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BOUNDARY TRIPLES FOR A FAMILY OF DEGENERATE ELLIPTIC OPERATORS OF KELDYSH TYPE

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We consider a one-parameter family of degenerately elliptic operators $\mathcal{L}_\gamma$ on the closed disk $\mathbb{D}$, of Keldysh (or Kimura) type, which appears in prior work by the authors and Mishra (2023), related to the geodesic X-ray transform. Depending on the value of a constant $\gamma \in \mathbb{R}$ in the subprincipal term, we prove that either the minimal operator is self-adjoint (case $|\gamma| \geq 1$), or that one may construct appropriate trace maps and Sobolev scales (on $\mathbb{D}$ and $\mathbb{S}^1 = \partial\mathbb{D}$) on which to formulate mapping properties, Dirichlet-to-Neumann maps, and extend Green's identities (case $|\gamma| < 1$). The latter can be reinterpreted in terms of a boundary triple for the maximal operator, or a generalized boundary triple for a distinguished restriction of it. The latter concepts, objects of interest in their own right, provide avenues to describe sufficient conditions for self-adjointness of extensions of $\mathcal{L}_{\gamma,\min}$ that are parametrized in terms of boundary relations, and we formulate some corollaries to that effect.

1. Introduction

This article is concerned with the study of boundary triples (or equivalently, the derivation of appropriate settings where generalized Green's identities[1] hold) of a one-parameter family of degenerate elliptic operators on the Euclidean unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| \leq 1\}$: for $\gamma \in \mathbb{R}$, using polar coordinates $z = \rho e^{i\omega}$ and using $x = 1 - \rho^2$ as the boundary defining function and with $\mathrm{d}V = \rho\, \mathrm{d}\rho\, \mathrm{d}\omega$ the Euclidean measure, we define

$$\mathcal{L}_\gamma := -\rho^{-1} x^{-\gamma} \partial_\rho (\rho x^{\gamma+1} \partial_\rho) - \rho^{-2} \partial_\omega^2 + (1+\gamma)^2 \tag{1}$$

$$= -x \partial_\rho^2 - (\rho^{-1} - (3+2\gamma)\rho)\partial_\rho - \rho^{-2}\partial_\omega^2 + (\gamma+1)^2\, \mathrm{id}. \tag{2}$$

Expression (1) shows that $\mathcal{L}_\gamma$ is formally self-adjoint for the space $L^2_\gamma := L^2(\mathbb{D}, x^\gamma \mathrm{d}V)$ (whose inner product we denote $(f, g)_{L^2_\gamma} := \int_{\mathbb{D}} f \bar{g}\, x^\gamma \mathrm{d}V$), while expression (2) shows the degenerate behavior of $\mathcal{L}_\gamma$, of first order in the top-degree term normal to the boundary. Although the coefficient γ only appears in subprincipal terms, the operator-theoretic properties of $\mathcal{L}_\gamma$ (e.g., number of self-adjoint extensions, associated traces and their regularity at the boundary) strongly depend on γ.

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[1] For the Laplacian on a bounded domain $\Omega \subset \mathbb{R}^2$ with smooth boundary and $f, g \in C^\infty(\bar{\Omega})$, Green's first identity reads $\int_\Omega ((-\Delta f) g + \nabla f \cdot \nabla g) = -\int_{\partial\Omega} \partial_\nu f g$, while Green's second identity is the skew-symmetrized version $\int_\Omega (-g\Delta f + f \Delta g) = \int_{\partial\Omega} (f \partial_\nu g - g \partial_\nu f)$.

There are many appearances of operators of this type in the literature:

• Such operators arise in the study of fluid flows, as operators that switch from being elliptic to hyperbolic across a curve, two prototypes of which are Keldysh (our case) and Tricomi operators; see, e.g., the book [Otway 2012]. Indeed, expression (1) of $\mathcal{L}_\gamma$ naturally extends past the unit circle, and the operator becomes hyperbolic outside the unit disk. Here we are interested in confining ourselves to the "elliptic region" $\mathbb{D}$ and studying the operator $\mathcal{L}_\gamma$ there.

• In recent advances on microlocal methods for asymptotically hyperbolic manifolds and spacetimes, Vasy [2013a; 2013b] initiated a series of works which leverage some properties of Keldysh-type operators. On asymptotically hyperbolic manifolds, after compactification and change of smooth structure, Laplace–Beltrami type of operators can be factored using model Keldysh operators, and the meromorphic continuation of the resolvent of the former uses crucially that the latter enjoy *radial point estimates*; see also [Zworski 2016]. Keldysh operators also arise as the restriction of the Minkowski Laplacian on the boundary of radially compactified Minkowski space-time. There, the elliptic regions correspond to polar caps limit points of timelike trajectories. See also [Lebeau and Zworski 2019; Galkowski and Zworski 2021].

• Going the opposite route, one may fit Keldysh-type operators into the framework of uniformly degenerate (or 0-) operators of Mazzeo [1991], a framework originally designed to study (Hodge-)Laplacians on asymptotically hyperbolic manifolds, providing a flexible context where such operators can be made Fredholm. Such operators also arise as (the spatial part of) Heston diffusions in mathematical finance [Feehan and Pop 2015], or Kimura diffusions in population genetics [Epstein and Mazzeo 2013; 2014]. In these cases, the study is further complicated by the fact that the model necessarily involves spatial domains with corners. Of study there is the diffusion associated with these operators, and regularity properties of solutions.

• The H^1-type spaces constructed below are example of Sobolev spaces associated with degenerate quadratic forms in the sense of [Cavalheiro 2008; Sawyer and Wheeden 2010], though they benefit from a slightly more specific degenerate behavior of the quadratic form (only at the boundary, uniform behavior in terms of the boundary point).

The authors' motivation for this work arises in the connection of the $\mathcal{L}_\gamma$ family with inverse problems on Riemannian manifolds with (geodesically) convex boundary, notably, the description of appropriate Hilbert scales which capture sharp mapping properties of the geodesic X-ray transform; see, e.g., [Mazzeo and Monard 2021; Mishra et al. 2023; Monard 2020; Monard et al. 2019; 2021]. In [Mishra et al. 2023] we proved that a one-parameter family of weighted geodesic X-ray transforms on the Euclidean disk had its normal operators be functions of (a distinguished self-adjoint realization of) $\mathcal{L}_\gamma$ and $i\partial_\omega$. As a step toward exploring the connections between degenerate elliptic operators and X-ray transform on more general Riemannian surfaces, this article endeavors to study the family $\mathcal{L}_\gamma$ in its own right, including the existence (or nonexistence) of trace operators and their regularity, Dirichlet-to-Neumann map, and appropriate Hilbert scales where mapping properties are sharply described.

The starting point of the work is the observation that for any $f, g \in C^\infty(\mathbb{D}^{\text{int}})$ and $R \in (0,1)$,

$$\int_{\mathbb{D}_R} \Big(\bar{g}\mathcal{L}_\gamma f - x\partial_\rho f\ \partial_\rho \bar{g} - \frac{1}{\rho}\partial_\omega f \frac{1}{\rho}\partial_\omega \bar{g} - (\gamma+1)^2 f\bar{g} \Big)\, x^\gamma\, \mathrm{d}V = \int_{\partial\mathbb{D}_R} \rho x^{\gamma+1} \bar{g}\partial_\rho f\, \mathrm{d}\omega, \tag{3}$$

$$\int_{\mathbb{D}_R} (\bar{g}\mathcal{L}_\gamma f - f\mathcal{L}_\gamma \bar{g})\, x^\gamma\, \mathrm{d}V = \int_{\partial\mathbb{D}_R} \rho x^{\gamma+1} (\bar{g}\partial_\rho f - f\partial_\rho \bar{g})\, \mathrm{d}\omega, \tag{4}$$

and the classical question becomes to understand in what sense these identities can be understood as Green's first and second identities: for what spaces for f, g can we send $R \to 1$ in the identities above, and make sense of the right-hand sides as boundary traces? Once this can be obtained on domains of definition of $\mathcal{L}_\gamma$ where it is a closed operator, the extension of identity (4) gives a measure of how close $\mathcal{L}_\gamma$ is to being self-adjoint, and self-adjoint realizations of $\mathcal{L}_\gamma$ can be understood in terms of restrictions of $\mathcal{L}_\gamma$ to subspaces with specified boundary constraints. Classically, one defines the *minimal* operator $\mathcal{L}_{\gamma,\min}$ to be the closure of $\mathcal{L}_\gamma$ equipped with domain $\dot{C}^\infty(\mathbb{D})$ (smooth functions vanishing at infinite order at the boundary). From (4), the operator $\mathcal{L}_{\gamma,\min}$ is easily seen to be symmetric, and a classical question is to understand and characterize all self-adjoint realizations of $\mathcal{L}_\gamma$ between $\mathcal{L}_{\gamma,\min}$ and $\mathcal{L}_{\gamma,\max} := \mathcal{L}^*_{\gamma,\min}$.

We now briefly describe the main results of the article presented in the next section.

In Section 2.1, we first fix notation and state preliminary properties of the $\mathcal{L}_\gamma$ family, while recalling some distinguished self-adjoint realizations given in [Wünsche 2005; Mishra et al. 2023].

In Section 2.2, we first characterize the Friedrichs extension of $\mathcal{L}_\gamma$ in terms of previously known extensions, and deduce in Theorem 3 that for $|\gamma| \geq 1$, the minimal operator $\mathcal{L}_{\gamma,\min}$ is in fact self-adjoint. In particular, there is only one self-adjoint extension of $\mathcal{L}_{\gamma,\min}$, and (3)–(4) can only have trivial right-hand side if extended to $R \to 1$ with f, g in a domain where $\mathcal{L}_\gamma$ is closed.

Section 2.3 then covers the case $|\gamma| < 1$, where the situation is markedly different: one may define domains where $\mathcal{L}_\gamma$ is closed, and where the right-hand sides of (3)–(4) can be extended into Dirichlet and Neumann trace operators whose precise mapping properties and tangential regularity are given in the main theorems, Theorems 5 and 8. In this case, one can also naturally define a Dirichlet-to-Neumann map; see Theorem 6. The main theorems provide ways of making sense of Green's identities (3)–(4) when f, g belong to the maximal domain (the domain of $\mathcal{L}^*_{\gamma,\min}$), see Theorem 8, or a subspace of it called W^2_γ (see (28)) in Theorem 5. Unlike in Theorem 3, the operator $\mathcal{L}_\gamma$ now has infinitely many self-adjoint realizations, whose domains of definition are obtained by prescribing certain boundary conditions.

To make this last point more precise, in Section 2.4, we reformulate our main results in the language of *boundary triples* and *generalized boundary triples*. The latter objects allow, via a general functional-analytic framework, to describe self-adjoint extensions of a given operator in terms of self-adjoint boundary relations, see, e.g., [Behrndt and Langer 2007; 2012; Behrndt et al. 2020], in terms of classical notions such as γ-fields (called "Poisson maps" here to avoid conflicts with the constant γ) and Weyl M-functions.

We end this discussion by briefly describing the methodology. The family $\mathcal{L}_\gamma$ is rotation-invariant and as such gives rise to countably many one-dimensional operators $\{\mathcal{L}_{\gamma,n}\}_{n\in\mathbb{Z}}$ on $[0, 1]_\rho$ defined by the relation $\mathcal{L}_\gamma(e^{in\omega} f(\rho)) = e^{in\omega}\mathcal{L}_{\gamma,n} f(\rho)$. Such operators can in principle be studied using Sturm–Liouville theory. In the latter language, the endpoint $\rho = 0$ is always singular, while the properties of the endpoint $\rho = 1$ depend on γ but not on n: the cases $\gamma \in (-1, 0)$, $\gamma \in [0, 1)$ and $|\gamma| \geq 1$ respectively correspond to $\rho = 1$ being a "regular point", a "singular point in the limit circle case", and a "singular point in the limit point case". The case of $\gamma = 0$ also involves a double indicial root, which requires refined analysis. We use this *a priori* knowledge to construct H^1-type function spaces (directly on $\mathbb{D}$ rather than on each separate angular Fourier mode) which are adapted to each $\mathcal{L}_\gamma$, some of which require a 2D version of "quasiderivative", and/or log-type tangential Sobolev regularity in the case of double indicial roots. We construct a number of trace operators (as well as their right-inverses when they exist), whose boundedness is obtained by combining angular Fourier analysis with continuous families of 1D trace estimates (see, e.g., Lemmas 21 and 22), or at other times make use of specific knowledge about generalized Zernike polynomials found in [Wünsche 2005]; see, e.g., Proposition 27. Once such trace estimates are established and their right inverses are constructed, many results follows by density, duality and functional-analytic arguments. One of the advantages of the current analysis is that it is direct and self-contained, not requiring change of smooth structure or factorization of the operator $\mathcal{L}_\gamma$. These results provide new families of function spaces where boundary pairings, Green's identities for $\mathcal{L}_\gamma$ and Fredholm settings should be naturally understood. It should also be expected that these functional settings should become robust to similar operators that may no longer be rotation-invariant.

2. Main results

2.1. *Preliminaries.*

Indicial roots and conormal spaces $\mathcal{A}_\gamma$. Let $\dot{C}^\infty(\mathbb{D})$ be the space of smooth functions on the closed unit disk $\mathbb{D}$, all of whose derivatives vanish on $\mathbb{S}^1 = \partial\mathbb{D}$, with topological dual denoted $C^{-\infty}(\mathbb{D})$. The latter is the space of extendible distributions and will be the largest space considered in what follows.

Let us first define natural "smooth" spaces of definition for $\mathcal{L}_\gamma$. Rewriting $\mathcal{L}_\gamma$ in the form[2]

$$\mathcal{L}_\gamma = -x\Delta + (2+2\gamma)\rho\partial_\rho - \partial_\omega^2 + (\gamma+1)^2\,\mathrm{id},$$

where $\rho\partial_\rho$ and ∂_ω are smooth vector fields on $\mathbb{D}$ we see that $\mathcal{L}_\gamma$ has smooth coefficients in $\mathbb{D}$. Hence we naturally have

$$\mathcal{L}_\gamma : \dot{C}^\infty(\mathbb{D}) \to \dot{C}^\infty(\mathbb{D}). \tag{5}$$

We now enlarge this definition to some distinguished conormal spaces. For all $\gamma \in \mathbb{R}$, $\alpha \in \mathbb{R}$ and $n \in \mathbb{Z}$, a direct calculation gives

$$\mathcal{L}_\gamma(\rho^\alpha e^{in\omega}) = (\alpha+\gamma+1)^2\rho^\alpha e^{in\omega} + (n^2-\alpha^2)\rho^{\alpha-2}e^{in\omega}. \tag{6}$$

Similarly,

$$\mathcal{L}_\gamma(x^\alpha) = (2\alpha+\gamma+1)^2 x^\alpha - 4\alpha(\gamma+\alpha)x^{\alpha-1}. \tag{7}$$

In particular, the indicial roots (independent of the boundary point) are 0 and $-\gamma$, since unless $\alpha \in \{0, -\gamma\}$, $\mathcal{L}_\gamma(x^\alpha)$ is more singular as $x \to 0$ than x^α.

In what follows, the following intertwining property, which can be checked directly, will allow us to translate what is known of $\mathcal{L}_\gamma$ to obtain properties on $\mathcal{L}_{-\gamma}$:

$$\mathcal{L}_\gamma \circ x^{-\gamma} = x^{-\gamma} \circ \mathcal{L}_{-\gamma} \quad \text{on } C^\infty(\mathbb{D}^{\mathrm{int}}), \quad \gamma \in \mathbb{R}. \tag{8}$$

In the case $\gamma = 0$, we also have the important property

$$\mathcal{L}_0(\log x\ f) = \log x\ \mathcal{L}_0 f + 4(\rho\partial_\rho f + f), \quad f \in C^\infty(\mathbb{D}^{\mathrm{int}}). \tag{9}$$

[2]Note that this expression differs from that appearing in [Mishra et al. 2023, Proof of Theorem 6], where the term $(2+2\gamma)\rho\partial_\rho$ was erroneously given as $(3+2\gamma)\rho\partial_\rho$, although this is inconsequential for the purposes of [Mishra et al. 2023, Theorem 6].

Identity (9) can be either checked directly, or derived exploiting (8) upon sending $\gamma \to 0$ in the identity

$$\mathcal{L}_\gamma \circ \frac{x^{-\gamma}-1}{-\gamma} = \frac{x^{-\gamma}-1}{-\gamma} \circ \mathcal{L}_{-\gamma} + \frac{\mathcal{L}_\gamma - \mathcal{L}_{-\gamma}}{\gamma},$$

using that $(x^{-\gamma}-1)/(-\gamma) \to \log x$ as $\gamma \to 0$.

The above discussion on indicial roots together with (9) motivates the definition of

$$\mathcal{A}_\gamma := \begin{cases} x^{-\gamma} C^\infty(\mathbb{D}) + C^\infty(\mathbb{D}) & \text{if } \gamma \neq 0, \\ \log x \; C^\infty(\mathbb{D}) + C^\infty(\mathbb{D}) & \text{if } \gamma = 0, \end{cases} \tag{10}$$

subspaces of $C^\infty(\mathbb{D}^{\text{int}})$ that encode boundary behavior, each stable under $\mathcal{L}_\gamma$, and some of whose subspaces form core domains of self-adjointness for $\mathcal{L}_\gamma$. For example, for $\gamma > -1$, [Mishra et al. 2023, Theorem 6] states that $(\mathcal{L}_\gamma, C^\infty(\mathbb{D}))$ is essentially self-adjoint (further, its full eigendecomposition is known in terms of generalized Zernike polynomials). This result, together with the intertwining property (8), also implies immediately the existence of other self-adjoint extensions, which we state without proof:

Lemma 1. *For any $\gamma < 1$, the operator $(\mathcal{L}_\gamma, x^{-\gamma} C^\infty(\mathbb{D}))$ acting on L^2_γ is essentially self-adjoint.*

As we will see below, Dirichlet and Neumann traces, when they exist, correspond to mechanisms aiming at extracting the most and second most singular terms in the expansion of functions in $\mathcal{A}_\gamma$ off of $\partial\mathbb{D}$. Here, "singular" ordering is among polyhomogeneous terms $\{x^\zeta \log^k x\}_{\zeta\in\mathbb{C},\, k\in\mathbb{N}_0}$, where we have

$$x^\zeta \log^k x = o(x^{\zeta'} \log^{k'} x) \quad \text{if and only if} \quad \begin{cases} \mathrm{Re}(\zeta) > \mathrm{Re}(\zeta') & \text{or} \\ \mathrm{Re}(\zeta) = \mathrm{Re}(\zeta') & \text{and } \; k < k'. \end{cases}$$

In this sense, Dirichlet and Neumann traces will be thought of as the coefficients in front of the terms given below:

γ	$(-\infty, 0)$	0	$(0, \infty)$
Dirichlet term	1	$\log x$	$x^{-\gamma}$
Neumann term	$x^{-\gamma}$	1	1

Distributional facts. Let ${}^t\mathcal{L}_\gamma : C^{-\infty}(\mathbb{D}) \to C^{-\infty}(\mathbb{D})$ the transpose operator of (5). There is a natural injection $\iota_\gamma : L^2_\gamma \cup \mathcal{A}_\gamma \to C^{-\infty}(\mathbb{D})$ given by

$$\langle \iota_\gamma f, \psi \rangle := (f, \psi)_{L^2_\gamma}, \quad f \in L^2_\gamma \cup \mathcal{A}_\gamma, \quad \psi \in \dot{C}^\infty(\mathbb{D}), \tag{11}$$

Note that $\mathcal{A}_\gamma \subset L^2_\gamma$ if and only if $|\gamma| < 1$. We will say that a distribution $u \in C^{-\infty}(\mathbb{D})$ "belongs to L^2_γ (resp. $\mathcal{A}_\gamma$)" if there is $f \in L^2_\gamma$ (resp. $\mathcal{A}_\gamma$) such that $u = \iota_\gamma f$.

As is visible through an integration by parts, we have that $(\mathcal{L}_\gamma f, \psi)_{L^2_\gamma} = (f, \mathcal{L}_\gamma \psi)_{L^2_\gamma}$ for all $f \in \mathcal{A}_\gamma$ and $\psi \in \dot{C}^\infty(\mathbb{D})$. This implies that

$${}^t\mathcal{L}_\gamma(\iota_\gamma f) = \iota_\gamma(\mathcal{L}_\gamma f) \quad \text{for all } f \in \mathcal{A}_\gamma. \tag{12}$$

In particular, the "restriction" of ${}^t\mathcal{L}_\gamma$ to $\dot{C}^\infty(\mathbb{D})$ or $\mathcal{A}_\gamma$ through the map ι_γ agrees with $\mathcal{L}_\gamma$. Thus, for $f \in L^2_\gamma$, we'll say that $\mathcal{L}_\gamma f \in L^2_\gamma$ if the distribution ${}^t\mathcal{L}_\gamma(\iota_\gamma f)$ belongs to L^2_γ.

Two natural operators. One may define two natural closed operators out of $\mathcal{L}_\gamma$:

(i) The minimal operator $\mathcal{L}_{\gamma,\min}$, closure of the operator defined in (5) (also called the preminimal operator), i.e., whose domain is the completion of $\dot{C}^\infty(\mathbb{D})$ for the graph norm

$$f \mapsto n_\gamma(f) := (\|f\|^2_{L^2_\gamma} + \|\mathcal{L}_\gamma f\|^2_{L^2_\gamma})^{1/2}. \tag{13}$$

(ii) The maximal operator $\mathcal{L}_{\gamma,\max}$, the adjoint of $\mathcal{L}_{\gamma,\min}$, with domain

$$\operatorname{dom}(\mathcal{L}_{\gamma,\max}) = \{f \in L^2_\gamma,\ \mathcal{L}_\gamma f \in L^2_\gamma\}. \tag{14}$$

2.2. ***Characterization of Dirichlet extensions. Self-adjointness of $\mathcal{L}_{\gamma,\min}$ for $|\gamma| \ge 1$.*** Let $\mathcal{L}_{\gamma,D}$ be the Friedrichs extension for the quadratic form $\alpha_\gamma : C_c^\infty(\mathbb{D}^{\text{int}}) \to \mathbb{R}$ defined by

$$\alpha_\gamma(f) := (\mathcal{L}_\gamma f, f)_{L^2_\gamma} \overset{(\star)}{=} \|\sqrt{x}\,\partial_\rho f\|^2_{L^2_\gamma} + \|\rho^{-1}\partial_\omega f\|^2_{L^2_\gamma} + (1+\gamma)^2\|f\|^2_{L^2_\gamma}, \quad f \in C_c^\infty(\mathbb{D}^{\text{int}}), \tag{15}$$

where $(\star)$ follows from an integration by parts with no boundary term. Then we have the following characterizations:

Lemma 2. *The operator $\mathcal{L}_{\gamma,D}$ coincides with the closure of the following essentially self-adjoint operators*:

$$(\mathcal{L}_\gamma, x^{-\gamma} C^\infty(\mathbb{D})) \quad \textit{if } \gamma < 0 \quad \textit{and} \quad (\mathcal{L}_\gamma, C^\infty(\mathbb{D})) \quad \textit{if } \gamma \ge 0.$$

The spectral decomposition of the above operators is well-known: for $\gamma \ge 0$, $\mathcal{L}_{\gamma,D}$ has full eigendecomposition

$$\{G^\gamma_{n,k}, (n+1+\gamma)^2\}_{n\ge 0,\, 0\le k\le n}, \quad G^\gamma_{n,k} := P^\gamma_{n-k,k}, \tag{16}$$

where $P^\gamma_{m,\ell}$ denotes the generalized Zernike polynomials in the convention of [Wünsche 2005]; for $\gamma < 0$, $\mathcal{L}_{\gamma,D}$ has full eigendecomposition $\{x^{-\gamma} G^{-\gamma}_{n,k}, (n+1-\gamma)^2\}_{n\ge 0,\ 0\le k\le n}$. In either case, we can define a functional calculus for $\mathcal{L}_{\gamma,D}$ and a Dirichlet Sobolev scale

$$\widetilde{H}^{s,\gamma}_D(\mathbb{D}) := \operatorname{dom}(\mathcal{L}^{s/2}_{\gamma,D}), \quad s \in \mathbb{R}, \tag{17}$$

so that the following operator makes sense and is in fact an isometry:

$$\mathcal{L}^{-1}_{\gamma,D} : \widetilde{H}^{s,\gamma}_D(\mathbb{D}) \to \widetilde{H}^{s+2,\gamma}_D(\mathbb{D}), \quad s \in \mathbb{R}. \tag{18}$$

As a result of further density lemmas proved in Section 3.1, we have the following:

Theorem 3. *If $|\gamma| \ge 1$, then $\mathcal{L}_{\gamma,\min}$ is self-adjoint.*

In particular, such a result precludes the existence of trace maps on the maximal domain, or a Dirichlet-to-Neumann map.

