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Well/Ill-Posedness of the Boltzmann Equation with Soft Potential

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Abstract: We consider the Boltzmann equation with the soft potential and angular cutoff. Inspired by the methods from dispersive PDEs, we establish its sharp local well-posedness and ill-posedness in H^s Sobolev space. We find the well/ill-posedness separation at regularity $s = \frac{d-1}{2}$, strictly $\frac{1}{2}$-derivative higher than the scaling-invariant index $s = \frac{d-2}{2}$, the usually expected separation point.

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1. Introduction

We consider the Boltzmann equation

$$\begin{cases} (\partial_t + v \cdot \nabla_x) f(t,x,v) = Q(f,f), \\ f(0,x,v) = f_0(x,v), \end{cases} \tag{1.1}$$

where $f(t,x,v)$ is the distribution function for the particles at time $t \geq 0$, position $x \in \mathbb{R}^d$ and velocity $v \in \mathbb{R}^d$. The collision operator Q is conventionally split into a gain term and a loss term

$$Q(f,g) = Q^+(f,g) - Q^-(f,g)$$

where the gain term is

$$Q^+(f,g) = \int_{\mathbb{R}^d}\int_{\mathbb{S}^{d-1}} f(v^*)g(u^*)B(u-v,\omega)du d\omega, \tag{1.2}$$

and the loss term is

$$Q^-(f,g) = \int_{\mathbb{R}^d}\int_{\mathbb{S}^{d-1}} f(v)g(u)B(u-v,\omega)du d\omega, \tag{1.3}$$

with the relation between the pre-collision and after-collision velocities that

$$u^* = u + \omega\cdot(v-u)\omega, \quad v^* = v - \omega\cdot(v-u)\omega.$$

The Boltzmann collision kernel function $B(u-v,\omega)$ is a non-negative function depending only on the relative velocity $|u-v|$ and the deviation angle θ through $\cos\theta := \frac{u-v}{|u-v|}\cdot\omega$. Throughout the paper, we consider

$$B(u-v,\omega) = |u-v|^{\gamma}\mathbf{b}(\cos\theta) \tag{1.4}$$

under the Grad's angular cutoff assumption

$$0 \leq \mathbf{b}(\cos\theta) \leq C|\cos\theta|.$$

The different ranges $\gamma < 0$, $\gamma = 0$, $\gamma > 0$ correspond to soft potentials, Maxwellian molecules, and hard potentials, respectively. See also [11,12,66] for a more detailed physics background. This collision kernel (1.4) comes from an important model case of inverse-power law potentials and there have been a large amount of literature devoted to various problems for this model, such as its hydrodynamics limits which provide a description between the kinetic theory and hydrodynamic equations. For a detailed presentation and the derivation of macroscopic equations from the fundamental laws of physics, see for example [59].

The Cauchy problem for the Boltzmann equation is one of the fundamental mathematical problems in kinetic theory, as it is of vital importance for the physical interpretation and practical application. For instance, in the absence of uniqueness or continuous dependence on the initial condition, numerical calculations and algorithms, even if they can be done, could present puzzling results. Despite the innovative work [34,39] and many nice developments, the well/ill-posedness of the Boltzmann equation remains largely open. So far, there have been many developed methods and techniques for well-posedness, see for example [2,3,6,7,35,37,41,43,44,61,64].

In the recent series of paper [13–15], by taking dispersive techniques on the study of the quantum many-body hierarchy dynamics, especially space-time collapsing/multi-linear estimates techniques (see for instance [16–24,28,45,46,52,54,60]), T. Chen, Denlinger, and Pavlović provided a new approach to prove the well-posedness of the Boltzmann equation and suggested the possibility of a systematic study of Boltzmann equation using dispersive tools. With the dispersive techniques, the regularity index for well-posedness, which is usually at least the continuity threshold $s > \frac{d}{2}$, has been relaxed to $s > \frac{d-1}{2}$ for both Maxwellian molecules and hard potentials with cutoff in [13]. It is of mathematical and physical interest to prove well-posedness at the optimal regularity. From the scaling point of view, the Boltzmann equation (1.1) is invariant under the scaling

$$f_\lambda(t, x, v) = \lambda^{\alpha+(d-1+\gamma)\beta} f\left(\lambda^{\alpha-\beta}t, \lambda^\alpha x, \lambda^\beta v\right), \tag{1.5}$$

for any $\alpha, \beta \in \mathbb{R}$ and $\lambda > 0$. Then in the L^2 setting, it holds that

$$\| |\nabla_x|^s |v|^r f_\lambda \|_{L^2_{xv}} = \lambda^{\alpha+(d-1+\gamma)\beta} \lambda^{\alpha s-\beta r} \lambda^{-\frac{d}{2}\alpha-\frac{d}{2}\beta} \| |\nabla_x|^s |v|^r f \|_{L^2_{xv}}.$$

This gives the scaling-critical index

$$s = \frac{d-2}{2}, \quad r = s + \gamma. \tag{1.6}$$

From the past experience of scaling analysis, it is believed that the well/ill-posedness threshold[1] in H^s Sobolev space is $s_c = \frac{d-2}{2}$ with $r \geq 0$. Surprisingly, for the 3D constant kernel case, X. Chen and Holmer in [27] prove the well/ill-posedness threshold in H^s Sobolev space is exactly at regularity $s = \frac{d-1}{2}$, and thus point out the actual optimal regularity for the global well-posedness problem.

On the one hand, while there are many well-known progress such as [31,32,49–51,56, 57,65] regarding the study of dispersive equations, the illposedness of the Boltzmann equation remains largely open away from [27]. One certainly would like to have the sharp problem resolved for the Boltzmann equations. On the other hand, to initiate a systematic study of a large project including sharp well-posedness, blow-up analysis, regularity criteria, etc, it is of priority to find out the well/ill-posedness separation point. In the paper, moving forward from the special case [27], we investigate the general kernel with soft potentials, for which both the sharp well-posedness and ill-posedeness are open. We settle this problem and provide the well/ill-posedness threshold. With the finding of this optimal regularity index, we deal with the sharp small data global well-posedness in another paper [30].[2]

We start with the connection between the analysis of (1.1) and the theory of nonlinear dispersive PDEs. Let $\widetilde{f}(t, x, \xi)$ be the inverse Fourier transform in the velocity variable, that is,

$$\widetilde{f}(t, x, \xi) = \mathcal{F}^{-1}_{v\mapsto\xi}(f). \tag{1.7}$$

[1] Instead of scaling invariance of equation, the critical regularity for the Boltzmann equation is sometimes believed at $s = \frac{d}{2}$ in the sense that the critical embedding $H^{\frac{d}{2}} \hookrightarrow L^\infty$ fails, see for example [3,36–38].

[2] The hard potential case is also interesting and the ill-posedness result remains open. Our approximation solution gives desired bad behaviors for the hard potential. But it needs a totally different work space to generate the exact solution. Hence, we put it for further work.

Then the linear part of (1.1) is changed into the symmetric hyperbolic Schrödinger equation

$$i\partial_t \widetilde{f} + \nabla_\xi \cdot \nabla_x \widetilde{f} = 0, \tag{1.8}$$

which, in the nonlinear context, enables the application of Strichartz estimates that

$$\|e^{it\nabla_\xi\cdot\nabla_x}\widetilde{f_0}\|_{L_t^q L_{x\xi}^p} \lesssim \|\widetilde{f_0}\|_{L_{x\xi}^2}, \quad \frac{2}{q}+\frac{2d}{p}=d, \quad q\geq 2,\ d\geq 2. \tag{1.9}$$

We introduce the Sobolev norms

$$\|\widetilde{f}\|_{H_x^s H_\xi^r} = \|\langle\nabla_x\rangle^s\langle\nabla_\xi\rangle^r \widetilde{f}\|_{L_{x\xi}^2} = \|\langle\nabla_x\rangle^s\langle v\rangle^r f\|_{L_{xv}^2} = \|f\|_{L_v^{2,r}H_x^s}, \tag{1.10}$$

and the Fourier restriction norms (see [8–10,53,58])

$$\|\widetilde{f}\|_{X^{s,r,b}} = \|\hat{f}(\tau,\eta,v)\langle\tau+\eta\cdot v\rangle^b\langle\eta\rangle^s\langle v\rangle^r\|_{L_{\tau,\eta,v}^2}, \tag{1.11}$$

where $\hat{f}(\tau,\eta,v)$ denotes the Fourier transform of $\widetilde{f}(t,x,\xi)$ in $(t,x,\xi)\mapsto(\tau,\eta,v)$, and is thus the Fourier transform of $f(t,x,v)$ itself in only $(t,x)\mapsto(\tau,\eta)$, that is,

$$\hat{f}(\tau,\eta,v) = \mathcal{F}(\widetilde{f}) = \mathcal{F}_{(t,x)\mapsto(\tau,\eta)}(f).$$

It is customary to define their finite time restrictions via

$$\|\widetilde{f}\|_{X_T^{s,r,b}} = \inf\left\{\|F\|_{X^{s,r,b}} : F|_{[-T,T]} = \widetilde{f}\right\}. \tag{1.12}$$

We recall the definition of well-posedness, see for example [47,63].

Definition 1.1. We say that (1.1) is well-posed in $L_v^{2,r}H_x^s$ if for each $R>0$, there exists a time $T=T(R)>0$, and a set X, such that all of the following are satisfied.

(a) (Existence and Uniqueness) For each $f_0\in L_v^{2,r}H_x^s$ with $\|f_0\|_{L_v^{2,r}H_x^s}\leqslant R$, there exists a unique solution $f(t,x,v)$ to the integral equation of (1.1) in

$$C\Big([-T,T];L_v^{2,r}H_x^s\Big)\bigcap X.$$

Moreover, $f(t,x,v)\geqslant 0$ if $f_0\geqslant 0$.[3]

(b) (Uniform Continuity of the Solution Map)[4] The map $f_0\mapsto f$ is uniform continuous with the $C([-T,T];L_v^{2,r}H_x^s)$ norm. Specifically, suppose f and g are two solutions to (1.1) on $[-T,T]$, $\forall\varepsilon>0$, $\exists\,\delta(\varepsilon)$ independent of f or g such that

$$\|f(t)-g(t)\|_{C([-T,T];L_v^{2,r}H_x^s)} < \varepsilon \text{ provided that } \|f(0)-g(0)\|_{L_v^{2,r}H_x^s} < \delta(\varepsilon). \tag{1.13}$$

We take X to be the Fourier restriction norm space $X_T^{s,s+\gamma,b}$ defined by (1.12) with $b\in(\frac{1}{2},1)$.

[3] If $f_0\in L_{x,v}^1$, the solution $f(t)$ should also have the $L_{x,v}^1$ integrability in terms of the mass conservation law. However, this is not a simple problem. We deal with it in [30] by using regularity criteria which are beyond the scope of this paper.

[4] One could replace (c) with the Lipschitz continuity which is usually the case as well.

Theorem 1.2 (Main Theorem). *Let $d = 2, 3$.*

(1) *For $s > \frac{d-1}{2}$, $\frac{1-d}{2} \le \gamma \le 0$, (1.1) is locally well-posed in $L_v^{2,s+\gamma} H_x^s$.*
(2) *For $0 \le s_0 < \frac{d-1}{2}$, $\frac{1-d}{2} \le \gamma \le 0$, $r_0 = \max\{0, s_0+\gamma\}$, (1.1) is ill-posed in $L_v^{2,r_0} H_x^{s_0}$ in the sense that the data-to-solution map is not uniformly continuous. In particular, for each $M \gg 1$, there exists a time sequence $\left\{t_0^M\right\}_M$ such that*

$$t_0^M < 0, \quad t_0^M \nearrow 0,$$

and two solutions $f^M(t)$, $g^M(t)$ in $\left[t_0^M, 0\right]$ with

$$\|f^M\left(t_0^M\right)\|_{L_v^{2,r_0} H_x^{s_0}} \sim \|g^M\left(t_0^M\right)\|_{L_v^{2,r_0} H_x^{s_0}} \sim 1,$$

such that they are initially close at $t = t_0^M$

$$\|f^M\left(t_0^M\right) - g^M\left(t_0^M\right)\|_{L_v^{2,r_0} H_x^{s_0}} \leqslant \frac{1}{\ln \ln M} \ll 1,$$

but become fully separated at $t = 0$

$$\|f^M(0) - g^M(0)\|_{L_v^{2,r_0} H_x^{s_0}} \sim 1.$$

Theorem 1.2 is the main novelty, which finds the well/ill-posedness threshold, by establishing the sharp local well-posedness, and proving the ill-posedness for the soft potential case. There have been many nice work on the well-posedness part by the energy method which requires higher regularity, see for example [3,41,42,62]. For both Maxwellian molecules and hard potentials, the regularity index $s > \frac{d-1}{2}$ for well-posedness was achieved in [13] without ill-posedness. Our well-posedness result solves the remaining soft potential case.

We remark that, as scaling (1.6) in L^2 setting gives the restriction that $s + \gamma \ge 0$, the range $\frac{1-d}{2} \le \gamma \le 0$ should be sharp if one seeks the optimal regularity $s > \frac{d-1}{2}$. In addition, the endpoint case $\gamma = -1$ with $d = 3$ plays an important role in the derivation of the Boltzmann equation from quantum many-body dynamics in [26], where the collision kernel is composed of part hard sphere and part inverse power potential:

$$B(u-v,\omega) = \left(1_{\{|u-v|\le 1\}}|u-v| + 1_{\{|u-v|\ge 1\}}|u-v|^{-1}\right) \mathbf{b}\left(\frac{u-v}{|u-v|}\cdot\omega\right), \tag{1.14}$$

which also provides yet another physical background to our problem here. Our proof for ill-posedness also works for kernel (1.14).

Corollary 1.3. *For $d = 3$, $0 \le s_0 < 1$, (1.1) is ill-posed in $L_v^2 H_x^{s_0}$ with the kernel (1.14) in the sense that the data-to-solution map is not uniformly continuous.*

1.1. Outline of the paper. In Sect. 2, we prove the well-posedeness of (1.1). The bilinear estimates for gain/loss terms are the key step to conclude the well-posedness and the proof highly relies on the techniques from dispersive PDEs.

In Sect. 2.1, we appeal to dispersive estimates to prove the loss term bilinear estimate. This can be directly handled because of the factorization of the kernel. In Sect. 2.2, we deal with the gain term, which requires a more subtle analysis due to the complicated partial convolution structure. One important observation is that the energy conservation provides a lower bound estimate for after-collision velocities, which enables the application of the Littlewood–Paley theory and frequency analysis techniques in multi-linear estimates. Then with a convolution type estimate in [4], we are able to establish the gain term bilinear estimate with the help of Strichartz estimates in the Fourier restriction norm space. Finally, in Sect. 2.3, we complete the proof of well-posedness after our built-up $X^{s,r,b}$ spaces and its related frequency analysis in this context.

In Sect. 3, we prove the ill-posedness of (1.1). The idea is to construct an approximation solution which has the norm deflation property and then perturb it into an exact solution. We improvise and sharpen the prototype approximation solution found in [27]. To overcome the singularities carried by the soft potentials, which were known to be the main difficulties, we introduce a new scaling on the approximation solution, create an elaborate Z-norm, which is used to prove a closed estimate for the gain and loss terms, that is,

$$\|Q^{\pm}(f_1, f_2)\|_Z \lesssim \|f_1\|_Z \|f_2\|_Z, \tag{1.15}$$

and conclude the existence of small corrections. With this new treatment, the extra restriction that $s_0 > \frac{1}{2}$ in [27] can now be removed.

In Sect. 3.1, we first construct the approximation solution f_a and prove its norm deflation. Then in Sect. 3.1.1, we give a discussion on the L^1-based spaces and the hard potentials case, for which our approximation solution also gives desired bad behaviors.[5] Therefore, a similar mechanism of norm deflation in different settings is possible and deserves further investigations.

In Sects. 3.2–3.4, we introduce the Z-norm space and perform a perturbation argument to turn the approximation solution into the exact solution. In Sect. 3.2, we first provide the Z-norm bounds on the approximation solution. Then in Sect. 3.3, we deal with the error terms and prove the Z-norm error estimates. Proving the error estimates, as it includes a large quantity of error terms involving singularities at which we need geometric techniques on the nonlinear interactions between frequencies, is the most intricate part which we treat in Sects. 3.3.1, 3.3.2, 3.3.3, 3.3.4,. After dealing with the error terms, we prove that there is an exact solution which is mostly f_a in Sect. 3.4, and thus conclude the ill-posedness result in Sect. 3.5.

After the proof of the main theorem, we put and review some tools in Appendix A and the Strichartz estimates in Appendix B.

2. Well-Posedness

To conclude the well-posedeness of (1.1), it suffices to prove the following bilinear estimates

$$\|\langle \nabla_x \rangle^s \langle v \rangle^{s+\gamma} Q^{\pm}(f, g)\|_{L_t^2 L_{x,v}^2} \lesssim \|\widetilde{f}\|_{X^{s,s+\gamma,b}} \|\widetilde{g}\|_{X^{s,s+\gamma,b}}. \tag{2.1}$$

[5] It then provides a formal answer to a question raised by Professor K. Nakanishi.

Note that no v-variable Fourier transform of the collision kernel in (2.1) is needed if we fully work in the $X^{s,s+\gamma,b}$ space. Here, we will work on the (x,ξ) side and prove (2.1) by use of the Fourier transform of the kernel.

Taking the inverse v-variable Fourier transform on both side of (1.1), we get

$$i\partial_t \widetilde{f} + \nabla_\xi \cdot \nabla_x \widetilde{f} = i\mathcal{F}^{-1}_{v\mapsto\xi}\left[Q(f,f)\right]. \tag{2.2}$$

By the well-known Bobylev identity in a more general case, see for example [1,33], it holds that (up to an unimportant constant)

$$\mathcal{F}^{-1}_{v\mapsto\xi}\left[Q^-(f,g)\right](\xi) = \|\mathbf{b}\|_{L^1(\mathbb{S}^{d-1})}\int \frac{\widetilde{f}(\xi-\eta)\widetilde{g}(\eta)}{|\eta|^{d+\gamma}}d\eta, \tag{2.3}$$

$$\mathcal{F}^{-1}_{v\mapsto\xi}\left[Q^+(f,g)\right](\xi) = \int_{\mathbb{R}^d\times\mathbb{S}^{d-1}} \frac{\widetilde{f}(\xi^+ + \eta)\widetilde{g}(\xi^- - \eta)}{|\eta|^{d+\gamma}}\mathbf{b}\left(\frac{\xi}{|\xi|}\cdot\omega\right)d\eta d\omega, \tag{2.4}$$

where $\xi^+ = \frac{1}{2}(\xi + |\xi|\omega)$ and $\xi^- = \frac{1}{2}(\xi - |\xi|\omega)$. For convenience, we take the notation that $\widetilde{Q}^{\pm}(\widetilde{f},\widetilde{g}) = \mathcal{F}^{-1}_{v\mapsto\xi}\left[Q^{\pm}(f,g)\right]$.

