Construction of a Dirichlet form on metric measure spaces of controlled geometry

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ABSTRACT. Given a compact doubling metric measure space X that supports a 2-Poincaré inequality, we construct a Dirichlet form on $N^{1,2}(X)$ that is comparable to the upper gradient energy form on $N^{1,2}(X)$. Our approach is based on the approximation of X by a family of graphs that is doubling and supports a 2-Poincaré inequality (see [18]). We construct a bilinear form on $N^{1,2}(X)$ using the Dirichlet form on the graph. We show that the Γ-limit $\mathcal{E}$ of this family of bilinear forms (by taking a subsequence) exists and that $\mathcal{E}$ is a Dirichlet form on X. Properties of $\mathcal{E}$ are established. Moreover, we prove that $\mathcal{E}$ has the property of matching boundary values on a domain $\Omega \subseteq X$. This construction makes it possible to approximate harmonic functions (with respect to the Dirichlet form $\mathcal{E}$) on a domain in X with a prescribed Lipschitz boundary data via a numerical scheme dictated by the approximating Dirichlet forms, which are discrete objects.

1. Introduction

The idea of approximating a compact doubling metric space by graphs now is well-established, with the work of Christ [13] containing a prototype of such an idea. This idea blossomed into the construction of hyperbolic fillings of compact doubling spaces to obtain a Gromov hyperbolic space whose boundary is (homeomorphic to) the given space (see for instance [7, 8, 10, 18]). Under the additional condition that the compact doubling metric space is equipped with a doubling measure that supports a Poincaré inequality, much can be said about the approximating graphs, see [18]. In the non-smooth setting of metric measure spaces equipped with a doubling measure supporting a Poincaré inequality, it is now known

that there is a rich theory of harmonic functions (see for instance [5, 6, 29]). What is missing in the current literature is a construction of a Dirichlet form which is compatible with the upper gradient structure on the underlying metric measure space and, in addition, a way of approximating the Dirichlet form via its discrete couterparts defined on graphs approximating the metric space so that solutions to a Dirichlet boundary value problem for functions that are harmonic with respect to the limit Dirichlet form can be approximated by solutions to Dirichlet boundary value problem for the approximating sequence of graph-Dirichlet forms. The goal of this present note is to demonstrate one way of constructing such an approximation.

The notion of Dirichlet forms has its roots in the work of Beurling and Deny [3, 4]. The theory of Dirichlet forms has been fleshed out in the textbook [17], and an excellent exposition of the notions of Γ-convergence and Mosco-convergence of Dirichlet forms can be found for instance in [9, 15, 25, 26, 27]. Of these expositions, the works of [9, 15, 27] consider sequences of Dirichlet forms on Euclidean spaces, while the works [25, 26] consider sequences of Dirichlet forms, each defined on a (potentially) different metric measure space, converging in a sense akin to measured Gromov-Hausdorff convergence (see [12] for more on Gromov-Hausdorff convergence) of Dirichlet forms.

The paper [25] discusses convergence of graphs to graphs and the convergence of corresponding Dirichlet forms, while [26] studies Gromov-Haudorff convergence of function classes and Dirichlet forms. The papers [24, 1] construct a Dirichlet form that is equivalent to the Korevaar-Schoen energy, but their approximations are not discrete and hence do not lend themselves to numerical schemes. Neither of them work for us as we consider graphs that approximate a potentially non-graph metric measure space, and we do not know Gromov-Hausdorff convergence of corresponding Sobolev classes. Hence we have instead used graphs X_r, $r > 0$, to construct a family of Dirichlet forms $\mathcal{E}_r$ on $N^{1,2}(X)$, see (3.2). We then show that this family of Dirichlet forms has a subsequence that Γ-converges to a Dirichlet form on X so that the domain of this limit form coincides with the Newton-Sobolev class $N^{1,2}(X)$, but with Dirichlet energy that is comparable to the upper gradient energy therein. The definition of Γ-convergence is given in Definition 3.15 below.

THEOREM 1.1. *Let μ be a doubling measure on the complete metric space (X, d) so that (X, d, μ) supports a 2-Poincaré inequality. Then for forms $\mathcal{E}_r$ defined by (3.2), the following is true for every $u \in N^{1,2}(X)$ and its minimal 2-weak upper gradient $g_u \in L^2(X)$:*

(1) *There is a constant $C > 0$ such that*
$$\frac{1}{C} \sup_{r>0} \mathcal{E}_r[u] \leq \int_X g_u^2 \, d\mu \leq C \liminf_{\varepsilon \to 0^+} \mathcal{E}_\varepsilon[u].$$

(2) *There is a sequence $\{r_k\}_{k \in \mathbb{N}}$ with $\lim_k r_k = 0$, such that $\mathcal{E}_{r_k}$ Γ-converges to a Dirichlet form $\mathcal{E}$ on X, defined on $N^{1,2}(X)$. Moreover, for each $u \in N^{1,2}(X)$ we have that*
$$\frac{1}{C} \mathcal{E}[u] \leq \int_X g_u^2 \, d\mu \leq C \, \mathcal{E}[u].$$

(3) *The form $\mathcal{E}$ is a positive-definite symmetric Markovian closed bilinear form on $N^{1,2}(X)$.*

This above theorem enables us to have a way of numerically computing Dirichlet energies on a compact doubling metric measure space supporting a 2-Poincaré inequality, and in so-doing, we improve on the goal set in [16]. In this goal, we rely on the type of Poincaré inequality on graphs as found in [18]. This forms the first part of the present note. While our construction of approximating Dirichlet forms is based on the work of [18], there are alternate constructions based on the notion of Korevaar-Schoen energy, see [1]. As the construction of [1] is not immediately related to the notion of Poincaré inequalities (via averages of functions on balls), and as numerical approximations are not relevant to the construction found in [1], we do not consider their construction in the present paper.

The primary reason for considering this specific construction of a Dirichlet form $\mathcal{E}$ on X is to have a notion of harmonic functions on X (related to this form) that is easily approximated by graph-harmonic functions. Since X is compact, each approximating graph for X is a finite graph, and then given boundary data for each of these graphs, it is an exercise in computational linear algebra together with the mean value property for harmonic functions on graphs [22, 23] to obtain harmonic functions on an approximating domain in the graph by considering the adjacency matrix of the domain in the graph. Here, given the boundary data $f \in N^{1,2}(X)$ for the Dirichlet problem for $\mathcal{E}$ on a domain Ω in X, we may consider a discrete mollification of f as the boundary data for the Dirichlet form $\mathcal{E}_r$ with respect to the approximating domain $\Omega_r \subset X$. Observe that given a function $v \in N^{1,2}(X)$, the Γ-convergence gives us some sequence $v_k \in N^{1,2}(X)$ with $v_k \to u$ in $N^{1,2}(X)$ such that $\mathcal{E}[v] = \lim_k \mathcal{E}_{r_k}[v_k]$. However, in solving the Dirichlet boundary value problem, we would need the approximating functions v_k to have the same boundary values as u. We establish this in the second main theorem of this paper.

THEOREM 1.2. *Let $u \in N^{1,2}(X)$ and Ω be a domain. If there exist $v_j \to u$ in $N^{1,2}(X)$ and $\mathcal{E}[u] = \lim_{j \to \infty} \mathcal{E}_{r_j}[v_j]$, then there exists a sequence $u_j \in N^{1,2}(X)$ such that*

$$\mathcal{E}[u] = \lim_{j \to \infty} \mathcal{E}_{r_j}[u_j]$$

and

$$u - u_j \in N_0^{1,2}(\Omega).$$

The rest of the paper is organized as follows. The next section, Section 2, will provide the background information regarding Sobolev spaces $N^{1,2}(X)$ and the related notions. Towards the beginning of this section we also provide the description of approximating graphs X_r of the metric space X. In Section 3 we give the construction of the approximating forms $\mathcal{E}_r$ on X. Sections 3 and 4 contain the proofs of Theorem 1.1 and Theorem 1.2.

Theorem 1.2 will be proved near the end of Section 3. Theorem 1.1 will be proved in stages. The first claim of Theorem 1.1 will follow from Theorem 3.13, and the existence of the Γ-limit will be established in Proposition 3.16. The last part of Theorem 1.1, listing the properties of the Γ-limit form $\mathcal{E}$, will be established via the lemmas in Section 4. Since the approximating forms $\mathcal{E}_r$ are not known to be Markovian on $N^{1,2}(X)$, the fact that $\mathcal{E}$ is Markovian does not follow directly from the theory of Γ-limits; instead, we establish this property in Proposition 4.8.

2. Background

In this note (X, d, μ) will denote a complete geodesic metric space (X, d), equipped with a Radon measure μ supported on X, such that μ is *doubling*; that is, there is a constant $C_d \geq 1$ such that for all $x \in X$ and $r > 0$ we have

$$\mu(B(x, 2r)) \leq C_d \, \mu(B(x, r)).$$

Here, $B(x, r) := \{y \in X \, : \, d(x, y) < r\}$ is the open metric ball, and $\overline{B}(x, r) := \{y \in X \, : \, d(x, y) \leq r\}$ is the closed ball; note that the topological closure of $B(x, r)$ is contained in $\overline{B}(x, r)$, but need not equal.

DEFINITION 2.1. With $D \subset L^2(X)$ a vector subspace, a positive-definite symmetric bilinear form $\mathcal{E} : D \times D \to \mathbb{R}$ is said to be a Dirichlet form on D if

(1) D is dense in $L^2(X)$ in the norm of $L^2(X)$,
(2) with $\mathcal{E}_1(u, v) := \mathcal{E}(u, v) + \int_X u \, v \, d\mu$ for $u, v \in D$, $(D, \mathcal{E}_1)$ is a Hilbert space (in [17] this property identified as the closedness property of $\mathcal{E}$, see Definition 4.3 below),
(3) $\mathcal{E}(u_T, u_T) \leq \mathcal{E}(u, u)$ whenever $u \in D$ and $u_T := \max\{0, \min\{u, 1\}\}$ (in [17] this property is identified as the Markovian property).

Associated to the Dirichlet form there is an energy form, also denoted $\mathcal{E}$, with $\mathcal{E}[u] := \mathcal{E}(u, u)$ for each $u \in D$.

While [17] sets a Dirichlet form to be $\mathcal{E} : L^2(X) \times L^2(X) \to [-\infty, \infty]$ and then sets D to be the collection of all functions $u \in L^2(X)$ for which $\mathcal{E}(u, u) < \infty$, this is not necessary and recent papers on Dirichlet forms do not require $\mathcal{E}$ to act on all of $L^2(X)$, see for instance [25, Section 5], [11], or [30, page 7, Section 2C].

