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75 Years of IEEE AP-S Research in Computational Electromagnetics

A view on the discipline and its history, current state, and future prospects.

This article presents an overview of 75 years of research in computational electromagnetics (CEM) within the IEEE Antennas and Propagation Society (AP-S) and the AP community at large on the occasion of the 75th anniversary of AP-S, where both CEM and AP-S have similar and interwoven histories of 75 years, a half of the history of Maxwell's equations. The article discusses the discoveries, developments, implementations, and applications of principal CEM methodologies and techniques for AP. While historically there were periods of time when certain CEM approaches seemed to dominate AP applications, it is indicative that CEM for AP of the 21st century has been constituted by a true expansion and/or renaissance of all approaches, with an emphasis on hybridizations and integrations. Overall, we have witnessed phenomenal progress and dramatic enhancements in the accuracy, efficiency, versatility, reliability, stability, and robustness of AP modeling, analysis, and design. However, CEM is still a work in progress, and many important research challenges are yet to be addressed and solved for the ever-growing simulation and design demands of the next generations of AP technologies. Hence, the next 75 years of CEM and AP-S are promising to be equally rich, fascinating, and intense!

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INTRODUCTION

Antenna, radio-frequency (RF), wireless, and other electromagnetics-related technologies are exploding! CEM is the interdisciplinary field aimed at numerically analyzing and computationally solving practical problems within these technologies involving electromagnetic fields and waves and their interactions with materials and existing or to-be-designed structures and systems. Naturally, CEM has been one of the main topics within the scope and activities of AP-S.

Indeed, the importance of CEM to AP technologies can hardly be overstated. CEM simulations are effectively used today at frequencies spanning dc to optics, for system sizes ranging from subatomic to intergalactic, and for a broad spectrum of AP application areas, including the design of antennas and RF/microwave/terahertz devices, components, and systems, electromagnetic scattering, indoor and outdoor radio propagation, wireless communication systems, radar systems, remote sensing, packaging, RF interference, new materials, quantum electrodynamics, and bioelectromagnetics. To cover such a wide variety of structures, technologies, and applications, a range of different CEM methodologies and numerical discretization techniques is needed and preferred over a single approach.

In his famous book, *A Treatise on Electricity and Magnetism* (1873) [1], James Clerk Maxwell provided a unified mathematical framework for all fundamental laws of electricity and magnetism discovered experimentally by his predecessors, and he

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compiled and completed the four fundamental equations of electromagnetics that bear his name [2]. Per Albert Einstein, "The formulation of these equations is the most important event in physics since Newton's time." These four equations provide the foundation for the general analysis of electromagnetic radiation and propagation and for all of the theoretical principles, analytical procedures, and numerical techniques that together constitute the knowledge and tools for the understanding, analysis, and design of AP structures and systems [3].

So, as a community, just last year we celebrated 150 years of Maxwell's equations, and CEM has a history of about 75 years [3]–[142]. CEM and AP researchers have made tremendous progress in devising and developing new methods and codes, both "in-house" (research or custom-designed) techniques and commercial solvers, often in synergy with AP application researchers and practitioners as CEM users. Holistically, CEM discoveries, advancements, and impacts over 75 years have tightly relied on principal breakthroughs and developments made in electromagnetic/physical formulations and contexts of engineering problems, mathematical and numerical foundations of methods and algorithms, and computing hardware and software infrastructure. Truly, CEM research and practice is a comprehensive combination of engineering, physics, mathematics, and computer science, with exciting potential and challenges.

This year, the IEEE Antennas and Propagation Society celebrates its 75th anniversary—the Diamond Jubilee Anniversary—as it was founded in 1949. AP-S is one of the three oldest IEEE societies (or professional groups as the societies were called then), and its first official name was the IRE Professional Group on Antennas and Propagation. Remarkably, the Society was founded with the same name (Antennas and Propagation) that it kept during the following 75 years and still does. Of course, it all started with Maxwell's equations 150 years ago, and a half of that history was with AP-S as one of the leaders in the field.

This article presents an overview of 75 years of research in CEM within AP-S and the AP community at large. It aims at providing representation that is as complete and unified as possible of fundamental developments across a spectrum of CEM formulations, methodologies, solution techniques, and applications related to AP. However, given the wealth and diversity of past ideas and efforts as well as their results and outcomes (that is, devised and available CEM methods and codes), this article is by no means meant to be an exhaustive survey. Rather it is a systematic and illustrative presentation and discussion of commonalities and unifying concepts of CEM approaches and tools, including selected specifics, in terms of both the history and the state of the art, with an eye toward their practical applicability and usefulness.

Because of space limitations, all presented or mentioned CEM methodologies, techniques, implementations, and applications, all explicitly mentioned names, all presented commercial tools, all figures and results, and all listed and cited references are only meant to be representative, illustrative, informative, and interesting, and by no means complete or exhaustive. Also, to economize space, when an author name is mentioned in the article, it is typically and rather consistently a group or project leader on multiauthor publications with authors from the same group. These, again, are subjective selections that could certainly have been done differently and have left many key contributors unmentioned.

Moreover, this is one author's overview of the activities and accomplishments of thousands of researchers, practitioners, and students in many generations over 75 years and one person's attempt to condense efforts and successes of an era and the entire discipline in one article. Thus, it is inevitable that some objectively important facts and contributions were omitted or misprioritized by mistake, lack of knowledge, or by a subjective and biased view of the author or just his imperfect and nonoptimal endeavor to "squeeze" everything into the page limit. Therefore, one thing is certain: the most rigorous and exact aspect of this article is that it is a personal view of the author on the discipline and its history, current state, and future prospects. It is just one possible view and an imperfect but honest attempt.

Three special issues of *IEEE Transactions on Antennas and Propagation* on CEM provided comprehensive accounts of developments and the state of the art in the field as well as archival materials for the respective time periods. These were the 1997 special issue, titled "Advanced Numerical Techniques in Electromagnetics" [4]; the one published in 2008, on "Large and Multiscale Computational Electromagnetics" [5]; and the just-published special issue (in two parts) on "Frontiers in Computational Electromagnetics" (December 2023/January 2024) [6].

However, computational electromagnetics is still a work in progress, and many important research challenges are yet to be addressed and solved. Namely, we are not yet done in solving these four 150-year-old equations, as we continue a quest to devise, develop, and use new and improved methods and tools to achieve more speed, accuracy, versatility, and reliability of CEM simulations for different classes of real-world AP engineering applications and problems [3].

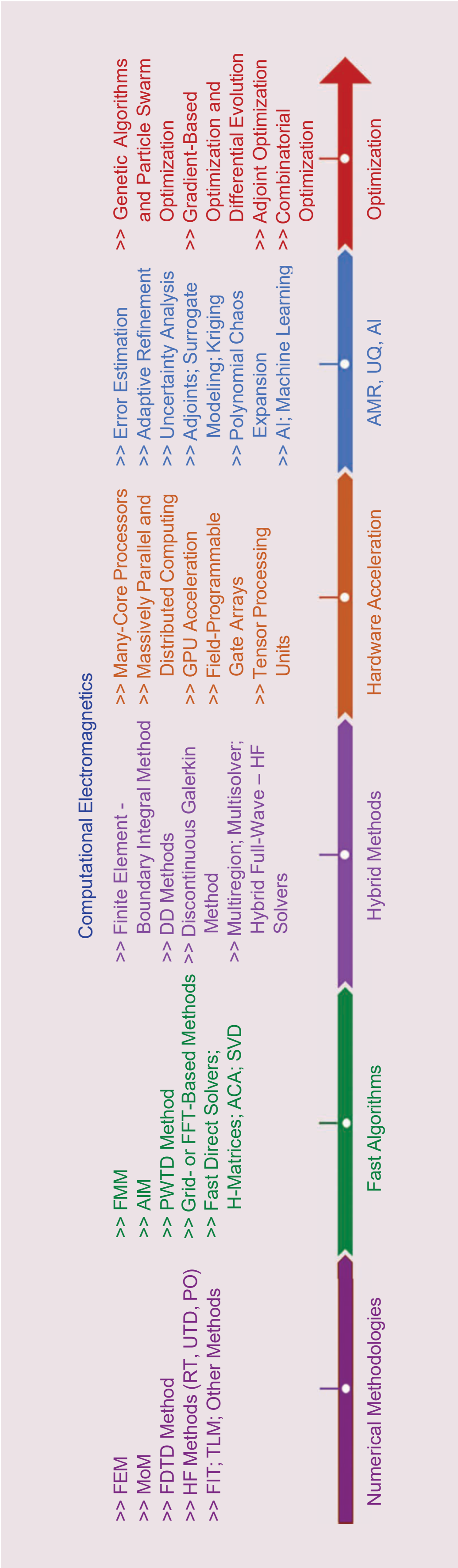

FIGURE 1. Overview of CEM for AP. RT: ray tracing; UTD: uniform theory of diffraction; PO: physical optics; FMM: fast multipole method; AIM: adaptive integral method; PWTD: plane-wave time-domain; FFT: fast Fourier transform; ACA: adaptive cross approximation; SVD: singular value decomposition; DD: domain decomposition; AMR: adaptive model refinement; UQ: uncertainty quantification; AI: artificial intelligence. (Figure adapted from [31], courtesy of Zhen Peng.)