In Sects. 2.1, 2.2, we establish the bilinear estimates for the loss and gain terms respectively. Then in Sect. 2.3, we complete the proof of the well-posedness of (1.1).

2.1. Bilinear estimate for loss term.

Lemma 2.1. *For $s > \frac{d-1}{2}$, it holds that*

$$\|\langle\nabla_x\rangle^s\langle\nabla_\xi\rangle^{s+\gamma}\widetilde{Q}^-(\widetilde{f},\widetilde{g})\|_{L^{2+}_t L^2_{x\xi}} \lesssim \|\widetilde{f}\|_{X^{s,s+\gamma,b}}\|\widetilde{g}\|_{X^{s,s+\gamma,b}}. \tag{2.5}$$

Proof. By the fractional Leibniz rule in Lemma A.1, we have

$$\begin{aligned}
&\|\widetilde{Q}^-(\widetilde{f},\widetilde{g})\|_{L^{2+}_t H^s_x H^{s+\gamma}_\xi} \\
&= \left\|\langle\nabla_x\rangle^s\int\langle\nabla_\xi\rangle^{s+\gamma}\widetilde{f}(t,x,\xi-\eta)\frac{\widetilde{g}(t,x,\eta)}{|\eta|^{d+\gamma}}d\eta\right\|_{L^{2+}_t L^2_{x\xi}} \\
&\lesssim \left\|\int\|\langle\nabla_x\rangle^s\langle\nabla_\xi\rangle^{s+\gamma}\widetilde{f}(t,x,\xi-\eta)\|_{L^2_x}\left\|\frac{\widetilde{g}(t,x,\eta)}{|\eta|^{d+\gamma}}\right\|_{L^\infty_x}d\eta\right\|_{L^{2+}_t L^2_\xi} \\
&\quad + \left\|\int\|\langle\nabla_\xi\rangle^{s+\gamma}\widetilde{f}(t,x,\xi-\eta)\|_{L^{2d+}_x}\left\|\frac{\langle\nabla_x\rangle^s\widetilde{g}(t,x,\eta)}{|\eta|^{d+\gamma}}\right\|_{L_x^{\frac{2d}{d-1}-}}d\eta\right\|_{L^{2+}_t L^2_\xi}.
\end{aligned}$$

Applying Sobolev inequalities that $W^{s,\frac{2d}{d-1}-} \hookrightarrow L^\infty$, $W^{s,2} \hookrightarrow L^{2d+}$ and Young's inequality,

$$\begin{aligned}
&\|\widetilde{Q}^-(\widetilde{f},\widetilde{g})\|_{L^{2+}_t H^s_x H^{s+\gamma}_\xi} \\
&\lesssim \|\langle\nabla_x\rangle^s\langle\nabla_\xi\rangle^{s+\gamma}\widetilde{f}(t,x,\xi)\|_{L^\infty_t L^2_\xi L^2_x}\left\|\frac{\langle\nabla_x\rangle^s\widetilde{g}(t,x,\eta)}{|\eta|^{d+\gamma}}\right\|_{L^{2+}_t L^1_\eta L_x^{\frac{2d}{d-1}-}} \\
&\quad + \|\langle\nabla_\xi\rangle^{s+\gamma}\widetilde{f}(t,x,\xi)\|_{L^\infty_t L^2_\xi L^{2d+}_x}\left\|\frac{\langle\nabla_x\rangle^s\widetilde{g}(t,x,\eta)}{|\eta|^{d+\gamma}}\right\|_{L^{2+}_t L^1_\eta L_x^{\frac{2d}{d-1}-}}
\end{aligned}$$

$$\lesssim \left\| \langle\nabla_x\rangle^s \left\langle\nabla_\xi\right\rangle^{s+\gamma} \widetilde{f}(t,x,\xi)\right\|_{L_t^\infty L_\xi^2 L_x^2} \left\| \frac{\langle\nabla_x\rangle^s \widetilde{g}(t,x,\eta)}{|\eta|^{d+\gamma}} \right\|_{L_t^{2+} L_\eta^1 L_x^{\frac{2d}{d-1}-}}. \tag{2.6}$$

We are left to deal with the last term on the right hand side of (2.6). Set

$$G(\eta) = \left\| \langle\nabla_x\rangle^s \widetilde{g}(t,x,\eta) \right\|_{L_x^{\frac{2d}{d-1}-}}.$$

Then by Hardy–Littlewood–Sobolev inequality (A.4) in Lemma A.3 with $\lambda = d+\gamma$, we obtain

$$\int \frac{G(\eta)}{|\eta|^{d+\gamma}} d\eta \lesssim \|G\|^{\alpha}_{L^{\frac{2d}{d-1}-}} \|G\|^{1-\alpha}_{L^{\frac{d}{-\gamma}+}}.$$

Therefore, we have

$$\begin{aligned} &\left\| \frac{\langle\nabla_x\rangle^s \widetilde{g}(t,x,\eta)}{|\eta|^{d+\gamma}} \right\|_{L_t^{2+} L_\eta^1 L_x^{\frac{2d}{d-1}-}} \\ &\lesssim \left\| \langle\nabla_x\rangle^s \widetilde{g}(t,x,\eta)\right\|^{\alpha}_{L_t^{2+} L_\eta^{\frac{2d}{d-1}-} L_x^{\frac{2d}{d-1}-}} \left\| \langle\nabla_x\rangle^s \widetilde{g}(t,x,\eta)\right\|^{1-\alpha}_{L_t^{2+} L_\eta^{\frac{d}{-\gamma}+} L_x^{\frac{2d}{d-1}-}} \\ &\le \left\| \langle\nabla_x\rangle^s \widetilde{g}(t,x,\eta)\right\|^{\alpha}_{L_t^{2+} L_x^{\frac{2d}{d-1}-} L_\eta^{\frac{2d}{d-1}-}} \left\| \langle\nabla_x\rangle^s \widetilde{g}(t,x,\eta)\right\|^{1-\alpha}_{L_t^{2+} L_x^{\frac{2d}{d-1}-} L_\eta^{\frac{d}{-\gamma}+}} \end{aligned}$$

where in the last inequality we have used the Minkowski inequality. Applying Sobolev inequality that $W^{s+\gamma,\frac{2d}{d-1}-} \hookrightarrow L^{\frac{d}{-\gamma}+}$ and Strichartz estimate (2.25), we arrive at

$$\begin{aligned} &\left\| \frac{\langle\nabla_x\rangle^s \widetilde{g}(t,x,\eta)}{|\eta|^{d+\gamma}} \right\|_{L_t^{2+} L_\eta^1 L_x^{\frac{2d}{d-1}-}} \\ &\le \left\| \langle\nabla_x\rangle^s \widetilde{g}(t,x,\eta)\right\|^{\alpha}_{L_t^{2+} L_x^{\frac{2d}{d-1}-} L_\eta^{\frac{2d}{d-1}-}} \left\| \langle\nabla_\eta\rangle^{s+\gamma} \langle\nabla_x\rangle^s \widetilde{g}(t,x,\eta)\right\|^{1-\alpha}_{L_t^{2+} L_x^{\frac{2d}{d-1}-} L_\eta^{\frac{2d}{d-1}-}} \\ &\le \left\| \langle\nabla_\eta\rangle^{s+\gamma} \langle\nabla_x\rangle^s \widetilde{g}(t,x,\eta)\right\|_{L_t^{2+} L_x^{\frac{2d}{d-1}-} L_\eta^{\frac{2d}{d-1}-}} \\ &\le \|\widetilde{g}\|_{X^{s,s+\gamma,b}}. \end{aligned}$$

Hence, we complete the proof of (2.5). □

2.2. Bilinear estimate for gain term. Before proving the bilinear estimate for the gain term, we first give a useful lemma as follows.

Lemma 2.2. *Let* $\frac{1}{p} + \frac{1}{q} = \frac{1}{2}$.

$$\left\| \int_{\mathbb{S}^{d-1}} \int_{\mathbb{R}^d} \frac{\widetilde{f}(\xi^+ + \eta)\widetilde{g}(\xi^- - \eta)}{|\eta|^{d+\gamma}} \boldsymbol{b}\left(\frac{\xi}{|\xi|}\cdot\omega\right) d\eta d\omega \right\|_{L_\xi^2} \lesssim \|\widetilde{f}\|_{L^{\frac{2pd}{2d-p\gamma}}} \|\widetilde{g}\|_{L^{\frac{2qd}{2d-q\gamma}}}. \tag{2.7}$$

In particular, we have

$$\|\widetilde{Q}^+(\widetilde{f},\widetilde{g})\|_{L_\xi^2} \lesssim \|\widetilde{f}\|_{L_\xi^2} \|\widetilde{g}\|_{L_\xi^{\frac{d}{-\gamma}}}, \tag{2.8}$$

$$\|\widetilde{Q}^+(\widetilde{f},\widetilde{g})\|_{L_\xi^2} \lesssim \|\widetilde{f}\|_{L_\xi^{\frac{d}{-\gamma}}} \|\widetilde{g}\|_{L_\xi^2}. \tag{2.9}$$

Proof. For the case of Maxwellian molecules, it holds that

$$\Big\| \int_{\mathbb{S}^{d-1}} \widetilde{f}(\xi^+)\widetilde{g}(\xi^-)\mathbf{b}(\frac{\xi}{|\xi|}\cdot\omega)d\omega\Big\|_{L^2_\xi} \lesssim \|\widetilde{f}\|_{L^p_\xi}\|\widetilde{g}\|_{L^q_\xi}, \quad \frac{1}{p}+\frac{1}{q}=\frac{1}{2}, \tag{2.10}$$

which is proved in [4, Theorem 1]. By Cauchy-Schwarz inequality and then (2.10), we have

$$\begin{aligned}
&\Big\| \int_{\mathbb{S}^{d-1}}\int_{\mathbb{R}^d} \frac{\widetilde{f}(\xi^+ + \eta)\widetilde{g}(\xi^- - \eta)}{|\eta|^{d+\gamma}}\mathbf{b}\Big(\frac{\xi}{|\xi|}\cdot\omega\Big)d\eta d\omega\Big\|_{L^2_\xi}\\
&\le \Big\| \int_{\mathbb{S}^{d-1}} \Big[\int_{\mathbb{R}^d}\frac{|\widetilde{f}(\xi^+ + \eta)|^2}{|\eta|^{d+\gamma}}d\eta\Big]^{\frac12}\Big[\int_{\mathbb{R}^d}\frac{|\widetilde{g}(\xi^- - \eta)|^2}{|\eta|^{d+\gamma}}d\eta\Big]^{\frac12}\mathbf{b}\Big(\frac{\xi}{|\xi|}\cdot\omega\Big)d\omega\Big\|_{L^2_\xi}\\
&\lesssim \Big\|\Big[\int_{\mathbb{R}^d}\frac{|\widetilde{f}(\xi+\eta)|^2}{|\eta|^{d+\gamma}}d\eta\Big]^{\frac12}\Big\|_{L^p_\xi}\Big\|\Big[\int_{\mathbb{R}^d}\frac{|\widetilde{g}(\xi-\eta)|^2}{|\eta|^{d+\gamma}}d\eta\Big]^{\frac12}\Big\|_{L^q_\xi}\\
&= \Big\|\int_{\mathbb{R}^d}\frac{|\widetilde{f}(\xi+\eta)|^2}{|\eta|^{d+\gamma}}d\eta\Big\|^{\frac12}_{L^{\frac{p}{2}}_\xi}\Big\|\int_{\mathbb{R}^d}\frac{|\widetilde{g}(\xi-\eta)|^2}{|\eta|^{d+\gamma}}d\eta\Big\|^{\frac12}_{L^{\frac{q}{2}}_\xi}\\
&\lesssim \|\widetilde{f}\|_{L^{\frac{2pd}{2d-p\gamma}}}\|\widetilde{g}\|_{L^{\frac{2qd}{2d-q\gamma}}},
\end{aligned} \tag{2.11}$$

where in the last inequality we have used Hardy–Littlewood–Sobolev inequality (A.3). This completes the proof of (2.7). Then by taking

$$(p,q)=\Big(\frac{2d}{d+\gamma},-\frac{2d}{\gamma}\Big),\quad (p,q)=\Big(-\frac{2d}{\gamma},\frac{2d}{d+\gamma}\Big),$$

we immediately obtain (2.8) and (2.9). □

To prove the bilinear estimate for the gain term, we need a detailed frequency analysis from Littlewood–Paley theory.[6] Let $\chi(x)$ be a smooth function and satisfy $\chi(x)=1$ for all $|x|\le 1$ and $\chi(x)=0$ for $|x|\ge 2$. Let N be a dyadic number and set $\varphi_N(x)=\chi(\frac{x}{N})-\chi(\frac{x}{2N})$. Define the Littlewood–Paley projector

$$\widehat{P_N u}(\eta)=\varphi_N(\eta)\widehat{u}(\eta). \tag{2.12}$$

We denote by P^x_N/P^ξ_M the projector of the x-variable and ξ-variable respectively. Now, we delve into the analysis of the bilinear estimate.

Lemma 2.3. *For $s>\frac{d-1}{2}$, we have*

$$\|\langle\nabla_x\rangle^s\langle\nabla_\xi\rangle^{s+\gamma}\widetilde{Q}^+(\widetilde{f},\widetilde{g})\|_{L^2_tL^2_{x\xi}} \lesssim \|\widetilde{f}\|_{X^{s,s+\gamma,b}}\|\widetilde{g}\|_{X^{s,s+\gamma,b}}. \tag{2.13}$$

Proof. By duality, (2.13) is equivalent to

$$\int \widetilde{Q}^+(\widetilde{f},\widetilde{g})h\,dx\,d\xi\,dt \lesssim \|\widetilde{f}\|_{X^{s,s+\gamma,b}}\|\widetilde{g}\|_{X^{s,s+\gamma,b}}\|h\|_{L^2_tH^{-s-\gamma}_\xi H^{-s}_x}. \tag{2.14}$$

[6] See [23,25,29] for some examples sharing similar critical flavor but carrying completely different structures.

We denote by I the integral in (2.14). Inserting a Littlewood–Paley decomposition gives that

$$I = \sum_{\substack{M,M_1,M_2\\N,N_1,N_2}} I_{M,M_1,M_2,N,N_1,N_2}$$

where

$$I_{M,M_1,M_2,N,N_1,N_2} = \int \widetilde{Q}^+\left(P^x_{N_1}P^\xi_{M_1}\widetilde{f},\, P^x_{N_2}P^\xi_{M_2}\widetilde{g}\right)P^x_N P^\xi_M h dx d\xi dt.$$

Note that $\widetilde{Q}^+$ commutes with P^x_N, so this gives the constraint that $N \lesssim \max(N_1, N_2)$ due to that

$$P^x_N\left(P^x_{N_1}\widetilde{f}P^x_{N_2}\widetilde{g}\right) = 0, \quad \text{if } N \geq 10\max(N_1, N_2). \tag{2.15}$$

In addition, we observe that such a property (2.15) is also hinted in the ξ-variable, that is,

$$P^\xi_M\widetilde{Q}^+\left(P^\xi_{M_1}\widetilde{f},\, P^\xi_{M_2}\widetilde{g}\right) = 0, \quad \text{if } M \geq 10\max(M_1, M_2). \tag{2.16}$$

Indeed, notice that

$$\begin{aligned}&\mathcal{F}_\xi\left(P^\xi_M\widetilde{Q}^+\left(P^\xi_{M_1}\widetilde{f},\, P^\xi_{M_2}\widetilde{g}\right)\right)\\&\quad= \varphi_M(v)\int_{\mathbb{R}^d}\int_{\mathbb{S}^{d-1}}(\varphi_{M_1}f)(v^*)(\varphi_{M_2}g)(u^*)B(u-v,\omega)dud\omega.\end{aligned} \tag{2.17}$$

Then from the energy conservation which implies the inequality $|v|^2 \leq |v^*|^2 + |u^*|^2$, we have the lower bound that

$$|u^*| \geq \frac{M}{4}, \text{ or } |v^*| \geq \frac{M}{4} \tag{2.18}$$

for all $(u,\omega) \in \mathbb{R}^d \times \mathbb{S}^{d-1}$ and $|v| \geq \frac{M}{2}$. Therefore, for $M \geq 10\max(M_1, M_2)$, the lower bound (2.18) forces the v^*-variable or u^*-variable off their own support set, which makes the integral on the right hand side of (2.17) vanish. Hence, this gives the constraint that $M \lesssim \max(M_1, M_2)$.

Now, we divide the sum into four cases as follows

Case A. $M_1 \geq M_2$, $N_1 \geq N_2$.

Case B. $M_1 \leq M_2$, $N_1 \geq N_2$.

Case C. $M_1 \geq M_2$, $N_1 \leq N_2$.

Case D. $M_1 \leq M_2$, $N_1 \leq N_2$.

We only need to treat Cases A and B, as Cases C and D follow similarly.

Case A. $M_1 \geq M_2$, $N_1 \geq N_2$.

