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Quantum q-series and mock theta functions

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Abstract

Our results investigate mock theta functions and quantum modular forms via quantum q-series identities. After Lovejoy, quantum q-series identities are such that they do not hold as an equality between power series inside the unit disc in the classical sense, but do hold at dense sets of roots of unity on the boundary. We establish several general (multivariable) quantum q-series identities and apply them to various settings involving (universal) mock theta functions. As a consequence, we surprisingly show that limiting, finite, universal mock theta functions at roots of unity for which their infinite counterparts do not converge are quantum modular. Moreover, we show that these finite limiting universal mock theta functions play key roles in (generalized) Ramanujan radial limits. A further corollary of our work reveals that the finite Kontsevich–Zagier series is a kind of "universal quantum mock theta function," in that it may be used to evaluate odd-order Ramanujan mock theta functions at roots of unity. (We also offer a similar result for even-order mock theta functions.) Finally, to complement the notion of a quantum q-series identity and the results of this paper, we also define what we call an "antiquantum q-series identity' and offer motivating general results with applications to third-order mock theta functions.

Keywords: Mock theta functions, Quantum modular form, q-series, q-hypergeometric series, Basic hypergeometric series, Quantum q-series

Mathematics Subject Classification: 11F37, 11F99, 33D15, 33D70, 33D99

1 Introduction

The universal mock theta functions $g_2(w;q)$ and $g_3(w;q)$ of Gordon and McIntosh [25] defined by

$$g_2(w;q) := \sum_{n=0}^{\infty} \frac{q^{\frac{1}{2}n(n+1)}(-q;q)_n}{(w;q)_{n+1}(q/w;q)_{n+1}},$$
$$g_3(w;q) := \sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(w;q)_{n+1}(q/w;q)_{n+1}},$$

are generalizations of the original mock theta functions of Ramanujan appearing in his notebooks and last letter to Hardy. The functions $g_2(w;q)$ and $g_3(w;q)$ are so-called "universal" due to the fact that all classical mock theta functions of Ramanujan and sub-

sequent natural generalizations may be written in terms of either $g_2(w;q)$ or $g_3(w;q)$ depending on the parity of their "order," a number assigned to each mock theta function. Here and throughout, the q-Pochhammer symbol is defined for $n \in \mathbb{N}_0 \cup \{\infty\}$ by $(a;q)_n := \prod_{j=0}^{n-1}(1-aq^j)$. For example, Ramanujan's popular third-order mock theta function

$$f(q) := \sum_{n=0}^{\infty} \frac{q^{n^2}}{(-q;q)_n^2}$$

satisfies $f(q) = 2 - 2g_3(-1;q)$. Both classically and in their more modern generalizations, mock theta functions have been objects of extensive research in the areas of number theory, combinatorics, q-hypergeometric series, and other areas including mathematical physics (see, e.g., [1,6,13,15,17,25]). One major question surrounding the mock theta functions was resolved relatively recently nearly 90 years after Ramanujan's death by Zwegers, whose important work revealed how exactly the mock theta functions fit into the theory of modular forms [39,40]. In particular, we now know that they are examples of mock modular forms [37]. Mock modular forms are holomorphic parts of harmonic Maass forms (see [6,9] for a precise definition and more information); the latter transform like modular forms under the action of an appropriate subgroup of $\mathrm{SL}_2(\mathbb{Z})$ on the upper half of the complex plane $\mathbb{H} := \{\tau \in \mathbb{C} \mid \mathrm{Im}(\tau) > 0\}$, but are also annihilated by a certain Laplacian operator and possess relaxed growth conditions at cusps.

Quantum modular forms, defined more recently by Zagier [38], transform similarly under the action of an appropriate $\Gamma \subseteq \mathrm{SL}_2(\mathbb{Z})$ not on $\mathbb{H}$, but rather on $\mathbb{Q}$—and up to the addition of suitably real analytic error functions. A priori mock and quantum modular forms need not be related; however, connections between the two topics have emerged and their study in tandem has been of interest to many [6,11,22,38]. For example, work of the first author, Rhoades, and Ono [22] revealed how quantum modular forms play a part in Ramanujan's original "definition" of a mock theta function [4,6] in hindsight, e.g., we have from [22, Theorem 1.1] the limit as q approaches even-order k roots of unity (ζ_k^h with h/k reduced and k even) radially from within the unit disk for Ramaujan's third-order $f(q)$

$$\lim_{q \to \zeta_k^h} (f(q) - (-1)^{k/2} b(q)) = -4U(-1;q), \tag{1.1}$$

which we owe some further explanation. Here and throughout, we let $\zeta_N := e^{2\pi i/N}$, and call a rational number r/s *reduced* if $r \in \mathbb{Z}, s \in \mathbb{N}$, and $\gcd(r,s) = 1$. The function $b(q)$ is a certain modular form, up to minor normalizing factors, and when viewed as a function of $\tau \in \mathbb{H}$, with $q = e^{2\pi i \tau}$. The even-order k roots of unity ζ_k^h appearing are singularities of both $f(q)$ and $b(q)$, while $U(-1;\zeta_k^h)$ converges (to a finite sum). The function $U(-1;q)$ is a combinatorial generating function related to ranks of strongly unimodal sequences—which, non-trivially, is also a quantum modular form (see [21]), after appropriate minor normalizing factors and when viewed as a function of $x \in \mathbb{Q}$ with $q = e^{2\pi i x}$. That is, (1.1) asymptotically relates mock modular, modular, and quantum modular forms.

A quantum q-series identity plays an important role in establishing the quantum modularity of $U(-1;q)$ as discussed above. That is, for roots of unity $q = \zeta_k^h$ with h/k reduced, we have that

$$U(w;q) := \sum_{n=0}^{\infty} q^{n+1}(wq;q)_n(w^{-1}q;q)_n =_{q^{-1}} \sum_{n=0}^{\infty} w^{n+1}(wq;q)_n =: F(w;q) \tag{1.2}$$

for suitable w (see [16,21,26]). Here and throughout, we adopt Lovejoy's [31] terminology *quantum q-series identity* and notation " $=''_{q^{-1}}$, which we now explain: a quantum q-series identity $A(q) =_{q^{-1}} B(q)$ is one between functions $A(q)$ and $B(q)$ such that as power series inside the unit disc $A(q)$ and $B(q)$ are not equal there in the classical sense, but are equal to one another on a dense set of roots of unity on the boundary of the disk, and with $q \mapsto q^{-1}$ for one of the functions ($A(\zeta_k^h) = B(\zeta_k^{-h})$ for a dense set of ζ_k^h). As Lovejoy explains in [31], quantum q-series identities have emerged not only in the context of identity (1.2) just described and its extensions, but also in older work of Cohen related to Ramanujan's σ and σ^* functions studied by Andrews–Dyson–Hickerson [2,12], and more, such as in the newer results of [31,32].

The specialization $F(1;q)$ (with $F(w;q)$ as defined in (1.2)) is a function first introduced by Kontsevich [29] and later studied by Zagier [36,38] who offered it as one of his first pioneering examples of a quantum modular form (up to a minor normalization) when viewed as a function of $x \in \mathbb{Q}$ with $q = e^{2\pi i x}$. Further quantum modular properties of $F(w;q)$ have since been established as have connections to colored Jones polynomials for torus knots in topology [16,21,26].

The radial limit (1.1) is generalized by [22, Theorem 1.2] of the first author with Ono and Rhoades as follows:

$$\lim_{q\to\zeta_k^h}(R(\zeta_b^a;q) - \zeta_{b^2}^{-a^2h'k}C(\zeta_b^a;q)) = -(1-\zeta_b^a)(1-\zeta_b^{-a})U(\zeta_b^a;\zeta_k^k) \tag{1.3}$$

where h/k and a/b are reduced, $b \mid k$, and $hh' \equiv -1 \pmod{k}$, in which the partition rank generating function $R(w;q)$ replaces $f(q)$ in (1.1) (note that $f(q) = R(-1;q)$) and in which the partition crank generating function

$$C(w;q) := \frac{(q;q)_\infty}{(wq;q)_\infty(w^{-1}q;q)_\infty}$$

(multiplied by an additional constant) replaces $(-1)^{k/2}b(q)$ (note that $b(q) = C(-1;q)$). As in (1.1), $U(\zeta_b^a;\zeta_k^h)$ converges while $R(\zeta_b^a;\zeta_k^h)$ and $C(\zeta_b^a;\zeta_k^h)$ do not. Subsequent follow-up work by Bringmann–Rolen [8] and Jang–Lobrich [27] related to questions and work of Choi–Lim–Rhoades [11] establishes radial limit results analogous to the work in [22] for the universal mock theta functions $g_2(w;q)$ and $g_3(w;q)$, respectively (with g_2 and g_3 assuming the role of the mock modular R (or f)). As a consequence, their works reveal the quantum modularity of $g_2(w;q)$ and $g_3(w;q)$ for certain fixed roots of unity w as a function of $x \in \mathbb{Q}$ with $q = e^{2\pi i x}$ where $g_2(w;q)$ and $g_3(w;q)$ naturally converge (see [8] and [27] for a precise statement of their results). Earlier work by others including [10,19,20,28] also study universal mock theta functions in the context of quantum modular and quantum Jacobi forms at (pairs of) roots of unity where the functions converge.

