y,L)\}\\
&\qquad + \sum_{i\neq j}\mathbb{E}\{M_u(X,J_i)M_u(X,J_j)M_u(Y,L)\}.
\end{aligned}
\tag{36}
$$

By Lemma 6.3, there exists $\alpha' > 0$ such that for u large enough,

$$
\sum_{i\neq j: d(J_i,J_j)>0} \mathbb{E}\{M_u(X,J_i)M_u(X,J_j)M_u(Y,L)\} \le \exp\Big\{-\frac{u^2}{1+R}-\alpha' u^2\Big\}.
\tag{37}
$$

If J_i and J_j are neighboring, that is, $d(J_i, J_j) = 0$, we have

$$
\begin{aligned}
&\mathbb{E}\{M_u(X,J_i\cup J_j)[M_u(X,J_i\cup J_j)-1]M_u(Y,L)\}\\
&\quad= \mathbb{E}\{[M_u(X,J_i)+M_u(X,J_j)][M_u(X,J_i)+M_u(X,J_j)-1]M_u(Y,L)\}\\
&\quad= 2\mathbb{E}\{M_u(X,J_i)M_u(X,J_j)M_u(Y,L)\} + \mathbb{E}\{M_u(X,J_i)[M_u(X,J_i)-1]M_u(Y,L)\}\\
&\qquad + \mathbb{E}\{M_u(X,J_j)[M_u(X,J_j)-1]M_u(Y,L)\}.
\end{aligned}
\tag{38}
$$

Applying (32) and (35) to the second last sum in (36) and (38), we see that for any $\varepsilon > 0$ and u large enough,

$$
\begin{aligned}
&\sum_i \mathbb{E}\{M_u(X,J_i)[M_u(X,J_i)-1]M_u(Y,L)\}\\
&\quad + \sum_{i\neq j: d(J_i,J_j)=0} \mathbb{E}\{M_u(X,J_i)M_u(X,J_j)M_u(Y,L)\} \le \exp\Big\{-\frac{u^2}{2\beta(K,L)+\varepsilon}\Big\},
\end{aligned}
\tag{39}
$$

where $\beta(K, L)$ is defined in (33) with J replaced by K. It is evident that

$$
\beta(K,L) \le \sup_{t\in K, s\in L} \operatorname{Var}\big((X(t)+Y(s))/2\big) = (1+R)/2.
$$

Moreover, we will show below that

$$
\beta(K,L) < (1+R)/2.
\tag{40}
$$

By the definition, if $\beta(K, L) = (1 + R)/2$, then there exist $(t, s) \in \bar{K} \times \bar{L}$ and $e \in \mathbb{S}^{N-1}$ such that

$$
\operatorname{Var}\big((X(t)+Y(s))/2 \,|\, \nabla X(t) = 0, \nabla^2 X(t)e = 0\big) = (1+R)/2,
\tag{41}
$$

implying $r(t,s)=R$ and $\mathbb{E}\{[X(t)+Y(s)]\nabla X(t)\}=\mathbb{E}\{Y(s)\nabla X(t)\}=0$. By Proposition 2.1, $\mathbb{E}\{Y(s)\nabla^2 X(t)\}\preceq 0$. Since $X(t)$ has unit variance, $\mathbb{E}\{X(t)\nabla^2 X(t)\}=-\mathrm{Cov}(\nabla X(t))\prec 0$. Therefore, $\mathbb{E}\{[X(t)+Y(s)]\nabla^2 X(t)e\}\neq 0$ for all $e\in\mathbb{S}^{N-1}$. This contradicts (41) and hence (40) holds. Applying this fact and plugging (37) and (39) into (36), we finish the proof. □

6.4. *Adjacent faces.* The following result shows that the last two sums involving the joint moment of two adjacent faces in (21) are super-exponentially small.

LEMMA 6.5. *Let* $\{(X(t),Y(s)):t\in T,s\in S\}$ *be an* $\mathbb{R}^2$*-valued, centered, unit-variance Gaussian vector field satisfying* (**H**1), (**H**2), *and* (**H**3). *Then there exists* $\alpha>0$ *such that as* $u\to\infty$,

$$\begin{aligned}
&\mathbb{E}\{M_u^E(X,K)M_u^E(X,K')M_u^E(Y,L)\}=o\left(\exp\left\{-\frac{u^2}{1+R}-\alpha u^2\right\}\right),\\
&\mathbb{E}\{M_u^E(X,K)M_u^E(Y,L)M_u^E(Y,L')\}=o\left(\exp\left\{-\frac{u^2}{1+R}-\alpha u^2\right\}\right),
\end{aligned}\tag{42}$$

where K *and* K' *are different faces of* T *with* $d(K,K')=0$, L *and* L' *are different faces of* S *with* $d(L,L')=0$.

PROOF. We only prove the first line in (42), since the proof for the second line is the same. Let $I:=\bar{K}\cap\bar{K}'$, which is nonempty since $d(K,K')=0$. Without loss of generality, assume

$$\begin{aligned}
\sigma(K)&=\{1,\dots,m,m+1,\dots,k\},\\
\sigma(K')&=\{1,\dots,m,k+1,\dots,k+k'-m\},\\
\sigma(L)&=\{1,\dots,l\},
\end{aligned}$$

where $0\le m\le k\le k'\le N$ and $k'\ge 1$. If $k=0$, we consider $\sigma(K)=\varnothing$ by convention. Under such assumption, $K\in\partial_k T$, $K'\in\partial_{k'}T$, $\dim(I)=m$ and $L\in\partial_l S$. We assume also that all elements in $\varepsilon(K)$ and $\varepsilon(K')$ are 1.

We first consider the case when $k\ge 1$ and $l\ge 1$. By the Kac–Rice metatheorem,

$$\begin{aligned}
&\mathbb{E}\{M_u^E(X,K)M_u^E(X,K')M_u^E(Y,L)\}\\
&\quad\le\int_K dt\int_{K'}dt'\int_L ds\int_u^\infty dx\int_u^\infty dx'\int_u^\infty dy\int_0^\infty dz_{k+1}\cdots\int_0^\infty dz_{k+k'-m}\\
&\qquad\int_0^\infty dw_{m+1}\cdots\int_0^\infty dw_k\mathbb{E}\{|\det\nabla^2 X_{|K}(t)||\det\nabla^2 X_{|K'}(t')||\det\nabla^2 Y_{|L}(s)||\\
&\qquad X(t)=x,X(t')=x',Y(s)=y,\\
&\qquad\nabla X_{|K}(t)=0,X_{k+1}(t)=z_{k+1},\dots,X_{k+k'-m}(t)=z_{k+k'-m},\\
&\qquad\nabla X_{|K'}(t')=0,X_{m+1}(t')=w_{m+1},\dots,X_k(t')=w_k,\nabla Y_{|L}(s)=0\}\\
&\qquad\times p_{t,t',s}(x,x',y,0,z_{k+1},\dots,z_{k+k'-m},0,w_{m+1},\dots,w_k,0)\\
&\quad:=\int\int\int_{K\times K'\times L}A(t,t',s)\,dt\,dt'\,ds,
\end{aligned}\tag{43}$$