PRINCIPAL CEM METHODOLOGIES AND APPROACHES

The CEM methodologies of the 75 years of CEM and AP-S can broadly be classified into those based on integral equation (IE) formulations, namely, on solving IEs for currents (field sources) [7], [8], [9], [10], [11], [12], [13], [14], [15], those based on partial differential equation (PDE) formulations, i.e., on solving PDEs for fields **E** and/or **H** as unknown quantities [12], [13], [14], [15], [16], [17], [18], [19], [20], [21], [22], [23], [24], [25], and high-frequency (HF) asymptotic techniques [26], [27], [28] as well as hybrid approaches [15], [16], [17], [18], [19], [20], [21]. IE methods included those for discretizing surface integral equations (SIEs) for the surface electric current, $\mathbf{J}_s$, on metallic AP structures (see, e.g., [8]), and for equivalent electric and magnetic (artificial) surface currents, $\mathbf{J}_s$ and $\mathbf{M}_s$, placed over surfaces of dielectric parts of a structure (see, e.g., [9]) as well as volume integral equations (VIEs) with volume electric current, **J**, inside dielectric volumes (see, e.g., [11]).

As general numerical solution procedures, CEM used the method of moments (MoM) or boundary element method (BEM) [7], [8], [9], [10], [11] to discretize SIEs and VIEs and both the finite-element method (FEM) [16], [17], [18], [19], [20], [21] and the finite-difference time-domain (FDTD) method [22], [23], [24] to discretize PDEs, in addition to other approaches, including the transmission-line modeling (TLM) method [25], multiresolution time-domain method [29], and finite integration technique (FIT) [30] (Figure 1).

Numerically approximate HF techniques were devised and used for the asymptotic CEM analysis of electrically very large AP structures and systems with smooth surfaces, such as a large antenna platform (e.g., a ship), a scatterer (e.g., an airplane illuminated by a radar beam), or an antenna reflector (e.g., a parabolic dish) [26], [27], [28]. These techniques included either ray-based approaches, such as geometrical optics, ray tracing (RT), the shooting and bouncing rays (SBR) method, and uniform geometric theory of diffraction (UTD), or current-based ones, for example, physical optics (PO); see Figure 1. In fact, historically, HF methods dominated the first few decades of the 75-year history of CEM and AP-S, with groundbreaking works by Kouyoumjian, Pathak, Burnside, Marhefka, Mittra, Rahmat-Samii, and others [32], [33],

[34], [35], [36]. For example, the first steps of the men on the Moon during the Apollo 11 mission in 1969 were TV-broadcasted live on Earth thanks to dish reflector antennas designed using HF tools.

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In terms of the particulars of the numerical discretizations, most of the frequently used CEM tools were low-order techniques, with the AP structure modeled by geometrical elements that are electrically very small (on the order of $m/10$ or $m/20$ in each dimension, m being the wavelength in the medium), and the currents and/or fields within the elements were approximated by low-order basis functions. This typically results in a significant burden on the functional (current/field) and geometrical discretization errors on the CEM solution accuracy and a low convergence rate of the solution, where the solution accuracy improves slowly with increasing the number of unknowns or degrees of freedom (DoFs) and the requirements in computational resources.

The CEM for AP also used higher order discretizations, featuring higher order basis functions defined in large curved geometrical elements (e.g., on the order of m, e.g., $2m$, in each dimension) [37], [8], [9], [10], [14]. The more complex elements, basis functions, and implementations are generally justified by both higher order geometrical flexibility and higher order current/field-approximation flexibility, preferably in the same method, which enables faster (higher order) convergence of the solution and more efficient model refinements.

THE METHOD OF MOMENTS

The basic electric and magnetic field IEs (EFIE and MFIE) were described by Maue in 1949 [38], with the understanding that they can be expressed in a variety of ways by shifting derivatives from one term to another and other transformations, and can be combined together, have remained the basis of most formulations [39], including, for example, the combined field integral equation (CFIE) [40], [41]; the Poggio, Miller, Chang, Harrington, Wu, and Tsai (PMCHWT) formulation [42], [43], [44], [45]; and Müller's formulation [46], [47].

Numerical solutions to EFIE/MFIE formulations for AP problems have proven promising and valuable since the early 1960s, when practitioners such as Kennaugh [48], Andreason [49], Richmond [50], Harrington [7], and others began to disseminate their research efforts in that area [39]. Since IE formulations typically involve dense matrix equations, all of these early approaches were severely limited by the available computational resources. Because of this limitation, the early formulations were most successful for 2D scattering problems [49], [50] and for the analysis of wire antennas and wire structures [7], [51], with one of the first higher order MoM techniques being that for wire-dipole antenna analysis by Popovic´ in [52]. It is noteworthy that a cell-by-cell parametric mapping was used for IE analysis of arbitrary metallic antennas and scatterers as early as 1972 [53]. Furthermore, the successful practice of the CEM discipline was often considered to be more of an art than a science [39].

The release of the famous Numerical Electromagnetic Code (NEC) for wire antennas and scatterers, by Burke and Poggio of the Lawrence Livermore National Laboratory, in 1980 [54] initiated a great deal of activity in utilization of the MoM/BEM for antennas and has had, over decades, a profound impact on AP design.

Surface patches were more challenging for MoM modeling than the wire segments in the NEC, primarily because of the difficulty in enforcing the current continuity across junctions between patches. The Rao–Wilton–Glisson (RWG) vector basis functions (named after their inventors) defined on flat triangular patches [55] solved this problem. Historically, the introduction of the RWG functions in 1982 [55] resulted in a large number of new MoM codes and is one of the most important advances in CEM for AP. While representing the state of the art in the early 1980s, remarkably, they remain the dominant approach in use today [39]. Currently, these are the most widely used MoM basis functions in the AP community.

A triangle is commonly represented in terms of simplex coordinates, p_1, p_2, and p_3, representing normal distances of a point on the triangle surface to the triangle edges [55], [11], [12], [3], as shown in Figure 2(a). For the approximation of the surface current density, $\mathbf{J}_s$, the RWG function $\mathbf{K}_1$ has a normal component only on edge 1, which, moreover, is constant along the edge, so that the current continuity condition for the normal component of $\mathbf{J}_s$ along the edge can be automatically adjusted with the accompanying RWG function on the adjacent patch across the edge, as in Figure 2(b). During the intervening years, higher order generalizations of RWG functions on curved triangular patches, as shown in Figure 2(c), have been proposed [10], [14].

Quadrilateral MoM–SIE models with bilinear quadrilateral patches, Figure 2(d), proposed by Kolundžija [9], or generalized curved quadrilateral elements, Figure 2(e), proposed by Notaroš [45], have also been used, with u- and v-components of $\mathbf{J}_s$ approximated using higher order polynomial vector basis functions in the parametric coordinates u and v. The lowest order yields the so-called rooftop functions, the version of RWG functions for quadrilaterals, which serve in automatic enforcement of the current continuity condition along quadrilateral edges.

More complex parameterizations based on rational polynomial functions, e.g., nonuniform rational B-splines (NURBS) [56], leading to what is called isogeometric analysis [57], [58], allowed for direct incorporation of CAD models into MoM/BEM (and FEM) computation.

