The graph limit for a pairwise competition model

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Abstract

This paper is aimed at extending the graph limit with time dependent weights obtained in [1] for the case of a pairwise competition model introduced in [10], in which the equation governing the weights involves a weak singularity at the origin. Well posedness for the graph limit equation associated with the ODE system of the pairwise competition model is also proved.

1   Introduction

General Background. In this work, we are concerned with analyzing the graph limit of the following system of $(d+1)N$ ODEs

$$\begin{cases} \dot{x}_i^N(t) = \frac{1}{N} \sum_{j=1}^{N} m_j^N(t)\mathbf{a}(x_j^N(t) - x_i^N(t)), & x_i^N(0) = x_i^{0,N} \\ \dot{m}_i^N(t) = \psi_i^N(\mathbf{x}_N(t), \mathbf{m}_N(t)), & m_i^N(0) = m_i^{0,N}. \end{cases} \tag{1.1}$$

The notation is as follows: the unknowns are $x_i^N \in \mathbb{R}^d$ and $m_i^N \in \mathbb{R}$ are referred to as the opinions and weights respectively. The evolution of the opinions is given in terms of the weights and a function $\mathbf{a} : \mathbb{R}^d \to \mathbb{R}^d$ which is called the influence. The evolution of the weights is given by means of functions $\psi_i^N : \mathbb{R}^{dN} \times \mathbb{R}^N \to \mathbb{R}$ where we apply the notation

$$\mathbf{x}_N(t) := (x_1^N(t), ..., x_N^N(t)), \ \mathbf{m}_N(t) := (m_1^N(t), ..., m_N^N(t)).$$

This model has been proposed in [10], along with several other models which are meant to idealize social dynamics. We refer to [10, 13] for more details of how these models originate from biology and social sciences. Mathematically, the system (1.1) is a weighted version of the first order $N-$body problem (simply by taking all the weights to be identically equal to 1). By now, the mean field limit of the $N-$body problem

$$\dot{x}_i^N(t) = \frac{1}{N} \sum_{j=1}^{N} \mathbf{a}(x_j^N(t) - x_i^N(t)), \ x_i^N(0) = x_i^{0,N} \tag{1.2}$$

is fairly well understood even for influence functions with strong singularities at the origin [14]. The mean field limit can be analysed in terms of the empirical measure defined by

$$\mu_N(t) := \frac{1}{N} \sum_{i=1}^{N} \delta_{x_i^N(t)}.$$

Thanks to the work of Dobrushin [4] it is possible to prove quantitative convergence of $\mu_N(t)$ to the solution μ of the (velocity free) Vlasov equation

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$$\partial_t \mu(t,x) - \text{div}(\mu \mathbf{a} \star \mu)(t,x) = 0, \ \mu(0,\cdot) = \mu^0 \qquad (1.3)$$

with respect to the Wasserstein metric (provided this is true initially of course). The mean field limit with time dependent weights has been investigated in [1, 5, 6] for Lipschitz continuous interactions and ψ_i^N which are at least Lipschitz in each variable, and more recently in [2] for the case of the 1D attractive Coulomb interaction (but still with ψ_i^N regular enough). There is a different regime, the so called graph limit, closely related to the mean-field limit. In the graph limit, we pass from a discrete system of ODEs to a "continuous" system in the following sense: we associate to $\mathbf{x}_N(t), \mathbf{m}_N(t)$ the following Riemman sums $\widetilde{x}_N : [0,T] \times I \to \mathbb{R}, \widetilde{m}_N : [0,T] \times I \to \mathbb{R}$ defined by

$$\widetilde{x}_N(t,s) := \sum_{i=1}^N x_i(t) \mathbf{1}_{[\frac{i-1}{N}, \frac{i}{N}]}(s), \ \widetilde{m}_N(t,s) := \sum_{i=1}^N m_i(t) \mathbf{1}_{[\frac{i-1}{N}, \frac{i}{N}]}(s).$$

Using the equation for the trajectories of the opinions and weights, one easily finds that $\widetilde{x}_N, \widetilde{m}_N$ are governed by the following equations

$$\begin{cases} \partial_t \widetilde{x}_N(t,s) = \displaystyle\int_I \widetilde{m}_N(t,s_*) \mathbf{a}(\widetilde{x}_N(t,s_*) - \widetilde{x}_N(t,s)) ds_*, & \widetilde{x}_N(0,s) = \widetilde{x}_N^0(s) \\ \partial_t \widetilde{m}_N(t,s) = N \displaystyle\int_{\frac{1}{N}\lfloor sN \rfloor}^{\frac{1}{N}(\lfloor sN \rfloor + 1)} \psi(s_*, \widetilde{x}_N(t,\cdot), \widetilde{m}_N(t,\cdot)) ds_*, & \widetilde{m}_N(0,s) = \widetilde{m}_N^0(s). \end{cases}$$

Lebesgue differentiation theorem leads us formally to the following integro-differential equation

$$\begin{cases} \partial_t x(t,s) = \displaystyle\int_I m(t,s_*) \mathbf{a}(x(t,s_*) - x(t,s)) ds_*, & x(0,s) = x^0(s) \\ \partial_t m(t,s) = \Psi(s, x(t,\cdot), m(t,\cdot)), & m(0,s) = m^0(s). \end{cases} \qquad (1.4)$$

Here $\Psi : I \times L^\infty(\mathbb{R}^d) \times L^\infty(\mathbb{R}^d) \to \mathbb{R}$ is a functional whose relation to ψ_i^N is given by the formula (2.5) in the next section. The formula relating $x^0(s), m^0(s)$ to $\widetilde{x}_N^0(s), \widetilde{m}_N^0(s)$ will be given in the next section as well (formula (2.6)). Hence, one expects that the sums $\widetilde{x}_N(t,s), \widetilde{m}_N(t,s)$ are an approximation of the solution $(x(t,s), m(t,s))$ of the Equation (1.4) .

Before going further, let us briefly comment on the origin of the terminology "graph limit". This name stems from the fact that the system (1.1) can be viewed as a nonlinear heat equation on a graph. For example, in the case where the weights are time independent and the $\mathbf{a}$ is taken to be the identity, then the system (1.1) can be rewritten as the linear heat equation with respect to the Laplacian associated to the underlying simple graph. This is the point of view which has been taken in [11]. However, this underlying combinatorial structure seems to come into play mostly when the weights may vary from one opinion to another, in which case methods from graph theory prove as highly useful. We also refer to the more recent work [9] for a demonstration of the power of graph theory techniques in the context of the mean field limit, and [3] in the context of convergence to consensus for the graph limit equation. See also [12] for a proof of the graph limit for metric valued labels, alongside an extensive explanation of the relation between the graph limit and the hydrodynamic and mean field limits. In our settings, which are very similar to the framework in [1], this graph structure is not as relevant, and we shall therefore not dwell on this matter. It is instructive to view the system (1.4) as continuous version of (1.1), in the sense that it is obtained by replacing averaged sums by integrals on the unit interval and summation indices by variables in the unit interval.

Relevant Literature and Contribution of the Present work. It appears that the graph limit point of view has not received as much attention as the mean-field limit. The study of this problem was initiated in [11], which as already remarked, considers time independent weights which may depend on the index of the opinion as well. This result has been extended in [1] to cover time dependent weights (although in [1] the weights depend only on the summation index). The evolution in time of the weights renders difficult the problem both at the microscopic and graph limit level- since

the corresponding ODE/integro-differential equation become coupled (compare for instance Equations (1.1) and (1.2)), and at the macroscopic level- since the mean field PDE includes a non-local source term (see Section 4 for more details). In both of these results, the functions ψ_i^N are assumed to be well behaved in terms of regularity. On the other hand, models corresponding to scenarios where the functions ψ_i^N exhibit singularities recently received attention in [10]. For instance, the following ODE has been studied in [10]:

$$
\begin{cases}
\dot{x}_i^N(t) = \frac{1}{N} \sum_{j=1}^{N} m_j^N(t) \mathbf{a}(x_j^N(t) - x_i^N(t)), & x_i^N(0) = x_i^{0,N} \\
\dot{m}_i^N(t) = \frac{1}{N} \sum_{j=1}^{N} m_i^N(t) m_j^N(t) \left\langle \frac{\dot{x}_i^N(t) + \dot{x}_j^N(t)}{2}, \mathbf{s}(x_i^N(t) - x_j^N(t)) \right\rangle, & m_i^N(0) = m_i^{0,N},
\end{cases}
\tag{1.5}
$$

where $\mathbf{a} : \mathbb{R}^d \to \mathbb{R}^d$ is Lipschitz and takes the form $\mathbf{a}(x) = a(|x|)x$ for some radial $a : \mathbb{R} \to \mathbb{R}$, and $\mathbf{s} : \mathbb{R}^d \to \mathbb{S}^{d-1}$ is the projection on the unit sphere, i.e.

$$
\mathbf{s}(x) := \begin{cases} \frac{x}{\|x\|}, & x \neq 0 \\ 0, & x = 0. \end{cases}
\tag{1.6}
$$

Of course, inserting the equation for x_i into the equation for m_i transfers the system to the form (1.1). System (1.5) is referred to as a pairwise competition model in [10], and its well posedness can be proved provided opinions are separated initially ($i \neq j \implies x_i^0 \neq x_j^0$). It is the aim of this work to investigate how to overcome the challenges created due to the singularity in the weight function in the context of the graph limit. The problem of the graph limit for singularities in the influence function is also interesting. As already remarked, for the mean field limit this has been successfully achieved in [2] for the 1D attractive Coulomb case. However, it is not clear how to study the graph limit regime in this whole generality. The 1D repulsive Coulomb interaction however is manageable, and can be handled by similar methods to the one demonstrated in the present work.

A first contribution of the present work is reflected on two levels, both of which are considered in 1D: the well posedness of the graph limit equation (1.4), and the derivation of (1.4) from the opinion dynamics (1.1) in the limit as $N \to \infty$. As for the first point, we note that in the case when $\mathbf{a}$ and ψ_i^N are well behaved then equation (1.4) can be viewed as a Banach valued ODE, and noting that at each time t our unknowns $(x(t, \cdot), m(t, \cdot))$ are functions of the variable s, and therefore there is a straightforward analogy between the well posedness of the discrete System (1.1) and equation (1.4). As already mentioned, the global well-posedness of the finite dimensional version of Equation (1.4), namely System (1.5) has been (among other things) proved in [10] using the theory of differential inclusions as developed by Fillipov [7]. Originally, Fillipov formulated his theory for unknowns taking values in a finite dimensional space in contrast to Equation (1.4). We follow a slightly different route which is in fact more elementary and does not require any familiarity with convex analysis. A second contribution of the present work, is studying the graph limit in arbitrary dimensions $d > 1$. In higher dimensions, a natural assumption to impose on the initial datum x^0 is that it is bi-Lipschitz in s - an assumption of this type is strictly stronger from what is needed in 1D. This in turn leads to considering Riemann sums whose labeling variable s varies on the d-dimensional unit cube rather on the unit interval, because cubes of different dimensions cannot be diffeomorphic. This labelling procedure does not have any modelling interpretation since particles (opinions) are still exchangeable or indistinguishable. It would in fact be possible to still work on the unit interval through a change of variable on the labeling variable, since all cubes (and most measurable spaces that one may use) are isomorphic to the unit interval per the Borel isomorphism theorem. However the corresponding analysis would be far more convoluted, and instead having the labeling variable on the d-dimensional unit cube make the various technical steps more transparent. These considerations are therefore detailed separately in Section 5. For both points it is crucial to observe the lower bound $|x(t, s_2) - x(t, s_1)| \gtrsim |x^0(s_2) - x^0(s_1)|$. In 1D the initial separation at the continuous level will be replaced by the assumption that x^0 is increasing, whereas in higher dimensions this assumption will be replaced by requiring that x^0 is bi-Lipschitz. Finally we remark that the method here extends the case of the main results in [1], in the sense that it

simultaneously covers functions **s** which are either Lipschitz or have a jump discontinuity at the origin. This last observation is simple but not obvious- for example in the case where the singularity emerges from the influence part, as mentioned earlier, it is not clear how to unify both results.

We organize the paper as follows: Section 2 reviews the terminology introduced in [1] in the specific context of system (1.5). In particular, Section 2 includes preliminaries such as the existence and uniqueness of classical solutions to the system (1.1) in the present settings and other basic properties of solutions (of course, uniqueness is not strictly needed for the purpose of the graph or mean field limit). Section 3 is a continuous adaptation of section 2, namely well posedness for the 1D graph limit equation for which uniqueness is essential. Section 4 includes the main evolution estimate leading to the 1D graph limit, and clarifies the link between the mean field and the graph limit. In Section 5 we introduce multi-dimensional Riemann sums and study the graph limit for arbitrary $d > 1$.

2 Preliminaries

2.1 The ODE system.

Recall that the system which will occupy us is

$$
\begin{cases}
\dot{x}_i^N(t) = \frac{1}{N} \sum_{j=1}^{N} m_j(t) \mathbf{a}(x_j(t) - x_i(t)) & x_i(0) = x_i^{0,N} \\
\dot{m}_i^N(t) = \psi_i^N(\mathbf{x}_N(t), \mathbf{m}_N(t)) & m_i(0) = m_i^{0,N}
\end{cases}
\tag{2.1}
$$

where

$$
\psi_i^N(\mathbf{x}_N, \mathbf{m}_N) := \frac{1}{2N^2} m_i \sum_{j,k} m_j m_k \left(\mathbf{a}(x_k - x_i) + \mathbf{a}(x_k - x_j) \right) \mathbf{s}(x_i - x_j).
\tag{2.2}
$$

When $d = 1$, which is the case of main interest here, we note that **s** identifies with the sign function. We start by reviewing the well-posedness theory which has been established for the System (2.1) in [10]. As usual with ODEs with weakly singular right hand sides, the argument in [10] rests on the theory of differential inclusions as developed by Fillipov [7] and the fact that opinions remain separated for all times provided this is true initially. Unless necessary, we omit the super index N in the opinions and weights.

Proposition 2.1. ([10, Proposition 3] *Suppose* $\mathbf{a} : \mathbb{R}^d \to \mathbb{R}^d$ *is Lipschitz with* $\mathbf{a}(0) = 0$ *and* $x_i^0 \neq x_j^0$ *for all* $i \neq j$. *Then there exists a unique classical solution* $(\mathbf{x}_N(t), \mathbf{m}_N(t))$ *to the System (2.1) with* $x_i(t) \neq x_j(t)$ *for all* $i \neq j$ *and* $t \geq 0$.

We also recap the following basic properties of solutions, which already appear implicitly or explicitly in [1, 10], and will appear in the course of the proof of the main theorems.

