A NEW APPROACH TO THE MEAN-FIELD LIMIT OF VLASOV-FOKKER-PLANCK EQUATIONS

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ABSTRACT. This article introduces a novel approach to the mean-field limit of stochastic systems of interacting particles, leading to the first ever derivation of the mean-field limit to the Vlasov-Poisson-Fokker-Planck system for plasmas in dimension 2 together with a partial result in dimension 3. The method is broadly compatible with second order systems that lead to kinetic equations and it relies on novel estimates on the BBGKY hierarchy. By taking advantage of the diffusion in velocity, those estimates bound weighted L^p norms of the marginals or observables of the system, uniformly in the number of particles. This allows to qualitatively derive the mean-field limit for very singular interaction kernels between the particles, including repulsive Poisson interactions, together with quantitative estimates for a general kernel in L^2.

1. INTRODUCTION

The rigorous derivation of kinetic models such as the Vlasov-Poisson system from many-particle systems has been a long standing open question, ever since the introduction of the Vlasov-Poisson system in [82, 83]. While our understanding of the mean-field limit for singular interactions has made significant progress for first-order dynamics, the mean-field limit for second-order systems has remained frustratingly less understood. This article proposed a new approach that is broadly applicable to second-order systems with repulsive interactions and diffusion in velocity. In particular this allows to derive for the first time the Vlasov-Poisson-Fokker-Planck system in dimension higher than one without any truncation or regularizing.

We more precisely consider the classical second-order Newton dynamics

$$
\frac{d}{dt}X_i(t) = V_i(t), \quad X_i(t=0) = X_i^0,
$$

(1)

$$
dV_i(t) = \frac{1}{N}\sum_{j\neq i} K(X_i - X_j)\,dt + \sigma\,dW_i, \quad V_i(t=0) = V_i^0,
$$

where the W_i are N independent Wiener processes. For simplicity we take the positions X_i on the torus Π^d, while the velocities lie in $\mathbb{R}^d$. The kernel K models the pairwise interaction between particles and is taken *repulsive* throughout this paper, in the basic sense that it derives from a potential $K = -\nabla\phi$ that is even and positive, $\phi \geq 0$.

Remark 1. *For simplicity, we denote $\phi(0) = 0$ and $K(0) = 0$, even if ϕ and K are not continuous at 0. This notation simplifies the writing in the equation by allowing to sum over all j in* (1) *since the term $j = i$ trivially vanishes.*

We naturally focus on singular kernels K with, as a main guiding example, the case of Coulombian interactions

$$(2) \qquad\qquad K = \alpha\, \frac{x}{|x|^d} + K_0(x),$$

with $\alpha > 0$ and K_0 a smooth correction to periodize K. This corresponds to the choice $\phi = \frac{d\alpha}{|x|^{d-2}} + \text{correction}$ if $d \geq 3$ or $\phi = -\alpha \ln|x| + \text{correction}$ if $d = 2$.

The Coulombian kernel (2) typically model electrostatic interactions between point charges, such as ions or electrons in a plasma when the velocities are small enough with respect to the speed of light. In that setting, diffusion in (1) may for example represent collisions against a random background, such as the collision of the faster electrons against the background of ions. Such random collisions may also involve some friction in velocity, which we did not include in (1) but could be added to our method without difficulty. This makes (1) with (2) one of the most classical and important starting point for the modeling of plasmas; we refer in particular to the classical [6, 7].

Coulombian interactions are also a natural scaling in many models. The obvious counterpart to plasmas concerns the Newtonian dynamics of point masses through gravitational interactions. This consists in taking $\alpha < 0$ in (2) and leads to attractive interactions with a negative potential and for this reason cannot be handled with the method presented here.

The system (1) usually involve a very large number of particles, typically up to $10^{20} - 10^{25}$ in plasmas for example. This makes the mean-field limit especially attractive. This is a kinetic, Vlasov-Fokker-Planck equation posed on the limiting 1-particle density $f(t, x, v)$

$$(3)\quad \partial_t f + v \cdot \nabla_x f + (K \star_x \rho) \cdot \nabla_v f = \frac{\sigma^2}{2}\, \Delta_v f \quad \text{with} \quad \rho = \int_{\mathbb{R}^d} f\, dv.$$

Well posedness for mean-field kinetic equations such as (3) is now reasonably well understood, including for singular Coulombian interactions such as (2) in dimension $d \leq 3$. For the non-diffusive case $\sigma = 0$, weak solutions were established in [1], while classical solutions were obtained in dimension 2 in [78]. The dimension 3 case is harder and obtaining classical solutions requires more difficult dispersive argument and were only obtained later in [58, 66, 73], see also the more recent [28, 43, 59, 65]. In the case with diffusion $\sigma > 0$, we refer to [80] for weak solutions, and to [10, 20, 63, 70, 81] for classical solutions.

Of course the mean-field scaling is not the only possible scaling on systems such as (1). One can in particular mention the likely even more critical Boltzmann-Grad limit, such as obtained in the classical [53] and the recent major results in [3, 4, 25, 68, 69]. We note as well that the derivation of macroscopic equations from mesoscopic systems such as (3) is another important and challenging questions. For example the passage to the fluid macroscopic system from Vlasov-Poisson-Fokker-Planck has been approached in different low-field (parabolic) or high-field (hyperbolic) regimes depending on the space dimension, see for example [67, 62, 34, 16] and references therein.

Mean-field limits have been rigorously derived for general systems, including second order dynamics such as (1), in the case of Lipschitz interaction kernels K. We refer to the classical works [60, 77] in the stochastic case and [11, 22] for the deterministic case. Uniform in time propagation of chaos has also been obtained in the locally Lipschitz case, notably in a close to convex case in [8] and more recently in a non-convex setting in [36].

There now exists a large literature on the question of the mean-field limits, see for example the survey in [29, 45, 49]. However in the specific case of second order systems such as (1), very little is known. In dimension $d = 1$, the Vlasov-Poisson-Fokker-Planck system was derived in [35, 42]. In dimension $d \geq 2$, the only results for unbounded interaction kernels were obtained in [40, 41]. But those are valid only in the deterministic case $\sigma = 0$, and for only mildly singulars kernels with $|K(x)| \lesssim |x|^{-\alpha}$, $|\nabla K| \lesssim |x|^{-\alpha-1}$ for $\alpha < 1$. [46] derived the mean-field limit with $K \in L^\infty$ and without extra derivative. Those cannot cover Coulombian interactions, even in dimension 2.

More is known for singular interaction kernels K that are smoothed or truncated at some N-dependent scale ε_N. In that truncated case, one can mention in particular [26, 27, 79, 84] for the convergence of so-called particle methods. The recent works [9, 54, 55] in the deterministic case and [44] in the stochastic case considerably extended the results for such truncated kernels and allowed to almost reach the critical physical

scale $\varepsilon_N \sim N^{-1/d}$. One can also mention [15] with polynomial cut-off. It is also possible to derive the Vlasov-Poisson system directly from many-particle quantum dynamics such as the Hartree equation, for which we briefly refer to [31, 52, 72].

The mean-field limits for first order systems with singular interactions appear to be more tractable. A classical example concerns the dynamics of point vortices or stochastic point vortices where the mean-field limit corresponds to the vorticity formulation of 2d incompressible Euler or Navier-Stokes. The interaction between vortices obey the Biot-Savart law which has the same singularity as the Coulombian kernel in dimension 2. In the deterministic case the mean-field limit was classically obtained for example in [32, 33] or [74, 75] for the 2d Euler and extended to remarkably essentially any Riesz kernels in [76]. In the stochastic case, we refer in particular to [24, 64, 48] for the limit to 2d Navier-Stokes, to [13, 14] for singular attractive kernels, or to [61] for multiplicative noise. Uniform in time propagation of chaos was even recently obtained in [37, 71].

One of the reason second order systems appear more difficult to handle stems from how the structure of the singularity interacts with the distribution of velocities. Because of the term $K(X_i - X_j)$, the singularity in pairwise interactions is typically localized on collisions $X_i = X_j$. For first-order systems this corresponds to a point singularity, while for second-order systems the presence of the additional velocity variables makes it into a plane. In that regard, we also note that the derivation of macroscopic system directly from 2nd order dynamics is in fact better understood than the derivation of kinetic equations like (3). We refer to the derivation of incompressible Euler in [38], or to the derivation of monokinetic solutions to (3) (which are essentially equivalent to a macroscopic system) in [76].

The main argument in our proof is a new quantitative estimate on the so-called marginals of the system through the BBGKY hierarchy. This leads to the propagation of some weighted L^p estimates on the marginals. It implies a weak propagation of chaos in the sense of [77] but it applies more broadly to initial data that are not chaotic or not close to being chaotic.

Recently new approaches have been introduced to bound marginals on 1st order systems with non-degenerate diffusion: in [51] using the relative entropy and in [47] using the L^2 norm of the marginals. Both take advantage of the regularizing provided by the diffusion to avoid "losing" a derivative in the hierarchy estimates but require interaction kernels K in L^∞ for [47] (or in some exponential Orlicz space for [51]).

Our method combines this general idea with a specific choice of weights for the L^p norms that are propagated. Those weights are based on a total energy reduced to k particles when dealing with the marginal of order k. They allow to take advantage of a further regularizing effect in the hierarchy to only require kernels K in some L^p with $p > 1$.

A direct consequence of our approach is the first ever derivation of the mean-field limit for the repulsive Vlasov-Poisson-Fokker-Planck over a finite time interval. This applies to any chaotic initial data in dimension $d = 2$ and for initial data with more restrictive energy bound in any dimension $d \geq 3$. We are expecting to extend this derivation in a future work to any chaotic initial data in any dimension $d \geq 2$ by decomposing appropriately the initial data.

The paper is structured as follows: We start in Section 2 with the notations and main results. We first state our main result, Theorem 1, that proves the convergence to the Vlasov-Fokker-Planck equation as N tend to infinity followed with a Theorem 2 proving quantitative estimates for singular kernels in L^2. We next introduce Proposition 1 which states the explicit propagation of weighted L^p bounds on the marginals. We in particular discuss more thoroughly the limitations and possible extensions of our approach after stating Proposition 1. Section 3 is devoted to the proof of Proposition 1 and Theorem 1 from the key technical contribution of the article around Lemma 1 and ends with the proof of Theorem 2.

2. Main results

2.1. **The new result.** We introduce the full N-particle joint law of the system f_N which satisfies the Liouville or forward Kolmogorov equation

$$
(4) \qquad
\begin{aligned}
&\partial_t f_N + \sum_{i=1}^{N} v_i \cdot \nabla_{x_i} f_N \\
&+ \sum_{i=1}^{N} \frac{1}{N} \sum_{j=1}^{N} K(x_i - x_j) \cdot \nabla_{v_i} f_N = \frac{\sigma^2}{2} \sum_i \Delta_{v_i} f_N,
\end{aligned}
$$

which is a linear advection-diffusion equation. However the marginals $f_{k,N}$ of f_N will also play a critical role in the analysis. They correspond to the law of k among N particles and are represented through

$$
(5) \qquad
\begin{aligned}
&f_{k,N}(t, x_1, v_1, \ldots, x_k, v_k) = \\
&\int_{\Pi^{d(N-k)} \times \mathbb{R}^{d(N-k)}} f_N(t, x_1, v_1 \ldots, x_N, v_N) \, dx_{k+1} \, dv_{k+1} \ldots dx_N \, dv_N.
\end{aligned}
$$

The question of well-posedness for Eq. (4) can be delicate and is separate from the issue of the mean-field limit that we consider here. For this reason, we consider a notion of entropy solution $f_N \in L^\infty(\mathbb{R}_+ \times \Pi^{dN} \times \mathbb{R}^{dN})$ to Eq. (4), that is fully described later in subsection 2.4, and to which we impose some Gaussian decay in velocity

$$(6) \qquad \sup_{t \leq 1} \int_{\Pi^{dN} \times \mathbb{R}^{dN}} e^{\beta \sum_{i \leq N} |v_i|^2} f_N \, dx_1 \, dv_1 \ldots dx_N \, dv_N \leq V^N,$$

$$\text{for some } \beta > 0, \ V > 0,$$

for which we refer to the short discussion in subsection 2.4.