2.3. ***Traces, Green's identities and Dirichlet-to-Neumann map for $|\gamma| < 1$.*** While Theorem 3 prevents the existence of more than one self-adjoint extension for $\mathcal{L}_{\gamma,\min}$ whenever $|\gamma| \ge 1$, we now describe a markedly different scenario for any value $\gamma \in (-1, 1)$. The construction has varying degrees of simplicity depending on whether $\gamma \in (-1, 0)$, $\gamma = 0$ or $\gamma \in (0, 1)$, though in the interest of conciseness, we will unify the presentation. For each $\gamma \in (-1, 1)$, there exists a radial function ϕ_γ (see (47)) nonvanishing on a neighborhood $[0, x_\gamma)_x \times \mathbb{S}^1_\omega$ of $\partial\mathbb{D}$, and satisfying $\mathcal{L}_\gamma \phi_\gamma = (\gamma+1)^2 \phi_\gamma$ on $[0, x_\gamma)_x \times \mathbb{S}^1_\omega$, with the relevant behavior

$$\begin{cases} \phi_\gamma \equiv 1 & \text{if } \gamma \in (-1, 0), \\ \lim_{\rho\to 1} x^\gamma \phi_\gamma = 1, \quad \lim_{\rho\to 1} x^{\gamma+1}\partial_\rho \phi_\gamma = 2\gamma & \text{if } \gamma \in (0, 1), \\ \lim_{\rho\to 1}(\phi_0/\log x) = 1, \quad \lim_{\rho\to 1} x\partial_\rho\phi_0 = -2 & \text{if } \gamma = 0. \end{cases} \tag{19}$$

We may then define "regularized" Dirichlet and Neumann traces $\tau_\gamma^{D,N} : \mathcal{A}_\gamma \to C^\infty(\mathbb{S}^1)$ as

$$\tau_\gamma^D f := (f/\phi_\gamma)|_{x=0}, \quad \tau_\gamma^N f := W(f, \phi_\gamma)|_{x=0}, \quad \text{where } W(f, g)(\rho, \omega) := \rho x^{\gamma+1}(f\partial_\rho g - g\partial_\rho f). \tag{20}$$

In particular, a direct calculation shows that, for $f \in \mathcal{A}_\gamma$, taking the form $f = f^{(0)} + \log x f^{(\log)}$ if $\gamma = 0$, or $f = f^{(0)} + x^{-\gamma} f^{(-\gamma)}$ if $|\gamma| \in (0, 1)$, with $f^{(0)}, f^{(\log)}, f^{(-\gamma)} \in C^\infty(\mathbb{D})$, we have

$$\tau_\gamma^D f = \begin{cases} f^{(0)}|_{x=0} & \text{if } \gamma \in (-1, 0), \\ f^{(\log)}|_{x=0} & \text{if } \gamma = 0, \\ f^{(-\gamma)}|_{x=0} & \text{if } \gamma \in (0, 1), \end{cases} \quad \text{and} \quad \tau_\gamma^N f = \begin{cases} -2\gamma f^{(-\gamma)}|_{x=0} & \text{if } \gamma \in (-1, 0), \\ -2f^{(0)} - 2c_0 f^{(\log)}|_{x=0} & \text{if } \gamma = 0, \\ 2\gamma f^{(0)}|_{x=0} & \text{if } \gamma \in (0, 1), \end{cases} \tag{21}$$

where c_0 is the constant appearing in the definition (47) of ϕ_0. In this sense, τ_γ^D extracts the most singular term and τ_γ^N extracts the second most singular term (or a linear combination of them for $\gamma = 0$). Moreover, the following intertwining property follows naturally: for $\gamma \in (0, 1)$ and $f \in \mathcal{A}_\gamma$, then $x^\gamma f \in \mathcal{A}_{-\gamma}$ and

$$\tau_\gamma^D f = \tau_{-\gamma}^D(x^\gamma f) \quad \text{and} \quad \tau_\gamma^N f = \tau_{-\gamma}^N(x^\gamma f). \tag{22}$$

To discuss Green's identities, we first need to define an "H^1" inner product where the Dirichlet trace extends boundedly. A first guess would be to extend the form α_γ defined in (15) to $\mathcal{A}_\gamma$, but this only makes sense for $\gamma \in (-1, 0)$. Indeed, for $\gamma \in (0, 1)$ for instance, the last right-hand side of (15) can become infinite when applied to an element of $x^{-\gamma} C^\infty(\mathbb{D})$, and the equality ($\star$) there no longer holds; see also the discussion in Section 4.1, titled "Case $0 \le \gamma < 1$".

To remedy this, we let $\rho_\gamma = \sqrt{1 - x_\gamma}$, and for $a, b \in [0, 1]$ with $a < b$, we let $\mathbb{D}_a$ be the centered disk of radius a, with boundary C_a, and $A_{a,b}$ be the annulus $\{a < \rho < b\}$. We then define, for $f, g \in \mathcal{A}_\gamma$, and $b \in (\rho_\gamma, 1)$,

$$\begin{aligned} \mathfrak{t}_{\gamma,b}[f, g] := \big(\sqrt{x}\, \phi_\gamma \partial_\rho(f/\phi_\gamma), \sqrt{x}\, \phi_\gamma \partial_\rho(g/\phi_\gamma)\big)_{x^\gamma, A_{b,1}} - b(x(b))^{\gamma+1} \frac{\partial_\rho \phi_\gamma}{\phi_\gamma}(b) \int_{C_b} f\bar{g} \\ + (\sqrt{x}\, \partial_\rho f, \sqrt{x}\, \partial_\rho g)_{x^\gamma, \mathbb{D}_b} + (\rho^{-1}\partial_\omega f, \rho^{-1}\partial_\omega g)_{L^2_\gamma} + (\gamma+1)^2 (f, g)_{L^2_\gamma}, \end{aligned} \tag{23}$$

where, here and below, $\int_{C_b} h$ is shorthand for $\int_0^{2\pi} h(b, \omega)\, d\omega$. It can be checked for any $f \in \mathcal{A}_\gamma$ that $\sqrt{x}\, \partial_\rho f$ is in L^2_γ away from the boundary, and that $\sqrt{x}\, \phi_\gamma \partial_\rho(f/\phi_\gamma)$ is in L^2_γ near the boundary, so that (23) is well-defined.

Lemma 4. (1) *For any $f, g \in \mathcal{A}_\gamma$, the definition of $\mathfrak{t}_{\gamma,b}$ does not depend on $b \in (b_\gamma, 1)$. We thus denote $(\cdot, \cdot)_{\widetilde{H}^{1,\gamma}}$ the value of $\mathfrak{t}_{\gamma,b}$ for any b.*

(2) *With α_γ the form defined in* (15), *we have for $\gamma \in [0, 1)$*

$$(f, f)_{\widetilde{H}^{1,\gamma}} = \alpha_\gamma(f), \quad f \in \mathcal{A}_{\gamma,D}, \quad \textit{where } \mathcal{A}_{\gamma,D} := \ker \tau_\gamma^D, \tag{24}$$

while for $\gamma \in (-1, 0)$, the above equality holds trivially true on all of $\mathcal{A}_\gamma$.

(3) *For any* $f, g \in \mathcal{A}_\gamma$, *the first Green's identity holds*:

$$(\mathcal{L}_\gamma f, g)_{L^2_\gamma} = (f, g)_{\widetilde{H}^{1,\gamma}} + (\tau_\gamma^N f, \tau_\gamma^D g)_{L^2(\mathbb{S}^1)}. \tag{25}$$

(4) *For any* $\gamma \in (-1, 1)$, *the form* $(\cdot,\cdot)_{\widetilde{H}^{1,\gamma}}$ *defined in* (23) *is positive definite on* $\mathcal{A}_\gamma$.

Skew-symmetrizing (25), the second Green's identity reads

$$(\mathcal{L}_\gamma f, g)_{L^2_\gamma} - (f, \mathcal{L}_\gamma g)_{L^2_\gamma} = (\tau_\gamma^N f, \tau_\gamma^D g)_{L^2(\mathbb{S}^1)} - (\tau_\gamma^N g, \tau_\gamma^D f)_{L^2(\mathbb{S}^1)}, \quad f, g \in \mathcal{A}_\gamma, \tag{26}$$

which is a way of quantifying the lack of self-adjointness of $\mathcal{L}_\gamma$. The question is then to find spaces where $\mathcal{L}_\gamma$ is closed and extend the traces (20) to those spaces.

Since by virtue of Lemma 4(4), the form $(\cdot,\cdot)_{\widetilde{H}^{1,\gamma}}$ is positive definite, we then let

$$\begin{aligned} &\widetilde{H}^{1,\gamma}(\mathbb{D}) : \text{the completion of } (\mathcal{A}_\gamma, (\cdot,\cdot)_{\widetilde{H}^{1,\gamma}}),\\ &\widetilde{H}^{1,\gamma}_0(\mathbb{D}) : \text{the completion of } (\dot{C}^\infty(\mathbb{D}), (\cdot,\cdot)_{\widetilde{H}^{1,\gamma}}). \end{aligned} \tag{27}$$

We have the obvious inclusions $\widetilde{H}^{1,\gamma}_0(\mathbb{D}) \subset \widetilde{H}^{1,\gamma}(\mathbb{D}) \subset L^2_\gamma \subset C^{-\infty}(\mathbb{D})$. An important subspace of $\widetilde{H}^{1,\gamma}$ for what follows is

$$W^2_\gamma := \{f \in \widetilde{H}^{1,\gamma}(\mathbb{D}),\ \mathcal{L}_\gamma f \in L^2_\gamma\} = \operatorname{dom}(\mathcal{L}_{\gamma,\max}) \cap \widetilde{H}^{1,\gamma}(\mathbb{D}), \tag{28}$$

equipped with the norm $\|f\|^2_{W^2_\gamma} := \|f\|^2_{\widetilde{H}^{1,\gamma}} + \|\mathcal{L}_\gamma f\|^2_{L^2_\gamma}$.

On $\mathbb{S}^1 = \partial\mathbb{D}$, we define $H_{(\gamma)}$ to be the completion of $(C^\infty(\mathbb{S}^1), \|\cdot\|_{(\gamma)})$, where for $f = \sum_{k\in\mathbb{Z}} f_k e^{ik\theta}$, we define

$$\|f\|^2_{(\gamma)} := \begin{cases} \sum_{k\in\mathbb{Z}} \langle k\rangle^{2|\gamma|} |f_k|^2 & \text{if } |\gamma| \in (0, 1),\\ \sum_{k\in\mathbb{Z}} (1 + \log\langle k\rangle) |f_k|^2 & \text{if } \gamma = 0. \end{cases} \tag{29}$$

For $|\gamma| \in (0, 1)$, $H_{(\gamma)}$ is the classical Sobolev space $H^{|\gamma|}(\mathbb{S}^1)$ with dual identified with $H^{-|\gamma|}(\mathbb{S}^1)$. For $\gamma = 0$, $H_{(0)}$ is a log-weighted Sobolev space, whose dual is identified with the completion of $C^\infty(\mathbb{S}^1)$ for the norm $f \mapsto \sum_{k\in\mathbb{Z}} (1 + \log\langle k\rangle)^{-1} |f_k|^2$. Below, we write $\langle \cdot,\cdot \rangle_{H'_{(\gamma)}, H_{(\gamma)}}$ for the corresponding duality pairing. Our first main theorem is the following:

Theorem 5. *Let* $\gamma \in (-1, 1)$. *The traces* $\tau_\gamma^{D,N}$ *defined in* (20) *extend as bounded operators*

$$\tau_\gamma^D : \widetilde{H}^{1,\gamma}(\mathbb{D}) \to H_{(\gamma)} \quad \text{and} \quad \tau_\gamma^N : W^2_\gamma \to H'_{(\gamma)},$$

with τ_γ^D *onto. Green's first identity* (25) *extends to* $f \in W^2_\gamma$ *and* $g \in \widetilde{H}^{1,\gamma}$:

$$(\mathcal{L}_\gamma f, g)_{L^2_\gamma} = (f, g)_{\widetilde{H}^{1,\gamma}} + \langle \tau_\gamma^N f, \tau_\gamma^D g\rangle_{H'_{(\gamma)}, H_{(\gamma)}}. \tag{30}$$

In particular, *Green's second identity* (26) *extends to* $f, g \in W^2_\gamma$:

$$(\mathcal{L}_\gamma f, g)_{L^2_\gamma} - (f, \mathcal{L}_\gamma g)_{L^2_\gamma} = \langle \tau_\gamma^N f, \tau_\gamma^D g\rangle_{H'_{(\gamma)}, H_{(\gamma)}} - \langle \tau_\gamma^N g, \tau_\gamma^D f\rangle_{H'_{(\gamma)}, H_{(\gamma)}}. \tag{31}$$

Furthermore, for $\gamma \in (-1, 1)$, one may define the analogue of a Dirichlet-to-Neumann map, defined with respect to a spectral parameter $\lambda \in \rho(\mathcal{L}_{\gamma,D})$ (where, for any operator B, $\rho(B)$ denotes its resolvent set).

Theorem 6. *For any $\gamma \in (-1, 1)$ and any $\lambda \in \rho(\mathcal{L}_{\gamma,D})$, there exists a bounded Dirichlet-to Neumann map operator*

$$\Lambda_\gamma(\lambda) : H_{(\gamma)} \to H'_{(\gamma)}, \tag{32}$$

such that for any $u \in W^2_\gamma \cap \ker(\mathcal{L}_\gamma - \lambda)$, $\Lambda_\gamma(\lambda)(\tau^D_\gamma u) = \tau^N_\gamma u$.

Our second main result consists in extending Green's second identity (26) to the maximal domain $\mathrm{dom}(\mathcal{L}_{\gamma,\max})$ defined in (14). This requires a splitting of the maximal domain in terms of a distinguished, well-understood self-adjoint extension of $\mathcal{L}_\gamma$, notably $\mathcal{L}_{\gamma,D}$ here. Here and below, we define $\mathcal{N}_\lambda(\mathcal{L}_{\gamma,\max}) := \{f \in \mathrm{dom}(\mathcal{L}_{\gamma,\max}),\ \mathcal{L}_\gamma f = \lambda f\}$.

Lemma 7. *With $\mathrm{dom}(\mathcal{L}_{\gamma,D}) = \widetilde{H}^{2,\gamma}_D(\mathbb{D})$, for any $\lambda \in \rho(\mathcal{L}_{\gamma,D})$, we have*

$$\mathrm{dom}(\mathcal{L}_{\gamma,\max}) = \mathrm{dom}(\mathcal{L}_{\gamma,D}) \oplus \mathcal{N}_\lambda(\mathcal{L}_{\gamma,\max}) \tag{33}$$

For $f \in \mathrm{dom}(\mathcal{L}_{\gamma,\max})$, upon fixing $\lambda \in \rho(\mathcal{L}_{\gamma,D})$, we write $f = f_D + f_\lambda$ the decomposition of f according to (33). Our second main theorem is as follows.

Theorem 8. *The traces $\tau^{D,N}_\gamma$ defined in (20) extend as bounded, surjective operators*

$$\tau^N_\gamma : \widetilde{H}^{2,\gamma}_D(\mathbb{D}) \to H^{1-|\gamma|}(\mathbb{S}^1), \quad \tilde{\tau}^D_\gamma : \mathrm{dom}(\mathcal{L}_{\gamma,\max}) \to H^{-1+|\gamma|}(\mathbb{S}^1).$$

Green's second identity (26) extends to $f, g \in \mathrm{dom}(\mathcal{L}_{\gamma,\max})$ as follows:

$$(\mathcal{L}_\gamma f, g)_{L^2_\gamma} - (f, \mathcal{L}_\gamma g)_{L^2_\gamma} = \langle \tilde{\tau}^D_\gamma g, \tau^N_\gamma f_D\rangle_{H^{-1+|\gamma|},H^{1-|\gamma|}} - \langle \tilde{\tau}^D_\gamma f, \tau^N_\gamma g_D\rangle_{H^{-1+|\gamma|},H^{1-|\gamma|}}. \tag{34}$$

2.4. ***Consequences: boundary triples, self-adjoint extensions and more.*** The theory of boundary triples is a functional-analytic framework to help describe all self-adjoint realizations of an operator and their spectrum, through the use of their Poisson maps and Weyl functions; see [Behrndt et al. 2020]. Refined concepts, such as *generalized boundary triples* and *quasiboundary triples* have been introduced (see, e.g., [Behrndt and Langer 2007; 2012; Derkach and Malamud 1995]), and we now explain how and why our main Theorems 5 and 8 fit this framework, along with some natural corollaries.

2.4.1. *Preliminaries.*

Boundary triples and their Weyl functions. Here and below, we follow the exposition in [Behrndt and Langer 2012] and refer to there for more details. Let S be a densely defined, closed, symmetric operator acting on a Hilbert space $(\mathcal{H}, (\cdot,\cdot)_\mathcal{H})$. A triple $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ is said to be a *boundary triple* for the adjoint operator S^* if $(\mathcal{G}, (\cdot,\cdot)_\mathcal{G})$ is a Hilbert space and $\Gamma_0, \Gamma_1 : \mathrm{dom}(S^*) \to \mathcal{G}$ are linear mappings such that the map $(\Gamma_0, \Gamma_1) : \mathrm{dom}(S^*) \to \mathcal{G} \times \mathcal{G}$ is surjective and we have the following abstract Lagrange/Green identity, true for all $f, g \in \mathrm{dom}(S^*)$,

$$(S^* f, g)_\mathcal{H} - (f, S^* g)_\mathcal{H} = (\Gamma_1 f, \Gamma_0 g)_\mathcal{G} - (\Gamma_0 f, \Gamma_1 g)_\mathcal{G}. \tag{35}$$

By virtue of [Behrndt and Langer 2012, Proposition 1.2], a boundary triple helps parametrize all closed extensions of S which are restrictions of S^* in terms of closed, linear relations[3] R in $\mathcal{G}$ via the bijection

$$R \mapsto A_R := S^*|(\Gamma_0, \Gamma_1)^{-1}(R).$$

[3] A closed linear relation R in $\mathcal{G}$ is a closed linear subspace of $\mathcal{G} \times \mathcal{G}$.

Given a relation R on $\mathcal{G}$, we define the adjoint relation

$$R^* := \{(f, f') \in \mathcal{G} \times \mathcal{G},\ (f', h)_{\mathcal{G}} = (f, h')_{\mathcal{G}} \text{ for all } (h, h') \in R\}.$$

The further identity $A_{R^*} = A_R^*$ implies that A_R is self-adjoint if and only if R is a self-adjoint relation in $\mathcal{G}$.

To say something further about the spectral properties of such extensions, we should mention that the results quoted below will sometimes constrain the relation R to be of the form $\{(f, \Theta f),\ f \in \mathcal{G}\}$ for some linear map $\Theta : \mathcal{G} \to \mathcal{G}$, in which we case we will denote the extension $A_\Theta := S^*|\ker(\Gamma_1 - \Theta\Gamma_0)$, or of the form $\{(Bf, f),\ f \in \mathcal{G}\}$ for some linear map $B : \mathcal{G} \to \mathcal{G}$, in which case we will denote the extension $A_{[B]} := S^*|\ker(B\Gamma_1 - \Gamma_0)$.

Denoting $A_0 := A_{\{0\}\times\mathcal{G}} = S^*|\ker\Gamma_0$, a boundary triple gives rise to a notion of *Poisson map*[4] P and a *Weyl function* M,

$$P(\lambda) := (\Gamma_0|\mathcal{N}_\lambda(S^*))^{-1}, \quad M(\lambda) := \Gamma_1 P(\lambda), \quad \lambda \in \rho(A_0).$$

By [Behrndt and Langer 2012, Propositions 1.5 and 1.6], P and M are holomorphic functions on $\rho(A_0)$ with values in $\mathcal{B}(\mathcal{G}, \mathcal{H})$ and $\mathcal{B}(\mathcal{G})$, respectively.

Further, M is a Herglotz function which allows to characterize the spectrum of any closed extension A_Θ of S in the following way: in the "Θ" convention mentioned above, [Behrndt and Langer 2012, Theorem 1.7(ii)] states that $\lambda \in \sigma_i(A_\Theta)$ if and only if $0 \in \sigma_i(\Theta - M(\lambda))$, $i = \mathrm{p, c, r}$, where $\sigma_{\mathrm{p}}, \sigma_{\mathrm{c}}, \sigma_{\mathrm{r}}$ denote the point, continuous and residual spectrum, respectively. The functions P and M also allow to write Krein's formula relating the resolvent of A_Θ with that of $A_0 := S^*|\ker\Gamma_0$: for all $\lambda \in \rho(A_0) \cap \rho(A_\Theta)$,

$$(A_\Theta - \lambda)^{-1} = (A_0 - \lambda)^{-1} + P(\lambda)(\Theta - M(\lambda))^{-1} P(\bar{\lambda})^*. \tag{36}$$

Given the usefulness of P and M, the attention then turns toward their computation, or how to relate them to quantities such as a Dirichlet-to-Neumann map. As such, it is known that boundary triples for maximal operators involve some operations which make this purpose more difficult, and this motivates the definition of slightly weaker notions below, namely *generalized boundary triples*, first introduced in [Derkach and Malamud 1995], or even weaker yet, *quasiboundary triples*, first introduced in [Behrndt and Langer 2007].