Let I_A denote the integral restricted to the Case A. By Cauchy-Schwarz,

$$I_A \lesssim \sum_{\substack{M,M_1\geq M_2\\N,N_1\geq N_2\\M_1\gtrsim M,N_1\gtrsim N}} \|\widetilde{Q}^+\left(P^x_{N_1}P^\xi_{M_1}\widetilde{f},\, P^x_{N_2}P^\xi_{M_2}\widetilde{g}\right)\|_{L^2_tL^2_{x\xi}}\|P^x_N P^\xi_M h\|_{L^2_tL^2_{x\xi}}.$$

By using the estimate (2.8) in Lemma 2.7 and then Hölder inequality, we have

$$
\begin{aligned}
I_A &\lesssim \sum_{\substack{M,M_1\geq M_2\\ N,N_1\geq N_2\\ M_1\geqslant M,N_1\geqslant N}} \left\| \|P^x_{N_1}P^\xi_{M_1}\widetilde{f}\|_{L^2_\xi}\|P^x_{N_2}P^\xi_{M_2}\widetilde{g}\|_{L_\xi^{\frac{d}{-\gamma}}}\right\|_{L^2_tL^2_x}\|P^x_NP^\xi_Mh\|_{L^2_tL^2_{x\xi}}\\
&\leq \sum_{\substack{M,M_1\geq M_2\\ N,N_1\geq N_2\\ M_1\geqslant M,N_1\geqslant N}} \|P^x_{N_1}P^\xi_{M_1}\widetilde{f}\|_{L^\infty_tL^2_xL^2_\xi}\|P^x_{N_2}P^\xi_{M_2}\widetilde{g}\|_{L^2_tL^\infty_xL_\xi^{\frac{d}{-\gamma}}}\|P^x_NP^\xi_Mh\|_{L^2_tL^2_{x\xi}}.
\end{aligned}
$$

By using Minkowski inequality, Sobolev inequality that $W^{\frac{d-1}{2}+\gamma,\frac{2d}{d-1}}\hookrightarrow L^{\frac{d}{-\gamma}}$, and Bernstein inequality that $\|P^x_{N_2}\widetilde{g}\|_{L^\infty_x}\lesssim\|\langle\nabla_x\rangle^{\frac{d-1}{2}}P^x_{N_2}\widetilde{g}\|_{L_x^{\frac{2d}{d-1}}}$, we obtain

$$
\begin{aligned}
I_A &\lesssim \sum_{\substack{M,M_1\geq M_2\\ N,N_1\geq N_2\\ M_1\geqslant M,N_1\geqslant N}} \frac{N^sM^{s+\gamma}}{N_1^sM_1^{s+\gamma}}\|P^x_{N_1}P^\xi_{M_1}\langle\nabla_x\rangle^s\langle\nabla_\xi\rangle^{s+\gamma}\widetilde{f}\|_{L^\infty_tL^2_xL^2_\xi}\\
&\quad\times\|P^x_{N_2}P^\xi_{M_2}\langle\nabla_x\rangle^{\frac{d-1}{2}}\langle\nabla_\xi\rangle^{\frac{d-1}{2}+\gamma}\widetilde{g}\|_{L^2_tL_x^{\frac{2d}{d-1}}L_\xi^{\frac{2d}{d-1}}}\|P^x_NP^\xi_M\langle\nabla_x\rangle^{-s}\langle\nabla_\xi\rangle^{-s-\gamma}h\|_{L^2_tL^2_{x\xi}}\\
&\lesssim \sum_{\substack{M,M_1\geq M_2\\ N,N_1\geq N_2\\ M_1\geqslant M,N_1\geqslant N}} \frac{N^sM^{s+\gamma}}{N_1^sM_1^{s+\gamma}}\frac{1}{N_2^{s-\frac{d-1}{2}}}\frac{1}{M_2^{s-\frac{d-1}{2}}}\|P^x_NP^\xi_Mh\|_{L^2_tH_\xi^{-s-\gamma}H_x^{-s}}\\
&\quad\times\|P^x_{N_1}P^\xi_{M_1}\langle\nabla_x\rangle^s\left\langle\nabla_\xi\right\rangle^{s+\gamma}\widetilde{f}\|_{L^\infty_tL^2_xL^2_\xi}\|P^x_{N_2}P^\xi_{M_2}\langle\nabla_x\rangle^s\left\langle\nabla_\xi\right\rangle^{s+\gamma}\widetilde{g}\|_{L^2_tL_x^{\frac{2d}{d-1}}L_\xi^{\frac{2d}{d-1}}},
\end{aligned}
$$

where in the last inequality we have used Bernstein inequality again. By Strichartz estimate (2.25),

$$
\begin{aligned}
I_A &\lesssim \sum_{\substack{M,M_1\geq M_2\\ N,N_1\geq N_2\\ M_1\geqslant M,N_1\geqslant N}} \frac{N^sM^{s+\gamma}}{N_1^sM_1^{s+\gamma}}\frac{1}{N_2^{s-\frac{d-1}{2}}}\frac{1}{M_2^{s-\frac{d-1}{2}}}\|P^x_NP^\xi_Mh\|_{L^2_tH_\xi^{-s-\gamma}H_x^{-s}}\\
&\quad\times\|P^x_{N_1}P^\xi_{M_1}\widetilde{f}\|_{X^{s,s+\gamma,b}}\|P^x_{N_2}P^\xi_{M_2}\widetilde{g}\|_{X^{s,s+\gamma,b}}.
\end{aligned}
$$

Note that $s>\frac{d-1}{2}$, so we use that $\|P^x_{N_2}P^\xi_{M_2}\widetilde{g}\|_{X^{s,s+\gamma,b}}\lesssim\|\widetilde{g}\|_{X^{s,s+\gamma,b}}$ and then carry out the N_2, M_2 sums to obtain

$$
I_A\lesssim\|\widetilde{g}\|_{X^{s,s+\gamma,b}}\sum_{\substack{M_1\geq M\\ N_1\geq N}}\frac{N^sM^{s+\gamma}}{N_1^sM_1^{s+\gamma}}\|P^x_{N_1}P^\xi_{M_1}\widetilde{f}\|_{X^{s,s+\gamma,b}}\|P^x_NP^\xi_Mh\|_{L^2_tH_\xi^{-s-\gamma}H_x^{-s}}.
$$

By Cauchy-Schwarz in M, M_1, N and N_1, we have

$$
I_A\lesssim\|\widetilde{g}\|_{X^{s,s+\gamma,b}}\left(\sum_{\substack{M_1\geq M\\ N_1\geq N}}\frac{N^sM^{s+\gamma}}{N_1^sM_1^{s+\gamma}}\|P^x_{N_1}P^\xi_{M_1}\widetilde{f}\|^2_{X^{s,s+\gamma,b}}\right)^{\frac{1}{2}}
$$

$$\left(\sum_{\substack{M_1\geq M\\ N_1\geq N}}\frac{N^s M^{s+\gamma}}{N_1^s M_1^{s+\gamma}}\|P_N^x P_M^\xi h\|^2_{L_t^2 H_\xi^{-s-\gamma} H_x^{-s}}\right)^{\frac{1}{2}}$$
$$\lesssim \|\widetilde{f}\|_{X^{s,s+\gamma,b}}\|\widetilde{g}\|_{X^{s,s+\gamma,b}}\|h\|_{L_t^2 H_\xi^{-s-\gamma} H_x^{-s}}, \tag{2.19}$$

which completes the proof of (2.14) for Case A.

Case B. $M_1 \leq M_2$, $N_1 \geq N_2$.

Following the same way as Case A, we have

$$I_B \lesssim \sum_{\substack{M,M_2\geq M_1\\ N,N_1\geq N_2\\ M_2\gtrsim M, N_1\gtrsim N}} \|\widetilde{Q}^+(P_{N_1}^x P_{M_1}^\xi \widetilde{f}, P_{N_2}^x P_{M_2}^\xi \widetilde{g})\|_{L_t^2 L_{x\xi}^2}\|P_N^x P_M^\xi h\|_{L_t^2 L_{x\xi}^2}.$$

By using the estimate (2.9) in Lemma 2.7 and then Hölder inequality, we have

$$I_B \lesssim \sum_{\substack{M,M_2\geq M_1\\ N,N_1\geq N_2\\ M_2\gtrsim M, N_1\gtrsim N}} \left\| \|P_{N_1}^x P_{M_1}^\xi \widetilde{f}\|_{L_\xi^{\frac{d}{-\gamma}}} \|P_{N_2}^x P_{M_2}^\xi \widetilde{g}\|_{L_\xi^2}\right\|_{L_t^2 L_x^2}\|P_N^x P_M^\xi h\|_{L_t^2 L_{x\xi}^2}$$
$$\leq \sum_{\substack{M,M_2\geq M_1\\ N,N_1\geq N_2\\ M_2\gtrsim M, N_1\gtrsim N}} \|P_{N_1}^x P_{M_1}^\xi \widetilde{f}\|_{L_t^2 L_x^{\frac{2d}{d-1}} L_\xi^{\frac{d}{-\gamma}}} \|P_{N_2}^x P_{M_2}^\xi \widetilde{g}\|_{L_t^\infty L_x^{2d} L_\xi^2}\|P_N^x P_M^\xi h\|_{L_t^2 L_{x\xi}^2}.$$

By Minkowski inequality, Sobolev inequality that $W^{\frac{d-1}{2}+\gamma,\frac{2d}{d-1}} \hookrightarrow L^{\frac{d}{-\gamma}}$, $W^{\frac{d-1}{2},2} \hookrightarrow L^{2d}$ and Bernstein inequality, we obtain

$$I_B \lesssim \sum_{\substack{M,M_2\geq M_1\\ N,N_1\geq N_2\\ M_2\gtrsim M, N_1\gtrsim N}} \|P_{N_1}^x P_{M_1}^\xi \langle\nabla_\xi\rangle^{\frac{d-1}{2}+\gamma}\widetilde{f}\|_{L_t^2 L_x^{\frac{2d}{d-1}} L_\xi^{\frac{2d}{d-1}}}$$
$$\times \|P_{N_2}^x P_{M_2}^\xi \langle\nabla_x\rangle^{\frac{d-1}{2}}\widetilde{g}\|_{L_t^\infty L_x^2 L_\xi^2}\|P_N^x P_M^\xi h\|_{L_t^2 L_{x\xi}^2}$$
$$\lesssim \sum_{\substack{M,M_2\geq M_1\\ N,N_1\geq N_2\\ M_2\gtrsim M, N_1\gtrsim N}} \frac{N^s M^{s+\gamma}}{N_1^s M_2^{s+\gamma}}\frac{1}{N_2^{s-\frac{d-1}{2}}}\frac{1}{M_1^{s-\frac{d-1}{2}}}\|P_N^x P_M^\xi h\|_{L_t^2 H_\xi^{-s-\gamma} H_x^{-s}}$$
$$\times \|P_{N_1}^x P_{M_1}^\xi \langle\nabla_x\rangle^s\langle\nabla_\xi\rangle^{s+\gamma}\widetilde{f}\|_{L_t^2 L_x^{\frac{2d}{d-1}} L_\xi^{\frac{2d}{d-1}}}\|P_{N_2}^x P_{M_2}^\xi \langle\nabla_x\rangle^s\left\langle\nabla_\xi\right\rangle^{s+\gamma}\widetilde{g}\|_{L_t^\infty L_x^2 L_\xi^2}.$$

By Strichartz estimate (2.25),

$$I_B \lesssim \sum_{\substack{M,M_2\geq M_1\\ N,N_1\geq N_2\\ M_2\gtrsim M, N_1\gtrsim N}} \frac{N^s M^{s+\gamma}}{N_1^s M_2^{s+\gamma}}\frac{1}{N_2^{s-\frac{d-1}{2}}}\frac{1}{M_1^{s-\frac{d-1}{2}}}\|P_N^x P_M^\xi h\|_{L_t^2 H_\xi^{-s-\gamma} H_x^{-s}}$$
$$\times \|P_{N_1}^x P_{M_1}^\xi \widetilde{f}\|_{X^{s,s+\gamma,b}}\|P_{N_2}^x P_{M_2}^\xi \widetilde{g}\|_{X^{s,s+\gamma,b}}.$$

Note that $s > \frac{d-1}{2}$, so we use that

$$\|P^x_{N_1} P^\xi_{M_1} \widetilde{f}\|_{X^{s,s+\gamma,b}} \lesssim \|P^x_{N_1} \widetilde{f}\|_{X^{s,s+\gamma,b}}, \quad \|P^x_{N_2} P^\xi_{M_2} \widetilde{g}\|_{X^{s,s+\gamma,b}} \lesssim \|P^\xi_{M_2} \widetilde{g}\|_{X^{s,s+\gamma,b}},$$

and then carry out the N_2, M_1 sums to obtain

$$I_B \lesssim \sum_{\substack{M_2 \geq M \\ N_1 \geq N}} \frac{N^s M^{s+\gamma}}{N_1^s M_2^{s+\gamma}} \|P^x_{N_1} \widetilde{f}\|_{X^{s,s+\gamma,b}} \|P^\xi_{M_2} \widetilde{g}\|_{X^{s,s+\gamma,b}} \|P^x_N P^\xi_M h\|_{L^2_t H^{-s-\gamma}_\xi H^{-s}_x}.$$

In a similar way to (2.19), we use Cauchy-Schwarz inequality to get

$$\begin{aligned} I_B \lesssim & \sum_{N_1 \geq N} \frac{N^s}{N_1^s} \|P^x_{N_1} \widetilde{f}\|_{X^{s,s+\gamma,b}} \left(\sum_{M_2 \geq M} \frac{M^{s+\gamma}}{M_2^{s+\gamma}} \|P^\xi_{M_2} \widetilde{g}\|^2_{X^{s,s+\gamma,b}} \right)^{\frac{1}{2}} \\ & \times \left(\sum_{M_2 \geq M} \frac{M^{s+\gamma}}{M_2^{s+\gamma}} \|P^x_N P^\xi_M h\|^2_{L^2_t H^{-s-\gamma}_\xi H^{-s}_x} \right)^{\frac{1}{2}} \\ \lesssim & \|\widetilde{g}\|_{X^{s,s+\gamma,b}} \sum_{N_1 \geq N} \frac{N^s}{N_1^s} \|P^x_{N_1} \widetilde{f}\|_{X^{s,s+\gamma,b}} \|P^x_N h\|_{L^2_t H^{-s-\gamma}_\xi H^{-s}_x} \\ \lesssim & \|\widetilde{f}\|_{X^{s,s+\gamma,b}} \|\widetilde{g}\|_{X^{s,s+\gamma,b}} \|h\|_{L^2_t H^{-s-\gamma}_\xi H^{-s}_x}. \end{aligned}$$

Hence, we complete the proof of of (2.14) for Case B. □

2.3. Well-posedness in Fourier restriction norm space. We first recall some standard results on the Fourier restriction norms and Strichartz estimates.

Lemma 2.4. *Let* $b \in \left(\frac{1}{2}, 1\right)$, $s \in \mathbb{R}$, $r \in \mathbb{R}$, *and* $\theta(t)$ *be a smooth cutoff function. Define*

$$U(t) := e^{it\nabla_x \cdot \nabla_\xi}, \quad D(F) := \int_0^t U(t-\tau)F(\tau)d\tau. \tag{2.20}$$

Then we have

$$\|\widetilde{f}\|_{C^0_t H^s_x H^r_\xi} \lesssim \|\widetilde{f}\|_{X^{s,r,b}}, \tag{2.21}$$

$$\|\theta(t)U(t)\widetilde{f_0}\|_{X^{s,r,b}} \lesssim \|\widetilde{f_0}\|_{H^s_x H^r_\xi}, \tag{2.22}$$

$$\|\theta(t)D(F)\|_{X^{s,r,b}} \lesssim \|F\|_{X^{s,r,b-1}}, \tag{2.23}$$

$$\|\widetilde{f}\|_{X^{s,r,b-1}} \lesssim \|\widetilde{f}\|_{L^p_t H^s_x H^r_\xi}, \quad p \in (1,2],\ b \leq \frac{3}{2} - \frac{1}{p}, \tag{2.24}$$

$$\|\widetilde{f}\|_{L^q_t L^p_{x\xi}} \lesssim \|\widetilde{f}\|_{X^{0,0,b}}, \quad \frac{2}{q} + \frac{2d}{p} = d, \quad q \geq 2,\ d \geq 2. \tag{2.25}$$

Proof. These type estimates are well-known in the dispersive literatures. The Strichartz estimate (2.25) follows from the linear Strichartz estimate (B.5) and the transference principle. See for example [63, Chapter 2.6]. □

We prove the existence, uniqueness, and the Lipschitz continuity of the solution map. The nonnegativity of f follows from the persistence of regularity (as shown in [14,55]) by use of the bilinear estimates (2.5) and (2.13) for the soft potential case.

Proof of Well-Posedness in Theorem 1.2. Let $\theta_T(t) = \theta(t/T)$. By estimate (2.24), Hölder inequality, and the bilinear estimates for $Q^{\pm}$, we have

$$\begin{aligned}
&\|\theta_T(t)\widetilde{Q}(\widetilde{f},\widetilde{g})\|_{X^{s,s+\gamma,b-1}}\\
&\lesssim \|\theta_T(t)\widetilde{Q}(\widetilde{f},\widetilde{g})\|_{L_t^{\frac{2}{3-2b}}H_x^s H_\xi^{s+\gamma}}\\
&\lesssim T^{1-b}\|\theta_T(t)\widetilde{Q}^{+}(\widetilde{f},\widetilde{g})\|_{L_t^2 H_x^s H_\xi^{s+\gamma}} + T^{1-b}\|\theta_T(t)\widetilde{Q}^{-}(\widetilde{f},\widetilde{g})\|_{L_t^{2+}H_x^s H_\xi^{s+\gamma}}\\
&\lesssim T^{1-b}\|\widetilde{f}\|_{X^{s,s+\gamma,b}}\|\widetilde{g}\|_{X^{s,s+\gamma,b}}. \qquad (2.26)
\end{aligned}$$

Let $B = \left\{\widetilde{f} : \|\widetilde{f}\|_{X^{s,s+\gamma,b}} \le R\right\}$ with $R = 2C\|\widetilde{f}_0\|_{H_x^s H_\xi^{s+\gamma}}$ and define the nonlinear map

$$\Phi(\widetilde{f}) := \theta_T(t)U(t)\widetilde{f}_0 + D(\widetilde{f},\widetilde{f}),$$

where

$$D(\widetilde{f},\widetilde{f}) := \theta_T(t)\int_0^t U(t-\tau)\theta_T(\tau)\widetilde{Q}(\widetilde{f}(\tau),\widetilde{f}(\tau))d\tau.$$

By estimates (2.22), (2.23), and (2.26), we obtain

$$\begin{aligned}
\|\Phi(\widetilde{f})\|_{X^{s,s+\gamma,b}} &\le \|\theta_T(t)U(t)\widetilde{f}_0\|_{X^{s,s+\gamma,b}} + \|D(\widetilde{f},\widetilde{f})\|_{X^{s,s+\gamma,b}}\\
&\le C\|\widetilde{f}_0\|_{H_x^s H_\xi^{s+\gamma}} + C\|\theta_T\widetilde{Q}(\widetilde{f},\widetilde{f})\|_{X^{s,s+\gamma,b-1}}\\
&\le \frac{R}{2} + \mathrm{CT}^{1-b}\|\widetilde{f}\|^2_{X^{s,s+\gamma,b}}\\
&\le R
\end{aligned}$$

where in the last inequality we have used that $CT^{1-b}R \le \frac{1}{2}$. Thus, Φ maps the set B into itself. In a similar way, for $\widetilde{f}$ and $\widetilde{g} \in B$ we have

$$\begin{aligned}
\|\Phi(\widetilde{f}) - \Phi(\widetilde{g})\|_{X^{s,s+\gamma,b}} &= \|D(\widetilde{f},\widetilde{f}) - D(\widetilde{g},\widetilde{g})\|_{X^{s,s+\gamma,b}}\\
&\le C\|\theta_T\widetilde{Q}(\widetilde{f}-\widetilde{g},\widetilde{f})\|_{X^{s,s+\gamma,b-1}} + C\|\theta_T\widetilde{Q}(\widetilde{g},\widetilde{f}-\widetilde{g})\|_{X^{s,s+\gamma,b-1}}\\
&\le \mathrm{CT}^{b-1}\left(\|\widetilde{f}\|_{X^{s,s+\gamma,b}} + \|\widetilde{g}\|_{X^{s,s+\gamma,b}}\right)\|\widetilde{f}-\widetilde{g}\|_{X^{s,s+\gamma,b}}\\
&\le \frac{1}{2}\|\widetilde{f}-\widetilde{g}\|_{X^{s,s+\gamma,b}}.
\end{aligned}$$

Therefore, Φ is a contraction mapping in $X^{s,s+\gamma,b}$ and has a unique fixed point $\widetilde{f}$ on the time scale $|T| \sim \langle R\rangle^{\frac{1}{b-1}}$.