where $p_{t,t',s}(x,x',y,0,z_{k+1},\dots,z_{k+k'-m},0,w_{m+1},\dots,w_k,0)$ is the density of

$$\begin{aligned}
&(X(t),X(t'),Y(s),\nabla X_{|K}(t),X_{k+1}(t),\dots,X_{k+k'-m}(t),\\
&\nabla X_{|K'}(t'),X_{m+1}(t'),\dots,X_k(t'),\nabla Y_{|L}(s))
\end{aligned}$$

evaluated at $(x, x', y, 0, z_{k+1}, \dots, z_{k+k'-m}, 0, w_{m+1}, \dots, w_k, 0)$. We define

$$\mathcal{M}_0 := \{(t,s) \in I \times \bar{L} : r(t,s) = R, \mathbb{E}\{X_i(t)Y(s)\} = \mathbb{E}\{X(t)Y_j(s)\} = 0, \forall i = 1, \dots, k+k'-m, j = 1, \dots, l\}, \tag{44}$$

and distinguish two cases for $\mathcal{M}_0$ in discussions below.

Case (i): $\mathcal{M}_0 = \varnothing$. Since I is a compact set, by the uniform continuity of conditional variance, there exist constants $\varepsilon_1, \delta_1 > 0$ such that

$$\sup_{t \in B(I,\delta_1), t' \in B'(I,\delta_1), s \in L} \mathrm{Var}\big(\big[X(t) + Y(s)\big]/2 | \nabla X_{|K}(t), \nabla X_{|K'}(t'), \nabla Y_{|L}(s)\big) \le \frac{1+R}{2} - \varepsilon_1, \tag{45}$$

where $B(I,\delta_1) = \{t \in K : d(t,I) \le \delta_1\}$ and $B'(I,\delta_1) = \{t \in K' : d(t,I) \le \delta_1\}$. Partitioning $K \times K'$ into $B(I,\delta_1) \times B'(I,\delta_1)$ and $(K \times K') \backslash (B(I,\delta_1) \times B'(I,\delta_1))$, and applying the Kac–Rice formula, we obtain

$$\begin{aligned} &\mathbb{E}\{M_u(X,K)M_u(X,K')M_u(Y,L)\} \\ &\le \int_{(K\times K')\backslash(B(I,\delta_1)\times B'(I,\delta_1))} dt\,dt' \int_L ds p_{\nabla X_{|K}(t), \nabla X_{|K'}(t'), \nabla Y_{|L}(s)}(0,0,0) \\ &\quad \times \mathbb{E}\{|\det \nabla^2 X_{|K}(t)||\det \nabla^2 X_{|K'}(t')||\det \nabla^2 Y_{|L}(s)| \mathbb{1}_{\{X(t)\ge u, X(t')\ge u, Y(s) \ge u\}} | \\ &\quad \nabla X_{|K}(t) = 0, \nabla X_{|K'}(t') = 0, \nabla Y_{|L}(s) = 0\} \\ &\quad + \int_{B(I,\delta_1)\times B'(I,\delta_1)} dt\,dt' \int_L ds p_{\nabla X_{|K}(t), \nabla X_{|K'}(t'), \nabla Y_{|L}(s)}(0,0,0) \\ &\quad \times \mathbb{E}\{|\det \nabla^2 X_{|K}(t)||\det \nabla^2 X_{|K'}(t')||\det \nabla^2 Y_{|L}(s)| \mathbb{1}_{\{X(t)\ge u, Y(s) \ge u\}} | \\ &\quad \nabla X_{|K}(t) = 0, \nabla X_{|K'}(t') = 0, \nabla Y_{|L}(s) = 0\} \\ &:= I_1 + I_2. \end{aligned} \tag{46}$$

Note that

$$\begin{aligned} (K \times K') \backslash (B(I,\delta_1) \times B'(I,\delta_1)) &= ((K \backslash B(I,\delta_1)) \times B'(I,\delta_1)) \cup (B(I,\delta_1) \times (K \backslash B(I,\delta_1))) \\ &\quad \cup ((K \backslash B(I,\delta_1)) \times (K \backslash B(I,\delta_1))), \end{aligned}$$

where each product on the right hand side consists of two sets with a positive distance. It then follows from Lemma 6.3 that I_1 is super-exponentially small. On the other hand, since $\mathbb{1}_{\{X(t)\ge u, Y(s)\ge u\}} \le \mathbb{1}_{\{[X(t)+Y(s)]/2 \ge u\}}$, one has

$$\begin{aligned} I_2 &\le \int_{B(I,\delta_1)\times B'(I,\delta_1)} dt\,dt' \\ &\quad \times \int_L ds \int_u^\infty dx p_{\frac{X(t)+Y(s)}{2}}\big(x | \nabla X_{|K}(t) = \nabla X_{|K'}(t') = \nabla Y_{|L}(s) = 0\big) \\ &\quad \times \mathbb{E}\{|\det \nabla^2 X_{|K}(t)||\det \nabla^2 X_{|K'}(t')||\det \nabla^2 Y_{|L}(s)|| \big[X(t)+Y(s)\big]/2 = x, \\ &\quad \nabla X_{|K}(t) = 0, \nabla X_{|K'}(t') = 0, \nabla Y_{|L}(s) = 0\} p_{\nabla X_{|K}(t), \nabla X_{|K'}(t'), \nabla Y_{|L}(s)}(0,0,0). \end{aligned} \tag{47}$$

Combining this with (45), we obtain that I_2 and hence $\mathbb{E}\{M_u^E(X,K)M_u^E(X,K')M_u^E(Y,L)\}$ are super-exponentially small.