Quasi/generalized-boundary triples and their Weyl functions. A triple $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ is said to be a *quasi-boundary triple* for S^* if $(\mathcal{G}, (\cdot,\cdot)_{\mathcal{G}})$ is a Hilbert space and there exists an operator T such that $\overline{T} = S^*$ and $\Gamma_0, \Gamma_1 : \mathrm{dom}(T) \to \mathcal{G}$ are linear mappings such that $(\Gamma_0, \Gamma_1) : \mathrm{dom}(T) \to \mathcal{G} \times \mathcal{G}$ has dense range, (35) holds for all $f, g \in \mathrm{dom}(T)$, and $A_0 := T|\ker\Gamma_0$ is self-adjoint in $\mathcal{H}$. If, in addition, Γ_0 is onto $\mathcal{G}$, then $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ is a *generalized boundary triple*.

Given R a linear relation in $\mathcal{G}$, we define the extension A_R of S in analogy to the above:

$$A_R := T|\{f \in \mathrm{dom}(T) : (\Gamma_0 f, \Gamma_1 f) \in R\},$$

although this no longer gives a bijective correspondence with self-adjoint extensions of S. Since the map (Γ_0, Γ_1) is no longer surjective, we define $\mathcal{G}_0 := \mathrm{ran}(\Gamma_0)$ and $\mathcal{G}_1 := \mathrm{ran}(\Gamma_1)$, and we may associate to the above quasiboundary triple the Poisson map P and Weyl function M, for any $\lambda \in \rho(A_0)$

$$P(\lambda) := (\Gamma_0|\mathcal{N}_\lambda(T))^{-1} : \mathcal{G}_0 \to \mathcal{H}, \quad M(\lambda) := \Gamma_1 P(\lambda) : \mathcal{G}_0 \to \mathcal{G}_1.$$

[4]Note that the interpretation as a Poisson map arises in PDE contexts. In abstract boundary triple theory, such a map is referred to as a γ-field, a terminology which we are avoiding here to avoid confusion with the constant γ in $\mathcal{L}_\gamma$.

In the "generalized" boundary triple situation, $\mathcal{G}_0=\mathcal{G}$, in which case $P(\lambda)$ and $M(\lambda)$ are defined on all of $\mathcal{G}$. Under additional assumptions, such functions share many properties of their "boundary triple" counterpart, and they are used to obtain, via [Behrndt and Langer 2012, Theorem 1.18] in the "Θ" convention above, sufficient conditions on Θ for $T|\ker(\Gamma_1-\Theta\Gamma_0)$ to be self-adjoint, allowing a parametrization of self-adjoint extensions of S which are restrictions of T. Specifically:

Theorem 9 [Behrndt and Langer 2012, Theorem 1.18]. *Let S be a densely defined closed symmetric operator in $\mathcal{H}$ and let $\{\mathcal{G},\Gamma_0,\Gamma_1\}$ be a quasiboundary triple for $\overline{T}=S^*$ with $A_i=T|\ker\Gamma_i$, $i=0,1$, and Weyl function M. Assume that A_1 is self-adjoint and that $\overline{M(\lambda_0)^{-1}}$ is a compact operator in $\mathcal{G}$ for some $\lambda_0\in\mathbb{C}\setminus\mathbb{R}$. If Θ is a bounded self-adjoint operator in $\mathcal{G}$ such that*

$$\Theta(\operatorname{dom}\overline{M(\lambda_\pm)})\subset\mathcal{G}_1 \tag{37}$$

holds for some $\lambda_+\in\mathbb{C}^+$ and $\lambda_-\in\mathbb{C}^-$, then $A_\Theta:=T|\ker(\Gamma_1-\Theta\Gamma_0)$ is a self-adjoint operator in $\mathcal{H}$. In particular, condition (37) *is satisfied if $\operatorname{ran}(\Theta)=\mathcal{G}_1$.*

A more recent refinement, dropping the above compactness requirement on $\overline{M(\lambda_0)^{-1}}$, is the following result, formulated in the "B" convention mentioned above.

Theorem 10 [Behrndt et al. 2017, Corollary 2.7]. *Let $\{\mathcal{G},\Gamma_0,\Gamma_1\}$ be a quasiboundary triple for $T\subset S^*$ with corresponding Poisson map P and Weyl function M. Let B be a bounded self-adjoint operator in $\mathcal{G}$ and assume that there exists a $\lambda_0\in\rho(A_0)\cap\mathbb{R}$ such that the following conditions are satisfied*:

(i) $1\in\rho(B\overline{M(\lambda_0)})$.

(ii) $B(\operatorname{ran}\overline{M(\lambda_0)}))\subset\operatorname{ran}\Gamma_0$.

(iii) $B(\operatorname{ran}\Gamma_1)\subset\operatorname{ran}\Gamma_0$ *or* $\lambda_0\in\rho(A_1)$.

Then the operator $A_{[B]}:=T|\ker(B\Gamma_1-\Gamma_0)$ is a self-adjoint extension of S such that $\lambda_0\in\rho(A_{[B]})$, and the resolvent formula

$$(A_{[B]}-\lambda)^{-1}=(A_0-\lambda)^{-1}+P(\lambda)(I-BM(\lambda))^{-1}BP(\bar{\lambda})^* \tag{38}$$

holds for all $\lambda\in\rho(A_{[B]})\cap\rho(A_0)$.

Further, if $\{\mathcal{G},\Gamma_0,\Gamma_1\}$ is a generalized boundary triple, then conditions (ii) and (iii) are automatically satisfied and $\overline{M(\lambda_0)}=M(\lambda_0)$. In that case, the conditions for self-adjointness reduce to the condition that $1\in\rho(BM(\lambda_0))$.

2.4.2. *Corollaries of Theorems 5 and 8.* Identity (34) implies that, upon defining isometric isomorphisms

$$\iota_{\pm(1-|\gamma|)}:H^{\pm(1-|\gamma|)}(\mathbb{S}^1)\to L^2(\mathbb{S}^1)$$

such that $\langle\varphi,\psi\rangle_{-1+|\gamma|,1-|\gamma|}=(\iota_-\varphi,\iota_+\psi)_{L^2(\mathbb{S}^1)}$, we have, for all $f,g\in\operatorname{dom}(\mathcal{L}_{\gamma,\max})$,

$$(\mathcal{L}_\gamma f,g)_{L^2_\gamma}-(f,\mathcal{L}_\gamma g)_{L^2_\gamma}=(\iota_{1-|\gamma|}\tau_\gamma^N f_D,\iota_{-1+|\gamma|}\tilde{\tau}_\gamma^D g)_{L^2(\mathbb{S}^1)}-(\iota_{-1+|\gamma|}\tilde{\tau}_\gamma^D f,\iota_{1-|\gamma|}\tau_\gamma^N g_D)_{L^2(\mathbb{S}^1)}, \tag{39}$$

where $f_D:=(\mathcal{L}_{\gamma,D}-\lambda)^{-1}(\mathcal{L}_\gamma-\lambda)f$ is the projection onto the left summand of (33) for any $\lambda\in\rho(\mathcal{L}_{\gamma,D})$. Following the definitions above, equation (39) exactly means the following:

Corollary 11. *Fix $\lambda \in \rho(\mathcal{L}_{\gamma,D})$ and let $f \mapsto f_D$ be the projection onto the left summand in* (39). *Upon defining the maps $\Gamma_{0,1} : \mathrm{dom}(\mathcal{L}_{\gamma,\max}) \to L^2(\mathbb{S}^1)$ by*

$$\Gamma_0 f := \iota_{-1+|\gamma|}\tilde{\tau}_\gamma^D f, \quad \Gamma_1 g := \iota_{1-|\gamma|}\tau_\gamma^N g_D, \quad f, g, \in \mathrm{dom}(\mathcal{L}_{\gamma,\max}), \tag{40}$$

where $\tilde{\tau}_\gamma^D$, τ_γ^N are defined in Theorem 5, the triple $\{L^2(\mathbb{S}^1), \Gamma_0, \Gamma_1\}$ is a boundary triple for the operator $\mathcal{L}_{\gamma,\max}$.

From the previous section, Corollary 11 then allows to parametrize all self-adjoint extensions of $\mathcal{L}_{\gamma,\min}$ in terms of self-adjoint boundary relations. As mentioned in the previous section, the appearance of the map $g \mapsto g_D$ changes the computation of the M function, as the latter is no longer a factorization of the Dirichlet-to-Neumann map. It could still in principle be computed, following ideas as in, e.g., [Behrndt et al. 2020, Lemma 8.4.5], applied to Schrödinger operators there.

A way to simplify the description is to introduce the intermediate extension $T := \mathcal{L}_{\gamma,\max}|W_\gamma^2$, with W_γ^2 defined in (28). Identity (31) implies that, upon defining isometric isomorphisms $\iota_{(\gamma)} : H_{(\gamma)} \to L^2(\mathbb{S}^1)$ and $\iota_{(\gamma)'} : H'_{(\gamma)} \to L^2(\mathbb{S}^1)$, we have for all $f, g \in W_\gamma^2$,

$$(\mathcal{L}_\gamma f, g)_{L^2_\gamma} - (f, \mathcal{L}_\gamma g)_{L^2_\gamma} = (\iota_{(\gamma)'}\tau_\gamma^N f, \iota_{(\gamma)}\tau_\gamma^D g)_{L^2(\mathbb{S}^1)} - (\iota_{(\gamma)}\tau_\gamma^D f, \iota_{(\gamma)'}\tau_\gamma^N g)_{L^2(\mathbb{S}^1)}.$$

Moreover, the operator $\Gamma_0 := \iota_{(\gamma)}\tau_\gamma^D|W_\gamma^2$ is surjective, i.e., $\mathcal{G}_0 = L^2(\mathbb{S}^1)$ (indeed, inspecting the proof of Theorem 6, for any $f \in H_{(\gamma)}$, one constructs a function $u_f \in W_\gamma^2$ such that $\tau_\gamma^D u_f = f$) and $\Gamma_1 := \iota_{(\gamma)'}\tau_\gamma^N$ has dense range in $L^2(\mathbb{S}^1)$ since $\mathcal{A}_\gamma \subset W_\gamma^2$ and $\tau_\gamma^N(\mathcal{A}_\gamma) = C^\infty(\mathbb{S}^1)$. Together with the fact that $T|\ker\Gamma_0 = \mathcal{L}_{\gamma,D}$ is self-adjoint, we can then conclude:

Corollary 12. *The triple $\{L^2(\mathbb{S}^1), \iota_{(\gamma)}\tau_\gamma^D, \iota_{(\gamma)'}\tau_\gamma^N\}$ is a generalized boundary triple for $S^* = \mathcal{L}_{\gamma,\max}$ via the extension $T = \mathcal{L}_{\gamma,\max}|W_\gamma^2$. Moreover, the Weyl function $M_\gamma(\lambda) : L^2(\mathbb{S}^1) \to L^2(\mathbb{S}^1)$ is given by $M_\gamma(\lambda) = \iota_{(\gamma)'}\Lambda_\gamma(\lambda)\iota_{(\gamma)}^{-1}$, where Λ_γ is the Dirichlet-to-Neumann map defined in Theorem 6.*

This result, combined with Theorem 10 and the remark right after it, allows to give a simple criterion for the self-adjointness of extensions of $\mathcal{L}_{\gamma,\min}$ with domain included in W_γ^2, for a wide range of boundary conditions including, e.g., conditions of Robin type relating both traces.

Corollary 13. *Let $\gamma \in (-1, 1)$, $\{L^2(\mathbb{S}^1), \iota_{(\gamma)}\tau_\gamma^D, \iota_{(\gamma)'}\tau_\gamma^N\}$, T and $M_\gamma(\lambda)$ be as in the previous result. Let B be a bounded, self-adjoint operator in $L^2(\mathbb{S}^1)$ and assume that there exists $\lambda_0 \in \rho(\mathcal{L}_{\gamma,D}) \cap \mathbb{R}$ such that*

$$1 \in \rho(BM_\gamma(\lambda_0)).$$

Then the operator $\mathcal{L}_{\gamma,[B]} := T|\ker(B\iota_{(\gamma)'}\tau_\gamma^N - \iota_{(\gamma)}\tau_\gamma^D)$ is a self-adjoint extension of $\mathcal{L}_{\gamma,\min}$ such that $\lambda_0 \in \rho(\mathcal{L}_{\gamma,[B]})$ and the resolvent formula

$$(\mathcal{L}_{\gamma,[B]} - \lambda)^{-1} = (\mathcal{L}_{\gamma,D} - \lambda)^{-1} + P_\gamma(\lambda)(I - BM_\gamma(\lambda))^{-1}BP_\gamma(\bar{\lambda})^*$$

holds for all $\lambda \in \rho(\mathcal{L}_{\gamma,[B]}) \cap \rho(\mathcal{L}_{\gamma,D})$, where $P_\gamma(\lambda) := (\iota_{(\gamma)}\tau_\gamma^D|\mathcal{N}_\lambda(\mathcal{L}_{\gamma,\max}))^{-1}$.

Refinements of the above results may leverage norm estimates of $M_\gamma(\lambda_0)$ to show that the remaining condition is always satisfied, see, e.g., [Behrndt et al. 2017] situations where $\|M(\lambda)\| \to 0$ as $\lambda \to -\infty$. A more in-depth inquiry of such aspects, particularizing to certain boundary conditions, will be the object of future work.

Outline of next sections. The remainder of the article is organized as follows. In Section 3, we treat the case $|\gamma| \geq 1$, first formulating some density results in Section 3.1, then showing the characterization of the Friedrichs extension (Lemma 2) in Section 3.2, finally showing that $\mathcal{L}_{\gamma,\min}$ is self-adjoint (Theorem 3) in Section 3.3. In Section 4, we cover the construction of a generalized boundary triple (Theorem 5) and a Dirichlet-to-Neumann map (Theorem 6) for $\mathcal{L}_{\gamma,\max}$ when $\gamma \in (-1, 1)$, with a refined roadmap of proofs at the beginning of the section. In Section 5, we cover the construction of a(n ordinary) boundary triple (Theorem 8) for $\mathcal{L}_{\gamma,\max}$ when $\gamma \in (-1, 1)$, with a refined roadmap of proofs at the beginning of the section. Some proofs of auxiliary lemmas are provided in the Appendix.

3. Density results, proofs of Lemma 2 and Theorem 3

3.1. ***Some preliminary density results.*** We begin by exploring when $C_c^\infty(\mathbb{D})$ is dense in spaces of the form $(\log x)^k x^\alpha C^\infty(\mathbb{D})$, with respect to the L^2_γ topology and the graph-norm topology n_γ defined in (13).

Theorem 14. (i) *$C_c^\infty(\mathbb{D})$ is dense in L^2_γ for all $\gamma \in \mathbb{R}$, and in particular it is dense in $(\log x)^k x^\alpha C^\infty(\mathbb{D})$ with respect to the L^2_γ topology for all $\alpha > (-\gamma - 1)/2$ and $k \in \mathbb{N}_0$.*

(ii) *$C_c^\infty(\mathbb{D})$ is dense in $(\log x)^k x^\alpha C^\infty(\mathbb{D})$ with respect to the graph norm n_γ for all $\gamma \in \mathbb{R}$, $\alpha > (1-\gamma)/2$ and $k \in \mathbb{N}_0$.*

To prove the density results, we use the following "cutoff" lemma:

Lemma 15. *Let $\chi \in C_c^\infty(\mathbb{R})$ and $g \in (\log x)^k x^\alpha C^\infty(\mathbb{D})$ for $k \in \mathbb{N}_0$ and $\alpha \in \mathbb{R}$. Then, if $2\alpha + \gamma > -1$, or if χ vanishes to infinite order at 0, we have*

$$\|\chi(x/\epsilon) g\|_{L^2_\gamma} = O(|\log \epsilon|^k \epsilon^{\alpha + (\gamma+1)/2}), \quad \text{as } \epsilon \to 0^+.$$

Proof of Lemma 15. Denote $C_g := \max_{\mathbb{D}} |(\log x)^{-k} x^{-\alpha} g|$, finite by assumption. Then we have

$$\begin{aligned}
\|\chi(x/\epsilon) g\|^2_{L^2_\gamma} &= \int_{\mathbb{D}} |\chi(x/\epsilon)|^2 |g|^2 x^\gamma \, \mathrm{d}V \\
&\leq C_g^2 \int_{\mathbb{S}^1} \int_0^1 |\chi(x/\epsilon)|^2 x^{2\alpha+\gamma} |\log x|^{2k} \, \mathrm{d}x \, \mathrm{d}\omega \\
&\overset{x=\epsilon y}{=} 2\pi C_g^2 \int_0^{1/\epsilon} |\chi(y)|^2 (\epsilon y)^{2\alpha+\gamma} |\log(\epsilon y)|^{2k} \epsilon \, \mathrm{d}y \\
&\leq \sum_{i=0}^{2k} C_i |\log \epsilon|^i \epsilon^{2\alpha+\gamma+1}, \quad C_i = 2\pi \binom{2k}{i} C_g \int_0^\infty |\chi(y)|^2 y^{2\alpha+\gamma} |\log(y)|^{2k-i} \, \mathrm{d}y,
\end{aligned}$$

where all constants are finite under the assumption that $2\alpha + \gamma > -1$ or χ vanishes at infinite order. □

Proof of Theorem 14. Proof of (i). Since the weight x^γ does not vanish on a set of positive measure on $\mathbb{D}$, it follows that functions in L^2_γ can be approximated by functions whose support is a compact subset of the *interior* of $\mathbb{D}$. Such functions can be approximated by $C_c^\infty(\mathbb{D})$ functions in the standard way, noting that x^γ is strictly positive on any compact subset of the interior of $\mathbb{D}$. The second part follows since $x^\alpha C^\infty(\mathbb{D}) \subset L^2_\gamma$ for all $\alpha > (-\gamma - 1)/2$.

To prove (ii), we argue by cutting off. That is, fix $\chi \in C_c^\infty(\mathbb{R})$ with $\chi \equiv 1$ in a neighborhood of 0, and for $f \in x^\alpha C^\infty(\mathbb{D})$, let

$$f_\epsilon = (1 - \chi(x/\epsilon)) f.$$

Then $f_\epsilon \in C_c^\infty(\mathbb{D})$ for $\epsilon > 0$. We have $f_\epsilon \to f$ in L^2_γ, so it suffices to check if $\mathcal{L}_\gamma f_\epsilon \to \mathcal{L}_\gamma f$ in L^2_γ. We have

$$\mathcal{L}_\gamma f_\epsilon = -\rho^{-1} x^{-\gamma} \partial_\rho(\rho\, x^{\gamma+1} \partial_\rho f_\epsilon) - \rho^{-2} \partial_\omega^2 f_\epsilon + (1+\gamma)^2 f_\epsilon.$$

The third term converges to $(1+\gamma)^2 f$ in L^2_γ, while the second term equals $\rho^{-2}(1-\chi(x/\epsilon))\partial_\omega^2 f$, which converges to $\rho^{-2}\partial_\omega^2 f$ in L^2_γ. It remains to check that $\|\rho^{-1}x^{-\gamma}\partial_\rho(\rho\, x^{\gamma+1}\partial_\rho[f - f_\epsilon])\|_{L^2_\gamma} \to 0$ as $\epsilon \to 0$. We thus write

$$\begin{aligned} -\rho^{-1}x^{-\gamma}\partial_\rho(\rho x^{\gamma+1}\partial_\rho[f - f_\epsilon]) &= -\rho^{-1}x^{-\gamma}\partial_\rho(\rho\, x^{\gamma+1}\partial_\rho(\chi(x/\epsilon) f)) \\ &= -\rho^{-1}x^{-\gamma}\partial_\rho(\rho\, x^{\gamma+1}\chi(x/\epsilon)\partial_\rho f - 2\rho^2\, x^{\gamma+1}\epsilon^{-1}\chi'(x/\epsilon) f) \\ &= \chi(x/\epsilon) g_0 + \epsilon^{-1}\chi'(x/\epsilon) g_1 + \epsilon^{-2}\chi''(x/\epsilon) g_2 \end{aligned} \tag{41}$$

where

$$\begin{aligned} g_0 &= -\rho^{-1}\, x^{-\gamma}\partial_\rho(\rho\, x^{\gamma+1}\partial_\rho f), \\ g_1 &= -\rho^{-1}\, x^{-\gamma}(-2\rho^2\, x^{\gamma+1})\partial_\rho f - \rho^{-1}\, x^{-\gamma}\partial_\rho(-2\rho^2\, x^{\gamma+1} f) \\ &= 4\rho\, x\partial_\rho f + (4x - 4(\gamma+1)\rho^2) f, \\ g_2 &= -4\rho^2 x\, f. \end{aligned} \tag{42}$$

We now observe that if $f \in (\log x)^k x^\alpha C^\infty(\mathbb{D})$, then the functions g_0, g_1, g_2 defined in (42) satisfy $g_i \in \sum_{j=\max(0,k-i)}^k (\log x)^j x^{\alpha-1+i} C^\infty(\mathbb{D})$, and hence by Lemma 15 we have

$$\|\epsilon^{-i}\chi^{(i)}(x/\epsilon) g_i\|_{L^2_\gamma} = O(|\log\epsilon|^k \epsilon^{-i+(\alpha-1+i)+(\gamma+1)/2}) = O(|\log\epsilon|^k\epsilon^{\alpha-(1-\gamma)/2})$$

for $i = 0, 1, 2$. It follows that if $\alpha > (1-\gamma)/2$, combining the above with (41), we arrive at

$$-\rho^{-1}\, x^{-\gamma}\partial_\rho(\rho\, x^{\gamma+1}\partial_\rho[f - f_\epsilon]) = O_{L^2_\gamma}(|\log\epsilon|^k\epsilon^{\alpha-(1-\gamma)/2}) = o_{L^2_\gamma}(1), \quad \text{as } \epsilon \to 0. \qquad \square$$

3.2. ***Friedrichs extensions and proof of Lemma 2.*** Recall the following result which holds for any symmetric and positive (more generally, semibounded) operator (see, e.g., [Helffer 2013, Theorem 4.4, p. 34]).

Theorem 16. *A symmetric, positive operator T_0 on $\mathcal{H}$ (with $D(T_0)$ dense in $\mathcal{H}$) admits a self-adjoint extension, called the Friedrichs extension.*

The construction goes as follows. To T_0 we associate the form $\alpha(u, v) := (T_0 u, v)_{\mathcal{H}}$, with domain $D(T_0)$. We then consider the Hilbert space completion V of $D(T_0)$ with respect to the norm $u \mapsto (\alpha(u, u))^{1/2}$, viewing V as injecting into $\mathcal{H}$, and we extend α by continuity to a form $\tilde{\alpha}$ with domain V (the form will be continuous with respect to the norm on V). Finally let

$$D_F = \{u \in V : \text{ there exists } C > 0 \text{ such that } |\tilde{\alpha}(u, v)| \le C\|v\|_{\mathcal{H}}\}, \tag{43}$$

i.e., D_F consists of $u \in V$ where $\tilde{\alpha}(u, \cdot)$ is a bounded antilinear functional on $\mathcal{H}$. For such u, there exists a unique element $T_F u \in \mathcal{H}$ such that $\tilde{\alpha}(u, \cdot) = (T_F u, \cdot)_{\mathcal{H}}$; then T_F with domain D_F is the Friedrichs extension. Note that $D(T_0) \subset D_F \subset D(T_0^*)$, with $T_0^*|_{D_F} = T_F$ and $T_F|_{D(T_0)} = T_0$.

