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The Small-Noise Limit of the Most Likely Element is the Most Likely Element in the Small-Noise Limit

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Abstract. In this paper, we study the Onsager-Machlup function and its relationship to the Freidlin-Wentzell function for measures equivalent to arbitrary infinite dimensional Gaussian measures. The Onsager-Machlup function can serve as a density on infinite dimensional spaces, where a uniform measure does not exist, and has been seen as the Lagrangian for the "most likely element". The Freidlin-Wentzell rate function is the large deviations rate function for small-noise limits and has also been identified as a Lagrangian for the "most likely element". This leads to a conundrum - what is the relationship between these two functions?

We show both pointwise and Γ-convergence (which is essentially the convergence of minimizers) of the Onsager-Machlup function under the small-noise limit to the Freidlin-Wentzell function - and give an expression for both. That is, we show that the small-noise limit of the most likely element is the most likely element in the small noise limit for infinite dimensional measures that are equivalent to a Gaussian. Examples of measures include the law of solutions to path-dependent stochastic differential equations and the law of an infinite system of random algebraic equations.

1. Introduction

The primary objective of this paper is an investigation of the relationship between the Onsager-Machlup (Onsager and Machlup, 1953) and Freidlin-Wentzell (Freidlin and Wentzell, 1984) functions for measures equivalent to an arbitrary Gaussian measure.

First, recall the setting of $\mathbb{R}^d$-valued diffusion processes. By Girsanov's theorem (Øksendal, 2003), the law of the solution to the stochastic differential equation (SDE)

$$dX^\varepsilon = b(X^\varepsilon)dt + \varepsilon dB(t)$$

is equivalent to the law of the Gaussian process $\varepsilon B(t)$, in the sense that both measures agree on the same null sets. In 'small noise' settings as $\varepsilon \to 0$, there has been extensive work deriving the so-called 'Friedlin-Wentzell' large deviations rate functions for diffusions (see Dembo and Zeitouni

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(1998), as well as Gao and Liu (2023); Budhiraja and Song (2020) for more recent work), and it can be shown that the Friedlin-Wentzell function for X^ε is given by

$$\mathrm{FW}_X(z) = \frac{1}{2}\int_0^T (z'(t) - b(z(t))^2 dt.$$

A well-known interpretation of the minimizer of this Friedlin-Wentzell function is as the "most likely" path the small noise process takes between fixed initial and final states. On the other hand, the minimizer of the Onsager-Machlup function (Dürr and Bach, 1978) for X^ε ($\varepsilon > 0$),

$$\mathrm{OM}_{X^\varepsilon}(z) = \frac{1}{2\varepsilon^2}\int_0^T [(z'(t) - b(z(t))^2 + \varepsilon^2 b'(z(t))]dt,$$

when z is sufficiently regular, is also interpreted as the most likely path between initial and final states. This immediately poses a dilemma - if both theories claim to produce the most likely path, how can they be reconciled?

In the SDE setting, the relationship between these two theories has been explored in Li and Li (2021); Lu et al. (2017). As can be immediately anticipated from the displays above, $\varepsilon^2\,\mathrm{OM}_{X^\varepsilon}$ converges pointwise to FW_X as $\varepsilon \to 0$. In these papers the authors further prove that the Onsager-Machlup function Γ-converges (Braides, 2002) to the Freidlin-Wentzell function. Thus, in particular for X^ε, their results imply that $\varepsilon^2\,\mathrm{OM}_{X^\varepsilon}(z)$ Γ-converges to the $\mathrm{FW}_X(z)$ as $\varepsilon \to 0$. Since Γ-convergence can be understood as essentially the convergence of minimizers, it follows, at least in the case of SDEs, that the Freidlin-Wentzell "most likely path" is the small noise limit of the Onsager-Machlup "most likely path".

However, the connection between Onsager-Machlup and Freidlin-Wentzell theories appears to be more subtle. For instance, in the paper Dutra et al. (2014) the authors study numerics for estimating the most likely path of a given stochastic differential equation. The authors show that the choice of discretization leads to differing behavior - an Euler-Maruyama discretization led to the Freidlin-Wentzell "most likely path", while a trapezoid discretization led to the Onsager-Machlup "most likely path". Further, connections between the Onsager-Machlup and Friedlin-Wentzell theories have been hinted at in more general settings. For instance, in the context of computing rare event paths for stochastic partial differential equations (SPDEs), E et al. (2004) observe that the Friedlin-Wentzell path is an "analogue" of the Onsager-Machlup path (see Liu et al. (2020) as well). However, to the authors' best knowledge, no work has been done to rigorously establish the relationship between the Onsager-Machlup and Freidlin-Wentzell functions (and their minimizers) at this level of generality.

By a sufficiently general version of Girsanov's theorem, in many settings the law of a solution to SDEs/SPDEs driven by a general Gaussian noise (which can be a stochastic process, random field or more general objects) is equivalent to the law of the underlying noise. Utilizing this, our primary result below encompasses the setting of SDEs/SPDEs, and extends well beyond to that of measures equivalent to arbitrary Gaussian reference measures.

Theorem 1.1. *Let $\mathcal{B}$ be a separable Banach space with centered Gaussian measure μ_0. Let μ be another Borel measure on $\mathcal{B}$ equivalent to μ_0, with density $\frac{d\mu}{d\mu_0} = \exp(\Phi)$, where Φ satisfies mild regularity conditions. Define the measures μ_0^ε by $\mu_0^\varepsilon(A) = \mu_0(\frac{1}{\varepsilon}A)$ for Borel $A \subset \mathcal{B}$ and $\mu^\varepsilon = \exp(\frac{1}{\varepsilon^2}\Phi)\mu_0^\varepsilon$. Then the Onsager-Machlup function for the measures μ^ε exist, denoted by $\mathrm{OM}_{\mu^\varepsilon}$. Additionally, we have that $\{\mu^\varepsilon\}$ satisfy a LDP with rate function $\mathrm{FW}(z) := \lim_{\varepsilon\to 0^+} \varepsilon^2\,\mathrm{OM}_{\mu^\varepsilon}(z)$ and speed ε^2.*

Furthermore, denoting by $\text{OM-Mode}(\mu^\varepsilon) := \arg\inf_{z\in\mathcal{B}} \mathrm{OM}_{\mu^\varepsilon}$ we have that every cluster point of the elements $\text{OM-Mode}(\mu^\varepsilon)$ is a minimizer of FW, denoted by $\text{FW-Mode}(\mu) := \arg\inf_{z\in\mathcal{B}} \mathrm{FW}(z)$. If $\varepsilon^2\,\mathrm{OM}_{\mu^\varepsilon}$ are equicoercive, then we also have that $\lim_{\varepsilon\to 0^+} \text{OM-Mode}(\mu^\varepsilon) = \text{FW-Mode}(\mu)$.

The proof of Theorem 1.1 boils down to two "tilting" lemmas - one for Onsager-Machlup in Corollary 2.6 and one for large deviations in Lemma 3.2. In large deviations analysis, the exponential tilting principle is a mechanism for transferring large deviations principles from a given sequence of measures to a sequence of equivalent measures. Lemma 3.2 proves a slightly generalized version of this principle that applies in the setting of measures equivalent to a Gaussian. In Corollary 2.6, we present the Onsager-Machlup function for measures equivalent to Gaussians, and represents a type of tilting. Our proof uses these two results to establish the main Theorem 1.1.