Lemma 2.1. *Let the assumptions of Proposition 2.1 hold. Assume also* $m_i^0 > 0, i = 1, ..., N$ *and* $\text{Lip}(\mathbf{a}) = L$. *Let* $(\mathbf{x}_N(t), \mathbf{m}_N(t))$ *be the solution of System (2.1) on* $[0, T]$. *Then*

i. *(Conservation of total mass).* $\frac{1}{N} \sum_{i=1}^{N} m_i^0 = 1 \implies \frac{1}{N} \sum_{i=1}^{N} m_i(t) = 1, \ t \in [0, T]$.

ii. *(Uniform bound in time on opinions).* *If* $\left| x_i^0 \right| \leq \overline{X}$ *then for all* $t \in [0, T]$ *it holds that*

$$
|x_i(t)| \leq \overline{X} e^{2LT}.
$$

iii. *(Uniform bound in time on weights).* $m_i(t) > 0$ *for all* $t \in [0, T]$ *with the estimate*

$$
m_i^0 e^{-2\overline{X} e^{2LT} t} \leq m_i(t) \leq m_i^0 e^{2\overline{X} e^{2LT} t}.
$$

4

iv. (Opinions are separated). There is a constant $C = C(L, T) > 1$ such for all $t \in [0, T]$ the following bound holds

$$\frac{1}{C} \left| x_i^0 - x_j^0 \right| \leq |x_i(t) - x_j(t)| \leq C \left| x_i^0 - x_j^0 \right|.$$

Proof. For i. see Proposition 2 in [10]. For ii., fix a time $\tau > 0$ such that $m_j(\tau) \geq 0, j = 1, ..., N$ for all $t \in [0, \tau]$ (such a time exists by continuity). We utilize i. and the assumption $\mathbf{a}(0) = 0$ to find that for each $t \in [0, \tau_0]$

$$|x_i(t)| \leq \left| x_i^0 \right| + \frac{2L}{N} \sum_{j=1}^{N} \int_0^t |m_j(\tau)| \max_{1 \leq k \leq N} |x_k(\tau)| \, d\tau = \overline{X} + 2L \int_0^t \max_{1 \leq k \leq N} |x_k(\tau)| \, d\tau$$

so that

$$\max_{1 \leq k \leq N} |x_k(t)| \leq \overline{X} + 2L \int_0^t \max_{1 \leq k \leq N} |x_k(\tau)| \, d\tau,$$

which by Gronwall's Lemma implies

$$\max_{1 \leq k \leq N} |x_k(t)| \leq \overline{X} e^{2LT}. \tag{2.3}$$

We prove iii., from which we will conclude ii. for all $t \in [0, T]$. We start by explaining why $m_i(t) > 0$. Indeed, if on the contrary $m_i(t) \leq 0$ for some $1 \leq i \leq N$ and $t \in [0, T]$ and let

$$\tau := \inf \left\{ t \in [0, T] \, | \exists 1 \leq i \leq N : m_i(t) \leq 0 \right\}.$$

Then the bound from ii. and preservation of total mass of i. imply that for all $t \in [0, \tau)$ we have

$$\left| \frac{d}{dt} \frac{1}{2} \log \left(m_i^2 \right) \right| = \left| \frac{\dot{m}_i(t)}{m_i} \right| = \left| \frac{1}{2N^2} \sum_{j,k} m_j m_k \left(\mathbf{a}(x_k - x_i) + \mathbf{a}(x_k - x_j) \right) \mathbf{s}(x_i - x_j) \right| \leq 2\overline{X} e^{2LT},$$

hence

$$-2\overline{X} e^{2LT} \leq \frac{1}{2} \frac{d}{dt} \log \left(m_i^2(t) \right) \leq 2\overline{X} e^{2LT}.$$

Integration in time yields that for all $t \in [0, \tau]$

$$-2\overline{X} e^{2LT} t + \log \left(m_i^0 \right) \leq \log \left(m_i \right) \leq 2\overline{X} e^{2LT} t + \log \left(m_i^0 \right),$$

and consequently

$$m_i^0 e^{-2\overline{X} e^{2LT} t} \leq m_i(t) \leq m_i^0 e^{2\overline{X} e^{2LT} t}.$$

Letting $t \nearrow \tau$ yields a contradiction. Therefore $m_i(t) > 0$ for all $t \in [0, T]$ which in turn implies that (2.3) holds for all $t \in [0, T]$. Remark also that the same estimate done on the interval $[0, T]$ yields the asserted bound on $[0, T]$. Point iv. is Proposition 7 in [10].

$\square$

2.2 The graph limit equation.

In the graph limit we attach to the flow of System (1.1) the following "Riemman sums"

$$\widetilde{x}_N(t,s) := \sum_{i=1}^{N} x_i(t) \mathbf{1}_{[\frac{i-1}{N}, \frac{i}{N}]}(s), \ \widetilde{m}_N(t,s) := \sum_{i=1}^{N} m_i(t) \mathbf{1}_{[\frac{i-1}{N}, \frac{i}{N}]}(s). \tag{2.4}$$

The functional $\Psi : I \times L^\infty(I) \times L^\infty(I) \to \mathbb{R}$ and the functions $x^0 : I \to \mathbb{R}^d, m^0 : I \to \mathbb{R}$ are given and the functions ψ_i^N and the initial data $x_i^{0,N}, m_i^{0,N}$ are defined in terms of these functions through the following formula

$$\psi_i^N(\mathbf{x}_N(t), \mathbf{m}_N(t)) := N \int_{\frac{i-1}{N}}^{\frac{i}{N}} \Psi(s, \widetilde{x_N}(t, \cdot), \widetilde{m_N}(t, \cdot)) ds \tag{2.5}$$

and

$$x_i^{0,N} := N \int_{\frac{i-1}{N}}^{\frac{i}{N}} x^0(s) ds, \ m_i^{0,N} := N \int_{\frac{i-1}{N}}^{\frac{i}{N}} m^0(s) ds. \tag{2.6}$$

If Ψ is given by

$$\Psi(s, x(\cdot), m(\cdot)) := m(s) \iint_{I^2} m(s_*) m(s_{**}) \left(\mathbf{a}(x(s_{**}) - x(s)) + \mathbf{a}(x(s_{**}) - x(s_*)) \right) \mathbf{s}(x(s) - x(s_*)) ds_* ds_{**},$$

then one readily checks that the ψ_i^N in Formula (2.2) are recovered via Formula (2.5). Notice that by Lebesgue's differentiation theorem $\widetilde{x}_N(0, s), \widetilde{m}_N(0, s)$ well approximate $x^0(s), m^0(s)$ because for a.e. s we have pointwise convergence

$$\widetilde{x}_N(0, s) = N \int_{\frac{\lfloor sN \rfloor}{N}}^{\frac{\lfloor sN \rfloor + 1}{N}} x^0(\sigma) d\sigma \underset{N \to \infty}{\to} x^0(s), \ \widetilde{m}_N(0, s) = N \int_{\frac{\lfloor sN \rfloor}{N}}^{\frac{\lfloor sN \rfloor + 1}{N}} m^0(\sigma) d\sigma \underset{N \to \infty}{\to} m^0(s).$$

Also, it is worthwhile remarking that unlike in the mean field limit regime, where the initial data realizing the initial convergence can be chosen from a set of full measure, here we use a very specific choice for the initial data, and in particular all initial data of the form specified by formula (2.6) constitute a set of measure 0, which means that the probabilistic methods that we have at our disposal in the mean field limit become useless in the graph limit. We will return to this point in Section 4. The functions $\widetilde{x}_N(t, s), \widetilde{m}_N(t, s)$ defined through Formula (5.2) are governed by the following equations, which should be compared with the graph limit Equation (1.4).

Proposition 2.2. *Let the assumptions of Proposition 2.1 hold and let $(\mathbf{x}_N(t), \mathbf{m}_N(t))$ be the solution to System (2.1) on $[0, T]$. Let $\widetilde{x}_N, \widetilde{m}_N$ be given by (5.2). Then*

$$\begin{cases} \partial_t \widetilde{x}_N(t, s) = \int_I \widetilde{m}_N(t, s_*) \mathbf{a}(\widetilde{x}_N(t, s_*) - \widetilde{x}_N(t, s)) ds_*, \\ \partial_t \widetilde{m}_N(t, s) = N \int_{\frac{\lfloor sN \rfloor}{N}}^{\frac{\lfloor sN \rfloor + 1}{N}} \Psi(s_*, \widetilde{x}_N(t, \cdot), \widetilde{m}_N(t, \cdot)) ds_*. \end{cases} \tag{2.7}$$

Proof. We start with the equation for $\widetilde{x}_N(t, s)$. Fix $s \in \left[\frac{i_0 - 1}{N}, \frac{i_0}{N} \right)$, we get

$$\partial_t \widetilde{x}_N(t, s) = \sum_{i=1}^{N} \dot{x}_i(t) \mathbf{1}_{[\frac{i-1}{N}, \frac{i}{N}]}(s) = \frac{1}{N} \sum_{i=1}^{N} \sum_{j=1}^{N} m_j(t) \mathbf{a}(x_j(t) - x_i(t)) \mathbf{1}_{[\frac{i-1}{N}, \frac{i}{N}]}(s)$$

$$= \frac{1}{N} \sum_{j=1}^{N} m_j(t) \mathbf{a}(x_j(t) - x_{i_0}(t)) = \frac{1}{N} \sum_{j=1}^{N} m_j(t) \mathbf{a}(x_j(t) - \widetilde{x}_N(t, s)).$$

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On the other hand, we have

$$
\begin{aligned}
\int_I \widetilde{m}_N(t, s_*)\mathbf{a}(\widetilde{x}_N(t, s_*) - \widetilde{x}_N(t, s))ds_* &= \int_I \sum_{j=1}^N m_j(t)\mathbf{1}_{[\frac{j-1}{N}, \frac{j}{N}]}(s_*)\mathbf{a}\left(\sum_{k=1}^N x_k(t)\mathbf{1}_{[\frac{k-1}{N}, \frac{k}{N}]}(s_*) - \widetilde{x}_N(t, s)\right)ds_* \\
&= \sum_{j=1}^N \int_I \mathbf{1}_{[\frac{j-1}{N}, \frac{j}{N}]}(s_*)m_j(t)\mathbf{a}\left(x_j(t) - \widetilde{x}_N(t, s)\right)ds_* \\
&= \frac{1}{N}\sum_{j=1}^N m_j(t)\mathbf{a}\left(x_j(t) - \widetilde{x}_N(t, s)\right).
\end{aligned}
$$

The equation for $\widetilde{m}_N$ is obtained due to the following identities

$$
\begin{aligned}
\partial_t \widetilde{m}_N(t, s) &= \sum_{i=1}^N \dot{m}_i(t)\mathbf{1}_{[\frac{i-1}{N}, \frac{i}{N}]}(s) = N \sum_{i=1}^N \mathbf{1}_{[\frac{i-1}{N}, \frac{i}{N}]}(s) \int_{\frac{i-1}{N}}^{\frac{i}{N}} \Psi(s_*, \widetilde{x}_N(t, \cdot), \widetilde{m}_N(t, \cdot))ds_* \\
&= N \int_{\frac{\lfloor sN \rfloor}{N}}^{\frac{\lfloor sN \rfloor + 1}{N}} \Psi(s_*, \widetilde{x}_N(t, \cdot), \widetilde{m}_N(t, \cdot))ds_*.
\end{aligned}
$$

$\square$

3 Well Posedness for the Graph Limit Equation

3.1 The decoupled equation

We first decouple the equation and prove well posedness for the two resulting equations separately as in [1]. To be more precise, the system considered in [1, Section 4] reads

$$
\begin{cases}
\partial_t x(t, s) = \int_I m(t, s_*)\mathbf{a}(x(t, s_*) - x(t, s))ds_*, & x(0, s) = x^0(s) \\
\partial_t m(t, s) = \psi_{S,k}(s, x(t, \cdot), m(t, \cdot)), & m(0, s) = m^0(s),
\end{cases}
\tag{3.1}
$$

where $\psi_{S,k}(s, x(t, \cdot), m(t, \cdot)) := m(t, s) \int_{I^k} m^{\otimes k}(s_1, ..., s_k)S(x(s), x(s_1), ..., x(s_k))ds_1...ds_k$. The hypothesis on the function S and the initial data x^0, m^0 in (3.1) assumed in [1] are:

H1' $d \geq 1, \mathbf{a}(0) = 0$ and $\mathbf{a} \in \text{Lip}(\mathbb{R}^d)$.

H2' $(x^0, m^0) \in L^\infty(I; \mathbb{R}^d) \times L^\infty(I; \mathbb{R}_{>0})$.

H3' The function $S \in C_b(\mathbb{R}^{(k+1)d}) \cap \text{Lip}(\mathbb{R}^{(k+1)d})$ and there are $(i, j) \in \{0, ..., k\}$ such that

$$
S(..., y_i, ..., y_j, ...) = -S(..., y_j, ..., y_i, ...).
\tag{3.2}
$$

The most restrictive assumption for the graph limit is **H3'** since S is not Lipschitz in problems of interest mentioned in the introduction, see [5, 1]. Furthermore, any solution to the graph limit equation (1.4) is expected to satisfy an estimate analogue to Inequality iv. in Lemma 2.1, namely

$$
\frac{1}{C}\left|x^0(s_1) - x^0(s_2)\right| \leq |x(t, s_1) - x(t, s_2)| \leq C\left|x^0(s_1) - x^0(s_2)\right|,
$$

which would imply Lipschitz continuity along the trajectories provided x^0 is one to one. This also leads us to remark that the initial separation in the microscopic system (1.5) can be replaced by the assumption that x^0 is one to one in the infinite dimensional case, which means that we need to be able to evaluate x^0 pointwise, and therefore a more natural assumption is $x^0 \in C(I)$ rather than $x^0 \in L^\infty$.

To summarize, in contrast to [1], we assume the hypotheses:

H1 $d = 1$, $\mathbf{a}(0) = 0$ and $\mathbf{a} \in \text{Lip}(\mathbb{R})$ with $L := \text{Lip}(\mathbf{a})$.

H2 i. $m^0 \in L^\infty(I)$, $\int_I m_0(s)ds = 1$ and $\frac{1}{M} \leq m^0 \leq M$ for some $M > 1$.

   ii. $x^0 \in C(I)$ is one to one and $\left|x^0\right| \leq X$ for some $X > 0$.

H3 i. The restrictions $\mathbf{s}|_{(0,\infty)}$ and $\mathbf{s}|_{(-\infty,0)}$ are Lipschitz, i.e. there is some $\mathbf{S} > 0$ such that

$$|\mathbf{s}(x_1) - \mathbf{s}(x_2)| \leq \mathbf{S}\,|x_1 - x_2|\,, \ \ x_1, x_2 \in (0, \infty)$$

   and

$$|\mathbf{s}(x_1') - \mathbf{s}(x_2')| \leq \mathbf{S}\,|x_1' - x_2'|\,, \ \ x_1', x_2' \in (-\infty, 0).$$

   ii. $\mathbf{s}$ is odd $(\mathbf{s}(-x) = -\mathbf{s}(x))$ and there is some $\mathbf{S}_\infty > 0$ such that $|\mathbf{s}(x)| \leq \mathbf{S}_\infty$, $x \in \mathbb{R}$.

Clearly, the sign function is a particular example of hypothesis **H3**. In the following Lemma, which is a variant of [1, Lemma 3], the new considerations discussed above will be taken into account.