We now give a criterion for when a self-adjoint extension is the Friedrichs extension, whose proof is relegated to Section A.1. Recall that given a symmetric form α, we say that a sequence u_n in the domain of α will *α-converge* to $u \in \mathcal{H}$ (not necessarily in the domain of α) if

$$u_n \xrightarrow{n\to\infty} u \text{ in } \mathcal{H} \quad \text{and} \quad \alpha(u_n - u_m, u_n - u_m) \xrightarrow{m,n\to\infty} 0.$$

Lemma 17 (characterization of the Friedrichs extension). *Let T_0 be a symmetric semibounded operator, and let T be a self-adjoint extension of T_0. Then T is the Friedrichs extension of T_0 if and only if $D(T)$ is contained in the α-closure of $D(T_0)$, that is,*

$$\textit{for all } u \in D(T), \textit{ there exists a sequence } u_n \in D(T_0) \textit{ such that } u_n \to_\alpha u.$$

Proof of Lemma 2. We aim to apply Lemma 17, working with the quadratic form α_γ defined in (15). The proof amounts to showing that the α_γ-closure of $C_c^\infty(\mathbb{D})$ contains

(i) $C^\infty(\mathbb{D})$ if $\gamma \geq 0$ and

(ii) $x^{-\gamma} C^\infty(\mathbb{D})$ if $\gamma < 0$.

The domain of the α_γ-closure is also closed with respect to the graph norm, hence it contains the graph norm closure of $C^\infty(\mathbb{D})$ (resp. $x^{-\gamma} C^\infty(\mathbb{D})$) for $\gamma \geq 0$ (resp. $\gamma < 0$).

We briefly explain why (i) implies (ii). For $\gamma < 0$ and $f \in x^{-\gamma} C^\infty(\mathbb{D})$, assuming (i) is true, $x^\gamma f$ can be approximated in the $\alpha_{-\gamma}$ norm by a sequence f_n in $C_c^\infty(\mathbb{D})$. Via the intertwining property (8), we then see that $\alpha_{-\gamma}(f_n - x^\gamma f) = \alpha_\gamma(x^{-\gamma} f_n - f) \to 0$, where $x^{-\gamma} f_n \in C_c^\infty(\mathbb{D})$. Thus $x^{-\gamma} C^\infty(\mathbb{D})$ is contained in the α_γ-closure of $C_c^\infty(\mathbb{D})$, and (ii) holds.

On to the proof of (i), we recall, for $\gamma \geq 0$, that the closure of $(\mathcal{L}_\gamma, C^\infty(\mathbb{D}))$ is a self-adjoint operator with eigenbasis $\{G_{n,k}^\gamma\}_{n\geq 0,\ 0\leq k\leq n}$ as defined in (16), satisfying

$$\mathcal{L}_\gamma G_{n,k}^\gamma = (n+1+\gamma)^2 G_{n,k}^\gamma, \qquad G_{n,k}^\gamma|_{\mathbb{S}^1}(e^{i\omega}) = e^{i(n-2k)\omega},$$

where the second property uses [Wünsche 2005, equation (2.10)]. The domain equals $\widetilde{H}_D^{2,\gamma}$ as defined in (17), also characterized as

$$\widetilde{H}_D^{s,\gamma} = \left\{ f = \sum_{n,k} f_{n,k}\, \widehat{G_{n,k}^\gamma} \;:\; \sum_{n,k} (n+\gamma+1)^{2s} |f_{n,k}|^2 < \infty \right\},$$

where $\widehat{G_{n,k}^\gamma} = G_{n,k}^\gamma / \|G_{n,k}^\gamma\|_{L_\gamma^2}$. In light of Lemma 17, it suffices to show that for each $f \in \widetilde{H}_D^{2,\gamma}$, there exists $f^{(j)}$ in the α_γ-closure of $C_c^\infty(\mathbb{D})$, such that

$$f^{(j)} \xrightarrow{j\to\infty}_{\alpha_\gamma} f.$$

Noting that every f can be approximated in $\widetilde{H}_D^{2,\gamma}$ by a finite linear combination of the Zernike polynomials $G_{n,k}^\gamma$, and that a sequence convergent in $\widetilde{H}_D^{2,\gamma}$ will also α_γ-converge, it suffices to show the above approximation when $f = G_{n,k}^\gamma$.

We note, in light of Theorem 14(ii), that $xC^\infty(\mathbb{D}) \subset \overline{C_c^\infty(\mathbb{D})}^{n_\gamma}$ for all $\gamma > -1$ (in particular for $\gamma \geq 0$). In particular, $xC^\infty(\mathbb{D})$ is contained in the α_γ-closure of $C_c^\infty(\mathbb{D})$. Thus, we now aim to show: if $f = G_{n,k}^\gamma$ for some n, k, then there exists a sequence $f^{(j)} \in xC^\infty(\mathbb{D})$ such that

$$f^{(j)} \xrightarrow{j\to\infty}_{\alpha_\gamma} f.$$

Note that $f \in C^\infty(\mathbb{D})$ belongs to $xC^\infty(\mathbb{D})$ if and only if $f|_{\mathbb{S}^1} = 0$. Furthermore, for a polynomial written in the form $\sum_{n,k} f_{n,k} G_{n,k}^\gamma$, we have

$$\sum_{n,k} f_{n,k} G_{n,k}^\gamma \big|_{\mathbb{S}^1}(e^{i\omega}) = \sum_{m\in\mathbb{Z}} \bigg(\sum_{n-2k=m} f_{n,k} \bigg) e^{im\omega}.$$

As such, for a fixed choice of (n, k), let $m = n - 2k$, and choose an ansatz

$$f^{(j)} = G_{n,k}^\gamma - \sum_{n'-2k'=m} f_{n',k'}^{(j)} G_{n',k'}^\gamma \tag{44}$$

where, for each j, only finitely many coefficients $f_{n',k'}^{(j)}$ are nonzero so that $f^{(j)} \in C^\infty(\mathbb{D})$, and we have

$$\sum_{n'-2k'=m} f_{n',k'}^{(j)} = 1 \tag{45}$$

so that $\tau_\gamma^N f^{(j)} = 0$, i.e., $f^{(j)} \in xC^\infty(\mathbb{D})$. We aim to choose the coefficients so that

$$\|f^{(j)} - G_{n,k}^\gamma\|_{\widetilde{H}_D^{1,\gamma}} \xrightarrow{j\to\infty} 0.$$

Note that

$$\|f^{(j)} - G_{n,k}^\gamma\|_{\widetilde{H}_D^{1,\gamma}} = \bigg\| \sum_{n'-2k'=m} f_{n',k'}^{(j)} G_{n',k'}^\gamma \bigg\|_{\widetilde{H}_D^{1,\gamma}}^2 = \sum_{n'-2k'=m} (n'+1+\gamma)^2 \|G_{n',k'}^\gamma\|_{L_\gamma^2}^2 |f_{n',k'}^{(j)}|^2.$$

We now use the following elementary lemma, whose proof appears in Appendix A.2:

Lemma 18. *Suppose $\{a_k\}_{k=1}^\infty$ is a sequence of positive numbers satisfying $\sum_{k=1}^\infty 1/a_k = \infty$. Then there exists a sequence of sequences $\{\{c_k^{(j)}\}_{k=1}^\infty\}_{j=1}^\infty$, with only finitely many numbers $c_k^{(j)}$ nonzero for any fixed j, satisfying*

$$\sum_{k=1}^\infty c_k^{(j)} = 1 \quad \textit{and} \quad \lim_{j\to\infty} \sum_{k=1}^\infty a_k |c_k^{(j)}|^2 = 0.$$

Thus, we aim to find $f_{n',k'}^{(j)}$ satisfying (45) and such that

$$\sum_{n'-2k'=m} (n'+1+\gamma)^2 \|G_{n',k'}^\gamma\|_{L_\gamma^2}^2 |f_{n',k'}^{(j)}|^2 \xrightarrow{j\to\infty} 0.$$

By Lemma 18, applied to $a_{k'} = (n'+1+\gamma)^2 \|G_{n',k'}^\gamma\|_{L_\gamma^2}^2$ and $c_{k'}^{(j)} = f_{n',k'}^{(j)}$ (where $n'-2k' = m$), we see that this is possible, provided that

$$\sum_{n'-2k'=m} \frac{1}{(n'+1+\gamma)^2 \|G_{n',k'}^\gamma\|_{L_\gamma^2}^2} = \infty.$$

Note that, for fixed m, we can parametrize the terms in the above sum by k', with $k' \in \mathbb{N}$, $k' \geq \max(0, -m)$, with

$$n' = 2k' + m \quad \Longrightarrow \quad n' - k' = k' + m.$$

Thus

$$\begin{aligned}(n'+1+\gamma)^2 \|G^{\gamma}_{n',k'}\|^2_{L^2_\gamma} &= (n'+1+\gamma)^2 \frac{\pi (n'-k')!\,(k')!\,(\gamma!)^2}{(n'+\gamma+1)(n'-k'+\gamma)!\,(k'+\gamma)!} \\ &= (2k'+m+1+\gamma)^2 \frac{\pi (k'+m)!\,(k')!\,(\gamma!)^2}{(2k'+m+\gamma+1)(k'+m+\gamma)!\,(k'+\gamma)!},\end{aligned}$$

and the latter is asymptotic to $2\pi(\gamma!)^2(k')^{1-2\gamma}$ as $k' \to \infty$. It follows that

$$\sum_{n'-2k'=m} \frac{1}{(n'+1+\gamma)^2 \|G^{\gamma}_{n',k'}\|^2_{L^2_\gamma}} = \infty$$

as long as $1 - 2\gamma \leq 1$, or equivalently, $\gamma \geq 0$. □

Inspecting the proof above, we can in fact prove the following result, which will be useful in later sections.

Corollary 19. *For $\gamma \in (-1, 1)$, $C^\infty_c(\mathbb{D}^{\text{int}})$ is dense in $\mathcal{A}_{\gamma,D}$ for the $\widetilde{H}^{1,\gamma}$ topology.*

Proof. Recall that the quadratic forms α_γ and $(\cdot,\cdot)_{\widetilde{H}^{1,\gamma}}$ agree on $C^\infty_c(\mathbb{D})$. For $\gamma \in (-1, 0)$, $\mathcal{A}_{\gamma,D} = \overline{xC^\infty(\mathbb{D}) + x^{-\gamma}C^\infty(\mathbb{D})}$, where $x^{-\gamma}C^\infty(\mathbb{D})$ is included in $\overline{C^\infty_c(\mathbb{D}^{\text{int}})}^{\alpha_\gamma}$ by the previous proof, and $xC^\infty(\mathbb{D})$ is included in $\overline{C^\infty_c(\mathbb{D}^{\text{int}})}^{n_\gamma}$ by Theorem 14(ii), which itself is included in $\overline{C^\infty_c(\mathbb{D}^{\text{int}})}^{\alpha_\gamma}$ (this is because, by the Cauchy–Schwarz inequality, we immediately have $\alpha_\gamma(f) \leq n_\gamma(f)^2$). The proofs for the cases $\gamma = 0$ and $\gamma \in (0, 1)$ are similar. □

3.3. ***Proof of Theorem 3.*** We are now ready to prove Theorem 3, namely that $\mathcal{L}_{\gamma,\min}$ is self-adjoint for $|\gamma| \geq 1$, by showing that $\mathcal{L}_{\gamma,\min} = \mathcal{L}_{\gamma,D}$. This amounts to proving that

(i) $\overline{C^\infty_c(\mathbb{D})}^{n_\gamma} = \overline{C^\infty(\mathbb{D})}^{n_\gamma}$ for $\gamma \geq 1$, and

(ii) $\overline{C^\infty_c(\mathbb{D})}^{n_\gamma} = \overline{x^{-\gamma}C^\infty(\mathbb{D})}^{n_\gamma}$ for $\gamma \leq -1$.

Assuming (i) holds, (ii) can be proved as follows. Fix $\gamma \leq -1$ and $f \in x^{-\gamma}C^\infty(\mathbb{D})$. By (i), $x^\gamma f \in C^\infty(\mathbb{D})$ can be approximated in $n_{-\gamma}$ by a sequence $f_n \in C^\infty_c(\mathbb{D}^{\text{int}})$, and then, using the intertwining property (8), $\|x^{-\gamma} f_n - f\|_{n_\gamma} = \|f_n - x^\gamma f\|_{n_{-\gamma}} \to 0$ as $n \to \infty$, i.e., the sequence $x^{-\gamma} f_n \in C^\infty_c(\mathbb{D}^{\text{int}})$ approximates f in the n_γ topology.

We now prove (i), fixing $\gamma \geq 1$. If $\gamma > 1$, then the conclusion follows from Theorem 14(ii). The limiting case $\gamma = 1$ requires special care, and the proof that follows works for all $\gamma \geq 1$ *a fortiori*. By Theorem 14(ii), we have that $xC^\infty(\mathbb{D}) \subset \overline{C^\infty_c(\mathbb{D}^{\text{int}})}^{n_\gamma}$, and hence it remains to show that $C^\infty(\mathbb{D}) \subset \overline{xC^\infty(\mathbb{D})}^{n_\gamma}$. Since each $f \in C^\infty(\mathbb{D})$ can be written as an n_γ-convergent sum of Zernike polynomials, it suffices to show that every Zernike polynomial $G^\gamma_{n,k}$ can be n_γ-approximated by elements of $xC^\infty(\mathbb{D})$. Thus fixing (n, k) and setting $m = n-2k$, we aim to approximate $G^\gamma_{n,k}$ with a sequence $\{f^{(j)}\}_j$ in $xC^\infty(\mathbb{D})$ of the form (44), whose coefficients are chosen to satisfy (45) to enforce $f^{(j)}|_{\mathbb{S}^1} = 0$, and so that

$$\|f^{(j)} - G^\gamma_{n,k}\|_{\widetilde{H}^{2,\gamma}_D} = \Big\| \sum_{n'-2k'=m} f^{(j)}_{n',k'} G^\gamma_{n',k'} \Big\|_{\widetilde{H}^{2,\gamma}_D} \xrightarrow{j\to\infty} 0.$$

In this case, we have

$$\Big\| \sum_{n'-2k'=m} f^{(j)}_{n',k'} G^\gamma_{n',k'} \Big\|^2_{\widetilde{H}^{2,\gamma}_D} = \sum_{n'-2k'=m} (n'+1+\gamma)^4 \|G^\gamma_{n',k'}\|^2_{L^2_\gamma} |f^{(j)}_{n',k'}|^2.$$

By Lemma 18, applied to $a_{k'} = (n'+1+\gamma)^4\|G^\gamma_{n',k'}\|^2_{L^2_\gamma}$ and $c^{(j)}_{k'} = f^{(j)}_{n',k'}$ (where $n'-2k' = m$), we can find our desired sequences, provided that

$$\sum_{n'-2k'=m} \frac{1}{(n'+1+\gamma)^4\|G^\gamma_{n',k'}\|^2_{L^2_\gamma}} = \infty.$$

Note that, for fixed m, we can parametrize the terms in the above sum by k', with $k' \in \mathbb{N}$, $k' \geq \max(0, -m)$, with

$$n' = 2k'+m \quad \Longrightarrow \quad n'-k' = k'+m.$$

Thus

$$\begin{aligned}(n'+1+\gamma)^4\|G^\gamma_{n',k'}\|^2_{L^2_\gamma} &= (n'+1+\gamma)^4 \frac{\pi(n'-k')!\,(k')!\,(\gamma!)^2}{(n'+\gamma+1)(n'-k'+\gamma)!\,(k'+\gamma)!} \\ &= (2k'+m+1+\gamma)^4 \frac{\pi(k'+m)!\,(k')!\,(\gamma!)^2}{(2k'+m+\gamma+1)(k'+m+\gamma)!\,(k'+\gamma)!},\end{aligned}$$

which is asymptotic to $2\pi(\gamma!)^2(k')^{3-2\gamma}$ as $k' \to \infty$. It follows that

$$\sum_{n'-2k'=m} \frac{1}{(n'+1+\gamma)^4\|G^\gamma_{n',k'}\|^2_{L^2_\gamma}} = \infty$$

as long as $3-2\gamma \leq 1$, i.e., $\gamma \geq 1$. Thus, we can construct $f^{(j)} \in xC^\infty(\mathbb{D}) \subset \mathrm{dom}\,(\mathcal{L}_{\gamma,\min})$ approximating any $G^\gamma_{n,k}$ in the graph norm, and this concludes the proof of the case $\gamma \geq 1$. The proof of Theorem 3 is complete. □

4. Proof of Lemma 4 and Theorems 5 and 6

This section aims at proving the first main theorem, Theorem 5. This is done through the following roadmap.

In Section 4.1, we first prove Lemma 4, which consists in making sense of the appropriate $\widetilde{H}^{1,\gamma}$ inner product for which Green's identities hold for functions in $\mathcal{A}_\gamma$. Section 4.2 then covers the surjective extension of τ^D_γ to $\widetilde{H}^{1,\gamma}$, as stated in Theorem 20 below. This requires elementary one-dimensional traces estimates as derived in Section 4.2.1, combined with Fourier expansions in the boundary coordinate to complete the proof of Theorem 20 in Section 4.2.2. In Section 4.3, we first prove preliminary properties in Lemma 25: characterizations of the space $\widetilde{H}^{1,\gamma}_0(\mathbb{D})$, Green's first identity without boundary term, and a weak setting where $\mathcal{L}_\gamma$ is bounded. Moving to the proof of Theorem 5, it remains to show the extension of the Neumann trace to W^2_γ by duality arguments, and the extension of Green's identities by density arguments. Finally, we prove Theorem 6 in Section 4.4.

4.1. *Proof of Lemma 4.*

Case $-1 < \gamma < 0$. This case is the "simplest" in that the $\widetilde{H}^{1,\gamma}$ space does not require the use of an auxiliary function ϕ_γ, or equivalently, $\phi_\gamma = 1$ can be used, definition (23) is equivalent to

$$\mathcal{A}_\gamma \ni f, g \mapsto (f, g)_{\widetilde{H}^{1,\gamma}} := (\sqrt{x}\, \partial_\rho f, \sqrt{x}\, \partial_\rho g)_{L^2_\gamma} + (\rho^{-1}\partial_\omega f, \rho^{-1}\partial_\omega g)_{L^2_\gamma} + (1+\gamma)^2 (f, g)_{L^2_\gamma}, \tag{46}$$

hence the independence on b. Note that $\tau^D_\gamma, \tau^N_\gamma : \mathcal{A}_\gamma \to C^\infty(\mathbb{S}^1)$ defined in (20) take the simplified form

$$\tau^D_\gamma f = f|_{x=0}, \quad \tau^N_\gamma f = \lim_{x\to 0} (-x^{\gamma+1}\partial_\rho f), \quad f \in \mathcal{A}_\gamma.$$

Then (25) follows by sending $R \to 1$ in (3).

Case $0 \le \gamma < 1$. We first briefly explain what further treatment is required, explaining what happens for $\gamma \in (0, 1)$, though the case $\gamma = 0$ is similar. One may think about defining a space $\widetilde{H}^{1,\gamma}$ to be the completion of $C^\infty(\mathbb{D})$ for the norm coming from the inner product (46), though one may find that such a space does not have a good trace in the sense that, e.g., as a direct application of Corollary 19, for any $\gamma \in (0, 1)$, $C^\infty_c(\mathbb{D})$ is dense in $C^\infty(\mathbb{D})$ for the norm coming from the inner product (46). Since the indicial root $r = -\gamma$ is the most singular, one may think of taking the completion of $\mathcal{A}_\gamma$ with respect to the same norm, though another issue arises: the norm (46) of elements in $x^{-\gamma}C^\infty$ is in general not finite. In Sturm–Liouville theory, this situation is characteristic of the case of singular endpoint in the limit circle case, for which a systematic remedy in 1D can be found in, e.g., [Behrndt et al. 2020, §6.9]. We adapt these ideas to the present two-dimensional context.

We first make the function ϕ_γ introduced in Section 2.3 explicit. For $x \in (0, 1)$, we define

$$\phi_\gamma(x) := \begin{cases} (-\gamma)\displaystyle\sum_{k=0}^{\infty} \frac{x^{k-\gamma}}{k-\gamma} & \text{if } \gamma \in (0,1), \\ \log x - \log(1-x) - c_0 & \text{if } \gamma = 0, \end{cases} \tag{47}$$

where $c_0 > \log(e^4 - 1)$ is a fixed constant. For any $\gamma \in [0, 1)$, ϕ_γ is a radial solution of $\mathcal{L}_\gamma \phi_\gamma = (\gamma+1)^2\phi_\gamma$ (this simplifies into $\partial_\rho(x^{\gamma+1}\rho\partial_\rho\phi_\gamma) = 0$), satisfying the relevant limits (19). Moreover, there exists $x_\gamma > 0$ such that ϕ_γ is nonzero on $(0, x_\gamma)$. If $x \in (0, 1)$, then x_γ solves the implicit equation $\gamma \sum_{k=1}^\infty x^k/(k-\gamma) = 1$, and if $\gamma = 0$, then $x_0 = 1/(1+e^{-c_0})$. Set $b_\gamma := \sqrt{1 - x_\gamma}$.

Let us fix $\gamma \in [0, 1)$, and write $\phi = \phi_\gamma$ for conciseness. Let us denote[5]

$$N_\phi f := \phi\, \partial_\rho\left(\frac{f}{\phi}\right) = \partial_\rho f - \frac{\partial_\rho \phi}{\phi} f. \tag{48}$$

[5]The definition differs philosophically from [Behrndt et al. 2020, equation (6.9.2), p. 443] in a couple of ways: the $\sqrt{p}$ is not factored in; the derivative is the ∂_ρ specific to our case.

Given $b \in (b_\gamma, 1)$, let us define the bilinear form as in (23)

$$\mathfrak{t}_{\gamma,b}[f,g] = (\sqrt{x} N_\phi f, \sqrt{x} N_\phi g)_{x^\gamma, A_{b,1}} - b(x(b))^{\gamma+1} \frac{\partial_\rho \phi}{\phi}(b) \int_{C_b} f \bar{g} + (\sqrt{x}\partial_\rho f, \sqrt{x}\partial_\rho g)_{x^\gamma, \mathbb{D}_b} + (\rho^{-1}\partial_\omega f, \rho^{-1}\partial_\omega g)_{L^2_\gamma} + (\gamma+1)^2 (f,g)_{L^2_\gamma}, \quad (49)$$

where for $0 \le a < c \le 0$, we denote $A_{a,c}$ for the annulus $\{a < \rho < c\}$.