1.1. *Literature Review.* We provide a short review of the literature related to the Onsager-Machlup and Friedlin-Wentzell theories; a full review is outside the scope of this short paper. The Onsager-Machlup theory was originally introduced in Onsager and Machlup (1953) to compute the probability that a system experiencing Gaussian (thermodynamic) fluctuations will pass through a succession of non-equilibrium states over time. This probability was expressed in terms of what is now recognized as the Friedlin-Wentzell function. The Onsager-Machlup theory for SDEs was originally studied in Dürr and Bach (1978), who generalized the results in Onsager and Machlup (1953) to non-Gaussian Markov diffusions. Durr and Bach provide a rigorous definition and derivation of the Onsager-Machlup function . Subsequently, the theory has been substantially developed and extended to random fields and SPDEs (Dembo and Zeitouni, 1991; Mayer-Wolf and Zeitouni, 1993) as well. A key motivation for the continued development of the Onsager-Machlup theory is its central role in metastability analysis (Cuff et al., 2012; Olivieri, 2003; Cassandro et al., 1986; Hongler and Desai, 1986; Davies, 1982), and the fact that the Onsager-Machlup function can be viewed as the Lagrangian yielding the most likely path connecting metastable states in a stochastic system.

The Friedlin-Wentzell function arises as the small noise large deviations rate function of a Markov diffusion process (Dembo and Zeitouni, 1998; Freidlin and Wentzell, 1984) . The Friedlin-Wentzell theory is also used in pathwise analyses of metastable behavior, with the large deviations rate function begin used to determine the time-scale of 'tunneling' behavior in the small noise setting (Olivieri and Vares, 2005). Metastable phenomena emerge outside the small noise setting, and therefore the relationship between the Onsager-Machlup and Friedlin-Wentzell theories is of significance and interest. The convergence of the Onsager-Machlup to Freidlin-Wentzell for SDEs, specifically, was studied in Li and Li (2021) and Du et al. (2021). This paper establishes the relationship between them for measures that are equivalent to Gaussian measures on Banach spaces.

2. The Onsager-Machlup Formalism

In finite dimensional probability, computations involving expectations, probabilities and related quantities can be greatly eased by working with a probability density function. These densities capture how a probability distribution compares with some kind of uniform measure - typically Lebesgue or counting measure. Densities are also the optimization objective to be maximized when computing the mode (or most likely element) of a distribution.

For instance, given a probability measure μ on $\mathbb{R}^n$ that is equivalent to the Lebesgue measure λ, we can express its Radon-Nikodym derivative by

$$\frac{d\mu}{d\lambda}(z) = \lim_{\varepsilon\to 0}\frac{\mu(B_\varepsilon(z))}{\lambda(B_\varepsilon(z))} = \lim_{\varepsilon\to 0}\frac{\mu(B_\varepsilon(z))}{\lambda(B_\varepsilon(0))} \propto \lim_{\varepsilon\to 0}\frac{\mu(B_\varepsilon(z))}{\mu(B_\varepsilon(0))},$$

where the final relation comes from the Lebesgue differentiation theorem. The last expression makes sense even on non-locally compact spaces where there might not be a uniform measure.

The notion of a density does not immediately transfer to infinite dimensional probability as there is no uniform measure on infinite dimensional spaces. However, the so-called Onsager-Machlup function can serve the role of a density in infinite dimensions. This function was introduced by Onsager and Machlup (1953), and the key insight in Onsager-Machlup theory is that one can compare a probability distribution to translations of itself rather than comparing a probability distribution to some translation invariant measure. This recovers the standard density in the finite dimensional case but allows for "densities" on infinite dimensional spaces.

Definition 2.1. Let (X, d) be a metric space. Let μ be a Borel probability measure on X. If the following limit exists

$$\lim_{\varepsilon \to 0} \frac{\mu(B_\varepsilon(z_1))}{\mu(B_\varepsilon(z_2))} = \exp\left(\mathrm{OM}_\mu(z_2) - \mathrm{OM}_\mu(z_1)\right), \tag{2.1}$$

then $\mathrm{OM}_\mu(z)$ is called the Onsager-Machlup function for μ.

Remark 2.2. The Onsager-Machlup function in Definition 2.1 can be thought of as the negative log "density" of μ. That is,

$$\text{“}\frac{d\mu}{d\lambda}(z)\text{”} = \exp\left(-\mathrm{OM}_\mu(z)\right)$$

for some possibly nonexistent uniform measure λ. In the case where $X = \mathbb{R}^d$ then the above equality is rigorous, by the Lebesgue differentiation theorem. Also, note that the Onsager-Machlup function is only defined up to an additive constant. We also define the "mode" or the most likely element of μ as the minimizer of OM_μ.

Remark 2.3. We also observe that the Onsager-Machlup formalism has found its way into Bayesian statistics and MAP estimation such as in Dashti et al. (2013); Stuart (2010). Additionally, in Selk et al. (2021) the authors prove a "portmanteau" theorem that relates the Onsager-Machlup function on an abstract Banach space equipped with a Gaussian measure to an information projection problem, to an "open loop" or state-independent KL-weighted control problem, and in the case of classical Wiener space to an Euler-Lagrange equation or variational form. Furthermore, using this Portmanteau theorem the authors in Selk and Honnappa (2021) prove a Feynman-Kac type result for systems of ordinary differential equations. They demonstrate that the solution to a system of second order and linear ordinary differential equations is the most likely path of a diffusion. This Feynman-Kac result, like the original Feynman-Kac for parabolic partial differential equations, can (in principle) be used to efficiently solve systems of ordinary differential equations via Monte Carlo methods.

The following proposition represents a "tilting" lemma, allowing for Onsager-Machlup functions to be transfered to equivalent measures.

Lemma 2.4. *Let μ_0 be a Borel measure on Banach space $\mathcal{B}$. Suppose that μ_0 has an associated Onsager-Machlup function $\mathrm{OM}_{\mu_0} : \mathcal{B} \to [-\infty, \infty]$. Consider the measure μ with density*

$$\frac{d\mu}{d\mu_0} = \frac{1}{E_{\mu_0}[e^{-\Phi}]} \exp(-\Phi).$$

Suppose that for each $\varepsilon_0 > 0$, for each $x \in \mathcal{B}$ and for all $\varepsilon < \varepsilon_0$ there is some continuous increasing $\phi_{x,\varepsilon_0} : [0, \varepsilon_0] \to [0, \infty)$ with $\phi(0) = 0$ so that $|\Phi(u) - \Phi(x)| \le \phi(\varepsilon)$ on $B_\varepsilon(x)$. Then μ has an associated Onsager-Machlup function

$$\mathrm{OM}_\Phi(z) = \Phi(z) + \mathrm{OM}_{\mu_0}(z).$$

Proof: We consider the ratio

$$\frac{\mu(B_\varepsilon(z_1))}{\mu(B_\varepsilon(z_2))} = \frac{\int_{B_\varepsilon(z_1)} \mu(du)}{\int_{B_\varepsilon(z_2)} \mu(du)}.$$