In particular, it can be shown that $g_3(\zeta_b^a;\zeta_k^h)$ converges (to an explicit value, see [27]), where h/k and a/b are reduced rationals with $b \nmid k$ (the complementary set as in the radial limit (1.3)), and as described above, is essentially quantum modular as a function of $h/k \in \mathbb{Q}$ for each fixed $w = \zeta_b^a$ with $b \nmid k$. Our first set of results (see Theorems 1.1, 1.2 and Corollary 1.3), rather surprisingly, shows how the limiting, *finite* universal mock theta functions

$$g^*_{3,\frac{a}{b}}(\tfrac{h}{k}) := \lim_{y\to\frac{a}{b}}(1-e(yk))^2(g_3)_{[k]}(e(y);\zeta_k^h)$$

(where $e(u) := e^{2\pi i u}$) play three important roles in the above contexts, namely:

(1) in Ramanujan's radial limits,
(2) as quantum dual to the generalized Kontsevich–Zagier function,
(3) as quantum modular forms—yet at the complementary set of roots of unity

$$Q_b := \left\{ \frac{h}{k} : h/k \text{ reduced, and } b \mid k \right\} \subset \mathbb{Q}$$

on which $g_3(\zeta_b^a; \zeta_k^h)$ (viewed as a function of $x = h/k$ for fixed ζ_b^a) does not converge.

Here and throughout, we use the notation $S_{[k]}$ to stand for the truncation of a series $S := \sum_{n=0}^{\infty} a_n$ as follows

$$S_{[k]} := \sum_{0 \le n \le k-1} a_n, \tag{1.4}$$

so that $\lim_{k\to\infty} S_{[k]} = S$. When S is a function of the form $S(x) := \sum_{n\ge 0} a_n(x)$, we will write $S_{[k]}(x)$ for $(S(x))_{[k]}$ (for ease of notation). We extend the notation in the obvious way to series of multiple variables.

To this end, our first result is as follows.

Theorem 1.1 *Let h/k and a/b be reduced, with $b \mid k$, and let h' be an integer such that $hh' \equiv -1 \pmod{k}$. Then as q approaches ζ_k^h radially from within the complex unit disk, we have*

$$\lim_{q\to\zeta_k^h} \left(g_3(\zeta_b^a; q) - \frac{\zeta_{b^2}^{-a^2h'k-ab}}{1-\zeta_b^a} C(\zeta_b^a; q) \right) = \lim_{w\to\zeta_b^a} (1-w^k)^2 (g_3)_{[k]}(w; \zeta_k^h). \tag{1.5}$$

Remark 1 We further explain this theorem as follows. Theorem 1.1 establishes a radial limit result for the universal mock theta function $g_3(w;q)$, highlighting separate limits in both variables q and w, and in terms of the *finite* universal mock theta $(g_3)_{[k]}(w;q)$. On the left-hand side of (1.5), we see the radial limit difference between $g_3(\zeta_b^a;q)$ and the crank function $C(\zeta_b^a;q)$ (up to multiplication by a constant) as $q \to \zeta_k^h$ radially from within the disk, noting that for fixed $w = \zeta_b^a$ with $b \mid k$, the universal mock theta function $g_3(\zeta_b^a;\zeta_k^h)$ and the crank function $C(\zeta_b^a;\zeta_k^h)$ have singularities. The right-hand side of (1.5) reveals that the growth of the crank function compensates for the growth of the universal mock theta function g_3 at such singularities, and their radial limit difference (in the variable q) may be realized again in terms of g_3, namely the finite $(g_3)_{[k]}(w;\zeta_k^h)$ upon a limit in the second variable $w \to \zeta_b^a$. Note also that the limit in w in (1.5) need not be radial, and that the truncation is forced in the sense that its infinite counterpart does not converge at these values. (See also Remark 2 (3).)

The following quantum q-series result for the finite universal mock theta function $(g_3)_{[k]}$ shows that at (toward) suitable roots of unity it may be evaluated in terms of the generalized Kontsevich–Zagier $F(w;q)$ function.

Theorem 1.2 *Let $q = \zeta_k^h$ with h/k reduced, and let a/b be reduced with $b \mid k$. Then*

$$\lim_{w\to\zeta_b^a} (1-w^k)^2 (g_3)_{[k]}(w;q) =_{q^{-1}} \zeta_b^{-2a}(1-\zeta_b^a)F(\zeta_b^a;q) - \zeta_b^{-a}.$$

Combining our Theorem 1.2 with the aforementioned work on the quantum modular properties of $F(w;q)$ (see [21,38]) yields the following result.

Corollary 1.3 *For each reduced a/b with $b \neq 1$, up to suitable normalizations, the function*

$$g^*_{3,\frac{a}{b}} : Q_b \to \mathbb{C}$$

is a weight $1/2$ quantum modular form.

Remark 2 (1) When $b = 1$ both sides of Theorem 1.2 evaluate to -1; moreover, work of Zagier [38] establishes the quantum modularity of (a suitably normalized) $F(1;q)$.

(2) At first glance, our corollary shows that the infinite family $g^*_{3,\frac{a}{b}}(\frac{h}{k})$ (indexed by reduced rationals a/b) of finite limiting universal mock theta functions are *imperfect* quantum modular forms (after suitable normalizations) due to separate dependence on the denominators of rationals in Q_b, i.e., the function $(g_3)_{[k]}$ is a sum up to $k-1$. Despite what the name might suggest, several families of such forms have been studied, e.g., in [5,18] and in relation to the Riemann Hypothesis in [30], following Zagier's original elegant prototype example of such a form given by the Dedekind sum in [38]. On the other hand, the identity of Theorem 1.2 reveals that this apparently separate dependence on k may be ignored.

(3) As noted above, an interesting feature of this set of results is that it establishes the quantum modularity of an infinite family of finite limiting universal mock theta functions at the complementary set of roots of unity $h/k \in Q_b$ for which the infinite series defining $g_3(\zeta_b^a;\zeta_k^h)$ is not defined; the analogous limits defining $g^*_{3,a/b}$ also do not exist if one replaces the finite universal mock theta function $(g_3)_{[k]}$ by its infinite counterpart g_3. We emphasize that the truncations at $k-1$ are "unnatural" (e.g., it is not true that the nth summands for $n \geq k$ of these q-hypergeometric series are all equal to 0). See also the preprint [33] for related work.

We also obtain a similar set of results for the universal mock theta function g_2 in Theorem 1.4 and Corollary 1.5, which are stated in terms of the finite limiting universal mock theta function

$$g^*_{2,\frac{a}{b}}(\tfrac{h}{k}) := \lim_{y\to\frac{a}{b}} (1-e(yk))^2 (g_2)_{[k]}(e(y);e(\tfrac{h}{k})),$$

the quantum set of rationals where $g_2(\zeta_b^a;\zeta_k^h)$ does not naturally converge

$$Q_b^o := \left\{ \frac{h}{k} : h/k \text{ reduced, } k \text{ odd, and } b \mid k \right\} \subset \mathbb{Q},$$

and the q-hypergeometric series

$$H(w;q) := \sum_{n=0}^{\infty} \frac{(wq;q)_n}{(-wq;q)_n} w^n,$$

which was studied in the context of quantum modular forms in [16,23] and [34]; see also the relevant quantum modular results in [8], noting that as a q-series ($|q| < 1$) [14],

$$(1-w)H(w;q) = 1 + 2\sum_{n\geq 1} (-w^2)^n q^{n^2}.$$

Theorem 1.4 *Let $q = \zeta_k^h$ with h/k reduced, k odd, and let a/b be reduced with $b \mid k$. Then*

$$\lim_{w \to \zeta_b^a} (1-w^k)^2 (g_2)_{[k]}(w;q) =_{q^{-1}} -\zeta_b^{-a} - (\zeta_b^{-a} - \zeta_b^{-2a}) H(\zeta_b^{-a};q). \tag{1.6}$$

Corollary 1.5 *For each reduced a/b with $b \neq 1$, up to suitable normalizations, the function*

$$g^*_{2,\frac{a}{b}} : Q_b^o \to \mathbb{C}$$

is a weight $1/2$ quantum modular form.