Since the doubling measure is supported on X, it follows that (X, d) is itself a *doubling metric space*; that is, there is some positive integer N_d such that for each $x \in X$ and $r > 0$, there are at most N_d number of points in $B(x, r)$ that are separated by a distance at least $r/2$ from each other; that is, whenever $A \subset B(x, r)$ such that for $x, y \in A$ we have $d(x, y) \geq r/2$ or $x = y$, we must have that A has at most N_d number of elements. It follows that for each $r > 0$ we can find a set $X_r \subset X$ such that

(a) $d(x_1, x_2) \geq r$ whenever $x_1, x_2 \in X_r$ with $x_1 \neq x_2$,
(b) $X = \bigcup_{\bar{x} \in X_r} B(\bar{x}, r)$.

The set X_r can be given the structure of a graph, with elements of X_r being the vertices of the graph and two vertices $\bar{x}, \bar{y} \in X_r$ neighbors if $\bar{x} \neq \bar{y}$ and $B(\bar{x}, 2r) \cap B(\bar{y}, 2r) \neq \emptyset$. We denote the neighborhood relationship by $\bar{x} \sim \bar{y}$. By gluing an interval of length r to the two neighboring vertices $\bar{x}, \bar{y}$, we obtain a metric graph, equipped with the path-metric d_r. Note that because X is a connected metric space, necessarily we must also have X_r be a connected metric space. We refer the interested reader to [20] for details regarding this construction.

Note that $\bar{x} \in X_r$ now has two identities; one as a point in the graph X_r, and one as a point in the metric space X. The following lemma compares the two metrics on X_r, namely the metrics d_r and d.

LEMMA 2.2. *Let (X, d) be a doubling geodesic metric space and (X_r, d_r) be defined as above. Then*

$$\frac{1}{2}d(\bar{x}, \bar{y}) \leq d_r(\bar{x}, \bar{y}) \leq 3d(\bar{x}, \bar{y})$$

for all $\bar{x}, \bar{y} \in X_r$.

PROOF. For $\bar{x}, \bar{y} \in X_r$ with $\bar{x} \neq \bar{y}$, we have that

$$d_r(\bar{x}, \bar{y}) := Nr,$$

where N is the smallest number for which there is a collection of points $\{\bar{x}_i\}_{i=0}^{N}$ so that $\bar{x}_0 = \bar{x}$, $\bar{x}_N = \bar{y}$ and $\bar{x}_{i-1} \sim \bar{x}_i$ for $i = 1, \ldots, N$. Such a collection is called an X_r–chain between $\bar{x}$ and $\bar{y}$. Given the shortest such chain, we see that

$$d(\bar{x}, \bar{y}) \leq \sum_{i=1}^{N} d(\bar{x}_{i-1}, \bar{x}_i) \leq \sum_{i=1}^{N} 2r = 2d_r(\bar{x}, \bar{y}).$$

Since X is a geodesic space, we also have a reverse relationship between d and d_r as well.

Let $\bar{x}, \bar{y} \in X_r$ be two distinct points, connected in X by a geodesic γ. If $d_r(\bar{x}, \bar{y}) = r$, then $r \leq d(\bar{x}, \bar{y}) \leq 4r$. If $d_r(\bar{x}, \bar{y}) \geq 2r$, then $d(\bar{x}, \bar{y}) \geq 4r$, and we can find points $z_0 := \bar{x}, z_1, \ldots, z_{N-1}, z_N := \bar{y}$ on γ with the property that

$$d(z_{N_0-1}, z_{N_0}) \leq r, \qquad d(z_{i-1}, z_i) = r, \text{ for } i = 1, \ldots, N_0 - 1,$$

and $r \leq (N_0 - 1)r \leq d(\bar{x}, \bar{y}) \leq N_0 r$ with $N_0 \geq 3$. Furthermore, each z_i lies in the ball $B(\bar{z}_i, r)$, for some $\bar{z}_i \in X_r$ for $i = 1, \cdots, N_0 - 1$, where we set $\bar{z}_0 = \bar{x}, \bar{z}_{N_0} = \bar{y}$. Then $d(\bar{z}_{i-1}, \bar{z}_i) \leq d(\bar{z}_{i-1}, z_{i-1}) + d(z_{i-1}, z_i) + d(z_i, \bar{z}_i) < 3r$, and so $d_r(\bar{z}_{i-1}, \bar{z}_i) \leq r$ (with the value either 0, if $\bar{z}_{i-1} = \bar{z}_i$, or r otherwise). Thus

$$d_r(\bar{x}, \bar{y}) \leq \sum_{i=1}^{N_0} d_r(\bar{z}_{i-1}, \bar{z}_i) \leq N_0 r \leq 3(N_0 - 1)r \leq 3d(\bar{x}, \bar{y}).$$

$\square$

The graph metric space X_r plays the role of a discrete approximation of the metric space (X, d) at scale r. By equipping X_r with a "lift" of the measure μ, given by

$$\mu_r(A) = \sum_{\bar{x} \in A} \mu(B(\bar{x}, 2r))$$

whenever $A \subset X_r$, we obtain a metric measure space (X_r, d_r, μ_r) with the measure μ_r a doubling Radon measure on X_r. For more on the approximation nature of (X_r, d_r, μ_r) we refer the reader to [18].

Given a function $u : X \to \mathbb{R}$, we say that a non-negative Borel measurable function g on X is an *upper gradient* of u if for every non-constant, compact, rectifiable curve γ in X we have

$$|u(y) - u(x)| \leq \int_{\gamma} g \, ds,$$

where x and y denote the two end points of γ. If u is a locally Lipschitz-continuous function on X, then the local Lipschitz-constant function $\mathrm{Lip}\,u$, given by

$$\mathrm{Lip}\,u(x) = \lim_{r \to 0^+} \sup_{x \neq y \in B(x,r)} \frac{|u(y) - u(x)|}{d(y,x)}$$

is an upper gradient of u.

For $1 < p < \infty$, we say that $u \in N^{1,p}(X)$ if u is measurable with $\int_X |u|^p \, d\mu < \infty$ such that u has an upper gradient $g \in L^p(X)$. We refer the reader to [21] for more on upper gradients and the Sobolev class $N^{1,p}(X)$.

In this note we will also assume that (X, d, μ) supports a p-*Poincaré inequality*, that is, there is a constant $C_P \geq 1$ such that for each ball $B \subset X$ and each function-upper gradient pair (u, g) on X we have

$$\fint_B |u - u_B| \, d\mu \leq C_P \, \mathrm{rad}(B) \left(\fint_B g^p \, d\mu \right)^{1/p}.$$

A more general version of Poincaré inequality would have the existence of a constant $\Theta \geq 1$ such that the right-hand side of the above inequality has ΘB rather than B. However, when X is a geodesic space (and a biLipschitz change in the metric does provide a geodesic space), it is possible to choose $\Theta = 1$ at the expense of increasing the value of C_P, and this is what we choose to do here. We refer the interested reader to [19, 21] for more on this. A theorem of Cheeger [12] tells us that there is a linear differential structure on such X, commensurate with a notion of Taylor approximation by Lipschitz charts on X, and a measurable inner product structure on this differential structure, so that the inner product norm of the differential of a Lipschitz function u on X is comparable to $\mathrm{Lip}\,u$. However, this proof, as given in [12], is that of existence.

In the rest of the paper we consider only the case $p = 2$. The goal of this note is to give a construction of a Dirichlet form on X, by using the ideas of [16, 18], together with a modification of the notion of Mosco convergence, by considering the natural Dirichlet forms on the approximating graphs.

REMARK 2.3. It was shown in [18] that (X, d, μ) is doubling and supports a 2-Poincaré inequality if and only if the family of approximating spaces (X_r, d_r, μ_r) also is doubling and supports a 2-Poincaré inequality, with the relevant doubling constant and the Poincaré constant independent of r. We will make use of this result in our paper.

PROPOSITION 2.4 (Gill-Lopez [18]). *Let (X, d, μ) be a doubling metric measure space and (X_r, d_r, μ_r) be defined as above. If (X, d, μ) admits a Poincare inequality with constants $C_{PI} > 0$ and $\lambda \geq 1$, then (X_r, d_r, μ_r) admits a Poincare inequality in the Holopainen-Soardi sense: there are constants $C_{HS} > 0$ and $\theta \geq 1$ independent of r such that for any $\bar{x}_0 \in X_r$, $R \geq 0$ and $\bar{u}$ defined on X_r, we have*

$$\sum_{\bar{x} \in G_R} |\bar{u}(\bar{x}) - (\bar{u})_{G_R}| \frac{\mu_r(\bar{x})}{\mu_r(G_R)} \leq C_{HS} \cdot R \left[\sum_{\bar{x} \in G_{\theta R}} \sum_{\bar{y} \sim \bar{x}} \left| \frac{\bar{u}(\bar{x}) - \bar{u}(\bar{y})}{r} \right|^p \frac{\mu_r(\bar{x})}{\mu_r(G_{\theta R})} \right]^{1/p},$$

where $G_R = G(\bar{x}_0; R) = \{\bar{x} \in X_r : d_r(\bar{x}, \bar{x}_0) \leq R\}$.

Given the graph X_r as above, the graph energy of a function $u_r : X_r \to \mathbb{R}$ is given by

$$\sum_{\bar{x} \in X_r} \sum_{\bar{y} \sim \bar{x}} \frac{|u_r(\bar{y}) - u_r(\bar{x})|^2}{r^2} \mu_r(\bar{x}). \tag{2.5}$$

We straddle three realms here; the graph energy on the graph X_r, the energy $\mathcal{E}_r$ induced on X by the graph energy (see (3.2) below), and the Γ-limit energy $\mathcal{E}$ on X. The graph energy is directly seen to be derived from a Dirichlet form, and we will see in the last section that the Γ-limit energy $\mathcal{E}$ is also derived from a Dirichlet form. The energy $\mathcal{E}_r$, however, is not known to be from a Dirichlet form, though it is derived from a positive-definite symmetric bilinear form.