Using the density, adding and subtracting $\Phi(z_i)$ for $i = 1, 2$ in both integrals yields that

$$\begin{aligned}\frac{\mu(B_\varepsilon(z_1))}{\mu(B_\varepsilon(z_2))} &= \frac{\int_{B_\varepsilon(z_1)} \exp(-\Phi(u))\mu_0(du)}{\int_{B_\varepsilon(z_2)} \exp(-\Phi(u))\mu_0(du)} \\ &= \frac{\int_{B_\varepsilon(z_1)} \exp(-\Phi(u) + \Phi(z_1) - \Phi(z_1))\mu_0(du)}{\int_{B_\varepsilon(z_2)} \exp(-\Phi(u) + \Phi(z_2) - \Phi(z_2))\mu_0(du)} \\ &= \exp\left(\Phi(z_2) - \Phi(z_1)\right) \frac{\int_{B_\varepsilon(z_1)} \exp(-\Phi(u) + \Phi(z_1))\mu_0(du)}{\int_{B_\varepsilon(z_2)} \exp(-\Phi(u) + \Phi(z_2))\mu_0(du)}.\end{aligned}$$

By assumption, there are some ϕ_i on $B_\varepsilon(z_i)$ so that

$$|\Phi(z_i) - \Phi(u)| \leq \phi_i(\varepsilon)$$

for $i = 1, 2$. Therefore for $\phi = \phi_1 - \phi_2$ we have that

$$\frac{\mu_0(B_\varepsilon(z_1))}{\mu_0(B_\varepsilon(z_2))} e^{-\phi(\varepsilon)} \leq \frac{\int_{B_\varepsilon(z_1)} \exp(-\Phi(u) + \Phi(z_1))\mu_0(du)}{\int_{B_\varepsilon(z_2)} \exp(-\Phi(u) + \Phi(z_2))\mu_0(du)} \leq \frac{\mu_0(B_\varepsilon(z_1))}{\mu_0(B_\varepsilon(z_2))} e^{\phi(\varepsilon)}.$$

Taking the limit $\varepsilon \to 0$ concludes. □

For the purposes of our paper, we are interested in Gaussian measures. For more information on Gaussian measure theory see Bogachev (1998); Hairer (2009). For measures equivalent to a Gaussian (Dashti et al., 2013, Theorem 3.2) derives an expression for the Onsager-Machlup function, which we recall in the next proposition.

Proposition 2.5. *Let μ_0 be a centered Gaussian measure on Banach space $\mathcal{B}$ with Cameron-Martin space $\mathcal{H}_{\mu_0}$ and Cameron-Martin norm $\|\cdot\|_{\mu_0}$. Let $\Phi : \mathcal{B} \to \mathbb{R}$ be a function that is locally Lipschitz and locally bounded. Define the measure μ with positive density $\frac{d\mu}{d\mu_0} = \frac{1}{E_{\mu_0}[e^{-\Phi}]} e^{-\Phi}$. Then the Onsager-Machlup function for μ exists and is equal to*

$$\mathrm{OM}_\mu(z) = \begin{cases} \Phi(z) + \frac{1}{2}\|z\|_{\mu_0}^2 & \text{if } z \in \mathcal{H}_\mu \\ \infty & \text{else .} \end{cases}$$

Proof: We recall (see e.g. Bogachev (1998), Section 4.7) that the Onsager-Machlup function for the Gaussian measure μ_0 is

$$\mathrm{OM}_{\mu_0}(z) = \begin{cases} \frac{1}{2}\|z\|_{\mu_0}^2 & if\ z \in \mathcal{H}_{\mu_0} \\ \infty & else. \end{cases}$$

The expression for OM_μ follows by Lemma 2.4. □

The following corollary can be straightforwardly proved by substitution in Proposition 2.5, and contains the assumptions we need for Theorem 1.1.

Corollary 2.6 (ε-dependent Tilting Lemma)**.** *Let μ_0 be a centered Gaussian measure on Banach space $\mathcal{B}$ with Cameron-Martin space $\mathcal{H}_{\mu_0}$ and Cameron-Martin norm $\|\cdot\|_{\mu_0}$. Consider the functions $F^\varepsilon(y) : \mathcal{B} \to \mathbb{R}$ and suppose that they satisfy the following expansion*

$$F^\varepsilon(y) = F_0(y) + \varepsilon F_1(y) + \frac{\varepsilon^2}{2} F_2(y) + \ldots + \varepsilon^n R_n(\varepsilon, y),$$

for some functions $F_i : \mathcal{B} \to \mathbb{R}$ with $\lim_{\varepsilon \to 0} R_n(\varepsilon, y) = 0$. Suppose that F_0 is continuous. Furthermore, assume that the functions F_i and R_n satisfy the following moment condition

$$\limsup_{\varepsilon \to 0} \varepsilon^2 \log E_{\mu_0^\varepsilon}\left[\exp\left(\gamma_i \frac{\max\{|F_i(y)|, |R_n(\varepsilon, y)|\}}{\varepsilon^2}\right)\right] < \infty,$$

for some $\gamma_i > 0$ and for all $0 \leq i \leq n$. Define the measures equivalent to μ_0^ε by

$$\mu^\varepsilon = \frac{1}{E_{\mu_0^\varepsilon}\left[\exp\left(-\frac{1}{\varepsilon^2}F^\varepsilon(y)\right)\right]} \exp\left(-\frac{1}{\varepsilon^2}F^\varepsilon(y)\right)\mu_0^\varepsilon.$$

Then the Onsager-Machlup function for μ^ε exists and is equal to

$$\mathrm{OM}_{\mu^\varepsilon}(z) = \begin{cases} \frac{1}{\varepsilon^2}F^\varepsilon(z) + \frac{1}{2\varepsilon^2}\|z\|^2_{\mu_0} & \text{if } z \in \mathcal{H}_{\mu_0} \\ \infty & \text{else .} \end{cases}$$

3. An ε-dependent Tilting Lemma and Small Noise LDPs

There are multiple ways of constructing new LDPs from existing ones. One principled approach is “tilting" - which passes large deviations principles from a sequence of reference measures to a sequence of measures equivalent to the reference measures. We direct the reader to den Hollander (2000, Theorem III.17) , for the standard tilting lemma. One form of the standard tilting lemma reads as follows.

Lemma 3.1. *Let μ_0^ε be a collection of Borel probability measures on a Banach space $\mathcal{B}$ satisfying a LDP with good rate function $I_0 : \mathcal{B} \to [0, \infty]$ and rate ε^2. Consider a continuous function $F : \mathcal{B} \to \mathbb{R}$. Assume that for some $\gamma > 0$ we have*

$$\limsup_{\varepsilon \to 0} \varepsilon \log E_{\mu_0^\varepsilon}\left[\exp\left(\gamma|F(y)|\right)\right] < \infty.$$

Define the measures equivalent to μ_0^ε by

$$\mu^\varepsilon = \frac{1}{E_{\mu_0^\varepsilon}[\exp\left(-\frac{1}{\varepsilon^2}F(y)\right)]} \exp\left(-\frac{1}{\varepsilon^2}F(y)\right)\mu_0^\varepsilon.$$

Then μ^ε satisfies a LDP with good rate function

$$I(y) := I_0(y) + F(y) - \inf_{z \in \mathcal{B}}\{F(z) + I_0(z)\}.$$

However, in many cases where one would like to apply the tilting lemma, the function F depends on ε. This is apparent in the case of Freidlin-Wentzell large deviations for stochastic differential equations as we will see shortly. Therefore, we need a generalized version of the tilting lemma which we provide.