Case (ii): $\mathcal{M}_0 \neq \varnothing$. Let

$$B(\mathcal{M}_0, \delta_2) := \{(t,t',s) \in K \times K' \times L : d((t,s), \mathcal{M}_0) \vee d((t',s), \mathcal{M}_0) \le \delta_2\},$$

where δ_2 is a small positive number to be specified. Note that, by the definitions of $\mathcal{M}_0$ and $B(\mathcal{M}_0, \delta_2)$, there exists $\varepsilon_2 > 0$ such that

$$\sup_{(t,t',s)\in(K\times K'\times L)\setminus B(\mathcal{M}_0,\delta_2)} \operatorname{Var}\big(\big[X(t)+Y(s)\big]/2|\nabla X_{|K}(t), \nabla X_{|K'}(t'), \nabla Y_{|L}(s)\big) \le \frac{1+R}{2} - \varepsilon_2. \tag{48}$$

Similar to (47), we obtain that $\int_{(K\times K'\times L)\setminus B(\mathcal{M}_0,\delta_2)} A(t,t',s)\,dt\,dt'\,ds$ is super-exponentially small. It suffices to show below that $\int_{B(\mathcal{M}_0,\delta_2)} A(t,t',s)\,dt\,dt'\,ds$ is super-exponentially small.

Due to (**H**3) and Proposition 2.1, we can choose δ_2 small enough such that for all $(t,t',s)\in B(\mathcal{M}_0,\delta_2)$,

$$\begin{aligned}\Lambda_{K\cup K'}(t,s) &:= -\mathbb{E}\big\{\big[X(t)+Y(s)\big]\nabla^2 X_{|K\cup K'}(t)\big\}\\ &= -\big(\mathbb{E}\big\{\big[X(t)+Y(s)\big]X_{ij}(t)\big\}\big)_{i,j=1,\dots,k+k'-m}\end{aligned}$$

are positive definite. Let $\{e_1, e_2, \dots, e_N\}$ be the standard orthonormal basis of $\mathbb{R}^N$. For $t\in K$, $t'\in K'$ and $s\in L$, let $e_{t,t'} = (t'-t)/\|t'-t\|$ and $\alpha_i(t,t',s) = \langle e_i, \Lambda_{K\cup K'}(t,s)e_{t,t'}\rangle$. Then

$$\Lambda_{K\cup K'}(t,s)e_{t,t'} = \sum_{i=1}^N \langle e_i, \Lambda_{K\cup K'}(t,s)e_{t,t'}\rangle e_i = \sum_{i=1}^N \alpha_i\big(t,t',s\big)e_i \tag{49}$$

and there exists $\alpha_0 > 0$ such that for all $(t,t',s)\in B(\mathcal{M}_0,\delta_2)$,

$$\big\langle e_{t,t'}, \Lambda_{K\cup K'}(t,s)e_{t,t'}\big\rangle \ge \alpha_0. \tag{50}$$

Since all elements in $\varepsilon(K)$ and $\varepsilon(K')$ are 1, we may write

$$\begin{aligned}t &= (t_1,\dots,t_m,t_{m+1},\dots,t_k,b_{k+1},\dots,b_{k+k'-m},0,\dots,0),\\ t' &= \big(t'_1,\dots,t'_m,b_{m+1},\dots,b_k,t'_{k+1},\dots,t'_{k+k'-m},0,\dots,0\big),\end{aligned}$$

where $t_i\in(a_i,b_i)$ for $i\in\sigma(K)$ and $t'_j\in(a_j,b_j)$ for $j\in\sigma(K')$. Therefore,

$$\begin{aligned}&\langle e_i, e_{t,t'}\rangle \ge 0, \quad \forall m+1\le i\le k,\\ &\langle e_i, e_{t,t'}\rangle \le 0, \quad \forall k+1\le i\le k+k'-m,\\ &\langle e_i, e_{t,t'}\rangle = 0, \quad \forall k+k'-m < i\le N.\end{aligned} \tag{51}$$

Let

$$\begin{aligned}&D_i = \big\{\big(t,t',s\big)\in B(\mathcal{M}_0,\delta_2) : \alpha_i\big(t,t',s\big)\ge\beta_i\big\}, \quad \text{if } m+1\le i\le k,\\ &D_i = \big\{\big(t,t',s\big)\in B(\mathcal{M}_0,\delta_2) : \alpha_i\big(t,t',s\big)\le-\beta_i\big\}, \quad \text{if } k+1\le i\le k+k'-m,\\ &D_0 = \left\{\big(t,t',s\big)\in B(\mathcal{M}_0,\delta_2) : \sum_{i=1}^m \alpha_i\big(t,t',s\big)\langle e_i, e_{t,t'}\rangle \ge \beta_0\right\},\end{aligned} \tag{52}$$

where $\beta_0,\beta_1,\dots,\beta_{k+k'-m}$ are positive constants such that $\beta_0 + \sum_{i=m+1}^{k+k'-m}\beta_i < \alpha_0$. It follows from (51) and (52) that, if (t,s) does not belong to any of $D_0, D_{m+1},\dots,D_{k+k'-m}$, then by (49),

$$\big\langle \Lambda_{K\cup K'}(t,s)e_{t,t'}, e_{t,t'}\big\rangle = \sum_{i=1}^N \alpha_i\big(t,t',s\big)\langle e_i, e_{t,t'}\rangle \le \beta_0 + \sum_{i=m+1}^{k+k'-m}\beta_i < \alpha_0,$$

which contradicts (50). Thus $D_0 \cup \bigcup_{i=m+1}^{k+k'-m} D_i$ is a covering of $B(\mathcal{M}_0, \delta_2)$. By (43),

$$\mathbb{E}\{M_u^E(X,K)M_u^E(X,K')M_u^E(Y,L)\} \le \int_{D_0} A(t,t',s)\,dt\,dt'\,ds + \sum_{i=m+1}^{k+k'-m} \int_{D_i} A(t,t',s)\,dt\,dt'\,ds.$$

By the Kac–Rice metatheorem and the fact $\mathbb{1}_{\{X(t)\ge u, Y(s)\ge u\}} \le \mathbb{1}_{\{[X(t)+Y(s)]/2 \ge u\}}$, we obtain

$$\begin{aligned}\int_{D_0} A(t,t',s)\,dt\,dt'\,ds &\le \int_{D_0} dt\,dt'\,ds \int_u^\infty dx\, p_{\nabla X_{|K}(t), \nabla X_{|K'}(t'), \nabla Y_{|L}(s)}(0,0,0) \\ &\quad \times p_{[X(t)+Y(s)]/2}(x \mid \nabla X_{|K}(t)=0, \nabla X_{|K'}(t')=0, \nabla Y_{|L}(s)=0) \\ &\quad \times \mathbb{E}\{|\det \nabla^2 X_{|K}(t)||\det \nabla^2 X_{|K'}(t')||\det \nabla^2 Y_{|L}(s)| \mid [X(t)+Y(s)]/2 = x, \\ &\quad \nabla X_{|K}(t) = \nabla X_{|K'}(t') = \nabla Y_{|L}(s) = 0\},\end{aligned} \tag{53}$$

and that for $i = m+1, \dots, k$,

$$\begin{aligned}\int_{D_i} A(t,t',s)\,dt\,dt'\,ds &\le \int_{D_i} dt\,dt'\,ds \int_u^\infty dx \int_0^\infty dw_i \mathbb{E}\{|\det \nabla^2 X_{|K}(t)||\det \nabla^2 X_{|K'}(t')||\det \nabla^2 Y_{|L}(s)| \mid \\ &\quad [X(t)+Y(s)]/2 = x, \nabla X_{|K}(t) = 0, X_i(t') = w_i, \nabla X_{|K'}(t') = \nabla Y_{|L}(s) = 0\} \\ &\quad \times p_{[X(t)+Y(s)]/2, \nabla X_{|K}(t), X_i(t'), \nabla X_{|K'}(t'), \nabla Y_{|L}(s)}(x, 0, w_i, 0, 0).\end{aligned} \tag{54}$$