3. Constructing approximating energy forms $\mathcal{E}_r$

In this section we use the structure of X_r to construct an approximating Dirichlet form $\mathcal{E}_r : N^{1,2}(X) \times N^{1,2}(X) \to [-\infty, \infty]$ on X as follows. For functions $u \in N^{1,2}(X)$ and $\bar{x} \in X_r$, we set $u_r : X_r \to \mathbb{R}$ by

$$u_r(\bar{x}) := \fint_{B(\bar{x}, r/4)} u \, d\mu, \tag{3.1}$$

and set the Dirichlet form by the formula

$$\mathcal{E}_r(u, v) := \sum_{\bar{x} \in X_r} \sum_{\bar{y} \sim \bar{x}} \frac{[u_r(\bar{y}) - u_r(\bar{x})][v_r(\bar{x}) - v_r(\bar{y})]}{r^2} \mu_r(\bar{x}) \tag{3.2}$$

whenever $u, v \in N^{1,2}(X)$. It is clear that $\mathcal{E}_r$ is a bilinear form on the Hilbert space $N^{1,2}(X)$. We set $\mathcal{E}_r[u] := \mathcal{E}_r(u, u)$.

REMARK 3.3. The form $\mathcal{E}_r$ is a bilinear energy form but is not known to satisfy the Markov property, and so should, according to the definition of [**3**, **17**], not be called a Dirichlet form. Moreover, $\mathcal{E}_r$ need not be closed as it acts only on $N^{1,2}(X)$, but it is closable in $L^2(X)$. The point here is that we can have a sequence $(u_n)_n$ of functions in $N^{1,2}(X)$ that converges to a function in $L^2(X) \setminus N^{1,2}(X)$ and so that $\mathcal{E}_r(u_n - u_m, u_n - u_m) \to 0$ as $n, m \to \infty$. However, $\mathcal{E}_r$ is induced by a bilinear energy form on the graph X_r which is indeed a Dirichlet form, and so we allow ourselves the liberty of calling $\mathcal{E}_r$ also a Dirichlet form as this does not give rise to any confusion. The Γ-limit energy form on $N^{1,2}(X)$ obtained from the forms $\mathcal{E}_r$ will indeed be a Dirichlet form, see Proposition 4.8 and Lemma 4.6 below.

We are also concerned with the natural energy form on X inherited from the upper gradient structure; for $u \in N^{1,p}(X)$ we set

$$E[u] := \inf_g \int_X g^2 \, d\mu,$$

where the infimum is over all upper gradients g of u. From the results in [**21**] we know that there is a minimal 2-weak upper gradient, g_u, associated with each $u \in N^{1,2}(X)$, such that

$$E[u] = \int_X g_u^2 \, d\mu.$$

When u is locally Lipschitz on X, it follows from [**12**] together with [**21**] that $g_u = \operatorname{Lip} u$. For functions $u \in L^2(X)$ that do not have (a modification on a set of measure zero) representative in $N^{1,2}(X)$ we set $E[u] := \infty$.

The following proposition gives us a comparison between the family $\mathcal{E}_r$ and E.

PROPOSITION 3.4. *Let X be a doubling metric measure space admitting a 2-Poincaré inequality. Then there exists $C > 0$ such that $\mathcal{E}_r[u] \leq CE[u]$ for all $r > 0$ and $u \in N^{1,2}(X)$. Moreover, whenever $\operatorname{Lip} u$ is continuous and belongs to $L^2(X, d\mu)$,we have*

$$E[u] \leq C \liminf_{r \to 0+} \mathcal{E}_r[u].$$

The constant of comparison, C, is dependent solely on the doubling constant C_d associated with μ, the metric doubling constant N_d associated with the doubling metric space (X, d), and the scaling constant θ from Proposition 2.4; in particular, it is independent of u and r.

This proposition will be proved using Lemma 3.5 below.

LEMMA 3.5. *Let $u \in N^{1,2}(X)$ be a locally Lipschitz function such that $\operatorname{Lip} u$ is a continuous function on X. Then for any $r_0 > 0$ and any $x \in X$ there exists a ball B_x, centered at x and of radius $r(B_x) \leq r_0$, such that for all $\varepsilon \leq \operatorname{rad}(B_x)/36$ we have*

$$\operatorname{Lip} u(x) \lesssim \sum_{\bar{x}, \bar{y} \in E(x, r(B_x))} \frac{|u_\varepsilon(\bar{x}) - u_\varepsilon(\bar{y})|}{r(B_x)} \frac{\mu_\varepsilon(\bar{x}) \mu_\varepsilon(\bar{y})}{\mu(B_x)^2}, \tag{3.6}$$

where we set

$$E(x, r(B_x)) = \{\bar{x} \in X_\varepsilon : B(\bar{x}; \varepsilon) \subset B_x\}.$$

PROOF. We fix $r_0 > 0$ and $x \in X$. Without loss of generality we may assume that $\operatorname{Lip} u(x) > 0$. Then by the continuity of $\operatorname{Lip} u$, there is some positive number $r_1 \leq r_0$ such that

$$\sup_{y \in B(x, r_1)} |\operatorname{Lip} u(y) - \operatorname{Lip} u(x)| \leq \frac{1}{2} \operatorname{Lip} u(x). \tag{3.7}$$

Next, by the definition of $\operatorname{Lip} u(x)$, there exists a positive number $r_2 \leq r_1/2$ and a point $y \in B(x, r_2)$ such that

$$\sup_{x' \in B(x, r_2)} \frac{|u(x') - u(x)|}{d(x, x')} \leq \frac{3}{2} \operatorname{Lip} u(x) \tag{3.8}$$

and

$$\frac{|u(y) - u(x)|}{r_2} \geq \frac{2}{3} \operatorname{Lip} u(x). \tag{3.9}$$

To see this claim, we argue as follows. By the definition of $\operatorname{Lip} u(x)$ we know that there is some $y \in B(x, r_1/2)$ such that $y \neq x$ and

$$\frac{|u(y) - u(x)|}{d(y, x)} \geq \frac{2}{3} \operatorname{Lip} u(x).$$

Setting $r_2 = d(x, y)$, we obtain (3.9). By (3.7) we know that

$$\sup_{w \in B(x, r_1)} \operatorname{Lip} u(w) \leq \frac{3}{2} \operatorname{Lip} u(x),$$

and as for each $x' \in B(x, r_2)$ we know that $B(x', r_2) \subset B(x, r_1)$, we know that u is $\frac{3}{2} \operatorname{Lip} u(x)$-Lipschitz continuous on $B(x, r_2)$, and so (3.8) follows.

We set $B_x := B(x, r_2)$ and will show that (3.6) holds for $\varepsilon \leq r_2/36$. In order to do this, consider the balls $B_1 = B(x, r_2/9)$ and $B_2 = B(y, r_2/9)$. Then by the choice of r_2 and by (3.8), for all $x' \in B_1$ we have

$$|u(x') - u(x)| \leq \frac{3}{2} \operatorname{Lip} u(x) \, d(x, x') < \frac{r_2}{6} \operatorname{Lip} u(x).$$

Therefore

$$\frac{|u(x') - u(x)|}{r_2} \leq \frac{1}{6} \operatorname{Lip} u(x).$$

Similarly, because X is a geodesic space and $\operatorname{Lip} u$ is an upper gradient of the Lipschitz function u (see [20], [21]) and by (3.7) we see that for all $y' \in B_2$,

$$|u(y') - u(y)| \leq \sup_{z \in B(y, r_2)} \operatorname{Lip} u(z) \, d(y, y') \leq \frac{3}{2} \operatorname{Lip} u(x) \, \frac{r_2}{9} = \frac{r_2}{6} \operatorname{Lip} u(x).$$

Note that $d(x, y) = r_2$, and so

$$E(x, r_2/9) := \{\bar{x} \in X_\varepsilon : B(\bar{x}, \varepsilon) \subset B_1\} \subset E(x, r_2),$$

$$E(y, r_2/9) := \{\bar{x} \in X_\varepsilon : B(\bar{x}, \varepsilon) \subset B_2\} \subset E(x, r_2).$$

For $\bar{x} \in E(x, r_2/9)$ and $\bar{y} \in E(y, r_2/9)$ we have

$$\frac{|u_\varepsilon(\bar{x}) - u(x)|}{r_2} \leq \fint_{B(\bar{x}, \varepsilon/4)} \frac{|u(x') - u(x)|}{r_2} \, d\mu(x') \leq \frac{1}{6} \operatorname{Lip} u(x)$$

and

$$\frac{|u_\varepsilon(\bar{y}) - u(y)|}{r_2} \leq \fint_{B(\bar{y}, \varepsilon/4)} \frac{|u(y') - u(y)|}{r_2} d\mu(y') \leq \frac{1}{6} \operatorname{Lip} u(x).$$

Hence by (3.9), for all $\bar{x} \in E(x, r_2/9)$ and $\bar{y} \in E(y, r_2/9)$,

$$\frac{|u_\varepsilon(\bar{x}) - u_\varepsilon(\bar{y})|}{r_2} \geq \frac{1}{3} \operatorname{Lip} u(x). \tag{3.10}$$

As the balls $B(\bar{x}, \varepsilon/2)$ are disjoint in X and the measure μ is doubling, we have

$$
\mu_\varepsilon(E(x, r_2/9)) = \sum_{\bar{x} \in E(x, r_2/9)} \mu_\varepsilon(\bar{x}) = \sum_{\bar{x} \in E(x, r_2/9)} \mu(B(\bar{x}, \varepsilon))
$$

$$
\lesssim \sum_{\bar{x} \in E(x, r_2/9)} \mu(B(\bar{x}; \varepsilon/2))
$$

$$
\leq \mu(B_1).
$$

Similarly we obtain

$$
\mu_\varepsilon(E(y, r_2/9)) \lesssim \mu(B_2).
$$

On the other hand, as $\varepsilon \leq r_2/36$ and X_ε is an ε-net, we have that

$$
\tfrac{1}{2}B_1 \subset \bigcup_{\bar{x} \in E(x, r_2/9)} B(\bar{x}; \epsilon) \quad \text{and} \quad \tfrac{1}{2}B_2 \subset \bigcup_{\bar{y} \in E(y, r_2/9)} B(\bar{y}; \epsilon).
$$

Therefore

$$
\mu(B_1) \lesssim \mu(\tfrac{1}{2}B_1) \leq \sum_{\bar{x} \in E(x, r_2/9)} \mu(B(\bar{x}; \varepsilon)) = \mu_\varepsilon(E(x, r_2/9)),
$$

$$
\mu(B_2) \lesssim \mu(\tfrac{1}{2}B_2) \leq \sum_{\bar{y} \in E(y, r_2/9)} \mu(B(\bar{y}; \varepsilon)) = \mu_\varepsilon(E(y, r_2/9)).
$$