The 3D generalization of RWG functions—the Schaubert–Wilton–Glisson (SWG) basis functions on tetrahedral VIE elements (proposed in 1984) [59]—have been to date the most popular MoM basis functions for volume current **J**, namely, for the electric flux density, **D**, in VIE modeling [60]. The hexahedral counterparts are those by Notaroš [61] and Volakis [11].

For transient MoM analysis, time-domain integral equations (TDIEs) have been used, with the SIE being discretized in both space, as in Figure 2(b), and time, using various time-marching schemes and temporal basis functions [62] and a breakthrough in the form of stable TDIE methods by Michielssen, Shanker, Weile, and others (see, e.g., [62]).

A major problem in MoM computation is the presence of Green's function, $g(R)$, in the SIE or VIE integrals, making these integrals singular or near singular when the source-to-field distance R is zero or small, e.g., in Figure 2(b), and the integrand becomes infinite or abruptly very large. Seminal progress has been made by the AP/CEM community, for example, by Wilton, Graglia, Ylä-Oijala, Duffy, and others, in computing these integrals based on singularity extraction or subtraction methods [63], [64], [65] and singularity cancellation or coordinate transformation methods [66], [67], [68].

Additional advances in IE modeling for AP included formulations that address the low-frequency breakdown problem, where Chew and others provided general solutions [69], [70] and fundamental matrix conditioning issues, e.g., Calderon preconditioners, with impactful works by Andriulli and others [71], [72], [41]; Nyström discretizations, such as locally corrected Nyström schemes, proposed by Canino and others [73], [74]; Green's functions for infinite multilayer dielectric media, with seminal contributions by Michalski and Mosig [75]; and multitrace SIE formulations [76].

THE FINITE ELEMENT METHOD

By its inherent features, the FEM for PDE discretization is especially suitable for 3D modeling and analysis of AP structures and systems that contain inhomogeneous and complex electromagnetic materials as well as geometrical irregularities [3]. Importantly, PDE operators do not imply direct action at a distance like integral operators do [e.g., R in Figure 2(b)] but at a point; hence, the sparsity of FEM matrices, as opposed to the MoM, and the absence of Green's functions and singular integrals. On the other hand, the FEM generally requires larger numbers of DoFs than the MoM or BEM since it solves for the fields and hence inherently needs mesh termination and volumetric discretization and is prone to numerical pollution error [77].

The first FEM techniques for CEM were based on a point-matched approach combined with nodal basis functions for each scalar field component. This approach was susceptible to spurious modes. A fundamental solution to this was the implementation of vector basis functions in the FEM. Namely, extending the triangular patch in Figure 2(a) to a volumetric element, a FEM tetrahedron, shown in Figure 2(f), is obtained, with a 3D version of RWG bases that automatically stipulates the continuity of the tangential **E** field across element faces, known as Whitney forms. Elements with such vector bases are referred to as vector finite elements.

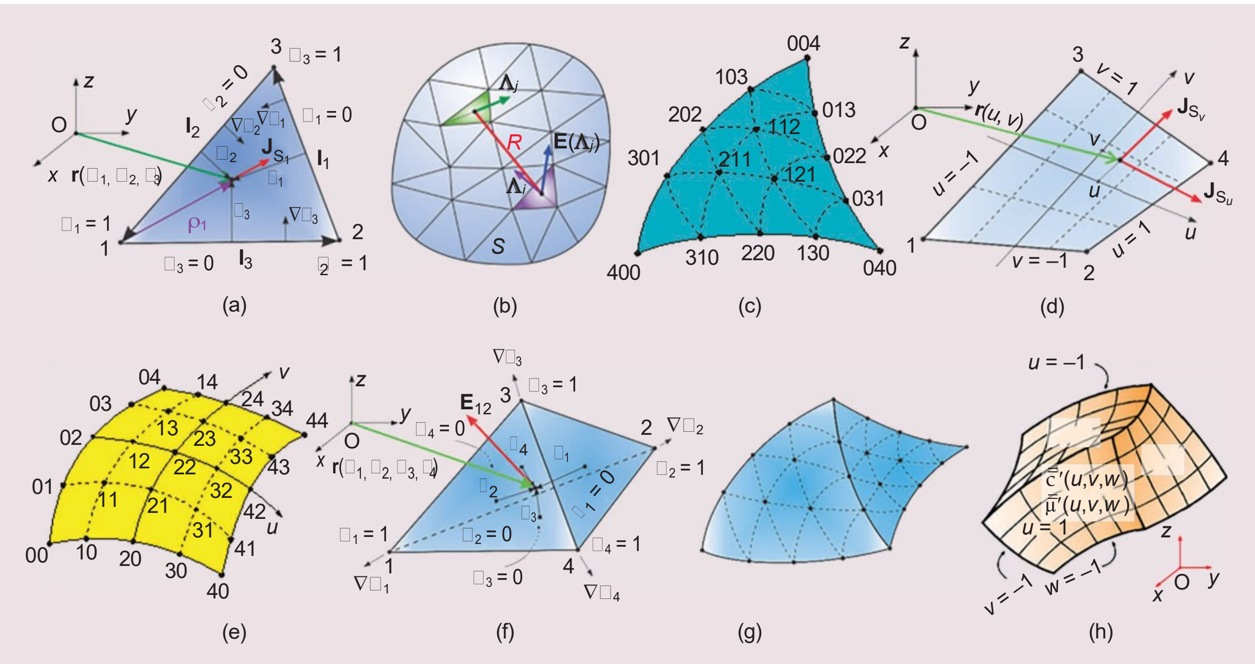

FIGURE 2. CEM patches and elements [3], [37]. (a) Triangular patch with RWG basis functions. (b) MoM-SIE triangular mesh of a metallic antenna using elements in (a). (c) Generalized curved parametric triangular MoM-SIE patch. (d) Bilinear quadrilateral patch with higher order polynomial vector basis functions for $\mathbf{J}_s$ in MoM-SIE. (e) Higher order generalized curved quadrilateral MoM-SIE patch. (f) Tetrahedral vector finite element with Whitney forms as basis functions for **E** modeling. (g) Generalized curved parametric tetrahedron for FEM. (h) Generalized curved hexahedral finite element with inhomogeneous anisotropic material.

Historically, the development of the 3D vector finite elements [78], [79], [80], [81], with seminal contributions by Raviart, Thomas, Nedelec, Lee, and others, was another major advancement in CEM. They can correctly and accurately model **E**- and **H**-field vectors, and, thanks to them as well as the multigrid preconditioners and efficient matrix solution processes [82], the FEM has become practical for AP applications. Currently, these elements are used in the leading commercial FEM solvers and in numerous research codes [3].

Higher order tetrahedral finite elements, proposed by Graglia, Wilton, and Peterson [83], Figure 2(g), were particularly useful because of their higher order convergence for large-scale AP problems, where the fields propagate over long electrical distances [84]. Equivalently, FEM modeling of AP structures was carried out using higher order hexahedral FEM modeling, Figure 2(h), proposed by Notaroš, as a volume generalization of quadrilateral patches in Figure 2(e) [85].

In AP analysis (open-region problems), the fields theoretically occupy an infinite region, and the FEM or FDTD computational domain therefore needed to be truncated by implementing a suitable mesh termination scheme [3]. The FEM termination based on approximate absorbing boundary conditions (ABCs) [21], while preserving the simplicity of the FEM method and sparsity of the final system, required placing the truncation surface sufficiently far away from the antenna (or other AP structure), which significantly increased the computational domain. The perfectly matched layer (PML) [21], an artificial lossy domain added around the computational domain, was born in the context of the FDTD method by Berenger [86] and further advanced by Gedney [87] and others as another breakthrough in CEM. The locally conformal PML conforming to the geometry of an antenna was implemented for the FEM.

As another important advancement in CEM for AP, with pivotal works by Jin, Volakis, Lee, and others, in a mesh termination scheme based on boundary integral (BI) equations [88], [20], giving rise to a hybrid FE–BI, also referred to as the FEM–MoM, methodology, the FEM region was truncated and numerically closed (completed) by SIE (MoM) solution outside the region, which was an exact termination. The method has been continuously developed for more than 35 years, resulting in a variety of different formulations, including the latest symmetric ones [89], [90], [84]. The computation of fully populated BI subblocks was accelerated using fast algorithms [89].