Comparing (53) and (54) with equations (4.33) and (4.36) respectively in the proof of Theorem 4.8 in Cheng and Xiao [5], the only essential difference is the additional effect of $\nabla Y_{|L}(s) = 0$, which however will not affect the desired super-exponentially small estimation since (X, Y) is nondegenerate under the condition (**H**2). Therefore, following similar arguments therein, we obtain that $\int_{D_0} A(t,t',s)\,dt\,dt'\,ds$ and $\int_{D_i} A(t,t',s)\,dt\,dt'\,ds$ ($i = m+1, \dots, k$) are super-exponentially small.

It is similar to show that $\int_{D_i} A(t,t',s)\,dt\,dt'\,ds$ are super-exponentially small for $i = k+1, \dots, k+k'-m$. For the case $k = 0$ or $l = 0$, the argument is even simpler when applying the Kac–Rice formula (see, e.g., (31)). Hence the details are omitted here. We have completed the proof. □

Notice that, in the proof of Lemma 6.5, we have shown in (46) that, if $\mathcal{M}_0 = \varnothing$, then $\mathbb{E}\{M_u(X,K)M_u(X,K')M_u(Y,L)\}$ is super-exponentially small. Under the boundary condition (7), which implies and generalizes the condition $\mathcal{M}_0 = \varnothing$ in terms of the correlation function $r(t,s)$, we have the following result.

LEMMA 6.6. *Let $\{(X(t), Y(s)) : t \in T, s \in S\}$ be an $\mathbb{R}^2$-valued, centered, unit-variance Gaussian vector field satisfying* (**H**1), (**H**2), *and the boundary condition* (7). *Then there exists a constant $\alpha > 0$ such that as $u \to \infty$,*

$$\mathbb{E}\{M_u(X,K)M_u(X,K')M_u(Y,L)\} = o\left(\exp\left\{-\frac{u^2}{1+R} - \alpha u^2\right\}\right),$$

$$\mathbb{E}\{M_u(X,K)M_u(Y,L)M_u(Y,L')\} = o\left(\exp\left\{-\frac{u^2}{1+R} - \alpha u^2\right\}\right),$$

where K and K' are adjacent faces of T, and L and L' are adjacent faces of S.

7. Estimation of the difference between EEC and the upper bound. In this section, we shall show that the difference between the expected number of extended outward local maxima, that is, the upper bound in (20), and the expected Euler characteristic of the excursion set is super-exponentially small.

PROPOSITION 7.1. *Let $\{(X(t), Y(s)) : t \in T, s \in S\}$ be an $\mathbb{R}^2$-valued, centered, unit-variance Gaussian vector field satisfying* (**H**1), (**H**2), *and* (**H**3). *Then there exists $\alpha > 0$ such that for any $K \in \partial_k T$ and $L \in \partial_l S$ with $k, l \geq 0$, as $u \to \infty$,*

$$\begin{aligned}
&\mathbb{E}\{M_u^E(X,K)M_u^E(Y,L)\} \\
&= (-1)^{k+l}\int_K\int_L \mathbb{E}\{\det\nabla^2 X_{|K}(t)\det\nabla^2 Y_{|L}(s)\mathbb{1}_{\{X(t)\geq u, \varepsilon_\ell^* X_\ell(t)\geq 0 \text{ for all } \ell\notin\sigma(K)\}} \\
&\quad \times \mathbb{1}_{\{Y(s)\geq u, \varepsilon_\ell^* Y_\ell(s)\geq 0 \text{ for all } \ell\notin\sigma(L)\}} | \nabla X_{|K}(t) = \nabla Y_{|L}(s) = 0\} \\
&\quad \times p_{\nabla X_{|K}(t), \nabla Y_{|L}(s)}(0,0)\,dt\,ds + o\left(\exp\left\{-\frac{u^2}{1+R} - \alpha u^2\right\}\right) \\
&= (-1)^{k+l}\mathbb{E}\left\{\left(\sum_{i=0}^k (-1)^i \mu_i(X,K)\right)\left(\sum_{j=0}^l (-1)^j \mu_j(Y,L)\right)\right\} \\
&\quad + o\left(\exp\left\{-\frac{u^2}{1+R} - \alpha u^2\right\}\right).
\end{aligned} \tag{55}$$

PROOF. The second equality in (55) follows from the application of the Kac–Rice theorem below:

$$\begin{aligned}
&\mathbb{E}\left\{\left(\sum_{i=0}^k (-1)^i \mu_i(X,K)\right)\left(\sum_{j=0}^l (-1)^j \mu_j(Y,L)\right)\right\} \\
&= \sum_{i=0}^k (-1)^i \sum_{j=0}^l (-1)^j \int_K\int_L dt\,ds\,p_{\nabla X_{|K}(t), \nabla Y_{|L}(s)}(0,0) \\
&\quad \times \mathbb{E}\{|\det\nabla^2 X_{|K}(t)||\det\nabla^2 Y_{|L}(s)|\mathbb{1}_{\{\text{index}(\nabla^2 X_{|K}(t))=i\}}\mathbb{1}_{\{\text{index}(\nabla^2 Y_{|L}(t))=j\}} \\
&\quad \times \mathbb{1}_{\{X(t)\geq u, \varepsilon_\ell^* X_\ell(t)\geq 0 \text{ for all } \ell\notin\sigma(K)\}}\mathbb{1}_{\{Y(s)\geq u, \varepsilon_\ell^* Y_\ell(s)\geq 0 \text{ for all } \ell\notin\sigma(L)\}} \\
&\quad |\nabla X_{|K}(t) = \nabla Y_{|L}(s) = 0\} \\
&= \int_K\int_L dt\,ds\,p_{\nabla X_{|K}(t), \nabla Y_{|L}(s)}(0,0)\mathbb{E}\{\det\nabla^2 X_{|K}(t)\det\nabla^2 Y_{|L}(s) \\
&\quad \times \mathbb{1}_{\{X(t)\geq u, \varepsilon_\ell^* X_\ell(t)\geq 0 \text{ for all } \ell\notin\sigma(K)\}}\mathbb{1}_{\{Y(s)\geq u, \varepsilon_\ell^* Y_\ell(s)\geq 0 \text{ for all } \ell\notin\sigma(L)\}}| \\
&\quad \nabla X_{|K}(t) = \nabla Y_{|L}(s) = 0\},
\end{aligned}$$

where the last step is because, for a matrix B, $(-1)^i|\det(B)| = \det(B)$ if $\text{index}(B) = i$.