Lemma 3.1. *Let hypothesis **H1-H3** hold.*
1. Suppose that
   • $m_1, m_2 \in C([0,T], L^\infty(I))$ are non-negative and $\int_I m_2(t,s)ds = \int_I m_1(t,s)ds = 1$ for all $t \in [0,T]$.
   • $x \in C([0,T] \times I)$ with $\sup_{[0,T]\times I} |x| \leq \overline{X}$.
   Then, for all $t \in [0,T]$ it holds that

$$\int_I |\Psi(s, x(t, \cdot), m_1(t, \cdot)) - \Psi(s, x(t, \cdot), m_2(t, \cdot))|\,ds \leq 12L\mathbf{S}_\infty \overline{X}\,\|m_1(t, \cdot) - m_2(t, \cdot)\|_1\,.$$

2. Suppose that
   • $m \in C([0,T], L^\infty(I))$ is non-negative and $\int_I m(t,s)ds = 1$ for all $t \in [0,T]$ and

$$\sup_{[0,T]} \|m(t, \cdot)\|_\infty \leq \overline{M}.$$

   • $x_1, x_2 \in C([0,T] \times I)$ are such that for all $t \in [0,T]$ the maps $s \mapsto x_1(t,s), s \mapsto x_2(t,s)$ are increasing.
   Then, for all $t \in [0,T]$ it holds that

$$\int_I |\Psi(s, x_1(t, \cdot), m(t, \cdot)) - \Psi(s, x_2(t, \cdot), m(t, \cdot))|\,ds \leq L\left(3\overline{M}\mathbf{S}_\infty + \mathbf{S}_\infty + 16\mathbf{S}\overline{X}\right) \sup_I |x_1(t, \cdot) - x_2(t, \cdot)|\,.$$

   Proof. **Step 1**. For readability, we suppress the time variable (unless unavoidable). Set $\mathbf{a}(s, s_*, s_{**}) := \mathbf{a}(x(s_{**}) - x(s)) + \mathbf{a}(x(s_{**}) - x(s_*))$, we have

$$\left| m_1(s) \iint_{I^2} m_1(s_*)m_1(s_{**})\mathbf{a}(s, s_*, s_{**})\mathbf{s}(x(s) - x(s_*))ds_*ds_{**} \right.$$

$$\left. -m_2(s) \iint_{I^2} m_2(s_*)m_2(s_{**})\mathbf{a}(s, s_*, s_{**})\mathbf{s}(x(s) - x(s_*))ds_*ds_{**} \right|$$

$$\leq m_1(s) \left| \iint_{I^2} \left(m_1(s_*)m_1(s_{**}) - m_2(s_*)m_2(s_{**})\right)\mathbf{a}(s, s_*, s_{**})\mathbf{s}(x(s) - x(s_*))ds_*ds_{**} \right|$$

$$+ |m_1(s) - m_2(s)| \left| \iint_{I^2} m_2(s_*)m_2(s_{**})\mathbf{a}(s, s_*, s_{**})\mathbf{s}(x(s) - x(s_*))ds_*ds_{**} \right|$$

$$\leq 4L\mathbf{S}_\infty \sup_{[0,T]\times I} |x|\, m_1(s) \iint_{I^2} |m_1(s_*)m_1(s_{**}) - m_2(s_*)m_2(s_{**})|\,ds_*ds_{**}$$

$$+ 4L\mathbf{S}_\infty \sup_{[0,T]\times I} |x|\, |m_1(s) - m_2(s)| \iint_{I^2} m_2(t, s_*)m_2(t, s_{**})ds_*ds_{**}. \tag{3.3}$$

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Using the assumption that $\int_I m_1(s)ds = \int_I m_2(s)ds = 1$, the first integral in the right hand side of (3.3) can be estimated as

$$\leq \iint_{I^2} |m_1(s_*)m_1(s_{**}) - m_1(s_*)m_2(s_{**})|\, ds_* ds_{**} + \iint_{I^2} |m_1(s_*)m_2(s_{**}) - m_2(s_*)m_2(s_{**})|\, ds_* ds_{**}$$

$$= \int_I |m_1(s_{**}) - m_2(s_{**})|\, ds_{**} + \int_I |m_1(s_*) - m_2(s_*)|\, ds_* = 2\,\|m_1(t,\cdot) - m_2(t,\cdot)\|_1\,.$$

Therefore, integrating (3.3) in s over I produces

$$\int_I |\Psi(s, x(t,\cdot), m_1(t,\cdot)) - \Psi(s, x(t,\cdot), m_2(t,\cdot))|\, ds \leq 12 L \mathbf{S}_\infty \sup_{[0,T]\times I} |x| \, \|m_1(t,\cdot) - m_2(t,\cdot)\|_1\,.$$

Step 2. Set $\mathbf{a}_i(s, s_*, s_{**}) := \mathbf{a}(x_i(s_{**}) - x_i(s)) + \mathbf{a}(x_i(s_{**}) - x_i(s_*))$, $i = 1, 2$, we can also estimate as

$$\left| \iint_{I^2} m(s_*)m(s_{**})(\mathbf{a}_1(s, s_*, s_{**})\mathbf{s}(x_1(s) - x_1(s_*))ds_* ds_{**} - \mathbf{a}_2(s, s_*, s_{**})\mathbf{s}(x_2(s) - x_2(s_*)))ds_* ds_{**} \right|$$

$$\leq L\mathbf{S}_\infty \iint_{I^2} m(s_*)m(s_{**})\left(2\,|x_1(s_{**}) - x_2(s_{**})| + |x_1(s_*) - x_2(s_*)| + |x_1(s) - x_2(s)|\right)ds_* ds_{**}$$

$$+ \iint_{I^2} m(s_*)m(s_{**})\,|\mathbf{a}_2(s, s_*, s_{**})|\,|\mathbf{s}(x_1(s) - x_1(s_*)) - \mathbf{s}(x_2(s) - x_2(s_*))|\, ds_* ds_{**}$$

$$:= J_1(t, s) + J_2(t, s).$$

Since $s \mapsto x_1(t, s), s \mapsto x_2(t, s)$ are increasing, we recognize from assumption **H3** that

$$J_2(t, s) = \int_I \int_0^s m(s_*)m(s_{**})\,|\mathbf{a}_2(s, s_*, s_{**})|\,|\mathbf{s}(x_1(s) - x_1(s_*)) - \mathbf{s}(x_2(s) - x_2(s_*))|\, ds_* ds_{**}$$

$$+ \int_I \int_s^1 m(s_*)m(s_{**})\,|\mathbf{a}_2(s, s_*, s_{**})|\,|\mathbf{s}(x_1(s) - x_1(s_*)) - \mathbf{s}(x_2(s) - x_2(s_*))|\, ds_* ds_{**}$$

$$\leq 8 L \overline{X} \mathbf{S} \sup_I |x_1(t,\cdot) - x_2(t,\cdot)| + 8 L \overline{X} \mathbf{S} \sup_I |x_1(t,\cdot) - x_2(t,\cdot)|$$

$$= 16 L \overline{X} \mathbf{S} \sup_I |x_1(t,\cdot) - x_2(t,\cdot)|\,.$$

We estimate $\int_I m(t, s)\,|J_1(t, s)|\, ds$.

$$\int_I m(t, s)\,|J_1(t, s)|\, ds \leq 3 L \mathbf{S}_\infty \int_I m(s) \sup_{[0,T]} \|m(t,\cdot)\|_\infty \sup_I |x_1(t,\cdot) - x_2(t,\cdot)|\, ds$$

$$+ L \mathbf{S}_\infty \int_I m(s) \sup_I |x_1(t,\cdot) - x_2(t,\cdot)|\, ds$$

$$= L \mathbf{S}_\infty \left(3 \sup_{[0,T]} \|m(t,\cdot)\|_\infty + 1 \right) \sup_I |x_1(t,\cdot) - x_2(t,\cdot)|\,.$$

As a result, we obtain

$$\int_I |\Psi(s, x_1(t,\cdot), m(t,\cdot)) - \Psi(s, x_2(t,\cdot), m(t,\cdot))|\, ds \leq \int_I m(s)\,|J_1(t, s)|\, ds + \int_I m(s)\,|J_2(t, s)|\, ds$$

$$\leq L\left(3\overline{M}\mathbf{S}_\infty + \mathbf{S}_\infty + 16\mathbf{S}\overline{X}\right) \sup_I |x_1(t,\cdot) - x_2(t,\cdot)|\,.$$

$\square$

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Lemma 3.2. *Let hypotheses **H1**-**H3** hold. Suppose also*

- $\overline{x} \in C([0,T] \times I)$ *is such that for each $t \in [0,T]$ the map $s \mapsto \overline{x}(t,s)$ is one to one.*

- $\overline{m} \in C([0,T]; L^\infty(I))$ *is non-negative such that $\int_I \overline{m}(t,s)ds = 1$ for each $t \in [0,T]$.*

Then, there exists a unique solution $(x,m) \in C^1([0,T]; C(I)) \oplus C^1([0,T]; L^\infty(I))$ to the decoupled system

$$
\begin{cases}
\partial_t x(t,s) = \int_I \overline{m}(t,s_*)\mathbf{a}(x(t,s_*) - x(t,s))ds_*, & x(0,s) = x^0(s) \\
\partial_t m(t,s) = \Psi(s, \overline{x}(t,\cdot), m(t,\cdot)), & m(0,s) = m^0(s).
\end{cases}
\tag{3.4}
$$

The solution x is such that $s \mapsto x(t,s)$ is one to one and the solution m is non-negative such that $\int_I m(t,s)ds = 1$.

Step 1. **Existence and uniqueness for the equation for x.** Fix $0 < \underline{T} < \frac{1}{2L\|\overline{m}\|_{\infty,\infty}}$. Let M_{x_0} be the metric space of functions in $C([0,\underline{T}] \times I)$ with $x(0,s) = x^0(s)$. Define the operator $K_{x_0} : M_{x_0} \to C([0,\underline{T}] \times I)$ by

$$
K_{x_0}(x)(t,s) := x^0(s) + \int_0^t \int_I \overline{m}(\tau,s_*)\mathbf{a}(x(\tau,s_*) - x(\tau,s))ds_*d\tau.
$$

We view M_{x_0} as a complete metric space. We then have

$$
\begin{aligned}
|(K_{x_0}x)(t,s) - (K_{x_0}y)(t,s)| &= \left| \int_0^t \int_I \overline{m}(\tau,s_*)\left(\mathbf{a}(x(\tau,s_*) - x(\tau,s)) - \mathbf{a}(y(\tau,s_*) - y(\tau,s))\right)ds_*d\tau \right| \\
&\leq \left| \int_0^t \int_I \overline{m}(\tau,s_*)\left(\mathbf{a}(x(\tau,s_*) - x(\tau,s)) - \mathbf{a}(x(\tau,s_*) - y(\tau,s))\right)ds_*d\tau \right| \\
&\quad + \left| \int_0^t \int_I \overline{m}(\tau,s_*)\left(\mathbf{a}(x(\tau,s_*) - y(t,s)) - \mathbf{a}(y(\tau,s_*) - y(t,s))\right)ds_*d\tau \right| \\
&\leq 2L\|\overline{m}\|_{\infty,\infty}\underline{T} \sup_{I \times [0,\underline{T}]} |x-y|.
\end{aligned}
$$

The choice of $\underline{T}$ ensures $2L\|\overline{m}\|_{\infty,\infty}\underline{T} < 1$, thereby making the Banach contraction principle available which implies there exist a unique solution $x \in C([0,\underline{T}] \times I)$ to the equation

$$
x(t,s) = x^0(s) + \int_0^t \int_I \overline{m}(\tau,s_*)\mathbf{a}(x(\tau,s_*) - x(\tau,s))ds_*d\tau.
$$

By a standard iteration argument we have existence and uniqueness on the whole interval $[0,T]$. Evidently the map $\tau \mapsto \int_I \overline{m}(\tau,s_*)\mathbf{a}(x(\tau,s_*) - x(\tau,s))ds_*$ is continuous so that by the fundamental theorem of calculus we conclude $x \in C^1([0,\underline{T}]; C(I))$. Next we claim that this solution must be one to one.

Claim 3.1. *Let $x \in C^1([0,T]; C(I))$ be a solution of*

$$
\partial_t x(t,s) = \int_I \overline{m}(t,s_*)\mathbf{a}(x(t,s_*) - x(t,s))ds_*.
$$

Then for all $t \in [0,T]$ and all $s_1, s_2 \in I$ it hold that

$$
|x(t,s_2) - x(t,s_1)|^2 \geq e^{-Lt}\left|x^0(s_2) - x^0(s_1)\right|^2.
$$

In particular, $s \mapsto x(t,s)$ is increasing.

Proof. We start by showing that $|x(t,s_2) - x(t,s_1)|^2 > 0$ for all $t \in [0,T]$. Assume to the contrary there is some $t \in [0,T]$ and $s_2 > s_1$ such that $x(t,s_2) = x(t,s_1)$ and set

$$\tau_0 := \inf \{ t \in [0,T] \,|\, x(t,s_2) = x(t,s_1) \} > 0.$$

Then for all $t \in [0,\tau_0)$ we have $|x(t,s_2) - x(t,s_1)| > 0$ and as a result

$$\frac{d}{dt} |x(t,s_2) - x(t,s_1)|^2 = (x(t,s_2) - x(t,s_1)) \int_I m(t,s_*) \left(\mathbf{a}(x(t,s_*) - x(t,s_2)) - \mathbf{a}(x(t,s_*) - x(t,s_1)) \right) ds_*$$

$$\geq -L \, |x(t,s_2) - x(t,s_1)|^2 \int_I m(t,s_*) ds_* = -L \, |x(t,s_2) - x(t,s_1)|^2. \tag{3.5}$$

Division by $|x(t,s_2) - x(t,s_1)|^2 \neq 0$ implies

$$\frac{d}{dt} \log \left(|x(t,s_2) - x(t,s_1)|^2 \right) \geq -L,$$

which in turn gives the inequality

$$|x(t,s_2) - x(t,s_1)|^2 \geq e^{-Lt} \left| x^0(s_2) - x^0(s_1) \right|^2 > 0.$$

Taking $t \nearrow \tau_0$ gives a contradiction. Repeating now the estimate (3.5) shows that in fact for all $t \in [0,T]$

$$|x(t,s_2) - x(t,s_1)|^2 \geq e^{-Lt} \left| x^0(s_2) - x^0(s_1) \right|^2.$$

By continuity and the assumption that x^0 is increasing it follows that $s \mapsto x(t,s)$ is increasing.

Step 2. **Existence and uniqueness for the equation for** m.

2.1. Short time. Put $\mathbf{X} := \sup_{I \times [0,T]} |\bar{x}| + 1$ and pick $0 < \underline{T} \leq \frac{1}{16LS_\infty \mathbf{X} M^4}$. Let M_{m_0} be the metric subspace of $C([0,\underline{T}]; L^1(I))$ of functions with $m(0,s) = m^0(s)$, $0 \leq m \leq 2M$ and $\int_I m(t,s) ds = 1$ for all $t \in [0,\underline{T}]$. Define $K_{m_0} : M_{m_0} \to C([0,\underline{T}]; L^1(I))$ by

$$K_{m_0}(m)(t,s) := m^0(s) + \int_0^t \Psi(s, \bar{x}(\tau,\cdot), m(\tau,\cdot)) d\tau.$$

We start by observing that K maps M_{m_0} into itself.

Claim 3.2. $0 \leq K_{m_0}(m) \leq 2M$ and $\int_I K_{m_0}(m)(t,s) ds = 1$ for all $t \in [0,\underline{T}]$.