THE FDTD METHOD

The FDTD method discretizes directly the two curl Maxwell's equations [2], which represent a system of simultaneous equations with two unknowns (**E** and **H**). The method relies on the approximation of time and space derivatives by central differences on a staggered grid. Figure 3 illustrates the Yee lattice (cell) for 3D FDTD field computation, where any of the E vector components is surrounded by only H components, and vice versa. The overall result is a set of six coupled difference equations that are solved by moving (marching) forward in space through the computational grid and time, the so-called leapfrog scheme. The FDTD implementation is inherently iterative and not a matrix method.

Historically, the discovery of the Yee cell [91] was what made the FDTD method later become a practical CEM tool [22], [23], [24], [92], [93] currently in a wide use by AP modelers and designers, as implemented in some of the most popular commercial codes and in a large number of "in-house" code realizations. Being made as early as in 1966, it was the first major advance in CEM for AP. However, the method was more the one of theoretical curiosity than a practical solution long after its inception. Early pioneers in the development of the FDTD method as a practical AP tool in the United States were Taflove, Holland, and Kunz. In Europe, Weiland independently developed a twin approach dubbed the finite integration technique (FIT) [92], [93], [94].

The early 1980s witnessed a surge in the development of ABCs that allowed for open-problem FDTD and FEM simulations, and in the early 1990s, the first FDTD applications to the modeling of antennas began to appear in the literature [92], [93], [94]. With the increase in the size of the problems being tackled, the problem of numerical dispersion error came to the forefront in the late 1980s, and a series of higher order FDTD algorithms was developed to mitigate this error using larger finite-difference stencils [92], [93]. The introduction of the PML in the mid-1990s [86], [87] provided a major improvement in the dynamic range of open-domain FDTD simulations, which under very mild computational costs could now achieve 80 dB absorption and beyond [94]. The development of unconditionally stable algorithms for the FDTD in the late 1990s and early 2000s was another major milestone since it lifted the Courant stability limit, with the time step not being bounded by the stability criterion anymore, which enabled highly refined spatial grids [93], [94].

Note that the respective FDTD equation agrees perfectly with Maxwell's first equation in integral form [2] applied to

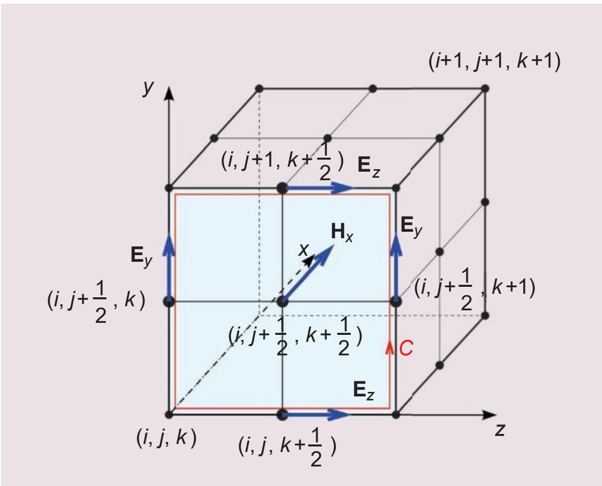

FIGURE 3. Yee lattice (cell) for a 3D FDTD technique for solving general curl Maxwell's equations [3].

contour C in Figure 3. In fact, the FIT method [30] explicitly solves Maxwell's integral equations based on such contours and the closed surface of a cell, in Figure 3.

Time-domain FEM techniques were based on a numerical discretization of the time-domain version of wave equations [93], [94], [95], [96], or, more recently, coupled curl Maxwell's equations [97], where, in both cases, marching in time similar to the process in the FDTD method was commonly used [16]. Historically, the development of the time-domain (TD)-FEM, with works by Lee, Jin, Teixeira, and collaborators, was yet another important advancement in CEM [84].

Generally, time-domain formulations, including the FDTD, FIT, TD-FEM, and TDIE approaches, enable effective modeling of time-varying and nonlinear problems and fast broadband simulations (providing broadband information in a single run) at the expense of the additional discretization—direct discretization in the time domain [3].

FAST SOLVERS, DOMAIN DECOMPOSITION METHODS, AND HYBRID APPROACHES

A main problem and grand challenge in CEM for AP applications has been the appearance of large matrices resulting from MoM/BEM and FEM discretizations of large and complex AP problems. Historically, the state of the art in terms of MoM solver performance in the 1980s was vividly summarized by Figure 2 of Miller's review article of 1988 [98], [39]. This figure showed the largest size problem that could be solved by lower–upper (LU) decomposition in 1 h on the state-of-the-art computer of that year. While fundamentally a plot of computer performance, that figure presented a rather pessimistic future for IE formulations [39]. A rather general conclusion was that the MoM would not play critical roles in solving real-life AP problems because of its unfavorable computational complexity and scaling [82]. However, the development of fast solvers shifted that trajectory in a dramatic way.

The fast multipole method (FMM), pioneered by Rokhlin [99], was an iterative fast solver that allowed for the rapid computation of long-range or "far-field" interactions between groups of unknowns in the MoM. The FMM performed fast matrix–vector multiplication, the speed of which was further increased by the multilevel FMM (MLFMM) [100], [12], with multiple levels of hierarchically defined groups of varying fineness and $O(N \log N)$ computational time complexity, where N is the number of DoFs, a dramatic acceleration relative to the $O(N^3)$ complexity of the MoM with LU decomposition. The time-domain counterpart of the MLFMM was the multilevel plane-wave time-domain (PWTD) algorithm, pioneered by Shanker and Michielssen [101]. Direct fast solvers based on H-matrices and matrix compressions provided kernel-independent algebraic methods with matrix blocks approximated by low-rank matrices, using singular value decomposition (SVD) [102], rank-revealing QR (RRQR) decomposition [103], or adaptive cross approximation (ACA) [104]. Low-rank compression schemes gained popularity through the works of Hackbusch [105] and Michielssen [106] as examples.

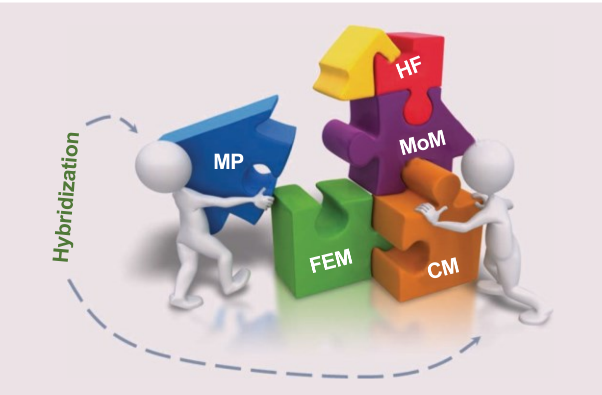

FIGURE 4. The hybridization of different CEM methodologies, e.g., MoM, FEM, and HF asymptotic techniques, together with a circuit model (CM) and multiphysics (MP) analysis, into a single code for AP/interdisciplinary simulation and design.

Historically, the discovery of fast solvers constituted by the FMM and similar accelerations [99], [12] was another major advancement in CEM for AP. Indeed, it completely changed the landscape of CEM [82]. Chew led a team of researchers who successfully implemented the FMM and MLFMM and demonstrated that they can be applied to solve AP problems with unprecedented electrical size and uncompromised accuracy [100], [12]. They proved that it was possible to drastically improve upon the trajectory of Miller's 1988 figure, and that IE formulations were not as limited by computer hardware as previously thought [39].

Another class of introduced fast methods included grid- or fast Fourier transform (FFT)-based algorithms, which relied on the block Toeplitz nature of Green's functions mapped onto regular grids. The adaptive integral method (AIM), pioneered by, among others, Bleszynski's et al. [107], and the precorrected-FFT (p-FFT) method, pioneered by Phillips [108], were among the most popular and well-documented techniques [82], [109] (Figure 1).

Great developments in the AP community also occurred in the class of domain decomposition (DD) methods, with seminal works by Lee, Jin, Vouvakis, and Peng, among others (see, e.g., [110], [111], and [112]), which allowed splitting of the original large problem into a number of smaller ones, analyzed independently. These were then stitched together by some sort of local or integral boundary conditions, tremendously reducing the computational burden, yet yielding in the process a rigorous solution of Maxwell's equations for the problem. Additionally, the DD methods were ideally suited for parallelization and the use of multiple machines and multiple processors. An alternative approach to handling large problems based on the principle of localization was the characteristic basis function method (CBFM), pioneered by Mittra [113].

