Reduction of Joule Losses in Memristive Switching Using Optimal Control

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***Abstract*—This theoretical study investigates strategies for minimizing Joule losses in resistive random access memory (ReRAM) cells, which are also referred to as memristive devices. Typically, the structure of ReRAM cells involves a nanoscale layer of resistance-switching material sandwiched between two metal electrodes. The basic question that we ask is what is the optimal driving protocol to switch a memristive device from one state to another. In the case of ideal memristors, in the most basic scenario, the optimal protocol is determined by solving a variational problem without constraints with the help of the Euler-Lagrange equation. In the case of memristive systems, for the same situation, the optimal protocol is found using the method of Lagrange multipliers. We demonstrate the advantages of our approaches through specific examples and compare our results with those of switching with constant voltage or current. Our findings suggest that voltage or current control can be used to reduce Joule losses in emerging memory devices.**

***Index Terms*—Memristors, memristive systems, switching, optimal control, functional optimization.**

I. Introduction

THE problem of memristive switching optimization has a few facets. Minimizing Joule heat, switching time, or a combination of the two are possibilities. Furthermore, memristive devices can be described using ideal models [1], memristive models [2], [3], [4], or probabilistic models [5], [6], [7], [8]. Lastly, the optimization problem can be formulated with application-specific constraints and/or physics-based constraints. The highest voltage that can be used in the circuit or the compliance current to prevent the device from being damaged are the constraint examples.

To proceed, we shall first introduce the memristive devices [2] and their certain subset known as ideal memristors [1]. The voltage-controlled memristive devices are defined by the set of equations [2]

$$I(t) = R_M^{-1}(x, V)\, V(t), \tag{1}$$

$$\dot{x} = \boldsymbol{f}(x, V), \tag{2}$$

Received 22 November 2024; revised 6 December 2024; accepted 10 December 2024. Date of publication 13 December 2024; date of current version 27 December 2024. The work of Yuriy V. Pershin was supported by NSF under Grant EFRI-2318139. The review of this article was arranged by Associate Editor Y. Hirai. *(Corresponding author: Yuriy V. Pershin.)*

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Digital Object Identifier 10.1109/TNANO.2024.3517161

where V and I are the voltage across and current through the device, respectively, $R_M(x, V)$ is the memristance (memory resistance), x is a vector of n internal state variables, and $\boldsymbol{f}(x, V)$ is a vector state evolution function. Current-controlled memristive devices are defined similarly [2]. In the above set, (1) is the generalized Ohm's law, while (2) is the state equation. The latter defines the evolution of the internal state variable or variables.

In ideal memristors [1], the state evolution function is often proportional to the current. Consequently, the internal state variable x is proportional to the charge q. In addition, it is common to set the proportionality coefficient to one. In this case, the internal state variable is simply the charge flown through the device from an initial moment of time. In what follows, the response of such *ideal* devices is described in terms of a generalized Ohm's law

$$V = R_M(q) I(t), \tag{3}$$

where the memristance, R_M, is a function of charge. Although this model is quite abstract (physical devices behave substantially differently from the ideal ones [9], [10], [11]), its straightforward structure is advantageous for analytical calculations.

In this paper, we apply the calculus of variations and optimal control theory to the problem of memristive switching. We have derived the optimal driving protocols for the following optimization problems:

- unconstrained switching of ideal memristors within a fixed interval of time (Section II-A1);
- unconstrained switching of ideal memristors within a variable interval of time (Section II-A3);
- unconstrained switching of memristive systems within a fixed interval of time (Section II-B1);
- switching of ideal memristors within a fixed interval of time in the presence of a constraint (Section III-A).

An interesting finding is that, in unconstrained ideal memristor problems, the optimal trajectory corresponds to Joule losses occurring at a consistent rate (see Theorems 1 and 2 below). We compare our derived switching protocols to the cases of constant voltage or current. Our results show that the voltage or current control can be used to reduce Joule losses in emerging ReRAM circuits and systems.

The numerical simulations reported in this work were performed using Mathematica ver. 14.0.0.0. The comparison of optimal switching protocols is made with switching by constant voltage and current, which are included solely for comparative purposes.

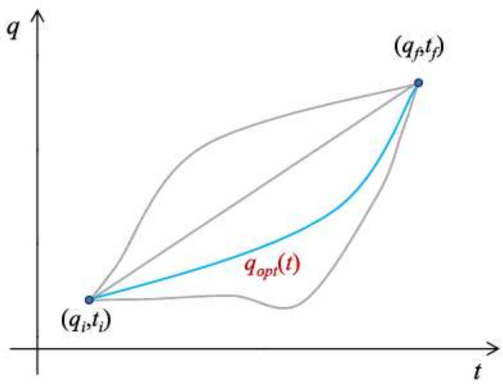


Fig. 1. Schematics of the optimal trajectory (blue curve labeled by $q_{opt}(t)$) and few non-optimal trajectories (grey curves) for the optimization problem in Section II-A1. The optimal trajectory minimizes the Joule heat functional given by (4).

This paper is structured as follows. Section II is devoted to unconstrained optimization problems. Within Section II, Section II-A focuses on ideal memristors, while Section II-B is dedicated to memristive systems. Constrained switching is discussed in Section III. Examples of optimal switching are given in Sections II-A2, II-A3, II-B2, and Section III-A2 where the calculus of variations and optimal control theory are applied to linear ideal memristors and memristive systems with a threshold. The paper concludes with a discussion of our results and future work.