Lemma 3.2 (ε-dependent Tilting Lemma)**.** *Let μ_0^ε be a collection of exponentially tight Borel measures on Banach space $\mathcal{B}$ satisfying a LDP with good rate function $I_0 : \mathcal{B} \to [0, \infty]$. Consider the functions $F^\varepsilon(y) : \mathcal{B} \to \mathbb{R}$ and suppose that they satisfy the following expansion*

$$F^\varepsilon(y) = F_0(y) + \varepsilon F_1(y) + \frac{\varepsilon^2}{2}F_2(y) + ... + \varepsilon^n R_n(\varepsilon, y),$$

for some functions $F_i : \mathcal{B} \to \mathbb{R}$ with $\lim_{\varepsilon\to 0} R_n(\varepsilon, y) = 0$. Suppose that F_0 is continuous. Furthermore, assume that the functions F_i and R_n satisfy the following moment condition

$$\limsup_{\varepsilon \to 0} \varepsilon^2 \log E_{\mu_0^\varepsilon}\left[\exp\left(\gamma_i \frac{\max\{|F_i(y)|, |R_n(\varepsilon, y)|\}}{\varepsilon^2}\right)\right] < \infty,$$

for some $\gamma_i > 0$ and for all $0 \leq i \leq n$. Define the measures equivalent to μ_0^ε by

$$\mu^\varepsilon = \frac{1}{E_{\mu_0^\varepsilon}\left[\exp\left(-\frac{1}{\varepsilon^2}F^\varepsilon(y)\right)\right]} \exp\left(-\frac{1}{\varepsilon^2}F^\varepsilon(y)\right)\mu_0^\varepsilon.$$

Assume that the measures μ^ε are exponentially tight. Then μ^ε satisfies a LDP with good rate function

$$I(y) := I_0(y) + F_0(y) - \inf_{z \in \mathcal{B}}\{F_0(z) + I_0(z)\}.$$

Proof: We will apply Bryc's lemma (see Dembo and Zeitouni (1998, Theorem 4.4.2)). To this end, consider a bounded and continuous function $\varphi : \mathcal{B} \to \mathbb{R}$. Then consider

$$L := \varepsilon^2 \log\left(E_{\mu^\varepsilon}\left[\exp\left(-\frac{\varphi(y)}{\varepsilon^2}\right)\right]\right).$$

Using the form of F^ε, we get that

$$L = \varepsilon^2 \log\left(E_{\mu_0^\varepsilon}\left[\exp\left(-\frac{\varphi(y)+F^\varepsilon(y)}{\varepsilon^2}\right)\right]\right) - \varepsilon^2 \log\left(E_{\mu_0^\varepsilon}\left[\exp\left(-\frac{F^\varepsilon(y)}{\varepsilon^2}\right)\right]\right).$$

In the limit $\varepsilon \to 0$, by Hölder's inequality, reverse Hölder's inequality, Varadhan's lemma and the assumptions on F_i and R_n, we have that

$$\lim_{\varepsilon\to 0} L = \lim_{\varepsilon\to 0} \varepsilon^2 \log\left(E_{\mu_0^\varepsilon}\left[\exp\left(-\frac{\varphi(y)+F_0(y)}{\varepsilon^2}\right)\right]\right) - \varepsilon^2 \log\left(E_{\mu_0^\varepsilon}\left[\exp\left(-\frac{F_0(y)}{\varepsilon^2}\right)\right]\right). \tag{3.1}$$

By Varadhan's lemma (Dembo and Zeitouni, 1998, Theorem 4.3.1) and the assumptions on F_0, we have that

$$\lim_{\varepsilon\to 0} L = -\inf_{y\in\mathcal{B}}\{\varphi(y)+F_0(y)+I_0(y)\} + \inf_{z\in\mathcal{B}}\{F_0(z)+I_0(z)\}.$$

By Bryc's lemma (Dembo and Zeitouni, 1998, Theorem 4.4.2) and the assumption of tightness of μ^ε, we conclude. □

Small noise large deviations for arbitrary Gaussian measures on Banach space are well known. In Bogachev (1998), for instance, it is shown that the Freidlin-Wentzell rate function for a general Gaussian measure μ_0 with Cameron-Martin space $\mathcal{H}_{\mu_0}$ is $\frac{1}{2}\|\cdot\|_{\mu_0}^2$. In particular, the small noise rate function for $\varepsilon B(t)$ is

$$\mathrm{FW}(z) = \begin{cases} \frac{1}{2}\int_0^T (z'(t))^2 dt & \text{for } z \in \mathcal{W}_0^{1,2} \\ \infty & \text{else ,} \end{cases}$$

where $\mathcal{W}_0^{1,2} = \{t \mapsto \int_0^t f(s)ds : \int_0^T f^2(s)ds < \infty\}$ is the Cameron-Martin space of the classical Wiener measure which consists of absolutely continuous functions with L^2 weak derivative. The first instance of small noise large deviations for infinite-dimensional non-Gaussian measures came in Freidlin and Wentzell (1984), where the authors studied small noise large deviations for the solution to the SDE

$$dX^\varepsilon(t) = b(X^\varepsilon(t))dt + \varepsilon dB(t), \tag{3.2}$$

where $b \in C^1$. In particular, Freidlin and Wentzell (1984) shows the following proposition.

Proposition 3.3. *The law of X^ε satisfies a LDP as $\varepsilon \to 0$, with speed ε^2 and with rate function*

$$\mathrm{FW}(z) = \begin{cases} \frac{1}{2}\int_0^T (b(z(t)) - z'(t))^2 dt & \textit{for } z \in \mathcal{W}_0^{1,2} \\ \infty & \textit{else .} \end{cases}$$

Below, we offer an alternate, considerably simpler, proof of this result by invoking the ε-tilting lemma above.

Proof: By Girsanov, the law of X^ε, μ^ε, has density with respect to the law of $\varepsilon B(t)$, μ_0^ε given by

$$\frac{d\mu^\varepsilon}{d\mu_0^\varepsilon} = \exp\left(\frac{1}{\varepsilon^2}\left(\int_0^T b(B(t))dB(t) - \frac{1}{2}\int_0^T b^2(B(t))dt\right)\right). \tag{3.3}$$

At first glance, it might appear from equation (3.3) that we might not need the full ε-dependent Lemma 3.2. However, note that the Itô integral is not defined pathwise. On the other hand, recall that the Itô integral is μ_0^ε-a.s. equal to a Stratonovich integral, which is defined pathwise. Applying Itô's lemma under μ_0^ε yields

$$\frac{d\mu^\varepsilon}{d\mu_0^\varepsilon} = \exp\left(\frac{1}{\varepsilon^2}\left(\int_0^T b(B(t)) \circ dB(t) - \frac{\varepsilon^2}{2}\int_0^T b'(B(t))dt - \frac{1}{2}\int_0^T b^2(B(t))dt\right)\right),$$

where $\int_0^T b(B(t)) \circ dB(t)$ represents the Stratonovich integral. As b is continuous it has an anti-derivative F, and the Stratonovich integral satisfies $\int_0^T b(B(t)) \circ dB(t) = F(B(T)) - F(B(0))$, which is indeed continuous.