Our main is the derivation of the mean-field limit for a broad class of singular kernels.

Theorem 1. *Assume that that there exists some constant $\theta > 0$ s.t. the potential ϕ satisfies*

$$(7) \qquad \int_\Pi e^{\theta \, \phi(x)} \, dx < +\infty,$$

and that

$$K = -\nabla\phi \in L^p(\Pi^d) \qquad \text{for some} \qquad p > 1.$$

Let f be the unique smooth solution to the Vlasov equation (3) with initial data $f^0 \in C^\infty(\Pi^d \times \mathbb{R}^d)$ such that $\int_{\Pi^d \times \mathbb{R}^d} f^0 \, e^{\beta \, |v|^2} < \infty$. Consider moreover an entropy solution f_N to (4) in the sense of subsection 2.4 and satisfying (6) with initial data $f_N^0 \in L^\infty(\Pi^{dN} \times \mathbb{R}^{dN})$. Assume that $f_{k,N}^0$ converges weakly in L^1 to $(f^0)^{\otimes k}$ for each fixed k and that

$$\|f_{k,N}^0\|_{L^\infty(\Pi^{dN} \times \mathbb{R}^{dN})} \leq M^k,$$

for some $M > 0$ and for all $k \leq N$. Then there exists T^* depending only on M, V, and $\|K\|_{L^p}$ such that $f_{k,N}$, given by (5), weakly converge to $f_k = f^{\otimes k}$ in $L^q_{loc}([0, T^\star] \times \Pi^{kd} \times \mathbb{R}^{kd})$ for any k, and any $2 < q < \infty$, with $1/q + 1/p \leq 1$.

Our estimates can also provide quantitative rates of convergence though we need to a stronger assumption, namely $K \in L^2$.

Theorem 2. *Assume the same conditions and hypotheses of Theorem 1, with moreover $p = 2$. We also assume that there exists a constant C independent of N and $\varepsilon_N \to 0$ such that*

$$\int_{\Pi^{kd} \times \mathbb{R}^{kd}} |f_{k,N}^0 - (f^0)^{\otimes k}|^2 e^{\lambda(0)e_k} \leq C^k \varepsilon_N,$$

for all k. Then, there exists T^* such that $f_{k,N}$ converges strongly to f_k in $L^2_{loc}([0, T^\star] \times \Pi^{kd} \times \mathbb{R}^{kd})$, for any k, and we have the following

quantitative estimate

$$\sup_{t \leq T^\star} \int_{\Pi^{kd} \times \mathbb{R}^{kd}} |f_{N,k} - f^{\otimes k}|^2 e^{\lambda(t)e_k} \leq \widetilde{C}^k \varepsilon_N,$$

for some $\widetilde{C}$ independent of N.

In addition to the mean-field limit, Theorem 1 implies the weak propagation of chaos in the sense of the famous [77], although with strong conditions on f_N^0. Theorem 1 also justifies for the first time the convergence to the Vlasov-Poisson-Fokker-Planck in two space dimension. More precisely, we highlight the following result

Corollary 1. *Let $d = 2$ and consider the Poisson kernel $K = -\nabla\phi$ with its associated potential $\phi(x) \simeq -\ln|x|$. Then, the convergence properties given by Theorem 1 hold true, leading to the Vlasov-Poisson-Fokker-Planck system.*

2.2. **New stability estimates.** Theorem 1 relies on a new approach to estimate on the BBGKY hierarchy solved by the marginals $f_{k,N}$ which is of significant interest in itself. In general deriving bounds on either the BBGKY or limiting Vlasov hierarchy is complex. We refer for example to [30] for the Vlasov hierarchy, to [23] for the study of long-time corrections to mean-field limits. Bounds on the hierarchy are critical for the derivation of collisional models such as the Boltzmann equation, ever since [53]. Even a partial discussion of the challenges in the collisional setting would go well beyond the scope of this paper and we simply refer again to [2, 3, 4, 25, 50, 53, 68, 69].

The main difficulty in handling the hierarchy consists in the term

$$(8) \qquad \nabla_{v_i} \int_{\Pi^d \times \mathbb{R}^d} K(x_i - x_{k+1}) \, f_{k+1,N} \, dx_{k+1} \, dv_{k+1},$$

as seen in (17), because this introduces the next order marginal $f_{k+1,N}$ into the equation for $f_{k,N}$. When treated naively as a source term, it leads to a loss of one derivative on each equation of the hierarchy.

However it was recently noticed in [47, 51] that one may avoid this loss of derivative in the stochastic case for non-degenerate diffusion: Any L^2 estimate then gains an additional H^1 dissipation which can be used to control the loss of one derivative. This idea still appears applicable in the present kinetic context: Even though we only have diffusion in velocity, the derivative in (8) is also only on the velocity variable.

Both [47, 51] require high integrability on the kernel: $K \in L^\infty$ for [47] and some sort of exponential Orlicz space of the type $\int e^{\lambda|K(x)|} \, dx < C$

for [51]. Paper [51] used relative entropy estimates to prove unique-
ness on the BBGKY hierarchy, while [47] proved uniqueness on a tree-
indexed limiting hierarchy through L^2 bounds. Hence in both case, the
corresponding bounds on the marginals was already known uniformly
in N and the challenge was to prove smallness.

This leads to a first key difference with respect to the present ap-
proach and to the first critical new idea introduced in this paper. In
essence, we note that the integral in (8) leads to a regularizing effect
that has the same scaling as the convolution at the limit: One has by
Hölder estimates that

$$(9) \qquad \left\| \int_{\Pi^d} K(x_i - x_{k+1}) \, f(x_1, \ldots, x_{k+1}) \, dx_{k+1} \right\|_{L^q(\Pi^{dk})}$$
$$\leq \|K\|_{L^p(\Pi^d)} \|f\|_{L^q(\Pi^{d(k+1)})},$$

provided that $1/p + 1/q \leq 1$.

Taking advantage of (9) for singular $K \in L^p$ with p small naturally
leads to try to propagate L^q norms of the marginals $f_{k,N}$ for large ex-
ponents q; at the opposite of [47, 51]. But it also leads to an additional
major difficulty, due to the velocity variable in the unbounded space $\mathbb{R}^d$
in (8). In fact, trying to use (9) in (8) as is would force the use of a
mixed norm $L_x^q L_v^1$ on the marginals. Unfortunately such mixed norms
are notoriously ill-behaved on kinetic equations.

Instead a more natural idea, from the point of view of kinetic equa-
tions, consists in using some moments or fast decay in velocity. Even if
they are less usual for kinetic equations, the use of Gaussian moments
is especially attractive in the current case because they are naturally
tensorized. For example, one has the extension of (9)

$$(10) \qquad \int_{\Pi^{dk} \times \mathbb{R}^{dk}} e^{|v_1|^2 + \ldots + |v_k|^2} \left| \int_{\Pi^d \times \mathbb{R}^d} K(x_i - x_{k+1}) \, f_{k+1,N} \, dx_{k+1} \, dv_{k+1} \right|^q$$
$$\leq C_d \, \|K\|_{L^p(\Pi^d)}^q \int_{\Pi^{d(k+1)} \times \mathbb{R}^{d(k+1)}} e^{|v_1|^2 + \ldots + |v_{k+1}|^2} \, |f_{k+1,N}|^q$$

still provided $1/p + 1/q \leq 1$.

However pure Gaussian moments in velocity does not seem to be
naturally propagated at the discrete level of the hierarchy, even though
they would trivially be propagated on the limiting mean-field equation
at least for short time. This leads to the final critical idea of the paper,
which is to incorporate the potential energy in the Gaussian: Namely
to consider $e^{\lambda(t) e_k}$ instead of a pure Gaussian with

$$(11) \qquad e_k(x_1, v_1, \ldots, x_k, v_k) = \sum_{i \leq k} (1 + |v_i|^2) + \frac{1}{N} \sum_{i,j \leq k} \phi(x_i - x_j).$$

We remark that the use dynamical weights argument has been recently developed in [14] for first order particle systems with singular kernels. We also note that Proposition 1, stated below, shows the propagation of weighted L^q bounds on the marginals, without requiring the initial data to be chaotic or close to chaotic as introduced in [50]. It hence applies to a broader framework than just the mean-field limit.

Proposition 1. *Let us assume $K \in L^p(\Pi^d)$, for some $p > 1$, and define*

$$\lambda(t) = \frac{1}{\Lambda(1+t)}, \qquad L = \frac{C}{\lambda(1)^\theta}\|K\|_{L^p}^q,$$

for positive constants Λ, C, θ depending only on q, d and σ and q and $1/q+1/p \le 1$. Consider a renormalized solution f_N to (4) satisfying (6) with initial data $f_N^0 \in L^\infty(\Pi^{dN} \times \mathbb{R}^{dN})$, satisfying

(12)
$$\int_{\Pi^{kd} \times \mathbb{R}^{kd}} |f_{k,N}^0|^q \, e^{\lambda(0) \, e_k} \le F_0^k,$$

$$\sup_{t \le 1} \int_{\Pi^{Nd} \times \mathbb{R}^{Nd}} |f_N|^q \, e^{\lambda(t) \, e_N} \le F^N,$$

for some $F > 0$, $F_0 > 0$ and q such that $2 \le q < \infty$, with $1/q+1/p \le 1$. Then, one has that

(13)
$$\sup_{t \le T} \int_{\Pi^{kd} \times \mathbb{R}^{kd}} |f_{k,N}|^q \, e^{\lambda(t) \, e_k} \le 2^k \, F_0^k + F^k \, 2^{2k-N-1},$$

where T is given by

$$T \le \min\left(1, \frac{1}{4 \, L \, \max(F_0, F)}\right).$$

Proposition 1 shows that the corresponding L^q norm of a marginal at order k behaves like C^k for some constant C. This is the expected scaling for propagation of chaos and tensorized marginals $f_k = f^{\otimes k}$.

However Proposition 1 also presents several intriguing features that we want to highlight.

• *Vlasov-Poisson-Fokker-Planck in higher dimensions.* Proposition 1 handles just as easily Coulombian interactions in any dimension d, and not only dimension $d = 2$ as Theorem 1. Therefore, as claimed, Proposition 1 already implies a result of propagation of chaos for the Vlasov-Poisson-Fokker-Planck system in any dimension. However, as is, this is not fully satisfying because we cannot take $f_N^0 = (f^0)^{\otimes N}$: Assumption (12) cannot hold in such a case as $e^{\lambda(0) \, e_k}$ is not integrable if K is the Poisson kernel in dimension $d > 2$. The issue is that by taking $f_N^0 = (f^0)^{\otimes N}$, we allow some configurations with high potential

energy. And roughly speaking the existence time T in the proposition vanishes as the starting potential energy increases in that case.

• *Repulsive potentials.* Proposition 1 does require repulsive potentials $\phi \geq 0$ as this assumption is critical in the proof. The repulsive assumption on the potential only appears to be needed to handle the discrete many-particle system. The extension to non-repulsive settings remains an open problem.

• *Extension to the stochastic case of mildly singular kernels.* A special case concerns mildly singular kernels K with $K \in L^p$ for some $p > 1$ s.t. $\phi \in L^\infty$. In that situation, by considering $\phi + \|\phi\|_{L^\infty}$ instead of ϕ, yielding the same interaction kernel K, we can always ensure that $\phi \geq 0$. For example this easily extends for the first time to the stochastic settings the results of [40, 41], that had been obtained only for deterministic second order systems with $|K| \lesssim |x|^{-\alpha}$ for $\alpha < 1$.

• *Convergence for finite times.* We finally emphasize that Proposition 1, just as Theorem 1, holds over a finite time interval, independent of N. This may initially appear puzzling since we are dealing with linear equations for any fixed N. However because those estimates are essentially independent of N, they also extend to the non-linear limiting Vlasov equation. Moreover Proposition 1 includes a propagation of Gaussian moments in velocity over the marginals, from the term $e^{\lambda(t)\, e_k}$ and the definition (11) of e_k. The propagation for all times of such moments for Vlasov–Poisson is only known in dimension $d = 2$, see [78, 20], and dimension $d = 3$, see [10, 28, 43, 58, 63, 65, 66, 70, 73, 81] as cited in the introduction; it also requires in dimension 3 the use of dispersion estimates that are not present in our proof. As we already noted, Proposition 1 is in fact valid in any dimension which naturally limits it to some given finite time interval.