Finally, it is extremely beneficial to hybridize different methodologies and approaches (Figures 1 and 4) to combine their distinctive features and advantages. Historically, a notable example were hybrid FE–BI or FEM–MoM techniques [20], [88], [89], [90]. HF asymptotic approaches were hybridized with numerically exact methods, such as the MoM or FEM, giving rise, for example, to hybrid PO–MoM solutions [114] and hybrid SBR–FEM solvers [115]. Hybrid solvers included incorporation of a circuit model (CM) and the cosimulation and co-design of integrated AP systems using field-circuit solvers as well as the integration of multiphysics (MP), such as electromagnetic, thermal, mechanic, photonic, fluidic, or elastic phenomena into tightly coupled mixed-physics modeling methods. This has become critical in some emerging AP designs because of the rise of mixed-technology systems [109]. DD methods can be used to combine completely different numerical techniques (e.g., FEM, MoM–SIE, MoM–VIE, and HF methods) in the same complex system. Indeed, the future of CEM will require and facilitate hybridizations of all approaches at once, as indicated in Figure 4.

PARALLELIZATION AND HIGH-PERFORMANCE COMPUTING

AP CEM researchers and practitioners have taken full advantage of available high-performance computing (HPC) platforms over time. Indeed, computing hardware and software infrastructure has provided, especially more recently, unparalleled opportunities, with the associated challenges, for the rapid growth of CEM capabilities and modeling technologies needed and desired by AP application researchers and practitioners. However, this has evolved in close synergy with the electromagnetic formulations and numerical foundations of CEM methods and algorithms (Figure 1). As a community, we have explored and devised innovative ways to maximally harness the advantages of the growing opportunities of the computing hardware and technology.

Historically, the most widely used parallel programming paradigms were open multiprocessing (OpenMP) and the message passing interface (MPI) [3]. For example, these protocols enabled truly unprecedented MLFMM scattering simulations with billions of unknowns [116], [117].

Now we can take a look back at Miller's 1988 figure [98] and note that the advancement over about 30 years (from the early 1950s to the mid-1980s) in terms of the largest problem that could be solved within 1 h was from $N = 100$ unknowns to about 6,000 unknowns (the maximum dimension of a scatterer being a few wavelengths), so an increase of about 60 times. The following ~30 years, from the mid-1980s to the late 2010s, however, saw the rise all the way to, as an example, $N = 10{,}666{,}680{,}960$ unknowns (a maximum dimension of 11,503*m*) within 7.5 h [117], so about a 1.7 million times improvement (note that with the $O(N^3)$ complexity in Miller's figure, 1 h versus 7.5 h computation time does not make a big difference in N). Indeed, what was once a rather pessimistic trajectory for the first half of the CEM history then became a blistering growth during the second half of the 75 years of CEM.

Enormous advances in graphics processing unit (GPU) technology, originally for graphics applications, have made these hardware accelerators widely useful outside the computer graphics world, including in CEM for AP [3]. For example, Table 1 illustrates how a speedup of an SBR method for RT propagation modeling, including electric field computation, as dramatic as four orders of magnitude faster could be achieved with massively parallel GPU optimization on an Nvidia OptiX Prime CUDA GPU parallelization framework, relative to the implementation on a central processing unit (CPU) in serial [118].

Currently and for the future, of greatest importance for HPC in CEM and AP is the hybridization of different parallelization paradigms and strategies, such as the MPI, OpenMP, and CUDA frameworks, to fully exploit the available and emerging hardware capabilities.

ERROR ESTIMATION, ADAPTIVE REFINEMENT, UNCERTAINTY QUANTIFICATION, OPTIMIZATION, AND DESIGN

Historically, it has always been clear that the task of proving the theoretical convergence of CEM solutions for AP is not trivial [39]. During the past several decades, a steady effort was directed toward a better understanding of theoretical error rates, and a knowledge base was generated (see, e.g., [119], [120], and [121]) that provided a foundation for a more rigorous scientific understanding of expected convergence rates, performance of error levels, etc. Another area where historically progress was slow is in the development of inexpensive error estimators for use with adaptive model refinement (AMR) algorithms. It is the convergence of these areas that could enable the realization of "dialable" accuracy in numerical solutions [39].

Indeed, of paramount importance is a synergistic combination of error estimation and control [119], [120], [121], meshing [122], AMR [123], and uncertainty quantification (UQ) [124] for CEM (Figure 1); see [125], [126], [127], [128], [129], and Figure 5, selected only as illustrative examples. A goal is to make CEM methodologies and techniques as accurate, efficient, accessible, usable, reliable, and robust as possible and thus maximally beneficial to a broad audience of AP researchers, practitioners, and students, with no need for expert user intervention.

Through effective and rigorous UQ, the quality of analyses and designs may be improved drastically, in terms of both effectiveness and reliability, since material parameters of an antenna/scatterer, object shape/size, mutual positions/orientations, etc., are all uncertain input parameters, which may be known only within a specific tolerance [124], [128], [129]. UQ in CEM involved studies of how the input parameter uncertainty resulted in uncertainty in the generated electric or magnetic field, for example, in assessing the sensitivity of the field to the uncertain input parameter [see, e.g., Figure 5(e) and (f)].

TABLE 1. COMPUTATION TIMES AND SPEEDUP OF A MASSIVELY GPU-PARALLELIZED SBR RT PROPAGATION ANALYSIS VERSUS SERIAL CPU IMPLEMENTATION [118].

Number of Rays	Serial Computation Time (s)	Parallel Computation Time (s)	Speedup (×)
1,600,000	86,214	6.5	13,263