II. Unconstrained Optimization Problems

A. Ideal Memristors

1) Minimization of Joule Losses: Let us first consider the problem of minimization of Joule losses in the switching of ideal memristors. Without loss of generality, the ideal memristors are described by (3). In this case, the energy cost to switch from one state to another is a functional [12] with respect to the charge trajectory $q(t)$. This trajectory links the initial state of the memristor, R_i, at the initial moment of time, t_i, with its final state, R_f, at the final moment of time, $t = t_f$. For a given $q(t)$, the Joule heat, $Q[q(t)]$, is expressed by

$$Q[q(t)] = \int_{(q_i,t_i)}^{(q_f,t_f)} \dot{q}^2 R_M(q) \mathrm{d}t, \tag{4}$$

where $q_{i/f}$ is the initial/final charge ($R_M(q_{i/f}) = R_{i/f}$), and it is assumed that $R_M(q)$ is known.[1] Our goal is to find the optimal trajectory, $q_{opt}(t)$, that minimizes the Joul heat functional, (4), see the illustration in Fig. 1.

In principle, this problem is similar to the principle of least action in classical mechanics [13]. Therefore, to determine the optimal trajectory we use the Euler-Lagrange equation that leads to the equation of motion

$$\ddot{q} R_M(q) + \frac{1}{2}\dot{q}^2 \frac{\mathrm{d}R_M(q)}{\mathrm{d}q} = 0. \tag{5}$$

[1] Equation (4) establishes a functional, which is an entity that produces a number when provided with a function.

The first integral of (5), corresponding to the energy conservation law in mechanics,

$$\dot{q}\sqrt{R_M(q)} = C_1, \tag{6}$$

can be integrated leading to

$$\int \sqrt{R_M(q)}\mathrm{d}q = C_1 \int \mathrm{d}t = C_1 t + C_2, \tag{7}$$

where C_1 and C_2 are constants.

Note that the first integral, (6), is nothing else than the conserved "kinetic energy", $R_M(q)\dot{q}^2$, for the Joule heat functional, (4). In the present context, it represents the power that is constant along the optimal trajectory. This observation is formulated in the following theorem.

Theorem 1: The optimal trajectory minimizing Joule losses in ideal memristors is characterized by constant power (in unconstrained problems).

2) Joule Losses in Linear Memristors: As an example of the above equations, consider the minimization of Joule losses in linear memristors. Let us assume that in the region of interest, the memristance $R_M(q)$ can be approximated by a linear function,

$$R_M(q) = a + bq. \tag{8}$$

Here, a and b are constants. In this case, the integral in (7) can be easily evaluated and (7) is rewritten as

$$\frac{2}{3b}\left(a + bq(t)\right)^{\frac{3}{2}} = C_1 t + C_2. \tag{9}$$

Next, the integration constants C_1 and C_2 are found using the initial and final point of the trajectory (see (4) and Fig. 1). Explicitly, the charge trajectory minimizing the Joule losses is

$$q_{opt}(t) = \frac{1}{b}\left[\frac{3b}{2}\left(C_1 t + C_2\right)\right]^{\frac{2}{3}} - \frac{a}{b} = \\ = \frac{1}{b}\left[\frac{R_i^{\frac{3}{2}}(t_f - t) + R_f^{\frac{3}{2}}(t - t_i)}{t_f - t_i}\right]^{\frac{2}{3}} - \frac{a}{b}. \tag{10}$$

The current corresponding to $q_{opt}(t)$ is

$$I_{opt}(t) = \frac{2}{3b}\frac{R_f^{\frac{3}{2}} - R_i^{\frac{3}{2}}}{t_f - t_i}\left[\frac{R_i^{\frac{3}{2}}(t_f - t) + R_f^{\frac{3}{2}}(t - t_i)}{t_f - t_i}\right]^{-\frac{1}{3}} \tag{11}$$

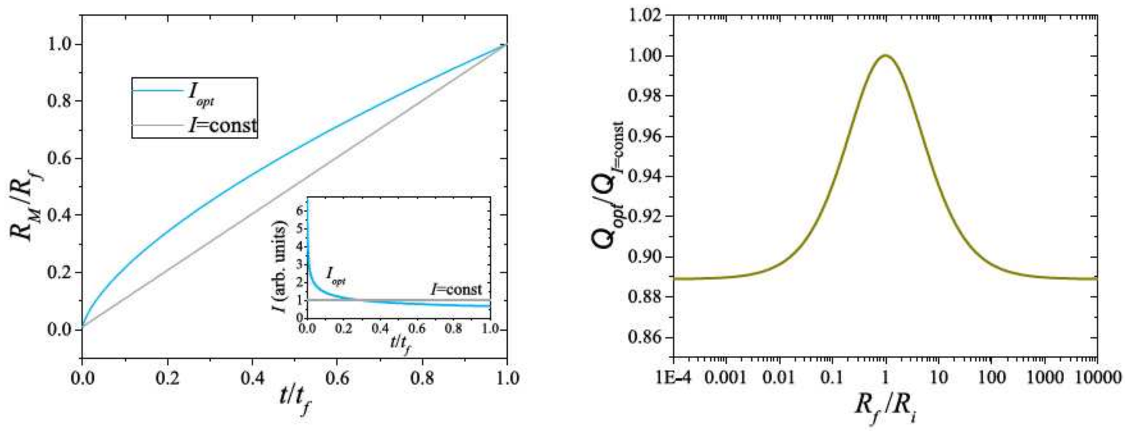
and total Joule heat generated during the switching is

$$Q_{opt} = \left(\frac{2}{3b}\right)^2 \frac{\left(R_f^{\frac{3}{2}} - R_i^{\frac{3}{2}}\right)^2}{t_f - t_i}. \tag{12}$$

Using (6) it is not difficult to verify that Theorem 1 is satisfied by $q_{opt}(t)$.