2.3. **The case of first order system.** While we focus on second order systems, we also emphasize that our method directly applies to first order systems on bounded domains (in a much simpler manner in fact) and provides the mean-field limit under very weak assumptions on the kernel K again.

Consider in that case

$$(14) \quad \frac{d}{dt}X_i(t) = \frac{1}{N} \sum_{j \neq i} K(X_i - X_j)\, dt + \sigma\, dW_i, , \quad X_i(t = 0) = X_i^0,$$

fully on the torus Π^d. The mean-field limit is similar to (3)

$$(15) \qquad\qquad \partial_t f + (K \star_x f) \cdot \nabla_x f = \frac{\sigma^2}{2}\, \Delta_x f.$$

Similarly the joint law $f_N(t, x_1, \ldots, x_N)$ solves an appropriately modified Liouville equation

$$(16) \qquad \partial_t f_N + \sum_{i=1}^{N} \frac{1}{N} \sum_{j=1}^{N} K(x_i - x_j) \cdot \nabla_{v_i} f_N = \frac{\sigma^2}{2} \sum_i \Delta_{x_i} f_N.$$

Because system (14) does not involve velocities, many technical difficulties in our proofs actually vanish. For example, we do not need anymore to add assumptions such as (6). We do not need either to impose that K derives from a potential and hence do not require assumptions like (7) either.

We then have the following equivalent of Theorem 1.

Theorem 3. *Assume that*

$$K \in L^p(\Pi^d) \quad \text{for some} \quad p > 1, \qquad (div\, K)_- \in L^\infty(\Pi^d),$$

where x_- denotes the negative part of x. Let f be the unique smooth solution to the Vlasov equation (15) with initial data $f^0 \in C^\infty(\Pi^d)$. Consider moreover an entropy solution f_N to (16), still in the sense of subsection 2.4, with initial data $f_N^0 \in L^\infty(\Pi^{dN})$. Assume that $f_{k,N}^0$ converges weakly in L^1 to $(f^0)^{\otimes k}$ for each fixed k and that

$$\|f_{k,N}^0\|_{L^\infty(\Pi^{dN})} \leq M^k,$$

for some $M > 0$ and for all $k \leq N$. Then there exists T^* depending only on M, $\|K\|_{L^p}$ and $\|(div\, K)_-\|_{L^\infty}$ such that $f_{k,N}$, given by (5), weakly converge to $f_k = f^{\otimes k}$ in $L_{loc}^q([0, T^\star] \times \Pi^{kd})$ for any k, and any $2 < q < \infty$, with $1/q + 1/p \leq 1$.

Because it is not our main focus, we do not give a distinct proof of Theorem 3.

2.4. **Our notion of entropy solution for the hierarchy; the well-posedness of Eq.** (4).

2.4.1. *The definition.* Being non-linear, our estimates cannot be performed on any weak solutions. Moreover, the concept of solution for f_N are carried over the marginals $f_{k,N}$ and not just the joint law f_N so that we also need an appropriate notion of entropy solutions on those marginals.

The hierarchy for the marginals from the Liouville equation. From Eq. (4), the $f_{k,N}$ solve the so-called BBGKY hierarchy

$$\partial_t f_{k,N} + \sum_{i=1}^{k} v_i \cdot \nabla_{x_i} f_{k,N} + \sum_{i \leq k} \frac{1}{N} \sum_{j \leq k} K(x_i - x_j) \cdot \nabla_{v_i} f_{k,N}$$

$$(17) \qquad + \frac{N-k}{N} \sum_{i \leq k} \nabla_{v_i} \cdot \int_{\Pi^d \times \mathbb{R}^d} f_{k+1,N} \, K(x_i - x_{k+1}) dx_{k+1} dv_{k+1}$$

$$= \frac{\sigma^2}{2} \sum_{i \leq k} \Delta_{v_i} f_{k,N}.$$

If f_N belongs to L^∞ and verifies (6), then all marginals $f_{k,N}$ belong to $L_t^\infty L_{x,v}^q$ for every $q < \infty$ with similar Gaussian decay. For simplicity, we denote here abstractly $L_{x,v}^q$ any space $L^q(\Pi^{kd} \times \mathbb{R}^{kd})$ when there is no confusion about the dimension k, as in our case. We also denote by $L_{\lambda e_k}^q$ the weighted L^q space

$$\|f\|_{L_{\lambda e_k}^q}^q = \int_{\Pi^{kd} \times \mathbb{R}^{kd}} |f|^q \, e^{\lambda e_k}.$$

Since $K \in L^p$, for some $p > 1$, then by using a direct Hölder inequality those bounds on the $f_{k,N}$ implies that

$$\int_{\Pi^d \times \mathbb{R}^d} f_{k+1,N} \, K(x_i - x_{k+1}) \, dx_{k+1} \, dv_{k+1} \in L_t^\infty L_{x,v}^q,$$

for all $q < \infty$. This allows to immediately and rigorously derive (17) from Eq. (4).

Definition of entropy solutions. We denote the advection component of (17)

$$(18) \qquad L_k = \sum_{i \leq k} v_i \cdot \nabla_{x_i} + \frac{1}{N} \sum_{i,j \leq k} K(x_i - x_j) \cdot \nabla_{v_i}.$$

The argument above implies that the only difficulties to propagate our estimates in (17) stem from L_k. Consequently we define our entropy solution in the following manner: A function $f_N \in L^\infty([0, 1] \times \Pi^{dN} \times \mathbb{R}^{dN})$ satisfying (6) is an entropy solution iff all marginals $f_{k,N}$ for $1 \leq k \leq N$, as defined by (5), satisfy that for any $T \in [0, 1]$, any $1 < q < \infty$ and any $\lambda < \lambda_0$

$$(19) \qquad \int_0^T \int_{\Pi^{dk} \times \mathbb{R}^{dk}} e^{\lambda e_k} \, |f_{k,N}|^{q-1}$$

$$\text{sign} \, f_{k,N} \, L_k \, f_{k,N} \, dx_1 \, dv_1 \ldots dx_k \, dv_k \, dt \geq 0.$$

Inequality (19) is still somewhat formal and should be understood in the following rigorous sense: For some smooth convolution kernel K_ε, one has that

$$(20) \quad \liminf_{\varepsilon \to 0} \int_0^T \int_{\Pi^{dk} \times \mathbb{R}^{dk}} e^{\lambda e_k} |K_\varepsilon^{\otimes k} \star f_{k,N}|^{q-1} \operatorname{sign}(K_\varepsilon^{\otimes k} \star f_{k,N})$$
$$K_\varepsilon^{\otimes k} \star (L_k f_{k,N}) \, dx_1 \, dv_1 \ldots dx_k \, dv_k \, dt \geq 0,$$

where we denote

$$K_\varepsilon^{\otimes k} \star g = \int_{\Pi^{dk} \times \mathbb{R}^{dk}} K_\varepsilon(x_1 - y_1, v_1 - w_1) \ldots K_\varepsilon(x_k - y_k, v_k - w_k)$$
$$g(y_1, w_1, \ldots, y_k, w_k) \, dy_1 \, dw_1 \ldots dy_k \, dw_k.$$

However it is usually more delicate to determine whether any weak solution f_N in L^∞ and with the bound (6) is an entropy solution according to our definition. For linear advection-diffusion equations such as (4), this is usually approached through the notion of renormalized solutions as introduced in [21]. In that context, (20) is obviously similar to the classical commutator estimate at the basis of many methods for renormalized solutions.

Remark 2. 1) *We first remark that* (19) *is automatically satisfied if we have classical solutions. Indeed L_k is an antisymmetric operator so that we expect it to propagate L^q norms so that if all terms are smooth*

$$|f_{k,N}|^{q-1} \, sign \, f_{k,N} \, L_k f_{k,N} = L_k \, |f_{k,N}|^q.$$

2) *We immediately observe that the reduced energy e_k is formally invariant under the advection component of* (17):

$$L_k \, e_k = \frac{2}{N} \sum_{i,j \leq k} v_i \cdot \nabla_{x_i} \phi(x_i - x_j) + \frac{2}{N} \sum_{i,j \leq k} K(x_i - x_j) \cdot v_i = 0,$$

since $K = -\nabla_x \phi$. In the same way, we have $L_k \, \Phi(e_k) = 0$, for any locally Lipschitz function Φ.
3) *If K is smooth and f_N is a classical solution to* (4), *we would hence immediately have equality in* (19). *With K only in L^p, it would be straightforward to obtain one entropy solution in the sense defined above, through passing to the limit in a sequence of solutions for a smoother kernel K.*

Remark 3. *There exists an extensive literature on renormalized solutions with a comparably large variety of potential assumptions that one may consider. While we cannot do justice to this question in this short discussion, we briefly mention for instance [39] that studies the specific*

case of the Liouville equation (4) *for second order systems without diffusion. In the present setting of a constant non-vanishing diffusion, we also refer to* [5, 56, 57] *that provide broad results of well-posedness for velocity fields in* L^p.

We in particular note that renormalized solutions apply to the case $K \in L^p$ *with* $p > 2$ *and* f_N *in* L^∞ *with* $\nabla_{v_i} f_N^{q/2} \in L^2$ *for any* $q < \infty$ *and satisfying the extension of* (6),

$$
\sup_{t \leq 1} \int_{\Pi^{dN} \times \mathbb{R}^{dN}} e^{\lambda_0 e_k} f_N \, dx_1 \, dv_1 \dots dx_N \, dv_N < \infty.
$$

The latter estimates are natural for the Liouville (4), *as demonstrated by Lemma* ?? *for the case* $k = N$ *in Section 3. In that situation, all marginals* $f_{k,N}$ *belong to* $L^\infty_t L^q_{x,v}$ *for every* $q < \infty$ *with similar exponential decay in* e_k *and with as well* $\nabla_{v_i} f_{k,N}^{q/2} \in L^r_{t,x,v}$ *for any* $r < 2$. *This regularity easily allows to prove that* (20) *holds for* $\lambda < \lambda_0$.

We also mention that so-called mild solutions can also offer a natural way to prove (20). *We simply refer to* [10, 17] *for such formulations through the Fokker–Planck kernel in whole space, or to* [18] *or* [20, 81] *for periodic conditions.*

2.4.2. *Strong solutions up to the first collision.* We also emphasize that, in the case of repulsive kernels smooth out of the origin but with singular potentials $\lim_{x \to 0} \phi(x) = +\infty$, a straightforward bound on the energy of the system can easily lead to strong solutions on the many-particle system (1), bypassing the need for entropy or renormalized solutions.

Very roughly, if $K \in C^\infty(\Pi^d \setminus \{0\})$, then up to the conditional time of first collision in (1), we may write that

$$
d\left(\sum_{i=1}^{N} |V_i|^2 + \frac{1}{N} \sum_{i \neq j} \phi(X_i - X_j) \right) = \sigma^2 \, dt + \sum_{i=1}^{N} 2 \, V_i \cdot dW_i.
$$

This implies that, with probability 1, the total energy remains finite if it was so initially. Because $\lim_{x \to 0} \phi(x) = +\infty$, it also implies that collisions almost surely never happen. This argument would in particular apply to the Coulombian case in any dimension $d \geq 2$.

To conclude this discussion of the well posedness of (4) or (1) for a fixed N, we emphasize the estimates that we described here cannot easily be made uniform in N. The previous discussion of the energy bound on the system (1) for the Coulombian interaction in dimension $d = 2$

is an excellent illustration: If we have the following bound

$$\sum_{i=1}^{N} |V_i|^2 + \frac{1}{N} \sum_{i \neq j} \phi(X_i - X_j) \leq E$$

with some large probability on some time interval, and for $\phi(x) = -\log|x|$ then this only proves that for any $i \neq j$

$$|X_i - X_j| \geq e^{-NE},$$

which is indeed finite for any fixed N but is completely unhelpful when considering the limit $N \to \infty$.