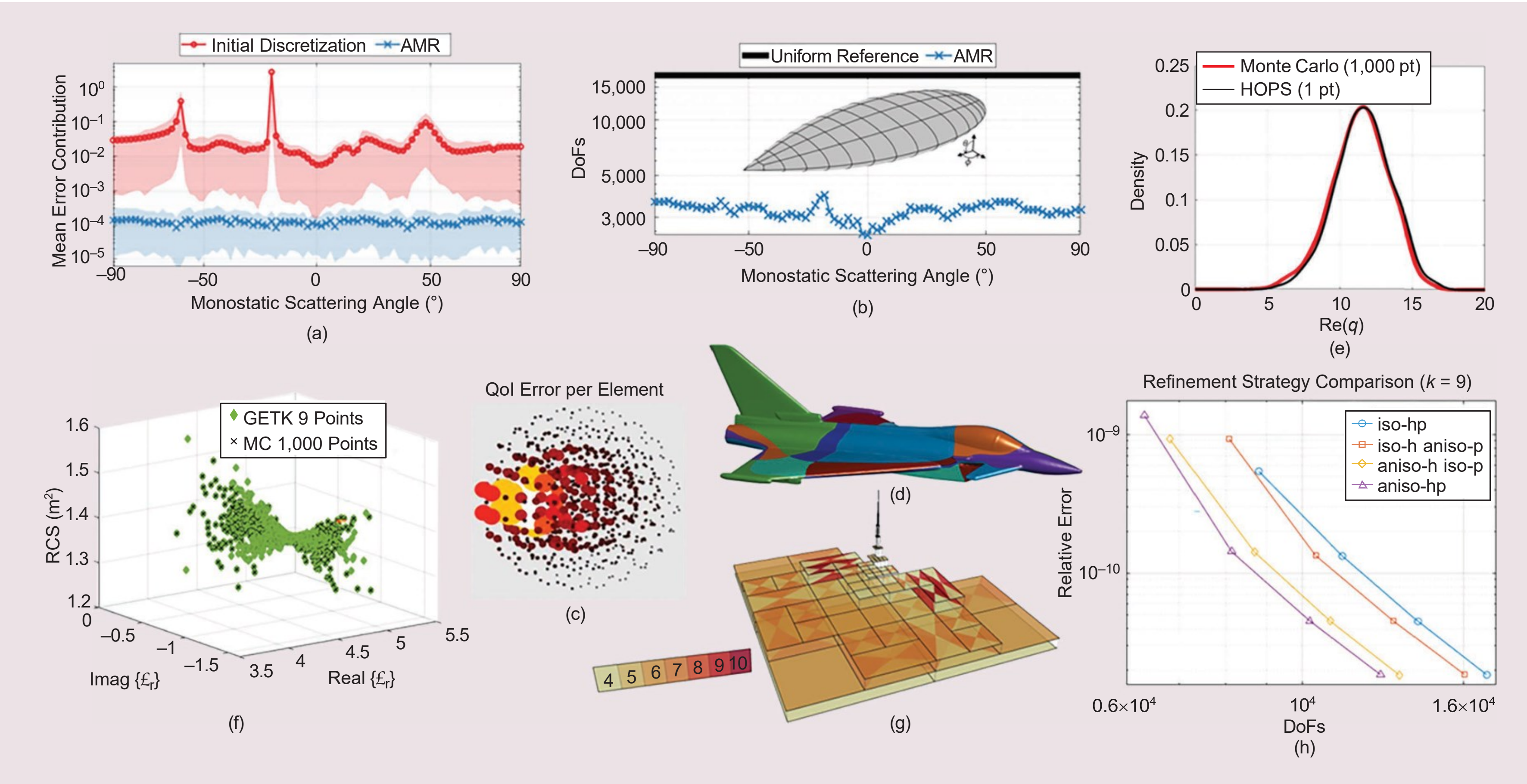

FIGURE 5. (a) Dramatic improvement in the monostatic RCS error magnitude and uniformity with adjoint-based adaptive error control and AMR for MoM-SIE analysis of a NASA almond [126]. (b) Drastic reduction of the number of DoFs with AMR [126]. (c) Adjoint-based per finite-element QoI error contribution estimates for FEM-PML analysis of a dielectric sphere scatterer enabling targeted AMR and dramatically improved resource allocation [125]. (d) Perfect curvature/detail modeling with hyperlarge hypercurved patches using a meshing method based on discrete surface Ricci flow with iterative adaptive refinement: fighter jet model featuring as few as 32 quadrilateral elements as in Figure 2(e) but of 64th order [122]. (e) Adjoint-based UQ with HOPS and FEM-PML: several orders of magnitude improvement in computation time with respect to a traditional MC simulation for RCS versus material uncertainty [128]. (f) Surrogate RCS reconstruction for 2D material uncertainty by the GETK UQ method: almost an exact match with MC solution even with an extremely small number of training input points [129]. (g) and (h) Multilevel fully anisotropic hp-refinement methodology for CEM based on a refinement-by-superposition approach, which uniquely drives perfect exponential convergence with respect to the number of DoFs even in the case of singular solution behaviors [127]. QoI: quantity of interest; RCS: radar cross section; GETK: gradient-enhanced Taylor kriging; HOPS: higher order parameter sampling; MC: Monte Carlo.

Again, it is key to have an accelerated design integrating automated adaptive error estimation and control, AMR, and UQ (see, e.g., Figures 1 and 5).

The AP CEM community made seminal advancements in developing and using electromagnetic optimizers for AP synthesis and design (Figure 1); see, e.g., [130]. Some of the most notable and impactful approaches included gradient-based optimization as well nature-inspired optimizers, such as genetic algorithm (GA) and particle swarm optimization (PSO), introduced in AP CEM design by Rahmat-Samii. The most recent advancements in CEM analysis and design invoked artificial intelligence (AI), machine learning, and data-enabled approaches. For example, various techniques in AI were applied for electromagnetic optimization and design; see, e.g., [131].

COMMERCIAL CEM CODES FOR AP

Perhaps the most significant change in the CEM for AP overall and its main signature for the start of the 21st century was the establishment of commercial CEM codes as essential and extremely widely used software tools for AP modeling, analysis, and design. Dramatically fewer AP research and development groups and units at academic, governmental, and industrial/commercial institutions and organizations are developing "in-house" CEM methods, algorithms, and codes. Most researchers, practitioners, and students are relying instead on commercial solvers for CEM simulation and design. For example, graduate students typically no longer write their own codes for their AP research.

In what follows, we list some of the most popular commercial CEM software tools used for AP simulations, highlight some of their numerical components, and outline some of their apparent capabilities and features relevant for AP modelers and designers [3]. Many other commercially available CEM codes are not listed, and many of the known characteristics of the codes listed are not included for brevity.

Ansys High Frequency Structure Simulator (HFSS) primarily employs the FEM to solve a wide range of AP problems in both the frequency and time domains. It uses rectilinear or curvilinear tetrahedral finite elements, Figure 2(f) and (g), with field expansions of first, second, and mixed order. The code carries out automatic and adaptive mesh refinement with these elements. Ansys HFSS is arguably the most widely used CEM tool for AP analysis and design.

The Computer Simulation Technology Microwave Studio (CST MWS) offers widely used CEM tools based on the FIT as well as the MoM, MLFMM, FEM, and RT, respectively. The software uses automatic meshing based on hexahedra or tetrahedra [Figure 2(f)–(h)] and mesh adaptation techniques implemented in cooperation with Dassault Systèmes. Antenna Magus is a software tool tailored specifically for antenna design and modeling.

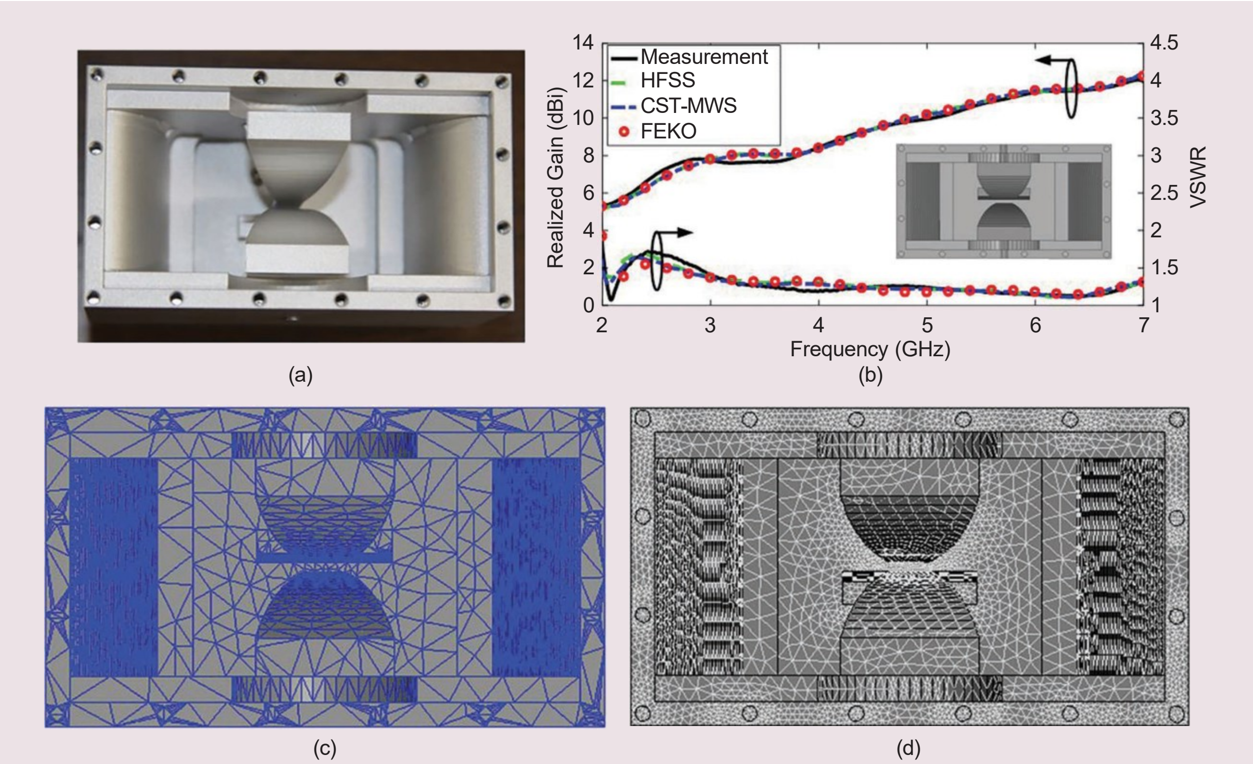

FIGURE 6. Design of a wideband compact double-ridged transverse electromagnetic (TEM) horn antenna embedded in a shaped rectangular cavity [3], [132]. (a) Photograph of the structure. (b) Comparison of numerical results obtained by ANSYS-HFSS (FEM, tetrahedral mesh, mixed-order basis functions), CST MWS (FIT, hexahedral mesh, time-domain solver), and FEKO (MoM-SIE, triangular mesh, RWG bases), and experimental results at 7 GHz. (c) HFSS mesh. (d) FEKO mesh. (Courtesy of Mohamed Elmansouri.)