Combining the above two sets of comparisons, we obtain

$$
\mu_\varepsilon(E(x, r_2/9)) \approx \mu(B_1) \quad \text{and} \quad \mu_\varepsilon(E(y, r_2/9)) \approx \mu(B_2),
$$

and as $r(B_1) = r(B_2) = r(B_x)/9$ with the centers x and y, of B_1 and B_2 respectively, in B_x, we obtain from the doubling property of μ that

$$
\mu_\varepsilon(E(x, r_2/9)) \approx \mu(B_x) \quad \text{and} \quad \mu_\varepsilon(E(y, r_2/9)) \approx \mu(B_x).
$$

Finally, by (3.10) we obtain

$$
\operatorname{Lip} u(x) \leq 3 \sum_{\bar{x} \in E(x, r_2/9)} \sum_{\bar{y} \in E(y, r_2/9)} \frac{|u_\varepsilon(\bar{x}) - u_\varepsilon(\bar{y})|}{r_2} \frac{\mu_\varepsilon(\bar{x})}{\mu_\varepsilon(E(x, r_2/9))} \frac{\mu_\varepsilon(\bar{y})}{\mu_\varepsilon(E(y, r_2/9))}
$$

$$
\leq 3 \sum_{\bar{x}, \bar{y} \in E(x, r(B_x))} \frac{|u_\varepsilon(\bar{x}) - u_\varepsilon(\bar{y})|}{r_2} \frac{\mu_\varepsilon(\bar{x})}{\mu_\varepsilon(E(x, r_2/9))} \frac{\mu_\varepsilon(\bar{y})}{\mu_\varepsilon(E(y, r_2/9))}
$$

$$
\lesssim \sum_{\bar{x}, \bar{y} \in E(x, r(B_x))} \frac{|u_\varepsilon(\bar{x}) - u_\varepsilon(\bar{y})|}{r_2} \frac{\mu_\varepsilon(\bar{x}) \, \mu_\varepsilon(\bar{y})}{\mu(B_x)^2}.
$$

$\square$

Now we are ready to prove Proposition 3.4.

PROOF OF PROPOSITION 3.4. Let $u \in N^{1,2}(X)$ be a locally Lipschitz continuous function on X such that its local Lipschitz constant function $\operatorname{Lip} u$ is also continuous on X. Without loss of generality, we assume that $\int_X (\operatorname{Lip} u)^2 \, d\mu$ is positive, for otherwise u is constant and all the relevant Dirichlet energies are zero. Now we introduce a large parameter

$\lambda > 1$ that we will fix at the end (indeed, it can be any choice of $\lambda > 12\theta + 1$ with θ from the Poincaré inequality given in Proposition 2.4 above). Let K be a closed ball of sufficiently large radius so that

$$\int_X (\operatorname{Lip} u)^2 \, d\mu \le 2 \int_K (\operatorname{Lip} u)^2 \, d\mu.$$

Note that as X is complete and doubling, it is proper; hence K is compact. By uniform continuity of $\operatorname{Lip} u$ on $10\lambda K$, we can find $r_1 \in (0,1)$ small enough so that $2r_1$ is less than the radius of K and

$$\sup_{\substack{x,y \in 10\lambda K \\ d(x,y) < 2r_1}} |(\operatorname{Lip} u(x))^2 - (\operatorname{Lip} u(y))^2| \le \frac{1}{\Lambda} \fint_K (\operatorname{Lip} u)^2 \, d\mu. \tag{3.11}$$

Here we have fixed a number $\Lambda > 2$, to be chosen later.

For each $x \in K$ we find a ball B_x satisfying the conclusions of Lemma 3.5 with the choice of $r_0 = r_1/(5\lambda)$. Such a family of balls is a cover of the compact set K, and so we can obtain a a finite subcover of K. Moreover, by the $5r$-covering lemma (see e.g. [20, Theorem 1.2] or [21, page 60]), we can further find a subcollection of balls $\{B_i\}_{i=1}^N$ such that

$$\{\lambda B_i\} \text{ are disjoint and } \bigcup_{i=1}^N 5\lambda B_i \supset K. \tag{3.12}$$

With this subcover, we get

$$\int_K (\operatorname{Lip} u)^2 \, d\mu \le \sum_{i=1}^N \left(\fint_{5\lambda B_i} (\operatorname{Lip} u)^2 \, d\mu \right) \cdot \mu(5\lambda B_i),$$

where the balls B_i satisfy the conclusions of Lemma 3.5. We denote the centers of these balls by x_i. Since the radii of each $5\lambda B_i$ is no more than r_1 and the center of $5\lambda B_i$ lies in K, it follows from (3.11) that

$$\fint_{5\lambda B_i} (\operatorname{Lip} u)^2 \, d\mu \le (\operatorname{Lip} u(x_i))^2 + \frac{1}{\Lambda} \fint_K (\operatorname{Lip} u)^2 \, d\mu.$$

We now obtain from the above observations that

$$\int_K (\operatorname{Lip} u)^2 \, d\mu \le \sum_{i=1}^N (\operatorname{Lip} u(x_i))^2 \cdot \mu(5\lambda B_i) + \left(\fint_K (\operatorname{Lip} u)^2 \, d\mu \right) \sum_{i=1}^N \mu(5\lambda B_i)$$

$$\lesssim \sum_{i=1}^N (\operatorname{Lip} u(x_i))^2 \cdot \mu(5\lambda B_i) + \frac{1}{\Lambda} \fint_K (\operatorname{Lip} u)^2 \, d\mu \sum_{i=1}^N \mu(B_i)$$

$$\le \sum_{i=1}^N (\operatorname{Lip} u(x_i))^2 \cdot \mu(5\lambda B_i) + \frac{\mu(2K)}{\Lambda} \fint_K (\operatorname{Lip} u)^2 \, d\mu$$

$$\le \sum_{i=1}^N (\operatorname{Lip} u(x_i))^2 \cdot \mu(5\lambda B_i) + \frac{1}{\Lambda} \int_K (\operatorname{Lip} u)^2 \, d\mu.$$

The last few steps in the above argument was arrived at by using the fact that the balls B_i are pairwise disjoint and are contained in $2K$. The comparison constant associated with the above series of inequalities, C_*, depends solely on the doubling constant of μ and the choice of λ. We choose $\Lambda \geq 2C_*$. It follows that $C_*/\Lambda \leq 1/2$, and so

$$\int_K (\operatorname{Lip} u)^2 \, d\mu \leq 2\, C_* \sum_{i=1}^N (\operatorname{Lip} u(x_i))^2 \cdot \mu(5\lambda B_i).$$

Recall that in Lemma 3.5, the parameter ε_0 depended on the radius of the ball B_x. We will indicate this dependence by denoting ε_0 by $\varepsilon_0(x)$. Therefore, choosing any $\varepsilon \leq \min_{1 \leq i \leq N} \varepsilon_0(x_i)$, we get

$$\int_K (\operatorname{Lip} u)^2 \, d\mu \lesssim \sum_{i=1}^N \left[\sum_{\bar{x},\bar{y} \in E(x_i, r(B_i))} \frac{|u_\varepsilon(\bar{x}) - u_\varepsilon(\bar{y})|}{r(B_i)} \frac{\mu_\varepsilon(\bar{x})\mu_\varepsilon(\bar{y})}{\mu(B_i)^2} \right]^2 \mu(B_i)$$

$$\lesssim \sum_{i=1}^N \left[\sum_{\bar{x} \in E(x_i, r(B_i))} \frac{|u_\varepsilon(\bar{x}) - (u_\varepsilon)_{G(x_i, r(B_i))}|}{r(B_i)} \frac{\mu_\varepsilon(\bar{x})}{\mu(B_i)} \right]^2 \mu(B_i),$$

where, as in Lemma 3.5, we have

$$\sum_{\bar{y} \in E(x_i, r(B_i))} \mu_\varepsilon(\bar{y}) \approx \mu(B_i)$$

and

$$E(x_i, r(B_i)) = \{\bar{x} \in X_\varepsilon \, : \, B(\bar{x}; \varepsilon) \subset B_i\} \subset G(x_i, r(B_i)) = \{\bar{x} \in X_\varepsilon \, : \, d_\varepsilon(x_i, \bar{x}) \leq r(B_i)\}.$$

By the doubling property of μ again, we have that $\mu_\varepsilon(G(x_i, r(B_i))) \approx \mu(B_i)$. Now we apply Proposition 2.4, to obtain

$$\int_K (\operatorname{Lip} u)^2 \, d\mu \lesssim \sum_{i=1}^N \left[\sum_{\bar{x} \in G(x_i, \theta r(B_i))} \sum_{\bar{y} \sim \bar{x}} \frac{|u_\varepsilon(\bar{x}) - u_\varepsilon(\bar{y})|^2}{\varepsilon^2} \frac{\mu_\varepsilon(\bar{x})}{\mu(B_i)} \right] \mu(B_i)$$

$$\lesssim \sum_{i=1}^N \left[\sum_{\bar{x} \in G(x_i, \theta r(B_i))} \sum_{\bar{y} \sim \bar{x}} \frac{|u_\varepsilon(\bar{x}) - u_\varepsilon(\bar{y})|^2}{\varepsilon^2} \mu_\varepsilon(\bar{x}) \right]$$

$$\lesssim \sum_{\bar{x} \in X_\varepsilon} \sum_{\bar{y} \sim \bar{x}} \left[\sum_{i=1}^N \chi_{G(x_i, \theta r(B_i))}(\bar{x}) \right] \frac{|u_\varepsilon(\bar{x}) - u_\varepsilon(\bar{y})|^2}{\varepsilon^2} \mu_\varepsilon(\bar{x}),$$

where we used Tonelli's theorem to obtain the last inequality. Since $\lambda > 12\theta + 1$ and the balls λB_i, $i = 1, \cdots, N$ are pairwise disjoint, and as for each i we have that $G(x_i, \theta r(B_i)) \subset 2\theta B_i \subset \lambda B_i$, we have that

$$\int_K (\operatorname{Lip} u)^2 \, d\mu \lesssim \sum_{\bar{x} \in X_\varepsilon} \sum_{\bar{y} \sim \bar{x}} \frac{|u_\varepsilon(\bar{x}) - u_\varepsilon(\bar{y})|^2}{\varepsilon^2} \mu_\varepsilon(\bar{x}) = \mathcal{E}_\varepsilon[u].$$

The above holds for all sufficiently small $\varepsilon > 0$, and so we have that

$$\int_X g_u^2 \, d\mu = \int_X (\operatorname{Lip} u)^2 \, d\mu \le 2 \int_K (\operatorname{Lip} u)^2 \, d\mu \lesssim \liminf_{\varepsilon \to 0^+} \mathcal{E}_\varepsilon[u],$$

which verifies the claimed latter inequality in the statement of the proposition.