Given two initial data $\widetilde{f}_0$ and $\widetilde{g}_0$, we set

$$R_1 = 2\max\left(\|\widetilde{f}_0\|_{H_x^s H_\xi^{s+\gamma}}, \|\widetilde{g}_0\|_{H_x^s H_\xi^{s+\gamma}}\right).$$

Let $\widetilde{f}, \widetilde{g}$ be the corresponding unique fixed points. Taking a difference gives that

$$\widetilde{f} - \widetilde{g} = \theta_T(t)U(t)(\widetilde{f}_0 - \widetilde{g}_0) + D(\widetilde{f}-\widetilde{g},\widetilde{f}) + D(\widetilde{g},\widetilde{f}-\widetilde{g}).$$

By estimates (2.22), (2.23), and (2.26), we have

$$\begin{aligned}\|\widetilde{f}-\widetilde{g}\|_{X^{s,s+\gamma,b}} \leq &\|\theta_T(t)U(t)(\widetilde{f}_0-\widetilde{g}_0)\|_{X^{s,s+\gamma,b}}+C\|\theta_T\widetilde{Q}(\widetilde{f}-\widetilde{g},\widetilde{f})\|_{X^{s,s+\gamma,b-1}}\\ &+C\|\theta_T\widetilde{Q}(\widetilde{g},\widetilde{f}-\widetilde{g})\|_{X^{s,s+\gamma,b-1}}\\ \leq &C\|\widetilde{f}_0-\widetilde{g}_0\|_{H_x^sH_\xi^{s+\gamma}}+CT^{1-b}\left(\|\widetilde{f}\|_{X^{s,s+\gamma,b}}+\|\widetilde{g}\|_{X^{s,s+\gamma,b}}\right)\|\widetilde{f}-\widetilde{g}\|_{X^{s,s+\gamma,b}},\end{aligned}$$

which together with $CT^{1-b}R_1\leq\frac{1}{2}$ gives that

$$\|\widetilde{f}-\widetilde{g}\|_{X^{s,s+\gamma,b}}\leq 2C\|\widetilde{f}_0-\widetilde{g}_0\|_{H_x^sH_\xi^{s+\gamma}}.$$

The Lipschitz continuity of the data-to-solution map on the time $[-T,T]$ follows from the embedding $X^{s,s+\gamma,b}\hookrightarrow C([-T,T];H_x^sH_\xi^{s+\gamma})$. □

3. Ill-Posedness

The idea is to first construct an approximation solution $f_{\mathrm{a}}(t)$ with the norm deflation property that

$$\|f_{\mathrm{a}}(0)\|_{L_v^{2,r_0}H_x^{s_0}}\ll 1,\quad \|f_{\mathrm{a}}(T_*)\|_{L_v^{2,r_0}H_x^{s_0}}\gtrsim 1,\tag{3.1}$$

with $T_*\nearrow 0$, and then use stability theory to perturb the approximation solution into an exact solution. Specifically, from the exact solution $f_{\mathrm{ex}}(t)$ to the Boltzmann equation (1.1)

$$\begin{cases}\partial_t f_{\mathrm{ex}}+v\cdot\nabla_x f_{\mathrm{ex}}=Q(f_{\mathrm{ex}},f_{\mathrm{ex}}),\\ f_{\mathrm{ex}}(t)=f_{\mathrm{a}}(t)+f_{\mathrm{c}}(t).\end{cases}\tag{3.2}$$

we have the equation for the correction term f_{c} that

$$\begin{cases}\partial_t f_{\mathrm{c}}+v\cdot\nabla_x f_{\mathrm{c}}=\pm Q^{\pm}(f_{\mathrm{c}},f_{\mathrm{a}})\pm Q^{\pm}(f_{\mathrm{a}},f_{\mathrm{c}})\pm Q^{\pm}(f_{\mathrm{c}},f_{\mathrm{c}})-F_{\mathrm{err}},\\ F_{\mathrm{err}}=\partial_t f_{\mathrm{a}}+v\cdot\nabla_x f_{\mathrm{a}}+Q^-(f_{\mathrm{a}},f_{\mathrm{a}})-Q^+(f_{\mathrm{a}},f_{\mathrm{a}}).\end{cases}\tag{3.3}$$

To prove the existence of f_{c}, we work with a Z-norm defined by (3.36) which is tailored to be stronger than the $L_v^{2,r_0}H_x^{s_0}$ norms. For the Z-norm, we are able to provide a closed bilinear estimate for the gain and loss terms in Lemma 3.13. Additionally, to work on the Z-norm space, we provide effective Z-norm bounds on the approximation solution f_{a}, which we conclude in Proposition 3.8, and then prove the Z-norm error estimates on the error term F_{err} that

$$\left\|\int_\tau^t e^{-(t-t_0)v\cdot\nabla_x}F_{\mathrm{err}}(t_0)\,dt_0\right\|_Z\ll 1,\tag{3.4}$$

which we set up in Proposition 3.9. Then by a perturbation argument in Proposition 3.14, we prove that the correction term indeed satisfies the smallness property that

$$\|f_{\mathrm{c}}(t)\|_{L^\infty([T_*,0];Z)}\ll 1.\tag{3.5}$$

Finally in Sect. 3.5, we conclude the ill-posedness results.

3.1. Norm deflation of the approximation solution. In the section, we get into the analysis of the construction of the approximation solution and its norm deflation property. Following the analysis of a prototype approximation solution in [27],[7] we decompose

$$f_{\mathrm{a}}(t) = f_{\mathrm{r}}(t) + f_{\mathrm{b}}(t).$$

For $d = 2, 3$, on the unit sphere $\mathbb{S}^{d-1}$, we call $J \sim N^{d-1}$ points $\{e_j\}_{j=1}^{J}$ are roughly equally spaced[8] if

$$\min_{i\neq j} |e_i - e_j| \gtrsim \frac{1}{N}, \quad \mathbb{S}^{d-1} \subset \bigcup_j B(e_j, \frac{100}{N}).$$

This just means that the distance between two points has a lower bound $\frac{1}{N}$ and the unit sphere can be covered by the union of the $\frac{100}{N}$-ball generated at each point. More specifically, for the case $d = 2$, we can directly choose the strictly uniform distribution on a unit circle

$$e_j = \left(\cos\left(\frac{2\pi j}{N}\right), \sin\left(\frac{2\pi j}{N}\right)\right), \quad 1 \le j \le N. \tag{3.6}$$

For the case $d = 3$, there are many choices of such a roughly uniform distribution on a unit sphere. Here, by the symmetry, an example for the upper hemisphere could be

$$e_{i,j} = \left(\sin\left(\frac{\pi j}{2N}\right)\cos\left(\frac{2\pi i}{j}\right), \sin\left(\frac{\pi j}{2N}\right)\sin\left(\frac{2\pi i}{j}\right), \cos\left(\frac{\pi j}{2N}\right)\right), \quad 1 \le i \le j \le N. \tag{3.7}$$

Equation (3.7) is not a direct 3D version of the 2D example (3.6), which would yield too many points near the north pole. In (3.7), from the north pole $(0, 0, 1)$ to the equator $|z| = 0$, the number of points grows from 1 to N. The total number of points is $\sum_{1\le i\le j\le N} \sim N^2$ and the distance between two points is at least $\frac{1}{N}$ up to a constant. Thus this example is a valid choice for our purposes here.

On the unit sphere, set $J \sim M^{d-1}N_2^{d-1}$ points $\{e_j\}_{j=1}^{J}$, where the points e_j are roughly equally spaced. Let P_{e_j} be the orthogonal projection onto the $1D$ subspace spanned by e_j and $P_{e_j}^{\perp}$ denote the orthogonal projection onto the orthogonal complement space $\{e_j\}^{\perp}$. Set

$$f_{\mathrm{b}}(t, x, v) = \frac{M^{\frac{d-1}{2}-s}}{N_2^{d+\gamma}} \sum_{j=1}^{J} K_j(x - vt) I_j(v), \tag{3.8}$$

where

$$K_j(x) = \chi(MP_{e_j}^{\perp}x)\chi\left(\frac{P_{e_j}x}{N_2}\right), \quad I_j(v) = \chi\left(\mathrm{MP}_{e_j}^{\perp}v\right)\chi\left(\frac{10P_{e_j}(v - N_2e_j)}{N_2}\right).$$

---

[7] One could see [27, Figure 1] for a picture of the approximation solutions there. They look like bullets hitting a rock in [27]. Our improvised and refined version is more like needles poking a rock through.

[8] Such a definition suffices for our purpose here. There can be different definitions.

In fact, $f_{\mathrm{b}}(t, x, v)$ is a linear solution to the transport equation:

$$\partial_t f_{\mathrm{b}} + v \cdot \nabla_x f_{\mathrm{b}} = 0. \tag{3.9}$$

Let $f_{\mathrm{r}}(t, x, v)$ be the solution to a drift-free linearized Boltzmann equation:

$$\partial_t f_{\mathrm{r}}(t, x, v) = -f_{\mathrm{r}}(t, x, v) \int \frac{f_{\mathrm{b}}(t, x, u)}{|u - v|^{-\gamma}}\, du = -Q^-(f_{\mathrm{r}}, f_{\mathrm{b}}), \tag{3.10}$$

with initial data $f_{\mathrm{r}}(0) = M^{\frac{d}{2}-s} N_1^{\frac{d}{2}} \chi(Mx)\chi(N_1 v)$. Therefore, we write out

$$f_{\mathrm{r}}(t, x, v) = M^{\frac{d}{2}-s} N_1^{\frac{d}{2}} \exp\left[-\int_0^t \int \frac{f_{\mathrm{b}}(\tau, x, u)}{|u - v|^{-\gamma}}\, du\, d\tau\right] \chi(Mx)\chi(N_1 v). \tag{3.11}$$

Recall that $0 \le s_0 < \frac{d-1}{2}$, $\frac{1-d}{2} \le \gamma \le 0$, and $r_0 = \max\{0, s_0 + \gamma\}$. In what follows, the parameters are set by

$$M \gg 1, \quad N_1 \ge N_2^{10} \ge M^{100}, \tag{3.12}$$

$$s = s_0 + \frac{\ln\ln\ln M}{\ln M}, \tag{3.13}$$

$$T_* = -M^{s-\frac{d-1}{2}} (\ln\ln\ln M). \tag{3.14}$$

Next, we give the Sobolev norm estimates on f_{b}, f_{r} and f_{a}.

Lemma 3.1 (Sobolev norm bounds on f_{b}). *We have for $t \le 0$,*

$$\| f_{\mathrm{b}}(t, x, v)\|_{L_v^{2,r_0} H_x^{s_0}} \lesssim M^{s_0 - s} N_2^{\max\{s_0, -\gamma\} - \frac{d-1}{2}} \le \frac{1}{\ln\ln M}. \tag{3.15}$$

Proof. Recall that

$$f_{\mathrm{b}}(t, x, v) = \frac{M^{\frac{d-1}{2}-s}}{N_2^{d+\gamma}} \sum_{j=1}^{J} K_j(x - vt) I_j(v). \tag{3.16}$$

Due to the v-support of f_{b}, the weight on v-variable produces a factor of $N_2^{r_0}$. Then expanding f_{b} gives that

$$\|\langle \nabla_x \rangle^{s_0} f_{\mathrm{b}}(t, x, v)\|_{L_v^{2,r_0} L_x^2}^2 \lesssim N_2^{2r_0} \frac{M^{d-1-2s}}{N_2^{2d+2\gamma}} \left\| \sum_{j=1}^{J} \langle \nabla_x \rangle^{s_0} K_j(x - vt) I_j(v) \right\|_{L_v^2 L_x^2}^2.$$

Due to the disjointness of the v-support, we have

$$\begin{aligned}
\|\langle \nabla_x \rangle^{s_0} f_{\mathrm{b}}(t, x, v)\|_{L_v^{2,r_0} L_x^2}^2 &\lesssim N_2^{2r_0} \frac{M^{d-1-2s}}{N_2^{2d+2\gamma}} \sum_{j=1}^{J} \|\langle \nabla_x \rangle^{s_0} K_j(x - vt) I_j(v)\|_{L_v^2 L_x^2}^2 \\
&\lesssim N_2^{2r_0} \frac{M^{d-1-2s}}{N_2^{2d+2\gamma}} \sum_{j=1}^{J} \|\langle \nabla_x \rangle^{s_0} K_j(x)\|_{L_x^2}^2 \|I_j(v)\|_{L_v^2}^2 \\
&\lesssim N_2^{2r_0} \frac{M^{d-1-2s}}{N_2^{2d+2\gamma}} (M N_2)^{d-1} (M^{2s_0+1-d} N_2)(M^{1-d} N_2)
\end{aligned}$$

$$= N_2^{2r_0} \frac{M^{2s_0-2s}}{N_2^{d-1+2\gamma}},$$

where in the second-to-last inequality we used that

$$\|\langle\nabla_x\rangle^{s_0} K_j(x)\|_{L_x^2}^2 \lesssim M^{2s_0} M^{1-d} N_2, \quad \|I_j(v)\|_{L_v^2}^2 \sim M^{1-d} N_2.$$

Notice that

$$r_0 = \max\{0, s_0+\gamma\}, \quad M^{s-s_0} = \ln\ln M, \quad \max\{s_0, -\gamma\} \leq \frac{d-1}{2}.$$

Hence, we complete the proof of (3.15). □

Before proceeding to the analysis of f_{r}, we give a useful pointwise bound on f_{b}.

Lemma 3.2 (Pointwise estimate on f_{b}). *Let* $-\frac{1}{4} \leq t \leq 0$ *and*

$$\beta(t,x,v) = \int_0^t \int \frac{f_{\mathrm{b}}(t_0,x,u)}{|u-v|^{-\gamma}}\, du dt_0 \leq 0.$$

For $k = 0, 1, 2$, *we have the pointwise upper bound*

$$\left|\chi(N_1 v)\nabla_x^k \beta(t,x,v)\right| \lesssim |t| M^{k+\frac{d-1}{2}-s}. \tag{3.17}$$

For the pointwise lower bound, we have

$$|\chi(N_1 v)\beta(t,x,v)| \gtrsim |t| M^{\frac{d-1}{2}-s}\chi(N_1 v), \quad \textit{for } |x| \leq M^{-1}. \tag{3.18}$$

Proof. For $-\frac{1}{4} \leq t \leq 0$, given the constraints on the u-variable, we have

$$\begin{aligned} &\frac{M^{\frac{d-1}{2}-s}}{N_2^{d+\gamma}} \sum_{j=1}^J \chi(10\mathrm{MP}_{e_j}^{\perp} x)\chi\left(\frac{10P_{e_j}x}{N_2}\right)\chi(\mathrm{MP}_{e_j}^{\perp} u)\chi\left(\frac{10P_{e_j}(u-N_2e_j)}{N_2}\right) \\ &\leq f_{\mathrm{b}}(t,x,u) \leq \frac{M^{\frac{d-1}{2}-s}}{N_2^{d+\gamma}} \sum_{j=1}^J \chi\left(\frac{\mathrm{MP}_{e_j}^{\perp} x}{10}\right)\chi\left(\frac{P_{e_j}x}{10N_2}\right)\chi(\mathrm{MP}_{e_j}^{\perp} u)\chi\left(\frac{10P_{e_j}(u-N_2e_j)}{N_2}\right). \end{aligned} \tag{3.19}$$

From the v-support and u-support, we have $|v| \sim N_1^{-1}$, $|u| \sim N_2$, and hence $|u-v| \sim N_2$. Then by using (3.19), we get

$$\begin{aligned} \chi(N_1 v)\int \frac{f_{\mathrm{b}}(t,x,u)}{|u-v|^{-\gamma}}\, du &\sim N_2^{\gamma}\chi(N_1 v)\int f_{\mathrm{b}}(t,x,u)\, du \\ &\sim N_2^{\gamma} \frac{M^{\frac{d-1}{2}-s}}{N_2^{d+\gamma}} M^{1-d} N_2 \chi(N_1 v) \sum_j^J \chi(\mathrm{MP}_{e_j}^{\perp} x)\chi\left(\frac{P_{e_j}x}{N_2}\right) \\ &= \frac{M^{\frac{1-d}{2}-s}}{N_2^{d-1}} \chi(N_1 v) \sum_j^J \chi(\mathrm{MP}_{e_j}^{\perp} x)\chi\left(\frac{P_{e_j}x}{N_2}\right). \end{aligned} \tag{3.20}$$

Thus, for the upper bound (3.17) with $k=0$, we use that $|J| \sim (MN_2)^{d-1}$ to obtain

$$|\chi(N_1 v)\beta(t,x,v)| \lesssim |t| \frac{M^{\frac{1-d}{2}-s}}{N_2^{d-1}}(MN_2)^{d-1}\chi(N_1 v) = |t| M^{\frac{d-1}{2}-s}\chi(N_1 v).$$

Notice that the upper bounds of estimates (3.19) and (3.20) remain true with the extra factor M^k if ∇_x^k is applied, for $k \geq 0$. Therefore, we conclude the pointwise upper bound (3.17) on $\beta(t,x,v)$ for $k = 0, 1, 2$.

For the lower bound (3.18), by noting that $\chi(MP_{e_j}^{\perp}x)\chi\left(\frac{P_{e_j}x}{N_2}\right) = 1$ for $|x| \leq M^{-1}$, we use (3.20) again to get

$$\begin{aligned}|\chi(N_1 v)\beta(t,x,v)| =& \chi(N_1 v)\int_t^0 \int \frac{f_{\mathrm{b}}(t_0,x,u)}{|u-v|^{-\gamma}}\,du\,dt_0 \\ \gtrsim& |t| \frac{M^{\frac{1-d}{2}-s}}{N_2^{d-1}}(MN_2)^{d-1}\chi(N_1 v) = |t| M^{\frac{d-1}{2}-s}\chi(N_1 v),\end{aligned}$$

which completes the proof of (3.18). □

Now, we are able to give the upper and lower bounds on f_{r}.

Lemma 3.3 (Sobolev norm bounds on f_{r}). *For $-\frac{1}{4} \leq t \leq 0$, we have the upper bound estimate*

$$\|f_{\mathrm{r}}(t)\|_{L_v^{2,r_0}H_x^{s_0}} \lesssim M^{s_0-s}\exp\left[|t|M^{\frac{d-1}{2}-s}\right]\langle |t| M^{\frac{d-1}{2}-s}\rangle, \tag{3.21}$$

and the lower bound estimate

$$\|f_{\mathrm{r}}(t)\|_{L_v^{2,r_0}H_x^{s_0}} \gtrsim M^{s_0-s}\exp\left[|t|M^{\frac{d-1}{2}-s}\right]. \tag{3.22}$$

In particular, we have

$$\|f_{\mathrm{r}}(0)\|_{L_v^{2,r_0}H_x^{s_0}} \lesssim \frac{1}{\ln\ln M}, \tag{3.23}$$

$$\|f_{\mathrm{r}}(T_*)\|_{L_v^{2,r_0}H_x^{s_0}} \gtrsim 1, \tag{3.24}$$

with $T_* = -M^{s-\frac{d-1}{2}}(\ln\ln\ln M)$.