It is interesting to compare Q_{opt} for the optimal trajectory (12) to the Joule heat for the trajectory connecting (q_i, t_i) to (q_f, t_f) linearly,

$$q_{I=\text{const}}(t) = q_i \frac{t_f - t}{t_f - t_i} + q_f \frac{t - t_i}{t_f - t_i}. \tag{13}$$



(a) (b)

Fig. 2. Minimization of Joule losses in the ideal linear memristors (for the model see (8)). (a) Memristance as a function of time for the cases of optimal current and constant current. The control currents are shown in the inset. This plot was obtained using $R_f/R_i = 100$ and $t_i = 0$. (b) Total Joule heat as a function of R_f/R_i. Asymptotically, $Q_{opt}/Q_{I(V)=\text{const}} \to 8/9$ as $R_f/R_i \to 0$ or ∞.

We note that this case corresponds to driving by a constant current. Using (4) one finds

$$Q_{I=\text{const}} = \frac{1}{2b^2}\frac{\left(R_f^2 - R_i^2\right)(R_f - R_i)}{t_f - t_i}. \tag{14}$$

It can be directly verified that the heat $Q_{I=\text{const}}$ is always greater than the optimal Joule heat, $Q_{I=\text{const}} > Q_{opt}$, for all values $R_f \neq R_i$ and $t_f > t_i$ as it should be.

When the memristor is driven by a constant voltage V, the Ohm's law is simply the differential equation $\dot{q}R_M(q) = V$ having the solution

$$q_{V=\text{const}}(t) = \frac{-a + \sqrt{R_i^2 + 2bV(t - t_i)}}{b}. \tag{15}$$

Interestingly, in this case the total Joule heat is the same as in the case of constant current, (14). As the voltage as a function of switching time interval can be written as

$$V = \frac{1}{2b}\frac{R_f^2 - R_i^2}{t_f - t_i}, \tag{16}$$

another expression for $Q_{V=\text{const}}$ is

$$Q_{V=\text{const}} = \frac{V(R_f - R_i)}{b}. \tag{17}$$

Certain results found for linear memristors are shown graphically in Fig. 2. In particular, switching linear memristors using optimal control reduces Joule losses by up to a factor of $8/9$ or approximately 11% compared to switching with constant voltage or current. Fig. 2(a) shows that reducing Joule losses is achieved by increasing the current I when the resistance R_M is smaller, and vice versa.

3) Simultaneous Minimization of Joule Losses and Switching Time: Equations (12) and (14) suggest that Joule losses can be minimized by increasing the switching time. However, in numerous applications, a high operating frequency is essential. In this part of the paper, we explore how to reconcile the conflicting demands of low losses and rapid switching.

Thus, we would like to find the trajectory that minimizes Joule losses and switching time simultaneously. For this purpose, the cost functional can be selected as

$$F = f\left(Q[q(t)]\right) + g\left(t_f - t_i\right), \tag{18}$$

where $f(x)$ and $g(x)$ are monotonously increasing functions, and $Q[q(t)]$ is given by (4). In the linear case, the cost functional is simply

$$F = w_1 Q[q(t)] + w_2 \cdot (t_f - t_i), \tag{19}$$

where w_i are the positive weights (distinct units).

To find the minimum value of the cost functional, we independently vary $q(t)$ and t_f. The variation of $q(t)$ gives the same (5), and the variation of t_f gives the additional equation

$$\left(w_2 - w_1\dot{q}^2 R_M(q)\right)\big|_{t=t_f} = 0, \tag{20}$$

which, according to (6), can be simplified to $w_2 = w_1 C_1^2$. As the first integral of (5) is given by the same (6), the power is constant along the optimal trajectory. Therefore, we state the following theorem.

Theorem 2: The optimal trajectory minimizing Joule losses and switching time in ideal memristors is characterized by constant power (in unconstrained problems).

Returning to the example in Section II-A2, we can use (12) to express the power in (20) and eventually obtain the optimal time

$$(t_f - t_i)_{opt} = \sqrt{\frac{w_1}{w_2}}\frac{2}{3b}\left(R_f^{\frac{3}{2}} - R_i^{\frac{3}{2}}\right). \tag{21}$$

At this value of $t_f - t_i$,

$$Q_{opt} = \sqrt{\frac{w_2}{w_1}}\frac{2}{3b}\left(R_f^{\frac{3}{2}} - R_i^{\frac{3}{2}}\right). \tag{22}$$

B. Memristive Systems

1) Lagrangian Function: Next, we consider a voltage-controlled memristive device satisfying (1)–(2). In the following discussion, we focus on a scenario with a single internal state variable. However, our primary equations can be adapted to more complex situations.

Our aim is to identify the most efficient driving protocol that minimizes Joule losses when the device switches from $x(t_i) = x_i$ to $x(t_f) = x_f$. Mathematically, we consider a model with the Joule heat functional defined by

$$Q = \int_{t_i}^{t_f} \frac{V^2(t)}{R_M(x)} \mathrm{d}t \;\rightarrow\; \min, \tag{23}$$

where x is the function of time according to (2).