Proof. To see why $K_{m_0}(m)$ is non-negative notice that because how $\underline{T}$ was chosen we have

$$K_{m_0}(m) \geq \frac{1}{M} - 4LS_\infty \underline{T} \mathbf{X} M^3 > \frac{1}{2M}.$$

Moreover

$$K_{m_0}(m) \leq M + 4LS_\infty \underline{T} \mathbf{X} M^3 \leq 2M.$$

To show that $K_{m_0}(m)$ has unit integral we use that $\mathbf{s}$ is odd.

$$\int_I \int_0^t \Psi(s, \bar{x}(\tau,\cdot), m(\tau,\cdot)) d\tau ds$$

$$= \frac{1}{2} \int_0^t \int_{I^3} m(\tau,s) m(\tau,s_*) m(\tau,s_{**}) \mathbf{a}(x(\tau,s_{**}) - x(\tau,s)) \mathbf{s}(\bar{x}(\tau,s) - \bar{x}(\tau,s_*)) ds ds_* ds_{**} d\tau$$

$$+ \frac{1}{2} \int_0^t \int_{I^3} m(\tau,s) m(\tau,s_*) m(\tau,s_{**}) \mathbf{a}(x(\tau,s_{**}) - x(\tau,s_*)) \mathbf{s}(\bar{x}(\tau,s) - \bar{x}(\tau,s_*)) ds ds_* ds_{**} d\tau.$$

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Changing variables $s \longleftrightarrow s^*$ and using that $\mathbf{s}$ is odd, the second integral in the right hand side is recast as

$$\frac{1}{2} \int_0^t \int_{I^3} m(\tau, s) m(\tau, s_*) m(\tau, s_{**}) \mathbf{a}(x(\tau, s_{**}) - x(\tau, s_*)) \mathbf{s}(\overline{x}(\tau, s_*) - \overline{x}(\tau, s)) ds ds_* ds_{**} d\tau$$

$$= -\frac{1}{2} \int_0^t \int_{I^3} m(\tau, s) m(\tau, s_*) m(\tau, s_{**}) \mathbf{a}(x(\tau, s_{**}) - x(\tau, s)) \mathbf{s}(\overline{x}(\tau, s) - \overline{x}(\tau, s_*)) d\tau,$$

which shows that

$$\int_I \int_0^t \Psi(s, \overline{x}(\tau, \cdot), m(\tau, \cdot)) d\tau ds = 0,$$

as wanted.

$\square$

We view M_{m_0} as a complete metric space. Let $m, n \in M_{m_0}$. Thanks to point 1. in Lemma 3.1 we have

$$\int_I |K_{m_0}(m)(t, s) - K_{m_0}(n)(t, s)| \, ds \leq \int_0^t \int_I |\Psi(s, \overline{x}(\tau, \cdot), m(\tau, \cdot)) - \Psi(s, \overline{x}(\tau, \cdot), n(\tau, \cdot))| \, ds d\tau$$

$$\leq 12 L \mathbf{S}_\infty \underline{T} \sup_{[0,T] \times I} |\overline{x}| \, \|m(t, \cdot) - n(t, \cdot)\|_1 \, ,$$

and thus

$$\sup_{t \in [0,T]} \int_I |K_{m_0}(m)(t, s) - K_{m_0}(n)(t, s)| \, ds \leq 12 L \mathbf{S}_\infty \underline{T} \sup_{[0,T] \times I} |\overline{x}| \sup_{[0,T]} \|m(t, \cdot) - n(t, \cdot)\|_1 \, .$$

The choice of $\underline{T}$ makes the Banach contraction theorem available thereby ensuring the existence of a unique solution $m \in M_{m_0}$ on $[0, \underline{T}]$ to the equation

$$m(t, s) = m^0(s) + \int_0^t \Psi(s, \overline{x}(\tau, \cdot), m(\tau, \cdot)) d\tau.$$

Moreover, from the choice of $\underline{T} > 0$ we evidently have

$$m(t, s) \geq \frac{1}{2M} > 0, \ t \in [0, \underline{T}].$$

2.2. long time. Let $m(t, s)$ be the unique solution on $[0, \underline{T}]$ to

$$\partial_t m(t, s) = m(t, s) \iint_{I^2} m(t, s_*) m(t, s_{**}) \left(\mathbf{a}(x(s_{**}) - x(s)) + \mathbf{a}(x(s_{**}) - x(s_*)) \right) \mathbf{s}(x(s) - x(s_*)) ds_* ds_{**}$$

given by step 2.1. Then we obtain

$$-4 L \mathbf{X} \mathbf{S}_\infty m(t, s) \leq \partial_t m(t, s) \leq 4 L \mathbf{X} \mathbf{S}_\infty m(t, s) \tag{3.6}$$

and as a result we deduce that

$$\left| \frac{d}{dt} \log(m(t, s)) \right| \leq 4 L \mathbf{X} \mathbf{S}_\infty$$

i.e.

$$\frac{1}{\exp\left(4 L \mathbf{X} \mathbf{S}_\infty T\right) M} \leq m(t, s) \leq M \exp\left(4 L \mathbf{X} \mathbf{S}_\infty T\right).$$

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Put $\tau = \tau(L, \mathbf{X}, \mathbf{S}_\infty, M, T) = \exp\left(4L\mathbf{X}\mathbf{S}_\infty T\right) M$. Then we get a solution on $\left[0, 2 \times \frac{1}{16L\mathbf{X}\mathbf{S}_\infty \tau}\right]$. Iterating the process $k > 16LT\mathbf{X}\mathbf{S}_\infty \tau$ times we get existence and uniqueness of a solution on $[0, T]$. We claim now to have the upgrade $m \in C^1([0, T]; L^\infty(I))$. Indeed, we have

$$|m(t, s)| \leq |m^0(s)| + 2 \sup_{[0,T]\times I} |\bar{x}| \, L\mathbf{S}_\infty \int_0^t |m(\tau, s)| \, d\tau,$$

so that

$$\|m(t, \cdot)\|_\infty \leq \|m^0\|_\infty + 2L\mathbf{S}_\infty \sup_{[0,T]\times I} |\bar{x}| \int_0^t \|m(\tau, \cdot)\|_\infty \, d\tau$$

which entails

$$\|m(t, \cdot)\|_\infty \leq \|m^0\|_\infty e^{2L\mathbf{S}_\infty \sup_{[0,T]\times I} |\bar{x}| t}$$

which upon maximizing in t yields

$$\sup_{[0,T]} \|m(t, \cdot)\|_\infty \leq \|m^0\|_\infty e^{2L\mathbf{S}_\infty \sup_{[0,T]\times I} |\bar{x}| T}. \tag{3.7}$$

Taking into account (3.6), we finally conclude $m \in C^1([0, T]; L^\infty(I))$.

$\square$

3.2 The coupled equation

The well posedness for the decoupled equation serves as the main tool for proving well posedness of the original system. We prove

Theorem 3.1. *Let hypothesis **H1-H3** hold. There exist a unique solution $(x, m) \in C^1([0, T]; C(I)) \oplus C^1([0, T]; L^\infty(I))$ to the system*

$$\begin{cases} \partial_t x(t, s) = \int_I m(t, s_*)\mathbf{a}(x(t, s_*) - x(t, s))ds_*, & x(0, s) = x^0(s) \\ \partial_t m(t, s) = \Psi(s, x(t, \cdot), m(t, \cdot)), & m(0, s) = m^0(s). \end{cases} \tag{3.8}$$

Proof. **Step 1. Existence**. We define recursively the following sequence of functions (x_n, m_n):

i. For all $t \in [0, T]$ and all $s \in I$ we set $x_0(t, s) = x^0(s)$ and for all $t \in [0, T]$ and a.e. $s \in I$ we set $m_0(t, s) = m^0(s)$.

ii. If (x_{n-1}, m_{n-1}) have been defined we define (x_n, m_n) to be the unique solution guaranteed by Lemma 3.2 to the equation

$$\begin{cases} \partial_t x_n(t, s) = \int_I m_{n-1}(t, s_*)\mathbf{a}(x_n(t, s_*) - x_n(t, s))ds_*, & x_n(0, s) = x_0(s) \\ \partial_t m_n(t, s) = \Psi(s, x_{n-1}(t, \cdot), m_n(t, \cdot)), & m_n(0, s) = m_0(s). \end{cases}$$

Start by noting that x_n is uniformly bounded (with respect to n) in the space $C([0, T] \times I)$. We have

$$|x_n(t, s)| = x^0(s) + \int_0^t \int_I m_{n-1}(\tau, s_*)\mathbf{a}(x_n(\tau, s_*) - x_n(\tau, s))ds_* d\tau \leq \sup_I |x^0| + 2L \int_0^t \sup_I |x_n(\tau, \cdot)| \, d\tau,$$

so that

$$\sup_I |x_n(t, \cdot)| \leq e^{2LT} \sup_I |x^0|,$$

hence

$$\sup_{I \times [0,T]} |x_n(t, s)| \leq e^{2LT} \sup_I |x^0| \leq \overline{X}(X, L, T) := \overline{X}.$$

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This also implies a uniform bound in n for the weights since in view of Inequality (3.7)

$$\|m_n(t,\cdot)\|_\infty \leq \|m^0\|_\infty e^{2\overline{X}T} \leq \overline{M}(M,X,L,T) := \overline{M}.$$

The proof of existence essentially boils down to proving that (x_n, m_n) is a Cauchy sequence in the space $C([0,T]; C(I)) \oplus C([0,T]; L^1(I))$.

Estimate for $\sup\limits_I |x_{n+1}(t,\cdot) - x_n(t,\cdot)|$:

$$|x_{n+1}(t,s) - x_n(t,s)| = \left| \int_0^t \int_I m_n(\tau, s_*) \mathbf{a}(x_{n+1}(\tau, s_*) - x_{n+1}(\tau, s)) ds_* d\tau \right.$$
$$\left. - \int_0^t \int_I m_{n-1}(\tau, s_*) \mathbf{a}(x_n(\tau, s_*) - x_n(\tau, s)) ds_* d\tau \right|$$
$$\leq \left| \int_0^t \int_I m_n(\tau, s_*) \left(\mathbf{a}(x_{n+1}(\tau, s_*) - x_{n+1}(\tau, s)) - \mathbf{a}(x_n(\tau, s_*) - x_n(\tau, s)) \right) ds_* d\tau \right|$$
$$+ \left| \int_0^t \int_I \left(m_n(\tau, s_*) - m_{n-1}(\tau, s_*) \right) \mathbf{a}(x_n(\tau, s_*) - x_n(\tau, s)) ds_* d\tau \right|$$

$$\leq L \int_0^t \int_I m_n(\tau, s_*) |x_{n+1}(\tau, s_*) - x_n(\tau, s_*)| ds_* d\tau$$
$$+ L \int_0^t \int_I m_n(\tau, s_*) |x_{n+1}(\tau, s) - x_n(\tau, s)| ds_* d\tau$$
$$+ \int_0^t \int_I |m_n(t, s_*) - m_{n-1}(t, s_*)| |\mathbf{a}(x_n(t, s_*) - x_n(t, s))| ds_* dt$$
$$\leq L \int_0^t \int_I m_n(\tau, s_*) |x_{n+1}(\tau, s_*) - x_n(\tau, s_*)| ds_* d\tau$$
$$+ L \int_0^t \int_I m_n(\tau, s_*) |x_{n+1}(\tau, s) - x_n(\tau, s)| ds_* d\tau$$
$$+ \int_0^t \int_I |m_n(t, s_*) - m_{n-1}(t, s_*)| |\mathbf{a}(x_n(t, s_*) - x_n(t, s))| ds_* dt$$
$$\leq 2L \int_0^t \sup_I |x_{n+1}(\tau, \cdot) - x_n(\tau, \cdot)| d\tau + 2\overline{X}L \int_0^t \|m_n(\tau, \cdot) - m_{n-1}(\tau, \cdot)\|_1 d\tau$$
$$\leq C_1(\overline{X}, L) \int_0^t \sup_I |x_{n+1}(\tau, \cdot) - x_n(\tau, \cdot)| + \|m_n(\tau, \cdot) - m_{n-1}(\tau, \cdot)\|_1 d\tau. \qquad (3.9)$$

Estimate for $\|m_{n+1}(t,\cdot) - m_n(t,\cdot)\|_1$.

$$|m_{n+1}(t,s) - m_n(t,s)| \leq \left| \int_0^t \Psi(s, x_n(\tau, \cdot), m_{n+1}(\tau, \cdot)) d\tau - \int_0^t \Psi(s, x_{n-1}(\tau, \cdot), m_n(\tau, \cdot)) d\tau \right|$$
$$\leq \left| \int_0^t \Psi(s, x_n(\tau, \cdot), m_{n+1}(\tau, \cdot)) - \Psi(s, x_n(\tau, \cdot), m_n(\tau, \cdot)) d\tau \right|$$
$$+ \left| \int_0^t \Psi(s, x_n(\tau, \cdot), m_n(\tau, \cdot)) - \Psi(s, x_{n-1}(\tau, \cdot), m_n(\tau, \cdot)) d\tau \right|$$
$$\leq \int_0^t |\Psi(s, x_n(\tau, \cdot), m_{n+1}(\tau, \cdot)) - \Psi(s, x_n(\tau, \cdot), m_n(\tau, \cdot))| d\tau$$
$$+ \int_0^t |\Psi(s, x_n(\tau, \cdot), m_n(\tau, \cdot)) - \Psi(s, x_{n-1}(\tau, \cdot), m_n(\tau, \cdot))| d\tau.$$

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Integrating in $s \in I$ gives

$$\int_I |m_{n+1}(t,s) - m_n(t,s)|\, ds \leq \int_0^t \int_I |\Psi(s, x_n(\tau,\cdot), m_{n+1}(\tau,\cdot)) - \Psi(s, x_n(\tau,\cdot), m_n(\tau,\cdot))|\, ds d\tau$$
$$+ \int_0^t \int_I |\Psi(s, x_n(\tau,\cdot), m_n(\tau,\cdot)) - \Psi(s, x_{n-1}(\tau,\cdot), m_n(\tau,\cdot))|\, ds d\tau.$$

Utilizing Lemma 3.1 shows that the first inner integral is

$$\leq 12L\mathbf{S}_\infty \sup_{[0,T]\times I} |x_n|\, \|m_{n+1}(\tau,\cdot) - m_n(\tau,\cdot)\|_1 \leq 12L\mathbf{S}_\infty \overline{X}\, \|m_{n+1}(\tau,\cdot) - m_n(\tau,\cdot)\|_1,$$

whereas the second inner integral is

$$\leq L\left(3\overline{M}\mathbf{S}_\infty + \mathbf{S}_\infty + 16\mathbf{S}\overline{X}\right) \sup_I |x_1(t,\cdot) - x_2(t,\cdot)|.$$

As a result, we get

$$\|m_{n+1}(t,\cdot) - m_n(t,\cdot)\|_1 \leq 12L\mathbf{S}_\infty \overline{X} \int_0^t \|m_{n+1}(\tau,\cdot) - m_n(\tau,\cdot)\|_1\, d\tau$$
$$+ L\left(3\overline{M}\mathbf{S}_\infty + \mathbf{S}_\infty + 16\mathbf{S}\overline{X}\right) \int_0^t \sup_I |x_n(\tau,\cdot) - x_{n-1}(\tau,\cdot)|\, d\tau$$
$$\leq C_2(T,X,L,M,\mathbf{S},\mathbf{S}_\infty)\, \Big(\int_0^t \|m_{n+1}(\tau,\cdot) - m_n(\tau,\cdot)\|_1$$
$$+ \sup_I |x_n(\tau,\cdot) - x_{n-1}(\tau,\cdot)|\, d\tau\, \Big). \qquad (3.10)$$

Estimate for $u_n(t) := \sup_I |x_{n+1}(t,\cdot) - x_n(t,\cdot)| + \|m_{n+1}(\tau,\cdot) - m_n(\tau,\cdot)\|_1$. Collecting the Inequalities (3.9) and (3.10) we find

$$u_n(t) \leq \overline{C}\left(\int_0^t u_n(\tau)d\tau + \int_0^t u_{n-1}(\tau)d\tau\right),$$

where $\overline{C} = C_1 + C_2$. Setting $U_0 := u_n(0)$ we find that

$$u_n(t) \leq e^{\overline{C}T} \int_0^t u_{n-1}(\tau),$$

which by easy induction implies

$$u_n(t) \leq \frac{(e^{\overline{C}T}t)^n}{n!} U_0.$$

It follows that

$$\sup_{[0,T]\times I} |x_{n+1} - x_n| + \sup_{[0,T]} \|m_{n+1}(\tau,\cdot) - m_n(\tau,\cdot)\|_1 \underset{n\to\infty}{\to} 0,$$

hence (x_n, m_n) is a Cauchy sequence in the Banach space $C([0,T]\times I)\oplus C([0,T]; L^1(I))$ and denote by $(x,m) \in C([0,T]\times I)\oplus C([0,T]; L^\infty(I))$ the limit point of (x_n, m_n). We finish by verifying that (x,m) is a $C^1([0,T]; C(I))\oplus C^1([0,T]; L^\infty(I))$ solution to Equation (3.8). Indeed the following equation is satisfied for each n

$$\begin{cases} x_n(t,s) = x^0(s) + \int_0^t \int_I m_{n-1}(\tau,s_*)\mathbf{a}(x_n(\tau,s_*) - x_n(\tau,s))ds_* d\tau \\ m_n(t,s) = m^0(s) + \int_0^t \Psi(s, x_{n-1}(\tau,\cdot), m_n(\tau,\cdot))d\tau. \end{cases}$$