Proof. Recall that

$$f_{\mathrm{r}}(t,x,v) = M^{\frac{d}{2}-s}N_1^{\frac{d}{2}}\exp\left[-\beta(t,x,v)\right]\chi(Mx)\chi(N_1 v). \tag{3.25}$$

Due to the v-support of f_{r}, we can discard the weight on the v-variable. For upper bound estimate (3.21) on f_{r}, we use the pointwise upper bound (3.17) to get

$$\begin{aligned}&\|\nabla_x f_{\mathrm{r}}(t)\|_{L_v^{2,r_0}L_x^2} \\ &\leq M^{1+\frac{d}{2}-s}N_1^{\frac{d}{2}}\left\|\exp\left[-\beta(t,x,v)\right](\nabla\chi)(Mx)\chi(N_1 v)\right\|_{L_v^2 L_x^2}\end{aligned}$$

$$
\begin{aligned}
&+ M^{\frac{d}{2}-s} N_1^{\frac{d}{2}} \|\nabla_x \beta(t,x,v) \exp\Big[-\beta(t,x,v)\Big]\chi(Mx)\chi(N_1 v)\|_{L_v^2 L_x^2} \\
&\lesssim M^{1+\frac{d}{2}-s} N_1^{\frac{d}{2}} \exp\Big[|t| M^{\frac{d-1}{2}-s}\Big] \|(\nabla\chi)(Mx)\|_{L_x^2} \|\chi(N_1 v)\|_{L_v^2} \\
&+ M^{\frac{d}{2}-s} N_1^{\frac{d}{2}} \langle |t| M^{1+\frac{d-1}{2}-s}\rangle \exp\Big[|t| M^{\frac{d-1}{2}-s}\Big] \|\chi(Mx)\|_{L_x^2} \|\chi(N_1 v)\|_{L_v^2} \\
&\lesssim M^{1-s} \exp\Big[|t| M^{\frac{d-1}{2}-s}\Big] \langle |t| M^{\frac{d-1}{2}-s}\rangle. \qquad (3.26)
\end{aligned}
$$

In the same way, we also have

$$
\|f_{\mathrm{r}}(t)\|_{L_v^{2,r_0} L_x^2} \lesssim M^{-s} \exp\Big[|t| M^{\frac{d-1}{2}-s}\Big].
$$

By the interpolation inequality, we obtain

$$
\|f_{\mathrm{r}}(t)\|_{L_v^{2,r_0} H_x^{s_0}} \leq \|f_{\mathrm{r}}(t)\|_{L_v^{2,r_0} H_x^1}^{s_0} \|f_{\mathrm{r}}(t)\|_{L_v^{2,r_0} L_x^2}^{1-s_0} \lesssim M^{s_0-s} \exp\Big[|t| M^{\frac{d-1}{2}-s}\Big] \langle |t| M^{\frac{d-1}{2}-s}\rangle.
$$

For the lower bound estimate (3.22) on f_{r}, we use the Sobolev inequality and lower bound estimate (3.18) to obtain

$$
\begin{aligned}
&\|\langle\nabla_x\rangle^{s_0} f_{\mathrm{r}}(t,x,v)\|_{L_v^{2,r_0} L_x^2} \\
&\quad \gtrsim \|f_{\mathrm{r}}(t,x,v)\|_{L_v^{2,r_0} L_x^{\frac{2d}{d-2s_0}}} \\
&\quad \gtrsim M^{\frac{d}{2}-s} N_1^{\frac{d}{2}} \|\exp[-\beta(t,x,v)]\,\chi(Mx)\chi(N_1 v)\|_{L_v^2 L_x^{\frac{2d}{d-2s_0}}} \\
&\quad \gtrsim M^{\frac{d}{2}-s} N_1^{\frac{d}{2}} \exp\Big[|t| M^{\frac{d-1}{2}-s}\Big] \|\chi(Mx)\|_{L_x^{\frac{2d}{d-2s_0}}} \|\chi(N_1 v)\|_{L_v^2} \\
&\gtrsim M^{s_0-s} \exp\Big[|t| M^{\frac{d-1}{2}-s}\Big].
\end{aligned}
$$

Hence, we have done the proof of estimate (3.22).

Inserting in $|T_*| = M^{s-\frac{d-1}{2}}(\ln\ln\ln M)$ and $M^{s_0-s} = \frac{1}{\ln\ln M}$, we have

$$
\|f_{\mathrm{r}}(T_*)\|_{L_v^{2,r_0} H_x^{s_0}} \gtrsim M^{s_0-s} \exp\Big[|T_*| M^{\frac{d-1}{2}-s}\Big] \gtrsim 1,
$$

which completes the proof of (3.24). □

Remark 3.4. The lower bound estimate (3.22) on $f_{\mathrm{r}}(t)$ also holds for the kernel ($d = 3$)

$$
B(u-v,\omega) = \Big(1_{\{|u-v|\leq 1\}}|u-v| + 1_{\{|u-v|\geq 1\}}|u-v|^{-1}\Big)\,\mathbf{b}\Big(\frac{u-v}{|u-v|}\cdot\omega\Big). \qquad (3.27)
$$

Indeed, in the proof of the lower bound estimate (3.18), the term $1_{\{|u-v|\leq 1\}}|u-v|$ would vanish due to that $|u-v| \sim N_2 \gg 1$.

In the end, we conclude the norm deflation property of the approximation solution f_{a}.

Proposition 3.5 (Norm deflation of f_{a}). *Let* $T_* = -M^{s-\frac{d-1}{2}}(\ln\ln\ln M)$. *We have*

$$\|f_{\mathrm{a}}(0)\|_{L_v^{2,r_0}H_x^{s_0}} \lesssim \frac{1}{\ln\ln M} \ll 1, \tag{3.28}$$

$$\|f_{\mathrm{a}}(T_*)\|_{L_v^{2,r_0}H_x^{s_0}} \gtrsim 1. \tag{3.29}$$

Proof. Since f_{b} and f_{r} have disjoint velocity supports, we get

$$\|f_{\mathrm{a}}(t)\|_{L_v^{2,r_0}H_x^{s_0}} \sim \|f_{\mathrm{r}}(t)\|_{L_v^{2,r_0}H_x^{s_0}} + \|f_{\mathrm{b}}(t)\|_{L_v^{2,r_0}H_x^{s_0}}. \tag{3.30}$$

Then by estimate (3.15) on f_{b} in Lemma 3.1 and estimates (3.23)–(3.24) on f_{r} in Lemma 3.3, we arrive at estimates (3.28) and (3.29). □

3.1.1. Discussion on the L^1-based space and hard potentials The Eq. (1.1) is invariant under the scaling

$$f_\lambda(t, x, , v) = \lambda^{\alpha+(d-1+\gamma)\beta} f(\lambda^{\alpha-\beta}t, \lambda^\alpha x, \lambda^\beta v), \tag{3.31}$$

for any $\alpha, \beta \in \mathbb{R}$ and $\lambda > 0$. Then

$$\||\nabla_x|^s|v|^r f_\lambda\|_{L^1_{xv}} = \lambda^{\alpha+(d-1+\gamma)\beta}\lambda^{\alpha s-\beta r}\lambda^{-d\alpha-d\beta}\||\nabla_x|^s|v|^r f\|_{L^1_{xv}},$$

which gives the L^1-based scaling-critical index

$$s_1 = d-1, \quad r_1 = 1+\gamma. \tag{3.32}$$

In the L^1 setting, we construct the approximation solution $f_{a,1} = f_{\mathrm{b},1} + f_{\mathrm{r},1}$, where

$$f_{\mathrm{b},1}(t, x, v) = \frac{M^{d-1-s}}{N_2^{d+2+\gamma}} \sum_{j=1}^{J} K_j(x - vt) I_j(v),$$
$$f_{\mathrm{r},1}(t, x, v) = M^{d-s} N_1^d \exp[-\beta(t, x, v)]\, \chi(Mx)\chi(N_1 v).$$

Repeating the proof of estimates (3.15) and (3.22), we also have

$$\|\langle\nabla\rangle^{s_0} f_{\mathrm{b},1}\|_{L_v^{1,r_1}L_x^1} \lesssim M^{s_0-s}, \tag{3.33}$$

$$\|\langle\nabla\rangle^{s_0} f_{\mathrm{r},1}\|_{L_v^{1,r_1}L_x^1} \gtrsim M^{s_0-s}\exp\left[|t|N_2^{-2}M^{d-1-s}\right]. \tag{3.34}$$

If $s_0 < s_1 = d-1$, a similar mechanism of norm deflation could be possible in the L^1 setting.

For the hard potential case that $\gamma > 0$, the norm deflation of the approximation solution $f_{\mathrm{a}}(t)$ also holds. But, to perturb it into the exact solution, it requires a much more different work space to prove the error bounds in Proposition 3.9 and provide a closed estimate in Lemma 3.13. We leave the problem for future work.

3.2. Z-norm bounds on the approximation solution. To perturb the approximation solution $f_{\rm a}(t)$ into an exact solution $f_{\rm ex}(t)$, we need to prove the existence of a small correction term $f_{\rm c}(t)$. As it satisfies a more complicated Eq. (3.3), some terms of (3.3) cannot be effectively treated using Strichartz estimates like (2.1). Hence, we tailor a Z-norm to provide a closed estimate for the gain and loss terms, that is,

$$\|Q^{\pm}(f_1, f_2)\|_Z \lesssim \|f_1\|_Z\|f_2\|_Z, \tag{3.35}$$

where the Z-norm is given by

$$\begin{aligned}\|f(t)\|_Z =& M^{\frac{d-3}{2}}\|\nabla_x f(t)\|_{L_v^{2,r_0}L_x^2} + M^{\frac{d-1}{2}}\|f(t)\|_{L_v^{2,r_0}L_x^2} + N_2^{\gamma}\|f(t)\|_{L_v^1L_x^\infty}\\ &+ N_2^{\frac{2d}{5}+\gamma}\|f(t)\|_{L_v^{\frac{5}{3}}L_x^\infty} + M^{-1}N_2^{\gamma}\|\nabla_x f(t)\|_{L_v^1L_x^\infty}\\ &+ M^{-1}N_2^{\frac{2d}{5}+\gamma}\|\nabla_x f(t)\|_{L_v^{\frac{5}{3}}L_x^\infty}.\end{aligned} \tag{3.36}$$

The closed estimate (3.35) which we will prove in Sect. 3.4 indeed plays a key role in the perturbation argument. In the section, we give Z-norm bounds on the approximation solution $f_{\rm a} = f_{\rm r} + f_{\rm b}$, which will be used to control the error term $F_{\rm err}$.

Lemma 3.6 (Z-norm bounds on $f_{\rm b}$). *For the Z-norm, we have*

$$\|f_{\rm b}(t)\|_{L^\infty([T_*,0];Z)} \lesssim M^{\frac{d-1}{2}-s}. \tag{3.37}$$

Proof. The $M^{\frac{d-3}{2}}\|\nabla_x \bullet\|_{L_v^{2,r_0}L_x^2}$ and $M^{\frac{d-1}{2}}\|\bullet\|_{L_v^{2,r_0}L_x^2}$ estimates.

This can be done in the same way as estimate (3.15) with the regularity index s_0 replaced by 1 and 0. Therefore, we obtain

$$M^{\frac{d-3}{2}}\|\nabla_x f_{\rm b}\|_{L_v^2L_x^2} \lesssim M^{\frac{d-1}{2}-s}N_2^{\max\{s_0,-\gamma\}-\frac{d-1}{2}} \le M^{\frac{d-1}{2}-s}, \tag{3.38}$$

$$M^{\frac{d-1}{2}}\|f_{\rm b}\|_{L_v^2L_x^2} \lesssim M^{\frac{d-1}{2}-s}N_2^{\max\{s_0,-\gamma\}-\frac{d-1}{2}} \le M^{\frac{d-1}{2}-s}. \tag{3.39}$$

The $N_2^{\gamma}\|\bullet\|_{L_v^1L_x^\infty}$ and $N_2^{\frac{2d}{5}+\gamma}\|\bullet\|_{L_v^{\frac{5}{3}}L_x^\infty}$ estimates.

$$\begin{aligned}N_2^{\gamma}\|f_{\rm b}\|_{L_v^1L_x^\infty} \lesssim& N_2^{\gamma}\frac{M^{\frac{d-1}{2}-s}}{N_2^{d+\gamma}}\Big\|\sum_{j=1}^{J}\|K_j(x-vt)\|_{L_x^\infty}I_j(v)\Big\|_{L_v^1}\\ \lesssim& \frac{M^{\frac{d-1}{2}-s}}{N_2^{d}}\Big\|\sum_{j=1}^{J}I_j(v)\Big\|_{L_v^1}\\ \lesssim& \frac{M^{\frac{d-1}{2}-s}}{N_2^{d}}N_2^{d} = M^{\frac{d-1}{2}-s},\end{aligned} \tag{3.40}$$

where in the last inequality we have used that

$$\sum_{j=1}^{J}I_j(v) \sim 1_{\left\{\frac{9N_2}{10}\le|v|\le\frac{11N_2}{10}\right\}}(v), \quad \Big\|1_{\left\{\frac{9N_2}{10}\le|v|\le\frac{11N_2}{10}\right\}}(v)\Big\|_{L_v^1} \lesssim N_2^{d}.$$

In the same way, we also have

$$
\begin{aligned}
N_2^{\frac{2d}{5}+\gamma}\|f_{\rm b}\|_{L_v^{\frac{5}{3}}L_x^\infty} &\lesssim N_2^{\frac{2d}{5}+\gamma}\frac{M^{\frac{d-1}{2}-s}}{N_2^{d+\gamma}}\Big\|\sum_{j=1}^J\|K_j(x-vt)\|_{L_x^\infty}I_j(v)\Big\|_{L_v^{\frac{5}{3}}} \\
&\lesssim N_2^{\frac{2d}{5}}\frac{M^{\frac{d-1}{2}-s}}{N_2^d}\Big\|\sum_{j=1}^J I_j(v)\Big\|_{L_v^{\frac{5}{3}}} \\
&\lesssim N_2^{\frac{2d}{5}}\frac{M^{\frac{d-1}{2}-s}}{N_2^d}N_2^{\frac{3d}{5}} = M^{\frac{d-1}{2}-s}.
\end{aligned}
$$

The same bound is obtained for $M^{-1}N_2^\gamma\|\nabla_x f_{\rm b}\|_{L_v^1L_x^\infty}$ and $M^{-1}N_2^{\frac{2d}{5}+\gamma}\|\nabla_x f_{\rm b}\|_{L_v^{\frac{5}{3}}L_x^\infty}$ with one x-derivative producing a factor of M. Therefore, we complete the proof of the Z-norm estimate (3.37). □

Lemma 3.7 (Z-norm bounds on $f_{\rm r}$). *For $T_*\le t\le 0$, we have*

$$\|f_{\rm r}(t)\|_Z \lesssim M^{\frac{d-1}{2}-s}\exp[|t|M^{\frac{d-1}{2}-s}]\langle |t|M^{\frac{d-1}{2}-s}\rangle. \tag{3.41}$$

In particular,

$$\|f_{\rm r}(t)\|_{L^\infty([T_*,0];Z)} \lesssim M^{\frac{d-1}{2}-s}(\ln\ln M)^2. \tag{3.42}$$

Proof. Recall

$$f_{\rm r}(t,x,v) = M^{\frac{d}{2}-s}N_1^{\frac{d}{2}}\exp\left[-\beta(t,x,v)\right]\chi(Mx)\chi(N_1v). \tag{3.43}$$

The $M^{\frac{d-3}{2}}\|\nabla_x\bullet\|_{L_v^{2,r_0}L_x^2}$ and $M^{\frac{d-1}{2}}\|\bullet\|_{L_v^{2,r_0}L_x^2}$ estimates.

The weight on v-variable plays no role due to the v-support set, so we can discard it. By the pointwise upper bound (3.17), we get

$$
\begin{aligned}
&M^{\frac{d-3}{2}}\|\nabla_x f_{\rm r}(t)\|_{L_v^{2,r_0}L_x^2} \\
&\le M^{\frac{d-3}{2}}M^{1+\frac{d}{2}-s}N_1^{\frac{d}{2}}\|\exp[-\beta(t,x,v)](\nabla\chi)(Mx)\chi(N_1v)\|_{L_v^2L_x^2} \\
&\quad + M^{\frac{d-3}{2}}M^{\frac{d}{2}-s}N_1^{\frac{d}{2}}\|\nabla_x\beta(t,x,v)\exp[-\beta(t,x,v)]\chi(Mx)\chi(N_1v)\|_{L_v^2L_x^2} \\
&\lesssim M^{\frac{d-1}{2}}M^{\frac{d}{2}-s}N_1^{\frac{d}{2}}\exp[|t|M^{\frac{d-1}{2}-s}]\|(\nabla\chi)(Mx)\|_{L_x^2}\|\chi(N_1v)\|_{L_v^2} \\
&\quad + M^{\frac{d-3}{2}}M^{\frac{d}{2}-s}N_1^{\frac{d}{2}}\langle |t|M^{1+\frac{d-1}{2}-s}\rangle\exp[|t|M^{\frac{d-1}{2}-s}]\|\chi(Mx)\|_{L_x^2}\|\chi(N_1v)\|_{L_v^2} \\
&\lesssim M^{\frac{d-1}{2}-s}\exp[|t|M^{\frac{d-1}{2}-s}]\langle |t|M^{\frac{d-1}{2}-s}\rangle.
\end{aligned}
$$

The $M^{\frac{d-1}{2}}\|f_{\rm r}\|_{L_v^{2,r_0}L_x^2}$ estimate can be handled in the same way.

The $N_2^\gamma\|\bullet\|_{L_v^1L_x^\infty}$ and $M^{-1}N_2^\gamma\|\nabla_x\bullet\|_{L_v^1L_x^\infty}$ estimates.

We only need to treat the $M^{-1}N_2^\gamma\|\nabla_x\bullet\|_{L_v^1L_x^\infty}$ norm, as the $N_2^\gamma\|\bullet\|_{L_v^1L_x^\infty}$ norm can be dealt with in a similar way. We use the pointwise upper bound (3.17) to obtain

$$M^{-1}N_2^\gamma\|\nabla_x f_{\rm r}(t)\|_{L_v^1L_x^\infty}$$

$$
\begin{aligned}
&\le M^{-1}N_2^{\gamma}M^{1+\frac{d}{2}-s}N_1^{\frac{d}{2}}\|\exp[-\beta(t,x,v)](\nabla\chi)(Mx)\chi(N_1v)\|_{L_v^1L_x^\infty}\\
&\quad+M^{-1}N_2^{\gamma}M^{\frac{d}{2}-s}N_1^{\frac{d}{2}}\|\nabla_x\beta(t,x,v)\exp[-\beta(t,x,v)]\chi(Mx)\chi(N_1v)\|_{L_v^1L_x^\infty}\\
&\lesssim M^{-1}N_2^{\gamma}M^{1+\frac{d}{2}-s}N_1^{\frac{d}{2}}\exp[|t|M^{\frac{d-1}{2}-s}]\|(\nabla\chi)(Mx)\|_{L_x^\infty}\|\chi(N_1v)\|_{L_v^1}\\
&\quad+M^{-1}N_2^{\gamma}M^{\frac{d}{2}-s}N_1^{\frac{d}{2}}\langle|t|M^{1+\frac{d-1}{2}-s}\rangle\exp[|t|M^{\frac{d-1}{2}-s}]\|\chi(Mx)\|_{L_x^\infty}\|\chi(N_1v)\|_{L_v^1}\\
&\lesssim M^{-1}N_2^{\gamma}M^{1+\frac{d}{2}-s}N_1^{-\frac{d}{2}}\exp[|t|M^{\frac{d-1}{2}-s}]\langle|t|M^{\frac{d-1}{2}-s}\rangle\\
&\lesssim N_1^{-\frac{d}{2}}N_2^{\gamma}M^{\frac{d}{2}-s}\exp[|t|M^{\frac{d-1}{2}-s}]\langle|t|M^{\frac{d-1}{2}-s}\rangle. \qquad (3.44)
\end{aligned}
$$

This bound is enough as it carries the smallness factor $N_1^{-\frac{d}{2}}$.