To prove the first approximation in (55) and address the main idea, we first deal with a special case when the two faces are both the interiors and then prove the general cases.

Case (i): $k = N$ and $l = N'$. By the Kac–Rice formula,

$$\begin{aligned}
&\mathbb{E}\{M_u^E(X,K)M_u^E(Y,L)\}\\
&\quad = \int_K \int_L p_{\nabla X(t),\nabla Y(s)}(0,0)\,dt\,ds \int_u^\infty \int_u^\infty dx\,dy p_{X(t),Y(s)}(x,y|\nabla X(t)=\nabla Y(s)=0)\\
&\qquad \times \mathbb{E}\{\det\nabla^2 X(t)\det\nabla^2 Y(s)\mathbb{1}_{\{\nabla^2 X(t)\prec 0\}}\mathbb{1}_{\{\nabla^2 Y(s)\prec 0\}}|\\
&\qquad X(t)=x, Y(s)=y, \nabla X(t)=\nabla Y(s)=0\}\\
&\quad := \int_K \int_L p_{\nabla X(t),\nabla Y(s)}(0,0)\,dt\,ds \int_u^\infty \int_u^\infty A(t,s,x,y)\,dx\,dy.
\end{aligned}$$

Let

$$\begin{aligned}
\mathcal{M}_1 &= \{(t,s)\in \bar{K}\times\bar{L} : r(t,s)=R, \mathbb{E}\{X(t)\nabla Y(s)\}=\mathbb{E}\{Y(s)\nabla X(t)\}=0\},\\
B(\mathcal{M}_1,\delta_1) &= \{(t,s)\in K\times L : d((t,s),\mathcal{M}_1)\le \delta_1\},
\end{aligned} \tag{56}$$

where δ_1 is a small positive number to be specified. Then, we only need to estimate

$$\int_{B(\mathcal{M}_1,\delta_1)} p_{\nabla X(t),\nabla Y(s)}(0,0)\,dt\,ds \int_u^\infty \int_u^\infty A(t,s,x,y)\,dx\,dy, \tag{57}$$

since the integral above with $B(\mathcal{M}_1,\delta_1)$ replaced by $(K\times L)\backslash B(\mathcal{M}_1,\delta_1)$ is super-exponentially small due to the fact

$$\sup_{(t,s)\in(K\times L)\backslash B(\mathcal{M}_1,\delta_1)} \mathrm{Var}([X(t)+Y(s)]/2|\nabla X(t)=\nabla Y(s)=0) < \frac{1+R}{2}.$$

Notice that, for all $(t,s)\in\mathcal{M}_1$, $\mathbb{E}\{X(t)\nabla^2 X(t)\}\prec 0$, and $\mathbb{E}\{Y(s)\nabla^2 Y(s)\}\prec 0$ since $X(t)$ and $Y(s)$ have unit-variance; and by (**H**3) and Proposition 2.1, $\mathbb{E}\{X(t)\nabla^2 Y(s)\}\preceq 0$ and $\mathbb{E}\{Y(s)\nabla^2 X(t)\}\preceq 0$. Thus there exists δ_1 small enough such that $\mathbb{E}\{[X(t)+Y(s)]\nabla^2 Y(s)\}\prec 0$ and $\mathbb{E}\{[X(t)+Y(s)]\nabla^2 X(t)\}\prec 0$ for all $(t,s)\in B(\mathcal{M}_1,\delta_1)$. In particular, let λ_0 be the largest eigenvalue of $\mathbb{E}\{[X(t)+Y(s)]\nabla^2 X(t)\}$ over $B(\mathcal{M}_1,\delta_1)$, then $\lambda_0<0$ by the uniform continuity. Also note that both $\mathbb{E}\{X(t)\nabla Y(s)\}$ and $\mathbb{E}\{Y(s)\nabla X(t)\}$ tend to 0 as $\delta_1\to 0$. Therefore, as $\delta_1\to 0$,

$$\begin{aligned}
&\mathbb{E}\{X_{ij}(t)|X(t)=x, Y(s)=y, \nabla X(t)=\nabla Y(s)=0\}\\
&\quad = (\mathbb{E}\{X_{ij}(t)X(t)\}, \mathbb{E}\{X_{ij}(t)Y(s)\}, \mathbb{E}\{X_{ij}(t)X_1(t)\},\dots,\mathbb{E}\{X_{ij}(t)X_N(t)\},\\
&\qquad \mathbb{E}\{X_{ij}(t)Y_1(s)\},\dots,\mathbb{E}\{X_{ij}(t)Y_N(s)\})[\mathrm{Cov}(X(t),Y(s),\nabla X(t),\nabla Y(s))]^{-1}\\
&\qquad \times (x,y,0,\dots,0,0,\dots,0)^T\\
&\qquad = (1+o(1))(\mathbb{E}\{X_{ij}(t)X(t)\},\mathbb{E}\{X_{ij}(t)Y(s)\})\begin{pmatrix}1 & R\\ R & 1\end{pmatrix}^{-1}\begin{pmatrix}x\\ y\end{pmatrix}\\
&\quad = (1+o(1))\frac{\mathbb{E}\{X_{ij}(t)X(t)\}[x-Ry]+\mathbb{E}\{X_{ij}(t)Y(s)\}[y-Rx]}{1-R^2};
\end{aligned} \tag{58}$$

and similarly,

$$\begin{aligned}
&\mathbb{E}\{Y_{ij}(s)|X(t)=x, Y(s)=y, \nabla X(t)=\nabla Y(s)=0\}\\
&\quad = (1+o(1))\frac{\mathbb{E}\{Y_{ij}(s)X(t)\}[x-Ry]+\mathbb{E}\{Y_{ij}(s)Y(s)\}[y-Rx]}{1-R^2}.
\end{aligned} \tag{59}$$