FEKO by Altair HyperWorks is one of the most popular commercially available CEM tools in the AP community. The primary computational method employed is the MoM with triangular patches [Figure 2(a) and (b)] and RWG basis functions, optionally with the MLFMM acceleration. However, FEKO provides a whole suite of codes, such as FEM, FDTD, PO, and UTD solutions, which can be used either independently or in hybrid arrangements.

WIPL-D Pro is a powerful design program capable of modeling AP structures comprising both metal (WIres and PLates) and Dielectric materials. It is becoming increasingly popular among antenna engineers for its economic and accurate solutions. WIPL-D implements the MoM and uses bilinear quadrilateral patches [Figure 2(d)] to model both metallic and dielectric surfaces. Currents are approximated by higher order polynomial vector basis functions.

TICRA software combines the PO, GTD, and RT techniques with the MoM using higher order quadrilateral patches [Figure 2(e)] and polynomial vector bases and is especially well known for the analysis and design of reflector antenna systems.

Remcom offers an FDTD (Figure 3) computer program for AP modeling and design, XFdtd, which comes with a well-developed GUI. It also features the Wireless InSite, a powerful RT tool for AP modeling in outdoor and indoor environments.

SEMCAD X is a 3D full-wave CEM solver based on the FDTD method (Figure 3). It is often utilized for RF modeling of biological tissues in AP applications.

COMSOL Multiphysics performs CEM analysis based on the FEM [Figure 2(f)], and its primary strength is seamless coupling of various MP effects into the CEM AP analysis.

CEM APPLICATIONS

This section shows some illustrative AP applications of the CEM methodologies and numerical discretization techniques presented in this article.

Figures 6 and 7 show results obtained by the first three commercial CEM codes discussed in the previous section—for two real-world antenna designs and applications [3]. The excellent agreements in cross-validations of CEM results observed in the figures are very relevant, given that the solution approaches used are completely different, both conceptually (in terms of the CEM methodology, meshed domain, and discretized quantity) and numerically (with respect to the geometrical elements, basis functions, and

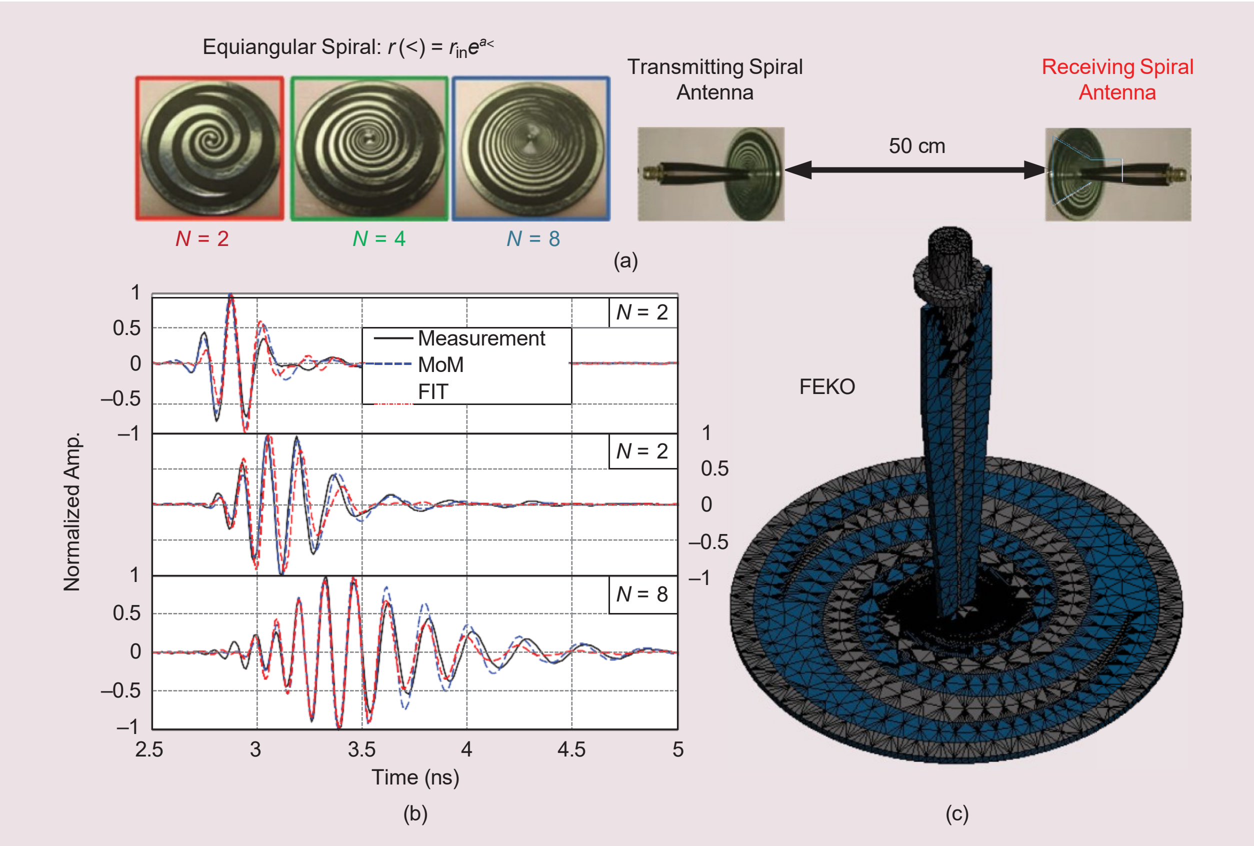

FIGURE 7. Mitigation of pulse distortion in spiral antenna-based ultra-wideband communication systems [3], [133]. (a) Antenna geometries. (b) Comparison of numerical results obtained by CST MWS (FIT, hexahedral mesh, time-domain solver) and FEKO (MoM-SIE, triangular mesh, RWG bases, indirect transient analysis using the discrete Fourier transform and its inverse), and experimental results for normalized received pulses of a two-arm planar equiangular spiral antenna-based communication link. (c) FEKO mesh. (Courtesy of Mohamed Elmansouri.)

solution techniques). For instance, the MoM–SIE (FEKO) method is a surface modeling technique that solves boundary IEs for currents, while the FEM (Ansys HFSS) and FIT (CST MWS) methods are volumetric modeling techniques that solve PDEs for fields. Discretizations are as different as triangular/low-order (FEKO), hexahedral/low-order (CST MWS), and tetrahedral/mixed-order (Ansys-HFSS) types, and include both solutions in the frequency domain (Ansys HFSS and FEKO) and time-domain techniques (CST MWS).

Ultimately, the confirmation and demonstration of every design and validation of every numerical method and approach is an agreement with experiment [3], and we see excellent agreements of the CEM results with the antenna measurements in Figures 6 and 7.

Figure 8 depicts 14 comprehensive CEM-AP applications with results provided by AP-S CEM researchers, which were obtained in their CEM and AP projects within the past 20 years, as illustration of the versatility and effectiveness of various CEM approaches. The applications and problems range from in-depth characterizations and evaluations to optimized designs, and from antennas, propagation, and scattering to radar meteorology and medical diagnostics.

CONCLUSIONS

This article has presented an overview of 75 years of research in computational electromagnetics within the IEEE Antennas and Propagation Society and the AP community at large, on the occasion of the 75th anniversary of AP-S, where both the CEM and AP-S have similar and interwoven histories of 75 years, half of the history of Maxwell's equations.