The $N_2^{\frac{2d}{5}+\gamma}\|\bullet\|_{L_v^{\frac{5}{3}}L_x^\infty}$ and $M^{-1}N_2^{\frac{2d}{5}+\gamma}\|\nabla_x\bullet\|_{L_v^{\frac{5}{3}}L_x^\infty}$ estimates.

These two norms can be controlled in the same manner as (3.44) with the L_v^1 norm replaced by the $L_v^{\frac{5}{3}}$ norm. As a result, we also have

$$
N_2^{\frac{2d}{5}+\gamma}\|f_r(t)\|_{L_v^{\frac{5}{3}}L_x^\infty}\lesssim N_1^{-\frac{d}{10}}N_2^{\frac{2d}{5}+\gamma}M^{\frac{d}{2}-s}\exp[|t|M^{\frac{d-1}{2}-s}]\langle|t|M^{\frac{d-1}{2}-s}\rangle, \qquad (3.45)
$$

$$
M^{-1}N_2^{\frac{2d}{5}+\gamma}\|\nabla_x f_r(t)\|_{L_v^{\frac{5}{3}}L_x^\infty}\lesssim N_1^{-\frac{d}{10}}N_2^{\frac{2d}{5}+\gamma}M^{\frac{d}{2}-s}\exp[|t|M^{\frac{d-1}{2}-s}]\langle|t|M^{\frac{d-1}{2}-s}\rangle. \qquad (3.46)
$$

By the condition (3.12) that $N_1\ge N_2^{10}\ge M^{100}$, it is sufficient to obtain the desired bound. Thus, we complete the proof of (3.41).

Inserting in $|T_*|=M^{s-\frac{d-1}{2}}(\ln\ln\ln M)$ and $M^{s-s_0}=\ln\ln M$, we obtain

$$
\|f_r(t)\|_{L^\infty([T_*,0];Z)}\lesssim M^{\frac{d-1}{2}-s}(\ln\ln M)^2,
$$

which completes the proof of (3.42). □

To the end, we conclude the Z-norm bounds on $f_a=f_r+f_b$.

Proposition 3.8 (Z-norm bounds on f_a). *For the Z-norm,*

$$
\|f_a(t)\|_{L^\infty([T_*,0];Z)}\lesssim M^{\frac{d-1}{2}-s}(\ln\ln M)^2. \qquad (3.47)
$$

Proof. By the triangle inequality, we have

$$
\|f_a(t)\|_Z\lesssim\|f_r(t)\|_Z+\|f_b(t)\|_Z.
$$

Then combining estimate (3.37) on f_b and estimate (3.41) on f_r, we complete the proof of estimate (3.47). □

3.3. Z-norm bounds on the error terms. In the section, we give the Z-norm bounds on the error term F_{err}. Recall the error term

$$\begin{aligned} F_{\text{err}} &= \partial_t f_{\text{a}} + v\cdot\nabla_x f_{\text{a}} + Q^-(f_{\text{a}}, f_{\text{a}}) - Q^+(f_{\text{a}}, f_{\text{a}}) \\ &= v\cdot\nabla_x f_{\text{r}} - Q^+(f_{\text{r}}, f_{\text{b}}) \mp Q^{\pm}(f_{\text{b}}, f_{\text{r}}) \mp Q^{\pm}(f_{\text{r}}, f_{\text{r}}) \mp Q^{\pm}(f_{\text{b}}, f_{\text{b}}), \end{aligned}$$

and thus the estimate on F_{err} highly relies on the Z-norm bounds of f_{r} and f_{b}. Recall the estimate (3.38) in Lemma 3.6 that

$$\begin{aligned} M^{\frac{d-3}{2}}\|\nabla_x f_{\text{b}}\|_{L_v^{2,r_0}L_x^2} &\lesssim M^{\frac{d-1}{2}-s} N_2^{\max\{s_0,-\gamma\}-\frac{d-1}{2}}, \\ M^{\frac{d-1}{2}}\|f_{\text{b}}\|_{L_v^{2,r_0}L_x^2} &\lesssim M^{\frac{d-1}{2}-s} N_2^{\max\{s_0,-\gamma\}-\frac{d-1}{2}}. \end{aligned}$$

For the case $\gamma\in(\frac{1-d}{2},0]$, the extra smallness comes from the factor $N_2^{\max\{s_0,-\gamma\}-\frac{d-1}{2}}$ as we have required that $s_0<\frac{d-1}{2}$ and $N_2\gg M$. Thus, it is enough to deal with the hardest endpoint case that $\gamma=\frac{1-d}{2}$, in which the $M^{\frac{d-3}{2}}\|\nabla_x\bullet\|_{L_v^{2,r_0}L_x^2}$ and $M^{\frac{d-1}{2}}\|\bullet\|_{L_v^{2,r_0}L_x^2}$ norms of f_{b} are the order of $M^{\frac{d-1}{2}-s}$ and hence would not give any smallness for $s<\frac{d-1}{2}$. Additionally, we only need to prove the $d=3$ case as the $d=2$ case follows from a similar way.

In the section, we set $d=3$, $\gamma=-1$ and hence $r_0=0$, for which the Z-norm is

$$\begin{aligned} \|f(t)\|_Z =&\|\nabla_x f(t)\|_{L_v^2L_x^2} + M\|f(t)\|_{L_v^2L_x^2} + N_2^{-1}\|f(t)\|_{L_v^1L_x^\infty} \\ &+ N_2^{\frac{1}{5}}\|f(t)\|_{L_v^{\frac{5}{3}}L_x^\infty} + M^{-1}N_2^{-1}\|\nabla_x f(t)\|_{L_v^1L_x^\infty} + M^{-1}N_2^{\frac{1}{5}}\|\nabla_x f(t)\|_{L_v^{\frac{5}{3}}L_x^\infty}. \end{aligned} \tag{3.48}$$

The following is the main result about the Z-norm bounds on the error term F_{err}.

Proposition 3.9 (Z-norm bounds on F_{err}). *For $T_*\le\tau\le t\le 0$,*

$$\left\|\int_\tau^t e^{-(t-t_0)v\cdot\nabla_x}F_{err}(t_0)\,dt_0\right\|_Z \lesssim M^{-1}. \tag{3.49}$$

We deal with all of the terms in the following separate sections. In Sect. 3.3.1, we give estimates on the term $v\cdot\nabla_x f_{\text{r}}$. In Sect. 3.3.2, we handle the bilinear terms which contain f_{r}. Finally, we deal with $Q^{\pm}(f_{\text{b}}, f_{\text{b}})$ in Sects. 3.3.3, 3.3.4, which are the most intricate parts.

The estimates are mainly achieved by moving the t_0 integration to the outside as follows:

$$\left\|\int_\tau^t e^{-(t-t_0)v\cdot\nabla_x}F_{\text{err}}(t_0)\,dt_0\right\|_Z \lesssim |T_*|\|F_{\text{err}}\|_{L_t^\infty Z}.$$

The only exception is the treatment of the bound on $L_v^1L_x^\infty$ of $Q^{\pm}(f_{\text{b}}, f_{\text{b}})$, where a substantial gain is captured by carrying out the t_0 integration first.

3.3.1. Analysis of $v \cdot \nabla_x f_{\mathrm{r}}$

Lemma 3.10. *For $T_* \le \tau \le t \le 0$,*

$$\left\| \int_\tau^t e^{-(t-t_0)v\cdot\nabla_x}(v \cdot \nabla_x f_{\mathrm{r}})(t_0)\, dt_0 \right\|_Z \lesssim M^{-1}. \tag{3.50}$$

Proof. As we have required that $N_1 \gg M$ in (3.12), the desired decay bound is achieved provided the upper bound carries the smallness factor $N_1^{-\delta}$ for some $\delta > 0$.

The $\|\nabla_x \bullet\|_{L_v^2 L_x^2}$ and $M\|\bullet\|_{L_v^2 L_x^2}$ estimates.

It suffices to deal with the $\|\nabla_x \bullet\|_{L_v^2 L_x^2}$ norm, as the estimate for the $M\|\bullet\|_{L_v^2 L_x^2}$ norm follows the same way. Noting that f_{r} is supported on $\left\{|v| \lesssim N_1^{-1}\right\}$, we have

$$\|\nabla_x(v \cdot \nabla_x f_{\mathrm{r}})\|_{L_v^2 L_x^2} \lesssim N_1^{-1}\|\Delta_x f_{\mathrm{r}}\|_{L_v^2 L_x^2} \lesssim N_1^{-1} M^{2-s} \exp\left[|t|M^{1-s}\right]\langle |t| M^{1-s}\rangle^2, \tag{3.51}$$

where the last inequality follows from the proof of (3.26) with one x-derivative producing a factor of M. We then insert in $|T_*| = M^{s-1}(\ln\ln\ln M)$ to get

$$\begin{aligned}
&\left\|\nabla_x \int_\tau^t e^{-(t-t_0)v\cdot\nabla_x}(v \cdot \nabla_x f)(t_0)\, dt_0\right\|_{L_v^2 L_x^2} \\
&\lesssim |T_*| \sup_{t_0 \in [T_*, 0]} \|\nabla_x(v \cdot \nabla_x f_{\mathrm{r}})\|_{L_v^2 L_x^2} \\
&\lesssim M^{s-1}(\ln\ln\ln M) N_1^{-1} M^{2-s}(\ln\ln M)^3 \\
&\lesssim N_1^{-1} M (\ln\ln M)^4.
\end{aligned}$$

The $N_2^{-1}\|\bullet\|_{L_v^1 L_x^\infty}$ and $M^{-1}N_2^{-1}\|\nabla_x \bullet\|_{L_v^1 L_x^\infty}$ estimates.

We only need to treat the $M^{-1}N_2^{-1}\|\nabla_x \bullet\|_{L_v^1 L_x^\infty}$ norm, as the $N_2^{-1}\|\bullet\|_{L_v^1 L_x^\infty}$ norm can be dealt with in a similar way. Recalling that ($d = 3$)

$$f_{\mathrm{r}}(t, x, v) = M^{\frac{3}{2}-s} N_1^{\frac{3}{2}} \exp\left[-\beta(t, x, v)\right] \chi(Mx)\chi(N_1 v), \tag{3.52}$$

we use the pointwise upper bound (3.17) to get

$$\begin{aligned}
&\|\nabla_x(v \cdot \nabla_x f_{\mathrm{r}})\|_{L_v^1 L_x^\infty} \\
&\lesssim N_1^{-1} M^{2+\frac{3}{2}-s} N_1^{\frac{3}{2}} \|\exp[-\beta(t, x, v)](\nabla^2\chi)(Mx)\chi(N_1 v)\|_{L_v^1 L_x^\infty} \\
&\quad + N_1^{-1} M^{1+\frac{3}{2}-s} N_1^{\frac{3}{2}} \|\nabla_x \beta(t, x, v) \exp[-\beta(t, x, v)](\nabla\chi)(Mx)\chi(N_1 v)\|_{L_v^1 L_x^\infty} \\
&\quad + N_1^{-1} M^{\frac{3}{2}-s} N_1^{\frac{3}{2}} \|\nabla_x^2 \beta(t, x, v) \exp[-\beta(t, x, v)]\chi(Mx)\chi(N_1 v)\|_{L_v^1 L_x^\infty} \\
&\quad + N_1^{-1} M^{\frac{3}{2}-s} N_1^{\frac{3}{2}} \||\nabla_x \beta(t, x, v)|^2 \exp[-\beta(t, x, v)]\chi(Mx)\chi(N_1 v)\|_{L_v^1 L_x^\infty} \\
&\lesssim N_1^{-1} N_1^{-\frac{3}{2}} M^{2-s} M^{\frac{3}{2}} \exp[M^{1-s}|t|]\langle M^{1-s}|t|\rangle^2.
\end{aligned}$$

When multiplied by $|T_*| = M^{s-1}(\ln\ln\ln M)$, this gives

$$M^{-1}N_2^{-1}\left\|\nabla_x \int_\tau^t e^{-(t-t_0)v\cdot\nabla_x}(v \cdot \nabla_x f)(t_0)\, dt_0\right\|_{L_v^1 L_x^\infty}$$

$$\lesssim |T_*| N_1^{-\frac{5}{2}} N_2^{-1} M^{1-s} M^{\frac{3}{2}} \exp[M^{1-s}|T_*|] \langle M^{1-s}|T_*|\rangle^2$$
$$\lesssim N_1^{-\frac{5}{2}} N_2^{-1} M^{\frac{3}{2}} (\ln \ln M)^4. \tag{3.53}$$

The $N_2^{\frac{1}{5}} \| \bullet \|_{L_v^{\frac{5}{3}} L_x^\infty}$ and $M^{-1} N_2^{\frac{1}{5}} \|\nabla_x \bullet \|_{L_v^{\frac{5}{3}} L_x^\infty}$ estimates.

These two norms can be estimated in the same manner as (3.53) with the L_v^1 norm replaced by the $L_v^{\frac{5}{3}}$ norm. Therefore, we also have

$$M^{-1} N_2^{\frac{1}{5}} \left\| \nabla_x \int_\tau^t e^{-(t-t_0)v\cdot\nabla_x} (v \cdot \nabla_x f)(t_0)\, dt_0 \right\|_{L_v^{\frac{5}{3}} L_x^\infty}$$
$$\lesssim |T_*| N_1^{-1} N_1^{-\frac{3}{10}} N_2^{\frac{1}{5}} M^{1-s} M^{\frac{3}{2}} \exp[M^{1-s}|T_*|] \langle M^{1-s}|T_*|\rangle^2$$
$$\lesssim N_1^{-1} N_1^{-\frac{3}{10}} N_2^{\frac{1}{5}} M^{\frac{3}{2}} (\ln \ln M)^4$$
$$\lesssim N_1^{-\frac{11}{10}} M^{\frac{3}{2}} (\ln \ln M)^4, \tag{3.54}$$

where in the last inequality we have used that $N_1 \geq N_2$. □

3.3.2. Analysis of $Q^+(f_r, f_b)$, $Q^\pm(f_b, f_r)$, and $Q^\pm(f_r, f_r)$ Before getting into the analysis of the terms, we recall some estimates on f_b and f_r, which are established in Lemma 3.6 and Lemma 3.7. That is,

$$\|f_b\|_{L^\infty([T_*,0];Z)} \lesssim M^{1-s}, \tag{3.55}$$
$$\|f_r\|_{L^\infty([T_*,0];Z)} \lesssim M^{1-s} (\ln \ln M)^2, \tag{3.56}$$
$$\|f_r\|_{L_v^1 L_x^\infty}^{\frac{1}{6}} \|f_r\|_{L_v^{\frac{5}{3}} L_x^\infty}^{\frac{5}{6}} \lesssim N_1^{-\frac{1}{2}} M^{\frac{3}{2}-s} (\ln \ln M)^2, \tag{3.57}$$
$$M^{-1} \|\nabla_x f_r\|_{L_v^1 L_x^\infty}^{\frac{1}{6}} \|\nabla_x f_r\|_{L_v^{\frac{5}{3}} L_x^\infty}^{\frac{5}{6}} \lesssim N_1^{-\frac{1}{2}} M^{\frac{3}{2}-s} (\ln \ln M)^2, \tag{3.58}$$

where the last two inequalities (3.57)–(3.58) follow from estimates (3.44), (3.46). In addition, during the proof of the bilinear estimate on $Q^\pm$ in Lemma 3.13 we postpone to Sect. 3.4, we actually have that

$$\|Q^-(f_b, f_r)\|_Z \lesssim \|f_b\|_Z \left(\|f_r\|_{L_v^1 L_x^\infty}^{\frac{1}{6}} \|f_r\|_{L_v^{\frac{5}{3}} L_x^\infty}^{\frac{5}{6}} + M^{-1} \|\nabla_x f_r\|_{L_v^1 L_x^\infty}^{\frac{1}{6}} \|\nabla_x f_r\|_{L_v^{\frac{5}{3}} L_x^\infty}^{\frac{5}{6}} \right),$$
$$\|Q^+(f_b, f_r)\|_Z \lesssim \|f_b\|_Z \left(\|f_r\|_{L_v^1 L_x^\infty}^{\frac{1}{6}} \|f_r\|_{L_v^{\frac{5}{3}} L_x^\infty}^{\frac{5}{6}} + M^{-1} \|\nabla_x f_r\|_{L_v^1 L_x^\infty}^{\frac{1}{6}} \|\nabla_x f_r\|_{L_v^{\frac{5}{3}} L_x^\infty}^{\frac{5}{6}} \right),$$
$$\|Q^+(f_r, f_b)\|_Z \lesssim \|f_b\|_Z \left(\|f_r\|_{L_v^1 L_x^\infty}^{\frac{1}{6}} \|f_r\|_{L_v^{\frac{5}{3}} L_x^\infty}^{\frac{5}{6}} + M^{-1} \|\nabla_x f_r\|_{L_v^1 L_x^\infty}^{\frac{1}{6}} \|\nabla_x f_r\|_{L_v^{\frac{5}{3}} L_x^\infty}^{\frac{5}{6}} \right),$$
$$\|Q^\pm(f_r, f_r)\|_Z \lesssim \|f_r\|_Z \left(\|f_r\|_{L_v^1 L_x^\infty}^{\frac{1}{6}} \|f_r\|_{L_v^{\frac{5}{3}} L_x^\infty}^{\frac{5}{6}} + M^{-1} \|\nabla_x f_r\|_{L_v^1 L_x^\infty}^{\frac{1}{6}} \|\nabla_x f_r\|_{L_v^{\frac{5}{3}} L_x^\infty}^{\frac{5}{6}} \right).$$

Note that such an estimate is not possible for $Q^-(f_{\mathrm{r}}, f_{\mathrm{b}})$, which is not contained in the error terms. Therefore, for

$$(sgn, 1, 2) \in \{(+, r, b), (\pm, b, r), (\pm, r, r)\},$$

by estimates (3.55)–(3.57), we have

$$\begin{aligned}
&\left\| \int_\tau^t e^{-(t-t_0)v\cdot\nabla_x} Q^{sgn}(f_1, f_2)(t_0)\, dt_0 \right\|_Z \\
&\lesssim |T_*| \left(\|f_{\mathrm{r}}\|_Z + \|f_{\mathrm{b}}\|_Z\right) \left(\|f_{\mathrm{r}}\|^{\frac{1}{6}}_{L_v^1 L_x^\infty} \|f_{\mathrm{r}}\|^{\frac{5}{6}}_{L_v^{\frac{5}{3}} L_x^\infty} + M^{-1} \|\nabla_x f_{\mathrm{r}}\|^{\frac{1}{6}}_{L_v^1 L_x^\infty} \|\nabla_x f_{\mathrm{r}}\|^{\frac{5}{6}}_{L_v^{\frac{5}{3}} L_x^\infty} \right) \\
&\lesssim |T_*| M^{1-s} (\ln\ln M)^2 N_1^{-\frac{1}{2}} M^{\frac{3}{2}-s} (\ln\ln M)^2 \\
&\lesssim N_1^{-\frac{1}{2}} M^{\frac{3}{2}-s} (\ln\ln M)^5,
\end{aligned}$$

where in the last inequality we have inserted in $|T_*| = M^{s-1}(\ln\ln\ln M)$. This bound suffices for our goal as it carries the smallness factor $N_1^{-\frac{1}{2}}$.