We now complete the proof of the proposition by verifying the first inequality stated in the claim. To this end, let $u \in N^{1,2}(X)$. We fix $r > 0$ such that $r < \operatorname{diam}(X)/2$. Recall that for distinct $\bar{x}, \bar{y} \in X_r$ with $\bar{x} \sim \bar{y}$ we have $d(\bar{x}, \bar{y}) \in [r, 4r)$. Thus

$$B(\bar{y}; r) \subset B(\bar{x}; 5r) \subset B(\bar{y}; 9r).$$

As μ is doubling, it follows that

$$\frac{\mu(B(\bar{x}, 5r))}{\mu(B(\bar{y}, r))} \le \frac{\mu(B(\bar{y}, 9r))}{\mu(B(\bar{y}, r))} \le C_d^4 \quad \text{and} \quad \frac{\mu(B(\bar{x}, 5r))}{\mu(B(\bar{x}, r))} \le C_d^3.$$

Hence, if $\bar{x} \sim \bar{y}$, then

$$|u_r(\bar{x}) - u_r(\bar{y})| \le \fint_{B(\bar{x}; r/4)} \fint_{B(\bar{y}; r/4)} |u(x) - u(y)| d\mu(y) d\mu(x)$$

$$\lesssim \fint_{B(\bar{x}; 5r)} \fint_{B(\bar{x}; 5r)} |u(x) - u(y)| d\mu(y) d\mu(x)$$

$$\lesssim \fint_{B(\bar{x}; 5r)} |u(x) - u_{B(\bar{x}; 5r)}| d\mu(x).$$

So

$$\mathcal{E}_r[u] = \sum_{\bar{x} \in X_r} \sum_{\bar{x} \sim \bar{y}} \left| \frac{u_r(\bar{x}) - u_r(\bar{y})}{r} \right|^2 \mu_r(\bar{x})$$

$$\lesssim N_d \cdot \sum_{\bar{x} \in X_r} \left| (5r)^{-1} \fint_{B(\bar{x}; 5r)} |u(x) - u_{B(\bar{x}; 5r)}| d\mu(x) \right|^2 \mu_r(\bar{x}),$$

where N_d is the doubling constant of the metric space (X, d). Next, applying the Poincaré inequality, we get

$$\mathcal{E}_r[u] \lesssim \sum_{\bar{x} \in X_r} \fint_{B(\bar{x}; 5r)} (\operatorname{Lip} u)^2 \, d\mu \cdot \mu(B(\bar{x}; r)) \lesssim \sum_{\bar{x} \in X_r} \int_{B(\bar{x}; 5r)} (\operatorname{Lip} u)^2 \, d\mu$$

$$\le N_d^3 \int_X (\operatorname{Lip} u)^2 \, d\mu,$$

where we have used the fact that two balls $B(\bar{x}, 5r)$ and $B(\bar{x}', 5r)$ intersect with $\bar{x} \ne \bar{x}'$ if and only if $r \le d(\bar{x}, \bar{x}') < 5r$. The constants of comparison in the above inequalities all depend only on the doubling constant C_d of the measure μ, and in particular are independent of u and r. $\qquad\square$

THEOREM 3.13. *Let X be a doubling metric measure space admitting a 2-Poincaré inequality. Then there is a constant $C > 0$ such that for each $u \in N^{1,2}(X)$ we have*

$$\frac{1}{C} \sup_{r>0} \mathcal{E}_r[u] \leq E[u] = \int_X g_u^2 \, d\mu \leq C \liminf_{\varepsilon \to 0^+} \mathcal{E}_\varepsilon[u]. \tag{3.14}$$

Moreover, if u_k, $k \in \mathbb{N}$, is a sequence of functions in $N^{1,2}(X)$ with $u_k \to u$ in $N^{1,2}(X)$, then $\lim_k \mathcal{E}_r[u_k] = \mathcal{E}_r[u]$ for each $r > 0$.

PROOF. The first inequality follows immediately from Proposition 3.4, and so only the second inequality needs verification. That is the focus of this proof.

Let $u \in N^{1,2}(X)$, and $(u_k)_k$ be a sequence of compactly supported Lipschitz functions on X with compactly supported Lipschitz upper gradients, such that $u_k \to u$ in $N^{1,p}(X)$; see [12, Theorem 5.3] (note that in [12], the continuous upper gradient $\hat{v}_{k,j(k)}$ used to construct the approximating function $u_k = \hat{f}_{k,j(k)}$, by the construction of u_k, necessarily has the property that $\mathrm{Lip}\, u_k = \hat{v}_{k,j(k)}$). By passing to a subsequence if necessary, we may assume that $\|u - u_k\|_{N^{1,2}(X)} \leq 1/k$. Then by the first part of Proposition 3.4 again, we have that

$$\mathcal{E}_r[u - u_k] \leq C \int_X g_{u-u_k}^2 \, d\mu \leq \frac{C}{k^2}.$$

Note that the constant C independent of r. Therefore, for each $\varepsilon > 0$ we can find a positive integer k_ε such that whenever $k_\varepsilon \leq k \in \mathbb{N}$, for all $r > 0$ we have $1/k < \varepsilon$ and

$$\mathcal{E}_r[u - u_k] \leq \varepsilon^2.$$

Now by the last inequality in the statement of Proposition 3.4, there is some $r_\varepsilon > 0$ such that for each $0 < r < r_\varepsilon$, we have

$$\int_X g_{u_k}^2 \, d\mu \leq C \, \mathcal{E}_r[u_k].$$

For such r and k, we have by triangle inequality,

$$\mathcal{E}_r[u]^{1/2} \geq \mathcal{E}_r[u_k]^{1/2} - \mathcal{E}_r[u - u_k]^{1/2} \geq \frac{1}{C} \left(\int_X g_{u_k}^2 \, d\mu \right)^{1/2} - \varepsilon.$$

The desired conclusion follows from letting $r \to 0^+$ and then $\varepsilon \to 0$.

We now show that if $(u_k)_k$ is a sequence in $N^{1,2}(X)$ such that $u_k \to u$ in $N^{1,2}(X)$, then for each $r > 0$ we have that $\mathcal{E}_r[u_k - u] \to 0$ as $k \to \infty$. We fix $r > 0$ and consider the balls B_j, $j \in \mathbb{N}$, related to the construction of X_r; namely the balls of the form $B(\bar{x}, r)$, $\bar{x} \in X_r$. Note that $\bar{x} \sim \bar{y}$ if and only if $B(\bar{x}; 2r) \cap B(\bar{y}; 2r)$ is nonempty. Now setting $v_k = u_k - u$,

we see that

$$\mathcal{E}_r[u_k - u] = \mathcal{E}_r[v_k] = \sum_{\bar{x} \in X_r} \sum_{\bar{y} \in X_r : \bar{x} \sim \bar{y}} \frac{|(v_k)_{B(\bar{x};r/4)} - (v_k)_{B(\bar{y};r/4)}|^2}{r^2} \mu(B(\bar{x};r))$$

$$\leq \frac{4}{r^2} \sum_{\bar{x} \in X_r} \sum_{\bar{y} \in X_r : \bar{x} \sim \bar{y}} \left((|v_k|)^2_{B(\bar{x};r/4)} + (|v_k|)^2_{B(\bar{y};r/4)} \right) \mu(B(\bar{x};r))$$

$$\leq \frac{4}{r^2} \sum_{\bar{x} \in X_r} \sum_{\bar{y} \in X_r : \bar{x} \sim \bar{y}} \left(\fint_{B(\bar{x};r/4)} |v_k|^2 \, d\mu + \fint_{B(\bar{y};r/4)} |v_k|^2 \, d\mu \right) \mu(B(\bar{x};r)).$$

Note that if $\bar{x} \sim \bar{y}$, then $B(\bar{y};r) \subset B(\bar{x};5r)$; moreover, there is at most N such $\bar{y} \in X_r$ that are neighbors of $\bar{x}$, with N independent of $\bar{x}$ and r (in fact, it depends solely on the metric doubling constant N_d). Therefore by the doubling property of μ we have

$$\mathcal{E}_r[u_k - u] = \mathcal{E}_r[v_k] \lesssim \frac{1}{r^2} \sum_{\bar{x} \in X_r} \int_{B(\bar{x};5r)} |v_k|^2 \, d\mu \lesssim \frac{1}{r^2} \int_X |v_k|^2 \, d\mu,$$

where in the last step we have used the bounded overlap property $\sum_{\bar{x} \in X_r} \chi_{B(\bar{x};5r)} \leq K$ on X with K depending solely on the doubling constant N_d. Thus, if $u_k \to u$ in $L^2(X)$, that is, $v_k \to 0$ in $L^2(X)$, it follows that $\mathcal{E}_r[u_k - u] \to 0$ as well as $k \to \infty$. Note that

$$\mathcal{E}_r[u]^{1/2} - \mathcal{E}_r[u_k - u]^{1/2} \leq \mathcal{E}_r[u_k]^{1/2} \leq \mathcal{E}_r[u]^{1/2} + \mathcal{E}_r[u_k - u]^{1/2}.$$

This completes the proof of the theorem; though note that the convergence of $\mathcal{E}_r[u_k - u]$ to zero is not uniform across $r > 0$ as we have r^2 in the denominator of the above estimate. $\square$

DEFINITION 3.15. A sequence $\mathcal{E}_{r_k}$ of Dirichlet forms on the Hilbert space $N^{1,2}(X)$ is said to Γ-converge to a Dirichlet form $\mathcal{E}$ if the following two conditions are satisfied:

(i) Whenever $(u_k)_k$ is a sequence in $N^{1,2}(X)$ and $u \in N^{1,2}(X)$ such that $u_k \to u$ in $N^{1,2}(X)$, we must have $\mathcal{E}[u] \leq \liminf_k \mathcal{E}_{r_k}[u_k]$;
(ii) For each $u \in N^{1,2}(X)$ there is a sequence $(u_k)_k$ in $N^{1,2}(X)$ such that $u_k \to u$ in $N^{1,2}(X)$ and $\mathcal{E}[u] \geq \limsup_k \mathcal{E}_{r_k}[u_k]$.