Proof of Lemma 4. We first derive some identities. We compute

$$\mathcal{L}_\gamma f \;\; \rho x^\gamma \bar{g} = -\partial_\rho (x^{\gamma+1} \rho \partial_\rho f) \bar{g} - \bar{g} \rho^{-2} \partial_\omega^2 f \rho x^\gamma + (\gamma+1)^2 f \bar{g} \rho x^\gamma.$$

The first term in the right-hand side is rewritten as

$$\begin{aligned}
-\partial_\rho (x^{\gamma+1}\rho\partial_\rho f)\bar{g} &= -\partial_\rho (x^{\gamma+1}\rho\partial_\rho(\phi f/\phi))\bar{g} \\
&= -\partial_\rho (x^{\gamma+1}\rho(\partial_\rho\phi) f/\phi))\bar{g} - \bar{g}\partial_\rho(x^{\gamma+1}\rho N_\phi f) \\
&= -\cancel{\partial_\rho (x^{\gamma+1}\rho\partial_\rho\phi)}(f/\phi)\bar{g} - x^{\gamma+1}\rho\frac{\partial_\rho\phi}{\phi}(N_\phi f)\bar{g} - \bar{g}\partial_\rho(x^{\gamma+1}\rho N_\phi f) \\
&= -x^{\gamma+1}\rho\frac{\partial_\rho\phi}{\phi}(N_\phi f)\bar{g} + \partial_\rho\bar{g} x^{\gamma+1}\rho N_\phi f - \partial_\rho(\bar{g} x^{\gamma+1}\rho N_\phi f) \\
&= (N_\phi \bar{g}) x^{\gamma+1}\rho(N_\phi f) - \partial_\rho(\bar{g} x^{\gamma+1}\rho N_\phi f).
\end{aligned}$$

Hence on an annulus $A_{a,b} = \{a < \rho < b\}$, we obtain the integration by parts formula

$$(\mathcal{L}_\gamma f, g)_{x^\gamma, A_{a,b}} = (\sqrt{x} N_\phi f, \sqrt{x} N_\phi g)_{x^\gamma, A_{a,b}} + (\rho^{-1}\partial_\omega f, \rho^{-1}\partial_\omega g)_{x^\gamma, A_{a,b}} \dots - \left[x(\rho)^{\gamma+1}\rho \int_{C_\rho} \bar{g} N_\phi f \right]_{\rho=a}^{\rho=b} + (\gamma+1)^2 (f,g)_{x^\gamma, A_{a,b}}. \quad (50)$$

We now rewrite the $N_\phi f N_\phi \bar{g}$ term:

$$\begin{aligned}
(N_\phi f)(N_\phi \bar{g})\rho x^{\gamma+1} &= \rho x^{\gamma+1}\left(\partial_\rho f - f\frac{\partial_\rho\phi}{\phi}\right)\left(\partial_\rho \bar{g} - \bar{g}\frac{\partial_\rho\phi}{\phi}\right) \\
&= \rho x^{\gamma+1}\partial_\rho f \partial_\rho \bar{g} - \rho x^{\gamma+1}\partial_\rho(f\bar{g})\frac{\partial_\rho\phi}{\phi} + \rho x^{\gamma+1} f\bar{g}\left(\frac{\partial_\rho\phi}{\phi}\right)^2 \\
&= \rho x^{\gamma+1}\partial_\rho f \partial_\rho \bar{g} - \partial_\rho\left(\rho x^{\gamma+1} f\bar{g}\frac{\partial_\rho\phi}{\phi}\right) + f\bar{g}\partial_\rho\left(\rho x^{\gamma+1}\frac{\partial_\rho\phi}{\phi}\right) + \rho x^{\gamma+1} f\bar{g}\left(\frac{\partial_\rho\phi}{\phi}\right)^2 \\
&\overset{(19)}{=} \rho x^{\gamma+1}\partial_\rho f \partial_\rho \bar{g} - \partial_\rho\left(\rho x^{\gamma+1} f\bar{g}\frac{\partial_\rho\phi}{\phi}\right),
\end{aligned}$$

and thus

$$(\sqrt{x} N_\phi f, \sqrt{x} N_\phi g)_{x^\gamma, A_{a,b}} = (\sqrt{x}\partial_\rho f, \sqrt{x}\partial_\rho g)_{x^\gamma, A_{a,b}} - \left[\rho x^{\gamma+1}\frac{\partial_\rho\phi}{\phi}\int_{C_\rho} f\bar{g}\right]_{\rho=a}^{\rho=b}. \quad (51)$$

Proof of (1). If $\sqrt{1-x_\gamma} < a < b < 1$, we compute

$$\begin{aligned}\mathfrak{t}_{\gamma,b}[f,g]-\mathfrak{t}_{\gamma,a}[f,g] &= -(\sqrt{x}N_\phi f, \sqrt{x}N_\phi g)_{x^\gamma,A_{a,b}} - b(x(b))^{\gamma+1}\frac{\partial_\rho\phi}{\phi}(b)\int_{C_b} f\bar{g} \\ &\qquad + a(x(a))^{\gamma+1}\frac{\partial_\rho\phi}{\phi}(a)\int_{C_a} f\bar{g} + (\sqrt{x}\partial_\rho f, \sqrt{x}\partial_\rho g)_{x^\gamma,A_{a,b}} \\ &\overset{(51)}{=} 0,\end{aligned}$$

and hence $\mathfrak{t}_{\gamma,b}$ does not depend on $b \in (b_\gamma, 1)$.

Proof of (2). We note that if $f \in \mathcal{A}_\gamma$ with $\gamma \in (-1,0)$ or if $f \in \mathcal{A}_{\gamma,D}$ with $\gamma \in [0,1)$ that

$$\lim_{b\to 1}\left[b(x(b))^{\gamma+1}\frac{\partial_\rho\phi}{\phi}(b)\int_{C_b}|f|^2\right] = 0.$$

Indeed, if $\gamma \in (-1,0)$ then $\partial_\rho\phi \equiv 0$, while otherwise

$$b(x(b))^{\gamma+1}\frac{\partial_\rho\phi}{\phi}(b) = \begin{cases} O(x(b)^\gamma) & \text{if } \gamma \in (0,1), \\ O\Big(\dfrac{1}{|\log(x(b))|}\Big) & \text{if } \gamma = 0 \end{cases}$$
$$= o(1)$$

as $b \to 1$, and $f \in \mathcal{A}_{\gamma,D}$ implies that f is uniformly bounded as $x \to 0$, as $\mathcal{A}_{\gamma,D} = C^\infty(\mathbb{D}) + x^{1-\gamma}C^\infty(\mathbb{D})$ if $\gamma \in (0,1)$, or $\mathcal{A}_{\gamma,D} = C^\infty(\mathbb{D}) + x(\log x)C^\infty(\mathbb{D})$ for $\gamma = 0$. As such, given that

$$\lim_{b\to 1}\|\sqrt{x}N_\phi f\|^2_{x^\gamma,A_{b,1}} = 0$$

since $\sqrt{x}N_\phi f \in L^2_\gamma$ near the boundary, it follows that we can take the limit as $b \to 1$ in (49) to obtain (24) as desired.

Proof of (3). Using that, for $b \in (0,1)$,

$$\begin{aligned}(\mathcal{L}_\gamma f, g)_{x^\gamma,\mathbb{D}_b} \\ = -\int_{C_b} x^{\gamma+1}\rho g\partial_\rho f + (\sqrt{x}\partial_\rho f, \sqrt{x}\partial_\rho g)_{x^\gamma,\mathbb{D}_b} + (\rho^{-1}\partial_\omega f, \rho^{-1}\partial_\omega g)_{x^\gamma,\mathbb{D}_b} + (\gamma+1)^2(f,g)_{x^\gamma,\mathbb{D}_b},\end{aligned}$$

combining with (49), we arrive at

$$\begin{aligned}\mathfrak{t}_\gamma[f,g] = (\sqrt{x}N_\phi f, \sqrt{x}N_\phi g)_{x^\gamma,A_{b,1}} + (\rho^{-1}\partial_\omega f, \rho^{-1}\partial_\omega g)_{x^\gamma,A_{b,1}} + (\gamma+1)^2(f,g)_{x^\gamma,A_{b,1}} \\ -\int_{C_b}\underbrace{\Big(\rho x^{\gamma+1}\frac{\partial_\rho\phi}{\phi}f\bar{g} - x^{\gamma+1}\rho\bar{g}\partial_\rho f\Big)}_{(\bar{g}/\phi)W(f,\phi)} + (\mathcal{L}_\gamma f, g)_{x^\gamma,\mathbb{D}_b}.\end{aligned}$$

Equation (25) then follows by sending $b \to 1$.

Proof of (4). We successively treat $\gamma \in (-1,0)$, $\gamma \in (0,1)$, and $\gamma = 0$.

For $\gamma \in (-1,0)$, we note that since $\phi_\gamma \equiv 1$, we in fact have for any $f \in \mathcal{A}_\gamma$

$$(f,f)_{\widetilde{H}^{1,\gamma}} = \|\sqrt{x}\partial_\rho f\|^2_{L^2_\gamma} + \|\rho^{-1}\partial_\omega f\|^2_{L^2_\gamma} + (\gamma+1)^2\|f\|^2_{L^2_\gamma} \geq (1+\gamma)^2\|f\|^2_{L^2_\gamma}. \tag{52}$$

The case $\gamma \in (0,1)$ is deduced from the case $\gamma \in (-1,0)$ by intertwining. First notice that if $f \in \mathcal{A}_\gamma$, then $x^\gamma f \in \mathcal{A}_{-\gamma}$. Assuming the following holds true:

$$(f,g)_{\widetilde{H}^{1,\gamma}} = (x^\gamma f, x^\gamma g)_{\widetilde{H}^{1,-\gamma}}, \quad f,g \in \mathcal{A}_\gamma. \tag{53}$$

The result then follows using the case $\gamma \in (-1,0)$: for any $f \in \mathcal{A}_\gamma$,

$$(f,f)_{\widetilde{H}^{1,\gamma}} = (x^{-\gamma} f, x^{-\gamma} f)_{\widetilde{H}^{1,-\gamma}} \geq (1+(-\gamma))^2 \|x^{-\gamma} f\|^2_{L^2_{-\gamma}} = (1-|\gamma|)^2 \|f\|^2_{L^2_\gamma},$$

where the inequality follows from (52). To prove (53), also notice that the differential intertwining property (8) implies $\mathcal{L}_\gamma f = x^{-\gamma}\mathcal{L}_{-\gamma}(x^\gamma f)$ for all $f \in \mathcal{A}_\gamma$. We now compute

$$\begin{aligned}(f,g)_{\widetilde{H}^{1,\gamma}} &\overset{(25)}{=} (\mathcal{L}_\gamma f, g)_{L^2_\gamma} - (\tau^N_\gamma f, \tau^D_\gamma g)_{L^2(\mathbb{S}^1)} \\ &= (x^{-\gamma}\mathcal{L}_{-\gamma}(x^\gamma f), x^{-\gamma}(x^\gamma g))_{L^2_\gamma} - (\tau^N_\gamma f, \tau^D_\gamma g)_{L^2(\mathbb{S}^1)} \\ &\overset{(22)}{=} (\mathcal{L}_{-\gamma}(x^\gamma f), x^\gamma g)_{L^2_{-\gamma}} - (\tau^N_{-\gamma}(x^\gamma f), \tau^D_{-\gamma}(x^\gamma g))_{L^2(\mathbb{S}^1)} \overset{(25)}{=} (x^\gamma f, x^\gamma g)_{\widetilde{H}^{1,-\gamma}}.\end{aligned}$$

Finally the case $\gamma = 0$ requires special care. Recall that $\phi_0(x) = \log x - \log(1-x) - c_0$ as defined in (47). Let $x_0 > 0$ satisfy the property that $\phi_0(x) < 0$ for all $x \in (0, x_0)$. Then, for any $b > b_0 := \sqrt{1-x_0}$, throwing away all first-order terms in (23) gives the crude bound

$$(f,f)_{\widetilde{H}^{1,0}} \geq \|f\|^2_{L^2} - \frac{b x(b)\partial_\rho \phi_0(x(b))}{\phi_0(x(b))} \int_{C_b} |f|^2.$$

Noting that $b x(b)\partial_\rho\phi_0(x(b)) = -2$ for all b, multiplying both sides by $-b\,\phi_0(x(b))/2$ and integrating from $b = b_0$ to $b = 1$, we obtain

$$c(f,f)_{\widetilde{H}^{1,0}} \geq c\|f\|^2_{L^2} - \|f\|^2_{A_{b_0,1}}, \quad \text{where } c := \int_{b_0}^1 \frac{-\phi_0(x(b))}{2}\, b\, \mathrm{d}b = \int_0^{x_0} \frac{-\phi_0(x)}{4}\,\mathrm{d}x.$$

We can relate the constant c to c_0 as follows: first note that the condition $\phi_0(x_0) = 0$ gives $x_0 = 1/(1+e^{-c_0})$. Integrating by parts, we then get

$$\int_0^{x_0} -\phi_0(x)\,\mathrm{d}x = \underbrace{[-x\phi_0(x)]_{x\to 0}^{x=x_0}}_{=0} + \int_0^{x_0} x\phi_0'(x)\,\mathrm{d}x = \int_0^{x_0} \frac{\mathrm{d}x}{1-x} = -\log(1-x_0) = \log(1+e^{c_0}),$$

and hence $c = \frac{1}{4}\log(1+e^{c_0})$. Returning to the estimate, we arrive at

$$(f,f)_{\widetilde{H}^{1,0}} \geq \|f\|^2_{L^2} - c^{-1}\|f\|^2_{A_{b_0,1}} \geq (1-c^{-1})\|f\|^2_{L^2}.$$

In particular $(\cdot,\cdot)_{\widetilde{H}^{1,0}}$ is positive definite if $c > 1$, which is then equivalent to the condition $c_0 > \log(e^4 - 1)$.

The proof of Lemma 4 is complete. □

4.2. ***Extension of the Dirichlet trace to $\widetilde{H}^{1,\gamma}$.*** The first part of Theorem 5 consists in extending the Dirichlet trace to $\widetilde{H}^{1,\gamma}(\mathbb{D})$, which we state as a separate result.

Theorem 20. *For any $\gamma \in (-1, 1)$ and with $\widetilde{H}^{1,\gamma}(\mathbb{D})$, $H_{(\gamma)}$ respectively defined in* (27) *and* (29), *the Dirichlet trace τ_γ^D defined in* (20) *extends to a bounded*, *surjective operator*

$$\tau_\gamma^D : \widetilde{H}^{1,\gamma}(\mathbb{D}) \to H_{(\gamma)}.$$

The right inverse $R : H_{(\gamma)} \to \widetilde{H}^{1,\gamma}(\mathbb{D})$ arises as the extension to $H_{(\gamma)}$ of a continuous operator $R : C^\infty(\mathbb{S}^1) \to \mathcal{A}_\gamma$.

In the case $\gamma \in (-1, 0)$, the construction of $\widetilde{H}^{1,\gamma}$ is associated with operators with regular boundary points, in which case this is relatively straightforward. The construction for $\gamma \in (0, 1)$ is associated with operators with singular points in the limit circle case, requiring a regularization approach, tying it with the previous family of cases $\gamma \in (-1, 0)$. Finally, the case $\gamma = 0$, linked with the case of double indicial roots and log-weighted Sobolev spaces, also requiring regularization, is treated separately.

4.2.1. *Some 1D trace estimates.* Let $a > 0$. Let us first prove a lemma that will be useful for the case $\gamma \in (-1, 0)$. Define $\widetilde{H}^{1,\gamma}[0, a]$ to be the completion of $C^\infty([0, a])$ for the norm

$$\|f\|^2_{\widetilde{H}^{1,\gamma}[0,a]} = \|\sqrt{x}\partial_x f\|^2_{L^2_\gamma[0,a]} + (1+\gamma)^2\|f\|^2_{L^2_\gamma[0,a]}, \tag{54}$$

where $\|f\|^2_{L^2_\gamma[0,a]} = \int_0^a |f(x)|^2 x^\gamma \, dx$. We prove the following trace estimate:

Lemma 21. *The evaluation map $C^\infty([0, a]) \ni f \mapsto f(0)$ extends by density to $\widetilde{H}^{1,\gamma}[0, a]$ and we have the following estimate for all $\ell \in (0, a)$:*

$$\ell^\gamma |f(0)|^2 \le (\gamma+1)\left(\ell^{-1}\|f\|^2_{L^2_\gamma[0,a]} + \frac{2}{-\gamma}\|\sqrt{x} f'\|^2_{L^2_\gamma[0,a]}\right). \tag{55}$$

Proof. From the relation $f(0) = f(x) - \int_0^x f'(t)\, dt$, we deduce

$$\begin{aligned} |f(0)|^2 &\le 2|f(x)|^2 + 2\left|\int_0^x f'(t)\, dt\right|^2 \\ &\le 2|f(x)|^2 + 2\underbrace{\int_0^x t^{-\gamma-1}\, dt}_{x^{-\gamma}/(-\gamma)} \int_0^x t|f'(t)|^2\, t^\gamma\, dt. \end{aligned}$$

Now multiply by x^γ and integrate from 0 to ℓ to obtain

$$\frac{\ell^{\gamma+1}}{\gamma+1}|f(0)|^2 \le 2\|f\|^2_{L^2_\gamma[0,a]} + 2\frac{\ell}{-\gamma}\|\sqrt{x} f'\|^2_{L^2_\gamma[0,a]},$$

equivalent to (55). □

Considering now a log-type weight that will be relevant to $\gamma = 0$, let us now define $\widetilde{H}^{1,0}_{\log}[0, a]$ the closure of $C^\infty[0, a] + \frac{1}{\log x} C^\infty[0, a]$ for the norm

$$\|f\|^2_{\widetilde{H}^{1,0}_{\log}} = \int_0^a (x|f'(x)|^2 + |f(x)|^2) \log^2 x\, dx. \tag{56}$$

Lemma 22. *The evaluation map* $C^\infty[0,a] + \frac{1}{\log x} C^\infty[0,a] \ni f \mapsto f(0)$ *extends to a bounded map on* $\widetilde{H}^{1,0}_{\log}[0,a]$. *More precisely, we have for every* $\ell \in (0,a]$,

$$\frac{\int_0^\ell \log^2 x \, dx}{\int_0^\ell - \log x \, dx} |f(0)|^2 \le \frac{2}{\int_0^\ell - \log x \, dx} \int_0^a (\log x)^2 |f(x)|^2 \, dx + 2 \int_0^a x \log^2 x |f'(x)|^2 \, dx. \tag{57}$$

Proof. If $f \in C^\infty + \frac{1}{\log x} C^\infty$, then f' is integrable near 0 and we may write, for any $x \le a$, $f(0) = f(x) - \int_0^x f'(t) \, dt$. We then compute

$$|f(0)|^2 \le 2|f(x)|^2 + 2 \left| \int_0^x f'(t) \, dt \right|^2 \le 2|f(x)|^2 + 2 \underbrace{\int_0^x \frac{dt}{t \log^2 t}}_{-1/\log x} \int_0^a t \log^2 t |f'(t)|^2 \, dt.$$

Multiply by $\log^2 x$ and integrate from 0 to $\ell \in (0,a)$ to obtain[6]

$$\int_0^\ell \log^2 x \, dx |f(0)|^2 \le 2 \int_0^a (\log x)^2 |f(x)|^2 \, dx + 2 \int_0^\ell (-\log x) \, dx \int_0^a t \log^2 t |f'(t)|^2 \, dt.$$

Divide by $\int_0^\ell - \log x \, dx$ to obtain (57). □

Lemma 23. *The equality* $\int_0^\ell - \log x \, dx = \epsilon$ *gives rise to a function* $\ell : [0,1] \to [0,1]$, *continuous, increasing with* $\ell(0) = 0$ *and* $\ell(1) = 1$, *satisfying* $\lim_{\epsilon \to 0} \frac{\log(\ell(\epsilon))}{\log \epsilon} = 1$.

Proof. The existence and increasing nature is obtained from inverse function theorem and differentiation. This equality can also be seen as $\ell(1 - \log \ell) = \epsilon$, which implies the obvious bound $\ell \le \epsilon$, hence the zero limit. To obtain the asymptotics: we already have that $\log \ell \le \log \epsilon$ by monotonicity of log. In addition, for every $\alpha \in (0,1)$, for ℓ small enough, we have $1 + \log \frac{1}{\ell} \le C_\alpha \frac{1}{\ell^\alpha}$, and thus

$$\epsilon = \ell(1 - \log \ell) \le \ell^{1-\alpha} C_\alpha \quad \Longrightarrow \quad \log \epsilon \le \log C_\alpha + (1-\alpha) \log \ell.$$

These inequalities imply

$$\frac{1}{1-\alpha} \le \liminf_{\epsilon \to 0} \frac{\log \ell}{\log \epsilon} \le \limsup_{\epsilon \to 0} \frac{\log \ell}{\log \epsilon} \le 1 \quad \text{for every } \alpha > 0,$$

hence the result follows. □

4.2.2. *Proof of Theorem 20.* The proof of Theorem 20 treats, in order, the cases $\gamma \in (-1,0)$, $\gamma \in (0,1)$, and $\gamma = 0$.

Case $\gamma \in (-1,0)$. In a tubular neighborhood of the boundary $[0,a]_x \times \mathbb{S}^1_\omega$ (with $\partial \mathbb{D} = \{x = 0\}$), one may expand a smooth function in the form $f = \sum_{n \in \mathbb{Z}} f_n(x) e^{in\omega}$. Apply (55) to each f_n with $\ell = a/\langle n \rangle^2$, sum over n to obtain

$$a^\gamma \sum_n \langle n \rangle^{-2\gamma} |f_n(0)|^2 \le (\gamma+1) \left(a^{-1} \sum_n \langle n \rangle^2 \|f_n\|^2_{L^2_\gamma[0,a]} + \frac{2}{-\gamma} \sum_n \|\sqrt{x} f_n'\|^2_{L^2_\gamma[0,a]} \right),$$

[6] One could be more explicit and use that $\int_0^\ell \log^2 t \, dt = \ell((1-\log \ell)^2 + 1)$ and $\int_0^\ell \log t \, dt = \ell(\log \ell - 1)$ but this may not be useful.

Observing that the right-hand side is equivalent to the squared $\widetilde{H}^{1,\gamma}(\mathbb{D})$ norm of f and the left-hand side is the squared $H^{-\gamma}(\mathbb{S}^1)$-norm of $\omega \mapsto f(0,\omega)$, this boundedness estimate extends by density to $\widetilde{H}^{1,\gamma}(\mathbb{D})$.