Applying the ε-dependent tilting Lemma 3.2 gives that the rate function for μ^ε is

$$\mathrm{FW}(z) = \begin{cases} -\left(\int_0^T b(z(t))dz(t) - \frac{1}{2}\int_0^T b^2(z(t))dt\right) + \frac{1}{2}\int_0^T (z'(t))^2 dt & \text{if } z \in \mathcal{W}_0^{1,2} \\ \infty & \text{else .} \end{cases}$$

□

4. Proof of Theorem 1.1

We begin this section with a motivating example for small noise large deviations for Gaussian measures in finite dimensions.

A Motivating Example. Consider a family of real valued normally distributed random variables X^ε where $X^\varepsilon \sim \mathcal{N}(0, \varepsilon^2)$. We are interested in the decay of the probability

$$P(X^\varepsilon \in A) = \int_A \frac{1}{\sqrt{2\pi\varepsilon^2}} e^{-x^2/2\varepsilon^2} dx,$$

for Borel $A \subset \mathbb{R}$. Thankfully the standard Laplace principle on $\mathbb{R}$ yields the appropriate scaling and gives the large deviations principle

$$\lim_{\varepsilon \to 0^+} \varepsilon^2 \log P(X^\varepsilon \in A) = -\operatorname{essinf}_{x \in A} \frac{x^2}{2}.$$

In this case, the Onsager-Machlup function for the law of the random variable X^ε is just the term in the exponent - $\mathrm{OM}_{\mu^\varepsilon}(x) = \frac{x^2}{2\varepsilon^2}$ and we have that $\varepsilon^2\,\mathrm{OM}_{\mu^\varepsilon}(x) = \frac{x^2}{2} = \mathrm{FW}(x)$.

Perhaps surprisingly, this equivalence (up to an ε^2 scaling) of the Onsager-Machlup and Freidlin-Wentzell functions holds true even for a Gaussian measure on a Banach space. More precisely, consider a Banach space $(\mathcal{B}, \|\cdot\|)$ with a Borel measure μ so that all the one dimensional projections are Gaussian. Consider the measure μ^ε defined by $\mu^\varepsilon(A) := \mu(\varepsilon^{-1}A)$ for Borel $A \subset \mathcal{B}$. Then it is shown in e.g. Bogachev (1998) section 4.9, that the measures μ^ε satisfy a large deviations principle (LDP) with rate function FW_μ and additionally for regular enough z we have that, for any $\varepsilon > 0$,

$$\varepsilon^2\,\mathrm{OM}_{\mu^\varepsilon}(z) = \mathrm{FW}_\mu(z) = \frac{1}{2}\|z\|_\mu^2,$$

where $\|\cdot\|_\mu$ is the so-called Cameron-Martin norm (distinct from the norm on the Banach space).

To obtain an analogous result on arbitrary Banach spaces, first recall the definition of Γ-convergence (for more information on Γ-convergence see Braides (2002)):

Definition 4.1. Let X be a topological space and $\mathcal{N}(x)$ denote the set of all neighborhoods of $x \in X$. Further, let $F_n : X \to \overline{\mathbb{R}}$ be a sequence of functions on X. The Γ-lower and Γ-upper limits are defined as

$$\Gamma - \liminf_{n\to\infty} F_n(x) = \sup_{N_x \in \mathcal{N}(x)} \liminf_{n\to\infty} \inf_{y \in N_x} F_n(y),$$

$$\Gamma - \limsup_{n\to\infty} F_n(x) = \sup_{N_x \in \mathcal{N}(x)} \limsup_{n\to\infty} \inf_{y \in N_x} F_n(y).$$

Then, the function $F : X \to \overline{\mathbb{R}}$ is a Γ-limit of F_n if $\Gamma - \liminf_{n\to\infty} F_n = \Gamma - \limsup_{n\to\infty} F_n = F$.

Recall the small-noise SDE in (3.2), whose measure μ^ε satisfies (3.3). Applying Proposition 2.6 shows that the Onsager-Machlup function for μ^ε is

$$\begin{aligned}\mathrm{OM}_{\mu^\varepsilon}(z) = &-\frac{1}{\varepsilon^2}\left(\int_0^T b(z(t))dz(t) - \frac{\varepsilon^2}{2}\int_0^T b'(z(t))dt - \frac{1}{2}\int_0^T b^2(z(t))dt\right)\\ &+\frac{1}{2\varepsilon^2}\int_0^T (z'(t))^2 dt,\end{aligned}$$

for $z \in \mathcal{W}_0^{1,2}$ and infinite otherwise. It is not hard to see that $\varepsilon^2 \,\mathrm{OM}_{\mu^\varepsilon}$ converges to FW both as a pointwise and as a Γ-limit.

The following proposition shows that this conclusion can be generalized considerably.

Proposition 4.2. *Let μ_0 be a centered Gaussian measure on a Banach space $\mathcal{B}$ with Cameron-Martin norm $\|\cdot\|_{\mu_0}$ (see Hairer (2009), Section 3.2). Let μ_0^ε denote the measure defined by $\mu_0^\varepsilon(A) = \mu_0(\varepsilon^{-1}A)$ for Borel $A \subset \mathcal{B}$. Let $F : \mathcal{B} \to \mathbb{R}$ be a function and suppose that $F(y) = F_0(y) + \varepsilon F_1(y) + \ldots + \varepsilon^n R_n(\varepsilon, y) := F^\varepsilon(y)$, μ_0^ε-a.s. where the F_i and R_n satisfy the assumptions of Lemma 3.2, and suppose that the F_i are locally bounded. Define the measures μ^ε with densities*

$$\frac{d\mu^\varepsilon}{d\mu_0^\varepsilon} = \frac{1}{E_{\mu_0^\varepsilon}[e^{-\frac{1}{\varepsilon^2}F^\varepsilon(y)}]}\exp\left(-\frac{1}{\varepsilon^2}F^\varepsilon(y)\right).$$

Then the Freidlin-Wentzell rate function for μ^ε exists and is

$$\mathrm{FW}(y) = F_0(y) + \frac{1}{2}\|y\|_{\mu_0}^2 - \inf_{z\in\mathcal{H}_{\mu_0}}(F_0(z) + \frac{1}{2}\|z\|_{\mu_0}^2).$$

Furthermore, denote by $\mathrm{OM}_\varepsilon(z)$ the Onsager-Machlup function for μ^ε. Then the limit

$$\lim_{\varepsilon\to 0^+} \varepsilon^2 \,\mathrm{OM}_\varepsilon(z) = \mathrm{FW}(z).$$

holds, both as pointwise and in the Γ-limit sense.