Hence the present discussion remains focused on renormalized solutions for a fixed N. Quantitative approach to renormalized solutions have for example been introduced in [19], which are based on the propagation of a sort of log-derivative on the characteristics; see also for example the discussion on Eulerian variants in [12]. This leads to an interesting and so far mostly fully open question as to whether it would be possible to obtain quantitative bounds that would combine the limit $N \to \infty$ with some regularity estimates on the solution for a fixed N.

3. PROOF OF THE MAIN RESULTS

3.1. **The BBGKY and Vlasov hierarchies.** Using (3), the tensorized limits $f_k = \overline{f}^{\otimes k}$ satisfy the following Vlasov hierarchy

(21)
$$\partial_t f_k + \sum_{i=1}^{k} v_i \cdot \nabla_{x_i} f_k$$
$$+ \sum_{i=1}^{k} (K \star \int_{\mathbb{R}^d} f \, dv) \cdot \nabla_{v_i} f_k = \frac{\sigma^2}{2} \sum_{i=1}^{k} \Delta_{v_i} f_k.$$

To avoid repeating the analysis working on (17) or (21), we introduce the generalized hierarchy equation

(22)
$$\partial_t F_{k,N} + \sum_{i=1}^{k} v_i \cdot \nabla_{x_i} F_{k,N} + \sum_{i \leq k} \frac{\gamma}{N} \sum_{j \leq k} K(x_i - x_j) \cdot \nabla_{v_i} F_{k,N}$$
$$+ \frac{N - \gamma k}{N} \sum_{i \leq k} \nabla_{v_i} \cdot \int_{\Pi^d \times \mathbb{R}^d} F_{k+1,N} \, K(x_i - x_{k+1}) dx_{k+1} dv_{k+1}$$
$$= \frac{\sigma^2}{2} \sum_{i \leq k} \Delta_{v_i} F_{k,N} + R_{k,N}.$$

Note that Eq. (22) is exactly Eq. (21) for $\gamma = 0$, $R_{k,N} = 0$ and exactly Eq. (17) for $\gamma = 1$, $R_{k,N} = 0$. In the same spirit we denote

$$e_{k,\gamma} = \sum_{i \leq k}(1 + |v_i|^2) + \frac{\gamma}{N} \sum_{i,j \leq k} \phi(x_i - x_j),$$

$$L_{k,\gamma} = \sum_{i \leq k} v_i \cdot \nabla_{x_i} + \frac{\gamma}{N} \sum_{i,j \leq k} K(x_i - x_j) \cdot \nabla_{v_i}$$

and observe that we of course still have $L_{k,\gamma}\, e_{k,\gamma} = 0$.

The main technical contribution of this section and of the paper is Lemma 1 stated in subsection 3.2, which provides estimates for the solutions to (17). We will then use the uniform bound on the k-marginals $f_{k,N}$ for the proof of Prop. 1. Prop 1 allows passing to the limit in the hierarchy (17) and a final use of Lemma 1 leads to prove uniqueness to the limiting hierarchy (21) to conclude result of Theorem 1.

3.2. The key technical lemma.
We first present the key technical lemma which links the k-marginal L_w^q control to the $(k + 1)$-marginal L_w^q estimate control.

Lemma 1. *Assume that $K \in L^p(\Pi^d)$, for some $p > 1$. There exist some constants Λ, C, θ depending only on q, d and σ s.t*

$$\|F_{k,N}\|_{L_{\lambda(t)\,e_k}^q}^q \leq \|F_{k,N}(t = 0)\|_{L_{\lambda(t)\,e_k}^q}^q$$

$$+ q \int_0^t \int |F_{k,N}|^{q-1}\, sign\, F_{k,N}\, R_{k,N}\, e^{\lambda(s)\,e_{k,\gamma}}\, ds$$

$$+ k\, \frac{N - \gamma\, k}{N}\, \frac{C}{\lambda^\theta(t)}\, \|K\|_{L^p}^q \int_0^t \|F_{k+1,N}(s)\|_{L_{\lambda(s)\,e_{k+1}}^q}^q\, ds.$$

for any entropy solution $F_{k,N}$ to (22) in the sense of subsection 2.4 and satisfying (6) with $F_{k,N} \in L_{\lambda(t)\,e_{k,\gamma}}^q$, and for any $2 \leq q < \infty$ such that $1/q + 1/p \leq 1$, with $\lambda(t)$ defined by $\lambda(t) = \frac{1}{\Lambda\,(1+t)}$.

Proof. To be made fully rigorous, many calculations in this proof should involve a convolution kernel K_ε, estimating

$$\frac{d}{dt} \int |K_\varepsilon^{\otimes k} \star F_{k,N}|^q\, e^{\lambda(t)\,e_{k,\gamma}},$$

and passing to the limit in $\varepsilon \to 0$ while using appropriately the entropy condition (20). For simplicity however, we will only present the corresponding formal calculations.

We hence calculate in a straightforward manner

$$\frac{d}{dt} \int |F_{k,N}|^q \, e^{\lambda(t)\,e_{k,\gamma}} = q \int |F_{k,N}|^{q-1} \operatorname{sign} F_{k,N} \, \partial_t F_{k,N} \, e^{\lambda(t)\,e_{k,\gamma}}$$

$$+ \lambda'(t) \int e_{k,\gamma} \, |F_{k,N}|^q \, e^{\lambda(t)\,e_{k,\gamma}}.$$

Inserting now in this identity the definition of $\lambda(t)$ and the equation (17) we find

$$\frac{d}{dt} \int |F_{k,N}|^q \, e^{\lambda(t)\,e_{k,\gamma}} = -q \int |F_{k,N}|^{q-1} \operatorname{sign} F_{k,N} \, (L_{k,\gamma} \, F_{k,N}) \, e^{\lambda(t)\,e_{k,\gamma}}$$

$$+ q \, \frac{\sigma^2}{2} \int |F_{k,N}|^{q-1} \operatorname{sign} F_{k,N} \left(\sum_{i \le k} \Delta_{v_i} F_{k,N} \right) e^{\lambda(t)\,e_{k,\gamma}}$$

$$- q \, \frac{N - \gamma \, k}{N} \sum_{i \le k} \int |F_{k,N}|^{q-1} \operatorname{sign} F_{k,N}$$

$$\nabla_{v_i} \cdot \int K(x_i - x_{k+1}) \, F_{k+1,N} \, dx_{k+1} \, dv_{k+1} \, e^{\lambda(t)\,e_{k,\gamma}}$$

$$- \Lambda \, \lambda^2(t) \int e_{k,\gamma} \, |F_{k,N}|^q \, e^{\lambda(t)\,e_{k,\gamma}} + q \int |F_{k,N}|^{q-1} \operatorname{sign} F_{k,N} \, R_{k,N} \, e^{\lambda(t)\,e_{k,\gamma}}.$$

Note that

$$q \, |F_{k,N}|^{q-1} \operatorname{sign} F_{k,N} \, (L_{k,\gamma} \, F_{k,N}) = L_{k,\gamma} \, |F_{k,N}|^q,$$

so that by integration by parts, we formally have that

$$q \int |F_{k,N}|^{q-1} \operatorname{sign} F_{k,N} \, (L_{k,\gamma} \, F_{k,N}) \, e^{\lambda(t)\,e_{k,\gamma}}$$

$$= - \int |F_{k,N}|^q \, L_{k,\gamma} \, e^{\lambda(t)\,e_{k,\gamma}} = 0.$$

On the other hand, again by integration by parts

$$q \, \frac{\sigma^2}{2} \int |F_{k,N}|^{q-1} \operatorname{sign} F_{k,N} \left(\sum_{i \le k} \Delta_{v_i} F_{k,N} \right) e^{\lambda(t)\,e_{k,\gamma}}$$

$$= -q \, (q-1) \sum_{i \le k} \frac{\sigma^2}{2} \int |F_{k,N}|^{q-2} \, |\nabla_{v_i} F_{k,N}|^2 \, e^{\lambda(t)\,e_{k,\gamma}}$$

$$- 2 \, q \, \lambda(t) \sum_{i \le k} \frac{\sigma^2}{2} \int |F_{k,N}|^{q-1} \operatorname{sign} F_{k,N} \, v_i \cdot \nabla_{v_i} F_{k,N} \, e^{\lambda(t)\,e_{k,\gamma}}.$$

By Cauchy-Schwartz, since $q \geq 2$, we obtain that

$$
q \frac{\sigma^2}{2} \int |F_{k,N}|^{q-1} \operatorname{sign} F_{k,N} \left(\sum_{i \leq k} \Delta_{v_i} F_{k,N} \right) e^{\lambda(t) e_{k,\gamma}}
$$

$$
\leq -q(q-1) \sum_{i \leq k} \frac{\sigma^2}{4} \int |F_{k,N}|^{q-2} |\nabla_{v_i} F_{k,N}|^2 e^{\lambda(t) e_{k,\gamma}}
$$

$$
+ \frac{q}{q-1} \lambda^2 \frac{\sigma^2}{2} \int |F_{k,N}|^q \sum_{i \leq k} |v_i|^2 e^{\lambda(t) e_{k,\gamma}}.
$$

Note that since $\phi \geq 0$, we have that $\sum_{i \leq k} |v_i|^2 \leq e_k$ and, therefore, combining all our estimates so far, we deduce that

$$
\frac{d}{dt} \int |F_{k,N}|^q e^{\lambda(t) e_{k,\gamma}} \leq -q(q-1) \sum_{i \leq k} \frac{\sigma^2}{4} \int |F_{k,N}|^{q-2} |\nabla_{v_i} F_{k,N}|^2 e^{\lambda(t) e_{k,\gamma}}
$$

$$
- q \frac{N - \gamma k}{N} \sum_{i \leq k} \int |F_{k,N}|^{q-1} \operatorname{sign} F_{k,N}
$$

$$
\nabla_{v_i} \cdot \int K(x_i - x_{k+1}) F_{k+1,N} \, dx_{k+1} \, dv_{k+1} \, e^{\lambda(t) e_{k,\gamma}}
$$

$$
- \frac{\Lambda}{2} \lambda^2(t) \int e_{k,\gamma} |F_{k,N}|^q e^{\lambda(t) e_{k,\gamma}} + q \int |F_{k,N}|^{q-1} \operatorname{sign} F_{k,N} R_{k,N} \, e^{\lambda(t) e_{k,\gamma}},
$$

provided that $\Lambda \geq \frac{q}{q-1} \sigma^2$.