Remark 3 When $b = 1$, both sides of (1.6) evaluate to -1; moreover, quantum modular properties of $H(1;q)$ are established in [34].

A further corollary of our work reveals that the finite Kontsevich–Zagier series $(F)_{[k]}(w;q)$ is a kind of "universal quantum mock theta function," in that it may be used to evaluate the odd-order Ramanujan mock theta functions at (suitable) roots of unity. We also offer a similar result for even-order mock theta functions in terms of the finite $(H)_{[k]}(w;q)$ function (see Proposition 4.1). We deduce these results from Proposition 3.1 (restated in Proposition 1.6) and Proposition 4.1 and the fact that $g_3(w;q)$ and $g_2(w;q)$ are universal mock theta functions (see e.g. [25] and [6, Appendix A]). Namely, the following proposition reveals the quantum universal mock nature of the finite Kontsevich–Zagier series:

Proposition 1.6 *Let $q = \zeta_k^h$ with h/k reduced. For $|2 - w^k - w^{-k}| > 1$, $|w^k - w^{2k}| > 1$, we have*

$$g_3(w;q) = g_{3,3}(w;q) =_{q^{-1}} -w^{-1} - w^{-k-2} \frac{1-w}{1-w^k-w^{-k}} (F)_{[k]}(w;q).$$

(The function $g_{3,3}$ is defined in (3.1).) Explicitly, for the third-order mock theta functions, our work reveals the following at suitable roots of unity q (unless otherwise indicated, $q = e^{2\pi i h/k}$). See Table 1.

Our results in this paper extend beyond Theorems 1.1, 1.2 and 1.4, Corollaries 1.3 and 1.5, and the quantum mock universality (e.g. Proposition 1.6) all described above. In the remainder of the paper, we state and prove these and several additional theorems and corollaries relating the universal mock theta functions $g_3(w;q)$ and $g_2(w;q)$ along with

Table 1 Quantum q-series: third-order mock theta functions evaluated as finite Kontsevich–Zagier series at suitable roots of unity (See Proposition 1.6)

$f(q)$	$=$	$-\frac{4}{3}(F)_{[k]}(-1;q^{-1})$,
$\phi(q)$	$=$	$\frac{2i^{-k}}{1-i^k-i^{-k}}(F)_{[k]}(i;q^{-1})$,
$\psi(q)$	$=$	$-1-q^{-k-1}\frac{1-q}{1-q^k-q^{-k}}(F)_{[k]}(q;q^{-4})$, $(q^4 = e^{2\pi i h/k})$,
$\chi(q)$	$=$	$-(-\zeta_3)^{-k-1}\frac{(1+\zeta_3)^2}{1-(-\zeta_3)^k-(-\zeta_3)^{-k}}(F)_{[k]}(-\zeta_3;q^{-1})$,
$\omega(q)$	$=$	$-q^{-1}-q^{-k-2}\frac{1-q}{1-q^k-q^{-k}}(F)_{[k]}(q;q^{-2})$, $(q^2 = e^{2\pi i h/k})$,
$\nu(q)$	$=$	$iq^{-1/2}+i^{-k}q^{-(k+2)/2}\frac{1-iq^{1/2}}{1-i^kq^{k/2}-i^{-k}q^{-k/2}}(F)_{[k]}(iq^{1/2};q^{-1})$,
$\rho(q)$	$=$	$-\zeta_3^{-1}q^{-1}-(\zeta_3 q)^{-k-2}\frac{1-\zeta_3 q}{1-(\zeta_3 q)^k-(\zeta_3 q)^{-k}}(F)_{[k]}(\zeta_3 q;q^{-2})$, $(q^2 = e^{2\pi i h/k})$.

their finite counterparts $(g_3)_{[k]}(w;q)$ and $(g_2)_{[k]}(w;q)$ and other related q-hypergeometric series to the generalized Kontsevich–Zagier quantum modular $F(w;q)$ and the quantum modular $H(w;q)$, respectively (see Propositions 3.1, 3.4, 4.1, and 4.4 in Sects. 3 and 4). In Sect. 2, we state and prove three general quantum q-series identities in Propositions 2.1 and 2.2, which are both of independent interest and used to prove our subsequent results on (universal) mock theta functions and quantum modular forms. Finally, in Sect. 5, complementary to Lovejoy's definition and study of quantum q-series identities in [31] and our other results in this paper, we define the notion of an *antiquantum q-series identity* and offer motivating general results in Propositions 5.1 and 5.2, with applications to third-order mock theta functions in Corollary 5.3.

2 Quantum *q*-series

In this section, we state and prove three general quantum q-series identities, which are of independent interest and are also used to prove results in the following sections on (universal) mock theta functions and quantum modular forms. To do so, we first recall the basic hypergeometric series [24]

$$ {}_r\phi_s\left(\begin{matrix} a_1 & a_2 & a_3 & \cdots & a_r \\ b_1 & b_2 & \cdots & & b_s \end{matrix}; q; t\right) := \sum_{n=0}^{\infty} \frac{(a_1;q)_n \cdots (a_r;q)_n}{(b_1;q)_n \cdots (b_s;q)_n (q;q)_n} \left((-1)^n q^{\binom{n}{2}}\right)^{1+s-r} t^n. $$

Using these series, we define for $r \in \mathbb{N}$

$$ \begin{aligned} &\phi_r\left(\begin{matrix} a_1 & a_2 & \cdots & a_r \\ b_1 & b_2 & \cdots & b_r \end{matrix}; q; t\right) \\ &\quad := {}_{r+1}\phi_r\left(\begin{matrix} a_1 q & a_2 q & \cdots & a_r q & q \\ b_1 q & b_2 q & \cdots & b_r q & \end{matrix}; q; t\right) = \sum_{n=0}^{\infty} \frac{(a_1 q;q)_n \cdots (a_r q;q)_n}{(b_1 q;q)_n \cdots (b_r q;q)_n} t^n. \end{aligned} \tag{2.1} $$

With the notation $\mathbf{x} = \mathbf{x}_r := (x_1, x_2, \cdots, x_r)$ $(r \in \mathbb{N})$, we define

$$ \begin{aligned} u_{r,k}(\mathbf{a}, \mathbf{b}, t) &:= \frac{(1-a_1^k)\cdots(1-a_r^k)}{(1-b_1^k)\cdots(1-b_r^k)} t^k, \\ \delta_{r,k}(\mathbf{a}, \mathbf{b}, t) &:= \big(1 - u_{r,k}(\mathbf{a}, \mathbf{b}, t)\big)^{-1}, \end{aligned} $$

and

$$ c_{r,k}(\mathbf{a}, \mathbf{b}, t) := \delta_{r,k}(\mathbf{a}, \mathbf{b}, t) \frac{u_{r,k}(\mathbf{a}, \mathbf{b}, t)}{u_{r,1}(\mathbf{a}, \mathbf{b}, t)}. $$

In what follows, when $r = 1$, we may write $\mathbf{x}$ as x for simplicity. Our first general quantum q-series identity is the following.

Proposition 2.1 *Let $r, k \in \mathbb{N}$, and $q = \zeta_k^h$ with h/k reduced. Then*

$$ \phi_r\left(\begin{matrix} a_1 & a_2 & \cdots & a_r \\ b_1 & b_2 & \cdots & b_r \end{matrix}; q; t\right) =_{q^{-1}} c_{r,k}(\mathbf{a}, \mathbf{b}, t) (\phi_r)_{[k]}\left(\begin{matrix} b_1 & b_2 & \cdots & b_r \\ a_1 & a_2 & \cdots & a_r \end{matrix}; q; t^{-1}\right). \tag{2.2} $$

Remark 4 (1) It is possible for some or all of the parameters t and a_j, b_j ($1 \leq j \leq r$) appearing in Proposition 2.1 to depend on q; however, we emphasize that our notation " $=_{q^{-1}}$" used above (which was first used by Lovejoy in [31]) should be read (in this case, and similar instances throughout the paper) as $= c_{r,k}(\mathbf{a}, \mathbf{b}, t)(\phi_r)_{[k]}\left(\begin{matrix} b_1 & b_2 & \cdots & b_r \\ a_1 & a_2 & \cdots & a_r \end{matrix}; q^{-1}; t^{-1}\right)$.