By (58) and (59), there exists $0 < \varepsilon_0 < 1 - R$ such that for δ_1 small enough and all $(x, y) \in [u, \infty)^2$ with $(\varepsilon_0 + R)x < y < (\varepsilon_0 + R)^{-1}x$ (so that $x - Ry \geq \varepsilon_0 u$ and $y - Rx \geq \varepsilon_0 u$),

$$\Sigma_1(t, s, x, y) := \mathbb{E}\{\nabla^2 X(t) | X(t) = x, Y(s) = y, \nabla X(t) = \nabla Y(s) = 0\} \prec 0 \quad \text{and}$$

$$\Sigma_2(t, s, x, y) := \mathbb{E}\{\nabla^2 Y(s) | X(t) = x, Y(s) = y, \nabla X(t) = \nabla Y(s) = 0\} \prec 0.$$

Let $\Delta_1(t, s, x, y) = \nabla^2 X(t) - \Sigma_1(t, s, x, y)$ and $\Delta_2(t, s, x, y) = \nabla^2 Y(s) - \Sigma_2(t, s, x, y)$. Due to the following decomposition,

$$\{u \leq x, y < \infty\} = \{x \geq u, y \geq (\varepsilon_0 + R)^{-1}x\} \cup \{y \geq u, x \leq (\varepsilon_0 + R)^{-1}y\}$$
$$\cup \{x \geq u, u \vee (\varepsilon_0 + R)x < y < (\varepsilon_0 + R)^{-1}x\},$$

we can write

$$\begin{aligned}
&\int_u^\infty \int_u^\infty A(t, s, x, y)\,dx\,dy \\
&\quad = \int_u^\infty dx \int_{(\varepsilon_0+R)^{-1}x}^\infty A(t, s, x, y)\,dy + \int_u^\infty dy \int_{(\varepsilon_0+R)^{-1}y}^\infty A(t, s, x, y)\,dx \\
&\qquad + \int_u^\infty dx \int_{u\vee(\varepsilon_0+R)x}^{(\varepsilon_0+R)^{-1}y} A(t, s, x, y)\,dx,
\end{aligned} \tag{60}$$

where the first two integrals on right are super-exponentially small since $(\varepsilon_0 + R)^{-1} > 1$ and $\mathbb{1}_{\{X(t)\geq u, Y(s)\geq(\varepsilon_0+R)^{-1}X(t)\}} \vee \mathbb{1}_{\{Y(s)\geq u, X(t)\geq(\varepsilon_0+R)^{-1}Y(s)\}} \leq \mathbb{1}_{\{[X(t)+Y(s)]/2\geq[1+(\varepsilon_0+R)^{-1}]u/2\}}$. For the last integral in (60), we have

$$\begin{aligned}
&\int_u^\infty dx \int_{u\vee(\varepsilon_0+R)x}^{(\varepsilon_0+R)^{-1}x} A(t, s, x, y)\,dy \\
&\quad = \int_u^\infty dx \int_{u\vee(\varepsilon_0+R)x}^{(\varepsilon_0+R)^{-1}x} dy p_{X(t),Y(s)}(x, y | \nabla X(t) = \nabla Y(s) = 0) \\
&\qquad \times \mathbb{E}\{\det(\Delta_1(t, s, x, y) + \Sigma_1(t, s, x, y))\det(\Delta_2(t, s, x, y) \\
&\qquad + \Sigma_2(t, s, x, y))\mathbb{1}_{\{\Delta_1(t,s,x,y)+\Sigma_1(t,s,x,y)\prec 0\}} \\
&\qquad \times \mathbb{1}_{\{\Delta_2(t,s,x,y)+\Sigma_2(t,s,x,y)\prec 0\}} | X(t) = x, Y(s) = y, \nabla X(t) = \nabla Y(s) = 0\} \\
&\quad := \int_u^\infty dx \int_{u\vee(\varepsilon_0+R)x}^{(\varepsilon_0+R)^{-1}x} dy p_{X(t),Y(s)}(x, y | \nabla X(t) = \nabla Y(s) = 0) E(t, s, x, y).
\end{aligned} \tag{61}$$

Note that the following are two centered Gaussian random matrices (free of x and y):

$$\Omega^X(t, s) = (\Omega^X_{ij}(t, s))_{1\leq i,j\leq N} = (\Delta_1(t, s, x, y) | X(t) = x, Y(s) = y, \nabla X(t) = \nabla Y(s) = 0),$$

$$\Omega^Y(t, s) = (\Omega^Y_{ij}(t, s))_{1\leq i,j\leq N'} = (\Delta_2(t, s, x, y) | X(t) = x, Y(s) = y, \nabla X(t) = \nabla Y(s) = 0).$$

Denote the density of the Gaussian vector $((\Omega^X_{ij}(t, s))_{1\leq i\leq j\leq N}, (\Omega^Y_{ij}(t, s))_{1\leq i\leq j\leq N})$ by $h_{t,s}(v, w)$, where $v = (v_{ij})_{1\leq i\leq j\leq N} \in \mathbb{R}^{N(N+1)/2}$, $w = (w_{ij})_{1\leq i\leq j\leq N'} \in \mathbb{R}^{N'(N'+1)/2}$. Then

$$\begin{aligned}
E(t, s, x, y) &= \mathbb{E}\{\det(\Omega^X(t, s) + \Sigma_1(t, s, x, y))\det(\Omega^Y(t, s) + \Sigma_2(t, s, x, y)) \\
&\qquad \times \mathbb{1}_{\{\Omega^X(t,s)+\Sigma_1(t,s,x,y)\prec 0\}}\mathbb{1}_{\{\Omega^Y(t,s)+\Sigma_2(t,s,x,y)\prec 0\}}\} \\
&= \int_{v:(v_{ij})+\Sigma_1(t,s,x,y)\prec 0} \int_{w:(w_{ij})+\Sigma_2(t,s,x,y)\prec 0} \det((v_{ij}) + \Sigma_1(t, s, x, y)) \\
&\qquad \times \det((w_{ij}) + \Sigma_2(t, s, x, y))h_{t,s}(v, w)\,dv\,dw,
\end{aligned} \tag{62}$$

where (v_{ij}) and (w_{ij}) are respectively the abbreviations of the matrices $v = (v_{ij})_{1\le i,j\le N}$ and $w = (w_{ij})_{1\le i,j\le N'}$. Recall that $x \wedge y \ge u$ and $(\varepsilon_0 + R)x < y < (\varepsilon_0 + R)^{-1}x$ implies $x - Ry \ge \varepsilon_0 u$ and $y - Rx \ge \varepsilon_0 u$. By (58), there exists a constant $0 < c < -\lambda_0\varepsilon_0/(1-R^2)$ such that for δ_1 small enough and all $(t,s) \in B(\mathcal{M}_1, \delta_1)$, $x \ge u$, and $u \vee (\varepsilon_0 + R)x < y < (\varepsilon_0 + R)^{-1}x$,

$$(v_{ij}) + \Sigma_1(t,s,x,y) \prec 0, \quad \forall \|(v_{ij})\| := \left(\sum_{i,j=1}^{N} v_{ij}^2\right)^{1/2} < cu.$$