We integrate by parts the second term in the right-hand side to obtain

$$
\sum_{i \leq k} \int |F_{k,N}|^{q-1} \operatorname{sign} F_{k,N}
$$

$$
\nabla_{v_i} \cdot \int K(x_i - x_{k+1}) F_{k+1,N} \, dx_{k+1} \, dv_{k+1} \, e^{\lambda(t) e_{k,\gamma}} = RH_1 + RH_2,
$$

with

$$
RH_1 = -(q-1) \sum_{i \leq k} \int |F_{k,N}|^{q-2} \nabla_{v_i} F_{k,N}
$$

$$
\times \int K(x_i - x_{k+1}) F_{k+1,N} \, dx_{k+1} \, dv_{k+1} \, e^{\lambda(t) e_{k,\gamma}},
$$

and

$$RH_2 = -2\lambda(t) \sum_{i \leq k} \int |F_{k,N}|^{q-1} \operatorname{sign} F_{k,N} \, v_i$$
$$\times \int K(x_i - x_{k+1}) \, F_{k+1,N} \, dx_{k+1} \, dv_{k+1} \, e^{\lambda(t) \, e_{k,\gamma}}.$$

We perform a straightforward Cauchy-Schwartz inequality on both terms to find that

$$RH_2 \leq \frac{\lambda^2(t)}{2} \sum_{i \leq k} \int |F_{k,N}|^q \, |v_i|^2 \, e^{\lambda(t) \, e_{k,\gamma}}$$
$$+ \sum_{i \leq k} \int \frac{|F_{k,N}|^{q-2}}{2} \left| \int K(x_i - x_{k+1}) F_{k+1,N} \, dx_{k+1} \, dv_{k+1} \right|^2 e^{\lambda(t) e_{k,\gamma}},$$

and similarly

$$RH_1 \leq \frac{\sigma^2}{4} \sum_{i \leq k} \int |F_{k,N}|^{q-2} \, |\nabla_{v_i} F_{k,N}|^2 \, e^{\lambda(t) \, e_{k,\gamma}}$$
$$+ \frac{(q-1)^2}{\sigma^2} \sum_{i \leq k} \int |F_{k,N}|^{q-2} \left| \int K(x_i - x_{k+1}) F_{k+1,N} dx_{k+1} dv_{k+1} \right|^2 e^{\lambda(t) e_{k,\gamma}}.$$

Note that by Hölder estimates

$$\int |F_{k,N}|^{q-2} \left| \int K(x_i - x_{k+1}) \, F_{k+1,N} \, dx_{k+1} \, dv_{k+1} \right|^2 e^{\lambda(t) \, e_{k,\gamma}}$$
$$\leq \frac{q-2}{q} \lambda^2 \int |F_{k,N}|^q \, e^{\lambda(t) \, e_{k,\gamma}}$$
$$+ \frac{2}{q \, \lambda^{q-2}} \int e^{\lambda(t) \, e_{k,\gamma}} \left| \int K(x_i - x_{k+1}) \, F_{k+1,N} \, dv_{k+1} \right|^q.$$

Therefore, combining together all those terms, we obtain the further estimate

$$\sum_{i \leq k} \int |F_{k,N}|^{q-1} \text{sign}\,(F_{k,N})\, \nabla_{v_i} \cdot \int K(x_i - x_{k+1})\, F_{k+1,N} dx_{k+1} dv_{k+1} e^{\lambda(t)\, e_{k,\gamma}}$$

$$\leq \frac{\sigma^2}{4} \sum_{i \leq k} \int |F_{k,N}|^{q-2}\, |\nabla_{v_i} F_{k,N}|^2\, e^{\lambda(t)\, e_{k,\gamma}}$$

$$+ \lambda^2(t) \left(\frac{1}{2} + \frac{(q-2)\,(q-1)^2}{q\,\sigma^2} \right) \sum_{i \leq k} \int |F_{k,N}|^q\, (1 + |v_i|^2)\, e^{\lambda(t)\, e_{k,\gamma}}$$

$$+ \frac{2}{q\,\lambda^{q-2}} \left(\frac{1}{2} + \frac{(q-1)^2}{\sigma^2} \right)$$

$$\times \sum_{i \leq k} \int e^{\lambda(t)\, e_{k,\gamma}} \left| \int K(x_i - x_{k+1})\, F_{k+1,N}\, dx_{k+1} dv_{k+1} \right|^q.$$

Hence, provided that

$$\Lambda \geq q \left(1 + 2\, \frac{((q-2)\,q - 1)^2}{q\,\sigma^2} \right),$$

we obtain that

$$\frac{d}{dt} \int |F_{k,N}|^q\, e^{\lambda(t)\, e_{k,\gamma}}$$

$$\leq C_{q,\sigma,d}\, k\, \frac{N - \gamma k}{\lambda^{q-2}\, N} \int e^{\lambda(t)\, e_{k,\gamma}} \left| \int K(x_1 - x_{k+1}) F_{k+1,N} dx_{k+1} dv_{k+1} \right|^q.$$

At this point is where we take advantage of the specific structure of the hierarchy. We bound

$$\left| \int K(x_1 - x_{k+1})\, F_{k+1,N}\, dx_{k+1}\, dv_{k+1} \right|^q$$

$$\leq \left(\int |K(x_1 - x_{k+1})|^{q^*}\, e^{-\frac{q^*}{q}\, \lambda(t)\, |v_{k+1}|^2}\, dx_{k+1}\, dv_{k+1} \right)^{q/q^*}$$

$$\times \int |F_{k+1,N}|^q\, e^{\lambda(t)\, |v_{k+1}|^2}\, dx_{k+1}\, dv_{k+1},$$

which implies

$$\left| \int K(x_1 - x_{k+1})\, F_{k+1,N}\, dx_{k+1}\, dv_{k+1} \right|^q$$

$$\leq \frac{C_{q,\sigma,d}}{\lambda^{q\,d/2q^*}(t)}\, \|K\|_{L^p}^q \int |F_{k+1,N}|^q\, e^{\lambda(t)\, |v_{k+1}|^2}\, dx_{k+1}\, dv_{k+1},$$

since $q \geq p^*$. Consequently

$$\int e^{\lambda(t)\, e_{k,\gamma}} \left| \int K(x_1 - x_{k+1})\, F_{k+1,N}\, dx_{k+1}\, dv_{k+1} \right|^q$$

$$\leq \frac{C_{q,\sigma,d}}{\lambda^{qd/2q^*}(t)} \|K\|_{L^p}^q \int |F_{k+1,N}|^q\, e^{\lambda(t)\,|v_{k+1}|^2 + \lambda(t)\, e_{k,\gamma}}\, dx_1 dv_1 \ldots dx_{k+1} dv_{k+1}.$$

Note that

$$e_{k+1,\gamma} = e_{k,\gamma} + 1 + |v_{k+1}|^2 + \frac{2\,\gamma}{N} \sum_{i \leq k} \phi(x_i - x_{k+1}) \geq e_{k,\gamma} + 1 + |v_{k+1}|^2,$$

so that

$$\int e^{\lambda(t)\, e_k} \left| \int K(x_i - x_{k+1})\, f_{k+1,N}\, dx_{k+1}\, dv_{k+1} \right|^q$$

$$\leq \frac{C_{q,\sigma,d}}{\lambda^{qd/2q^*}(t)} \|K\|_{L^p}^q \int |f_{k+1,N}|^q\, e^{\lambda(t)\, e_{k+1}}\, dx_1\, dv_1 \ldots dx_{k+1}\, dv_{k+1}.$$

This finally lets us conclude, as claimed, that

$$\frac{d}{dt} \int |f_{k,N}|^q\, e^{\lambda(t)\, e_{k,\gamma}} \leq k\, \frac{N - \gamma\, k}{N}\, \frac{C_{q,\sigma},d}{\lambda^{\theta_{q,d}}(t)}\, \|K\|_{L^p}^q \int |f_{k+1,N}|^q\, e^{\lambda(t)\, e_{k+1,\gamma}}$$

$$+ q \int |F_{k,N}|^{q-1} \operatorname{sign} F_{k,N}\, R_{k,N}\, e^{\lambda(t)\, e_{k,\gamma}}.$$

$\square$

3.3. Proof of technical results. We start this subsection with the proof of the Proposition 1.

Proof of Proposition 1. From the analysis in subsection 3.1 and the assumptions (6) and (12) of Proposition 1, we have that $F_{k,N} = f_{k,N}$ is a renormalized solution to (17) and thus (22) with $\gamma = 1$. Moreover, $f_{k,N}$ satisfies the other assumptions in Lemma 1 with $R_{k,N} = 0$. Denoting

$$X_k(t) = \int |f_{k,N}|^q\, e^{\lambda(t)\, e_k},$$

we hence observe that, by Lemma 1, we have the coupled dynamical inequality system

$$X_k(t) \leq X_k(0) + k\, L \int_0^t X_{k+1}(s)\, ds,$$

for any $t \in [0,\ 1]$, where

$$L = \frac{C}{\lambda^\theta(1)}\, \|K\|_{L^p}^q.$$

From the assumptions of Proposition 1, we immediately have that

$$(23) \qquad X_k(t) \le F_0^k + k\, L \int_0^t X_{k+1}(s)\, ds.$$

We now invoke the following simple lemma

Lemma 2. *Consider any sequence $X_k(t)$ satisfying* (23) *then one has that*

$$(24) \qquad \begin{aligned} X_k(t) &\le \sum_{l=k}^m F_0^l\, L^{l-k}\, t^{l-k}\, \frac{(l-1)!}{(k-1)!\,(l-k)!} \\ &\quad + L^{m+1-k} \int_0^t X_{m+1}(s)\,(t-s)^{m-k}\, \frac{m!}{(k-1)!\,(m-k)!}\, ds. \end{aligned}$$

Assuming that Lemma 2 holds, we use (24) up to $m+1 = N$ to derive through the assumptions on f_N that

$$\begin{aligned} X_k(t) &\le \sum_{l=k}^{N-1} F_0^l\, L^{l-k}\, t^{l-k}\, \frac{(l-1)!}{(k-1)!\,(l-k)!} \\ &\quad + L^{N-k} \int_0^t F^N\,(t-s)^{N-1-k}\, \frac{(N-1)!}{(k-1)!\,(N-1-k)!}\, ds, \end{aligned}$$

that is

$$(25) \qquad \begin{aligned} X_k(t) &\le \sum_{l=k}^{N-1} F_0^l\, L^{l-k}\, t^{l-k}\, \frac{(l-1)!}{(k-1)!\,(l-k)!} \\ &\quad + F^N\, L^{N-k}\, t^{N-k}\, \frac{(N-1)!}{(k-1)!\,(N-k)!}. \end{aligned}$$

Note that

$$\frac{(l-1)!}{(k-1)!\,(l-k)!} = \binom{l-1}{k-1} \le 2^{l-1}.$$

Hence, (25) implies that

$$\begin{aligned} X_k(t) &\le \sum_{l=k}^N F_0^l\, L^{l-k}\, t^{l-k}\, 2^{l-1} + F^N\, L^{N-k}\, t^{N-k}\, 2^{N-1} \\ &= 2^{k-1}\, F_0^k \sum_{l=k}^{N-1} F_0^{l-k}\, 2^{l-k}\, L^{l-k}\, t^{l-k} + F^k 2^{k-1} F^{N-k}\, L^{N-k}\, t^{N-k}\, 2^{N-k} \\ &\le 2^{k-1}\, F_0^k (2 - 2^{k-N+1}) + F^k 2^{k-1} 2^{k-N} \\ &\le F_0^k 2^k + F^k 2^{2k-N-1} \end{aligned}$$

provided that $4\,L\,t\,\max(F_0, F) < 1$, which concludes the proof of the proposition. $\square$

We finish with the quick proof of Lemma 2.

Proof of Lemma 2. Taking $m = k$ in (24), we get

$$X_k(t) \leq F_0^k + L \int_0^t X_{k+1}(s) \frac{k!}{(k-1)!\,(k-k)!}\,ds,$$

which is our starting point. Moreover assuming that (24) holds for m, we may use (23) to find

$$X_k(t) \leq \sum_{l=k}^m F_0^l\,L^{l-k}\,t^{l-k}\,\frac{(l-1)!}{(k-1)!\,(l-k)!}$$
$$+ L^{m+1-k} \int_0^t \left(F_0^{m+1} + L\,(m+1) \int_0^s X_{m+2}(s)\,ds \right)$$
$$(t-s)^{m-k}\,\frac{m!}{(k-1)!\,(m-k)!}\,ds.$$

This yields

$$X_k(t) \leq \sum_{l=k}^m F_0^l\,L^{l-k}\,t^{l-k}\,\frac{(l-1)!}{(k-1)!\,(l-k)!}$$
$$+ L^{m+1-k}\,F_0^{m+1}\,\frac{m!}{(k-1)!\,(m-k)!} \int_0^t (t-s)^{m-k}\,ds$$
$$+ L^{m+2-k} \int_0^t X_{m+2}(r) \int_r^t (t-s)^{m-k}\,ds\,dr\,\frac{(m+1)!}{(k-1)!\,(m-k)!}\,ds,$$

or

$$X_k(t) \leq \sum_{l=k}^m F_0^l\,L^{l-k}\,t^{l-k}\,\frac{(l-1)!}{(k-1)!\,(l-k)!}$$
$$+ L^{m+1-k}\,F_0^{m+1}\,\frac{m!}{(k-1)!\,(m+1-k)!}\,t^{m+1-k}$$
$$+ L^{m+2-k} \int_0^t X_{m+2}(r)(t-r)^{m+1-k}\,ds\,dr\,\frac{(m+1)!}{(k-1)!\,(m+1-k)!}\,ds,$$

as claimed. $\square$

3.4. **Proof of Theorem 1.**

The proof of Theorem 1 follows closely the steps in the proof of Proposition 1, once appropriate bounds have been derived.