Proof: By Lemma 3.2, we have that the Freidlin-Wentzell rate function for μ^ε exists and is $\mathrm{FW}(z) = F_0(z) + \frac{1}{2}\|z\|_{\mu_0}^2 - \inf_{z\in\mathcal{H}_{\mu_0}}(F_0(z) + \frac{1}{2}\|z\|_{\mu_0}^2)$. Then we just need to check the pointwise and Γ convergence. Without loss of generality, we may assume that $\inf_{z\in\mathcal{H}_{\mu_0}}(F_0(z) + \frac{1}{2}\|z\|_{\mu_0}^2) = 0$. This is because the Onsager-Machlup function is only defined up to an additive constant and we may add $-\frac{1}{\varepsilon^2}\inf_{z\in\mathcal{H}_{\mu_0}}(F_0(z) + \frac{1}{2}\|z\|_{\mu_0}^2)$ to OM_ε. Proceeding with that, using Corollary 2.6 we have that

$$\varepsilon^2 \,\mathrm{OM}_\varepsilon(z) = F_0(z) + \varepsilon F_1(z) + \ldots + \varepsilon^n R_n(\varepsilon, z) + \frac{1}{2}\|z\|_{\mu_0}^2,$$

as the Onsager-Machlup function for μ_0^ε is $\frac{1}{2\varepsilon^2}\|z\|_{\mu_0}^2$. Clearly the pointwise limit of this is $F_0(z) + \frac{1}{2}\|z\|_{\mu_0}^2$. Now we just have to check Γ convergence. To this aim, note that $F_0(z) + \frac{1}{2}\|z\|_{\mu_0}^2$ is continuous and Γ convergence is stable under continuous pertubations. Therefore we just need to show that

$$\Gamma - \lim_{\varepsilon\to 0} \varepsilon F_1(z) + \ldots + \varepsilon^n R_n(\varepsilon, z) = 0.$$

Note that the F_i are locally bounded and thus on the neighborhood N_z of the point z, we have that

$$\lim_{\varepsilon\to 0}\inf_{x\in N_z} \varepsilon F_1(x) + \ldots + \varepsilon^n R_n(\varepsilon, x) = 0.$$

Since N_z is an arbitrary neighborhood of z, it follows by definition that OM_ε Γ-converges to FW. □

Proof of Theorem 1.1: Following Proposition 4.2, by Γ convergence, we have that every cluster point of the minimizers of $\varepsilon^2\,\mathrm{OM}_{\mu^\varepsilon}$ is a minimizer of FW (see Braides (2002), section 1.5). If additionally we know that the functions $\varepsilon^2\,\mathrm{OM}_{\mu^\varepsilon}$ are equicoercive (which is the case with SDEs with C^1 drift), by the fundamental theorem of Γ convergence (see Braides (2002) section 1.5), then we have the full version of Theorem 1.1. □

As a consequence of Theorem 1.1, we can specialize to the case of the generalized Girsanov theorem given in e.g. Nualart (2006), Theorem 4.1.2.

Proposition 4.3. *Let $(\mathcal{B}, \mu_0)$ be a Gaussian Banach space with Cameron-Martin space $\mathcal{H}_{\mu_0}$ and Cameron-Martin norm $\|\cdot\|_{\mu_0}$. Define μ_0^ε as above and consider white noise process $\{W(h) : h \in \mathcal{H}_{\mu_0}\}$ associated to μ_0. Suppose that $H : \mathcal{B} \to \mathcal{H}_{\mu_0}$ is a continuous function so that $W(H)$ is defined and suppose that for all $\varepsilon > 0$ we have*

$$E_{\mu_0^\varepsilon}\left[\exp\left(\frac{1}{\varepsilon^2}\left(W(H) - \frac{1}{2}\|H\|_{\mu_0}^2\right)\right)\right] = 1.$$

Define the collection of measures

$$\mu^\varepsilon = \exp\left(\frac{1}{\varepsilon^2}\left(W(H) - \frac{1}{2}\|H\|_{\mu_0}^2\right)\right)\mu_0^\varepsilon.$$

Then the Friedlin-Wentzell rate function for μ^ε exists and is equal to

$$\mathrm{FW}(z) = \begin{cases} \frac{1}{2}\|z - H(z)\|_{\mu_0}^2 & \text{if } z \in \mathcal{H}_{\mu_0} \\ \infty & \text{else.} \end{cases}$$

Furthermore we have that $\lim_{\varepsilon\to 0}\varepsilon^2\,\mathrm{OM}_\varepsilon(z) = \mathrm{FW}(z)$ both pointwise and in sense of Γ convergence.

Proof: Let $i : \mathcal{H}_{\mu_0} \to L^2([0, T], \mathbb{R})$ be an isomorphic isometry of separable Hilbert spaces. Denote by $z_t^* = i^{-1}(\chi_{[0,t]})$. Then $W(z_t^*)$ is a standard Brownian motion. Furthermore, one can verify that for all $h \in \mathcal{H}_{\mu_0}$ we have

$$W(\omega, h) = \int_0^T (ih)(t)dW(\omega, z_t^*).$$

Therefore we have that

$$W(\omega, H(\omega)) = \int_0^T (iH(\omega))(t)dW(\omega, z_t^*).$$

Under the measure μ_0^ε, we can change to Stratonovich integration to get that

$$\int_0^T (iH(\omega))(t)dW(\omega, z_t^*) = \int_0^T (iH(\omega))(t)\circ dW(\omega, z_t^*) - \frac{\varepsilon^2}{2}[(iH)(\omega), W(\omega, z_t^*)](T).$$

The Stratonovich integral is a continuous function of ω so long as H is, and so is $\frac{1}{2}\|H\|_{\mu_0}^2$, so therefore Proposition 4.2 applies and

$$\mathrm{FW}(z) = \begin{cases} -\int_0^T (iH(z))(t)\circ dW(z, z_t^*) + \frac{1}{2}\|H(z)\|_{\mu_0}^2 + \frac{1}{2}\|z\|_{\mu_0}^2 & \text{if } z \in \mathcal{H}_{\mu_0} \\ \infty & \text{else.} \end{cases}$$

One may note that for $z \in \mathcal{H}_{\mu_0}$ we have

$$\begin{aligned}\int_0^T (iH(z))(t)\circ dW(z, z_t^*) &= -\langle (iH)(z), (iz)\rangle_{L^2} \\ &= -\langle H(z), z\rangle_{\mu_0}.\end{aligned}$$

Therefore we arrive at

$$\mathrm{FW}(z) = \begin{cases} \frac{1}{2}\|z - H(z)\|_{\mu_0}^2 & \text{if } z \in \mathcal{H}_{\mu_0} \\ \infty & \text{else.} \end{cases}$$

Finally, note that

$$\mathrm{OM}_\varepsilon(z) = \begin{cases} \frac{1}{2\varepsilon^2}\|z - H(z)\|_{\mu_0}^2 + \frac{1}{2}[(iH)(z), W(z, z_t^*)](T) & \text{if } z \in \mathcal{H}_{\mu_0} \\ \infty & \text{else.} \end{cases}$$

□

Remark 4.4. Note that the remainder term

$$\varepsilon^2 \,\mathrm{OM}_\varepsilon(z) - \mathrm{FW}(z) = \frac{\varepsilon^2}{2}[(iH)(z), W(z, z_t^*)](T)$$

could be seen as a test of whether μ^ε is Gaussian or not. For example, if $iH(z) = b(z(t))$ for some sufficiently regular $b : \mathbb{R} \to \mathbb{R}$, as is the case with SDEs, then

$$\frac{\varepsilon^2}{2}[(iH)(z), W(z, z_t^*)](T) = \frac{\varepsilon^2}{2}\int_0^T b'(z(t))dt.$$

Which is constant as a function of z if and only if b' is constant. That is, if $W(H) - \frac{1}{2}\|H\|_{\mu_0}^2$ is a quadratic function.