According to the general scheme [14], the Lagrangian function is written as

$$\mathcal{L} = \lambda_0 \int_{t_i}^{t_f} \frac{V^2(t)}{R_M(x)} \mathrm{d}t + \int_{t_i}^{t_f} \Lambda(t) \left[\dot{x}(t) - f(x, V)\right] \mathrm{d}t + l, \tag{24}$$

where λ_0 and l are constants, and $\Lambda(t)$ is a function of time known as the Lagrange multiplier. From (24), the Lagrangian is

$$L\left(x, \dot{x}, V, \Lambda\right) = \lambda_0 \frac{V^2(t)}{R_M(x)} + \Lambda(t) \left[\dot{x}(t) - f(x, V)\right]. \tag{25}$$

The necessary conditions for an extrema are the following [14]:

- Euler-Lagrange equation: $\mathrm{d}(\partial L/\partial \dot{x})/\mathrm{d}t = \partial L/\partial x$;
- Stationary condition with respect to V: $\partial L/\partial V = 0$;
- Transversality conditions $L'_{\dot{x}}(t_i) = l'_{x(t_i)}$ and $L'_{\dot{x}}(t_f) = -l'_{x(t_f)}$, where $l = \lambda_i x(t_i) + \lambda_f x(t_f)$ is the terminant.

The last condition leads to $\Lambda(t_i) = \lambda_i$ and $\Lambda(t_f) = -\lambda_f$. As λ_i and λ_f are arbitrary constants, these conditions can be omitted. Therefore, our optimization problem is defined by the following set of equations:

$$\Lambda(t) = \lambda_0 \frac{2V(t)}{R_M(x) f'_V(x, V)}, \tag{26}$$

$$\frac{\mathrm{d}\Lambda(t)}{\mathrm{d}t} = \lambda_0 V^2(t) \frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{1}{R_M(x)}\right] - \Lambda(t) f'_x(x, V), \tag{27}$$

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = f(x, V), \tag{28}$$

$$x(t_i) = x_i \;,\; x(t_f) = x_f. \tag{29}$$

2) Derivation of Ideal Memristor Equations: Here, we show that (5) for ideal memristors in Section II-A1 can be derived directly from (26)–(29). For this purpose, using $f(x, V) = V/R_M(x)$ we first obtain the partial derivatives $f'_V = 1/R_M(x)$ and $f'_x = -R'_M(x)V/R^2_M(x)$. Substituting these partial derivatives into (26) yields

$$\Lambda(t) = 2\lambda_0 \frac{V(t)}{R_M(x)/R_M(x)} = 2\lambda_0 V(t). \tag{30}$$

Using (30), (27) can be rewritten as

$$\dot{V}(t) = \frac{1}{2} V^2(t) \frac{R'_M(x)}{R^2_M(x)}. \tag{31}$$

Taking into account that $V(t) = R_M(q)\dot{q}$, (31) leads to

$$\ddot{q} R_M(q) + \frac{1}{2}\left(\dot{q}\right)^2 \frac{\mathrm{d}R_M}{\mathrm{d}q} = 0, \tag{32}$$

which is precisely (5) in Section II-A1. Therefore, the results presented in Section II-A1 and II-A1 remain consistent when applying our comprehensive optimization theory formulated for the memristive devices in the current subsection.

3) Optimal Control of a Threshold-Type Memristive Device: Traditional experimental memristive devices [15], [16] demonstrate switching with a threshold, which is essential for nonvolatile information storage. In the following, we apply the optimization approach from Section II-B1 to a memristive device described by a model featuring a threshold. Specifically, we consider a device described by the equations [3]

$$R_M(x) = R_{on} + x\left(R_{off} - R_{on}\right), \tag{33}$$

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \begin{cases} k(V - V_{off}), & V > V_{off} > 0, \\ 0, & V_{on} \leq V \leq V_{off}, \\ k(V - V_{on}), & V < V_{on} < 0, \end{cases} \tag{34}$$

where $k > 0$ is the constant, R_{on} and R_{off} are the on-state and off-state resistances, and V_{on} and V_{off} are the thresholds. According to the above equations, when the voltage is greater than V_{off}, the memristive system switches to the high-resistance state, R_{off}.

For the sake of definiteness, let us consider the transition $x_i \rightarrow x_f$ *assuming* that $V > V_{off}$. As $f(x, V) = k(V - V_{off})$, (26)–(28) take the form

$$\Lambda(t) = 2\lambda_0 \frac{V(t)}{kR_M(x)}, \tag{35}$$

$$\frac{\mathrm{d}\Lambda(t)}{\mathrm{d}t} = -\lambda_0 V^2(t) \frac{R'_M(x)}{R^2_M(x)}, \tag{36}$$

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = k(V - V_{off}). \tag{37}$$

Next, without the loss of generality, we set $\lambda_0 = 1$. Substituting $V(t)/R_M(x)$ from (35) into (36) we arrive at

$$\frac{\mathrm{d}\Lambda(t)}{\mathrm{d}t} = -R'_M(x)\left(\frac{k}{2}\right)^2 \Lambda^2(t). \tag{38}$$

The solution of (38) can be expressed as

$$\Lambda(t) = \frac{1}{\frac{k^2}{4} R'_M(x) t + C_1}, \tag{39}$$

where C_1 is the integration constant.

Using (35) we obtain the following expression for the voltage

$$V(t) = \frac{k}{2} \frac{R_M(x)}{\frac{k^2}{4} R'_M(x) t + C_1} \tag{40}$$

that can be further simplified using the linear dependence of R_M on x, (33). The substitution of (40) and (33) into the state equation (37) leads to

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{2R_{on}}{R'_M(x)(t+t_0)} - kV_{off} + \frac{2}{t+t_0}x, \tag{41}$$

where $t_0 = 4C_1/(k^2 R'_M(x))$ is employed for compactness in place of C_1.