To prove surjectivity, let us construct an explicit right-inverse for τ_γ^D. Fix $g \in C_c^\infty([0,a),\mathbb{R})$ with $g(0)=1$ and consider the map

$$R\bigg(\sum_n a_n e^{in\omega}\bigg) = \sum_n a_n e^{in\omega} g(n^2 x).$$

It is easy to see that $R(C^\infty(\mathbb{S}^1)) \subset C^\infty([0,a]_x \times \mathbb{S}^1_\omega)$ and that $\tau_\gamma^D R = \mathrm{id}\,|_{C^\infty(\mathbb{S}^1)}$. To show that R extend to a bounded map $H^{-\gamma}(\mathbb{S}^1) \to \widetilde{H}^{1,\gamma}([0,a]_x \times \mathbb{S}^1_\omega)$, it is enough to show that for $f_n(x) := g(n^2 x)$, $n^2 \|f_n\|^2_{L^2_\gamma([0,a])} \sim \|\sqrt{x} f_n'(x)\|^2_{L^2_\gamma([0,a])} \lesssim n^{-2\gamma}$ with uniform constants in $n \in \mathbb{Z}$. To see this, notice that $f_n'(x) = n^2 g'(n^2 x)$, then

$$n^2 \int_0^a f^2(x) x^\gamma \,\mathrm{d}x \overset{u=n^2x}{=} n^{-2\gamma} \int_0^{n^2} g(u)^2 u^\gamma \,\mathrm{d}u = n^{-2\gamma} \int_0^a g(u)^2 u^\gamma \,\mathrm{d}u,$$

$$\int_0^a (f'(x))^2 x^{\gamma+1} \,\mathrm{d}x \overset{u=n^2x}{=} n^{-2\gamma} \int_0^{n^2} g'(u)^2 u^{\gamma+1} \,\mathrm{d}u = n^{-2\gamma} \int_0^a g'(u)^2 u^{\gamma+1} \,\mathrm{d}u,$$

This completes the proof of Theorem 20 in the case $\gamma \in (-1,0)$.

Case $\gamma \in (0,1)$. The proof is based on combining the previous case with the convenient intertwining property (8). Fix $\gamma \in (0,1)$, and let ψ be a radial function equal to ϕ_γ on $[0, x_\gamma/2)_x$, of class C^∞ on $\mathbb{D}^{\mathrm{int}}$ and bounded away from zero. The map $m_{1/\psi} : \mathcal{A}_\gamma \ni f \mapsto f/\psi \in \mathcal{A}_{-\gamma}$ extends by density into a homeomorphism $\widetilde{H}^{1,\gamma}(\mathbb{D}) \to \widetilde{H}^{1,-\gamma}(\mathbb{D})$. Indeed, a direct calculation gives

$$\begin{aligned}\left\|\frac{f}{\psi}\right\|^2_{\widetilde{H}^{1,-\gamma}} &= \left\|\sqrt{x}\,\partial_\rho \frac{f}{\psi}\right\|^2_{L^2_{-\gamma}} + \left\|\rho^{-1}\partial_\omega \frac{f}{\psi}\right\|^2_{L^2_{-\gamma}} + (1-\gamma)^2 \left\|\frac{f}{\psi}\right\|^2_{L^2_{-\gamma}} \\ &= \left\|\sqrt{x}\frac{x^{-\gamma}}{\psi} N_\psi f\right\|^2_{L^2_\gamma} + \left\|\rho^{-1}\frac{x^{-\gamma}}{\psi}\partial_\omega f\right\|^2_{L^2_\gamma} + (1-\gamma)^2 \left\|\frac{x^{-\gamma}}{\psi} f\right\|^2_{L^2_\gamma}.\end{aligned}$$

The function $\frac{x^{-\gamma}}{\psi} \in C^\infty(\mathbb{D})$ is bounded above and below by positive constants. The upper bound gives the boundedness of the map $m_{1/\psi}$ and the extension by density. The bound from below gives coercivity and by the open mapping theorem, the homeomorphism property. Combining this with the previous case, and noticing that $\tau_\gamma^D = \tau_{-\gamma}^D \circ m_{1/\psi}$ gives the result.

The case $\gamma = 0$. While the singular case $\gamma \in (0,1)$ could make use of the space $\widetilde{H}^{1,-\gamma}$ associated with a "regular" quadratic form, we need to construct the analogue of a "preregularized" quadratic form with good trace estimates in the case $\gamma = 0$. To this end, on a neighborhood of the boundary $\mathbb{S}^1_a = [0,a]_x \times \mathbb{S}^1_\omega$, let us define the norm on $C^\infty(\mathbb{S}^1_a)$

$$\|f\|^2_{\widetilde{H}^{1,0}_{\log}(\mathbb{S}^1_a)} := \|\sqrt{x}\log x\,\partial_x f\|^2_{L^2(\mathbb{S}^1_a)} + \|\log x\,\partial_\omega f\|^2_{L^2(\mathbb{S}^1_a)} + \|\log x f\|^2_{L^2(\mathbb{S}^1_a)}. \tag{58}$$

Writing $f(x,\omega) = \sum_{k\in\mathbb{Z}} f_n(x) e^{in\omega}$ near $x=0$, this norm looks like

$$\|f\|^2_{\widetilde{H}^{1,0}_{\log}(\mathbb{S}^1_a)} = 2\pi \sum_{k\in\mathbb{Z}} \big(\|\sqrt{x}\log x\,\partial_x f_n\|^2_{L^2[0,a]} + (n^2+1)\|\log x f_n\|^2_{L^2[0,a]}\big).$$

Upon defining $H_{(0)}$ as in (29), we now prove the following:

Lemma 24. *The evaluation map* $C^1(\mathbb{S}^1_a) \ni f \mapsto f(0,\omega) \in C^\infty(\mathbb{S}^1)$ *extends to a bounded, surjective trace map* $\widetilde{H}^{1,0}_{\log}(\mathbb{S}^1_a) \to H_{(0)}$.

Proof. To prove continuity, writing $f(x,\omega) = \sum_{k\in\mathbb{Z}} f_n(x) e^{in\omega}$ near $x = 0$, and use (57) on f_n with $\ell_n \in (0, a]$ (assume $a \le 1$ without loss of generality) chosen such that

$$\int_0^{\ell_n} -\log x \, \mathrm{d}x = \frac{a}{\langle n\rangle^2}.$$

By Lemma 23, ℓ_n exists, is unique, and

$$1 = \lim_{n\to\infty} \frac{\log \ell_n}{\log(a/\langle n\rangle^2)} = \lim_{n\to\infty} \frac{\log \ell_n}{-2\log\langle n\rangle}.$$

Further, using l'Hôpital's rule to show that

$$\lim_{t\to 0} \frac{\int_0^t \log^2 x \, \mathrm{d}x}{\log t \int_0^t \log x \, \mathrm{d}x} = 1,$$

we have

$$\frac{\int_0^{\ell_n} \log^2 x \, \mathrm{d}x}{\int_0^{\ell_n} -\log x \, \mathrm{d}x} \sim -\log \ell_n \sim 2\log\langle n\rangle, \qquad \text{as } n \to \infty.$$

On to the surjectivity, it remains to construct a right inverse. To this end, fix $h \in C_c^\infty([0,a), \mathbb{R})$ with $h(0) = 1$, and define $R : C^\infty(\mathbb{S}^1) \to C^\infty(\mathbb{S}^1_a) + \frac{1}{\log x} C^\infty(\mathbb{S}^1_a)$ as

$$R : \sum_{n\in\mathbb{Z}} a_n e^{in\omega} \mapsto \sum_{n\in\mathbb{Z}} a_n e^{in\omega} \underbrace{h(n^2 x)\left(1 + \frac{2\log n}{\log x}\right)}_{f_n(x)} \tag{59}$$

To show that $R : H_{(0)} \to \widetilde{H}^{1,0}_{\log}(\mathbb{S}^1_a)$ is bounded, it suffices to show that for $|n|$ large enough, the quantity

$$\frac{1}{\log n}\left(\|\sqrt{x}\log x\,\partial_x f_n\|^2_{L^2[0,a]} + n^2 \|\log x f_n\|^2_{L^2[0,a]}\right) \tag{60}$$

is uniformly bounded by a constant independent of n. To that end, and upon noting that

$$f_n(x) = h(n^2 x)\frac{\log(n^2 x)}{\log x} \quad \text{and} \quad f_n'(x) = n^2\left(h'(n^2 x)\frac{\log(n^2 x)}{\log x} - 2\log n \frac{h(n^2 x)}{n^2 x \log^2 x}\right),$$

we now compute

$$\begin{aligned}
&\int_0^a (n^2 f_n(x)^2 + x f_n'(x)^2)\log^2 x \, \mathrm{d}x \\
&\overset{u=n^2x}{=} \int_0^{an^2} \left(h^2(u)\log^2 u + u\left(h'(u)\log u - 2\log n \frac{h(u)}{u\log(u/n^2)}\right)^2\right)\mathrm{d}u \\
&= \int_0^{an^2} \left((h^2(u) + uh'(u)^2)\log^2 u - 4\frac{\log n\; h'(u)h(u)\log u}{\log(u/n^2)} + 4\log^2 n \frac{h^2(u)}{u\log^2(u/n^2)}\right)\mathrm{d}u \\
&= I_{n,1} + I_{n,2} + I_{n,3}.
\end{aligned}$$

The term $I_{n,1}$ is asymptotic to a constant, thus $o(\log n)$. On to $I_{n,2}$, we rewrite

$$-4\log n \frac{h'(u)h(u)\log u}{\log(u/n^2)} = 2\frac{h'(u)h(u)\log u}{1-\frac{\log u}{2\log n}},$$

which is dominated by the integrable function $2|h'(u)\,h(u)\log u|$, and converges pointwise to $2h'(u)\,h(u)\log u$. By dominated convergence, $I_{n,2}$ is also asymptotically constant, thus $o(\log n)$. On to $I_{n,3}$, we write

$$\begin{aligned}
I_{n,3} &= 4\log^2 n \int_0^{n^2 a} \frac{h^2(u)}{u\log^2(u/n^2)}\,\mathrm{d}u \\
&\overset{u=n^2x}{=} 4\log^2 n \int_0^{a} \frac{h^2(n^2x)}{x\log^2 x}\,\mathrm{d}x \\
&= 4\log^2 n\left[\int_0^a \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{h^2(n^2x)}{-\log x}\right)\mathrm{d}x + 2\int_0^a \frac{n^2h'(n^2x)h(n^2x)}{\log x}\,\mathrm{d}x\right] \\
&\overset{u=n^2x}{=} 8\log^2 n \int_0^{n^2 a} \frac{h'(u)h(u)}{\log(u/n^2)}\,\mathrm{d}u \\
&= -4\log n \int_0^{n^2 a} \frac{h'(u)h(u)}{1-\frac{\log u}{2\log n}}\,\mathrm{d}u.
\end{aligned}$$

Similarly to the analysis of $I_{n,2}$, this is asymptotic to $-4\log n\int_0^a h'(u)h(u)\,\mathrm{d}u = 2\log n$. As a conclusion, (60) is uniformly bounded as claimed, and the map R defined in (59) is bounded from $H_{(0)}$ to $\widetilde{H}^{1,0}_{\log}(\mathbb{S}^1_a)$. Lemma 24 is proved. □

On to the proof of Theorem 20, let ψ be a radial function equal to ϕ_0 on $[0, x_0/2)_x$, of class C^∞ on $\mathbb{D}^{\text{int}}$ and bounded away from zero. One may view τ_0^D as the composition of

$$\log x\, C^\infty(\mathbb{D}) + C^\infty(\mathbb{D}) \ni f \mapsto \frac{f}{\psi} \in C^\infty + \frac{1}{\log x}C^\infty$$

(which extends to an homeomorphism $\widetilde{H}^{1,0}(\mathbb{D}) \to \widetilde{H}^{1,0}_{\log}(\mathbb{D})$), with the restriction map from Lemma 24.

The proof of Theorem 20 is complete. □

4.3. *Proof of Theorem 5.* The following lemma gathers a few important preliminary facts.

Lemma 25. (1) *With the extended Dirichlet trace τ_D^γ defined in Theorem 20, we have*

$$\widetilde{H}^{1,\gamma}_0(\mathbb{D}) = \widetilde{H}^{1,\gamma}_D = \widetilde{H}^{1,\gamma}(\mathbb{D}) \cap \ker \tau_D^\gamma.$$

(2) *For every $f \in W^2_\gamma$, and $g \in \widetilde{H}^{1,\gamma}_0(\mathbb{D})$, we have*

$$(\mathcal{L}_\gamma f, g)_{L^2_\gamma} = (f, g)_{\widetilde{H}^{1,\gamma}(\mathbb{D})}. \tag{61}$$

(3) *The operator $\mathcal{L}_\gamma : \widetilde{H}^{1,\gamma}(\mathbb{D}) \to (\widetilde{H}^{1,\gamma}_0(\mathbb{D}))'$ is bounded.*

Proof of Lemma 25. Proof of (1). We first show that $\widetilde{H}_0^{1,\gamma}(\mathbb{D}) = \widetilde{H}_D^{1,\gamma}(\mathbb{D})$. We use Lemma 2, namely that the Dirichlet domain $\widetilde{H}_D^{2,\gamma}$ coincides with the domain of the Friedrichs extension.

By the construction of the Friedrichs extension, each element in the domain is the limit of functions in $C_c^\infty(\mathbb{D})$ with respect to the norm $f \mapsto ((\mathcal{L}_\gamma f, f)_{L^2_\gamma})^{1/2}$, which for functions in $C_c^\infty(\mathbb{D})$ coincides with the $\widetilde{H}^{1,\gamma}$ norm. It follows that the Dirichlet domain $\widetilde{H}_D^{2,\gamma}(\mathbb{D})$ is contained in $\widetilde{H}_0^{1,\gamma}(\mathbb{D})$. On the other hand, since $C_c^\infty(\mathbb{D}) \subset \widetilde{H}_D^{2,\gamma}(\mathbb{D})$, taking closures yields that $\widetilde{H}_0^{1,\gamma}(\mathbb{D})$ is precisely the closure of $\widetilde{H}_D^{2,\gamma}(\mathbb{D})$ with respect to the $\widetilde{H}^{1,\gamma}$ norm. However, since

$$\|f\|^2_{\widetilde{H}_D^{s,\gamma}} = \sum_{n,k} (n+1+|\gamma|)^{2s} |a_{n,k}|^2 \quad \text{if } f = \sum_{n,k} a_{n,k} e_{n,k}$$

where $e_{n,k}$ are the L^2_γ-normalized multiples of $G_{n,k}^\gamma$ if $\gamma \ge 0$ and $x^{-\gamma} G_{n,k}^{-\gamma}$ if $\gamma < 0$, and

$$\|f\|^2_{\widetilde{H}^{1,\gamma}} = (\mathcal{L}_\gamma f, f)_{L^2_\gamma} = \sum_{n,k} (n+1+|\gamma|)^2 |a_{n,k}|^2 = \|f\|^2_{\widetilde{H}_D^{1,\gamma}},$$

it follows that the closure of $\widetilde{H}_D^{2,\gamma}(\mathbb{D})$ with respect to the $\widetilde{H}^{1,\gamma}$ norm is precisely $\widetilde{H}_D^{1,\gamma}(\mathbb{D})$, as desired.

Next, the inclusion $\widetilde{H}_0^{1,\gamma}(\mathbb{D}) \subset \widetilde{H}^{1,\gamma}(\mathbb{D}) \cap \ker \tau_D^\gamma$ follows from the continuity of τ_D^γ on $\widetilde{H}^{1,\gamma}(\mathbb{D})$, as well as the fact that each element of $\widetilde{H}_0^{1,\gamma}(\mathbb{D})$ is a limit of functions in $C_c^\infty(\mathbb{D})$, all of which have trace zero.

Finally, to show that $\widetilde{H}^{1,\gamma}(\mathbb{D}) \cap \ker \tau_D^\gamma \subset \widetilde{H}_0^{1,\gamma}(\mathbb{D})$, pick $f \in \widetilde{H}^{1,\gamma}(\mathbb{D}) \cap \ker \tau_D^\gamma$ and let $f_n \in \mathcal{A}_\gamma$ a sequence approximating f in $\widetilde{H}^{1,\gamma}$. By continuity of τ_D^γ, we must have that $\tau_D^\gamma f_n \to \tau_D^\gamma f = 0$ in $H_{(\gamma)}$. With R a right-inverse for τ_D^γ as in Theorem 20, then $R\tau_D^\gamma f_n \to 0$ in $\widetilde{H}^{1,\gamma}$ and hence $g_n := f_n - R\tau_D^\gamma f_n$ is a sequence of elements in $\mathcal{A}_{\gamma,D}$, converging to f in $\widetilde{H}^{1,\gamma}$. Finally, by Corollary 19, $C_c^\infty(\mathbb{D})$ is $\widetilde{H}^{1,\gamma}$-dense in $\mathcal{A}_{\gamma,D}$, and hence for each n, there is $h_n \in C_c^\infty(\mathbb{D})$ such that $\|g_n - h_n\|_{\widetilde{H}^{1,\gamma}} < \frac{1}{n}$. The sequence h_n converges in $\widetilde{H}^{1,\gamma}$ to f, which in turn belongs to $\widetilde{H}_0^{1,\gamma}$.

Proof of (2). It is enough to show (61) for $f \in W^2_\gamma$ and $g \in \dot{C}^\infty$, since (61) then extends to $\widetilde{H}_0^{1,\gamma}$ by density. For $f \in W^2_\gamma \subset \widetilde{H}^{1,\gamma}$, let f_n be a sequence in $\mathcal{A}_\gamma$ converging to f in $\widetilde{H}^{1,\gamma}$, and fix $g \in \dot{C}^\infty$. We have, for every n,

$$(f_n, g)_{\widetilde{H}^{1,\gamma}} = (\mathcal{L}_\gamma f_n, g)_{L^2_\gamma} = \langle \iota_\gamma \mathcal{L}_\gamma f_n, g\rangle = \langle {}^t\mathcal{L}_\gamma \iota_\gamma f_n, g\rangle.$$

Since $f_n \to f$ in $\widetilde{H}^{1,\gamma}$, hence in L^2_γ, $\iota_\gamma f_n \to f$ in $C^{-\infty}$, and by sequential continuity, ${}^t\mathcal{L}_\gamma \iota_\gamma f_n \to {}^t\mathcal{L}_\gamma \iota_\gamma f$ in $C^{-\infty}$. Hence we may send $n \to \infty$ in the above equality to obtain

$$(f, g)_{\widetilde{H}^{1,\gamma}} = \langle {}^t\mathcal{L}_\gamma \iota_\gamma f, g\rangle \quad \text{for all } g \in \dot{C}^\infty(\mathbb{D}).$$

Finally, the assumption $\mathcal{L}_\gamma f \in L^2_\gamma$ gives that the right-side equals $(\mathcal{L}_\gamma f, g)_{L^2_\gamma}$, and (61) follows for $f \in W^2_\gamma$ and $g \in \dot{C}^\infty$.

Proof of (3). By (61), we have the identity

$$|(\mathcal{L}_\gamma f, g)_{L^2_\gamma}| = |(f, g)_{\widetilde{H}^{1,\gamma}(\mathbb{D})}| \le \|f\|_{\widetilde{H}^{1,\gamma}} \|g\|_{\widetilde{H}^{1,\gamma}}, \quad f \in \mathcal{A}_\gamma, \quad g \in \widetilde{H}_0^{1,\gamma}(\mathbb{D}).$$

This implies that $\mathcal{L}_\gamma : \mathcal{A}_\gamma \to (\widetilde{H}_0^{1,\gamma})'$ is bounded, and that

$$\|\mathcal{L}_\gamma f\|_{(\widetilde{H}_0^{1,\gamma})'} \le \|f\|_{\widetilde{H}^{1,\gamma}} \quad \text{for all } f \in \mathcal{A}_\gamma.$$

Since $\mathcal{A}_\gamma$ is dense in $\widetilde{H}^{1,\gamma}$, the latter inequality allows to extend $\mathcal{L}_\gamma$ to $\widetilde{H}^{1,\gamma}$ as a bounded $(\widetilde{H}_0^{1,\gamma})'$-valued map. □

Proof of Theorem 5. We first show that the Neumann trace extends into a bounded operator $\tau_\gamma^N : W_\gamma^2 \to H'_{(\gamma)}$. Following ideas in [Helffer 2013, §4.4.4], we combine Theorem 20 with the second Green's identity. By virtue of Theorem 20, let $R : H_{(\gamma)} \to \widetilde{H}^{1,\gamma}$ be a bounded right-inverse for τ_γ^D arising from a continuous operator $R : C^\infty(\mathbb{S}^1) \to \mathcal{A}_\gamma$. For $f \in W_\gamma^2$, define the map

$$\psi_f(h) := (\mathcal{L}_\gamma f, Rh)_{L^2_\gamma} - (f, Rh)_{\widetilde{H}^{1,\gamma}}, \quad h \in C^\infty(\mathbb{S}^1). \tag{62}$$

It is easily seen to satisfy an estimate of the form

$$|\psi_f(h)| \le (\|\mathcal{L}_\gamma f\|_{L^2_\gamma} + \|f\|_{\widetilde{H}^{1,\gamma}})\|Rh\|_{\widetilde{H}^{1,\gamma}} \le C(\|\mathcal{L}_\gamma f\|_{L^2_\gamma} + \|f\|_{\widetilde{H}^{1,\gamma}})\|h\|_{H_{(\gamma)}},$$

with C the operator norm of R. By Riesz representation, this defines a unique element $\tau_\gamma^N f \in H'_{(\gamma)}$, which further satisfies the estimate $\|\tau_\gamma^N f\|_{H'_{(\gamma)}} \le C(\|\mathcal{L}_\gamma f\|_{L^2_\gamma} + \|f\|_{\widetilde{H}^{1,\gamma}})$. Moreover, for $f \in \mathcal{A}_\gamma$ and $h \in C^\infty(\mathbb{S}^1)$ such that $Rh \in \mathcal{A}_\gamma$, the first Green's identity (25) gives

$$\psi_f(h) = (\tau_\gamma^N f, \tau_\gamma^D Rh)_{L^2(\mathbb{S}^1)} = (\tau_\gamma^N f, h)_{L^2(\mathbb{S}^1)} = \langle \tau_\gamma^N f, h\rangle_{H'_{(\gamma)}, H_{(\gamma)}},$$

hence the definition of τ_γ^N on W_γ^2 extends the original definition (20) on $\mathcal{A}_\gamma$.