(2) When $r = 1$, under the hypotheses given, Proposition 2.1 establishes a quantum q-series identity for Fine's basic hypergeometric series [14] $F_q(a_1, b_1; t) = \phi_1\left(\begin{matrix} a_1 \\ b_1 \end{matrix}; q; t\right)$ as follows:

$$F_q(a_1, b_1; t) =_{q^{-1}} c_{1,k}(a_1, b_1, t)(F_q)_{[k]}(b_1, a_1; t^{-1}).$$

(3) As is common with q-hypergeometric series identities in the literature (see, e.g., the books [14] including p2 Sec. 3 and [24], and numerous related papers including the recent [31] and [3]), we state Proposition 2.1 without enforced conditions on the additional parameters $\mathbf{a}$, $\mathbf{b}$, and t for maximum applicability, with the understanding that the identity may be used with any values of these parameters such that both the left- and right-hand sides simultaneously converge, or in other appropriate limiting settings, e.g., when certain parameters tend to 0 or ∞ as is common in the subject. We state other identities throughout the paper (including Proposition 2.2) similarly. For example, the right-hand side of (2.2) is a rational function and will converge for any complex $\mathbf{a}$, $\mathbf{b}$, and t that do not produce poles; it will also converge with certain poles after taking suitable limits. We illustrate some specific applications of interest including limiting ones in Sects. 3–5.

(4) The proof techniques used to prove Propositions 2.1, 2.2 and related results below are not limited to these settings; it would be of interest to use the techniques of this paper to establish quantum-type identities for other q-hypergeometric series of interest, including but not limited to mock modular and quantum modular forms.

Our second and third general quantum q-series identities are as follows.

Proposition 2.2 *Let $q = \zeta_k^h$ with h/k reduced. Then*

$$-z^{-1}(1-b^k)(1-z^k)\sum_{n=0}^{k-1}\frac{(bz^{-1})^n q^{n^2+n}}{(bq;q)_{n+1}(z^{-1};q)_{n+1}} =_{q^{-1}} \sum_{n=0}^{k-1}(b;q)_n z^n, \tag{2.3}$$

and

$$-z^{-1}\frac{(1-cq)(1-b^k)(1-z^k)}{(1-c^k)}\sum_{n=0}^{k-1}\frac{(bc^{-1};q)_n\left(-cqz^{-1}\right)^n q^{\frac{n^2+n}{2}}}{(bq;q)_{n+1}(z^{-1};q)_{n+1}} =_{q^{-1}} \sum_{n=0}^{k-1}\frac{(b;q)_n}{(c;q)_n}z^n. \tag{2.4}$$

Proof of Proposition 2.1 We use that for $q = \zeta_k^h$ (where h/k is reduced), we have $(xq;q)_{s+mk} = (1-x^k)^m(xq;q)_s$ ($s, m \in \mathbb{N}_0$), and hence,

$$\phi_r\left(\begin{matrix} a_1 & a_2 & \cdots & a_r \\ b_1 & b_2 & \cdots & b_r \end{matrix}; q; t\right) = \sum_{n\geq 0}\frac{(a_1q;q)_n\cdots(a_rq;q)_n}{(b_1q;q)_n\cdots(b_rq;q)_n}t^n$$

$$= \sum_{\substack{0\le s\le k-1\\ m\ge 0}} \frac{(a_1q;q)_{s+mk}\cdots(a_rq;q)_{s+mk}}{(b_1q;q)_{s+mk}\cdots(b_rq;q)_{s+mk}} t^{s+mk}$$

$$= \left(\sum_{m\ge 0}(u_{r,k}(\mathbf{a},\mathbf{b},t))^m\right)\left(\sum_{0\le s\le k-1}\frac{(a_1q;q)_s\cdots(a_rq;q)_s}{(b_1q;q)_s\cdots(b_rq;q)_s}t^s\right).$$

Using the given hypotheses, as well as that $(xq;q)_k = (xq;q)_s(xq^{s+1};q)_{k-s}$ and $(xq^{1-n};q)_n = (x;q^{-1})_n$, we find that this equals

$$\delta_{r,k}(\mathbf{a},\mathbf{b},t)u_{r,k}(\mathbf{a},\mathbf{b},1)\sum_{0\le s\le k-1}\frac{(b_1q^{s+1};q)_{k-s}\cdots(b_rq^{s+1};q)_{k-s}}{(a_1q^{s+1};q)_{k-s}\cdots(b_rq^{s+1};q)_{k-s}}t^s$$

$$= \delta_{r,k}(\mathbf{a},\mathbf{b},t)u_{r,k}(\mathbf{a},\mathbf{b},1)\sum_{0\le s\le k-1}\frac{(b_1;q^{-1})_{k-s}\cdots(b_r;q^{-1})_{k-s}}{(a_1;q^{-1})_{k-s}\cdots(a_r;q^{-1})_{k-s}}t^s$$

$$= c_{r,k}(\mathbf{a},\mathbf{b},t)\sum_{0\le s\le k-1}\frac{(b_1q^{-1};q^{-1})_s\cdots(b_rq^{-1};q^{-1})_s}{(a_1q^{-1};q^{-1})_s\cdots(a_rq^{-1};q^{-1})_s}t^{-s}$$

$$=_{q^{-1}} c_{r,k}(\mathbf{a},\mathbf{b},t)(\phi_r)_{[k]}\left(\begin{matrix}b_1 & b_2 & \cdots & b_r\\ a_1 & a_2 & \cdots & a_r\end{matrix};q;t^{-1}\right)$$

as claimed. □

Proof of Proposition 2.2 To prove (2.3), we begin with [31, (2.3)] with $N\mapsto k$, $q=\zeta_k^h$ and $q\mapsto q^{-1}$, so that

$$\sum_{n=0}^{k-1}(b;q^{-1})_n z^n = b^{k-1}\sum_{n=0}^{k-1}(b;q^{-1})_n(z^{-1}q^{-1};q^{-1})_n(z/b)^n q^{n^2+n}.$$

This equals

$$\frac{b^{k-1}}{(1-bq)(1-z^{-1})}\sum_{n=0}^{k-1}(bq;q^{-1})_{n+1}(z^{-1};q^{-1})_{n+1}(z/b)^n q^{n^2+n}$$

$$= \frac{b^{k-1}}{(1-bq)(1-z^{-1})}\sum_{n=0}^{k-1}(bq;q^{-1})_{k-n}(z^{-1};q^{-1})_{k-n}(z/b)^{k-n-1}q^{n^2+n}$$

$$= \frac{b^{k-1}}{(1-bq)(1-z^{-1})}\sum_{n=0}^{k-1}(bq^{n+2-k};q)_{k-n}(z^{-1}q^{n+1-k};q)_{k-n}(z/b)^{k-n-1}q^{n^2+n}$$

$$= \frac{b^{k-1}}{(1-bq)(1-z^{-1})}(bq^2;q)_k(z^{-1}q;q)_k\sum_{n=0}^{k-1}(z/b)^{k-n-1}\frac{q^{n^2+n}}{(bq^2;q)_n(z^{-1}q;q)_n}$$

$$= b^{k-1}(1-(bq)^k)(1-z^{-k})\sum_{n=0}^{k-1}(z/b)^{k-n-1}\frac{q^{n^2+n}}{(bq;q)_{n+1}(z^{-1};q)_{n+1}}$$

as claimed.