1) *Uniform bounds on f_N in $L_{e_N}^q$*. First of all, note that from the assumptions of Theorem 1, we can easily obtain a bound on f_N^0 in $L_{\lambda^0 e_N}^q$ for Λ large enough. Indeed

$$\int_{\Pi^{dN} \times \mathbb{R}^{dN}} |f_N^0|^q \, e^{\lambda^0 e_N}$$

$$= e^N \int_{\Pi^{dN} \times \mathbb{R}^{dN}} |f_N^0|^q \, e^{2\lambda^0 \sum_{i \leq N} |v_i|^2} \, e^{\frac{\lambda^0}{N} \sum_{i,j \leq N} \phi(x_i - x_j) - \lambda^0 \sum_{i \leq N} |v_i|^2}.$$

We have straightforward L^r estimates on $e^{\frac{\lambda^0}{N} \sum_{i,j \leq N} \phi(x_i - x_j) - \lambda^0 \sum_{i \leq N} |v_i|^2}$ as by Hölder inequality,

$$\int_{\Pi^{dN} \times \mathbb{R}^{dN}} e^{\frac{r\lambda^0}{N} \sum_{i,j \leq N} \phi(x_i - x_j) - r\lambda^0 \sum_{i \leq N} |v_i|^2}$$

$$= \frac{C^N}{\lambda_0^{N/2}} \int_{\Pi^{dN}} e^{\frac{r\lambda^0}{N} \sum_{i,j \leq N} \phi(x_i - x_j)}$$

$$\leq \frac{C^N}{\lambda_0^{N/2}} \left(\Pi_{i \leq N} \int_{\Pi^{dN}} e^{r\lambda^0 \sum_{j \leq N} \phi(x_i - x_j)} \right)^{1/N} \leq \frac{C^N}{\lambda_0^{N/2}},$$

from some constant C and by assumption (7) in Theorem 1, provided that $r \lambda^0 \leq 1/\theta$. This implies, again by Hölder inequality

$$\int_{\Pi^{dN} \times \mathbb{R}^{dN}} |f_N^0|^q \, e^{\lambda^0 e_N} \leq \frac{C^N}{\lambda_0^{N/2}} \int_{\Pi^{dN} \times \mathbb{R}^{dN}} |f_N^0|^{r^* q} \, e^{2 r^* \lambda^0 \sum_{i \leq N} |v_i|^2}$$

$$\leq \frac{C^N}{\lambda_0^{N/2}} \|f_N^0\|_{L^\infty}^{q \, r^* - 1} \int_{\Pi^{dN} \times \mathbb{R}^{dN}} |f_N^0|^{r^* q} \, e^{2 r^* \lambda^0 \sum_{i \leq N} |v_i|^2}.$$

Using now assumption (6), provided that $2 r^* \lambda_0 \leq \beta$, we conclude that

$$(26) \qquad \int_{\Pi^{dN} \times \mathbb{R}^{dN}} |f_N^0|^q \, e^{\lambda^0 e_N} \leq \left(\frac{C V M}{\lambda_0} \right)^N,$$

for any $q < \infty$. We now choose any fixed $2 < q < \infty$ such that $1/p + 1/q < 1$ and we remark that the Liouville Eq. (4) is included in Eq (22) for $\gamma = 1$, $R_{k,N} = 0$, and $k = N$. Thus, we next invoke Lemma 1 for f_N with $k = N$ and $\gamma = 1$, to find that f_N solves

$$\frac{d}{dt} \int_{\Pi^{dN} \times \mathbb{R}^{dN}} |f_N(t, ., .)|^q \, e^{\lambda(t) e_N} \leq 0,$$

so that from (26), we obtain that

$$\sup_{t \leq 1} \int_{\Pi^{dN} \times \mathbb{R}^{dN}} |f_N(t, ., .)|^q \, e^{\lambda(t) e_N} \leq \left(\frac{C V M}{\lambda_0} \right)^N.$$

This finally implies that there exists some constant $F > 0$ such that

$$(27) \qquad \sup_{t \leq 1} \int_{\Pi^{dN} \times \mathbb{R}^{dN}} |f_N(t, ., .)|^q \, e^{\lambda(t) \, e_N} \leq F^N.$$

2) *Uniform estimates on the marginals and passing the limit in the hierarchy* (17). First of all we can perform the same bounds on each $f_{k,N}^0$ to find similarly to (26) that

$$\int_{\Pi^{kd} \times \mathbb{R}^{kd}} |f_{k,N}^0|^q \, e^{\lambda^0 \, e_k} \leq \left(\frac{C \, V \, M}{\lambda_0} \right)^k.$$

As a consequence, every assumptions of Proposition 1 hold, and, in particular, assumption (12). This implies that for some time $T^* > 0$, depending only on V, M, $\|K\|_{L^p}$ and the choice of q, we have that

$$\sup_N \sup_{t \leq T^*} \int_{\Pi^{kd} \times \mathbb{R}^{kd}} |f_{k,N}|^q \, e^{\lambda(t) \, e_k} \leq \bar{M}^k,$$

for some constant $\bar{M}$. At this point, we will not need anymore the potential in the reduced energy e_k, which was required to handle the L_k operator that vanishes at the limit. For this reason, and since $\phi \geq 0$, we deduce from the previous inequality

$$(28) \qquad \sup_N \sup_{t \leq T^*} \int_{\Pi^{kd} \times \mathbb{R}^{kd}} |f_{k,N}|^q \, e^{\lambda(T^*) \sum_{i \leq k} |v_i|^2} \leq \bar{M}^k.$$

These uniform bounds let us extract a converging subsequence such that all $f_{k,N}$ converge weak-$\star$ to some $\bar{f}_k$ in $L^\infty([0, T^*], L^q_{x,v})$ that also satisfies

$$(29) \qquad \sup_{t \leq T^*} \int_{\Pi^{kd} \times \mathbb{R}^{kd}} |\bar{f}_k|^q \, e^{\lambda(T^*) \sum_{i \leq k} |v_i|^2} \leq \bar{M}^k,$$

where we have used classical convex estimates. We emphasize that for the moment we only have convergence of a subsequence, though we still denote it by N for simplicity. We eventually obtain the convergence of the whole sequence only after the uniqueness of the limit is proved in the next step.

From estimate (28), and since $1/q + 1/p \leq 1$, we may simply bound

$$\left\| \sum_{i \leq k} \frac{1}{N} \sum_{j \leq k} K(x_i - x_j) \cdot \nabla_{v_i} f_{k,N} \right\|_{L^\infty_t L^1_{x,v,loc}} \lesssim \frac{k^2}{N} \|K\|_{L^p} \|f_{k,N}\|_{L^\infty_t L^q_{x,v}}.$$

For any fixed k, the corresponding term vanishes as $N \to \infty$. Similarly estimate (28) allows to pass to the limit

$$\int_{\Pi^d \times \mathbb{R}^d} K(x_i - x_{k+1}) \, f_{k+1,N} \, dx_{k+1} \, dv_{k+1}$$

$$\longrightarrow \int_{\Pi^d \times \mathbb{R}^d} K(x_i - x_{k+1}) \, \bar{f}_{k+1} \, dx_{k+1} \, dv_{k+1},$$

for the weak-$\star$ topology of $L^\infty([0, \ T^*], \ L^q_{x,v})$. It is straightforward to pass to the limit in the sense of distributions in all other terms of the hierarchy (17) so that we deduce that $\bar{f}_k$ is a solution to the limiting hierarchy (21) in the sense of distributions.

We can also easily identify the initial value of $\bar{f}_k$. From (17) and the bounds derived from (28), we immediately obtain a uniform bound on $\partial_t f_{k,N}$ in $L^\infty_t W^{-1,q}_{x,v,loc}$. By the assumption of Theorem 1, $f^0_{k,N}$ converges weakly to $(f^0)^{\otimes k}$ so that we have $\bar{f}_k(t = 0) = (f^0)^{\otimes k}$.

3) *Uniqueness on the limiting hierarchy and conclusion.* We first argue that $\bar{f}_k$ is automatically a renormalized solution to (21). Indeed Eq. (21) can be seen as a linear advection-diffusion equation with a locally Lipschitz velocity field $(v_1, \ldots, v_k)$ and a remainder

$$\nabla_{v_i} \cdot \int_{\Pi^d \times \mathbb{R}^d} K(x_i - x_j) \, \bar{f}_{k+1} \, dx_{k+1} \, dv_{k+1},$$

that belongs to $L^\infty_t L^q_{x,v}$ with $q > 2$ per our prior estimates.

Next we note that since f is a classical solution to the vlasov equation (3), the $f^{\otimes k}$ also yield renormalized solutions to the Vlasov hierarchy (21) for every $k \geq 1$. Due to the linearity in terms of the sequence $\{f_k\}_{k \in \mathbb{N}^\star}$ of the Vlasov hierarchy, we get that each $F_k = \bar{f}_k - f^{\otimes k}$ is also a renormalized solution to the Vlasov Hierarchy (21) for every k. Moreover by the identification above of the initial with zero initial data.

Furthermore, by (29) and the assumption of Gaussian decay on f^0, we have that

$$(30) \qquad \sup_{t \leq T^*} \int_{\Pi^{kd} \times \mathbb{R}^{kd}} |F_k|^q \, e^{\tilde{\beta} \sum_{i \leq k}(1+|v_i|)^2} \leq \tilde{M}^k,$$

for some $\tilde{\beta}$ and some $\tilde{M}$. Eq. (21) corresponds to Eq. (22) in the case $\gamma = 0$, where $e_{k,\gamma}$ reduces to $e_{k,0} = \sum_{i \leq k}(1+|v_i|)^2$. Hence, provided we choose some $\tilde{\Lambda}$, possibly lower than Λ we satisfy all assumptions from Lemma 1.

Denoting by $Y_k = \int |F_k|^q e^{\tilde{\lambda}(t)e_{k,0}}$, we get for all $k \in \mathbb{N}^\star$

$$Y_k(t) \leq k \, \tilde{L} \int_0^t Y_{k+1} \, ds.$$

We can then use Lemma 2 with $F_0 = 0$ up to any arbitrary m to show, together with (30), that

$$
\begin{aligned}
(31) \quad Y_k(t) &\leq \tilde{L}^{m+1-k}\, \tilde{M}^{m+1} \int_0^t (t-s)^{m-k}\, \frac{m!}{(k-1)!\,(m-k)!}\, ds \\
&\leq \tilde{L}^{m+1-k}\, \tilde{M}^{m+1}\, t^{m+1-k} \binom{m}{k-1} \leq 2^k\, \tilde{M}^k\, (2\,\tilde{L}\,\tilde{M}\,t)^{m+1-k}.
\end{aligned}
$$

By taking $t < T_0$ with T_0 small enough and sending m to ∞, we obtain that $Y_k(t) = 0$ and hence $\bar{f}_k = f^{\otimes k}$ on $[0,\,T_0]$. This allows to repeat the argument starting from $t = T_0$ instead of $t = 0$ until we reach the maximum time T^*. This finally allows to conclude as claimed that $\bar{f}_k = f^{\otimes k}$ over the whole interval $[0,\,T^*]$.

Coming back to our extracted subsequence on $f_{k,N}$, since all such subsequences have the same limit, we have convergence of the whole sequence to the $f^{\otimes k}$ concluding the proof.