The solution of (41) can be written as

$$x(t) = C(t+t_0)^2 + kV_{off}(t+t_0) - \frac{R_{on}}{R'_M(x)}, \tag{42}$$

where C is an arbitrary constant. Consequently,

$$R_M(t) = CR'_M(x)(t+t_0)^2 + kR'_M(x)V_{off}(t+t_0). \tag{43}$$

The values of the arbitrary constants C and t_0 can be determined by setting $R_M(t_i)$ equal to R_i and $R_M(t_f)$ equal to R_f.

The voltage as a function of time is found by utilizing (40), which yields

$$V(t) = 2V_{off} + \frac{2C}{k}(t+t_0), \tag{44}$$

$$I(t) = \frac{2}{kR'_M(x)}\frac{1}{t+t_0}. \tag{45}$$

According to (44) the control voltage is a linear function of time.

Using the initial and final resistances, one can obtain

$$C = \frac{R_f + R_i - \sqrt{4R_iR_f + [kV_{off}R'_M(x)(t_f-t_i)]^2}}{R'_M(x)(t_f-t_i)^2} \tag{46}$$

and

$$t_0 = \frac{1}{2C}\left\{\frac{R_f - R_i}{(t_f-t_i)R'_M(x)} - kV_{off}\right\} - \frac{t_f+t_i}{2}. \tag{47}$$

It should be noted that the solution involving the positive root in (46) has been discarded, as it does not meet the condition $V(t) > V_{off}$, which we have assumed from the beginning of this subsection. Furthermore, the sum $t_0 + t_i > 0$, since we are operating under the assumption that $R_f > R_i$ in all cases. That is important because it guarantees the finiteness of the switching current $I(t)$ defined by (45) for all moments of time.

It is also observed that C is positive if $R_f > R_i + kV_{off}R'_M(x)(t_f - t_i)$. According to (44), this implies that the optimal control voltage rises over time, surpassing the double threshold voltage $2V_{off}$ even at the initial moment t_i. When $R_f = R_i + kV_{off}R'_M(x)(t_f - t_i)$, the parameter C becomes zero, and as per (44), the optimal control voltage remains constant at $V(t) = 2V_{off}$ throughout the entire switching interval, from t_i to t_f.

In the parameter domain $R_f < R_i + kV_{off}R'_M(x)(t_f - t_i)$, the solution given by (43)–(47), ceases to be optimal. In addition, this solution does not even satisfy the assumption that $V(t_f) > V_{off}$, when $R_f \to R_i$. It is clear that in this case, the optimal strategy is to quickly switch the memristor from the initial state R_i to the final state R_f during a finite time interval Δt by applying the voltage exceeding the threshold voltage V_{off} and then applying zero voltage for the remaining time. It turns out that the magnitude of optimal voltage is the same as in the case when $R_f = R_i + kV_{off}R'_M(x)(t_f - t_i)$,

$$V(t) = \begin{cases} 2V_{off}, & t_i < t < t_i + \Delta t, \\ 0, & t_i + \Delta t < t < t_f, \end{cases} \tag{48}$$

while in accordance with (33)–(34) memristance is a piecewise linear function of time

$$R(t) = \begin{cases} R_i + kV_{off}R'_M(x)(t - t_i), & t_i < t < t_i + \Delta t, \\ R_f, & t_i + \Delta t < t < t_f, \end{cases} \tag{49}$$

where $\Delta t = (R_f - R_i)/(kR'_M(x)V_{off})$.

Finally, the minimal Joule heat is expressed by

$$Q_{opt} = \frac{4}{k^2R'_M(x)}\left[C\,(t_f - t_i) + kV_{off}\ln\left(\frac{t_f+t_0}{t_i+t_0}\right)\right], \tag{50}$$

when $R_f - R_i \geq kV_{off}R'_M(x)(t_f - t_i)$,
and

$$Q_{opt} = \frac{4V_{off}}{kR'_M(x)}\ln\frac{R_f}{R_i}, \tag{51}$$

when $R_f - R_i \leq kV_{off}R'_M(x)(t_f - t_i)$.

It is interesting to compare these minimal Joule losses with the Joule losses for other possible switching regimes: switching at constant voltage $V = V_0$ and constant current $I = I_0$. For the constant voltage switching protocol by integrating in (23) over time with (33) and (34) taking into account, we get

$$Q_{V=\text{const}} = V_0^2\frac{t_f - t_i}{R_f - R_i}\ln\frac{R_f}{R_i}, \tag{52}$$

where $V_0 = V_{off} + (R_f - R_i)/(kR'_M(x)(t_f - t_i))$.