To prove (2.4), we begin with [31, (1.21)] with $N\mapsto k-1$, with $q=\zeta_k^h$ and $q\mapsto q^{-1}$ to obtain

$$\sum_{0\le n\le k-1}\frac{(b;q^{-1})_n}{(c;q^{-1})_n}z^n$$

$$= \frac{(c/b;q^{-1})_{k-1}}{(c;q^{-1})_{k-1}} b^{k-1} \sum_{0\le n\le k-1} \frac{(-1)^n (b;q^{-1})_n (z^{-1}q^{-1};q^{-1})_n (z/c)^n q^{n(n-1)/2}}{(bq^{-2}/c;q^{-1})_n}.$$

Now we proceed similarly to the above proof of (2.3) using that $(xq;q)_k = (xq;q)_s (xq^{s+1};q)_{k-s}$ and $(xq^{1-n};q)_n = (x;q^{-1})_n$, to eventually rewrite this as

$$\begin{aligned}
&- \frac{(c/b;q^{-1})_{k-1}}{(c;q^{-1})_{k-1}} b^{k-1} \frac{(b;q^{-1})_k (z^{-1}q^{-1};q^{-1})_k}{(bq^{-2}/c;q^{-1})_k} (z/c)^k \\
&\quad \times \sum_{0\le n\le k-1} \frac{(-1)^{k-n} (bq^{-1}/c;q)_{k-n} (c/z)^{k-n} q^{(k-n)(k-n+1)/2}}{(bq;q)_{k-n} (z^{-1};q)_{k-n}} \\
&= - \frac{(c/b;q^{-1})_{k-1}}{(c;q^{-1})_{k-1}} b^{k-1} \frac{(b;q^{-1})_k (z^{-1}q^{-1};q^{-1})_k}{(bq^{-2}/c;q^{-1})_k} (z/c)^k (1 - bq^{-1}/c) \\
&\quad \times \sum_{0\le n\le k-1} \frac{(-1)^{n+1} (b/c;q)_n (c/z)^{n+1} q^{(n+1)(n+2)/2}}{(bq;q)_{n+1} (z^{-1};q)_{n+1}}.
\end{aligned}$$

With a little more simplifying and using that $(xq;q)_k = (1-x^k)$ and $(xq;q)_{k-1} = (1-x^k)/(1-x)$, we obtain the result. □

3 The universal mock theta function $g_3(w;q)$ and the generalized Kontsevich–Zagier quantum modular form $F(w;q)$.

In this section, we state and prove, using in part our general results from the previous section, several quantum q-series results relating the universal mock theta function $g_3(w;q)$ and affiliate

$$g_{3,3}(w;q) := \sum_{n=0}^{\infty} \frac{q^n w^{-n}}{(w;q)_{n+1}} \tag{3.1}$$

studied in [7], to the generalized Kontsevich–Zagier quantum modular $F(w;q)$. We begin with a quantum q-series identity at roots of unity $q = \zeta_k^h$ and complex values of w for which $g_3(w;q)$ and $g_{3,3}(w;q)$ naturally converge, in terms of the finite generalized Kontsevich–Zagier $(F)_{[k]}(w;q)$, exhibiting a kind of universal quantum modularity of $(F)_{[k]}(w;q)$. (Note that Proposition 3.1 is also stated in Sect. 1 as Proposition 1.6 for convenience.) See also Table 1.

Proposition 3.1 *Let $q = \zeta_k^h$ with h/k reduced. For $|2 - w^k - w^{-k}| > 1$, $|w^k - w^{2k}| > 1$, we have*

$$g_3(w;q) = g_{3,3}(w;q) =_{q^{-1}} -w^{-1} - w^{-k-2} \frac{1-w}{1-w^k-w^{-k}} (F)_{[k]}(w;q).$$

In Fig. 1, we plot the set of complex w with $-2 \le \mathrm{Re}(w), \mathrm{Im}(w) \le 2$ satisfying the hypotheses of Proposition 3.1 with $k = 11$.

We also establish a type of dual to the above proposition, trading the non-finite g_3 and $g_{3,3}$ for the finite $(g_3)_{[k]}$ and $(g_{3,3})_{[k]}$ and trading the finite $(F)_{[k]}$ with the non-finite F. That is, in Theorem 3.2 (also stated as Theorem 1.2 in Sect. 1 for the function $(g_3)_{[k]}$ only) we establish a limiting quantum q-series identity for the finite $(g_3)_{[k]}$ and $(g_{3,3})_{[k]}$ at roots of unity $q = \zeta_k^h$ as w approaches roots of unity ζ_b^a with $b \mid k$ in terms of the generalized Kontsevich–Zagier $F(\zeta_b^a;q)$. We point out that unlike the previous proposition, for $b \mid k$,

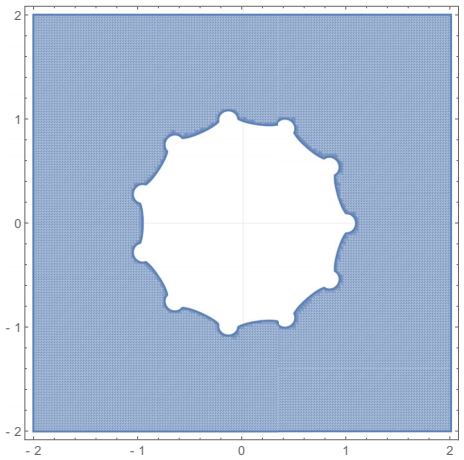

Fig. 1 $\{|2 - w^{11} - w^{-11}| > 1\} \cap \{|w^{11} - w^{22}| > 1\} \cap \{-2 \leq \mathrm{Re}(w), \mathrm{Im}(w) \leq 2\}$

$g_3(\zeta_b^a; \zeta_k^h)$ and $g_{3,3}(\zeta_b^a; \zeta_k^h)$ do not converge, but their finite limiting expressions presented below do—and may be evaluated in terms of the Kontsevich Zagier quantum modular form $F(\zeta_b^a; \zeta_k^{-h})$.

Theorem 3.2 *Let $q = \zeta_k^h$ with h/k reduced, and let a/b be reduced with $b \mid k$. Then*

$$\begin{aligned}\lim_{w \to \zeta_b^a} (1 - w^k)^2 (g_3)_{[k]}(w; q) &= -\lim_{w \to \zeta_b^a} (1 - w^k)(g_{3,3})_{[k]}(w; q) \\ &=_{q^{-1}} \zeta_b^{-2a}(1 - \zeta_b^a) F(\zeta_b^a; q) - \zeta_b^{-a}.\end{aligned}$$

As discussed in Sect. 1 (and as stated there as Corollary 1.3), we obtain

Corollary 3.3 *For each reduced a/b with $b \neq 1$, up to suitable normalizations, the function*

$$g^*_{3,\frac{a}{b}} : Q_b \to \mathbb{C}$$

is a weight $1/2$ quantum modular form.

Remark 5 The quantum modularity of $g^*_{3,3;a/b}(h/k) := \lim_{y \to a/b}(1 - e(ky))(g_{3,3})_{[k]}(e(y)); \zeta_k^h)$ on Q_b is similarly deduced from Theorem 3.2.

Next we offer a similar quantum q-series result to Theorem 3.2 for $(g_{3,3})_{[k]}$ for suitable w in terms of the generalized Kontsevich–Zagier quantum modular $F(w; q)$.

Proposition 3.4 *Let $q = \zeta_k^h$ with h/k reduced. For $|w^k - w^{2k}| < 1$, $w^k \neq 1$, we have*

$$(g_{3,3})_{[k]}(w; q) =_{q^{-1}} w^{-1-k} \frac{(1 - w^k + w^{2k})}{1 - w^k} - w^{-2-k} \frac{(1 - w^k + w^{2k})(1 - w)}{1 - w^k} F(w; q).$$

Remark 6 We also establish a non-limiting version of Theorem 3.2 for $(g_3)_{[k]}(w; q)$ for suitable w in the course of its proof (see equation (3.2)) similar to Proposition 3.4.

3.1 Proof of Theorem 1.1, Proposition 3.1, Theorem 1.2 (Thm. 3.2), and Proposition 3.4

Proof of Theorem 1.1 We use [7, Theorem 3.1] by the first author with Bringmann and Rhoades to write

$$R(\zeta_b^a;q)=\zeta_b^a(1-\zeta_b^a)\left(\zeta_b^{-a}+g_3(\zeta_b^a;q)\right)$$

for $|q|<1$. We also use the quantum q-series identity (1.2) for the strongly unimodal rank function and the generalized Kontsevich–Zagier function as well as our Theorem 3.2 (Thm. 1.2) to obtain

$$\lim_{w\to\zeta_b^a}(1-w^k)^2(g_3)_{[k]}(w;q)=\zeta_b^{-2a}(1-\zeta_b^a)U(\zeta_b^a;q)-\zeta_b^{-a}.$$

We use the above, together with the generalized Ramanujan radial limit [22, Theorem 1.2] by work of the first author with Ono and Rhoades restated in (1.3), to obtain Theorem 1.1 after some algebra and simplifications. □

Proof of Proposition 3.1 Using [7, Theorem 3.1] and [14, (2.4)], we find that

$$g_3(w;q)=g_{3,3}(w;q)=\frac{1}{1-w}\phi_1\left(\begin{matrix}0\\ w\end{matrix};q;w^{-1}q\right)=-\frac{1}{w}-\frac{1}{w^2}\phi_1\left(\begin{matrix}0\\ w\end{matrix};q;w^{-1}\right).$$