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We explain how the passage to the limit as $n \to \infty$ in the equation for m_n is done, and the passage for the equation of x_n is a standard verfication left to the reader. By Claim 3.1 we have for all $(\tau, s_*, s) \in [0, T] \times I^2$

$$|x_{n-1}(\tau, s_*) - x_{n-1}(\tau, s)|^2 \geq e^{-LT} |x^0(s_*) - x^0(s)|^2$$

so that

$$|x(\tau, s_*) - x(\tau, s)|^2 \geq e^{-LT} |x^0(s_*) - x^0(s)|^2.$$

Therefore, Lemma 3.1 is applicable and entails

$$\int_I |\Psi(s, x_{n-1}(\tau, \cdot), m_n(\tau, \cdot)) - \Psi(s, x(t, \cdot), m(t, \cdot))| \, ds$$

$$\leq \int_I |\Psi(s, x_{n-1}(\tau, \cdot), m_n(\tau, \cdot)) - \Psi(s, x(t, \cdot), m_n(t, \cdot))| \, ds$$

$$+ \int_I |\Psi(s, x(\tau, \cdot), m_n(\tau, \cdot)) - \Psi(s, x(t, \cdot), m(t, \cdot))| \, ds$$

$$\lesssim \|m_n(\tau, \cdot) - m(\tau, \cdot)\|_1 + \sup_I |x_{n-1}(\tau, \cdot) - x(\tau, \cdot)| \xrightarrow[n \to \infty]{} 0.$$

Hence, it follows that the right hand side in the equation for m_n convergence in $C([0, T]; L^1(I))$ to

$$\int_0^t \Psi(s, x(\tau, \cdot), m(\tau, \cdot)) d\tau,$$

which by uniqueness of the limit implies that for all $t \in [0, T]$ and a.e. $s \in I$ we have

$$m(t, s) = m^0(s) + \int_0^t \Psi(s, x(\tau, \cdot), m(\tau, \cdot)) d\tau.$$

The upgrade $(x, m) \in C^1([0, T] \times I) \oplus C^1([0, T]; L^\infty(I))$ is exactly by the same reasoning of Lemma 3.2.

Step 2. Uniqueness. Suppose we are given 2 solutions (x_1, m_1) and (x_2, m_2) with the same initial data. We have

$$|x_1(t, s) - x_2(t, s)| = \left| \int_0^t \int_I m_1(\tau, s_*) \mathbf{a}(x_1(\tau, s_*) - x_1(\tau, s)) ds_* d\tau \right.$$

$$\left. - \int_0^t \int_I m_2(\tau, s_*) \mathbf{a}(x_2(\tau, s_*) - x_2(\tau, s)) ds_* d\tau \right|$$

$$\leq \left| \int_0^t \int_I m_1(\tau, s_*) \mathbf{a}(x_1(\tau, s_*) - x_1(\tau, s)) ds_* d\tau \right.$$

$$\left. - \int_0^t \int_I m_2(\tau, s_*) \mathbf{a}(x_1(\tau, s_*) - x_1(\tau, s)) ds_* d\tau \right|$$

$$+ \left| \int_0^t \int_I m_2(\tau, s_*) \mathbf{a}(x_1(\tau, s_*) - x_1(\tau, s)) ds_* d\tau \right.$$

$$\left. - \int_0^t \int_I m_2(\tau, s_*) \mathbf{a}(x_2(\tau, s_*) - x_2(\tau, s)) ds_* d\tau \right|$$

$$\lesssim \int_0^t \|m_1(\tau, \cdot) - m_2(\tau, \cdot)\|_1 + \sup_I |x_1(\tau, \cdot) - x_2(\tau, \cdot)| \, d\tau.$$

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In addition, Lemma 3.1 yields

$$\|m_1(t,\cdot) - m_2(t,\cdot)\|_1 \leq \int_0^t \int_I |\Psi(s, x_1(\tau,\cdot), m_1(\tau,\cdot)) - \Psi(s, x_2(\tau,\cdot), m_2(\tau,\cdot))| \, ds d\tau$$

$$\leq \int_0^t \int_I |\Psi(s, x_1(t,\cdot), m_1(t,\cdot)) - \Psi(s, x_2(t,\cdot), m_1(t,\cdot))| \, ds d\tau$$

$$+ \int_0^t \int_I |\Psi(s, x_2(t,\cdot), m_1(t,\cdot)) - \Psi(s, x_2(t,\cdot), m_2(t,\cdot))| \, ds d\tau$$

$$\lesssim \int_0^t \|m_1(\tau,\cdot) - m_2(\tau,\cdot)\|_1 + \sup_I |x_1(\tau,\cdot) - x_2(\tau,\cdot)| \, d\tau.$$

It follows that

$$\|m_1(t,\cdot) - m_2(t,\cdot)\|_1 + \sup_I |x_1(t,\cdot) - x_2(t,\cdot)| \lesssim \int_0^t \|m_1(\tau,\cdot) - m_2(\tau,\cdot)\|_1 + \sup_I |x_1(\tau,\cdot) - x_2(\tau,\cdot)| \, d\tau,$$

from which we infer

$$\|m_1(t,\cdot) - m_2(t,\cdot)\|_1 + \sup_I |x_1(t,\cdot) - x_2(t,\cdot)| = 0,$$

and this a fortiori forces $m_1 = m_2$ and $x_1 = x_2$.

$\square$

4   The Graph Limit and consequences.

This section is devoted to obtaining a Gronwall estimate on the time dependent quantity $\xi_N(t) + \zeta_N(t)$, where

$$\xi_N(t) := \|\widetilde{x}_N(t,\cdot) - x(t,\cdot)\|_{L^2(I)}^2, \quad \zeta_N(t) := \|\widetilde{m}_N(t,\cdot) - m(t,\cdot)\|_{L^2(I)}^2,$$

where $\widetilde{x}_N, \widetilde{m}_N$ are given by Formula (5.2) and x, m are the corresponding solutions to Equation (3.8). We modify the argument demonstrated in Theorem 1 in [1] to our weakly singular settings. The estimate for $\zeta_N(t)$ reflects the main novelty of this section. The estimates we obtain are locally uniform in time. The symbol $\lesssim$ stands for inequality up to a constant which may depend only on $L, M, X, T, \mathbf{S}, \mathbf{S}_\infty$. The main theorem is

Theorem 4.1. *Let the hypotheses **H1-H3** hold. Let $(x, m) \in C^1([0,T]; C(I)) \oplus C^1([0,T]; L^\infty(I))$ be the solution to Equation (3.8). Let $(\mathbf{x}_N, \mathbf{m}_N) \in C^1([0,T]; \mathbb{R}^{2N})$ be the solution to the system (2.1). Then*

$$\|\widetilde{x}_N(t,\cdot) - x(t,\cdot)\|_{C([0,T]; L^2(I))} + \|\widetilde{m}_N(t,\cdot) - m(t,\cdot)\|_{C([0,T]; L^2(I))} \underset{N \to \infty}{\to} 0.$$

Proof. **Step 1. The time derivative of $\zeta_N(t)$.** The estimate for the time derivative of $\zeta_N(t)$ reflects the main difference with the argument in [1]. The time derivative of ζ_N is computed as follows.

$$\dot{\zeta}_N(t) = \int_I (\widetilde{m}_N(t,s) - m(t,s)) \left(N \int_{\frac{1}{N}\lfloor sN \rfloor}^{\frac{1}{N}(\lfloor sN \rfloor + 1)} \Psi(s_*, \widetilde{x}_N(t,\cdot), \widetilde{m}_N(t,\cdot)) ds_* - \Psi(s, x(t,\cdot), m(t,\cdot)) \right) ds$$

$$= \int_I (\widetilde{m}_N(t,s) - m(t,s)) \underbrace{\left(N \int_{\frac{1}{N}\lfloor sN \rfloor}^{\frac{1}{N}(\lfloor sN \rfloor + 1)} \Psi(s_*, \widetilde{x}_N(t,\cdot), \widetilde{m}_N(t,\cdot)) - \Psi(s_*, x(t,\cdot), m(t,\cdot)) ds_* \right)}_{:= h_N(t,s)} ds$$

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$$+ \int_I (\widetilde{m}_N(t,s) - m(t,s)) \underbrace{\left(N \int_{\frac{1}{N}\lfloor sN \rfloor}^{\frac{1}{N}(\lfloor sN \rfloor + 1)} \Psi(s_*, x(t,\cdot), m(t,\cdot)) ds_* - \Psi(s, x(t,\cdot), m(t,\cdot)) \right)}_{:= g_N(t,s)} ds. \quad (4.1)$$

By Lebesgue differentiation theorem, for each $t \in [0,T]$ it holds that $g_N(t,s) \underset{N \to \infty}{\to} 0$ pointwise a.e. $s \in I$. In addition, $\|x\|_{C([0,T] \times I)}, \|m\|_{C([0,T];L^\infty(I))}$ are bounded which implies that $g_N(t,s)$ is uniformly bounded (with respect to N) so that by the dominated convergence theorem we find that the second integral in (4.1) is

$$\leq \frac{1}{2} \zeta_N(t) + \frac{1}{2} \|g_N(t,\cdot)\|_2^2 \quad (4.2)$$

where for each $t \in [0,T]$ it holds that

$$\|g_N(t,\cdot)\|_2^2 \underset{N \to \infty}{\to} 0. \quad (4.3)$$

For the first integral, note

$$\|h_N(t,\cdot)\|_2^2 = N^2 \int_I \left| \int_{\frac{1}{N}\lfloor sN \rfloor}^{\frac{1}{N}(\lfloor sN \rfloor + 1)} \Psi(s_*, \widetilde{x}_N(t,\cdot), \widetilde{m}_N(t,\cdot)) - \Psi(s_*, x(t,\cdot), m(t,\cdot)) ds_* \right|^2 ds$$

$$= N^2 \sum_{i=1}^N \int_{\frac{i-1}{N}}^{\frac{i}{N}} \left| \int_{\frac{1}{N}\lfloor sN \rfloor}^{\frac{1}{N}(\lfloor sN \rfloor + 1)} \Psi(s_*, \widetilde{x}_N(t,\cdot), \widetilde{m}_N(t,\cdot)) - \Psi(s_*, x(t,\cdot), m(t,\cdot)) ds_* \right|^2 ds$$

$$\leq N \sum_{i=1}^N \int_{\frac{i-1}{N}}^{\frac{i}{N}} \int_{\frac{1}{N}\lfloor sN \rfloor}^{\frac{1}{N}(\lfloor sN \rfloor + 1)} |\Psi(s_*, \widetilde{x}_N(t,\cdot), \widetilde{m}_N(t,\cdot)) - \Psi(s_*, x(t,\cdot), m(t,\cdot))|^2 ds_* ds$$

$$= N \sum_{i=1}^N \int_{\frac{i-1}{N}}^{\frac{i}{N}} \int_{\frac{i-1}{N}}^{\frac{i}{N}} |\Psi(s_*, \widetilde{x}_N(t,\cdot), \widetilde{m}_N(t,\cdot)) - \Psi(s_*, x(t,\cdot), m(t,\cdot))|^2 ds_* ds$$

$$= \sum_{i=1}^N \int_{\frac{i-1}{N}}^{\frac{i}{N}} |\Psi(s_*, \widetilde{x}_N(t,\cdot), \widetilde{m}_N(t,\cdot)) - \Psi(s_*, x(t,\cdot), m(t,\cdot))|^2 ds_*$$

$$= \int_I |\Psi(s_*, \widetilde{x}_N(t,\cdot), \widetilde{m}_N(t,\cdot)) - \Psi(s_*, x(t,\cdot), m(t,\cdot))|^2 ds_*$$

$$\leq 2 \int_I |\Psi(s_*, \widetilde{x}_N(t,\cdot), \widetilde{m}_N(t,\cdot)) - \Psi(s_*, \widetilde{x}_N(t,\cdot), m(t,\cdot))|^2 ds_*$$

$$+ 2 \int_I |\Psi(s_*, \widetilde{x}_N(t,\cdot), m(t,\cdot)) - \Psi(s_*, x(t,\cdot), m(t,\cdot))|^2 ds_*$$

$$= 2 \int_I |\Psi(s, \widetilde{x}_N(t,\cdot), \widetilde{m}_N(t,\cdot)) - \Psi(s, \widetilde{x}_N(t,\cdot), m(t,\cdot))|^2 ds$$

$$+ 2 \int_I |\Psi(s, \widetilde{x}_N(t,\cdot), m(t,\cdot)) - \Psi(s, x(t,\cdot), m(t,\cdot))|^2 ds.$$

Note that at this stage we cannot quite appeal to the Estimate (3.1) since it was formulated for x which are one to one in the variable s. The main difference is in the estimate of the second integral, which is now bounded by $\|\widetilde{x}_N(t,\cdot) - x(t,\cdot)\|_2^2$ up to an error term which decays to 0 as $N \to \infty$. Precisely put

Lemma 4.1. *It holds that*

$$i. \int_I |\Psi(s, \widetilde{x}_N(t,\cdot), \widetilde{m}_N(t,\cdot)) - \Psi(s, \widetilde{x}_N(t,\cdot), m(t,\cdot))|^2 ds \lesssim \|\widetilde{m}_N(t,\cdot) - m(t,\cdot)\|_2^2.$$

$$ii. \int_I |\Psi(s, \widetilde{x}_N(t,\cdot), m(t,\cdot)) - \Psi(s, x(t,\cdot), m(t,\cdot))|^2 ds \lesssim \|\widetilde{x}_N(t,\cdot) - x(t,\cdot)\|_2^2 + o_N(1).$$

Proof. Unless unavoidable, we supress the time variable.