Thus $\{v : (v_{ij}) + \Sigma_1(t,s,x,y) \not\prec 0\} \subset \{v : \|(v_{ij})\| \ge cu\}$. This implies that the last integral in (62) with the integration domain replaced by $\{(v,w) : (v_{ij}) + \Sigma_1(t,s,x,y) \not\prec 0, w \in \mathbb{R}^{N'(N'+1)/2}\}$ is $o(e^{-\alpha' u^2})$ uniformly for all $(t,s) \in B(\mathcal{M}_1, \delta_1)$, where α' is a positive constant. The same result holds when replacing the integration domain by $\{(v,w) : v \in \mathbb{R}^{N(N+1)/2}, (w_{ij}) + \Sigma_2(t,s,x,y) \not\prec 0\}$. Therefore, we have that, uniformly for all $(t,s) \in B(\mathcal{M}_1, \delta_1)$, $x \ge u$, and $u \vee (\varepsilon_0 + R)x < y < (\varepsilon_0 + R)^{-1}x$,

$$\begin{aligned} E(t,s,x,y) = &\int_{\mathbb{R}^{N(N+1)/2}} \int_{\mathbb{R}^{N(N+1)/2}} \det\bigl((v_{ij}) + \Sigma_1(t,s,x,y)\bigr) \\ &\times \det\bigl((w_{ij}) + \Sigma_2(t,s,x,y)\bigr) h_{t,s}(v,w)\, dv\, dw + o\bigl(e^{-\alpha' u^2}\bigr). \end{aligned}$$

Plugging this into (61) and (60), we obtain that the indicator functions $\mathbb{1}_{\{\nabla^2 X(t) \prec 0\}}$ and $\mathbb{1}_{\{\nabla^2 Y(s) \prec 0\}}$ in (57) can be removed, causing only a super-exponentially small error. Therefore, there exists $\alpha > 0$ such that for u large enough,

$$\begin{aligned} &\mathbb{E}\{M_u^E(X,K) M_u^E(Y,L)\} \\ &= \int_K \int_L p_{\nabla X(t), \nabla Y(s)}(0,0)\, dt\, ds \int_u^\infty \int_u^\infty p_{X(t),Y(s)}\bigl(x,y \mid \nabla X(t) = \nabla Y(s) = 0\bigr) \\ &\quad \times \mathbb{E}\bigl\{\det \nabla^2 X(t) \det \nabla^2 Y(s) \mid X(t) = x, Y(s) = y, \nabla X(t) = \nabla Y(s) = 0\bigr\}\, dx\, dy \\ &\quad + o\left(\exp\left\{-\frac{u^2}{1+R} - \alpha u^2\right\}\right). \end{aligned}$$

Case (ii): $k, l \ge 0$. Note that, if $k = 0$ or $l = 0$, then by the Kac–Rice formula, the terms in (55) involving the Hessian will vanish, making the proof easier. Therefore, without loss of generality, let $k, l \ge 1$, $\sigma(K) = \{1, \dots, k\}$, $\sigma(L) = \{1, \dots, l\}$, and assume all the elements in $\varepsilon(K)$ and $\varepsilon(L)$ are 1. By the Kac–Rice formula,

$$\begin{aligned} &\mathbb{E}\{M_u^E(X,K) M_u^E(Y,L)\} \\ &= (-1)^{k+l} \int_K \int_L p_{\nabla X_{|K}(t), \nabla Y_{|L}(s)}(0,0)\, dt\, ds \\ &\quad \times \int_u^\infty \int_u^\infty p_{X(t),Y(s)}\bigl(x,y \mid \nabla X_{|K}(t) = \nabla Y_{|L}(s) = 0\bigr) \\ &\mathbb{E}\bigl\{\det \nabla^2 X_{|K}(t) \det \nabla^2 Y_{|L}(s) \mathbb{1}_{\{\nabla^2 X_{|K}(t) \prec 0\}} \mathbb{1}_{\{\nabla^2 Y_{|L}(s) \prec 0\}} \mathbb{1}_{\{X_{k+1}(t) > 0, \dots, X_N(t) > 0\}} \\ &\quad \times \mathbb{1}_{\{Y_{l+1}(s) > 0, \dots, Y_N(s) > 0\}} \mid X(t) = x, Y(s) = y, \nabla X_{|K}(t) = \nabla Y_{|L}(s) = 0\bigr\}\, dx\, dy \\ &:= (-1)^{k+l} \int_K \int_L p_{\nabla X_{|K}(t), \nabla Y_{|L}(s)}(0,0)\, dt\, ds \int_u^\infty \int_u^\infty A'(t,s,x,y)\, dx\, dy. \end{aligned}$$

Let

$$\begin{aligned} \mathcal{M}_2 = \bigl\{&(t,s) \in \bar{K} \times \bar{L} : \\ &r(t,s) = R, \mathbb{E}\{X(t) \nabla Y_{|L}(s)\} = \mathbb{E}\{Y(s) \nabla X_{|K}(t)\} = 0\bigr\}, \end{aligned} \tag{63}$$

$$B(\mathcal{M}_2, \delta_2) = \bigl\{(t,s) \in K \times L : d\bigl((t,s), \mathcal{M}_2\bigr) \le \delta_2\bigr\},$$

where δ_2 is a small positive number to be specified. Then, we only need to estimate

$$\int_{B(\mathcal{M}_2,\delta_2)} p_{\nabla X_{|K}(t),\nabla Y_{|L}(s)}(0,0)\,dt\,ds \int_u^\infty \int_u^\infty A'(t,s,x,y)\,dx\,dy, \tag{64}$$

since the integral above with $B(\mathcal{M}_2,\delta_2)$ replaced by $(K\times L)\backslash B(\mathcal{M}_2,\delta_2)$ is super-exponentially small due to the fact

$$\sup_{(t,s)\in(K\times L)\backslash B(\mathcal{M}_2,\delta_2)} \operatorname{Var}\big(\big[X(t)+Y(s)\big]/2|\nabla X(t)=\nabla Y(s)=0\big) < \frac{1+R}{2}.$$

On the other hand, following similar arguments in the proof for Case (i), we verify that removing the indicator functions $\mathbb{1}_{\{\nabla^2 X_{|K}(t)\prec 0\}}$ and $\mathbb{1}_{\{\nabla^2 Y_{|L}(s)\prec 0\}}$ in (64) will only cause a super-exponentially small error. Combining these results, we have shown that the first approximation in (55) holds, completing the proof. □

From the proof of Proposition 7.1, we see that the same arguments can be applied to $\mathbb{E}\{M_u(X,K)M_u(Y,L)\}$, yielding the following result.