The article has discussed the discoveries, developments, and implementations of principal IE and PDE CEM methodologies, HF asymptotic techniques, and general MoM, FEM,

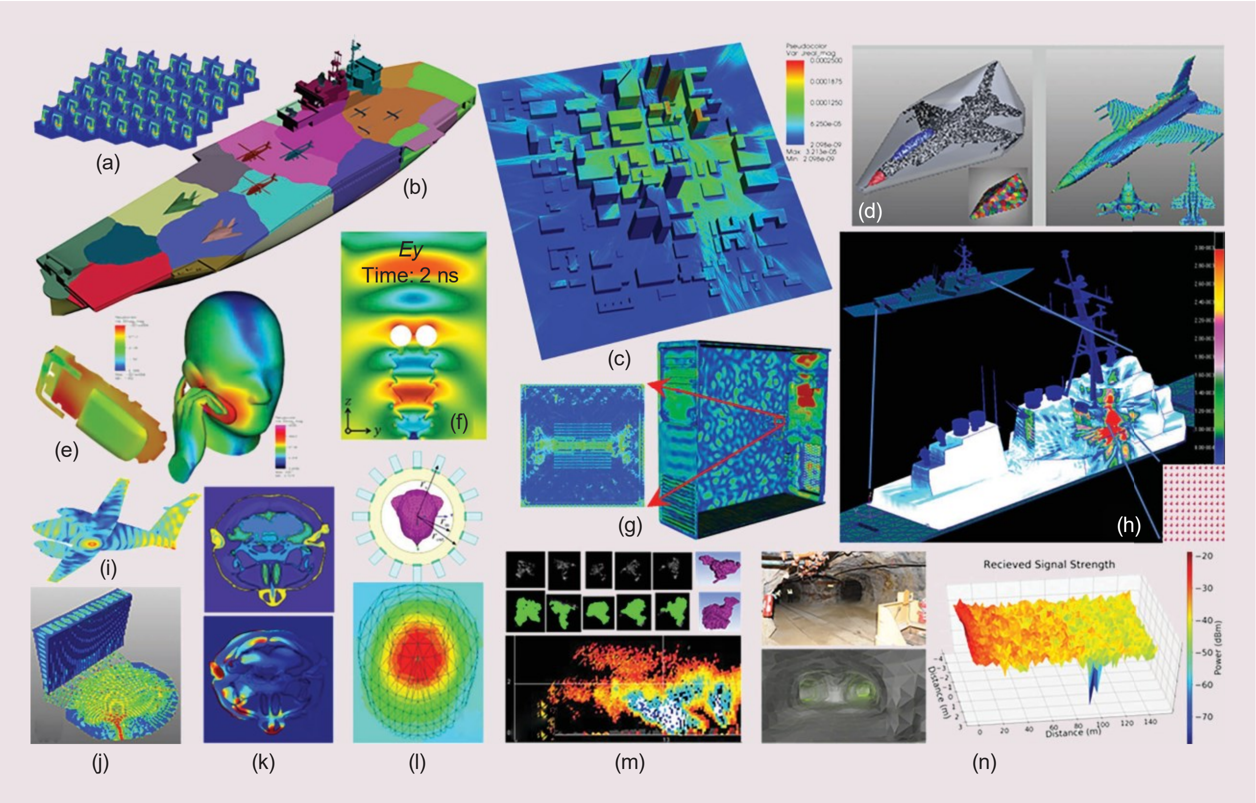

FIGURE 8. Collage of 14 comprehensive CEM-AP applications [31], [134], with the analyses and designs conducted using FE-BI, FEM-ABC, MoM-SIE, FDTD, DD, MLFMM, RT-SBR, and MoM/FEM-PO methods, and both frequency-domain and time-domain computations. The collage presents CEM characterizations of (a) large antenna array with a radome using frequency-selective surfaces [135]; (b) aircraft carrier with airplanes, helicopters, unmanned aerial vehicles, and antennas at 1.2 GHz; (c) 5G MIMO wireless channel in a city block; (d) scattering from an F16 aircraft at 1 GHz; (e) electric field distribution within a human head, hand, and cell phone, including the antenna, PCB, and battery, at 900 MHz [136]; (f) MP CEM-plasma interaction with air breakdown and plasma filamentary array formation near two metallic cylinders with an air gap, excited by high-power microwaves [137]; (g) wave propagation effects inside a desktop computer at 10 GHz; (h) antenna-platform interaction on a ship at 1 GHz; (i) transient currents on an aircraft with a mounted Vivaldi antenna at 270 MHz [138]; (j) horn antenna array fed with a Rotman lens; (k) specific absorption rate in a human head model at 835 MHz [139]; (l) RF coil/antenna array for next-generation, ultrahigh field magnetic resonance imaging [140]; (m) precipitation scattering [141]; and (n) wireless signal propagation in underground mine tunnels [142]. MIMO: multiple-input, multiple-output; PCB: printed circuit board. [(a) and (h) courtesy of John Volakis and Kubilay Sertel; (b), (c), and (g) courtesy of Zhen Peng; (d), (e), and (j) courtesy of Jin-Fa Lee; (f) and (i) courtesy of Jian-Ming Jin and Su Yan; (k) courtesy of Cynthia Furse; (l), (m), and (n) courtesy of Branislav Notaroš.]

and FDTD numerical discretization procedures for AP. These have included parametric triangular and quadrilateral patches and tetrahedral and hexahedral volume elements and characteristic types of current/field vector basis functions, such as RWG functions, Whitney forms, and polynomial vector bases as well as the Yee lattice. The article has also reviewed the history, state of the art, and future outlook of various important components of CEM modeling and computation, such as fast solvers, DD algorithms, and hybrid methods; parallel programming paradigms and HPC; and error estimation and control, meshing, AMR, and UQ for CEM-AP. Finally, we have overviewed some of the most popular commercial CEM software tools used for AP simulations and design.

Most researchers, practitioners, and students are relying instead on commercial solvers for CEM simulation and design.

Illustrative numerical results obtained by some of the leading commercial CEM codes have demonstrated excellent agreements, accuracy, and efficiency in comparisons of the solutions by multiple tools and against antenna measurements for diverse real-world antenna designs and applications [3]. The presented CEM results for several comprehensive AP applications have signified the power, versatility, and reliability of CEM and its impact on AP research and practice.

The article has shown a great diversity of formulations, elements, bases, and solution techniques within CEM for AP. Although all of these components as well as their many working combinations resulting in CEM codes for AP analysis and design seem to be completely different, they all have a lot in common, as has also been shown in the article. However, they all show some advantages and deficiencies. The choice of the " best" method depends on the particular problem that needs to be solved. Therefore, all presented and/or referenced CEM formulations, elements, bases, solutions, and implementations as well as those that could not be mentioned are important and constitute a body of knowledge in this area.

Historically, however, there were periods of time during the past 75 years of history of AP-S and history of CEM when certain CEM approaches seemed to dominate the AP applications. The first few decades saw the dominant practical use of HF methods. The 1980s were dominated by the MoM/BEM, along with HF approaches. The early 1990s were the beginning of practical AP applications of the FEM and the FDTD method, with the practical debut of fast methods in the late 1990s. By the early 2000s, it became obvious that commercial CEM software tools would play an incredibly important role in AP research and practice, and would essentially dominate AP simulations and design.

CEM for AP of the 21st century has been constituted by a true expansion and/or renaissance of all methodologies and techniques, with some highlights being commercial codes; fast solvers; hybridizations; DD; CEM– CM–MP cosimulation, co-design, and mixed technology; parallelization and HPC; higher order modeling; AMR; UQ; and design optimization. Overall, we have witnessed dramatic enhancements in the accuracy, efficiency, versatility, reliability, stability, and robustness of AP modeling, analysis, and design, for an unprecedented variety of AP/interdisciplinary applications.

The article has demonstrated the phenomenal progress that CEM and AP researchers and developers have made over the past 75 years. Of course, it was impossible to accurately and fully represent the tremendous successes and accomplishments of an era in one article. Yet, it is hoped that the article is still representative of the CEM history as well as the state of the CEM art for AP, which is truly outstanding. However, progress is still being made, and many innovative CEM approaches are yet to come in new computational methodologies, hybridization strategies, discretization techniques, and application-driven implementations as well as new ways to better harness the ever-growing HPC power. The next 75 years of CEM and AP-S are promising to be equally rich, fascinating, and intense! Are those four, now 150-year-old, Maxwell's equations ever going to be solved?

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