i. Thanks to Lemma 2.1 we have

$$\int_I \widetilde{m}_N(t,s)ds = 1, \ \widetilde{m}_N(t,s) \geq 0.$$

The estimate i. is almost identical to the estimate demonstrated in (3.1), the only minor difference being that here we take the L^2-norm in s. Setting $\mathbf{a}_N(s,s_*,s_{**}) := \mathbf{a}(\widetilde{x}_N(s_{**}) - \widetilde{x}_N(s)) + \mathbf{a}(\widetilde{x}_N(s_{**}) - \widetilde{x}_N(s_*))$, we have

$$\left| \widetilde{m}_N(s) \iint_{I^2} \widetilde{m}_N(s_*)\widetilde{m}_N(s_{**})\mathbf{a}_N(s,s_*,s_{**})\mathbf{s}(\widetilde{x}_N(s) - \widetilde{x}_N(s_*))ds_*ds_{**} \right.$$

$$\left. - m(s) \iint_{I^2} m(s_*)m(s_{**})\mathbf{a}_N(s,s_*,s_{**})\mathbf{s}(\widetilde{x}_N(s) - \widetilde{x}_N(s_*))ds_*ds_{**} \right|$$

$$\leq \widetilde{m}_N(s) \left| \iint_{I^2} (\widetilde{m}_N(s_*)\widetilde{m}_N(s_{**}) - m(s_*)m(s_{**})) \mathbf{a}(\widetilde{x}_N(s_{**}) - \widetilde{x}_N(s))\mathbf{s}(\widetilde{x}_N(s) - \widetilde{x}_N(s_*))ds_*ds_{**} \right.$$

$$\left. + \iint_{I^2} (\widetilde{m}_N(s_*)\widetilde{m}_N(s_{**}) - m(s_*)m(s_{**})) \mathbf{a}(\widetilde{x}_N(s_{**}) - \widetilde{x}_N(s_*))\mathbf{s}(\widetilde{x}_N(s) - \widetilde{x}_N(s_*))ds_*ds_{**} \right|$$

$$+ |\widetilde{m}_N(s) - m(s)| \left| \iint_{I^2} m(s_*)m(s_{**})\mathbf{a}(\widetilde{x}_N(s_{**}) - \widetilde{x}_N(s))\mathbf{s}(\widetilde{x}_N(s) - \widetilde{x}_N(s_*))ds_*ds_{**} \right.$$

$$\left. + \iint_{I^2} m(s_*)m(s_{**})\mathbf{a}(\widetilde{x}_N(s_{**}) - \widetilde{x}_N(s_*))\mathbf{s}(\widetilde{x}_N(s) - \widetilde{x}_N(s_*))ds_*ds_{**} \right|$$

$$\lesssim \iint_{I^2} |\widetilde{m}_N(s_*)\widetilde{m}_N(s_{**}) - m(s_*)m(s_{**})| \, ds_*ds_{**}$$

$$+ |\widetilde{m}_N(s) - m(s)| \iint_{I^2} m(s_*)m(s_{**})ds_*ds_{**}. \tag{4.4}$$

The first integral is

$$\lesssim \iint_{I^2} |\widetilde{m}_N(s_{**}) (\widetilde{m}_N(s_*) - m(s_*))| \, ds_*ds_{**} + \iint_{I^2} |m(s_*) (\widetilde{m}_N(s_{**}) - m(s_{**}))| \, ds_*ds_{**}$$

$$\lesssim \int_I |\widetilde{m}_N(s_{**}) - m(s_{**})| \, ds_{**} + \int_I |\widetilde{m}_N(s_*) - m(s_*)| \, ds_*.$$

Therefore squaring and integrating in s over I, Inequality (4.4) produces

$$\int_I |\Psi(s,\widetilde{x}_N(t,\cdot),\widetilde{m}_N(t,\cdot)) - \Psi(s,\widetilde{x}_N(t,\cdot),m(t,\cdot))|^2 \, ds \lesssim \|\widetilde{m}_N(t,\cdot) - m(t,\cdot)\|_2^2.$$

ii. We have

$$\left| \iint_{I^2} m(s_*)m(s_{**})\mathbf{a}_N(s,s_*,s_{**})\mathbf{s}(\widetilde{x}_N(s) - \widetilde{x}_N(s_*))ds_*ds_{**} \right.$$

$$\left. - \iint_{I^2} m(s_*)m(s_{**})\mathbf{a}(s,s_*,s_{**})\mathbf{s}(x(s) - x(s_*))ds_*ds_{**} \right|$$

$$\leq L \iint_{I^2} m(s_*)m(s_{**}) \left(2|\widetilde{x}_N(s_{**}) - x(s_{**})| + |\widetilde{x}_N(s_*) - x(s_*)| + |\widetilde{x}_N(s) - x(s)| \right) ds_*ds_{**}$$

$$+ \iint_{I^2} m(s_*)m(s_{**}) |\mathbf{a}(x(s_{**}) - x(s)) + \mathbf{a}(x(s_{**}) - x(s_*))|$$

$$\times |\mathbf{s}(\widetilde{x}_N(s) - \widetilde{x}_N(s_*)) - \mathbf{s}(x(s) - x(s_*))| \, ds_*ds_{**}$$

$$:= J_1(t,s) + J_2(t,s).$$

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We estimate separately $\int_I m^2(t,s)\,|J_1(t,s)|^2\,ds$ and $\int_I m^2(t,s)\,|J_2(t,s)|^2\,ds$. It is straightforward to check

$$\int_I m^2(t,s)\,|J_1(t,s)|^2\,ds \lesssim \|\widetilde{x}_N(t,\cdot) - x(t,\cdot)\|_2^2. \tag{4.5}$$

Let us concentrate on the estimate of $\int_I m^2(t,s)\,|J_2(t,s)|^2\,ds$. For each $s \in I$ set

$$A_N(t,s) := \{s_* \in I \,|\, \widetilde{x}_N(t,s_*) - \widetilde{x}_N(t,s) > 0\}, \quad B_N(t,s) := A_N^c(t,s)$$

and

$$A(t,s) := \{s_* \in I \,|\, x(t,s_*) - x(t,s) > 0\}, \quad B(t,s) := A^c(t,s).$$

We abbreviate

$$\mathbf{s}_N(s,s_*) := \mathbf{s}(\widetilde{x}_N(t,s) - \widetilde{x}_N(t,s_*)) - \mathbf{s}(x(t,s) - x(t,s_*)).$$

We estimate the integral as follows

$$\begin{aligned}
\int_I m^2(t,s)\,|J_2(t,s)|^2\,ds &\lesssim \iint_{I^2} |\mathbf{s}_N(s,s_*)|^2\,ds_*ds\\
&= \iint_{I^2} \mathbf{1}_{A_N(t,s)}(s_*)\mathbf{1}_{A(t,s)}(s_*)\,|\mathbf{s}_N(s,s_*)|^2\,ds_*ds\\
&\quad + \iint_{I^2} \mathbf{1}_{B_N(t,s)}(s_*)\mathbf{1}_{B(t,s)}(s_*)\,|\mathbf{s}_N(s,s_*)|^2\,ds_*ds\\
&\quad + \iint_{I^2} \mathbf{1}_{A_N(t,s)}(s_*)\left(\mathbf{1}_{B(t,s)} - \mathbf{1}_{B_N(t,s)}\right)(s_*)\,|\mathbf{s}_N(s,s_*)|^2\,ds_*ds\\
&\quad + \iint_{I^2} \left(\mathbf{1}_{A(t,s)} - \mathbf{1}_{A_N(t,s)}\right)(s_*)\mathbf{1}_{B_N(t,s)}(s_*)\,|\mathbf{s}_N(s,s_*)|^2\,ds_*ds\\
&=: \mathbf{I} + \mathbf{II} + \mathbf{III} + \mathbf{IV}.
\end{aligned}$$

By the assumption **H3** we have

$$\mathbf{I} + \mathbf{II} \lesssim \|\widetilde{x}_N(t,\cdot) - x(t,\cdot)\|_2^2 = \xi_N(t). \tag{4.6}$$

Recall that by Lemma 2.1 and Claim 3.1 there exist a constant $C > 1$ such that

$$\frac{1}{C}\,|\widetilde{x}_N(0,s_*) - \widetilde{x}_N(0,s)| \le |\widetilde{x}_N(t,s_*) - \widetilde{x}_N(t,s)| \le C\,|\widetilde{x}_N(0,s_*) - \widetilde{x}_N(0,s)|, \tag{4.7}$$

$$\frac{1}{C}\,|x^0(s_*) - x^0(s)| \le |x(t,s_*) - x(t,s)| \le C\,|x^0(s_*) - x^0(s)|.$$

Hence $\mathbf{1}_{A_N(t,s)} = \mathbf{1}_{A_N(0,s)}, \mathbf{1}_{A(t,s)} = \mathbf{1}_{A(0,s)}$, so that

$$\mathbf{IV} \lesssim \iint_{I^2} \left|\mathbf{1}_{A_N(t,s)}(s_*) - \mathbf{1}_{A(t,s)}(s_*)\right|\,ds_*ds = \iint_{I^2} \left|\mathbf{1}_{A_N(0,s)}(s_*) - \mathbf{1}_{A(0,s)}(s_*)\right|\,ds_*ds.$$

By Lebesgue differentiation theorem, for a.e. $s \in I$ it holds that

$$\widetilde{x_N}(0,s_*) - \widetilde{x_N}(0,s) \underset{N\to\infty}{\to} x^0(s_*) - x^0(s)$$

a.e. s_*. For all $s \in I$ the set $\left\{s_* \,\middle|\, x^0(s_*) = x^0(s)\right\}$ (being an atom) is null due to **H2**, and therefore for a.e. s it holds that

$$\mathbf{1}_{A_N(0,s)}(s_*) \underset{N\to\infty}{\to} \mathbf{1}_{A(0,s)}(s_*)$$

a.e. s_*. By dominated convergence we obtain

$$\iint_{I^2} \left| \mathbf{1}_{A_N(0,s)} - \mathbf{1}_{A(0,s)} \right| ds_* ds \underset{N \to \infty}{\to} 0,$$

which shows

$$\mathbf{IV} \lesssim \iint_{I^2} \left| \mathbf{1}_{A_N(t,s)} - \mathbf{1}_{A(t,s)} \right| ds_* ds = o_N(1). \tag{4.8}$$

The same reasoning also shows that

$$\mathbf{III} \lesssim \iint_{I^2} \left| \mathbf{1}_{B_N(t,s)} - \mathbf{1}_{B(t,s)} \right| ds_* ds = o_N(1). \tag{4.9}$$

The combination of (4.5), (4.6), (4.8) and (4.9) implies the announced claim.

□

Gathering i.,ii. and (4.2) gives

$$\dot{\zeta}_N(t) \lesssim \xi_N(t) + \zeta_N(t) + \|g_N(t, \cdot)\|_2^2 + o_N(1). \tag{4.10}$$

Step 2. The time derivative of $\xi_N(t)$. The time derivative of $\dot{\xi}_N(t)$ is mastered exactly as in [1]. Following the argument in [1] one finds that

$$\dot{\xi}_N(t) \lesssim \xi_N(t) + \zeta_N(t). \tag{4.11}$$

Step 3. Conclusion. The combination of Inequalities (4.10) and (4.11) yields

$$\dot{\xi}_N(t) + \dot{\zeta}_N(t) \lesssim \xi_N(t) + \zeta_N(t) + \frac{1}{2} \|g_N(t, \cdot)\|_2^2 + o_N(1),$$

i.e.

$$\xi_N(t) + \zeta_N(t) \lesssim \xi_N(0) + \zeta_N(0) + \int_0^t (\xi_N(\tau) + \zeta_N(\tau)) d\tau + \int_0^T \|g_N(\tau, \cdot)\|_2^2 d\tau + o_N(1)T.$$

Applying Gronwall's lemma entails

$$\xi_N(t) + \zeta_N(t) \le C \left(\xi_N(0) + \zeta_N(0) + \int_0^T \|g_N(\tau, \cdot)\|_2^2 d\tau + o_N(1)T \right) \exp\left(Ct\right).$$

Since $\|g_N(\tau, \cdot)\|_2^2$ is uniformly bounded, by (4.3) and dominated convergence $\int_0^T \|g_N(\tau, \cdot)\|_2^2 d\tau \underset{N \to \infty}{\to} 0$, which concludes the proof.

□

In the last part of this section, we recall how to obtain as a consequence a special version of the mean field limit for the empirical measure associated with the System (2.1). We start by pointing out that currently the existing literature does not cover the well posedness theory of the mean field equation, namely the non-local non-homogeneous transport equation

$$\partial_t \mu(t, x) + \partial_x \left(\mu(t, x) \mathbf{a} \star \mu(t, x) \right) = h\left[\mu\right](t, x), \ \mu(0, \cdot) = \mu^0,$$

where

$$h\left[\mu\right](t, x) := d\mu(t, x) \int_{\mathbb{R}^2} S(x, y, z) d\mu(t, y) d\mu(t, z), \ S(x, y, z) = \frac{1}{2} \left(\mathbf{a}(z - y) + \mathbf{a}(z - x) \right) \mathbf{s}(x - y).$$

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Indeed, in both works [5, 15] the well posedness theory for measure valued solutions made an extensive use of the fact that the source term satisfies some kind of Lipschitz continuity with respect to a generalized Wasserstein distance ρ defined in [5, 15], namely a bound of the type $\rho(h\,[\mu]\,,h\,[\nu]) \leq C\rho(\mu,\nu)$ with $C > 0$. In fact, the graph limit method already allows to get a clean characterization of the mean field limit for the special choice of initial data (2.6). With Theorem 4.1 the proof of this mean field limit can be deduced exactly by the same argument as in [1]. The following theorem is the first step in understanding the mean field limit of the pairwise competition model, which was a question raised in [10]. We denote by W_1 the Wasserstein distance of exponent 1 (see e.g. [16, Definition 6.1] for general background on Wasserstein distances).

Theorem 4.2. *Let the hypothesis **H1-H3** hold. Let $\mu_N(t) := \frac{1}{N} \sum_{j=1}^{N} m_j(t)\delta(x - x_j(t))$ and $\mu(t) := \int_I m(t,s)\delta(x - x(t,s))ds$[1]. Then $W_1(\mu_N(t), \mu(t)) \underset{N\to\infty}{\to} 0$ for all $t \in [0,T]$.*

Proof. Fix some $\varphi \in C^{0,1}(\mathbb{R}^d)$. Let

$$\overline{\mu_N}(t) := \int_I \widetilde{m_N}(t,s)\delta(x - \widetilde{x_N}(t,s))ds.$$

We split the integral $\int_{\mathbb{R}} \varphi(x)\,(\mu_N(t,dx) - \mu(t,dx))$ as follows.

$$\int_{\mathbb{R}} \varphi(x)\,(\mu_N(t,dx) - \mu(t,dx)) = \int_{\mathbb{R}} \varphi(x)\,(\mu_N(t,dx) - \overline{\mu_N}(t,dx)) + \int_{\mathbb{R}} \varphi(x)\,(\overline{\mu_N}(t,dx) - \mu(t,dx))\,. \tag{4.12}$$

The first integral in the right hand side of Equation (4.12) vanishes identically. Indeed

$$\int_{\mathbb{R}} \varphi(x)\,(\mu_N(t,dx) - \overline{\mu_N}(t,dx)) = \frac{1}{N} \sum_{j=1}^{N} m_j(t)\varphi(x_j(t))$$

$$- \sum_{i=1}^{N} \int_I \varphi\left(\sum_{j=1}^{N} \mathbf{1}_{\left[\frac{i-1}{N},\frac{j}{N}\right]}(s)x_j(t)\right) m_i(t)\mathbf{1}_{\left[\frac{i-1}{N},\frac{i}{N}\right]}(s)ds,$$

and the second term is

$$= \sum_{i=1}^{N} \int_I \varphi\left(\sum_{j=1}^{N} \mathbf{1}_{\left[\frac{j-1}{N},\frac{j}{N}\right]}(s)x_j(t)\right) m_i(t)\mathbf{1}_{\left[\frac{i-1}{N},\frac{i}{N}\right]}(s)ds$$

$$= \sum_{i=1}^{N} \int_{\left[\frac{i-1}{N},\frac{i}{N}\right]} \varphi\left(\sum_{j=1}^{N} \mathbf{1}_{\left[\frac{j-1}{N},\frac{j}{N}\right]}(s)x_j(t)\right) m_i(t)ds$$

$$= \sum_{i=1}^{N} \int_{\left[\frac{i-1}{N},\frac{i}{N}\right]} \varphi\left(x_i(t)\right) m_i(t)ds$$

$$= \frac{1}{N} \sum_{i=1}^{N} m_i(t)\varphi\left(x_i(t)\right).$$

---

[1]We view $\int_I m(t,s)\delta(x - x(t,s))ds$ as a measure valued integral. By definition $\left(\int_I m(t,s)\delta(x - x(t,s))ds\right)\varphi :=$ $\int_I m(t,s)\varphi(x(t,s))ds$ for all $\varphi \in C_b(\mathbb{R}^d)$.