PROPOSITION 7.2. *Let $\{(X(t),Y(s)) : t\in T, s\in S\}$ be an $\mathbb{R}^2$-valued, centered, unit-variance Gaussian vector field satisfying* (**H**1), (**H**2), *and* (**H**3). *Then there exists a constant $\alpha>0$ such that for any $K\in\partial_k T$ and $L\in\partial_l S$, as $u\to\infty$,*

$$\begin{aligned}&\mathbb{E}\big\{M_u(X,K)M_u(Y,L)\big\}\\ &\quad=(-1)^{k+l}\int_K\int_L \mathbb{E}\big\{\det\nabla^2 X_{|K}(t)\det\nabla^2 Y_{|L}(s)\mathbb{1}_{\{X(t)\ge u,Y(s)\ge u\}}|\nabla X_{|K}(t)=\nabla Y_{|L}(s)=0\big\}\\ &\qquad\times p_{\nabla X_{|K}(t),\nabla Y_{|L}(s)}(0,0)\,dt\,ds+o\bigg(\exp\bigg\{-\frac{u^2}{1+R}-\alpha u^2\bigg\}\bigg).\end{aligned}$$

8. Proofs of the main results. PROOF OF THEOREM 3.1. By Lemmas 6.4, 6.3, and 6.5, together with the fact that $M_u^E(X,K)\le M_u(X,K)$, we obtain that the factorial moments and the last two sums in (21) are super-exponentially small. It then follows from (20) and (21) that, there exists a constant $\alpha>0$ such that as $u\to\infty$,

$$\begin{aligned}&\mathbb{P}\Big\{\sup_{t\in T}X(t)\ge u, \sup_{s\in S}Y(s)\ge u\Big\}\\ &\quad=\sum_{k,l}\sum_{K\in\partial_k T, L\in\partial_l S}\mathbb{E}\big\{M_u^E(X,K)M_u^E(Y,L)\big\}+o\bigg(\exp\bigg\{-\frac{u^2}{1+R}-\alpha u^2\bigg\}\bigg).\end{aligned}$$

The desired result is thus an immediate consequence of Proposition 7.1 and (5). □

PROOF OF COROLLARY 3.1. By Remark 5.1, both inequalities (20) and (21) still hold with $M_u^E(\cdot)$ replaced by $M_u(\cdot)$. Therefore, the corresponding factorial moments and the last two sums in (21) with $M_u^E(\cdot)$ replaced by $M_u(\cdot)$ are super-exponentially small by Lemmas 6.4, 6.3, and 6.6. Consequently, there exists a constant $\alpha>0$ such that as $u\to\infty$,

$$\begin{aligned}&\mathbb{P}\Big\{\sup_{t\in T}X(t)\ge u, \sup_{s\in S}Y(s)\ge u\Big\}\\ &\quad=\sum_{k,l}\sum_{K\in\partial_k T, L\in\partial_l S}\mathbb{E}\big\{M_u(X,K)M_u(Y,L)\big\}+o\bigg(\exp\bigg\{-\frac{u^2}{1+R}-\alpha u^2\bigg\}\bigg).\end{aligned}$$

The desired result is thus an immediate consequence of Proposition 7.2. □

PROOF OF THEOREM 3.2. Note that in the proof of Theorem 3.1 we have seen that the points in $\mathcal{M}_2$ defined in (63) make the major contribution to the joint excursion probability. That is, with up to a super-exponentially small error, we can focus only on those product faces, say $J \times F$, whose closure $\bar{J} \times \bar{F}$ contains the unique point (t^*, s^*) with $r(t^*, s^*) = R$ and satisfying $\sigma(J) \subset \mathcal{I}_X^R(t^*, s^*)$ and $\sigma(F) \subset \mathcal{I}_Y^R(t^*, s^*)$ (i.e., the partial derivatives of r are 0 at (t^*, s^*) restricted on J and F). Specifically, let

$$T^* = \{J \in \partial_k T : t^* \in \bar{J}, \sigma(J) \subset \mathcal{I}_X^R(t^*, s^*), k = 0, \dots, N\},$$
$$S^* = \{F \in \partial_\ell S : s^* \in \bar{F}, \sigma(F) \subset \mathcal{I}_Y^R(t^*, s^*), \ell = 0, \dots, N\};$$

and for each $J \in T^*$ and $F \in S^*$, let

$$\begin{aligned} M_u^{E^*}(X, J) &:= \#\{t \in J : X(t) \geq u, \nabla X_{|J}(t) = 0, \nabla^2 X_{|J}(t) \prec 0, \\ &\qquad \varepsilon_j^* X_j(t) \geq 0 \text{ for all } j \in \mathcal{I}_X^R(t^*, s^*) \setminus \sigma(J)\}, \\ M_u^{E^*}(Y, F) &:= \#\{s \in F : Y(s) \geq u, \nabla Y_{|F}(s) = 0, \nabla^2 Y_{|F}(s) \prec 0, \\ &\qquad \varepsilon_j^* Y_j(s) \geq 0 \text{ for all } j \in \mathcal{I}_Y^R(t^*, s^*) \setminus \sigma(F)\}. \end{aligned}$$

Note that both inequalities (20) and (21) hold with $M_u^E(\cdot)$ replaced by $M_u^{E^*}(\cdot)$ when the corresponding face therein belongs to T^* or S^*, and replaced by $M_u(\cdot)$ otherwise. Following similar arguments in deriving Theorem 3.1 and Corollary 3.1, we obtain that, there exists $\alpha > 0$ such that as $u \to \infty$,

$$\begin{aligned} &\mathbb{P}\Big\{\sup_{t \in T} X(t) \geq u, \sup_{s \in S} Y(s) \geq u\Big\} \\ &\quad = \sum_{J \in T^*, F \in S^*} \mathbb{E}\{M_u^{E^*}(X, J) M_u^{E^*}(Y, F)\} + o\bigg(\exp\bigg\{-\frac{u^2}{1+R} - \alpha u^2\bigg\}\bigg). \end{aligned}$$

The desired result then follows from Proposition 7.1. □

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