3.3.3. Analysis of $Q^-(f_{\mathrm{b}}, f_{\mathrm{b}})$

Lemma 3.11. *For $T_* \le \tau \le t \le 0$,*

$$\left\| \int_\tau^t e^{-(t-t_0)v\cdot\nabla_x} Q^-(f_{\mathrm{b}}, f_{\mathrm{b}})(t_0)\, dt_0 \right\|_Z \lesssim M^{-1}. \tag{3.59}$$

Proof. As we have required that $N_2 \gg M$ in (3.12), the desired smallness comes from the factor $N_2^{-\delta}$ for some $\delta > 0$. As the x-derivative, which is put on f_{b}, produces a factor of M, it is sufficient to estimate the $L_v^2 L_x^2$, $L_v^1 L_x^\infty$ and $L_v^{\frac{5}{3}} L_x^\infty$ norms.

The $M\|\bullet\|_{L_v^2 L_x^2}$ estimate.

Note that

$$M \left\| \int_\tau^t e^{-(t-t_0)v\cdot\nabla_x} Q^-(f_{\mathrm{b}}, f_{\mathrm{b}})(t_0)\, dt_0 \right\|_{L_v^2 L_x^2} \lesssim |T_*| M \|Q^-(f_{\mathrm{b}}, f_{\mathrm{b}})\|_{L_t^\infty(T_*,0;L_v^2 L_x^2)}. \tag{3.60}$$

We only need to control $M\|Q^-(f_{\mathrm{b}}, f_{\mathrm{b}})\|_{L_t^\infty L_v^2 L_x^2}$. Recall the upper bound (3.19) that

$$f_{\mathrm{b}}(t, x, u) \lesssim \frac{M^{1-s}}{N_2^2} \sum_{j=1}^J \widetilde{K}_j(x) I_j(u), \tag{3.61}$$

where

$$\widetilde{K}_j(x) = \chi\left(\frac{\mathrm{MP}_{e_j}^\perp x}{10}\right) \chi\left(\frac{P_{e_j} x}{10 N_2}\right), \quad I_j(u) = \chi(\mathrm{MP}_{e_j}^\perp u)\, \chi\left(\frac{10 P_{e_j}(u - N_2 e_j)}{N_2}\right).$$

Then we have

$$
\begin{aligned}
&Q^{-}(f_{\mathrm{b}}, f_{\mathrm{b}})(t,x,v)\\
&\lesssim \left(\frac{M^{1-s}}{N_2^2}\right)^2 \left(\sum_{|j-k|\lesssim 1} \widetilde{K}_j(x)\widetilde{K}_k(x) Q^{-}(I_j, I_k)(v) + \sum_{|j-k|\gtrsim 1} \widetilde{K}_j(x)\widetilde{K}_k(x) Q^{-}(I_j, I_k)(v) \right).
\end{aligned} \tag{3.62}
$$

Case I: $|j-k| \lesssim 1$.

For the case that $|j-k| \lesssim 1$, the summands in the double sum $\sum_k^J \sum_j^J$ are reduced to $(MN_2)^2$. By Hölder and Hardy–Sobolev–Littlewood inequality (A.5), we obtain

$$
\begin{aligned}
&\left(\frac{M^{1-s}}{N_2^2}\right)^2 \Big\| \sum_{|j-k|\lesssim 1} \widetilde{K}_j(x)\widetilde{K}_k(x) Q^{-}(I_j, I_k)(v) \Big\|_{L_x^2 L_v^2}\\
&\lesssim \left(\frac{M^{1-s}}{N_2^2}\right)^2 \Big\| \sum_j \widetilde{K}_j(x) I_j(v) \Big\|_{L_x^2 L_v^2} \|\widetilde{K}_k(x)\|_{L_x^\infty} \Big\| \int \frac{I_k(u)}{|u-v|} du \Big\|_{L_v^\infty}\\
&\lesssim \left(\frac{M^{1-s}}{N_2^2}\right)^2 \Big\| \sum_j \widetilde{K}_j(x) I_j(v) \Big\|_{L_x^2 L_v^2} \|I_k\|_{L_v^1}^{\frac{1}{3}} \|I_k\|_{L_v^2}^{\frac{2}{3}}\\
&\lesssim \left(\frac{M^{1-s}}{N_2^2}\right)^2 (M^{-1}N_2^2)(M^{-2}N_2)^{\frac{2}{3}}\\
&= M^{-\frac{1}{3}-2s} N_2^{-\frac{4}{3}}
\end{aligned} \tag{3.63}
$$

where in the second-to-last inequality we have used the disjointness of the v-support to get

$$
\Big\| \sum_j \widetilde{K}_j(x) I_j(v) \Big\|_{L_x^2 L_v^2}^2 \lesssim \sum_j \|\widetilde{K}_j(x)\|_{L_x^2}^2 \|I_j(v)\|_{L_v^2}^2 \lesssim (MN_2)^2 (M^{-2}N_2)^2 = M^{-2}N_2^4. \tag{3.64}
$$

Case II: $|j-k| \gtrsim 1$.

For the case that $|j-k| \gtrsim 1$, this implies that $\sin \alpha_{j,k} \gtrsim (MN_2)^{-1}$, where $\alpha_{j,k}$ denotes the angle between e_j and e_k. Then we have

$$
\begin{aligned}
Q^{-}(I_j, I_k)(v) =& I_j(v) \int \frac{I_k(u)}{|u-v|} du\\
=& I_j(v) \int \frac{1}{|u-v|} \chi(\mathrm{MP}_{e_k}^{\perp} u) \chi\left(\frac{10 P_{e_k}(u - N_2 e_k)}{N_2}\right) du\\
\lesssim& I_j(v) \int \frac{1}{|P_{e_k}(u-v)| + |P_{e_k}^{\perp}(u-v)|} \chi(\mathrm{MP}_{e_k}^{\perp} u) \chi\left(\frac{10 P_{e_k}(u - N_2 e_k)}{N_2}\right) du.
\end{aligned}
$$

Due to the v-support and u-support, we write

$$
v = a e_j + c e_j^{\perp}, \quad u = b e_k + d e_k^{\perp}
$$

where $a \sim b \sim N_2$ and $c \sim d \sim M^{-1}$. Therefore, this gives

$$\begin{aligned}|P_{e_k}^{\perp}(u-v)| =&|P_{e_k}^{\perp}(be_k + de_k^{\perp} - ae_j - ce_j^{\perp})|\\ \gtrsim& a|P_{e_k}^{\perp}e_j| - d - c\\ \gtrsim& N_2 \sin\alpha_{j,k} - M^{-1} \gtrsim M^{-1}\end{aligned} \tag{3.65}$$

where in the last inequality we have used that $\sin\alpha_{j,k} \gtrsim (\mathrm{MN}_2)^{-1}$. By the estimate (3.65), we then set $\xi = \langle u, e_k\rangle$ to get

$$\begin{aligned}I_j(v)\int \frac{I_k(u)}{|u-v|}du \lesssim& I_j(v)\int \frac{1}{|P_{e_k}(u-v)| + M^{-1}}\chi(\mathrm{MN}_{e_k}^{\perp}u)\chi\left(\frac{10P_{e_k}(u-N_2e_k)}{N_2}\right)du\\ \lesssim& I_j(v)\int \frac{1}{|\xi - \langle e_k, v\rangle| + M^{-1}}\chi(M\xi^{\perp})\chi\left(\frac{10(\xi - N_2)}{N_2}\right)d\xi d\xi^{\perp}\\ \lesssim& I_j(v)M^{-2}\int \frac{1}{|\xi| + M^{-1}}\chi\left(\frac{10(\xi + \langle e_k, v\rangle - N_2)}{N_2}\right)d\xi\\ =& I_j(v)M^{-2}N_2\int_{-1}^{1}\frac{M}{\mathrm{MN}_2|\xi| + 1}\chi\left(\frac{10(N_2\xi + \langle e_k, v\rangle - N_2)}{N_2}\right)d\xi\\ \lesssim& I_j(v)M^{-2}N_2\int_{-1}^{1}\frac{M}{\mathrm{MN}_2|\xi| + 1}d\xi\\ \lesssim& I_j(v)\frac{\ln(\mathrm{MN}_2)}{M^2}.\end{aligned} \tag{3.66}$$

Consequently, we arrive at

$$\begin{aligned}&\left(\frac{M^{1-s}}{N_2^2}\right)^2 \sum_{|j-k|\gtrsim 1}\widetilde{K}_j(x)\widetilde{K}_k(x)Q^-(I_j, I_k)\\ &= \left(\frac{M^{1-s}}{N_2^2}\right)^2 \sum_{|j-k|\gtrsim 1}\widetilde{K}_j(x)\widetilde{K}_k(x)I_j(v)\int \frac{I_k(u)}{|u-v|}du\\ &\lesssim \left(\frac{M^{1-s}}{N_2^2}\right)^2 \sum_{j,k}\widetilde{K}_j(x)\widetilde{K}_k(x)I_j(v)\frac{\ln(MN_2)}{M^2}\\ &\le \left(\frac{M^{1-s}}{N_2^2}\right)^2 \frac{\ln(MN_2)}{M^2}\left(\frac{N_2}{|x| + M^{-1}}\right)^2 \chi\left(\frac{x}{N_2}\right)\sum_j \widetilde{K}_j(x)I_j(v)\end{aligned} \tag{3.67}$$

where in the last inequality we have used that

$$\sum_k \widetilde{K}_k(x) = \sum_k^J \chi\left(\frac{MP_{e_k}^{\perp}x}{10}\right)\chi\left(\frac{P_{e_k}x}{N_2}\right) \lesssim \left(\frac{N_2}{|x| + M^{-1}}\right)^2 \chi\left(\frac{x}{N_2}\right). \tag{3.68}$$

To see (3.68), we might as well take $x = (0, 0, |x|)$ with $M^{-1} \le |x| \le N_2$. Let θ_j be the angle between e_j and $(0, 0, 1)$. Then, we have

$$\sum_j^J \chi\left(\frac{MP_{e_j}^{\perp}x}{10}\right)\chi\left(\frac{P_{e_j}x}{N_2}\right) = \sum_j^J \chi\left(\frac{M|x|\sin\theta_j}{10}\right) = \sum_{j:\sin\theta_j \lesssim \frac{1}{|x|M}} 1 \sim \frac{(MN_2)^2}{(|x|M)^2} = \frac{N_2^2}{|x|^2}.$$

Applying the $L^2_v L^2_x$ norm, we have

$$\begin{aligned}
&\left(\frac{M^{1-s}}{N_2^2}\right)^2 \Big\| \sum_{|j-k|\gtrsim 1} \widetilde{K}_j(x)\widetilde{K}_k(x) Q^-(I_j, I_k) \Big\|_{L^2_v L^2_x} \\
&\lesssim \left(\frac{M^{1-s}}{N_2^2}\right)^2 \frac{\ln(MN_2)}{M^2} \Big\| \sum_j \widetilde{K}_j(x) I_j(v) \Big\|_{L^2_v L^\infty_x} \Big\| \left(\frac{N_2}{|x|+M^{-1}}\right)^2 \chi(\frac{x}{N_2}) \Big\|_{L^2_x} \\
&\lesssim \left(\frac{M^{1-s}}{N_2^2}\right)^2 \frac{\ln(MN_2)}{M^2} N_2^{\frac{3}{2}} (M^{\frac{1}{2}} N_2^2) \\
&= M^{\frac{1}{2}-2s} N_2^{-\frac{1}{2}} \ln(MN_2)
\end{aligned} \tag{3.69}$$

where we have used that

$$\Big\| \sum_j \widetilde{K}_j(x) I_j(v) \Big\|_{L^2_v L^\infty_x} \lesssim \Big\| \sum_j I_j(v) \Big\|_{L^2_v} \sim N_2^{\frac{3}{2}},$$

and

$$\Big\| \left(\frac{N_2}{|x|+M^{-1}}\right)^2 \chi\left(\frac{x}{N_2}\right) \Big\|_{L^2_x} = M^{\frac{1}{2}} N_2^2 \Big\| \left(\frac{1}{|x|+1}\right)^2 \chi\left(\frac{x}{MN_2}\right) \Big\|_{L^2_x} \lesssim M^{\frac{1}{2}} N_2^2. \tag{3.70}$$

Combining estimates (3.63) and (3.69) in the two cases, we finally reach

$$M \| Q^-(f_b, f_b) \|_{L^2_v L^2_x} \lesssim M^{\frac{3}{2}-2s} N_2^{-\frac{1}{2}} \ln(MN_2).$$

Together with (3.60), we insert in $|T_*| = M^{s-1}(\ln\ln\ln M)$ to obtain

$$M \Big\| \int_\tau^t e^{-(t-t_0)v\cdot\nabla_x} Q^-(f_b, f_b)(t_0)\, dt_0 \Big\|_{L^2_v L^2_x} \lesssim N_2^{-\frac{1}{2}} M^{\frac{1}{2}-s} \ln(MN_2)(\ln\ln\ln M),$$

which suffices for our goal.

The $N_2^{-1}\| \bullet \|_{L^1_v L^\infty_x}$ estimate.

For convenience, we use the notation

$$D^- = \int_\tau^t e^{-(t-t_0)v\cdot\nabla_x} Q^-(f_b, f_b)(t_0) dt_0.$$

From the analysis on $Q^-(f_b, f_b)$ in estimates (3.63) and (3.69), we actually get a pointwise estimate on $Q^-(f_b, f_b)$ that

$$\begin{aligned}
&Q^-(f_b, f_b) \\
&\lesssim \left(\frac{M^{1-s}}{N_2^2}\right)^2 \sum_j \widetilde{K}_j(x) I_j(v) \left[\frac{\ln(MN_2)}{M^2} \left(\frac{N_2}{|x|+M^{-1}}\right)^2 \chi\left(\frac{x}{N_2}\right) + (M^{-2}N_2)^{\frac{2}{3}} \right].
\end{aligned}$$

Expanding D^- gives that

$$
\begin{aligned}
D^- =& \int_\tau^t Q^-(f_b, f_b)(t_0, x - v(t - t_0), v)dt_0 \\
\lesssim& \int_\tau^t \left(\frac{M^{1-s}}{N_2^2}\right)^2 \sum_j \widetilde{K}_j(x - v(t - t_0))I_j(v) \\
&\times \left[\frac{\ln(MN_2)}{M^2}\left(\frac{N_2}{|x - v(t - t_0)| + M^{-1}}\right)^2 \chi\left(\frac{x - v(t - t_0)}{N_2}\right) + (M^{-2}N_2)^{\frac{2}{3}}\right] dt_0 \\
\lesssim& \frac{M^{-2s}}{N_2^4} I(v) \left[\ln(MN_2) \int_{T_*}^0 \left(\frac{N_2}{|x - v(t - t_0)| + M^{-1}}\right)^2 \chi\left(\frac{x - v(t - t_0)}{N_2}\right) dt_0 + (MN_2)^{\frac{2}{3}}\right],
\end{aligned}
$$

where in the last inequality we have used that

$$
\sum_j \widetilde{K}_j(x - v(t - t_0))I_j(v) \le \sum_j I_j(v) =: I(v) \sim 1_{\left\{\frac{9N_2}{10} \le |v| \le \frac{11N_2}{10}\right\}}(v).
$$

We then deal with the time integral. By change of variable, we have

$$
\begin{aligned}
& I(v) \int_{T_*}^0 \left(\frac{N_2}{|x - v(t - t_0)| + M^{-1}}\right)^2 \chi\left(\frac{x - v(t - t_0)}{N_2}\right) dt_0 \\
& \le I(v) \int_{T_*}^{|T_*|} \left(\frac{\mathrm{MN}_2}{|Mx - Mv\sigma| + 1}\right)^2 \chi\left(\frac{x - v\sigma}{N_2}\right) d\sigma \\
& \le \frac{(\mathrm{MN}_2)^2 I(v)}{M|v|} \int_{-M|T_*||v|}^{M|T_*||v|} \left(\frac{1}{\big|\sigma - M|x|\big| + 1}\right)^2 d\sigma \\
& \lesssim \mathrm{MN}_2 I(v),
\end{aligned}
$$

where in the last inequality we have used that $|v| \sim N_2$ and $\int \frac{d\tau}{\langle\tau\rangle^2} \lesssim 1$. Hence, after carrying out the 1D dt_0 integral, we arrive at

$$
\begin{aligned}
N_2^{-1} \|D^-\|_{L_v^1 L_x^\infty} \lesssim& N_2^{-1} \frac{M^{-2s}}{N_2^4} \|I(v)\|_{L_v^1} \left[\ln(\mathrm{MN}_2)\mathrm{MN}_2 + (\mathrm{MN}_2)^{\frac{2}{3}}\right] \\
\lesssim& N_2^{-1} \frac{M^{-2s}}{N_2^4} N_2^3 \ln(\mathrm{MN}_2)\mathrm{MN}_2 \\
=& N_2^{-1} M^{1-2s} \ln(\mathrm{MN}_2). \qquad (3.71)
\end{aligned}
$$

The $N_2^{\frac{1}{5}} \| \bullet \|_{L_v^{\frac{5}{3}} L_x^\infty}$ estimate.