Applying Proposition 2.1 in the case that $r=1$ (see Remark 2 (2)), we find after some simplification that

$$\phi_1\left(\begin{matrix}0\\ w\end{matrix};q;w^{-1}\right)=_{q^{-1}}c_{1,k}(0,w,w^{-1})\,(\phi_1)_{[k]}\left(\begin{matrix}w\\ 0\end{matrix};q;w\right),$$

which is equivalent to

$$\phi_1\left(\begin{matrix}0\\ w\end{matrix};q;w^{-1}\right)=_{q^{-1}}\frac{(1-w)w^{-k}}{1-w^k-w^{-k}}(F)_{[k]}(w;q).$$

Combining the above results and simplifying completes the proof. □

Proof of Theorem 3.2 (and Thm. 1.2) We begin with (2.3) from Proposition 2.2, with $z=b=w^{-1}$, which reveals that

$$\sum_{n=0}^{k-1}(w^{-1};q^{-1})_n w^{-n}=w^{1-k}(1-w^{-k}q^k)(1-w^k)(g_3)_{[k]}(w;q).$$

Now, we observe that $(g_3)_{[k]}(w;q)$ is invariant under $w\mapsto qw^{-1}$. Using this, and the fact that $q^k=1$, we find that

$$\sum_{n=0}^{k-1}(q^{-1}w;q^{-1})_n q^{-n}w^n=qw^{k-1}(1-w^k)(1-w^{-k})(g_3)_{[k]}(w;q). \tag{3.2}$$

that $w=\zeta_b^a$ with $b\mid k$ then we have that the function on the left-hand side of (3.2) equals $\phi_1\left(\begin{matrix}w\\ 0\end{matrix};q^{-1};wq^{-1}\right)$. Using [14, (2.4)], we see that this equals

$$\left(\phi_1\left(\begin{matrix}w\\ 0\end{matrix};q^{-1};w\right)-\frac{1}{1-w}\right)\frac{w-1}{w^2q^{-1}}.$$

After some simple algebraic manipulations, the result follows for $(g_3)_{[k]}(w;q)$ after establishing that the claimed limits exists. To show this, it suffices to show that for each $0 \leq n \leq k-1$, the limits

$$\lim_{w \to \zeta_b^a} \frac{(1-w^k)^2}{(w;q)_{n+1}(q/w;q)_{n+1}}$$

exist, where $q = \zeta_k^h$, a/b is reduced, and $b \mid k$. Indeed, we have that

$$\begin{aligned}\frac{(1-w^k)^2}{(w;q)_{n+1}(q/w;q)_{n+1}} &= -w^k \frac{((w;q)_k(q/w;q)_k}{(w;q)_{n+1}(q/w;q)_{n+1}} \\ &= -w^k (wq^{n+1};q)_{k-n-1}(w^{-1}q^{n+2};q)_{k-n-1},\end{aligned}$$

from which it follows that the limit pertaining to $(g_3)_{[k]}$ exists and equals the given expression in terms of $F(w;q)$, along with the easy to verify fact that $F(\zeta_b^a;\zeta_k^{-h})$ converges.

To obtain the result and limiting expression pertaining to $(g_{3,3})_{[k]}$, we multiply the identity of Proposition 3.4 by $(1-w^k)$. As above, we similarly justify that the limit as $w \to \zeta_b^a$ of $(1-w^k)(g_{3,3})_{[k]}(w;q)$ exists under the hypotheses given, recall that $F(\zeta_b^a;\zeta_k^{-h})$ converges, and simplify to obtain the result. □

Proof of Proposition 3.4 We apply Proposition 2.1 with $r=1$, $a_1 = w, b_1 = 0$ and $t = wq^{-1}$. After some algebra and simplifying, this gives

$$(g_{3,3})_{[k]}(w;q) = w^{1-k}q^{-1}\frac{(1-w^k+w^{2k})}{1-w^k}\phi_1\left(\begin{matrix} w \\ 0 \end{matrix}; q^{-1}; wq^{-1}\right).$$

Applying [14, (2.4)], after some simplifying, we find that this equals

$$=_{q^{-1}} w^{-1-k}\frac{(1-w^k+w^{2k})}{1-w^k} - w^{-2-k}\frac{(1-w^k+w^{2k})(1-w)}{1-w^k}F(w;q)$$

as claimed. □

4 The universal mock theta function $g_2(w;q)$ and the quantum modular form $H(w;q)$.

Analogous to the results of the previous section $g_3(w;q)$ and $F(w;q)$, in this section we state and prove (using in part our general results from Sect. 2) several quantum q-series results relating the universal mock theta function $g_2(w;q)$ and affiliate

$$g_{2,3}(w;q) := -\frac{1+w}{2w^2}\sum_{n=0}^{\infty}\frac{(-wq;q)_n}{(wq;q)_n}w^{-n} - \frac{1}{2w}$$

studied in [7], to the quantum modular $H(w;q)$. We begin with a quantum q-series identity at roots of unity $q = \zeta_k^h$ and complex values of w for which $g_2(w;q)$ and $g_{2,3}(w;q)$ naturally converge in terms of the finite H-function $(H)_{[k]}(w;q)$, which similar to the previous section is, interestingly, unnaturally truncated, and analogously reveals a kind of quantum universal mock property.

Proposition 4.1 *Let $q = \zeta_k^h$ with h/k reduced. For $\left|\frac{1-w^k}{(-w)^k-w^{2k}}\right| < 1$, $\left|\frac{1-(-1)^k}{2-(-w)^k-(-w)^{-k}}\right| < 1$, $w^k \neq 1$, we have*

$$g_2(-w;q) = g_{2,3}(-w;q) =_{q^{-1}} \frac{1}{2w} + \frac{(-1)^k(1+w)(1-w^k)w^{-k-1}}{2(1-(-w)^k+(-1)^k(1-w^{-k}))}(H)_{[k]}(-w;q).$$

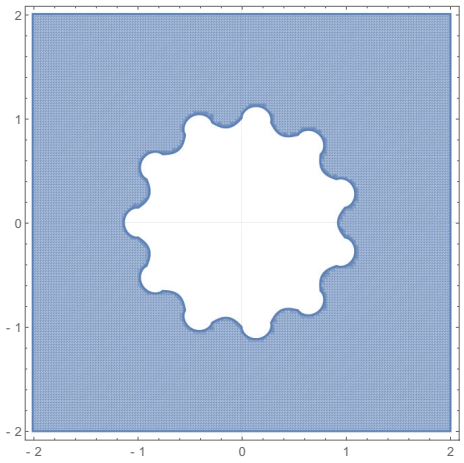

Fig. 2 $\left\{\left|\frac{1-w^{11}}{(-w)^{11}-w^{22}}\right|<1\right\}\cap\left\{\left|\frac{1-(-1)^{11}}{2-(-w)^{11}-(-w)^{-11}}\right|<1\right\}\cap\{-2\le \mathrm{Re}(w),\,\mathrm{Im}(w)\le 2\}$

Moreover, with k odd, and a/b reduced with $b \mid k$, we have

$$g_2(-\zeta_b^a;q)=g_{2,3}(-\zeta_b^a;q)=_{q^{-1}}\lim_{w\to\zeta_b^a}\left(\frac{1}{2w}-\frac{(1+w)(1-w^k)w^{-k-1}}{2(w^k+w^{-k})}(H)_{[k]}(-w;q)\right).$$

In Fig. 2, we plot the set of complex w with $-2\le \mathrm{Re}(w),\mathrm{Im}(w)\le 2$ satisfying the hypotheses of Proposition 4.1 with $k=11$.

The analogous dual-type result to the above proposition as seen in the previous section for $g_3(w;q)$ is the following theorem (also stated as Theorem 1.4 in Sect. 1), a quantum q-series identity for the limiting finite universal mock theta function $(g_2)_{[k]}(w;q)$ toward pairs of roots of unity $(w;q)=(\zeta_b^a;\zeta_k^h)$ for which the (infinite) series $g_2(w;q)$ does not converge.