Similarly, for the constant current switching protocol, we find for the Joule losses

$$Q_{I=\text{const}} = I_0\left[V_{off}(t_f - t_i) + \frac{R_f - R_i}{kR'_M(x)}\right], \tag{53}$$

where the constant current I_0 is a unique solution of the following equation

$$e^{kR'_M(x)(t_f-t_i)I_0} = \frac{R_fI_0 - V_{off}}{R_iI_0 - V_{off}}, \tag{54}$$

in the domain $I_0 > V_{off}/R_i$, and the voltage depends exponentially on time as

$$V(t) = V_{off} + (R_iI_0 - V_{off})\,e^{kR'_M(x)I_0(t-t_i)}. \tag{55}$$

Fig. 3(a) exemplifies the time dependence of the memristance for several driving protocols. We note that in the cases of optimal control, constant voltage control, and constant current control, the memristance increases quadratically (43), linearly, and exponentially with time, respectively. Based on the information presented in Fig. 3(b), the use of optimal control has the potential to decrease Joule losses by approximately 27% compared to switching based on constant voltage and

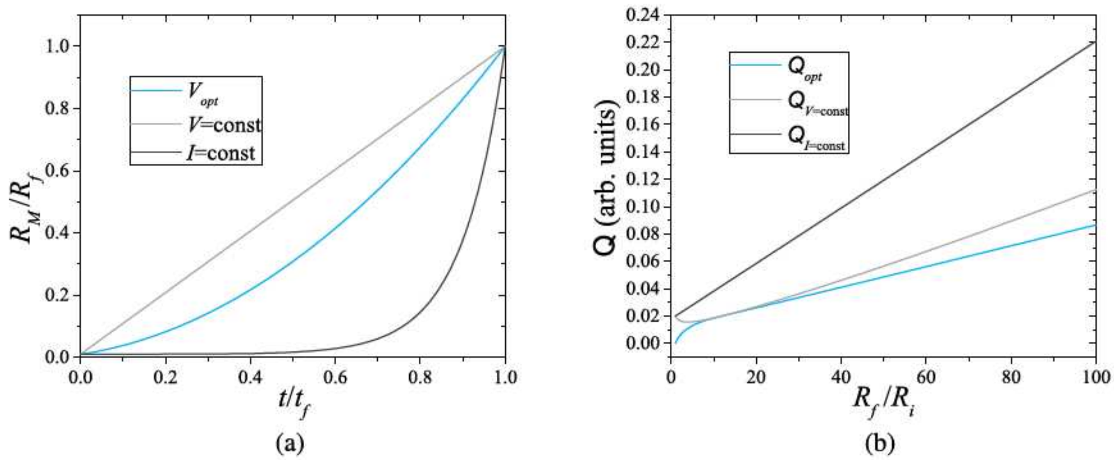


Fig. 3. Minimization of Joule losses in memristive devices. (a) Memristance as a function of time for the cases of optimal control, constant voltage control, and constant current control. (b) Joule heat as a function of R_f/R_i. These figures were obtained using the following set of parameter values: $R_{on} = 1$ k Ω, $R_{off} = 100$ kΩ, $k = 0.5$ $(\mathrm{V}\cdot\mu\mathrm{s})^{-1}$, $V_{off} = 0.1$ V, $t_i = 0$, $R_i = R_{on}$, $t_f = 2\,\mu$s, and (a) $R_f = R_{off}$, (b) $R_{on} \le R_f \le R_{off}$.

by approximately 35% in contrast to switching based on constant current. Qualitatively, the superior efficiency of constant voltage switching relative to constant current switching can be attributed to the dynamics in the constant voltage case that is closer to the optimal one. In particular, this can be recognized in Fig. 3(a).

III. Constrained Optimization Problems

A. Ideal Memristors

1) Pontryagin's Principle: Let us now address a more complex case involving the optimal control of memristive switching while taking into account inequality constraints. Specifically, we will focus on the case of an ideal memristor with a current constraint, where we assume that the current $I(t)$ passing through the device is limited by a critical (compliance) current I_c, such that $|I(t)| \le I_c$. Consequently, we are interested in minimizing Joule losses, which can be expressed through the functional (4), while also imposing the current limiting constraint $|\dot{q}| \le I_c$.

It is evident that when the critical current I_c is sufficiently large, the solution of the problem is determined by the same Euler-Lagrange equation (5). The situation becomes more complicated when the current, calculated using the solution derived from (7) with specified boundary conditions, exceeds the critical current at least at a certain point in time. In such a situation, it becomes necessary to incorporate Pontryagin's principle of maximum (minimum) to determine the optimal control $q(t)$. To facilitate this, it is convenient to introduce a new variable u following the definition[2]

$$u = \dot{q}. \tag{56}$$

This way we transfer the constraints into the configuration space

$$|u| \le I_c. \tag{57}$$

[2] It is evident that the definition of u is identical to the current. Nevertheless, a distinct notation is required since, in Pontryagin's principle, u is treated as an independent variable.

The Lagrangian function for this optimal problem is written as (see [14])

$$\mathcal{L} = \int_{t_i}^{t_f} \left[\lambda_0 R_M(q)u^2 + p(t)\left(\dot{q}(t) - u\right)\right] \mathrm{d}t + l, \tag{58}$$

where constant $\lambda_0 \ge 0$ and function $p(t)$ are the unknown Lagrange multipliers. In the subsequent discussion, we omit the term $l = \lambda_i q(t_i) + \lambda_f q(t_f)$ from the Lagrangian function (58) since the transversality conditions associated with it do not impose any limitations on the solution of the problem under consideration.

The optimal control must satisfy not only Euler-Lagrange equation with respect to q,

$$\lambda_0 R'_M(q)u^2 - \frac{\mathrm{d}p(t)}{\mathrm{d}t} = 0, \tag{59}$$

but also the Pontryagin's principle of maximum (minimum) with respect to u,

$$\min_{|\eta| \le I_c} \left[\lambda_0 R_M(q(t))\eta^2 - p(t)\eta\right] = \lambda_0 R_M(q(t))u^2 - p(t)u. \tag{60}$$

The last condition implies that, at any instant t, the optimal control $u(t)$ minimizes the expression in (58) within the range $-I_c \le u \le I_c$ for the optimal solution $q(t)$ and $p(t)$.