3.5. **Proof of Theorem 2.** The aim of this result is to provide a quantitative estimate between $f_{k,N}$ and f_k that satisfy (17) and (21), respectively, for the tensorized limits $f_k = \overline{f}^{\otimes k}$. First let us note that $F_k^N = f_{k,N} - f_k$ satisfies,

$$
\begin{aligned}
(32) \quad &\partial_t F_k^N + L_k F_k^N \\
&\quad + \frac{N-k}{N} \sum_{i=1}^k \nabla_{v_i} \cdot \int_{\Pi^d \times \mathbb{R}^d} F_{k+1}^N K(x_i - x_{k+1})\, dx_{k+1} dv_{k+1} \\
&= \frac{\sigma^2}{2} \sum_{i=1}^k \Delta_{v_i} F_{k,N} + R_{k,N}
\end{aligned}
$$

where L_k is defined in (18), and

$$
\begin{aligned}
(33) \quad R_{k,N} &= \sum_{i=1}^k \left[\left(K \star \int_{\mathbb{R}^d} \overline{f} \right)(t, x_i) - \frac{1}{N} \sum_{j=1}^k K(x_i - x_j) \right] \cdot \nabla_{v_i} f_k. \\
&\quad - \frac{N-k}{N} \sum_{i=1}^k \nabla_{v_i} \cdot \int_{\Pi^d \times \mathbb{R}^d} f_{k+1} K(x_i - x_{k+1}) dx_{k+1} dv_{k+1}.
\end{aligned}
$$

We again use Lemma 1 with $q = 2$ to deduce

$$
\begin{aligned}
(34) \quad & \frac{d}{dt} \int_{\Pi^{kd} \times \mathbb{R}^{kd}} |F_{k,N}|^2 e^{\lambda(t)e_{k,\gamma}} + \frac{\sigma^2}{4} \sum_{i \leq k} \int_{\Pi^{kd} \times \mathbb{R}^{kd}} |\nabla_{v_i} F_{k,N}|^2 e^{\lambda(t)e_k} \\
& \qquad \leq k \frac{N-k}{N} \frac{C_{2,\sigma,d}}{\lambda^{\theta_{2,d}}(t)} \|K\|_{L^2}^2 \int_{\Pi^{kd} \times \mathbb{R}^{kd}} |F_{k+1,N}|^2 e^{\lambda(t)e_{k+1}} \\
& \qquad\quad + \lambda'(t) \int_{\Pi^{kd} \times \mathbb{R}^{kd}} e_k |F_{k,N}|^2 e^{\lambda(t)e_k} \\
& \qquad\quad + \int_{\Pi^{kd} \times \mathbb{R}^{kd}} R_{k,N} F_{k,N} e^{\lambda(t)e_k}.
\end{aligned}
$$

Note that $R_{k,N}$ may be written as follows

$$
\begin{aligned}
(35) \quad R_{k,N} = {} & \sum_{i=1}^{k} \frac{1}{N} \sum_{j=1}^{k} \left[(K \star \int_{\mathbb{R}^d} \overline{f})(t, x_i) - K(x_i - x_j) \right] \cdot \nabla_{v_i} f_k \\
& - \frac{N-k}{N} \sum_{i=1}^{k} \left[\nabla_{v_i} \cdot \int_{\Pi^d \times \mathbb{R}^d} f_{k+1} K(x_i - x_{k+1}) dx_{k+1} dv_{k+1} \right. \\
& \qquad\qquad \left. - \left(K \star \int_{\mathbb{R}^d} \overline{f} \right)(t, x_i) \cdot \nabla_{v_i} f_k \right].
\end{aligned}
$$

Then, using that $f_k = \overline{f}^{\otimes k}$, we have

$$
\begin{aligned}
\int_{\Pi^{kd} \times \mathbb{R}^{kd}} R_{k,N} F_{k,N} e^{\lambda(t)e_k} = {} & \int_{\Pi^{kd} \times \mathbb{R}^{kd}} \frac{k}{N} \sum_{i=1}^{k} \left[\left(K \star \int_{\mathbb{R}^d} \overline{f} \right)(t, x_i) \right. \\
& \left. - K(x_i - x_1) \right] \cdot \nabla_{v_i} f_k \, F_{k,N} e^{\lambda(t)e_k},
\end{aligned}
$$

where we have used the fact that the particles are interchangeable. Integrating by parts with respect to v_i and using Young inequality, we

obtain

$$
\begin{aligned}
\int_{\Pi^{kd}\times\mathbb{R}^{kd}} R_{k,N}F_{k,N}e^{\lambda(t)e_k} \leq & \frac{\sigma^2}{4}\frac{k}{N}\sum_{i=1}^{k}\int_{\Pi^{kd}\times\mathbb{R}^{kd}}|\nabla_{v_i}F_{k,N}|^2 e^{\lambda(t)e_k} \\
& + \frac{1}{\sigma^2}\frac{k}{N}\sum_{i=1}^{k}\int_{\Pi^{kd}\times\mathbb{R}^{kd}}|\widetilde{R}^1_{k,N}|^2 e^{\lambda(t)e_k} \\
& + \lambda(t)\int_{\Pi^{kd}\times\mathbb{R}^{kd}} e_k\,|F_{k,N}|^2\,e^{\lambda(t)e_k} \\
& + \frac{1}{2}\int_{\Pi^{kd}\times\mathbb{R}^{kd}}|\widetilde{R}^2_{k,N}|^2\,e^{\lambda(t)e_k},
\end{aligned}
$$

(36)

where

$$
\widetilde{R}^1_{k,N} = \left[\left(K\star\int_{\mathbb{R}^d}\overline{f}\,dx\right)(t,x_i) - K(x_i - x_1)\right]f_k\,,
$$

$$
\widetilde{R}^2_{k,N} = \sum_{i=1}^{k}\left[\left(K\star\int_{\mathbb{R}^d}\overline{f}\,dx\right)(t,x_i) - K(x_i - x_1)\right]f_k\,.
$$

We observe that

$$
\|\widetilde{R}^i_{k,N}\|^2_{L^2_{\lambda(t)e_k}} \leq C\,k\int_{\Pi^{kd}\times\mathbb{R}^{kd}}|f_k|^p e^{\lambda(t)e_k}\,,
$$

with a constant C that does not depend on k. We have also used the fact that, in particular, $K\in L^2(\Pi^d)$ and $\overline{f}\in L^\infty(\Pi^d\times\mathbb{R}^d)$.

Then, using (13) and letting $N\to+\infty$, we get

$$
\sup_{t\leq T^*}\int_{\Pi^{kd}\times\mathbb{R}^{kd}}|f_k|^p e^{\lambda(t)e_{k,\gamma}} \leq 2^k F_0^k.
$$

We can insert this estimate into (36), for $p=2$, to derive

$$
\begin{aligned}
\int_{\Pi^{kd}\times\mathbb{R}^{kd}} R_{k,N}F_{k,N}e^{\lambda(t)e_k} \leq & \frac{\sigma^2}{4}\frac{k}{N}\sum_{i=1}^{k}\int_{\Pi^{kd}\times\mathbb{R}^{kd}}|\nabla_{v_i}F_{k,N}|^2 e^{\lambda(t)e_k} \\
& + \lambda(t)\int_{\Pi^{kd}\times\mathbb{R}^{kd}} e_k\,|F_{k,N}|^2\,e^{\lambda(t)e_k} \\
& + C\,k\,2^k F_0^k.
\end{aligned}
$$

Once this estimate is incorporated into (34) and using that $\lambda'(t) = -\frac{\lambda(t)}{1+t}$, we can, following the same lines of the proof of Proposition 1, repeat the estimate on the ODE inequality with the extra term coming

from the interaction of $F_{k,N}$ with rest term $R_{k,N}$. This provides the conclusion that there exists T^* such that

$$\sup_{t \leq T^\star} \int_{\Pi^{kd} \times \mathbb{R}^{kd}} |f_{N,k} - f_k|^2 e^{\lambda(t) e_{k,\gamma}}$$

$$\leq \widetilde{C}^k \varepsilon_N + \widetilde{C}^k \int_{\Pi^{kd} \times \mathbb{R}^{kd}} |f_{N,k}^0 - f_k^0|^2 e^{\lambda(0) e_{k,\gamma}},$$

where $\widetilde{C}$ is a positive constant that does not depend on N, and $\varepsilon_N = O(\varepsilon^N)$, where $\varepsilon < 1$ depends on T^* that is small enough. This expression can be deduced in a similar way as (31) in the proof of Theorem 1.

We finally emphasize that the quantitative bounds of Theorem 2 would allow to recover the optimal convergence rate in $O(1/N)$ that was recently obtained in [51].

References

[1] A. A. Arsenév. Existence in the large of a weak solution of Vlasovs system of equations. *Z. Vycisl. Mat. i Mat. Fiz.*, **15** (1975), 136–147, 276.

[2] T. Bodineau, I. Gallagher, L. Saint-Raymond. From hard sphere dynamics to the StokesFourier equations: an L^2 analysis of the Boltzmann-Grad limit. *Ann. PDE* **3** (2017), 2.

[3] T. Bodineau, I. Gallagher, L. Saint-Raymond, S. Simonella. One-sided convergence in the Boltzmann-Grad limit. *Ann. Fac. Sci.* **27**(5) (2018), 985–1022.

[4] T. Bodineau, I. Gallagher, L. Saint-Raymond, S. Simonella. Fluctuation Theory in the BoltzmannGrad Limit. *Journal of Statistical Physics* **180** (2020), 873–895.

[5] V.I. Bogachev, N.V. Krylov, M. Röckner, S.V. Shaposhnikov. *Fokker-Planck-Kolomogorv equations.* Mathematical Surveys and Monographs, **207**, AMS, 2010.

[6] N. N. Bogoliubov. Kinetic equations. *Journal of Experimental and Theoretical Physics* (in Russian), **16**(8) (1946), 691–702.

[7] N. N. Bogoliubov. Kinetic equations. *Journal of Physics USSR*, **10**(3) (1946), 265–274.

[8] F. Bolley, A. Guillin, F. Malrieu. Trend to equilibrium and particle approximation for a weakly selfconsistent Vlasov-Fokker-Planck equation *ESAIM: Mathematical Modelling and Numerical Analysis - Modélisation Mathématique et Analyse Numérique* **44** (2010), 867–884.

[9] N. Boers and P. Pickl. On Mean Field Limits for Dynamical Systems. *Journal of Statistical Physics*, **164** (2015), 1–16.

[10] F. Bouchut, Existence and uniqueness of a global smooth solution for the Vlasov-Poisson- Fokker-Planck system in three dimensions. *J. Funct. Anal.*, **111** (1993), 239–258.

[11] W. Braun and K. Hepp. The Vlasov dynamics and its fluctuations in the $1/N$ limit of interacting classical particles. *Communications in Mathematical Physics*, **56** (1977), 101–113.

[12] D. Bresch, P.-E. Jabin. Quantitative regularity estimates for compressible transport equations. New trends and results in mathematical description of fluid flows, 77113, Necas Center Ser., Birkhuser/Springer, Cham, 2018.

[13] D. Bresch, P.–E. Jabin, Z. Wang. Modulated free energy and mean field limit. *Séminaire Laurent Schwartz - EDP et applications,* 1–22, 2019.

[14] D. Bresch, P.–E. Jabin, Z. Wang, Mean-field limit and quantitative estimates with singular attractive kernels. arXiv:2011.08022 (2020).

[15] J.A. Carillo, Y.-P. Choi, S. Salem. Propagation of chaos for the Vlasov-Poisson-Fokker-Planck equation with a polynomial cut-off. *Communications in Contemporary Mathematics.* **21** (2019), 1850039.

[16] J.A. Carrillo,Y.-P. Choi,Y. Peng, Large friction-high force fields limit for the nonlinear Vlasov-Poisson-Fokker-Planck system, *Kinetic & Related Models*, doi: 10.3934/krm.2021052

[17] A.A. Carrillo, J. Soler, On the Vlasov-Poisson-Fokker-Planck equations with measures in Morrey spaces as initial data, *J. Math. Anal. Appl.* **207** (1997), 475–495.

[18] M. L. Clark, The analysis of periodic Vlasov-Poisson-Fokker-Planck equations, Master Thesis, Texas Tech University, 1993.

[19] G. Crippa, C. De Lellis. Estimates and regularity for the DiPerna-Lions flow. *J. Reine Angew. Math.* **616** (2008), 15–46,.

[20] P. Degond. Global existence of smooth solutions for the Vlasov-Fokker-Planck equation in 1 and 2 dimensions. *Ann. Scientifiques École Normale Supérieure,* Série **4**(9) (1986), 519–542.

[21] R.J. DiPerna, P.L. Lions, Ordinary differential equations, transport theory and Sobolev spaces. *Invent. Math.* **98** (1989), 511–547.