Theorem 4.2 *Let $q=\zeta_k^h$ with h/k reduced, k odd, and let a/b be reduced with $b\mid k$. Then*

$$\lim_{w\to\zeta_b^a}(1-w^k)^2(g_2)_{[k]}(w;q)=_{q^{-1}}-\zeta_b^{-a}-(\zeta_b^{-a}-\zeta_b^{-2a})H(\zeta_b^{-a};q).$$

As discussed in Sect. 1, the above theorem leads to the quantum modularity of the limiting finite universal mock theta functions g_2^* for roots of unity at which $g_2(\zeta_b^a;\zeta_k^h)$ does not converge (see also Corollary 1.5).

Corollary 4.3 *For each reduced a/b with $b\neq 1$, up to suitable normalizations, the function*

$$g^*_{2,\frac{a}{b}}:Q_b^o\to\mathbb{C}$$

is a weight $1/2$ quantum modular form.

Next we offer a similar quantum q-series result to Theorem 4.2 for $(g_{2,3})_{[k]}$ for suitable w in terms of the quantum modular $H(w;q)$.

Proposition 4.4 *Let $q = \zeta_k^h$ with h/k reduced. For $\left|\frac{w^k - w^{2k}}{1-(-w)^k}\right| < 1$, $w^k \neq 1$, we have*

$$(g_{2,3})_{[k]}(w;q) =_{q^{-1}} -\frac{1}{2w} - \frac{1}{2w^{k+1}} \frac{(1-w)(1-(-w)^k - w^k + w^{2k})}{(1-w^k)} H(w;q). \tag{4.1}$$

Moreover, for k odd and a/b reduced with $b \mid k$, we have

$$\lim_{w \to \zeta_b^a} (1-w^k)(g_{2,3})_{[k]}(w;q) =_{q^{-1}} (1-\zeta_b^{-a}) H(w;q). \tag{4.2}$$

Remark 7 We also establish a non-limiting version of Theorem 4.2 for $(g_2)[k](w;q)$ for suitable w in the course of its proof (see equation (4.3)) similar to Proposition 4.4.

4.1 Proof of Proposition 4.1, Theorem 4.2 (Thm. 1.4), and Proposition 4.4

Proof of Proposition 4.1 We begin with the fact that

$$-\frac{2w^2}{1-w}\left(g_{2,3}(-w;q) - \frac{1}{2w}\right) = \phi_1\left(\begin{matrix} w \\ -w \end{matrix}; q; -w^{-1}\right).$$

To this, we apply Proposition 2.1, also using that

$$(H)_{[k]}(-w;q) = (\phi_1)_{[k]}\left(\begin{matrix} -w \\ w \end{matrix}; q; -w\right).$$

The proposition follows after simplifying $c_{1,k}(w, -w, -w^{-1})$ and some straightforward algebraic manipulations, along with [7, Theorem 4.1]. □

Proof of Theorem 4.2 (Thm. 1.4) We set $b = z = w^{-1}$ and $c = -w^{-1}q^{-1}$ in equation (2.4) from Proposition 2.2 to obtain after some algebra and simplifying

$$(1-w^k)^2 (g_2)_{[k]}(w;q) = -\frac{w^{2k-1}(1-(-1)^k w^{-k})}{(1+w^{-1})} (\phi_1)_{[k]}\left(\begin{matrix} w^{-1}q \\ -w^{-1} \end{matrix}; q^{-1}; w^{-1}\right). \tag{4.3}$$

Now, for k odd and $w = \zeta_b^a$ (a/b reduced) with $b \mid k$, we have that

$$\begin{aligned}(\phi_1)_{[k]}\left(\begin{matrix} w^{-1}q \\ -w^{-1} \end{matrix}; q^{-1}; w^{-1}\right) &= \phi_1\left(\begin{matrix} w^{-1}q \\ -w^{-1} \end{matrix}; q^{-1}; w^{-1}\right) \\ &= \frac{(1+w^{-1})}{2} + \frac{(1-w^{-1})(1+w^{-1})}{2} H(w^{-1};q^{-1}),\end{aligned}$$

where we have also used [14, (4.5)], and thus, the right-hand side of (4.3) with $w = \zeta_b^a$, $b \mid k$, and k odd becomes

$$-\zeta_b^{-a} - \zeta_b^{-a}(1-\zeta_b^{-a}) H(\zeta_b^{-a}; \zeta_k^{-h}). \tag{4.4}$$

Similar to the proof of Theorem 3.2 (Thm. 1.2), it is not difficult to check (with k odd, $b \mid k$ and a/b reduced) that the limit as $w \to \zeta_b^a$ of the left-hand side of (4.3) exists and thus equals (4.4). □

Proof of Proposition 4.4 Proposition 4.4 (4.1) follows from Proposition 2.1 with $r = 1, a_1 = w, b_1 = -w$, and $t = w$, after some algebra and simplifications. Next we prove (4.2). It is not difficult to verify that $(1 - w^k)$ multiplied the right-hand side of (4.1) evaluated at $(w, q) = (\zeta_b^a, \zeta_k^h)$ for k odd and $b \mid k$ (and $a/b, h/k$ reduced) exists and equals $(1 - \zeta_b^{-a})H(\zeta_b^a; \zeta_k^{-h})$. Moreover, it can be shown as in the proofs of Theorem 3.2 (Thm. 1.2) and Theorem 4.2 (Thm. 1.4) that (for the same $a/b, h/k$) the limit $\lim_{w\to\zeta_b^a}(1-w^k)(g_{2,3})_{[k]}(w;q)$ exists, and thus via (4.1) and the above equals the right-hand side of (4.2) as claimed. □

5 Antiquantum *q*-series

In this section, we define and investigate what we call *antiquantum q-series identities,* again inspired by Lovejoy [31], to complement his notion of a quantum q-series and our results in the previous sections. Our antiquantum q-series identities are between series which converge and are equal to one another inside the disk $|q| < 1$, but our identities hold at (dense) sets of roots of unity on the boundary for which one of the series diverges and is "unnaturally" truncated. To illustrate this more precisely, we first define the q-series

$$m(w;q) := \sum_{n=0}^{\infty} \frac{(-w)^n q^{n^2}}{(wq;q)_n(-q;q)_n}$$

and

$$\widetilde{m}(w;q) := \sum_{n=0}^{\infty} (w^{-1}q^2;q^2)_n (wq)^n.$$

Specializations of these functions are related to Ramanujan's 3rd-order mock theta functions

$$f(q) := \sum_{n=0}^{\infty} \frac{q^{n^2}}{(-q;q)_n^2}, \quad \psi(q) := \sum_{n=1}^{\infty} \frac{q^{n^2}}{(q;q^2)_n}, \quad \text{and} \quad \phi(q) := \sum_{n=0}^{\infty} \frac{q^{n^2}}{(-q^2;q^2)_n}.$$

For example, we have

$$\begin{aligned} m(-1;q) &= f(q), \\ -q\widetilde{m}(-1;q) &= \widetilde{\psi}(-q), \\ q\widetilde{m}(-q;q) &= 1 - \widetilde{\phi}(-q), \end{aligned}$$

where

$$\widetilde{\psi}(-q) := \sum_{n=0}^{\infty} (-q^2;q^2)_n (-q)^{n+1} \quad \text{and} \quad \widetilde{\phi}(-q) := 1 - \sum_{n\geq 0} (-q;q^2)_n (-1)^n q^{2n+1}$$

are q-series which are equal to the mock theta functions $\psi(-q)$ and $\phi(-q)$, respectively, inside the disk $|q| < 1$, i.e., there,

$$\psi(-q) = \widetilde{\psi}(-q) \quad \text{and} \quad \phi(-q) = \widetilde{\phi}(-q).$$

However, these identities do not in general hold at roots of unity on the boundary (see Remark 8 for more).

Let $\kappa_{w,q,k} := \dfrac{2 - w^k}{1 - w} \cdot \dfrac{q^{k-1}}{2}$. We first give a general result (proved in part from our results in the previous sections) from which we deduce antiquantum q-series identities for mock theta functions.

Proposition 5.1 *We have that*

$$m(w;q) = \kappa_{w,q,k}^{-1} \cdot \widetilde{m}_{[k]}(w;q).$$

From this, we obtain the following antiquantum q-series identities for the third-order mock theta functions ϕ and ψ.