Let us first examine the specific case of $\lambda_0 = 0$. Considering the Euler-Lagrange equation (59), it follows that $p(t)$ is constant. Applying Pontryagin's principle (60) with $\lambda_0 = 0$, we can determine the optimal control for the function $u(t) = I_c \mathrm{sign}\, p(t) = \pm I_c$. Therefore, in the case where $\lambda_0 = 0$, the constant maximum current $\dot{q} = u = \pm I_c$ is maintained throughout the entire duration of the memristor switching.

This special solution is only possible under certain boundary conditions, specifically when $(q_f - q_i)/(t_f - t_i) = \pm I_c$. If this condition is not satisfied, this particular solution is not viable, and $\lambda_0 > 0$. Furthermore, we can assume $\lambda_0 = 1$ in the following discussion without loss of generality.

The minimization of the quadratic function in the LHS of (60) gives the following optimal control for the function $u(t)$:

$$u(t) = \begin{cases} \frac{p(t)}{2R_M(q)}, & \text{if } \frac{|p(t)|}{2R_M(q)} < I_c, \\ I_c \text{sign}\, p(t), & \text{otherwise.} \end{cases} \quad (61)$$

The system of equations (56), (59), with $\lambda_0 = 1$, and (61) fully determines the optimal switching of the ideal memristor. It is evident that the current $u(t)$ satisfies the constraints specified in (57). Furthermore, by eliminating the Lagrange multiplier $p(t)$ from Euler-Lagrange (59) using (61) for $u(t) < I_c$, we obtain the optimal trajectory for the ideal memristor without constraints, as expected. Therefore, it is apparent that achieving optimal control for the ideal memristor under current restrictions generally involves a smooth and continuous connection of the solution characterized by constant power (refer to (7)) with the solution represented by the maximum possible current $q = \pm I_c t + C_3$, where C_3 is an arbitrary constant.

2) Joule Losses in Linear Memristors: We will illustrate the method discussed above by considering the switching of a linear memristor, considered previously in Section II-A2. However, now we also assume that the maximum possible current is limited by the critical (compliance) value I_c. It is also assumed that during the time period from $t_i = 0$ to t_f, the memristance changes from $R_M(0) = R_i$ to $R_M(t_f) = R_f > R_i$. Furthermore, we consider the most interesting case where at the initial moment of time the optimal unrestricted current $I_{opt}(t = t_i)$, given by (11), greater than I_c. This corresponds to the following inequality:

$$2\frac{R_f^{3/2} - R_i^{3/2}}{3bt_f\sqrt{R_i}} > I_c. \quad (62)$$

Following the previous discussion, to minimize Joule heat, it is necessary to apply the highest current I_c for a duration of up to t_c. In the case of the linear memristor described by (8), this leads to the following resistance change:

$$R_M(t) = R_i + bI_c t, \quad \text{for } 0 \le t \le t_c. \quad (63)$$

Starting at $t = t_c$, the optimal solution is given by (10):

$$R_M(t) = \left[\frac{R_c^{\frac{3}{2}}(t_f - t) + R_f^{\frac{3}{2}}(t - t_c)}{t_f - t_c}\right]^{\frac{2}{3}}, \quad \text{for } t_c \le t \le t_f, \quad (64)$$

where resistance $R_c = R_M(t_c)$ calculated with the use of (63).

The values of the parameters t_c and R_c need to be determined from the continuity condition of the memristance, $R_M(t)$, and the current, $I(t) = \dot{R}_M/b$, at the instant t_c when the control change occurs. This leads to the following equations for R_c and t_c:

$$\sqrt{R_c^3} - 3(R_i + bI_c t_f)\sqrt{R_c} + 2\sqrt{R_f^3} = 0, \quad (65)$$

and

$$t_c = \frac{R_c - R_i}{bI_c}. \quad (66)$$

Note that an obvious condition must be met for the final state, R_f, to be reachable from the initial state, R_i, during time interval t_f due to the existance of the maximum possible current I_c:

$$R_f < R_i + bI_c t_f\ . \quad (67)$$

This inequality guarantees the existence of two positive roots of (65), the smallest of which is smaller than R_f, while the other is larger. This smallest positive root of (65) determines the resistance R_c and moment of time t_c (see (66)), where the optimal control changes. Note that the inequality (62) guarantees that $R_c > R_i$.

Fig. 4 illustrates our results on the optimal control of an ideal linear memristor with a current constraint. Fig. 4(b) represents memristance as a function of time calculated by using (63), (64) for the case of constrained switching and by using (8), (10) for the case of unconstrained switching. The corresponding currents, which are proportional to the derivatives of the memristances with respect to time, are presented in Fig. 4(a). Clearly, for the used set of parameters, inequalities (62) and (67) are satisfied. Thus, it allows us to find the only solution of (65) and (66), $R_c/R_f = 0.34$ and $t_c/t_f = 0.26$, which determines the resistance and time moment of control change for this set of parameters.

B. Memristive Systems

In principle, the general approach developed for memristive systems in Section II-B combined with the Pontryagin principle (which we used for the analysis of the switching of ideal memristors with a current constraint in Section III-A) can be employed to optimize the switching of memristive devices ((1) and (2)) in the presence of constraints. However, our derivations in Section III-A suggest that optimal switching protocols, in fact, can be obtained using a simplified approach, at least in certain cases. Such cases include (but are not limited to) some simple (first-order) memristive models and unidirectional switching.