[22] R. L. Dobrushin. Vlasov equations. *Functional Analysis and Its Applications*, **13** (1079), 115–123.

[23] M. Duerinckx, L. Saint-Raymond. LenardBalescu correction to mean-field theory. *Probability and Mathematical Physics* **2**(1) (2021), 27–69.

[24] N. Fournier, M. Hauray, and S. Mischler. Propagation of chaos for the 2d viscous vortex model. *J. Eur. Math. Soc.*, **16**(7) (2014), 1425–1466.

[25] I. Gallagher, L. Saint Raymond, B. Texier. From Newton to Boltzmann: Hard Spheres and Short-Range Potentials. *EMS*, Zurich, 2014.

[26] K. Ganguly and H. D. Victory, Jr. On the convergence of particle methods for multidimensional Vlasov-Poisson systems. *SIAM J. Numer. Anal.* **26**(2) (1989), 249–288.

[27] K. Ganguly, J. Todd Lee, and H. D. Victory, Jr. On simulation methods for Vlasov-Poisson systems with particles initially asymptotically distributed. *SIAM J. Numer. Anal.* **28**(6) (1991), 1574–1609.

[28] I. Gasser, P.-E. Jabin, and B. Perthame. Regularity and propagation of moments in some nonlinear Vlasov systems, *Proc. Roy. Soc. Edinburgh Sect. A* **130**(6) (2000), 1259–1273.

[29] F. Golse, On the dynamics of large particle systems in the mean field limit. In: Macroscopic and Large Scale Phenomena: Coarse Graining, Mean Field limits and Ergodicity. In: Lecture Notes Appl. Math. Mech., Vol. **3**, 1–144. Springer, 2016.

[30] F. Golse, C. Mouhot, and V. Ricci. Empirical measures and Vlasov hierarchies. *Kinet. Relat. Models* **6**(4) (2013), 919–943.

[31] F. Golse and T. Paul. Empirical Measures and Quantum Mechanics: Application to the Mean-Field Limit. *Communications in Mathematical Physics*, **369** (2019), 1021–1053.

[32] Goodman, J. and Hou, T.Y.: New stability estimates for the 2-D vortex method. Comm. Pure Appl. Math. **44** (1991), 1015–1031.

[33] J. Goodman, T. Y. Hou, and J. Lowengrub. Convergence of the point vortex method for the 2-D Euler equations. *Comm. Pure Appl. Math.* **43** (1990), 415–430.

[34] T. Goudon, J. Nieto, F. Poupaud, J. Soler, Multidimensional high-field limit of the electrostatic Vlasov-Poisson-Fokker-Planck system, *J. Differential Equations* **213** (2005), 418–442.

[35] A. Guillin, P. Le Bris. On systems of particles in singular repulsive interaction in dimension one : log and Riesz gas. arXiv:2204.10653

[36] A. Guillin, P. Le Bris, P. Monmarché. Convergence rates for the Vlasov-Fokker-Planck equation and uniform in time propagation of chaos in non convex cases. See arXiv:2105.09070v2

[37] A. Guillin, P. Le Bris, P. Monmarché. Uniform in time propagation of chaos for the 2D vortex model and other singular stochastic systems. arXiv:2108.08675

[38] D. Han-Kwan, M. Iacobelli, From Newton's second law to Euler's equations of perfect fluids. *Proc. Amer. Math. Soc.* **149**(7) (2021), 3045–3061.

[39] M. Hauray. On the Liouville transport equation with force field in BV_{loc}. *Comm. Partial Diff Eqs* **29** (2005), 207–2017.

[40] M. Hauray and P.-E. Jabin. N-particles Approximation of the Vlasov Equations with Singular Potential. *Archive for Rational Mechanics and Analysis* **183** (2007), 489–524.

[41] M. Hauray and P.-E. Jabin. Particle approximation of Vlasov equations with singular forces: propagation of chaos. *Annales Scientifiques de l'École Normale Supérieure. Quatrième Série* **48** (2015), 891–940.

[42] M. Hauray, S. Salem. Propagation of chaos for the Vlasov-Poisson-Fokker-Planck system in 1D. *AIMS* **12** (2019), 2303.

[43] T. Holding, E. Miot. Uniqueness and stability for the Vlasov-Poisson system with spatial density in Orlicz spaces. Mathematical analysis in fluid mechanicsselected recent results, 145–162, Contemp. Math., **710**, Amer. Math. Soc., Providence, RI, 2018.

[44] H. Huang, J. G. Liu, P. Pickl. On the mean-field limit for the Vlasov-Poisson-Fokker-Planck system. J. of Statistical Physics volume **181** (2020), 1915–1965.

[45] P.–.E Jabin. A review of the Mean-Field limits for Vlasov equations, *Kinet. Relat. Models* **7** (2014), 661-711.

[46] P.–E. Jabin, Z. Wang. Mean field limit and propagation of chaos for Vlasov Systems with bounded forces. *J. Funct. Anal.* **271** (2016), 3588–3627.

[47] P.–E. Jabin, D. Poyato, J. Soler. Mean field limit of non-exchangeable systems. See arXiv:2112.15406

[48] P.–E. Jabin, Z. Wang. Quantitative estimates of propagation of chaos for stochastic systems with $W^{-1,\infty}$ kernels. *Inventiones Math.* **214**(1) (2018), 523–591.

[49] P.–E. Jabin, Z. Wang, Mean field limit for stochastic particle systems. In: Active Particles, Volume 1, Theory, Models, Applications, pp 379–402. Birkhauser-Springer, Boston, 2017.

[50] M. Kac, Foundations of kinetic theory. In *Proceedings of the Third Berkeley Symposium on Mathematical Statistics and Probability, 1954–1955, vol. III*, pages 171–197. University of California Press. Berkeley and Los Angeles, 1956.

[51] D. Lacker, Hierarchies, entropy and quantitative propagation of chaos for mean field diffusions. arXiv:2105.02983

[52] L. Laflèche. Global Semiclassical Limit from Hartree to Vlasov Equation for Concentrated Initial Data. *Annales de l'Institut Henri Poincaré, Analyse Non Linéaire,* **38**(6) (2021), 1739–1762.

[53] O.E. Lanford. Time evolution of large classical systems. In: Moser, J. (ed.) Dynamical Systems, Theory and Applications. Lecture Notes in Physics. Springer-Verlag, Berlin, 1975.

[54] D. Lazarovici. The Vlasov-Poisson Dynamics as the Mean Field Limit of Extended Charges. *Communications in Mathematical Physics* **347** (2016), 271–289.

[55] D. Lazarovici and P. Pickl. A Mean Field Limit for the Vlasov-Poisson System. *Archive for Rational Mechanics and Analysis* **225** (2017), 1201–1231.

[56] C. Le Bris, P.–L. Lions. Existence and uniqueness of solutions to Fokker-Planck type equations with irregular coefficients. *Comm. Partial Differ. Equ* **33** (2008), 1272–1317.

[57] C. Le Bris, P.–L. Lions. Parabolic Equations with irregular data and related issues. Application to Stochastic Differential Equations., De Gruyter Series in Applied and Numerical Mathematics, 2019.

[58] P.-L. Lions and B. Perthame. Propagation of moments and regularity for the 3-dimensional Vlasov-Poisson system, *Invent. Math.* **105**(2) (1991), 415–430.

[59] G. Loeper. Uniqueness of the solution to the Vlasov-Poisson system with bounded density. J. Math. Pures Appl. **86**(1) (2006), 68-79.

[60] McKean, H.P. Jr.: Propagation of chaos for a class of non-linear parabolic equations. In: Stochastic Differential Equations, Lecture Series in Differential Equations, Session 7, 41–57, Catholic Univ, 1967.

[61] Q.H. Nguyen, M. Rosenzweig, S. Serfaty. Mean-field limits of Riez-type singular flows with possible multiplicative transport noise. arxiv:2107.02592 (2021).

[62] J. Nieto, F. Poupaud and J. Soler, High-field limit for the Vlasov-Poisson-Fokker-Planck system, *Arch. Ration. Mech. Anal.* **158** (2001), 29–59.

[63] K. Ono and W. A. Strauss, Regular solutions of the Vlasov-Poisson-Fokker-Planck system. *Discrete Contin. Dynam. Systems* **6** (2000), 751–772.

[64] H. Osada. Propagation of chaos for the two-dimensional Navier-Stokes equation. In *Probabilistic methods in mathematical physics (Katata/Kyoto, 1985)*, pages 303–334. Academic Press, Boston, MA, 1987.

[65] C. Pallard. Space moments of the Vlasov-Poisson system: propagation and regularity, *SIAM J. Math. Anal.* **46**(3) (2014), 1754–1770.

[66] K. Pfaffelmoser. Global classical solutions of the Vlasov-Poisson system in three dimensions for general initial data, *J. Differential Equations* **95**(2) (1992), 281–303.

[67] F. Poupaud and J. Soler, Parabolic limit and stability of the Vlasov-Poisson-Fokker-Planck system,*Math. Models Methods Appl. Sci.* **10** (2000), 1027–1045.

[68] M. Pulvirenti, C. Saffirio, S. Simonella. On the validity of the Boltzmann equation for short range potentials. *Rev. Math. Phys.* **26**(2) (2014), 1450001.

[69] M. Pulvirenti, S. Simonella. The Boltzmann-Grad limit of a hard sphere system: analysis of the correlation error. *Inventiones* **207**(3) (2017), 1135–1237.

[70] G. Rein and J. Weckler, Generic global classical solutions of the Vlasov-Fokker-Planck- Poisson system in three dimensions. *J. Differential Equations* **99** (1992), 59–77.

[71] M. Rosenzweig, S. Serfaty. Global-in-time mean-field convergence for singular Riesz-type diffusive flows. arXiv:2108.09878 (2021).

[72] C. Saffirio. From the Hartree equation to the Vlasov-Poisson system: strong convergence for a class of mixed states. *SIAM Journal on Mathematical Analysis* **52** (2020), 5533–5553.

[73] J. Schaeffer, Global existence of smooth solutions to the Vlasov-Poisson system in three dimensions. *Comm. Partial Differential Equations* **16** (1991), 1313–1335.

[74] S. Schochet. The weak vorticity formulation of the 2-D Euler equations and concentration-cancellation. *Comm. Partial Differential Equations* **20**(5-6) (1995),1077–1104.

[75] S. Schochet, The point-vortex method for periodic weak solutions of the 2-D Euler equations. *Comm. Pure Appl. Math.* **49**(9) (1996), 911–965.

[76] S. Serfaty. Mean field limit for Coulomb type flows. *Duke Math J.* **169** (2020), 2887–2935.

[77] A.S. Sznitman. Topics in propagation of chaos. In *École d'Été de Probabilités de Saint-Flour XIX—1989*, vol. 1464 of *Lecture Notes in Math.*, 165–251. Springer, Berlin, 1991.

[78] S. Ukai, T. Okabe. On classical solutions in the large in time of two-dimensional Vlasov's equation, *Osaka J. Math.* **15**(2) (1978), 245–261.

[79] H.D. Victory, jr., E.J. Allen, *The convergence theory of particle-in-cell methods for multidimensional Vlasov-Poisson systems*, SIAM J. Numer. Anal. **28** (1991), 1207–1241.

[80] H. D. Victory, On the existence of global weak solutions for Vlasov-Poisson-Fokker-Planck systems. *J. Math. Anal. Appl.* **160** (1991), 525–555.

[81] H. D. Victory, B. P. ODwyer, On classical solutions of Vlasov-Poisson Fokker-Planck systems. *Indiana Univ. Math. J.* **39** (1990), 105–156.

[82] A. A. Vlasov. On vibration properties of electron gas. *J. Exp. Theor. Phys.* (in Russian) **8** (1938), 291.

[83] A. A. Vlasov. The vibrational properties of an electron gas. *Sov. Phys. Usp.* **10** (1968), 721–733.

[84] S. Wollman, On the approximation of the Vlasov-Poisson system by particle methods. *SIAM J. Numer. Anal.*, **37**(4) (2000), 1369–1398.

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