Given the first condition above, the second condition is equivalent to knowing the existence of a sequence $(u_k)_k$ in $N^{1,2}(X)$ such that $\mathcal{E}[u] = \lim_k \mathcal{E}_{r_k}[u_k]$ and $u_k \to u$ in $N^{1,2}(X)$.

PROPOSITION 3.16. *There exists a Γ-convergent subsequence $\mathcal{E}_{r_k}$ of $\mathcal{E}_r$.*

PROOF. From [2] we know that $N^{1,2}(X)$ is separable. Therefore, by [14, Theorem 8.5], there exists a Γ-convergent subsequence $\mathcal{E}_{r_k}$ of $\mathcal{E}_r$. From [14, Proposition 6.8] we know that the Γ-limit is a closable bilinear energy form. $\square$

We denote the Γ-limit of $\mathcal{E}_{r_k}$ from the previous proposition by $\mathcal{E}$. We now show that the energy form induced by the Γ-limit $\mathcal{E}$ is comparable to the Newton-Sobolev energy form, thus proving Condition (2) of Theorem 1.1.

PROPOSITION 3.17. *There is a constant $C > 0$ such that the following is true for every $u \in N^{1,2}(X)$ and its minimal 2-weak upper gradient $g_u \in L^2(X)$:*

$$\frac{1}{C}\,\mathcal{E}[u] \le \int_X g_u^2\,d\mu \le C\,\mathcal{E}[u]. \tag{3.18}$$

PROOF. If u_k, $k \in \mathbb{N}$, is the sequence guaranteed by the Γ-limit so that $\lim_k \mathcal{E}_{r_k}[u_k] = \mathcal{E}[u]$ and $\|u_k - u\|_{N^{1,2}(X)} \to 0$ as $k \to \infty$, then we have that $\int_X g_u^2\,d\mu = \lim_k \int_X g_{u_k}^2\,d\mu$. By [14, Proposition 6.8], we know that the Γ-limit is lower semicontinuous. Then by Theorem 3.13 we have that

$$\mathcal{E}[u] \le \liminf_k \mathcal{E}[u_k] \le C \liminf_k \int_X g_{u_k}^2\,d\mu = C \int_X g_u^2\,d\mu,$$

and so $\mathcal{E}[u] \le C \int_X g_u^2\,d\mu$, which gives the left side of (3.18). On the other hand, by Theorem 3.13, we have

$$\mathcal{E}[u]^{1/2} = \lim_k \mathcal{E}_{r_k}[u_k]^{1/2} \ge \liminf_k \mathcal{E}_{r_k}[u]^{1/2} - \limsup_k \mathcal{E}_{r_k}[u - u_k]^{1/2} \ge \left(\frac{1}{C}\int_X g_u^2\,d\mu\right)^{1/2},$$

where we have used the fact that (Again from Theorem 3.13),

$$0 \le \limsup_k \mathcal{E}_{r_k}[u - u_k] \le \limsup_k \int_X g_{u_k - u}^2\,d\mu = 0. \tag{3.19}$$

This proves Claim (2) of Theorem 1.1. $\qquad\qquad\square$

One of the key properties needed with respect to the Γ-convergence is that if the function being approximated vanishes outside a bounded domain $\Omega \subset X$ with $\mu(X \setminus \Omega) > 0$, then the sequence approximating this function in the Γ-convergence condition (ii) also vanishes outside the approximating domain. This property is essential in the discussion on approximating solutions to the Dirichlet problem on Ω, and is called the *matching boundary values* in [9, Section 4.2.1]. However, the method of De Giorgi, as outlined in [9], uses a decomposition of the domain Ω into suitable annular rings and constructing a partition of unity subordinate to this decomposition. An alternate method is to construct different Γ-limits for different boundary data as in [14, Chapter 21], but this again is computationally involved (see [14, Theorem 21.1]). In the second main theorem, Theorem 1.2, we give a more direct proof of the existence of optimal sequences with matching boundary values in the sense of [9]. We now prove this theorem.

PROOF OF THEOREM 1.2. For $x \in X$ set $w(x) = \operatorname{dist}(x, X \setminus \Omega)$. For $j \in \mathbb{N}$ let

$$u_j = \min\{\max\{u - w, v_j\}, u + w\}.$$

Then,

$$u_j - u = \min\{\max\{-w, v_j - u\}, w\},$$

and therefore $u_j - u \in N_0^{1,2}(\Omega)$. Further, note that as $w \ge 0$, when $x \in X$ we have that $u_j(x) - u(x) > 0$ if and only if $v_j(x) - u(x) > 0$, and in which case we also have $u_j - u(x) \le v_j - u(x)$. If $u_j(x) - u(x) \le 0$, then necessarily $v_j(x) - u(x) \le 0$ and

$u_j - u(x) \geq v_j - u(x)$. Thus, $|u_j(x) - u(x)| \leq |v_j(x) - u(x)|$ for all $x \in X$. Therefore, as $v_j \to u$ in $L^2(X)$, we have $u_j \to u$ in $L^2(X)$.

To see the $N^{1,2}$-convergence, we use [21, Proposition 6.3.23]. Using this proposition and the fact that $g_w \leq \chi_\Omega$, we get

$$g_{u_j - u} \leq \chi_{\{|v_j - u| > w\} \cap \Omega} + g_{v_j - u} \cdot \chi_{\{|v_j - u| \leq w\} \cap \Omega}.$$

Note that for any $\epsilon > 0$,

$$\int_X g_{u_j - u}^2 \, d\mu = \int_\Omega g_{u_j - u}^2 d\mu = \int_{\Omega_\epsilon} g_{u_j - u}^2 d\mu + \int_{\Omega \setminus \Omega_\epsilon} g_{u_j - u}^2 d\mu,$$

where $\Omega_\epsilon = \{x \in \Omega : \text{dist}(x, X \setminus \Omega) > \epsilon\}$. So we get

$$\int_X g_{u_j - u}^2 \, d\mu \leq \mu(\{|v_j - u| > \epsilon\} \cap \Omega_\epsilon) + \int_{\{|v_j - u| \leq w\} \cap \Omega} g_{v_j - u}^2 d\mu + \mu(\Omega \setminus \Omega_\epsilon).$$

Note that

$$\mu(\{|v_j - u| > \epsilon\} \cap \Omega_\epsilon) \leq \frac{1}{\epsilon^2} \int_{\Omega_\epsilon} |v_j - u|^2 \, d\mu.$$

Since $v_j \to u$ in $L^2(X)$, it follows that $\limsup_{j \to \infty} \mu(\{|v_j - u| > \epsilon\} \cap \Omega_\epsilon) = 0$, and moreover, as $v_j \to u$ in the Hilbert space $N^{1,2}(X)$, it also follows that $\limsup_{j \to \infty} \int_\Omega g_{v_j - u}^2 \, d\mu = 0$. Therefore we have

$$\limsup_{j \to \infty} \int_X g_{u_j - u}^2 \, d\mu \leq \mu(\Omega \setminus \Omega_\epsilon).$$

As $\epsilon > 0$ can be chosen arbitrarily small, and $\bigcap_{\epsilon > 0} \Omega \setminus \Omega_\epsilon$ is empty, it follows that $\limsup_{\epsilon \to 0^+} \mu(\Omega \setminus \Omega_\epsilon) = 0$, and so

$$\lim_{j \to \infty} \int_X g_{u_j - u}^2 \, d\mu = 0.$$

Thus we have that $u_j \to u$ in $N^{1,2}(X)$.

Finally, note that $|\mathcal{E}_{r_j}^{1/2}[u_j] - \mathcal{E}_{r_j}^{1/2}[v_j]| \leq \mathcal{E}_{r_j}^{1/2}[u_j - v_j]$. By Theorem 3.13 we therefore have that

$$|\mathcal{E}_{r_j}^{1/2}[u_j] - \mathcal{E}_{r_j}^{1/2}[v_j]| \leq C \int_X g_{u_j - v_j}^2 \, d\mu = C \int_\Omega g_{v_j - u_j}^2 \, d\mu \to 0 \text{ as } j \to \infty.$$

It follows that $\mathcal{E}[u] = \lim_{j \to \infty} \mathcal{E}_{r_j}[v_j] = \lim_{j \to \infty} \mathcal{E}_{r_j}[u_j]$ as desired. $\square$

4. Properties of $\mathcal{E}$

In this section we will consider some key basic properties of the limit form $\mathcal{E}$. There is a property of $\mathcal{E}$ that would be essential in order to call $\mathcal{E}$ a Dirichlet form; namely, that $\mathcal{E}$ is Markovian. In this section we will also prove that $\mathcal{E}$ is Markovian and that it is local. The first lemma follows from [14, Proposition 11.9(e) and Theorem 11.10].

DEFINITION 4.1. Let $\mathcal{E}$ be a symmetric bilinear form from a Hilbert space H to $[-\infty, \infty]$. Let $\mathcal{D}(\mathcal{E})$ be a dense linear subspace of H such that $\mathcal{E}$ is a symmetric bilinear form mapping $\mathcal{D}(\mathcal{E})$ to $\mathbb{R}$. Then we say $\mathcal{D}(\mathcal{E})$ is the *domain* of $\mathcal{E}$.

DEFINITION 4.2. Let $C_c(X)$ be the collection of all real-valued continuous functions with compact support on X. A *core* of a symmetric biliear form $\mathcal{E}$ is a subset $\mathcal{C}$ of $\mathcal{D}(\mathcal{E}) \cap C_c(X)$ such that $\mathcal{C}$ is dense in $\mathcal{D}(\mathcal{E})$ under the norm $\sqrt{(\cdot, \cdot)_H + \mathcal{E}(\cdot)}$ and dense in $C_c(X)$ under the uniform norm.

DEFINITION 4.3. We say that $\mathcal{E}$ is *closed* if for any $(u_n)_n \subseteq \mathcal{D}(\mathcal{E})$ satisfying $(u_n - u_m, u_n - u_m)_H + \mathcal{E}[u_n - u_m] \to 0$ as $m, n \to \infty$, there exists $u \in \mathcal{D}(\mathcal{E})$ such that $(u_n - u, u_n - u)_H + \mathcal{E}[u_n - u] \to 0$ as $n \to \infty$.