This is not always the case, as we can show. Letting μ_0 be the law of a Brownian motion on the space of continuous functions, we can consider the density defined by

$$\Psi(B) = \exp\left(\frac{1}{\varepsilon^2}\left(\int_0^T \phi(t)dB(t) - \frac{1}{2}\int_0^T \phi^2(t)dt\right)\right),$$

where ϕ is the adapted process defined by

$$\phi(t) = \begin{cases} B(t) & \text{if } t \in [0, T/2] \\ -B(T-t) & \text{if } t \in [T/2, T]. \end{cases}$$

Then converting from Itô to Stratonovich yields zero quadratic covariation with $B(t)$, and thus the functions are equal but the measures $\mu^\varepsilon := \Psi(B)\mu_0^\varepsilon$ are not Gaussian.

4.1. *Examples.*

4.1.1. *Stochastic Differential Equations.*

The principal examples of measures satisfying Theorem 1.1 are the laws of solutions to stochastic differential equations. There is the classical theory of SDEs driven by Brownian motion that motivates our proof, but our result also extends to stochastic differential equations driven by more general Gaussian processes (see e.g. Budhiraja and Song (2020)), path dependent SDEs (see e.g. Ma et al. (2016)), the solution to stochastic PDEs driven by Gaussian fields (see e.g. Liu et al. (2020)), among other SDEs.

We provide one example here for a path-dependent SDE to demonstrate the potential utility of our result. In order to compute explicitly, we restrict to a linear path-dependent SDE which is Gaussian so our full machinary isn't necessary. Nonetheless, modulo solving nonlinear ODEs as in Ma et al. (2016), this general approach should in principle be able to be applied to more general path-dependent SDEs. Consider the stochastic process

$$X^\varepsilon(t) = \int_0^t a(s)\varepsilon B(s)ds + \varepsilon B(t),$$

where a is continuous. As $X^\varepsilon = \varepsilon X^1$ is Gaussian we know that its Onsager-Machlup and Freidlin-Wentzell functions satisfy $\mathrm{FW}(\cdot) = \varepsilon^2 \,\mathrm{OM}_\varepsilon(\cdot) = \frac{1}{2}\|\cdot\|_\mu^2$, where $\|\cdot\|_\mu$ is the Cameron-Martin norm associated to X^1, but we can still apply Girsanov and our Theorem 1.1.

An application of Girsanov shows that the law of $X^\varepsilon(t)$, μ^ε with respect to the law of $\varepsilon B(t)$, μ_0^ε is given by

$$\begin{aligned}\frac{d\mu^\varepsilon}{d\mu_0^\varepsilon} =\exp\Bigg(\frac{1}{\varepsilon^2}\Bigg(\int_0^T\int_0^t e^{-(A(t)-A(s))}dB(s)dB(t) \\ -\frac{1}{2}\int_0^T\left(\int_0^t e^{-(A(t)-A(s))}dB(s)\right)^2 ds\Bigg)\Bigg),\end{aligned} \tag{4.1}$$

where $A' = a$. Converting to Stratonovich integration gives that μ_0^ε-a.s. we have

$$\begin{aligned}\frac{d\mu^\varepsilon}{d\mu_0^\varepsilon} = \exp\Bigg(\frac{1}{\varepsilon^2}\Bigg(\int_0^T\int_0^t e^{-(A(t)-A(s))}\circ dB(s)\circ dB(t) \\ -\frac{1}{2}\int_0^T\left(\int_0^t e^{-(A(t)-A(s))}\circ dB(s)\right)^2 ds\Bigg)\Bigg),\end{aligned}$$

so the Onsager-Machlup function is

$$\mathrm{OM}_\varepsilon(z) = \frac{1}{2\varepsilon^2}\int_0^T\left(z'(t)-\int_0^t e^{-(A(t)-A(s))}z'(s)ds\right)^2 dt = \frac{1}{2\varepsilon^2}\|z\|_\mu^2.$$

Our Theorem 1.1 shows that $\varepsilon^2\,\mathrm{OM}_\varepsilon(\cdot) \to \mathrm{FW}(\cdot)$ both pointwise and as a Γ-limit (which for emphasis we repeat is known from the fact that X^ε is Gaussian). For a more general path-dependent SDE

$$dX(t) = b(X(t), B(t), t)dt + dB(t)$$

the density in equation (4.1) is more complicated and in general cannot be written down explicitly. However Girsanov ensures that it exists and a similar procedure would yield the same result.

4.1.2. *System of Random Algebraic Equations.* We conclude with an example demonstrating that the utility of our result extends beyond the situation of SDEs. Let a_n be a sequence of real numbers that is square summable. Let ξ_n be a sequence of i.i.d. standard normal random variables. Then by Hairer (2009), exercise 3.5 we have that the law of $\mathbf{g} := (a_1\xi_1, x_2\xi_2, ...)$, μ_0, is a Gaussian measure on the Banach (Hilbert) space $\mathcal{B}$ of square summable sequences. The Cameron-Martin space of μ_0, $\mathcal{H}_{\mu_0}$, is the collection of all sequences $\mathbf{z} = \{a_n^2\phi_n\}$ for some square summable sequence $\phi = \{\phi_n\}$. For each $\mathbf{z} \in \mathcal{H}_{\mu_0}$, we have that $W(\mathbf{z}) = \langle \phi, \mathbf{g}\rangle = \sum_{n=1}^\infty \phi_n a_n \xi_n$ where $W(\mathbf{z})$ is understood as in Proposition 4.3. The Onsager-Machlup for μ_0 by Theorem 2.6 is $\mathrm{OM}_{\mu_0}(\mathbf{z}) = \frac{1}{2}\sum_{n=1}^\infty \phi_n^2 a_n^2 = \frac{1}{2}\|\mathbf{z}\|_{\mu_0}^2$. Let f_n be measurable functions so that $f_n(\xi_n)$ satisfy for all $\varepsilon > 0$

$$E_{\mu_0}\left[e^{\frac{1}{2\varepsilon^2}\sum f_n^2(\varepsilon\xi_n)a_n^2}\right] < \infty.$$

Denote the law of $\mathbf{g}^\varepsilon = (a_1\varepsilon\xi_1, a_2\varepsilon\xi_2, ...)$ by μ_0^ε and define the measures

$$\mu^\varepsilon = \exp\left(\frac{1}{\varepsilon^2}\left(\sum_{n=1}^\infty f_n(\xi_n)a_n\xi_n - \frac{1}{2}\sum_{n=1}^\infty f_n^2(\xi_n)a_n^2\right)\right)\mu_0^\varepsilon.$$

Suppose that there exist solutions x_n to the random algebraic equations $x_n^\varepsilon = f_n(x_n^\varepsilon) + \varepsilon a_n\xi_n$. Then μ^ε is the law of the random sequence $\mathbf{x}^\varepsilon = \{x_n^\varepsilon\}$. We have that $\mathbf{x}^\varepsilon$ satisfies a LDP with rate function $\mathrm{FW}_\mu(\mathbf{z}) = \frac{1}{2}\sum_{n=1}^\infty(\phi_n - f(\phi_n))^2 a_n^2 = \varepsilon^2\,\mathrm{OM}_{\mu^\varepsilon}(\mathbf{z})$.