As for the second integral, it is

$$
\begin{aligned}
&= \int_{\mathbb{R}} \varphi(x) \left(\int_I \widetilde{m_N}(t,s)\delta(x - \widetilde{x_N}(t,s))ds - \int_I m(t,s)\delta(x - x(t,s))ds \right) \\
&= \int_I \widetilde{m_N}(t,s)\varphi(\widetilde{x_N}(t,s))ds - \int_I m(t,s)\varphi(x(t,s))ds \\
&\leq \left| \int_I \left(\widetilde{m_N}(t,s) - m(t,s) \right) \varphi(\widetilde{x_N}(t,s))ds \right| \\
&\quad + \left| \int_I m(t,s) \left(\varphi(\widetilde{x_N}(t,s)) - \varphi(x(t,s)) \right) ds \right| \\
&\leq \|\varphi\|_\infty \|\widetilde{m}_N(t,\cdot) - m(t,\cdot)\|_{L^2(I)} \\
&\quad + \overline{M} \|\varphi\|_{\mathrm{Lip}} \|\widetilde{x}_N(t,\cdot) - x(t,\cdot)\|_{L^2(I)}.
\end{aligned}
$$

The last estimate together with Theorem 4.1 entails the weak convergnce $\mu_N(t) \rightharpoonup \mu(t)$. Since $\mu_N(t)$ and $\mu(t)$ are compactly supported, it follows that $W_1(\mu_N(t), \mu(t)) \underset{N\to\infty}{\to} 0$ (see e.g. [16, Theorem 6.9]).

$\square$

5 The case $d > 1$

In this section we explain how to extend the graph limit for arbitrary higher dimensions $d > 1$. In some places the proof requires only minor modifications and we therefore concentrate only on the parts which require special treatment.

5.1 The graph limit equation $d > 1$

The first notable difference in comparison to the case $d = 1$ (or the work [1]) is reflected in the definition of the Riemann sums. Instead of labeling the opinions along a multi-index of length d we label them along a multi-d-dimensional matrix of indices. This is a particular case of the metric valued labelling procedure introduced in [12] when the labelling space is $[0,1]^d$. At the level of the graph limit equation this choice corresponds to considering the equation posed on $[0,T] \times I^d$ rather than $[0,T] \times I$. Indeed, the fact that $x(t,\cdot)$ is a map from I^d to itself enables to consider bi-Lipschitz initial data, which is crucial for the sake of properly analyzing the singularity in $\mathbf{s}$ as is clarified in Lemma 5.1. This labelling procedure does not have any modelling interpretation since particles (opinions) are still exchangeable or indistinguishable, it is solely needed for pure technical reasons. As we mentioned at the beginning of the paper, it would still be possible to go back to using $[0,1]$ as a labeling space through a change of variable, since $[0,1]$ and $[0,1]^d$ are isomorphic as measurable spaces per the Borel isomorphism theorem. But, obviously, this would lead to painful technical assumptions to replace the bi-Lipschitz condition on $x(t,.)$, while the analysis is otherwise much more transparent when considering $[0,1]^d$. Precisely put, we take the number of opinions to be perfect powers of d in which case the opinion dynamics system becomes the following system of $(d+1)N^d$ ODEs

$$
\begin{cases}
\dot{x}_i^N(t) = \frac{1}{N^d} \sum_{j=1}^{N^d} m_j^N(t)\mathbf{a}(x_j^N(t) - x_i^N(t)), & x_i^N(0) = x_i^{0,N} \\
\dot{m}_i^N(t) = \psi_i^N(\mathbf{x}_N(t), \mathbf{m}_N(t)), & m_i^N(0) = m_i^{0,N}.
\end{cases}
\tag{5.1}
$$

Let $\mathcal{M} = \{1,\dots,N\}^d$ be the set of sequences of length d with elements from $\{1,\dots,N\}$ and fix a bijection $\sigma : \mathcal{M} \to \{1,\dots,N^d\}$. For each $\mathbf{i} = (i_1,\dots,i_d) \in \mathcal{M}$ consider the cube

$$
\mathcal{Q}_\mathbf{i} := \prod_{k=1}^d \left[\frac{i_k - 1}{N}, \frac{i_k}{N} \right].
$$

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We attach to the flow of System (5.1) the following "Riemman sums" like quantities, as in the one dimensional case, defined by

$$\widetilde{x}_N(t,s) := \sum_{\mathbf{i}\in\mathcal{M}} x_{\sigma(\mathbf{i})}(t)\mathbf{1}_{\mathcal{Q}_{(\mathbf{i})}}(s) = \sum_{i_d=1}^{N}\cdots\sum_{i_1=1}^{N} x_{\sigma(i_1,\ldots,i_d)}(t)\mathbf{1}_{\mathcal{Q}_{(i_1,\ldots,i_d)}}(s), \qquad (5.2)$$

$$\widetilde{m}_N(t,s) := \sum_{\mathbf{i}\in\mathcal{M}} m_{\sigma(\mathbf{i})}(t)\mathbf{1}_{\mathcal{Q}_{(\mathbf{i})}}(s) = \sum_{i_d=1}^{N}\cdots\sum_{i_1=1}^{N} m_{\sigma(i_1,\ldots,i_d)}(t)\mathbf{1}_{\mathcal{Q}_{(i_1,\ldots,i_d)}}(s). \qquad (5.3)$$

Here the labeling variable s varies on the d-dimensional unit cube I^d. Generalizing the constructions of Section 2.2, the functional $\Psi : I^d \times L^\infty(I^d) \times L^\infty(I^d) \to \mathbb{R}$ and the functions $x^0 : I^d \to \mathbb{R}^d, m^0 : I^d \to \mathbb{R}$ are given and the functions ψ_i^N and the initial data $x_i^{0,N}, m_i^{0,N}$ ($1 \le i \le N^d$) are defined in terms of these functions through the following formulas

$$\psi_i^N(\mathbf{x}_N(t),\mathbf{m}_N(t)) := N^d \int_{\mathcal{Q}_{\sigma^{-1}(i)}} \Psi(s,\widetilde{x}_N(t,\cdot),\widetilde{m}_N(t,\cdot))ds \qquad (5.4)$$

and

$$x_i^{0,N} := N^d \int_{\mathcal{Q}_{\sigma^{-1}(i)}} x^0(s)ds, \quad m_i^{0,N} = N^d \int_{\mathcal{Q}_{\sigma^{-1}(i)}} m^0(s)ds. \qquad (5.5)$$

If Ψ is given by

$$\Psi(s,x(\cdot),m(\cdot)) := m(s)\iint_{I^{2d}} m(s_*)m(s_{**})\left(\mathbf{a}(x(s_{**}) - x(s)) + \mathbf{a}(x(s_{**}) - x(s_*))\right)\mathbf{s}(x(s)-x(s_*))ds_*ds_{**},$$

then one readily checks that the ψ_i^N in Formula (2.5) are recovered via Formula (5.4). Notice that $\widetilde{x}_N(0,s)$ and $\widetilde{m}_N(0,s)$ well approximate $x^0(s), m^0(s)$ because by Lebesgue's differentiation theorem for a.e. $s \in I^d$ we have pointwise convergence

$$\widetilde{x}_N(0,s) = N^d \int_{\mathcal{Q}_{\lfloor s_1 N\rfloor,\ldots,\lfloor s_d N\rfloor}} x^0(\sigma)d\sigma \underset{N\to\infty}{\to} x^0(s)$$

and

$$\widetilde{m}_N(0,s) = N^d \int_{\mathcal{Q}_{\lfloor s_1 N\rfloor,\ldots,\lfloor s_d N\rfloor}} m^0(\sigma)d\sigma \underset{N\to\infty}{\to} m^0(s).$$

The functions $\widetilde{x}_N(t,s), \widetilde{m}_N(t,s)$ defined through formulas (5.2), (5.3) are governed by the following equation, which is the obvious higher dimensional version of Equation (1.4).

Proposition 5.1. *Let the assumptions of Proposition 2.1 hold and let $(\mathbf{x}_N(t), \mathbf{m}_N(t))$ be the solution to System (5.1) on $[0,T]$. Let $\widetilde{x}_N, \widetilde{m}_N$ be given by (5.2) and (5.3) respectively. Then*

$$\begin{cases} \partial_t \widetilde{x}_N(t,s) = \displaystyle\int_{I^d} \widetilde{m}_N(t,s_*)\mathbf{a}(\widetilde{x}_N(t,s_*) - \widetilde{x}_N(t,s))ds_*, \\ \partial_t \widetilde{m}_N(t,s) = N^d \displaystyle\int_{\mathcal{Q}_{\lfloor s_1 N\rfloor,\ldots,\lfloor s_d N\rfloor}} \Psi(s_*,\widetilde{x}_N(t,\cdot),\widetilde{m}_N(t,\cdot))ds_*. \end{cases} \qquad (5.6)$$

Proof. We start with the equation for $\widetilde{x}_N(t,s)$. Fix $s \in \mathcal{Q}_{\mathbf{i}_0}$. Then

$$\partial_t \widetilde{x}_N(t,s) = \dot{x}_{\sigma(\mathbf{i}_0)}(t) = \frac{1}{N^d}\sum_{j=1}^{N^d} m_j(t)\mathbf{a}(x_j(t) - x_{\sigma(\mathbf{i}_0)}(t)) = \frac{1}{N^d}\sum_{j=1}^{N^d} m_j(t)\mathbf{a}(x_j(t) - \widetilde{x}_N(t,s)).$$

On the other hand

$$
\int_{I^d} \widetilde{m}_N(t,s_*)\mathbf{a}(\widetilde{x}_N(t,s_*) - \widetilde{x}_N(t,s))ds_* = \int_{I^d} \sum_{\mathbf{i}\in\mathcal{M}} m_{\sigma(\mathbf{i})} \mathbf{1}_{\mathcal{Q}_{\mathbf{i}}}(s_*)\mathbf{a}\left(\sum_{\mathbf{k}\in\mathcal{M}} x_{\sigma(\mathbf{k})}(t)\mathbf{1}_{\mathcal{Q}_{\mathbf{k}}}(s_*) - \widetilde{x}_N(t,s)\right)ds_*
$$

$$
= \frac{1}{N^d} \sum_{\mathbf{i}\in\mathcal{M}} m_{\sigma(\mathbf{i})}\mathbf{a}\left(x_{\sigma(\mathbf{i})}(t) - \widetilde{x}_N(t,s)\right)
$$

$$
= \frac{1}{N^d} \sum_{j=1}^{N^d} m_j(t)\mathbf{a}\left(x_j(t) - \widetilde{x}_N(t,s)\right).
$$

The equation for $\widetilde{m}_N$ is due to the following identities

$$
\partial_t \widetilde{m}_N(t,s) = \sum_{\mathbf{i}\in\mathcal{M}} \dot{m}_{\sigma(\mathbf{i})}\mathbf{1}_{\mathcal{Q}_{\mathbf{i}}}(s) = N^d \int_{\mathcal{Q}_{\lfloor s_1 N\rfloor,\ldots,\lfloor s_d N\rfloor}} \Psi(s_*,\widetilde{x}_N(t,\cdot),\widetilde{m}_N(t,\cdot))ds_*.
$$

$\square$

5.2 Well posedness for $d > 1$

The point which requires most care for the proof of well posedness is point 2. in Lemma 3.1. Let us first state the assumptions we impose on the initial data and the other functions involved.

A1 $d > 1, \mathbf{a}(0) = 0$ and $\mathbf{a} \in \mathrm{Lip}(\mathbb{R}^d)$ with $L := \mathrm{Lip}(\mathbf{a})$.

A2 i. $m^0 \in L^\infty(I^d)$, $\int_{I^d} m_0(s)ds = 1$ and $\frac{1}{M} \le m^0 \le M$ for some $M > 1$.

ii. $\left|x^0\right| \le X$ for some $X > 0$ and $x^0 : I^d \to \mathbb{R}^d$ is bi-Lipschitz, i.e. there is some $L_0 > 0$ such that for all $s_1, s_2 \in I^d$

$$
\frac{1}{L_0}\left|s_1 - s_2\right| \le \left|x^0(s_1) - x^0(s_2)\right| \le L_0\left|s_1 - s_2\right|.
$$

A3 i. $\mathbf{s} : \mathbb{R}^d \to \mathbb{R}$ is a measurable odd function $(\mathbf{s}(-x) = -\mathbf{s}(x))$.

ii. There is some locally L^1 function $\mathbf{S} \ge 0$ such that

$$
\left|\mathbf{s}(x_1) - \mathbf{s}(x_2)\right| \le \mathbf{S}(x_2)\left|x_1 - x_2\right|, \quad x_1, x_2 \in \mathbb{R}^d.
$$

Remark 5.1. *The assumption **A2** that x^0 is bi-Lipschitz is strictly stronger than the assumption that it is 1-1 when $d = 1$.*

Remark 5.2. *If $d > 1$, then $\mathbf{s}(x) := \begin{cases} \frac{x}{\|x\|}, & x \ne 0 \\ 0, & x = 0. \end{cases}$ is a particular example of hypothesis **A3** as can be seen from through the following elementary inequalities*

$$
\left|\frac{x_1}{|x_1|} - \frac{x_2}{|x_2|}\right| = \left|\frac{x_1|x_2| - |x_1|x_2}{|x_1||x_2|}\right| = \left|\frac{x_1(|x_2| - |x_1|) + |x_1|(x_1 - x_2)}{|x_1||x_2|}\right|
$$

$$
\le \frac{||x_1| - |x_2||}{|x_2|} + \frac{|x_1 - x_2|}{|x_2|} \le \frac{2|x_1 - x_2|}{|x_2|}.
$$

*Furthermore, it is clear that the condition i. in **A3** is more general than the assumption $\mathbf{s} \in \mathrm{Lip}(\mathbb{R}^d)$.*

Lemma 5.1. *Let hypotheses **A1**-**A3** hold. Suppose that*
• There is some $\overline{X}$ such that $\sup_{[0,T]\times I} |x_i| \le \overline{X}$, $i = 1, 2$.

- The function x_2 is bi-Lipschitz in the labeling variable, i.e. there is some $C > 1$ such that for all $(t, s, s_*) \in [0, T] \times I^d \times I^d$ it holds that

$$\frac{1}{C} |s - s_*| \leq |x_2(t, s) - x_2(t, s_*)| \leq C |s - s_*|, \ i = 1, 2.$$

Then

$$\int_{I^d} |\Psi(s, x_1(t, \cdot), m(t, \cdot)) - \Psi(s, x_2(t, \cdot), m(t, \cdot))| \, ds \leq \mathcal{C} \sup_{I^d} |x_1(t, \cdot) - x_2(t, \cdot)|,$$

with $\mathcal{C} = L \, \overline{M} (6 + 8 \, \overline{X} \, C)$.