Corollary 5.2 *For odd-order roots of unity $q = \zeta_k^h$ (h/k reduced and k odd), we have*

$$-1 + \phi(-q) = \frac{1}{3}(-1 + \widetilde{\phi}_{[k]}(-q)), \tag{5.1}$$

and

$$\psi(-q) = \frac{1}{3}\widetilde{\psi}_{[k]}(-q). \tag{5.2}$$

Remark 8 We explicitly point out the antiquantum nature of the identities in Corollary 5.2 as follows. We have the mock theta identity $\phi(-q) = \widetilde{\phi}(-q)$ inside the disk $|q| < 1$; the function $\phi(-q)$ converges at odd-order roots of unity ($q = \zeta_k^h$ with h/k reduced and k odd) while $\widetilde{\phi}(-q)$ diverges at odd-order roots of unity. Corollary 5.2 (5.1) shows that a forced truncation of $\widetilde{\phi}(-q)$ indeed equals $\phi(-q)$ at odd-order roots of unity, after a modest normalization (subtracting a constant and multiplying by a constant). A similar explanation of the antiquantum nature of (5.2) for the mock theta function ψ also holds.

The proof of Proposition 5.1 uses Proposition 2.1 as well as the following general identity.

Proposition 5.3 *For $q = \zeta_k^h$ with h/k reduced and k odd, we have that*

$$\sum_{s=0}^{k-1} q^{2s}(w;q)_{2s} = w^{k-1}\sum_{s=0}^{k-1}(-1)^s(w;q)_s.$$

Proof of Proposition 5.1 From [31, (2.3)] (with $q \mapsto q^2$, $b = w^{-1}q^2$, $z = wq$), we have that

$$\begin{aligned}
\widetilde{m}_{[k]}(w;q) &= \left(\frac{q^2}{w}\right)^{k-1} \sum_{0\le s\le k-1} (q^2/w;q^2)_s (q/w;q^2)_s \left(\frac{w^2}{q}\right)^s q^{-2s^2-2s} \\
&= \left(\frac{q^2}{w}\right)^{k-1} \sum_{0\le s\le k-1} (wq^{-2};q^{-2})_s (wq^{-1};q^{-2})_s q^{-2s} \\
&= \left(\frac{q^2}{w}\right)^{k-1} \sum_{0\le s\le k-1} (wq^{-1};q^{-1})_{2s} q^{-2s}.
\end{aligned}$$

Applying Proposition 5.3 (with $q \mapsto q^{-1}$, $w \mapsto wq^{-1}$), we find that this equals

$$\left(\frac{q^2}{w}\right)^{k-1}(wq^{-1})^{k-1}\sum_{0\le s\le k-1}(-1)^s(wq^{-1};q^{-1})_s$$
$$=\left(\frac{q^2}{w}\right)^{k-1}(wq^{-1})^{k-1}(\phi_1)_{[k]}\left(\begin{matrix}w\\0\end{matrix};q^{-1};-1\right).$$

Applying Proposition 2.1, we see that this equals

$$\left(\frac{q^2}{w}\right)^{k-1}(wq^{-1})^{k-1}\cdot\frac{2-w^k}{1-w}\phi_1\left(\begin{matrix}0\\w\end{matrix};q;-1\right).$$

Using [14, (12.3)], this is

$$\left(\frac{q^2}{w}\right)^{k-1}(wq^{-1})^{k-1}\cdot\frac{2-w^k}{1-w}\cdot\frac{1}{2}m(w;q)=\kappa_{w,q,k}\cdot m(w;q).$$

□

Proof of Corollary 5.2 We first prove (5.1). We apply Proposition 5.1 with $w=-q$ and $q=\zeta_k^h$, h/k reduced and k odd, which establishes that

$$2(1+q)g_3(-1;q)=\frac{2}{3}(1+q)(1-\widetilde{\phi}(-q))$$

or equivalently

$$1-\frac{1}{2}f(q)=\frac{1}{3}(1-\widetilde{\phi}(-q)). \tag{5.3}$$

Now we apply the Ramanujan–Watson identity [35, p63] between the third-order mock theta functions ϕ and f, which is equivalent to

$$2\phi(-q)-f(q)=\frac{(q;q)_\infty}{(-q;q)_\infty^2}. \tag{5.4}$$

We verify that the product on the right-hand side of (5.4) vanishes at odd-order roots of unity $q=\zeta_k^h$ (and that $\phi(-q)$ and $f(q)$ converge for such q). The result follows from this and (5.3).

Next we prove (5.2). We let $w=-1$ in Proposition 5.1 to obtain

$$f(q)=-\frac{4}{3}\widetilde{\psi}_{[k]}(-q) \tag{5.5}$$

at odd-order k roots of unity $q=\zeta_k^h$. Now we employ another Ramanujan-Watson identity [35, p63] between the third-order mock theta functions ψ and f, which is equivalent to

$$f(q)+4\psi(-q)=\frac{(q;q)_\infty}{(-q;q)_\infty^2}.$$

Using this, similar to the proof of (5.1), we find that $f(q)=-4\psi(-q)$ at odd-order roots of unity $q=\zeta_k^h$; the result follows from this and (5.5). □

Proof of Proposition 5.3 Let $q = \zeta_k^h$, a primitive odd-order k root of unity. We have that

$$\begin{aligned}\sum_{s=0}^{k-1}(-1)^s(w;q)_s &= \sum_{s=0}^{\frac{k-1}{2}}(w;q)_{2s} - \sum_{s=0}^{\frac{k-3}{2}}(w;q)_{2s+1}\\ &= (w;q)_{k-1} + \sum_{s=0}^{\frac{k-3}{2}}((w;q)_{2s} - (w;q)_{2s+1})\\ &= (w;q)_{k-1} + \sum_{s=0}^{\frac{k-3}{2}}(w;q)_{2s}(1-(1-wq^{2s}))\\ &= (w;q)_{k-1} + w\sum_{s=0}^{\frac{k-3}{2}}q^{2s}(w;q)_{2s}.\end{aligned}$$

On the other hand,

$$\begin{aligned}\sum_{s=0}^{k-1}q^{2s}(w;q)_{2s} &= \sum_{s=0}^{\frac{k-1}{2}}q^{2s}(w;q)_{2s} + \sum_{s=\frac{k+1}{2}}^{k-1}q^{2s}(w;q)_{2s}\\ &= \sum_{s=0}^{\frac{k-1}{2}}q^{2s}(w;q)_{2s} + \sum_{s=0}^{\frac{k-3}{2}}q^{k+2s+1}(w;q)_{k+2s+1}\\ &= \sum_{s=0}^{\frac{k-1}{2}}q^{2s}(w;q)_{2s} + (1-w^k)\sum_{s=0}^{\frac{k-3}{2}}q^{2s+1}(w;q)_{2s+1}\\ &= q^{k-1}(w;q)_{k-1} + \sum_{s=0}^{\frac{k-3}{2}}q^{2s}(w;q)_{2s} + (1-w^k)\sum_{s=0}^{\frac{k-3}{2}}q^{2s+1}(w;q)_{2s+1}\\ &= q^{-1}(w;q)_{k-1} + \sum_{s=0}^{\frac{k-3}{2}}q^{2s}(w;q)_{2s} + (1-w^k)\sum_{s=0}^{\frac{k-3}{2}}q^{2s+1}(w;q)_{2s+1}.\end{aligned}$$

Thus, to prove the result, it suffices to show that

$$(w^k-1)\sum_{s=0}^{k-2}q^s(w;q)_s = (q^{-1}-w^{k-1})(w;q)_{k-1}, \tag{5.6}$$

which we now establish. From [31, (2.3)], we deduce that

$$\sum_{n=0}^{k-1}(w;q)_n q^n = w^{k-1}$$

or equivalently that

$$\sum_{n=0}^{k-2}(w;q)_n q^n = w^{k-1} - (w;q)_{k-1}q^{-1}$$

so that the left-hand side of (5.6) equals

$$(w^k-1)(w^{k-1}-(w;q)_{k-1}q^{-1}) = w^{2k-1} - w^{k-1} + \frac{(1-w^k)^2}{(1-wq^{-1})}q^{-1}. \tag{5.7}$$

The right-hand side of (5.6) equals

$$(q^{-1} - w^{k-1})\frac{(1-w^k)}{(1-wq^{-1})}. \tag{5.8}$$

Subtracting (5.8) from (5.7), we obtain

$$\begin{aligned} w^{2k-1} - w^{k-1} + \frac{(1-w^k)^2}{(1-wq^{-1})}q^{-1} - (q^{-1} - w^{k-1})\frac{(1-w^k)}{(1-wq^{-1})} \\ = \frac{(1-w^k)}{(1-wq^{-1})}(-w^k q^{-1} + w^{k-1}) + w^{2k-1} - w^{k-1} \\ = (1-w^k)w^{k-1} + w^{2k-1} - w^{k-1} \\ = 0 \end{aligned}$$

as wanted. □

Data Availability

Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

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