In our setting, we have chosen the Hilbert space $H = N^{1,2}(X)$, and consequent to Proposition 3.17 (or Theorem 1.1), we also have that $N^{1,2}(X) = \mathcal{D}(\mathcal{E})$. This is because the comparability of $\mathcal{E}$ and the energy form on $N^{1,2}(X)$ (Proposition 3.17) guarantees that $\mathcal{E}$ should always be finite on $N^{1,2}(X)$.

We can choose the core $\mathcal{C}$ to be the collection of all Lipschitz functions on X. Since the measure μ on X is doubling and supports a 2-Poincaré inequality, it follows that Lipschitz functions are dense in $N^{1,2}(X)$, see for instance [21, Section 8.2].

LEMMA 4.4. *$\mathcal{E}$ is a symmetric bilinear form on $N^{1,2}(X)$.*

The above lemma follows from [14, Proposition 11.9 and Theorem 11.10], where symmetric bilinear forms are called quadratic forms.

LEMMA 4.5. *The form $\mathcal{E}$ is a local form on $N^{1,2}(X)$. Moreover, for each $u \in N^{1,2}(X)$ we have that*

$$\mathcal{E}[u] = \lim_{k \to \infty} \mathcal{E}_{r_k}[u].$$

PROOF. We first prove the locality property of $\mathcal{E}$. Let $u, v \in N^{1,2}(X)$ such that the support K_u of u and the support K_v of v are disjoint. Then, as X is compact, there is some $\rho > 0$ such that $\text{dist}(K_u, K_v) \geq 10\rho$. By the bilinearity and symmetry properties of $\mathcal{E}$, we have $\mathcal{E}[u+v] = \mathcal{E}[u-v] + 4\mathcal{E}(u, v)$. To show that $\mathcal{E}(u, v) = 0$ (the definition of locality for a symmetric bilinear form), it suffices to show that $\mathcal{E}[u+v] = \mathcal{E}[u-v]$. Note that $u+v, u-v$ are both zero in $X \setminus (K_u \cup K_v)$. Let $(\widehat{f}_k)_k \subset N^{1,2}(X)$ be an approximating sequence for $u + v$ as guaranteed by the Γ-convergence. Then, thanks to Theorem 1.2, we can also assume that each $\widehat{f}_k = 0$ on $X \setminus (K_u \cup K_v)$. Then for each positive integer k, the functions $f_k = \widehat{f}_k \chi_{K_u}$ and $h_k = \widehat{f}_k \chi_{K_v}$ both belong to $N^{1,2}(X)$ and $\widehat{f}_k = f_k + h_k$. Indeed, as there is a positive distance between K_u and K_v, we can find two compactly supported Lipschitz functions φ_u and φ_v so that $\varphi_u = 1$ on K_u, $\varphi_v = 1$ on K_v, and $f_k = \varphi_u \widehat{f}_k$, $h_k = \varphi_v \widehat{f}_k$. Thus $f_k, h_k \in N^{1,2}(X)$, as seen from [21, Proof of Proposition 7.1.35]. Moreover, $f_k \to u$ in $N^{1,2}(X)$ and $h_k \to v$ in $N^{1,2}(X)$. It follows that $f_k - h_k \to u - v$ in $N^{1,2}(X)$. It follows that

$$\mathcal{E}[u - v] \leq \liminf_{k \to \infty} \mathcal{E}_{r_k}[f_k - h_k].$$

On the other hand, when $r_k < \rho$, we have that

$$\mathcal{E}_{r_k}[f_k - h_k] = \mathcal{E}_{r_k}[f_k] + \mathcal{E}_{r_k}[h_k] = \mathcal{E}_{r_k}[f_k + h_k],$$

and so as $\lim_{k\to\infty} \mathcal{E}_{r_k}[\widehat{f}_k] = \mathcal{E}[u+v]$, we have that

$$\mathcal{E}[u-v] \le \mathcal{E}[u+v].$$

Replacing v with $-v$ in the above argument also gives that $\mathcal{E}[u+v] \le \mathcal{E}[u-v]$, and so we have that $\mathcal{E}[u+v] = \mathcal{E}[u-v]$. It follows that $\mathcal{E}$ is a local form.

Next, for $u \in N^{1,2}(X)$, let $(u_k)_k$ be a sequence in $N^{1,2}(X)$ such that $u_k \to u$ in $N^{1,2}(X)$ and $\lim_{k\to\infty} \mathcal{E}_{r_k}[u_k] = \mathcal{E}[u]$; existence of such sequence is guaranteed by the definition of Γ-convergence. Then

$$\mathcal{E}[u]^{1/2} \le \liminf_{k\to\infty} \mathcal{E}_{r_k}[u]^{1/2} \le \liminf_{k\to\infty} \left(\mathcal{E}_{r_k}[u-u_k]^{1/2} + \mathcal{E}_{r_k}[u_k]^{1/2} \right).$$

By Theorem 3.13 and by the fact that $u_k \to u$ in $N^{1,2}(X)$ we have $\lim_{k\to\infty} \mathcal{E}_{r_k}[u-u_k] = 0$, see for example (3.19). It follows that $\mathcal{E}[u] \le \liminf_{k\to\infty} \mathcal{E}_{r_k}[u_k] = \mathcal{E}[u]$, from which the desired conclusion now follows. $\qquad\square$

In [27, page 372], it is shown that the Γ-limit of bilinear forms on $L^2(X)$ is closed. In our setting, we consider a Γ-convergence with the ambient Hilbert space itself being $N^{1,2}(X)$. Therefore for completeness of the exposition, we include our proof here.

LEMMA 4.6. *The symmetric bilinear form $\mathcal{E}$ is closed.*

PROOF. We first show that if $(f_n)_n$ is a sequence in $N^{1,2}(X)$ and $f \in L^2(X)$ such that $f_n \to f$ in $L^2(X)$ and $\mathcal{E}[f_n - f_m] \to 0$ as $m, n \to \infty$, then by modifying f on a set of μ-measure zero on X if necessary, we have that $f \in N^{1,2}(X)$ and $\|f_n - f_m\|_{N^{1,2}(X)} \to 0$ as $n, m \to \infty$. Indeed, as $\mathcal{E}[f_n - f_m] \to 0$ as $m, n \to \infty$, it follows from Theorem 3.13 (or more specifically, from its limit version given in (2) of Theorem 1.1), we have that $\int_X g^2_{f_n - f_m} \, d\mu \to 0$ as $n, m \to \infty$. It follows that $(f_n)_n$ is Cauchy in the Banach space $N^{1,2}(X)$, and so such modification of f exists.

Now let $(f_n)_n \subset \mathcal{D}(\mathcal{E})$ be a sequence such that $\|f_n - f_m\|_{N^{1,2}(X)} + \mathcal{E}[f_n - f_m] \to 0$ as $m, n \to \infty$. Then necessarily $(f_n)_n$ is a Cauchy sequence in $N^{1,2}(X)$, and as $N^{1,2}(X)$ is a Banach space (see [28]), there exists $f \in N^{1,2}(X) = \mathcal{D}(\mathcal{E})$ such that $\|f_n - f\|_{N^{1,2}(X)} \to 0$ as $n \to \infty$. By Lemma 4.5, we know that $f \in \mathcal{D}(\mathcal{E})$. To complete the proof of closedness of $\mathcal{E}$ we now show that $\mathcal{E}[f_n - f] \to 0$ as $n \to \infty$.

By [14, Proposition 6.8], we know that the Γ-limit is lower semicontinuous. Therefore, for each fixed n, using the lower semicontinuity of $\mathcal{E}$, we have

$$\mathcal{E}[f_n - f] \le \liminf_{m\to\infty} \mathcal{E}[f_n - f_m].$$

Hence, we have

$$0 \le \limsup_{n\to\infty} \mathcal{E}[f_n - f] \le \limsup_{n\to\infty} \liminf_{m\to\infty} \mathcal{E}[f_n - f_m] = \lim_{m,n\to\infty} \mathcal{E}[f_n - f_m] = 0. \qquad (4.7)$$

$$\square$$

PROPOSITION 4.8. *The bilinear form $\mathcal{E}$ is Markovian.*

PROOF. To show that the form is Markovian, it suffices to show that if $u \in N^{1,2}(X)$, setting $v = \max\{0, \min\{1, u\}\}$, we have $\mathcal{E}[v] \leq \mathcal{E}[u]$. Since (X, d, μ) is doubling and supports a 2-Poincaré inequality, we know that Lipschitz functions form a dense subclass of $N^{1,2}(X)$; hence it suffices to prove this property for the case that u is Lipschitz continuous.

So let $u \in N^{1,2}(X)$ be Lipschitz on X, and set $E_- = \{u < 0\}$, $E_+ = \{u > 1\}$, and $E_0 = \{0 \leq u \leq 1$. Then as X is compact, and as $\overline{E_-}$ and $\overline{E_+}$ are disjoint, we have that $\rho := \text{dist}(\overline{E_-}, \overline{E_+}) > 0$. So if $0 < r < \rho/16$, then we have that for $\bar{x} \in X_r$ and $\bar{y} \in X_r$ with $\bar{y} \sim \bar{x}$, the set $B(\bar{x}, r/4) \cup B(\bar{y}, r/4)$ cannot intersect both $\overline{E_+}$ and $\overline{E_-}$.