Consider, for example, situations that involve current compliance (see (57)). The idea is to link the two operational conditions, specifically, with and without current compliance. First of all, one can consider the unconstrained scenario (using the approach outlined in Section II-B). If (57) holds consistently, then the solutions for both constrained and unconstrained problems coincide. Otherwise, the solutions with and without the constraint should be combined while adhering to the criteria of (i) maintaining the continuity of the control parameter and (ii) minimizing Joule losses.

IV. Discussion

In this theoretical study, novel strategies have been devised to reduce Joule losses that arise when memristive devices are switched. Various hypothetical scenarios have been examined, and substantial energy savings have been shown to be possible by following specific driving protocols proposed in this research.

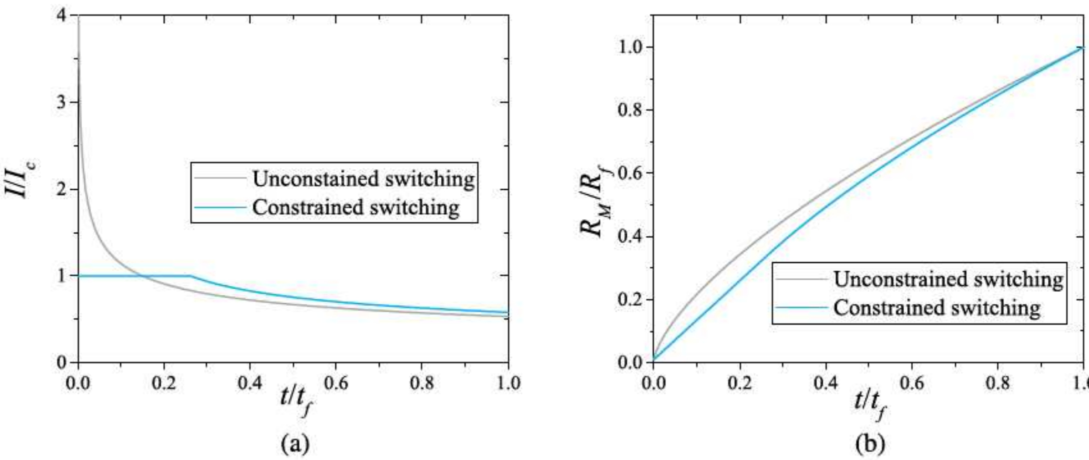


Fig. 4. Constrained versus unconstrained control of memristive switching. (a) Control current as a function of time. (b) Memristance as a function of time. This figure was obtained using the following set of parameter values: $R_{on} = R_i = 1$ k Ω, $R_{off} = R_f = 100$ kΩ, $I_c = 0.25$ mA, $b = 10^2$ $k\Omega/$(mA $\cdot$ μs), $t_i = 0$ and $t_f = 5$ μs.

It is important to note that our findings are model-dependent, implying that no universal protocol can be applied to all memristive devices. To implement our method(s), first, the device model must be identified. Subsequently, our optimization techniques can be utilized to determine the optimal driving protocol for that specific device model. The final step involves verification. Based on our experience, it is always beneficial to compare the derived protocol with other easily verifiable scenarios, such as switching by constant voltage or current. This comparison is crucial because the solution to the Euler-Lagrange equation represents an extremum, either a minimum or a maximum.

One might wonder about the experimental significance of the findings reported in this article, and indeed they are relevant. Our understanding of ideal devices (Section II-A) indicates that their optimal switching involves constant power. In threshold-type memristive systems (Section II-B3), although constant power conditions do not strictly apply, a similar pattern remains: lower voltages are paired with lower resistances and vice-versa. These observations could be utilized in an empirical optimization.

Furthermore, the model described by (33) and (34) in Section II-B3 represents a particular instance of the VTEAM model, which can accurately describe experimental devices [17]. Within the VTEAM, the function that describes the evolution of the state (the right-hand side of (34)), as a function of voltage, is $\propto (V - V_{on/off})^{\alpha_{on/off}}$, where $\alpha_{on/off}$ is a constant parameter. According to [17], (34) is relevant to the transition from the "on" state to the "off" state in the Pt-Hf-Ti memristor [18] and thus our theory can be verified on this experimental platform. Separate investigations are necessary for different $\alpha_{on/off}$ values and more complex circuits. We anticipate that when $\alpha_{on/off} > 2$, the most effective approach could be the use of the maximum permissible voltage, followed by a zero voltage.

Finally, pulse control is a commonly employed technique in memristive switching. Notably, the time-dependent current/voltage waveforms introduced in this paper can be utilized as pulses, which reduces the energy needed for memristive switching compared to using pulses with a fixed amplitude. Furthermore, in digital circuit architectures, continuous voltage waveforms can be replaced by a finite set of voltage levels.

V. Conclusion

The production of electricity from fossil fuels is one of the main contributors to global warming. To suppress climate change, energy-efficient systems, devices, and technologies must be implemented. Our study lays the foundation for future research in optimizing the operational conditions of memristive devices, which could potentially be adapted for other memory circuit elements [19] with suitable modifications. The experimental validation of our methods on single devices [18], [20] and their application to memristive circuits, including crossbar arrays [21], [22], is of significant interest. The practical implications of this work can be significant, as implementing the protocols outlined here could enhance the efficiency of memristive systems and related technologies such as in-memory computing and neuromorphic computing, to name a few.

Acknowledgment

The authors would like to thank Alon Ascoli and Fernando Corinto for useful comments and discussions.

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