On to extending Green's first identity to (30), given $g \in \widetilde{H}^{1,\gamma}$, we decompose $g = R\tau_\gamma^D g + g_0$, with $g_0 \in \widetilde{H}_0^{1,\gamma}$, and write, for $f \in W_\gamma^2$,

$$\begin{aligned}(\mathcal{L}_\gamma f, g)_{L^2_\gamma} - (f, g)_{\widetilde{H}^{1,\gamma}} &= (\mathcal{L}_\gamma f, R\tau_\gamma^D g)_{L^2_\gamma} - (f, R\tau_\gamma^D g)_{\widetilde{H}^{1,\gamma}} + (\mathcal{L}_\gamma f, g_0)_{L^2_\gamma} - (f, g_0)_{\widetilde{H}^{1,\gamma}} \\ &\overset{(62)}{=} \langle \tau_\gamma^N f, \tau_\gamma^D g\rangle_{H'_{(\gamma)}, H_{(\gamma)}} + (\mathcal{L}_\gamma f, g_0)_{L^2_\gamma} - (f, g_0)_{\widetilde{H}^{1,\gamma}},\end{aligned}$$

and the last two terms cancel out by virtue of (61). Theorem 5 is proved. □

Remark 26. Similarly to [Behrndt et al. 2020, Remark 8.2.5], the extension of τ_γ^N could be made to the larger space $\widetilde{W}_\gamma^2 = \{u \in \widetilde{H}^{1,\gamma}(\mathbb{D}),\ \mathcal{L}_\gamma u \in (\widetilde{H}^{1,\gamma}(\mathbb{D}))'\}$ into a bounded operator $\tau_\gamma^N : \widetilde{W}_\gamma^2 \to H'_{(\gamma)}$, and one would have a slightly more general first Green's identity

$$\langle \mathcal{L}_\gamma f, g\rangle_{(\widetilde{H}^{1,\gamma})', \widetilde{H}^{1,\gamma}} = (f, g)_{\widetilde{H}^{1,\gamma}} + \langle \tau_\gamma^N f, \tau_\gamma^D g\rangle_{H'_{(\gamma)}, H_{(\gamma)}}, \quad f \in \widetilde{W}_\gamma^2, \quad g \in \widetilde{H}^{1,\gamma}.$$

Upon skew-symmetrizing, Green's second identity generalizes to $f, g \in \widetilde{W}_\gamma^2$ in the obvious way.

4.4. ***DN map — Proof of Theorem 6.*** We end this section with the construction of the Dirichlet-to-Neumann map, i.e., the proof of Theorem 6.

Proof of Theorem 6. The construction of the DN map is done as follows. Consider $R_\gamma^D : H_{(\gamma)} \to \widetilde{H}^{1,\gamma}(\mathbb{D})$ a continuous right inverse to τ_γ^D as in Theorem 20. For $f \in H_{(\gamma)}$, the construction of a unique solution $u_f \in W_\gamma^2 \cap \ker(\mathcal{L}_\gamma - \lambda)$ to

$$(\mathcal{L}_\gamma - \lambda)u = 0 \quad (\mathbb{D}), \quad u|_{\mathbb{S}^1} = f \tag{63}$$

is done as follows: Setting $\tilde{f} = R_\gamma^D f \in \widetilde{H}^{1,\gamma}(\mathbb{D})$, write $u_f = w + \tilde{f}$ for some unknown w, which in turn should solve

$$(\mathcal{L}_\gamma - \lambda)w = -(\mathcal{L}_\gamma - \lambda)\tilde{f} \quad (\mathbb{D}), \quad w|_{\mathbb{S}^1} = 0, \tag{64}$$

where the right-hand side belongs to $(\widetilde{H}_0^{1,\gamma}(\mathbb{D}))'$ by Lemma 25(3), a space which coincides with $(\widetilde{H}_D^{1,\gamma}(\mathbb{D}))' = \widetilde{H}_D^{-1,\gamma}(\mathbb{D})$ by Lemma 25(1). By setting up a weak formulation and invoking Riesz representation theorem on $\widetilde{H}_D^{1,\gamma}$, or simply using $(\mathcal{L}_{\gamma,D}-\lambda)^{-1}$ which is well-understood, this gives a unique solution to (64) in $\widetilde{H}_D^{1,\gamma}$, given by

$$w := -(\mathcal{L}_{\gamma,D}-\lambda)^{-1}(\mathcal{L}_\gamma-\lambda)\tilde{f}.$$

Then $u_f = w+\tilde{f}$ is a solution to (63) (which can also be proved to be unique since $\mathcal{L}_{\gamma,D}-\lambda$ is injective). Originally, it belongs to $\widetilde{H}^{1,\gamma}$, so in particular in L^2_γ, so (63) also implies that $\mathcal{L}_\gamma u_f \in L^2_\gamma$, and hence $u_f \in W^2_\gamma \cap \ker(\mathcal{L}_\gamma-\lambda)$.

More succinctly, the DN map is then defined by

$$\Lambda_\gamma(\lambda) f := \tau_\gamma^N u_f = \tau_\gamma^N (\mathrm{id} - (\mathcal{L}_{\gamma,D}-\lambda)^{-1}(\mathcal{L}_\gamma-\lambda)) R_\gamma^D f,$$

which boundedly lands into $H'_{(\gamma)}$ by Theorem 5. The uniqueness of the solution to (63) makes it independent of the choice of right-inverse for τ_γ^D. □

5. Proof of Theorem 8

The idea is to follow the template of [Behrndt et al. 2020, Theorem 8.4.1, p. 601] to construct boundary triples for $\mathcal{L}_{\gamma,\max}$ when $\gamma\in(-1,1)$. In proving Theorem 8, we first prove in Section 5.1 how to extend the Neumann trace on the Dirichlet Sobolev scale (17), this is formulated as Proposition 27 below, and makes crucial use of facts about generalized Zernike polynomials. In Section 5.2, we then show how to extend the Dirichlet trace, a result formulated in Proposition 30. Finally, we complete the proof of Theorem 8 in Section 5.3, first proving Lemma 7 on the decompositions of the maximal domain, then using these decompositions to extend Green's second identity to (34).

5.1. *Extension of the Neumann trace.*

Proposition 27. *For $\gamma\in(-1,1)$ and any $s>1+|\gamma|$, the map τ_γ^N defined in* (20) *extends to a bounded, surjective map*

$$\tau_\gamma^N : \widetilde{H}_D^{s,\gamma}(\mathbb{D}) \to H^{s-1-|\gamma|}(\mathbb{S}^1), \tag{65}$$

with a right inverse arising as the extension to $H^{s-1-|\gamma|}(\mathbb{S}^1)$ of a continuous operator $C^\infty(\mathbb{S}^1)\to C^\infty(\mathbb{D})$. In particular, for $s=2$, where $\widetilde{H}_D^{2,\gamma}(\mathbb{D}) = \mathrm{dom}(\mathcal{L}_{\gamma,D})$,

$$\tau_\gamma^N : \mathrm{dom}(\mathcal{L}_{\gamma,D}) \to H^{1-|\gamma|}(\mathbb{S}^1) \tag{66}$$

is a bounded and surjective map, and we have the further characterization

$$\mathrm{dom}(\mathcal{L}_{\gamma,\min}) = \widetilde{H}_D^{2,\gamma} \cap \ker \tau_\gamma^N. \tag{67}$$

Proof. Case $0\le\gamma<1$. With $\{G_{n,k}^\gamma\}_{n,k}$ defined in (16), the main two fundamental properties needed are

$$\tau_\gamma^N(G_{n,k}^\gamma) = c_\gamma G_{n,k}^\gamma|_{\mathbb{S}^1} = c_\gamma e^{i(n-2k)\beta}, \tag{68}$$

$$\|G_{n,k}^\gamma\|^2_{L^2_\gamma} = \frac{\pi}{n+\gamma+1}\frac{(n-k)!\,\gamma!\,k!\,\gamma!}{(k+\gamma)!\,(n-k+\gamma)!} =: (n_{n,k}^\gamma)^2, \tag{69}$$

where we write for short $x! := \Gamma(x+1)$, and where $c_\gamma = 2\gamma$ for $\gamma \in (0,1)$, and $c_0 = -2$. The first equality uses [Wünsche 2005, equation (2.10)]. Hence, for $u \in \widetilde{H}_D^{s,\gamma}$, of the form

$$u = \sum_{n=0}^{\infty} \sum_{k=0}^{n} u_{n,k} \frac{G_{n,k}^{\gamma}}{n_{n,k}^{\gamma}}$$

with $\|u\|^2_{\widetilde{H}_D^{s,\gamma}} = \sum_{n=0}^{\infty} \sum_{k=0}^{n} (n+1+\gamma)^{2s} |u_{n,k}|^2 < \infty$, we have

$$\tau_\gamma^N u = c_\gamma \sum_{n,k} \frac{u_{n,k}}{n_{n,k}^{\gamma}} e^{i(n-2k)\beta} = c_\gamma \sum_{m \in \mathbb{Z}} e^{im\beta} [\tau_\gamma^N u]_m, \quad [\tau_\gamma^N u]_m := \sum_{n-2k=m} \frac{u_{n,k}}{n_{n,k}^{\gamma}}.$$

Then

$$\begin{aligned} \|\tau_\gamma^N u\|^2_{H^{s-\gamma-1}} &= 2\pi c_\gamma^2 \sum_{m \in \mathbb{Z}} \langle m \rangle^{2s-2\gamma-2} |[\tau_\gamma^N u]_m|^2 \\ &\le 2\pi c_\gamma^2 \sum_{m \in \mathbb{Z}} \langle m \rangle^{2s-2\gamma-2} \Bigg(\sum_{n-2k=m} (n+1+\gamma)^{2s} |u_{n,k}|^2 \Bigg) \Bigg(\sum_{n-2k=m} \frac{(n+1+\gamma)^{-2s}}{(n_{n,k}^{\gamma})^2} \Bigg) \end{aligned} \tag{70}$$

and the proof is complete if we can show that

$$C_{m;s,\gamma} := \langle m \rangle^{2s-2\gamma-2} \sum_{n-2k=m} \frac{(n+1+\gamma)^{-2s}}{(n_{n,k}^{\gamma})^2}$$

is uniformly bounded for all $m \in \mathbb{Z}$. Note that $C_{m;s,\gamma} = C_{-m;s,\gamma}$ so it enough to check it for $m \ge 0$, for which we have

$$\begin{aligned} C_{m;s,\gamma} &= \langle m \rangle^{2s-2\gamma-2} \sum_{\ell \ge 0} \frac{(m+2\ell+1+\gamma)^{-2s}}{(n_{m+2\ell,\ell}^{\gamma})^2} \\ &= \frac{\langle m \rangle^{2s-2\gamma-2}}{\pi (\gamma!)^2} \sum_{\ell \ge 0} (m+2\ell+1+\gamma)^{1-2s} \frac{(m+\ell+\gamma)!}{(m+\ell)!} \frac{(\ell+\gamma)!}{\ell!}. \end{aligned}$$

To this end, we state the following estimate, which is relegated to Section A.3:

Lemma 28. *There exist $C', C'' > 0$ such that $C' \le C_{m;s,\gamma} \le C''$ for all $m \ge 0$.*

Using Lemma 28, we thus see that

$$\|\tau_\gamma^N u\|^2_{H^{s-\gamma-1}} \le 2\pi c_\gamma^2 \sum_{m \in \mathbb{Z}} C_{m;s,\gamma} \Bigg(\sum_{n-2k=m} (n+1+\gamma)^{2s} |u_{n,k}|^2 \Bigg) \le 2\pi c_\gamma^2 (\sup_{m \in \mathbb{Z}} C_{m;s,\gamma}) \|u\|^2_{\widetilde{H}_D^{s,\gamma}(\mathbb{D})},$$

thus establishing the forward trace estimate.

To establish surjectivity, it suffices to construct a bounded right inverse $R : H^{s-\gamma-1}(\mathbb{S}^1) \to \widetilde{H}_D^{s,\gamma}(\mathbb{D})$. We construct the following map:

$$R\Bigg(\sum_{m \in \mathbb{Z}} a_m e^{im\omega} \Bigg) = \frac{1}{c_\gamma} \sum_{m \in \mathbb{Z}} \frac{a_m}{1+|m|} \sum_{n-2k=m, n \le 3|m|} G_{n,k}^{\gamma}. \tag{71}$$

Note that the set $\{(n,k) : n-2k = m,\ 0\le k\le n\}$ can be parametrized by

$$n = |m| + 2\ell, \quad k = \frac{|m|-m}{2} + \ell = \max(0,-m) + \ell, \quad \ell \ge 0,$$

so the inner sum is over $1+|m|$ elements. Specifically, R sends $e^{im\omega}$ to the average over the $|m|+1$ weighted Zernike polynomials of lowest degree whose trace equals $e^{im\omega}$. In particular, that R is a right-inverse directly follows from property (68).

Moreover we have

$$\begin{aligned}\left\| R\left(\sum_{m\in\mathbb{Z}} a_m e^{im\omega}\right)\right\|^2_{\widetilde{H}^{s,\gamma}_D(\mathbb{D})} &= \frac{1}{c_\gamma^2}\sum_{m\in\mathbb{Z}} \frac{|a_m|^2}{(1+|m|^2)} \sum_{n-2k=m,\, n\le 3|m|} (n+1+\gamma)^{2s}(n^\gamma_{n,k})^2 \\ &= \frac{1}{c_\gamma^2}\sum_{m\in\mathbb{Z}} \langle m\rangle^{2s-2\gamma-2}|a_m|^2 \frac{\langle m\rangle^2}{(1+|m|)^2} C'_m \\ &\le \left(\sup_{m\in\mathbb{Z}} \frac{\langle m\rangle^2 C'_m}{c_\gamma^2(1+|m|)^2}\right)\left\|\sum_{m\in\mathbb{Z}} a_m e^{im\omega}\right\|^2_{H^{s-1-\gamma}},\end{aligned}$$

where we have defined

$$C'_m := \langle m\rangle^{-2s+2\gamma} \sum_{n-2k=m,\, n\le 3|m|} (n+1+\gamma)^{2s}(n^\gamma_{n,k})^2.$$

Thus, that $R : H^{s-1-|\gamma|}(\mathbb{S}^1) \to \widetilde{H}^{s,\gamma}_D(\mathbb{D})$ is bounded will follow upon showing that C'_m is uniformly bounded in m. This is implied by the asymptotics

$$\sum_{n-2k=m,\, n\le 3|m|} (n+1+\gamma)^{2s}(n^\gamma_{n,k})^2 = O(\langle m\rangle^{2s-2\gamma}), \quad \text{as } |m|\to\infty, \tag{72}$$

which we prove now. As before, $C'_{-m} = C'_m$, so it suffices to assume $m\ge 0$, in which case we parametrize $n = m+2\ell$ and $k=\ell$, with $0\le \ell\le m$. Similarly as before we assume m is sufficiently large, say $m>\gamma$. Then the sum then becomes

$$\sum_{n-2k=m,\, n\le 3|m|} (n+1+\gamma)^{2s}(n^\gamma_{n,k})^2 \overset{(69)}{=} \frac{\pi}{(\gamma!)^2}\sum_{\ell=0}^{m} (m+2\ell+1+\gamma)^{2s-1}\frac{(m+\ell)!}{(m+\ell+\gamma)!}\frac{\ell!}{(\ell+\gamma)!}.$$

Using the asymptotic

$$\lim_{x\to\infty} \frac{x!}{(x+\gamma)!\,(x+1)^{-\gamma}} = 1,$$

and using that $m+2\ell+1+\gamma \le 4(m+1)$ and $2s-1\ge 1+2\gamma\ge 1$, the above sum is bounded above by some multiple of

$$\sum_{\ell=0}^{m} (m+2\ell+1+\gamma)^{2s-1}(m+\ell+1)^{-\gamma}(\ell+1)^{-\gamma} \lesssim (m+1)^{2s-\gamma-1}\sum_{\ell=0}^{m}(\ell+1)^{-\gamma}.$$

Finally, the function $x \mapsto (x+1)^{\gamma}$ is decreasing, so we can estimate

$$\sum_{\ell=0}^{m} (\ell+1)^{-\gamma} = 1 + \sum_{\ell=1}^{m} (\ell+1)^{-\gamma} \le 1 + \int_0^m (x+1)^{-\gamma}\, \mathrm{d}x = 1 + \frac{(m+1)^{1-\gamma} - 1}{1-\gamma}.$$

It follows that

$$\sum_{n-2k=m, n\le 3|m|} (n+1+\gamma)^{2s} (n_{n,k}^{\gamma})^2 \lesssim (m+1)^{2s-\gamma-1} \sum_{\ell=0}^{m} (\ell+1)^{\gamma} \lesssim (m+1)^{2s-2\gamma},$$

i.e., (72) follows.

Case $\gamma < 0$. Notice that the map $m_{x^\gamma} : x^{-\gamma} C^{\infty}(\mathbb{D}) \ni f \mapsto x^{\gamma} f \in C^{\infty}(\mathbb{D})$ extends into an isometry $\widetilde{H}_D^{s,\gamma} \mapsto \widetilde{H}_D^{s,-\gamma}$, and that we have $\tau_\gamma^N = \tau_{-\gamma}^N \circ m_{x^\gamma}$. The result for $\gamma \in (-1, 0)$ follows.

Finally, we prove the characterization (67) of the minimal domain. The second Green's identity (26) extends, for $f \in \widetilde{H}_D^{2,\gamma}(\mathbb{D})$, to the equation

$$(\mathcal{L}_\gamma f, g)_{L^2_\gamma} - (f, \mathcal{L}_\gamma g)_{L^2_\gamma} = (\tau_\gamma^N f, \tau_\gamma^D g)_{L^2(\mathbb{S}^1)}, \quad g \in \mathcal{A}_\gamma. \tag{73}$$

Since $\widetilde{H}_D^{2,\gamma}(\mathbb{D})$ is the domain of the Dirichlet realization $\mathcal{L}_{\gamma,D}$, it follows the minimal domain is contained in $\widetilde{H}_D^{2,\gamma}(\mathbb{D})$. As such, let $f \in \mathrm{dom}\,(\mathcal{L}_{\gamma,\min})$; then $f \in \widetilde{H}_D^{2,\gamma}(\mathbb{D})$. Moreover, since $\mathcal{L}_{\gamma,\min} = (\mathcal{L}_{\gamma,\max})^*$, it follows that for all $g \in \mathcal{L}_{\gamma,\max}$ we have

$$(\mathcal{L}_\gamma f, g)_{L^2_\gamma} - (f, \mathcal{L}_\gamma g)_{L^2_\gamma} = 0.$$

In particular this holds for all $g \in \mathcal{A}_\gamma$, in which case this combined with (73) gives

$$(\tau_\gamma^N f, \tau_\gamma^D g)_{L^2(\mathbb{S}^1)} = 0 \text{ for all } g \in \mathcal{A}_\gamma,$$

i.e., $\tau_\gamma^N f$ is orthogonal (in $L^2(\mathbb{S}^1)$) to $\tau_\gamma^D(\mathcal{A}_\gamma) = C^\infty(\mathbb{S}^1)$. By density of the latter, this forces $\tau_\gamma^N f \equiv 0$. It follows that $\mathrm{dom}\,(\mathcal{L}_{\gamma,\min}) \subset \widetilde{H}_D^{2,\gamma}(\mathbb{D}) \cap \ker \tau_\gamma^N$.

To show the other inclusion, we will use density arguments. First considering the case $\gamma \in [0, 1)$, first notice that by Theorem 14(ii), we have $xC^\infty(\mathbb{D}) \subset \overline{C_c^\infty(\mathbb{D})}^{n_\gamma} = \mathrm{dom}(\mathcal{L}_{\gamma,\min})$, and upon taking n_γ-closures, $\overline{xC^\infty(\mathbb{D})}^{n_\gamma} \subset \mathrm{dom}(\mathcal{L}_{\gamma,\min})$. Then $\widetilde{H}_D^{2,\gamma}(\mathbb{D}) \cap \ker \tau_\gamma^N \subset \mathrm{dom}(\mathcal{L}_{\gamma,\min})$ will hold provided that

$$\widetilde{H}_D^{2,\gamma}(\mathbb{D}) \cap \ker \tau_\gamma^N = \overline{xC^\infty(\mathbb{D})}^{n_\gamma}. \tag{74}$$

The inclusion $\supset$ in (74) is clear since the space on the left is n_γ-closed and contains $xC^\infty(\mathbb{D})$. On to the inclusion $\subset$, for any $f \in \widetilde{H}_D^{2,\gamma}(\mathbb{D})$, there is a sequence $f_n \in C^\infty(\mathbb{D})$ with $f_n \to f$ in $\widetilde{H}_D^{2,\gamma}(\mathbb{D})$. If f also satisfies $\tau_\gamma^N f \equiv 0$, then $\tau_\gamma^N f_n \to 0$ in $H^{1-|\gamma|}(\mathbb{S}^1)$. With R the right inverse for τ_γ^N defined in (71), $R\tau_\gamma^N f_n \in C^\infty(\mathbb{S}^1)$ and $R\tau_\gamma^N f_n \to 0$ in $\widetilde{H}_D^{2,\gamma}(\mathbb{D})$. Considering

$$g_n = f_n - R\tau_\gamma^N f_n \in C^\infty(\mathbb{D}),$$

since $\tau_\gamma^N g_n = \tau_\gamma^N f_n - \tau_\gamma^N R \tau_\gamma^N f_n = 0$, g_n in fact belongs to $xC^\infty(\mathbb{D})$ and converges to f in $\widetilde{H}_D^{2,\gamma}(\mathbb{D})$. This gives (74).

For $\gamma \in (-1, 0)$, we follow a similar strategy to show that $\widetilde{H}_D^{2,\gamma}(\mathbb{D}) \cap \ker \tau_\gamma^N = \overline{x^{1-\gamma} C^\infty(\mathbb{D})}^{n_\gamma}$, and noting that $C_c^\infty(\mathbb{D})$ is n_γ-dense in $x^{1-\gamma} C^\infty(\mathbb{D})$ for all $\gamma \in (-1, 0)$ (in fact for all $\gamma < 1$). ☐

From this, we can also show:

Lemma 29. *For $\gamma \in (-1,1)$, we have* $\mathrm{dom}(\mathcal{L}_{\gamma,\max}) = \overline{\mathcal{A}_\gamma}^{n_\gamma}$.