Now let $\bar{x}, \bar{y} \in X_r$ with $\bar{x} \sim \bar{y}$. If $B(\bar{x}, r/4) \cup B(\bar{y}, r/4) \subset \overline{E_-}$ or $B(\bar{x}, r/4) \cup B(\bar{y}, r/4) \subset \overline{E_+}$ or $B(\bar{x}, r/4) \cup B(\bar{y}, r/4) \subset E_0$, then we have $|v_r(\bar{x}) - v_r(\bar{y})| \leq |u_r(\bar{x}) - u_r(\bar{y})|$. If either $B(\bar{x}, r/4) \cup B(\bar{y}, r/4)$ intersects both E_- and E_0 (the case that $B(\bar{x}, r/4) \cup B(\bar{y}, r/4)$ intersects both E_+ and E_0 will be handled in s very similar manner). In this case, we have

$$|v_r(\bar{x}) - v_r(\bar{y})| \leq |v_r(\bar{x}) - u_r(\bar{x})| + |u_r(\bar{x}) - u_r(\bar{y})| + |u_r(\bar{y}) - v_r(\bar{y})|.$$

For such $\bar{x}$, we have to estimate $|v_r(\bar{x}) - u_r(\bar{x})|$ and $|v_r(\bar{y}) - u_r(\bar{y})|$. Note that

$$|v_r(\bar{x}) - u_r(\bar{x})|^2 = \left| \fint_{B(\bar{x}, r/4)} (u - v)\, d\mu \right|^2 \leq \frac{1}{\mu(B(\bar{x}, r/4))} \int_{B(\bar{x}, r/4) \cap E_-} |u|^2\, d\mu$$

$$|v_r(\bar{y}) - u_r(\bar{y})|^2 = \left| \fint_{B(\bar{y}, r/4)} (u - v)\, d\mu \right|^2 \leq \frac{1}{\mu(B(\bar{y}, r/4))} \int_{B(\bar{y}, r/4) \cap E_-} |u|^2\, d\mu.$$

Setting L to be a Lipschitz constant of u and that $B(\bar{x}, r/4) \cup B(\bar{y}, r/4)$ intersects E_- and E_0, and $d(\bar{x}, \bar{y}) \leq 4r$, necessarily $|u| \leq 5Lr$ on $B(\bar{x}, r/4) \cup B(\bar{y}, r/4)$. Hence

$$|v_r(\bar{x}) - u_r(\bar{x})|^2 \leq 25L^2 r^2 \quad \text{and} \quad |v_r(\bar{y}) - u_r(\bar{y})|^2 \leq 25L^2 r^2.$$

Similar estimates hold if $B(\bar{x}, r/4) \cup B(\bar{y}, r/4)$ intersects both E_+ and E_0, now with $|u|^2$ replaced by $|u - 1|^2$. Note also that when $B(\bar{x}, r/4) \cup B(\bar{y}, r/4)$ intersects both E_- and E_0 and $\bar{x} \sim \bar{y}$, necessarily we must have $B(\bar{x}, r/4) \cup B(\bar{y}, r/4) \subset \bigcup_{z \in \partial E_-} B(z, 6r)$ because X is a geodesic space and u is continuous. We set

$$(\partial E_-)_r = \bigcup_{z \in \partial E_-} B(z, 6r), \qquad \text{and} \qquad (\partial E_+)_r = \bigcup_{z \in \partial E_+} B(z, 6r),$$

and let

$$X_r[v] := \{\bar{x} \in X_r \,:\, B(\bar{x}, r/4) \subset (\partial E_-)_r \cup (\partial E_+)_r\}.$$

Note that

$$\mathcal{E}_r[v] = \sum_{\bar{x} \in X_r} \sum_{\bar{y} \sim \bar{x}} \frac{|v_r(\bar{x}) - v_r(\bar{y})|^2}{r^2}\, \mu_r(\bar{x})$$

$$\leq \sum_{\bar{x} \in X_r[v]} \sum_{\bar{y} \sim \bar{x}} \frac{(|v_r(\bar{x}) - u_r(\bar{x})| + |u_r(\bar{x}) - u_r(\bar{y})| + |u_r(\bar{y}) - v_r(\bar{y})|)^2}{r^2}\, \mu_r(\bar{x})$$

$$+ \sum_{\bar{x} \in X_r \setminus X_r[v]} \sum_{\bar{y} \sim \bar{x}} \frac{|u_r(\bar{x}) - u_r(\bar{y})|^2}{r^2}\, \mu_r(\bar{x}).$$

It follows that

$$\mathcal{E}_r[v] \leq \sum_{\bar{x}\in X_r} \sum_{\bar{y}\sim\bar{x}} \frac{|u_r(\bar{x}) - u_r(\bar{y})|^2}{r^2} \mu_r(\bar{x})$$

$$+ \sum_{\bar{x}\in X_r[v]} \sum_{\bar{y}\sim\bar{x}} \frac{|v_r(\bar{x}) - u_r(\bar{x})|^2}{r^2} \mu_r(\bar{x}) + \sum_{\bar{x}\in X_r[v]} \sum_{\bar{y}\sim\bar{x}} \frac{|v_r(\bar{y}) - u_r(\bar{y})|^2}{r^2} \mu_r(\bar{x})$$

$$+ 2 \sum_{\bar{x}\in X_r[v]} \sum_{\bar{y}\sim\bar{x}} \frac{|v_r(\bar{x}) - u_r(\bar{x})| \cdot |u_r(\bar{x}) - u_r(\bar{y})|}{r^2} \mu_r(\bar{x})$$

$$+ 2 \sum_{\bar{x}\in X_r} \sum_{\bar{y}\sim\bar{x}} \frac{|v_r(\bar{y}) - u_r(\bar{y})| \cdot |u_r(\bar{x}) - u_r(\bar{y})|}{r^2} \mu_r(\bar{x})$$

$$+ 2 \sum_{\bar{x}\in X_r[v]} \sum_{\bar{y}\sim\bar{x}} \frac{|v_r(\bar{x}) - u_r(\bar{x})| \cdot |v_r(\bar{y}) - u_r(\bar{y})|}{r^2} \mu_r(\bar{x}).$$

Using Hölder's inequality, we have

$$\mathcal{E}_r[v] \leq \mathcal{E}_r[u] + \sum_{\bar{x}\in X_r[v]} \sum_{\bar{y}\sim\bar{x}} \frac{|v_r(\bar{x}) - u_r(\bar{x})|^2}{r^2} \mu_r(\bar{x}) + \sum_{\bar{x}\in X_r[v]} \sum_{\bar{y}\sim\bar{x}} \frac{|v_r(\bar{y}) - u_r(\bar{y})|^2}{r^2} \mu_r(\bar{x})$$

$$+ 2\mathcal{E}_r[u]^{1/2} \cdot \left(\sum_{\bar{x}\in X_r[v]} \sum_{\bar{y}\sim\bar{x}} \frac{|v_r(\bar{x}) - u_r(\bar{x})|^2}{r^2} \mu_r(\bar{x}) \right)^{1/2}$$

$$+ 2\mathcal{E}_r[u]^{1/2} \cdot \left(\sum_{\bar{x}\in X_r[v]} \sum_{\bar{y}\sim\bar{x}} \frac{|v_r(\bar{y}) - u_r(\bar{y})|^2}{r^2} \mu_r(\bar{x}) \right)^{1/2}$$

$$+ 2 \left[\left(\sum_{\bar{x}\in X_r[v]} \sum_{\bar{y}\sim\bar{x}} \frac{|v_r(\bar{x}) - u_r(\bar{x})|^2}{r^2} \mu_r(\bar{x}) \right) \left(\sum_{\bar{x}\in X_r[v]} \sum_{\bar{y}\sim\bar{x}} \frac{|v_r(\bar{y}) - u_r(\bar{y})|^2}{r^2} \mu_r(\bar{x}) \right) \right]^{1/2} .$$

It now suffices to show that both the term $\sum_{\bar{x}\in X_r[v]} \sum_{\bar{y}\sim\bar{x}} \frac{|v_r(\bar{y}) - u_r(\bar{y})|^2}{r^2} \mu_r(\bar{x})$ and the term $\sum_{\bar{x}\in X_r[v]} \sum_{\bar{y}\sim\bar{x}} \frac{|v_r(\bar{y}) - u_r(\bar{y})|^2}{r^2} \mu_r(\bar{x})$ tend to 0 as $r \to 0$. From the discussion prior to the above estimates, we see that

$$\sum_{\bar{x}\in X_r[v]} \sum_{\bar{y}\sim\bar{x}} \frac{|v_r(\bar{x}) - u_r(\bar{x})|^2}{r^2} \mu_r(\bar{x}) \leq N \sum_{\bar{x}\in X_r[v]} \frac{|v_r(\bar{x}) - u_r(\bar{x})|^2}{r^2} \mu_r(\bar{x})$$

$$\leq N\, 25 L^2 \sum_{\bar{x}\in X_r[v]} \mu_r(\bar{x})$$

$$\leq C\, 25 L^2\, \mu((\partial E_-)_r \cup (\partial E_+)_r),$$

where we have used the fact that for each $\bar{x} \in X_r$ there are at most N number of vertices $\bar{y} \in X_r$ such that $\bar{x} \sim \bar{y}$, with N depending only on the doubling constant associated with

μ; and we also used fact that there is at most bounded overlap property of the balls $B(\bar{y}, r)$, $\bar{y} \in X_r$, to obtain the last inequality. A similar argument also gives

$$\sum_{\bar{x} \in X_r[v]} \sum_{\bar{y} \sim \bar{x}} \frac{|v_r(\bar{y}) - u_r(\bar{y})|^2}{r^2} \, \mu_r(\bar{x}) \leq C \, 25L^2 \, \mu((\partial E_-)_r \cup (\partial E_+)_r).$$

It follows that

$$\mathcal{E}_r[v] \leq \mathcal{E}_r[u] + 10C \, L \, \mathcal{E}_r[u]^{1/2} \mu((\partial E_-)_r \cup (\partial E_+)_r)^{1/2} + 25 \, C \, L^2 \, \mu((\partial E_-)_r \cup (\partial E_+)_r).$$

So if $\mu(\partial E_-) + \mu(\partial E_+) = 0$, then we have that

$$\mathcal{E}[v] \leq \liminf_{k \to \infty} \mathcal{E}_{r_k}[v] \leq \liminf_{k \to \infty} \mathcal{E}_{r_k}[u] = \mathcal{E}[u].$$

The last equality follows from Lemma 4.5 above. The assumption of $\mu(\partial E_-) + \mu(\partial E_+) = 0$ is not a serious obstacle, as we know that for almost every $t \in \mathbb{R}$ we have that $\mu(\{u = t\}) = 0$, and so we can approximate 0 from below by such $t_- < 0$, and approximate 1 from above by such $t_+ > 1$, and consider $v_{t_-, t_+} = \max\{t_-, \min\{t_+, u\}\}$ and obtain the Markov property for these, and then let $t_- \to 0$, $t_+ \to 1$ and note that $v_{t_-, t_+} \to v$ in $N^{1,2}(X)$; we know that $\mathcal{E}$ is bounded (from Theorem 3.13 above) bilinear form on $N^{1,2}(X)$. The fact that $\mathcal{E}_r[v_{t_-, t_+}]$ converges to $\mathcal{E}_r[v]$ can be obtained from the closability of $\mathcal{E}$ (lemma 4.6 above) and Theorem 3.13, together with the fact that $v_{t_-, t_+} \to v$ in $N^{1,2}(X)$ as $t_- \to 0$ and $t_+ \to 1$. □

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