References

Bogachev, V. I. *Gaussian measures*, volume 62 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI (1998). ISBN 0-8218-1054-5. MR1642391.

Braides, A. Γ*-convergence for beginners*, volume 22 of *Oxford Lecture Series in Mathematics and its Applications*. Oxford University Press, Oxford (2002). ISBN 0-19-850784-4. MR1968440.

Budhiraja, A. and Song, X. Large deviation principles for stochastic dynamical systems with a fractional Brownian noise. *ArXiv Mathematics e-prints* (2020). arXiv: 2006.07683.

Cassandro, M., Olivieri, E., and Picco, P. Small random perturbations of infinite-dimensional dynamical systems and nucleation theory. *Ann. Inst. H. Poincaré Phys. Théor.*, **44** (4), 343–396 (1986). MR850897.

Cuff, P., Ding, J., Louidor, O., Lubetzky, E., Peres, Y., and Sly, A. Glauber dynamics for the mean-field Potts model. *J. Stat. Phys.*, **149** (3), 432–477 (2012). MR2992796.

Dashti, M., Law, K. J. H., Stuart, A. M., and Voss, J. MAP estimators and their consistency in Bayesian nonparametric inverse problems. *Inverse Problems*, **29** (9), 095017, 27 (2013). MR3104933.

Davies, E. B. Metastability and the Ising model. *J. Statist. Phys.*, **27** (4), 657–675 (1982). MR661682.

Dembo, A. and Zeitouni, O. Onsager-Machlup functionals and maximum a posteriori estimation for a class of non-Gaussian random fields. *J. Multivariate Anal.*, **36** (2), 243–262 (1991). MR1096669.

Dembo, A. and Zeitouni, O. *Large deviations techniques and applications*, volume 38 of *Applications of Mathematics (New York)*. Springer-Verlag, New York, second edition (1998). ISBN 0-387-98406-2. MR1619036.

den Hollander, F. *Large deviations*, volume 14 of *Fields Institute Monographs*. American Mathematical Society, Providence, RI (2000). ISBN 0-8218-1989-5. MR1739680.

Du, Q., Li, T., Li, X., and Ren, W. The graph limit of the minimizer of the Onsager-Machlup functional and its computation. *Sci. China Math.*, **64** (2), 239–280 (2021). DOI: 10.1007/s11425-019-1650-7.

Dürr, D. and Bach, A. The Onsager-Machlup function as Lagrangian for the most probable path of a diffusion process. *Comm. Math. Phys.*, **60** (2), 153–170 (1978). MR489608.

Dutra, D. A., Teixeira, B. O. S., and Aguirre, L. A. Maximum *a posteriori* state path estimation: discretization limits and their interpretation. *Automatica J. IFAC*, **50** (5), 1360–1368 (2014). MR3198774.

E, W., Ren, W., and Vanden-Eijnden, E. Minimum action method for the study of rare events. *Comm. Pure Appl. Math.*, **57** (5), 637–656 (2004). MR2032916.

Freidlin, M. I. and Wentzell, A. D. *Random perturbations of dynamical systems*, volume 260 of *Grundlehren der mathematischen Wissenschaften*. Springer-Verlag, New York (1984). ISBN 0-387-90858-7. MR722136.

Gao, Y. and Liu, J.-G. A selection principle for weak KAM solutions via Freidlin-Wentzell large deviation principle of invariant measures. *SIAM J. Math. Anal.*, **55** (6), 6457–6495 (2023). MR4662405.

Hairer, M. An introduction to stochastic PDEs. *ArXiv Mathematics e-prints* (2009). arXiv: 0907.4178.

Hongler, M.-O. and Desai, R. C. Decay of unstable states in presence of fluctuations. *Helv. Phys. Acta*, **59** (3), 367–389 (1986). MR845434.

Li, T. and Li, X. Gamma-Limit of the Onsager–Machlup Functional on the Space of Curves. *SIAM J. Math. Anal.*, **53** (1), 1–31 (2021). DOI: 10.1137/20M1310539.

Liu, W., Tao, C., and Zhu, J. Large deviation principle for a class of SPDE with locally monotone coefficients. *Sci. China Math.*, **63** (6), 1181–1202 (2020). MR4119552.

Lu, Y., Stuart, A., and Weber, H. Gaussian approximations for transition paths in Brownian dynamics. *SIAM J. Math. Anal.*, **49** (4), 3005–3047 (2017). MR3684411.

Ma, J., Ren, Z., Touzi, N., and Zhang, J. Large deviations for non-Markovian diffusions and a path-dependent Eikonal equation. *Ann. Inst. Henri Poincaré Probab. Stat.*, **52** (3), 1196–1216 (2016). MR3531706.

Mayer-Wolf, E. and Zeitouni, O. Onsager Machlup functionals for non-trace-class SPDEs. *Probab. Theory Related Fields*, **95** (2), 199–216 (1993). MR1214087.

Nualart, D. *The Malliavin calculus and related topics*. Probability and its Applications (New York). Springer-Verlag, Berlin, second edition (2006). ISBN 978-3-540-28328-7; 3-540-28328-5. MR2200233.

Øksendal, B. *Stochastic differential equations. An introduction with applications*. Universitext. Springer-Verlag, Berlin, sixth edition (2003). ISBN 3-540-04758-1. MR2001996.

Olivieri, E. Metastability and entropy. In *Entropy*, Princeton Ser. Appl. Math., pp. 233–250. Princeton Univ. Press, Princeton, NJ (2003). MR2035824.

Olivieri, E. and Vares, M. E. *Large deviations and metastability*, volume 100 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge (2005). ISBN 0-521-59163-5. MR2123364.

Onsager, L. and Machlup, S. Fluctuations and irreversible processes. *Phys. Rev. (2)*, **91**, 1505–1512 (1953). MR57765.

Selk, Z., Haskell, W., and Honnappa, H. Information projection on Banach spaces with applications to state independent KL-weighted optimal control. *Appl. Math. Optim.*, **84** (suppl. 1), S805–S835 (2021). MR4316802.

Selk, Z. and Honnappa, H. A Feynman-Kac type theorem for ODEs: Solutions of second order ODEs as modes of diffusions. *ArXiv Mathematics e-prints* (2021). arXiv: 2106.08525.

Stuart, A. M. Inverse problems: a Bayesian perspective. *Acta Numer.*, **19**, 451–559 (2010). MR2652785.