Proof. Setting $\mathbf{a}_i(s, s_*, s_{**}) := \mathbf{a}(x_i(s_{**}) - x_i(s)) + \mathbf{a}(x_i(s_{**}) - x_i(s_*))$, $i = 1, 2$, we can estimate A2

$$\left| \iint_{I^{2d}} m(s_*) m(s_{**}) (\mathbf{a}_1(s, s_*, s_{**}) \mathbf{s}(x_1(s) - x_1(s_*)) ds_* ds_{**} - \mathbf{a}_2(s, s_*, s_{**}) \mathbf{s}(x_2(s) - x_2(s_*))) ds_* ds_{**} \right|$$

$$\leq J_1(t, s) + J_2(t, s),$$

where we define

$$J_1(t, s) = L \iint_{I^{2d}} m(s_*) m(s_{**}) \left(2 |x_1(s_{**}) - x_2(s_{**})| + |x_1(s_*) - x_2(s_*)| + |x_1(s) - x_2(s)| \right)$$
$$\left(|\mathbf{s}(x_1(s) - x_1(s_*))| + |\mathbf{s}(x_2(s) - x_2(s_*))| \right) ds_* ds_{**},$$

and

$$J_2(t, s) = \iint_{I^{2d}} m(s_*) m(s_{**}) |\mathbf{a}_2(s, s_*, s_{**})| |\mathbf{s}(x_1(s) - x_1(s_*)) - \mathbf{s}(x_2(s) - x_2(s_*))| \, ds_* ds_{**}.$$

We start with the estimate on $J_2(t, s)$. Using assumption **A3**, we have that

$$\left| \iint_{I^{2d}} m(s_*) m(s_{**}) \mathbf{a}_2(s, s_*, s_{**}) \mathbf{s}(x_1(s) - x_1(s_*)) ds_* ds_{**} \right.$$

$$\left. - \iint_{I^{2d}} m(s_*) m(s_{**}) \mathbf{a}_2(s, s_*, s_{**}) \mathbf{s}(x_2(s) - x_2(s_*)) ds_* ds_{**} \right|$$

$$\leq L \sup_{[0,T] \times I^d} \iint_{I^{2d}} (2 |x_2(s_{**})| + |x_2(s)| + |x_2(s_*)|) \, m(s_*) m(s_{**}) |\mathbf{s}(x_1(s) - x_1(s_*)) - \mathbf{s}(x_2(s) - x_2(s_*))| \, ds_* ds_{**}$$

$$\leq 4 L \, \overline{X} \iint_{I^{2d}} m(s_*) m(s_{**}) \, \mathbf{S}(x_2(s) - x_2(s_*)) \left(|x_1(s) - x_2(s)| + |x_1(s_*) - x_2(s_*)| \right) ds_* \, ds_{**}$$

$$\leq 8 L \, \overline{X} \, \overline{M} \sup_{I^d} |x_1(t, \cdot) - x_2(t, \cdot)| \sup_{s \in I^d} \int_{I^d} \mathbf{S}(x_2(s) - x_2(s_*)) \, ds_*.$$

From the bi-Lipschitz assumption on x_2 we have $|(\nabla_s x_2)^{-1}| \leq C$ so that

$$\int_{I^d} \mathbf{S}(x_2(s) - x_2(s_*)) \, ds_* \leq C \int_{I^d} \mathbf{S}(x_2(s) - y) \, dy \leq C \, \|S\|_{L^1(K)},$$

for some compact set $K \subset \mathbb{R}^d$. Therefore

$$\int_{I^d} m(t, s) |J_2(t, s)| \, ds \leq 8 L \, \overline{X} \, \overline{M} \, C \, \|\mathbf{S}\|_{L^1(K)} \sup_{I^d} |x_1(t, \cdot) - x_2(t, \cdot)|.$$

The estimate for $J_1(t, s)$ follows in a similar way,

$$\int_{I^d} m(t, s) |J_1(t, s)| \, ds$$

$$\leq 3L \sup_{I^d} |x_1(t, \cdot) - x_2(t, \cdot)| \int_{I^{2d}} m(s) \, m(s_*) \left(|\mathbf{s}(x_1(s) - x_1(s_*))| + |\mathbf{s}(x_2(s) - x_2(s_*))| \right) ds \, ds_*$$

$$\leq 6 L \, \overline{M} \|\mathbf{S}\|_{L^1(K)} \sup_{I^d} |x_1(t, \cdot) - x_2(t, \cdot)|.$$

□

5.3 Graph limit for $d > 1$

Consider

$$\xi_N(t) := \|\widetilde{x}_N(t,\cdot) - x(t,\cdot)\|_{L^1(I^d)}, \ \zeta_N(t) := \|\widetilde{m}_N(t,\cdot) - m(t,\cdot)\|_{L^1(I^d)}.$$

The symbol $\lesssim$ stands for inequality up to a constant which may depend only on $L, L_0, M, X, T, \mathbf{S}, A2\mathbf{S}_\infty$.

Theorem 5.1. *Let the hypotheses **A1**-**A3** hold. Let $(x, m) \in C^1\left([0,T]; C(I^d)\right) \oplus C^1\left([0,T]; L^\infty(I^d)\right)$ be the solution to Equation (3.8). Assume that $\mathbf{S}$ is chosen so that the system (2.1) has a well defined flow. Let $(\mathbf{x}_N, \mathbf{m}_N) \in C^1\left([0,T]; \mathbb{R}^{dN^d} \times \mathbb{R}^{N^d}\right)$ be the solution to the system (2.1). Then*

$$\|\widetilde{x}_N - x\|_{C([0,T];L^1(I^d))} + \|\widetilde{m}_N - m\|_{C([0,T];L^1(I^d))} \underset{N\to\infty}{\to} 0.$$

Proof. **Step 1**. **Time derivative of $\zeta_N(t)$.** The computation of the time derivative of ζ_N is essentially identical to the case $d = 1$, but we include it for clarity. For readability we set $\mathbf{i}_N(s) := (\lfloor s_1 N \rfloor, \ldots, \lfloor s_d N \rfloor)$.

$$\dot{\zeta}_N(t) \leq \int_{I^d} \left| N^d \int_{\mathcal{Q}_{\mathbf{i}_N(s)}} \Psi(s_*, \widetilde{x}_N(t,\cdot), \widetilde{m}_N(t,\cdot)) ds_* - \Psi(s, x(t,\cdot), m(t,\cdot)) \right| ds$$

$$\leq \int_{I^d} \underbrace{\left| N^d \int_{\mathcal{Q}_{\mathbf{i}_N(s)}} \Psi(s_*, \widetilde{x}_N(t,\cdot), \widetilde{m}_N(t,\cdot)) - \Psi(s_*, x(t,\cdot), m(t,\cdot)) ds_* \right|}_{:=h_N(t,s)} ds$$

$$+ \int_{I^d} \underbrace{\left| N^d \int_{\mathcal{Q}_{\mathbf{i}_N(s)}} \Psi(s_*, x(t,\cdot), m(t,\cdot)) ds_* - \Psi(s, x(t,\cdot), m(t,\cdot)) \right|}_{:=g_N(t,s)} ds.$$

By Lebesgue differentiation theorem, for each $t \in [0,T]$ it holds that $g_N(t,s) \to 0$ as $N \to \infty$ pointwise a.e. $s \in I^d$. In addition, $\|x\|_{C([0,T]\times I^d)}, \|m\|_{C([0,T];L^\infty(I^d))}$ are bounded which implies that $g_N(t,s)$ is uniformly bounded (with respect to N) so that by the dominated convergence theorem we find that for each $t \in [0,T]$ it holds that $\|g_N(t,\cdot)\|_1 \underset{N\to\infty}{\to} 0$. We now estimate the first integral as

$$\|h_N(t,\cdot)\|_1 = N^d \int_{I^d} \left| \int_{\mathcal{Q}_{\mathbf{i}_N(s)}} \Psi(s_*, \widetilde{x}_N(t,\cdot), \widetilde{m}_N(t,\cdot)) - \Psi(s_*, x(t,\cdot), m(t,\cdot)) ds_* \right| ds$$

$$= N^d \sum_{\mathbf{i}\in\mathcal{M}} \int_{\mathcal{Q}_{\mathbf{i}}} \left| \int_{\mathcal{Q}_{\mathbf{i}_N(s)}} \Psi(s_*, \widetilde{x}_N(t,\cdot), \widetilde{m}_N(t,\cdot)) - \Psi(s_*, x(t,\cdot), m(t,\cdot)) ds_* \right| ds$$

$$\leq N^d \sum_{\mathbf{i}\in\mathcal{M}} \int_{\mathcal{Q}_{\mathbf{i}}} \int_{\mathcal{Q}_{\mathbf{i}_N(s)}} |\Psi(s_*, \widetilde{x}_N(t,\cdot), \widetilde{m}_N(t,\cdot)) - \Psi(s_*, x(t,\cdot), m(t,\cdot))| ds_* ds$$

$$= N^d \sum_{\mathbf{i}\in\mathcal{M}} \int_{\mathcal{Q}_{\mathbf{i}}} \int_{\mathcal{Q}_{\mathbf{i}}} |\Psi(s_*, \widetilde{x}_N(t,\cdot), \widetilde{m}_N(t,\cdot)) - \Psi(s_*, x(t,\cdot), m(t,\cdot))| ds_* ds$$

$$= \sum_{\mathbf{i}\in\mathcal{M}} \int_{\mathcal{Q}_{\mathbf{i}}} |\Psi(s_*, \widetilde{x}_N(t,\cdot), \widetilde{m}_N(t,\cdot)) - \Psi(s_*, x(t,\cdot), m(t,\cdot))| ds_*$$

$$= \int_{I^d} |\Psi(s_*, \widetilde{x}_N(t,\cdot), \widetilde{m}_N(t,\cdot)) - \Psi(s_*, x(t,\cdot), m(t,\cdot))| ds_*.$$

This in turn leads to the following bound

$$\|h_N(t,\cdot)\|_1 \leq \int_{I^d} |\Psi(s_*, \widetilde{x}_N(t,\cdot), \widetilde{m}_N(t,\cdot)) - \Psi(s_*, \widetilde{x}_N(t,\cdot), m(t,\cdot))|\, ds_*$$

$$+ \int_{I^d} |\Psi(s_*, \widetilde{x}_N(t,\cdot), m(t,\cdot)) - \Psi(s_*, x(t,\cdot), m(t,\cdot))|\, ds_*$$

$$= \int_{I^d} |\Psi(s, \widetilde{x}_N(t,\cdot), \widetilde{m}_N(t,\cdot)) - \Psi(s, \widetilde{x}_N(t,\cdot), m(t,\cdot))|\, ds$$

$$+ \int_{I^d} |\Psi(s, \widetilde{x}_N(t,\cdot), m(t,\cdot)) - \Psi(s, x(t,\cdot), m(t,\cdot))|\, ds.$$

At this stage we state and prove the following simple adaption of Lemma 5.1. We precisely need the following result.

Lemma 5.2. *It holds that*

$$i. \int_{I^d} |\Psi(s, \widetilde{x}_N(t,\cdot), \widetilde{m}_N(t,\cdot)) - \Psi(s, \widetilde{x}_N(t,\cdot), m(t,\cdot))|\, ds \lesssim \|\widetilde{m}_N(t,\cdot) - m(t,\cdot)\|_1.$$

and

$$ii. \int_{I^d} |\Psi(s, \widetilde{x}_N(t,\cdot), m(t,\cdot)) - \Psi(s, x(t,\cdot), m(t,\cdot))|\, ds \lesssim \|\widetilde{x}_N(t,\cdot) - x(t,\cdot)\|_1.$$

Proof. Point i. follows from the same argument of 1. in Lemma 3.1, so let us concentrate on point ii. We estimate

$$\left| \iint_{I^{2d}} m(s_*)m(s_{**})\mathbf{a}_N(s, s_*, s_{**})\mathbf{s}(\widetilde{x}_N(s) - \widetilde{x}_N(s_*))ds_* ds_{**} \right.$$

$$\left. - \iint_{I^{2d}} m(s_*)m(s_{**})\mathbf{a}(s, s_*, s_{**})\mathbf{s}(x(s) - x(s_*))ds_* ds_{**} \right|$$

$$\lesssim \iint_{I^{2d}} m(s_*)m(s_{**}) \left(2\, |\widetilde{x}_N(s_{**}) - x(s_{**})| + |\widetilde{x}_N(s_*) - x(s_*)| + |\widetilde{x}_N(s) - x(s)| \right) ds_* ds_{**}$$

$$+ \iint_{I^{2d}} m(s_*)m(s_{**}) \left| \mathbf{a}(x(s_{**}) - x(s)) + \mathbf{a}(x(s_{**}) - x(s_*)) \right|$$

$$\times |\mathbf{s}(\widetilde{x}_N(s) - \widetilde{x}_N(s_*)) - \mathbf{s}(x(s) - x(s_*))|\, ds_* ds_{**}$$

$$:= J_1(t,s) + J_2(t,s).$$

We estimate seperately $\int_{I^d} m(t,s)\, |J_1(t,s)|\, ds$ and $\int_{I^d} m(t,s)\, |J_2(t,s)|\, ds$. It is straightforward to check

$$\int_{I^d} m(t,s)\, |J_1(t,s)|\, ds \lesssim \|\widetilde{x}_N(t,\cdot) - x(t,\cdot)\|_1. \tag{5.7}$$

We further estimate

$$\int_{I^d} m(t,s)\, |J_2(t,s)|\, ds \lesssim \iint_{I^{2d}} \mathbf{S}(x(t,s) - x(t,s_*)) \left(|\widetilde{x}_N(t,s) - x(t,s)| + |\widetilde{x}_N(t,s_*) - x(t,s_*)| \right) ds_* ds$$

$$\leq 2\, \|\mathbf{S}\|_{L^1(K)}\, \|\widetilde{x}_N(t,\cdot) - x(t,\cdot)\|_1,$$

for some compact set $K \subset \mathbb{R}^d$.

$$\square$$

Step 2. The time derivative of $\xi_N(t)$. The time derivative of $\dot{\xi}_N(t)$ is mastered exactly as in [1]. Following the argument in [1] one finds that

$$\dot{\xi}_N(t) \lesssim \xi_N(t) + \zeta_N(t). \tag{5.8}$$

Step 3. Conclusion. The combination of Inequality (5.8) and Lemma 5.2 yields

$$\dot{\xi}_N(t) + \dot{\zeta}_N(t) \lesssim \xi_N(t) + \zeta_N(t) + \frac{1}{2}\left\|g_N(t,\cdot)\right\|_1,$$

i.e.

$$\xi_N(t) + \zeta_N(t) \lesssim \xi_N(0) + \zeta_N(0) + \int_0^t (\xi_N(\tau) + \zeta_N(\tau))d\tau + \int_0^T \left\|g_N(\tau,\cdot)\right\|_1 d\tau.$$

Applying Gronwall's lemma entails

$$\xi_N(t) + \zeta_N(t) \leq C\left(\xi_N(0) + \zeta_N(0) + \int_0^T \left\|g_N(\tau,\cdot)\right\|_1 d\tau + o_N(1)T\right)\exp\left(Ct\right).$$

Since $\left\|g_N(\tau,\cdot)\right\|_2^2$ is uniformly bounded, by dominated convergence $\int_0^T \left\|g_N(\tau,\cdot)\right\|_1 d\tau \underset{N\to\infty}{\to} 0$, which concludes the proof.

$\square$

Remark 5.3. *Note that Theorem 5.1 proves convergence with respect to the L^1 norm, whereas Theorem 4.1 proves convergence with respect to the L^2 norm. This minor difference is because when $d = 2$ the L^2 norm of $\frac{1}{|s|}$ blows up, which prevents getting the inequality ii. in Lemma 5.2. Notice that for any $d \geq 3$, the L^2 approach is perfectly valid.*

Remark 5.4. *Essentially the same argument of Theorem 4.2 allows one to conclude the weak mean field limit from the graph limit in higher dimension.*

Acknowledgments

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