Proof. Consider the operator $\mathcal{L}_{\gamma,\mathcal{A}_\gamma}$ whose domain is $\mathcal{A}_\gamma$. It suffices to show that

$$\mathrm{dom}\,((\mathcal{L}_{\gamma,\mathcal{A}_\gamma})^*) \subset \mathrm{dom}\,(\mathcal{L}_{\gamma,\min}),$$

since by taking adjoints we would have

$$\mathrm{dom}\,(\mathcal{L}_{\gamma,\max}) = \mathrm{dom}\,((\mathcal{L}_{\gamma,\min})^*) \subset \mathrm{dom}\,((\mathcal{L}_{\gamma,\mathcal{A}_\gamma})^{**}) = \overline{\mathcal{A}_\gamma}^{n_\gamma}.$$

Note that

$$\widetilde{H}^{2,\gamma}_D(\mathbb{D}) \subset \overline{\mathrm{dom}\,(\mathcal{L}_{\gamma,\mathcal{A}_\gamma})}^{n_\gamma}$$

by taking closures on either $C^\infty(\mathbb{D})$ or $x^{-\gamma}C^\infty(\mathbb{D})$, and hence taking adjoints gives

$$\mathrm{dom}\,((\mathcal{L}_{\gamma,\mathcal{A}_\gamma})^*) \subset \widetilde{H}^{2,\gamma}_D(\mathbb{D}).$$

As such, suppose $f \in \mathrm{dom}\,((\mathcal{L}_{\gamma,\mathcal{A}_\gamma})^*)$; then $f \in \widetilde{H}^{2,\gamma}_D(\mathbb{D})$. For all $g \in \mathcal{A}_\gamma$, equation (73) gives that

$$(\mathcal{L}_\gamma f, g)_{L^2_\gamma} - (f, \mathcal{L}_\gamma g)_{L^2_\gamma} = (\tau^N_\gamma f, \tau^D_\gamma g)_{L^2(\mathbb{S}^1)}.$$

But we also have $(f, \mathcal{L}_\gamma g) = (f, (\mathcal{L}_{\gamma,\mathcal{A}_\gamma})g) = ((\mathcal{L}_{\gamma,\mathcal{A}_\gamma})^* f, g) = (\mathcal{L}_\gamma f, g)$, so we have

$$(\tau^N_\gamma f, \tau^D_\gamma g)_{L^2(\mathbb{S}^1)} = 0 \quad \text{for all } g \in \mathcal{A}_\gamma.$$

Similar reasoning as above yields $\tau^N_\gamma f \equiv 0$, except now that the characterization (67) has been proven, we can conclude that $f \in \mathrm{dom}\,(\mathcal{L}_{\gamma,\min})$, as desired. □

5.2. *Extension of the Dirichlet trace.*

Proposition 30. *For $\gamma \in (-1,1)$, the restriction to W^2_γ of the Dirichlet trace τ^D_γ defined in Theorem 20 extends to a bounded, surjective map*

$$\tilde{\tau}^D_\gamma : \mathrm{dom}(\mathcal{L}_{\gamma,\max}) \to H^{-1+|\gamma|}(\mathbb{S}^1).$$

Moreover, we have, for any $f \in \mathrm{dom}(\mathcal{L}_{\gamma,\max})$ and $g \in \widetilde{H}^{2,\gamma}_D(\mathbb{D})$,

$$(\mathcal{L}_\gamma f, g)_{L^2_\gamma} - (f, \mathcal{L}_\gamma g)_{L^2_\gamma} = -\langle \tilde{\tau}^D_\gamma f, \tau^N_\gamma g\rangle_{H^{-1+|\gamma|}, H^{1-|\gamma|}}. \tag{75}$$

Proof of Proposition 30. Let $R : H^{1-|\gamma|}(\mathbb{S}^1) \to \widetilde{H}^{2,\gamma}_D(\mathbb{D})$ a right inverse for the map (66), in particular $\tau^N_\gamma(Rh) = h$ for all $h \in H^{1-|\gamma|}(\mathbb{S}^1)$. For $f \in \mathrm{dom}(\mathcal{L}_{\gamma,\max})$, define the functional ϕ_f on $H^{1-|\gamma|}(\mathbb{S}^1)$ by

$$\phi_f(h) = -(\mathcal{L}_\gamma f, Rh)_{L^2_\gamma} + (f, \mathcal{L}_\gamma Rh)_{L^2_\gamma}, \quad h \in H^{1-|\gamma|}(\mathbb{S}^1). \tag{76}$$

The map ϕ_f obviously satisfies

$$\begin{aligned}|\phi_f(h)| &\le \|\mathcal{L}_\gamma f\|_{L^2_\gamma}\|Rh\|_{L^2_\gamma} + \|f\|_{L^2_\gamma}\|\mathcal{L}_\gamma Rh\|_{L^2_\gamma}\\ &\le (\|f\|_{L^2_\gamma} + \|\mathcal{L}_\gamma f\|_{L^2_\gamma})(\|Rh\|_{L^2_\gamma} + \|\mathcal{L}_\gamma Rh\|_{L^2_\gamma}) \le C(\|f\|_{L^2_\gamma} + \|\mathcal{L}_\gamma f\|_{L^2_\gamma})\|h\|_{H^{1-|\gamma|}(\mathbb{S}^1)},\end{aligned}$$

by boundedness of R. By Riesz representation, this defines a unique element $\tilde{\tau}_\gamma^D f$ in $H^{-1+|\gamma|}(\mathbb{S}^1)$. If $f \in W_\gamma^2$, then (31) implies that $\phi_f(h) = \langle \tau_\gamma^N(Rh), \tau_\gamma^D f\rangle_{H'_{(\gamma)}, H_{(\gamma)}}$ where, since $\tau_\gamma^D f \in H_{(\gamma)} \subset H^{-1+|\gamma|}$, and since $\tau_\gamma^N(Rh) = h \in H^{1-|\gamma|}$, the latter pairing also equals to $\langle \tau_\gamma^D f, h\rangle_{H^{-1+|\gamma|}, H^{1-|\gamma|}}$. In particular, if $f \in W_\gamma^2$, then $\tilde{\tau}_\gamma^D f = \tau_\gamma^D f$ as elements of $H^{-1+|\gamma|}$.

On to proving (75), pick $f \in \mathrm{dom}(\mathcal{L}_{\gamma,\max})$ and $g \in \widetilde{H}_D^{2,\gamma}$. Write $g = R\tau_\gamma^N g + g_{00}$, where $\tau_\gamma^N g_{00} = 0$ hence $g_{00} \in \mathrm{dom}(\mathcal{L}_{\gamma,\min})$. Then, writing $(\cdot,\cdot)$ for the L_γ^2 inner-product for conciseness,

$$\begin{aligned}(\mathcal{L}_\gamma f, g) - (f, \mathcal{L}_\gamma g) &= (\mathcal{L}_\gamma f, R\tau_\gamma^N g) - (f, \mathcal{L}_\gamma R\tau_\gamma^N g) + (\mathcal{L}_\gamma f, g_{00}) - (f, \mathcal{L}_\gamma g_{00}) \\ &\overset{(76)}{=} -\langle \tilde{\tau}_\gamma^D f, \tau_\gamma^N g\rangle_{H^{-1+|\gamma|}, H^{1-|\gamma|}} + (\mathcal{L}_\gamma f, g_{00}) - (f, \mathcal{L}_\gamma g_{00})\end{aligned}$$

and the last two terms cancel out since $\mathcal{L}_{\gamma,\max} = (\mathcal{L}_{\gamma,\min})^*$.

To show surjectivity of $\tilde{\tau}_\gamma^D$, let us now construct a right inverse. As a combination of Proposition 27 and (18), the operator $\Upsilon := \tau_\gamma^N \mathcal{L}_{\gamma,D}^{-1} : L_\gamma^2 \to H^{1-|\gamma|}(\mathbb{S}^1)$ is continuous and onto (with R defined in this proof, $\mathcal{L}_\gamma R$ is a right inverse for Υ), hence the transpose $\Upsilon' : H^{-1+|\gamma|}(\mathbb{S}^1) \to L_\gamma^2$ is bounded.

We first claim that Υ' is valued in $\mathcal{N}_0(\mathcal{L}_{\gamma,\max})$, whose topology is captured by L_γ^2. This makes Υ' a continuous $\mathrm{dom}(\mathcal{L}_{\gamma,\max})$-valued map. To see this, for $f \in H^{-1+|\gamma|}(\mathbb{S}^1)$ so that $\Upsilon' f \in L_\gamma^2$, let us check that $\mathcal{L}_\gamma \Upsilon' f = 0$ (in the distributional sense that ${}^t\mathcal{L}_\gamma \iota_\gamma(\Upsilon' f) = 0$): for $\psi \in \dot{C}^\infty$,

$$\begin{aligned}\langle {}^t\mathcal{L}_\gamma \iota_\gamma(\Upsilon' f), \psi\rangle &= (\Upsilon' f, \mathcal{L}_\gamma \psi)_{L_\gamma^2} \\ &= \langle f, \tau_\gamma^N \mathcal{L}_{\gamma,D}^{-1} \mathcal{L}_\gamma \psi\rangle_{H^{-1+|\gamma|}, H^{1-|\gamma|}} \\ &= 0,\end{aligned}$$

since $\psi \in \mathrm{dom}(\mathcal{L}_{\gamma,D})$ and $\tau_\gamma^N \psi = 0$. Hence Υ' is a continuous $\mathrm{dom}(\mathcal{L}_{\gamma,\max})$-valued map.

We finally show that $\tilde{\tau}_\gamma^D \Upsilon' = \mathrm{id}|_{H^{-1+|\gamma|}(\mathbb{S}^1)}$. For $f \in H^{-1+|\gamma|}(\mathbb{S}^1)$ and $h \in H^{1-|\gamma|}(\mathbb{S}^1)$, we have

$$\begin{aligned}\langle \tilde{\tau}_\gamma^D \Upsilon' f, h\rangle_{H^{-1+|\gamma|}, H^{1-|\gamma|}} &\overset{(76)}{=} -(\cancel{\mathcal{L}_\gamma \Upsilon' f}, Rh)_{L_\gamma^2} + (\Upsilon' f, \mathcal{L}_\gamma Rh)_{L_\gamma^2} \\ &= \langle f, \tau_\gamma^N \mathcal{L}_{\gamma,D}^{-1} \mathcal{L}_\gamma Rh\rangle_{H^{-1+|\gamma|}, H^{1-|\gamma|}},\end{aligned}$$

and since $Rh \in \mathrm{dom}(\mathcal{L}_{\gamma,D})$, $\tau_\gamma^N \mathcal{L}_{\gamma,D}^{-1} \mathcal{L}_\gamma Rh = \tau_\gamma^N Rh = h$, and hence $\tilde{\tau}_\gamma^D \Upsilon' f = f$ as elements of $H^{-1+|\gamma|}(\mathbb{S}^1)$.

The proof of Proposition 30 is complete. ☐

5.3. *Extension* (34) *of Green's second identity.*

We first prove Lemma 7.

Proof of Lemma 7. With $\mathcal{L}_{\gamma,D}^{-1}$ defined in (18), given $f \in \mathrm{dom}(\mathcal{L}_{\gamma,\max})$, since $(\mathcal{L}_\gamma - \lambda) f \in L_\gamma^2$, we may define $f_D := (\mathcal{L}_{\gamma,D} - \lambda)^{-1}(\mathcal{L}_\gamma - \lambda) f \in \widetilde{H}_D^{2,\gamma}(\mathbb{D}) \subset \mathrm{dom}(\mathcal{L}_{\gamma,\max})$ such that $(\mathcal{L}_\gamma - \lambda) f_D = (\mathcal{L}_\gamma - \lambda) f$ and $\tau_\gamma^D f_D = 0$, and set $f_\lambda := f - f_D$. Clearly $f_\lambda \in \mathrm{dom}(\mathcal{L}_{\gamma,\max})$ and $(\mathcal{L}_\gamma - \lambda) f_\lambda = 0$, i.e., $f_\lambda \in \mathcal{N}_\lambda(\mathcal{L}_{\gamma,\max})$.

To show uniqueness, if $f \in \mathrm{dom}(\mathcal{L}_{\gamma,D}) \cap \mathcal{N}_\lambda(\mathcal{L}_{\gamma,\max})$, then $(\mathcal{L}_\gamma - \lambda) f = 0$ with $\tau_D f = 0$ which, by injectivity of $\mathcal{L}_{\gamma,D} - \lambda$, implies $f = 0$. Lemma 7 is proved. ☐

We finally prove how to extend Green's second identity to (34). In this paragraph, inner products with no subscripts are implicitly L^2_γ-inner products. Using Lemma 7 with $\lambda = 0$, we compute

$$\begin{aligned}
&(\mathcal{L}_\gamma f, g) - (f, \mathcal{L}_\gamma g)\\
&\quad \overset{(33)}{=} (\mathcal{L}_\gamma f, g_D) + (\mathcal{L}_\gamma f, g_0) - (f_D, \mathcal{L}_\gamma g) - (f_0, \mathcal{L}_\gamma g)\\
&\quad \overset{(75)}{=} (f, \mathcal{L}_\gamma g_D) - \langle \tilde{\tau}^D_\gamma f, \tau^N_\gamma g\rangle_{-1+|\gamma|,1-|\gamma|} + (\mathcal{L}_\gamma f, g_0) \dots\\
&\qquad\quad - (\mathcal{L}_\gamma f_D, g) + \langle \tilde{\tau}^D_\gamma g, \tau^N_\gamma f_D\rangle_{-1+|\gamma|,1-|\gamma|} - (f_0, \mathcal{L}_\gamma g)\\
&\quad \overset{(33)}{=} (f_D, \mathcal{L}_\gamma g_D) + \cancel{(f_0, \mathcal{L}_\gamma g_D)} - \langle \tilde{\tau}^D_\gamma f, \tau^N_\gamma g\rangle_{-1+|\gamma|,1-|\gamma|} + \cancel{(\mathcal{L}_\gamma f_D, g_0)} + (\cancel{\mathcal{L}_\gamma f_0}, g_0) \dots\\
&\qquad\quad - (\mathcal{L}_\gamma f_D, g_D) - \cancel{(\mathcal{L}_\gamma f_D, g_0)} + \langle \tilde{\tau}^D_\gamma g, \tau^N_\gamma f_D\rangle_{-1+|\gamma|,1-|\gamma|} - \cancel{(f_0, \mathcal{L}_\gamma g_D)} - (f_0, \cancel{\mathcal{L}_\gamma g_0}).
\end{aligned}$$

Finally, $(f_D, \mathcal{L}_\gamma g_D) = (\mathcal{L}_\gamma f_D, g_D)$ because $(\mathcal{L}_\gamma, \widetilde{H}^{2,\gamma}_D(\mathbb{D}))$ is self-adjoint, and (34) follows.

The proof of Theorem 8 is complete. □

Appendix: Proofs of auxiliary lemmas

A.1. *Characterization of the Friedrichs extension — proof of Lemma 17.*

Proof of Lemma 17. After adjusting T_0 by a constant multiple of I (noting that this does not affect any domains involved), we may assume that $T_0 \geq 1$. The α-closure of $D(T_0)$ is precisely the space V described above, i.e., the Hilbert space closure of $D(T_0)$ with respect to the norm $u \mapsto (\alpha(u, u))^{1/2}$. Since the domain of the Friedrichs extension is D_F as defined in (43), with $D_F \subset V$, it follows that if T is the Friedrichs extension of T_0, then $D(T) = D_F$ must be contained in V, i.e., the α-closure of $D(T_0)$.

Conversely, suppose that $D(T)$ is contained in the α-closure of dom α, i.e., in V. We now claim that, with $\tilde{\alpha}$ the extension of α to V,

$$\tilde{\alpha}(u, v) = (Tu, v)_{\mathcal{H}}, \quad u \in D(T), \quad v \in V.$$

We first do so when $v \in D(T_0)$. Since $u \in D(T) \subset V$, it follows that there is a sequence $u_n \in D(T_0)$ such that $u_n \to_\alpha u$. In that case, we also have $\tilde{\alpha}(u, v) = \lim_{n\to\infty} \tilde{\alpha}(u_n, v) = \lim_{n\to\infty} \alpha(u_n, v)$. But we also have

$$\lim_{n\to\infty} \alpha(u_n, v) = \lim_{n\to\infty} (T_0 u_n, v)_{\mathcal{H}} = \lim_{n\to\infty} (u_n, T_0 v)_{\mathcal{H}} = (u, T_0 v)_{\mathcal{H}} = (u, Tv)_{\mathcal{H}} = (Tu, v)_{\mathcal{H}}.$$

The second equality follows from the symmetry of T_0. The third equality follows since $u_n \to_\alpha u$ necessarily implies $u_n \to u$ in $\mathcal{H}$. The fourth equality follows from T being an extension of T_0. Finally, the last equality follows from the symmetry of T on $D(T)$, noting that both u and v belong to $D(T)$. Thus, $\tilde{\alpha}(u, v) = (Tu, v)_{\mathcal{H}}$ for $u \in D(T)$ and $v \in \operatorname{dom} \alpha$. To extend the result to $v \in V$, we note that $v \in V$ implies the existence of $v_n \in D(T_0)$ such that $v_n \to_\alpha v$. Then

$$\tilde{\alpha}(u, v) = \lim_{n\to\infty} \tilde{\alpha}(u, v_n) = \lim_{n\to\infty} (Tu, v_n)_{\mathcal{H}} = (Tu, v)_{\mathcal{H}}.$$

The second equality follows from the case previously considered, while the third follows since $v_n \to_\alpha v$ necessarily implies $v_n \to v$ in $\mathcal{H}$. Hence, we have $\tilde{\alpha}(u, v) = (Tu, v)_{\mathcal{H}}$ for $u \in D(T)$ and $v \in V$. It follows, for $u \in D(T)$, that

$$|\tilde{\alpha}(u, v)| \leq \|Tu\|_{\mathcal{H}} \|v\|_{\mathcal{H}},$$

i.e., $u \in D_F$. Thus, $D(T)$ is contained in the domain of the Friedrichs extension. Since T is also self-adjoint, it follows that T must be the Friedrichs extension itself, since the domain of a self-adjoint extensions cannot be properly contained in the domain of another self-adjoint extension (this follows by taking adjoints). □

A.2. ***Proof of Lemma 18.***

Proof of Lemma 18. Let $s_j = \sum_{j=1}^{k} 1/a_k$; then $s_j \to \infty$ as $j \to \infty$ by assumption. Let

$$c_k^{(j)} = \begin{cases} \dfrac{1}{s_j a_k} & \text{if } 1 \le k \le j, \\ 0 & \text{if } k > j. \end{cases}$$

Then

$$\sum_{k=1}^{\infty} c_k^{(j)} = \frac{1}{s_j} \sum_{k=1}^{j} \frac{1}{a_k} = 1$$

by construction, and

$$\sum_k a_k |c_k^{(j)}|^2 = \sum_{k=1}^{j} \frac{1}{s_j^2 a_k} = \frac{1}{s_j} \to 0$$

since $s_j \to \infty$ by assumption. □

A.3. ***Proof of Lemma 28.*** Recall the expression

$$C_{m;s,\gamma} = \frac{\langle m\rangle^{2s-2\gamma-2}}{\pi(\gamma!)^2} \sum_{\ell \ge 0} (m+2\ell+1+\gamma)^{1-2s} \frac{(m+\ell+\gamma)!}{(m+\ell)!} \frac{(\ell+\gamma)!}{\ell!} \tag{77}$$

We note, for $\gamma \ge 0$ and $s > \gamma + 1$, that

$$1 - 2s \le 1 - 2s + 2\gamma = -1 - 2(s - \gamma - 1) < -1.$$

Noting the asymptotic

$$\lim_{x\to\infty} \frac{(x+\gamma)!}{x!(x+1)^\gamma} = 1,$$

it follows that the summand in (77) is bounded from above and from below by a multiple of

$$(m+2\ell+1+\gamma)^{1-2s}(m+\ell+1)^\gamma(\ell+1)^\gamma.$$

Thus, it suffices to establish upper and lower bounds on the sum

$$\langle m\rangle^{2s-2\gamma-2} \sum_{\ell\ge 0} (m+2\ell+1+\gamma)^{1-2s}(m+\ell+1)^\gamma(\ell+1)^\gamma.$$

For the upper bound, note that

$$m+2\ell+1+\gamma \ge m+\ell+1 \quad \Longrightarrow \quad (m+2\ell+1+\gamma)^{1-2s} \le (m+\ell+1)^{1-2s}$$

since $1 - 2s < 0$, and

$$\ell+1 \le m+\ell+1 \quad \Longrightarrow \quad (\ell+1)^\gamma \le (m+\ell+1)^\gamma$$

since $\gamma > 0$. Hence, we can estimate

$$\begin{aligned} C_{m;s,\gamma} &\le C\langle m\rangle^{2s-2\gamma-2} \sum_{\ell\ge 0} (m+2\ell+1+\gamma)^{1-2s}(m+\ell+1)^{\gamma}(\ell+1)^{\gamma} \\ &\le C\langle m\rangle^{2s-2\gamma-2} \sum_{\ell\ge 0} (m+\ell+1)^{1-2s+2\gamma}, \end{aligned}$$

where C does not depend on m. Note that the sum does converge since $1-2s+2\gamma < -1$. Since $x \mapsto (m+x)^{1-2s+2\gamma}$ is decreasing, we have

$$\sum_{\ell\ge 0} (m+\ell+1)^{1-2s+2\gamma} \le \int_0^\infty (m+x)^{1-2s+2\gamma}\,\mathrm{d}x = \frac{m^{2-2s+2\gamma}}{2s-2\gamma-2}.$$

It follows that

$$C_{m;s,\gamma} \le C\langle m\rangle^{2s-2\gamma-2} \frac{m^{2-2s+2\gamma}}{2s-2\gamma-2} \le \frac{C}{2s-2\gamma-2},$$

thus establishing the upper bound. For the lower bound, we assume without loss of generality that m is sufficiently large, say $m > \gamma$, and discard the (positive) terms for $0 \le \ell < m$. For the remaining terms, we have

$$\begin{aligned} 2\ell+2 > m+\ell+1 \quad &\Longrightarrow \quad \ell+1 > \tfrac{1}{2}(m+\ell+1) \quad \Longrightarrow \quad (\ell+1)^{\gamma} > 2^{\gamma}(m+\ell+1)^{\gamma}, \\ m+2\ell+1+\gamma < 2(m+\ell+1) \quad &\Longrightarrow \quad (m+2\ell+1+\gamma)^{1-2s} > 2^{1-2s}(m+\ell+1)^{1-2s}. \end{aligned}$$

We conclude that

$$\begin{aligned} C_{m;s,\gamma} &\ge c\langle m\rangle^{2s-2\gamma-2} \sum_{\ell\ge m} (m+2\ell+1+\gamma)^{1-2s}(m+\ell+1)^{\gamma}(\ell+1)^{\gamma} \\ &\ge 2^{1-2s+\gamma} c\langle m\rangle^{2s-2\gamma-2} \sum_{\ell\ge m} (m+\ell+1)^{1-2s+2\gamma}, \end{aligned}$$

where c does not depend on m. Again using that $x \mapsto (m+x)^{1-2s+2\gamma}$ is decreasing, we have

$$\sum_{\ell=m}^{\infty} (m+\ell+1)^{1-2s+2\gamma} \ge \int_{m+1}^\infty (m+x)^{1-2s+2\gamma}\,\mathrm{d}x = \frac{(2m+1)^{2-2s+2\gamma}}{2s-2\gamma-2}.$$

It follows that

$$C_{m;s,\gamma} \ge 2^{1-2s+\gamma} c\langle m\rangle^{2s-2\gamma-2} \frac{(2m+1)^{2-2s+2\gamma}}{2s-2\gamma-2} \ge c'$$

for all $m > \gamma$, where

$$c' = \frac{2^{1-2s+\gamma}c}{2s-2\gamma-2}\left(\inf_{m\ge 0} \frac{\langle m\rangle}{2m+1}\right)^{2s-2\gamma-2} = \frac{2^{1-2s+\gamma}5^{1-s+\gamma}c}{2s-2\gamma-2} > 0.$$

This establishes the desired lower bound.

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Cover image: The figure shows the outgoing scattered field produced by scattering a plane wave, coming from the northwest, off of the (stylized) letters P A A. The total field satisfies the homogeneous Dirichlet condition on the boundary of the letters. It is based on a numerical computation by Mike O’Neil of the Courant Institute.

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