By the interpolation inequality, we have

$$
N_2^{\frac{1}{5}} \|D^-\|_{L_v^{\frac{5}{3}} L_x^\infty} \le \left(N_2^{-1} \|D^-\|_{L_v^1 L_x^\infty}\right)^{\frac{1}{5}} \left(N_2^{\frac{1}{2}} \|D^-\|_{L_v^2 L_x^\infty}\right)^{\frac{4}{5}}. \qquad (3.72)
$$

For the $L_v^2 L_x^\infty$ norm on D^-, by Hölder inequality, we have

$$N_2^{\frac{1}{2}} \|D^-\|_{L_v^2 L_x^\infty} \lesssim N_2^{\frac{1}{2}} |T_*| \|Q^-(f_b, f_b)\|_{L_v^2 L_x^\infty} \lesssim N_2^{\frac{1}{2}} |T_*| \left\| \int \frac{f_b(x,u)}{|u-v|} du \right\|_{L_v^\infty L_x^\infty} \|f_b\|_{L_v^2 L_x^\infty}.$$

We then use the L^∞ estimate (A.5) in Lemma A.3 and interpolation inequality to get

$$\left\| \int \frac{f_b(x,u)}{|u-v|} du \right\|_{L_v^\infty L_x^\infty} \lesssim \|f_b\|_{L_v^1 L_x^\infty}^{\frac{1}{6}} \|f_b\|_{L_v^{\frac{5}{3}} L_x^\infty}^{\frac{5}{6}} \lesssim N_2^{-1} \|f_b\|_{L_v^1 L_x^\infty} + N_2^{\frac{1}{5}} \|f_b\|_{L_v^{\frac{5}{3}} L_x^\infty} \le \|f_b\|_Z.$$

By the Z-norm bound on f_b in Lemma 3.6, we have that

$$\|f_b\|_Z \lesssim M^{1-s}, \quad N_2^{\frac{1}{2}} \|f_b\|_{L_v^2 L_x^\infty} \lesssim M^{1-s}.$$

Thus, inserting in $|T_*| = M^{s-1}(\ln\ln\ln M)$, we obtain

$$N_2^{\frac{1}{2}} \|D^-\|_{L_v^2 L_x^\infty} \lesssim M^{1-s}(\ln\ln\ln M). \tag{3.73}$$

Combining estimates (3.71), (3.72) and (3.73), we reach

$$N_2^{\frac{1}{5}} \|D^-\|_{L_v^{\frac{5}{3}} L_x^\infty} \lesssim N_2^{-\frac{1}{5}} M^{1-\frac{6}{5}s} \ln(\mathrm{MN_2})(\ln\ln\ln M). \tag{3.74}$$

This bound is enough as it carries the smallness parameter $N_2^{-\frac{1}{5}}$. □

3.3.4. Analysis of $Q^+(f_b, f_b)$

Lemma 3.12. *For $T_* \le \tau \le t \le 0$,*

$$\left\| \int_\tau^t e^{-(t-t_0)v\cdot\nabla_x} Q^+(f_b, f_b)(t_0)\, dt_0 \right\|_Z \lesssim M^{-1}. \tag{3.75}$$

Proof. In a similar way to estimate $Q^-(f_b, f_b)$, we obtain the desired estimate provided the upper bound carries the smallness factor $N_2^{-\delta}$ for some $\delta > 0$. For convenience, we use the notation

$$D^+ = \int_\tau^t e^{-(t-t_0)v\cdot\nabla_x} Q^+(f_b, f_b)(t_0) dt_0.$$

The x-derivative produces the factor of M, so we only need to estimate the $L_v^2 L_x^2$, $L_v^1 L_x^\infty$ and $L_v^{\frac{5}{3}} L_x^\infty$ norms.

The $M\|\bullet\|_{L_v^2 L_x^2}$ estimate.

We use again the upper bound (3.19) that

$$f_b(t,x,u) \lesssim \frac{M^{1-s}}{N_2^2} \sum_{j=1}^J \widetilde{K}_j(x) I_j(u), \tag{3.76}$$

where

$$\widetilde{K}_j(x) = \chi\left(\frac{\mathrm{MP}_{e_j}^{\perp} x}{10}\right)\chi\left(\frac{P_{e_j} x}{10N_2}\right), \quad I_j(u) = \chi(\mathrm{MP}_{e_j}^{\perp} u)\chi\left(\frac{10P_{e_j}(u - N_2 e_j)}{N_2}\right).$$

Then we expand $Q^+(f_b, f_b)$ to get

$$\|Q^+(f_b, f_b)\|_{L_v^2 L_x^2}^2 \lesssim \frac{M^{4-4s}}{N_2^8} \sum_{j_1, j_2, j_3, j_4} \int \widetilde{K}_{j_1}(x)\widetilde{K}_{j_2}(x)\widetilde{K}_{j_3}(x)\widetilde{K}_{j_4}(x) Q^+(I_{j_1}, I_{j_2})(v) Q^+(I_{j_3}, I_{j_4})(v) dx dv.$$

By using that $\widetilde{K}_{j_2}(x) \lesssim 1$ and $\widetilde{K}_{j_4}(x) \lesssim 1$, we obtain

$$\begin{aligned}
&\sum_{j_1, j_2, j_3, j_4} \int \widetilde{K}_{j_1}(x)\widetilde{K}_{j_2}(x)\widetilde{K}_{j_3}(x)\widetilde{K}_{j_4}(x) Q^+(I_{j_1}, I_{j_2})(v) Q^+(I_{j_3}, I_{j_4})(v) dx dv \\
&\lesssim \sum_{j_1, j_2, j_3, j_4} \int \widetilde{K}_{j_1}(x)\widetilde{K}_{j_3}(x) dx \int Q^+(I_{j_1}, I_{j_2})(v) Q^+(I_{j_3}, I_{j_4})(v) dv \\
&= \sum_{j_1, j_3} \int \widetilde{K}_{j_1}(x)\widetilde{K}_{j_3}(x) dx \int Q^+(I_{j_1}, I)(v) Q^+(I_{j_3}, I)(v) dv
\end{aligned}$$

where

$$I(v) = \sum_j^J I_j(v) \sim 1_{\left\{\frac{9N_2}{10} \le |v| \le \frac{11N_2}{10}\right\}}(v).$$

By Hölder inequality and bilinear estimate (A.6) for Q^+ in Lemma A.4,

$$\begin{aligned}
\|Q^+(f_b, f_b)\|_{L_v^2 L_x^2}^2 &\lesssim \frac{M^{4-4s}}{N_2^8} \sum_{j_1, j_3} \int \widetilde{K}_{j_1}(x)\widetilde{K}_{j_3}(x) dx \|Q^+(I_{j_1}, I)\|_{L_v^2} \|Q^+(I_{j_3}, I)\|_{L_v^2} \\
&\lesssim \frac{M^{4-4s}}{N_2^8} \|I\|_{L_v^3}^2 \sum_{j_1, j_3} \int \widetilde{K}_{j_1}(x)\widetilde{K}_{j_3}(x) dx \|I_{j_1}\|_{L_v^{\frac{6}{5}}} \|I_{j_3}\|_{L^{\frac{6}{5}}}.
\end{aligned}$$

Using $\|I_j\|_{L_v^{\frac{6}{5}}} \lesssim (M^{-2}N_2)^{\frac{5}{6}}$, $\|I\|_{L_v^3} \lesssim N_2$, estimates (3.68) and (3.70) for the sum, we obtain

$$\begin{aligned}
\|Q^+(f_b, f_b)\|_{L_v^2 L_x^2}^2 &\lesssim \frac{M^{4-4s}}{N_2^8} (M^{-2}N_2)^{\frac{5}{3}} N_2^2 \Big\| \sum_j \widetilde{K}_j \Big\|_{L_x^2}^2 \\
&\lesssim \frac{M^{4-4s}}{N_2^8} (M^{-2}N_2)^{\frac{5}{3}} N_2^2 \Big\| \left(\frac{N_2}{|x| + M^{-1}}\right)^2 \chi\left(\frac{x}{N_2}\right) \Big\|_{L_x^2}^2 \\
&\lesssim \frac{M^{4-4s}}{N_2^8} (M^{-2}N_2)^{\frac{5}{3}} N_2^2 (M N_2^4) \\
&= M^{\frac{5}{3}-4s} N_2^{-\frac{1}{3}}.
\end{aligned}$$

Thus, we arrive at

$$M\|Q^+(f_b, f_b)\|_{L_v^2 L_x^2} \lesssim N_2^{-\frac{1}{6}} M^{\frac{11}{6}-2s}. \tag{3.77}$$

Upon multiplying by the time factor $|T_*| = M^{s-1}(\ln\ln\ln M)$, this yields a desired bound

$$M\|D^+\|_{L_v^2 L_x^2} \lesssim N_2^{-\frac{1}{6}} M^{\frac{5}{6}-s}(\ln\ln\ln M). \tag{3.78}$$

The $N_2^{-1}\|\bullet\|_{L_v^1 L_x^\infty}$ estimate.
Recall that

$$f_b(t) = \frac{M^{1-s}}{N_2^2}\sum_{j=1}^{J} K_j(x-vt)I_j(v),$$

where $K_j(x) = \chi(MP_{e_j}^\perp x)\chi\left(\frac{P_{e_j}x}{N_2}\right)$, $I_j(v) = \chi(MP_{e_j}^\perp v)\chi\left(\frac{10P_{e_j}(v-N_2e_j)}{N_2}\right)$. The gain term is

$$Q^+(f,g) = \int_{\mathbb{S}^2}\int_{\mathbb{R}^3} B(u-v,\omega)f(v^*)g(u^*)\,du\,d\omega,$$

with the relationship that

$$\begin{aligned} v^* &= P_\omega^\parallel u + P_\omega^\perp v, \quad u^* = P_\omega^\parallel v + P_\omega^\perp u,\\ v &= P_\omega^\perp v^* + P_\omega^\parallel u^*, \quad u = P_\omega^\parallel v^* + P_\omega^\perp u^*. \end{aligned}$$

Then, expanding D^+ gives

$$\begin{aligned} D^+ &= \frac{M^{2-2s}}{N_2^4}\sum_k^J\sum_j^J\int_\tau^t e^{-(t-t_0)v\cdot\nabla_x} Q^+(K_k(x-tv)I_k(v), K_j(x-tv)I_j(v))(t_0)dt_0\\ &= \frac{M^{2-2s}}{N_2^4}\sum_k^J\sum_j^J\int_{\mathbb{S}^2}\int_{\mathbb{R}^3} B(u-v,\omega)S_{j,k}(t,x,\omega,u^*,v^*)dud\omega, \end{aligned}$$

where

$$\begin{aligned} &S_{j,k}(t,x,\omega,u^*,v^*)\\ &\quad = \int_\tau^t K_k(x-v(t-t_0)-v^*t_0)I_k(v^*)K_j(x-v(t-t_0)-u^*t_0)I_j(u^*)dt_0. \end{aligned} \tag{3.79}$$

We estimate by

$$\begin{aligned} &\left\|D^+\right\|_{L_v^1 L_x^\infty}\\ &\quad\leqslant \frac{M^{2-2s}}{N_2^4}\left\|\sum_k^J\sum_j^J\int_{\mathbb{S}^2}\int_{\mathbb{R}^3} B(u-v,\omega)S_{j,k}(t,x,\omega,u^*,v^*)dud\omega\right\|_{L_v^1 L_x^\infty}\\ &\quad\leqslant \frac{M^{2-2s}}{N_2^4}\sum_k^J\sum_j^J\int_{\mathbb{S}^2}\int_{\mathbb{R}^3\times\mathbb{R}^3} B(u-v,\omega)\left\|S_{j,k}(t,x,\omega,u^*,v^*)\right\|_{L_x^\infty} dudvd\omega \end{aligned}$$

$$
\begin{aligned}
&= \frac{M^{2-2s}}{N_2^4} \sum_k^J \sum_j^J \int_{\mathbb{S}^2} \int_{\mathbb{R}^3\times\mathbb{R}^3} B(u-v,\omega) \left\| S_{j,k}(t,x,\omega,u^*,v^*) \right\|_{L_x^\infty} du^* dv^* d\omega \\
&\lesssim \frac{M^{2-2s}}{N_2^4} \sum_k^J \sum_j^J \int du^* \int dv^* \int_{\mathbb{S}^2} d\omega \frac{1}{|u^*-v^*|} \left\| S_{j,k}(t,x,\omega,u^*,v^*) \right\|_{L_x^\infty}
\end{aligned}
\tag{3.80}
$$

where in the second-to-last equality we used the change of variable, and in the last inequality we used that $B(u-v,\omega) = |u-v|^{-1}\mathbf{b}(\cos\theta) \lesssim |u-v|^{-1}$ and $|u-v| = |u^*-v^*|$. We note that

$$
v - u^* = P_\omega^\perp(v^*-u^*), \quad v - v^* = -P_\omega^\parallel(v^*-u^*),
$$

and hence get

$$
\begin{aligned}
x - v(t-t_0) - v^* t_0 &= x - vt - P_\omega^\parallel(v^*-u^*)t_0, \\
x - v(t-t_0) - u^* t_0 &= x - vt + P_\omega^\perp(v^*-u^*)t_0.
\end{aligned}
$$

For fixed u^* and v^*, we get

$$
\begin{aligned}
&S_{j,k}(t,x,\omega,u^*,v^*) \\
&\lesssim \int_{T_*}^0 \chi\left(\mathrm{MP}_{e_k}^\perp(x-vt-P_\omega^\parallel(v^*-u^*)t_0)\right) \chi\left(\frac{P_{e_k}(x-vt-P_\omega^\parallel(v^*-u^*)t_0)}{N_2}\right) I_k(v^*) \\
&\quad \chi\left(\mathrm{MP}_{e_j}^\perp(x-vt+P_\omega^\perp(v^*-u^*)t_0)\right) \chi\left(\frac{P_{e_j}(x-vt+P_\omega^\perp(v^*-u^*)t_0)}{N_2}\right) I_j(u^*) dt_0 \\
&\le I_k(v^*) I_j(u^*) E_k(t,x,v,\omega,u^*,v^*),
\end{aligned}
\tag{3.81}
$$

where

$$
E_k(t,x,v,\omega,u^*,v^*) =: \int_{T_*}^0 \chi\left(\mathrm{MP}_{e_k}^\perp(x-vt-P_\omega^\parallel(v^*-u^*)t_0)\right) dt_0.
$$

We split into two cases in terms of the angle $\alpha_{j,k}$ between e_j and e_k.

Case I: $\alpha_{j,k} \neq 0$. (In this case, we have that $\sin\alpha_{j,k} \gtrsim \frac{1}{\mathrm{MN}_2}$.)

Now, we get into the analysis of $E_k(t,x,v,\omega,u^*,v^*)$. First of all, it gives a trial upper bound that

$$
E_k(t,x,v,\omega,u^*,v^*) \le |T_*| \le 1. \tag{3.82}
$$

By the radial symmetry and monotonicity of the cutoff function χ, we obtain

$$
\int_{\mathbb{R}} \chi(\vec{n} t_0 + \vec{m}) dt_0 \le \frac{1}{|\vec{n}|} \int_{\mathbb{R}} \chi(t_0) dt_0. \tag{3.83}
$$

To see (3.83), without loss of generality, we take $\vec{n} = (0,0,1)$ and $\vec{m} = (m_1, m_2, m_3)$ to get

$$
\int_{\mathbb{R}} \chi(\vec{n} t_0 + \vec{m}) dt_0 = \int_{\mathbb{R}} \chi\left(\sqrt{m_1^2 + m_2^2 + (t_0+m_3)^2}\right) dt_0
$$

$$\leq \int_{\mathbb{R}} \chi\left(|t_0 + m_3|\right) dt_0 = \int_{\mathbb{R}} \chi\left(t_0\right) dt_0.$$

Thus, by (3.83) we arrive at

$$\begin{aligned} E_k(t, x, v, \omega, u^*, v^*) &= \int_{T_*}^{0} \chi\left(\mathrm{MP}_{e_k}^{\perp}(x - vt - P_{\omega}^{\|}(v^* - u^*)t_0)\right) dt_0 \\ &\lesssim \frac{\int \chi(t_0) dt_0}{M|P_{e_k}^{\perp} P_{\omega}^{\|}(v^* - u^*)|} \\ &\lesssim \frac{1}{M \sin\phi_k} \frac{1}{|P_{\omega}^{\|}(v^* - u^*)|}, \end{aligned} \tag{3.84}$$

where ϕ_k is the angle between ω and e_k. Due to the v^*-support and u^*-support, we have

$$\begin{aligned} |v^* - u^*|^2 \sim |ae_k - be_j|^2 =& (a-b)^2 + 2ab(1 - \cos\alpha_{j,k}) \\ \geq& N_2^2(1 - \cos\alpha_{j,k}) \gtrsim N_2^2(\sin\alpha_{j,k})^2. \end{aligned} \tag{3.85}$$

Let θ be the angle between ω and $v^* - u^*$. Then we obtain

$$|P_{\omega}^{\|}(v^* - u^*)| = |v^* - u^*| \cos\theta \gtrsim N_2 \sin\alpha_{j,k} \cos\theta. \tag{3.86}$$

Therefore, we get a useful upper bound that

$$I_k(v^*) I_j(u^*) E_k(t, x, v, \omega, u^*, v^*) \lesssim \frac{I_k(v^*) I_j(u^*)}{\mathrm{MN}_2 \sin\phi_k \cos\theta \sin\alpha_{j,k}}. \tag{3.87}$$

Now, we are able to establish the effective bound on $E_k(t, x, v, \omega, u^*, v^*)$. Set

$$A = \left\{\omega \in \mathbb{S}^2 : \phi_k \leq \frac{1}{\mathrm{MN}_2}\right\} \bigcup \left\{\frac{\pi}{2} - \theta \leq \frac{1}{\mathrm{MN}_2}\right\},$$

and denote by A^c the complementary set of A. With the trivial bound that $E_k \leq 1$ on the set A, we have

$$\begin{aligned} \int_{\mathbb{S}^2} \|E_k(t, x, v, \omega, u^*, v^*)\|_{L_x^\infty} d\omega &\leq \int_A 1 d\omega + \int_{A^c} \|E_k(t, x, v, \omega, u^*, v^*)\|_{L_x^\infty} d\omega \\ &\lesssim \frac{1}{(\mathrm{MN}_2)^2} + \int_{A^c} \|E_k(t, x, v, \omega, u^*, v^*)\|_{L_x^\infty} d\omega. \end{aligned} \tag{3.88}$$

For the second term on the right hand side of (3.88), by the upper bound (3.87), we get

$$\begin{aligned} & I_k(v^*) I_j(u^*) \int_{A^c} \|E_k(t, x, v, \omega, u^*, v^*)\|_{L_x^\infty} d\omega \\ & \leq \frac{I_k(v^*) I_j(u^*)}{\mathrm{MN}_2 \sin\alpha_{j,k}} \int_{A^c} \frac{1}{\sin\phi_k \cos\theta} d\omega \\ & \lesssim \frac{I_k(v^*) I_j(u^*)}{\mathrm{MN}_2 \sin\alpha_{j,k}} \left[\int_{A^c} \frac{1}{(\sin\phi_k)^2} d\omega + \int_{A^c} \frac{1}{(\cos\theta)^2} d\omega\right]. \end{aligned} \tag{3.89}$$

For the last two terms on the right hand side of (3.89), by the rotational symmetry, we might as well to consider

$$\int_{\mathbb{S}^2 \bigcap \left\{|\phi| \geq \frac{1}{MN_2}\right\}} \frac{1}{(\sin\phi)^2} d\omega,$$

where ϕ is the angle between the z-vector $(0, 0, 1)$ and ω. Using the surface integral formula,

$$\int_{\mathbb{S}^2 \bigcap \left\{|\phi| \geq \frac{1}{\mathrm{MN}_2}\right\}} \frac{1}{(\sin\phi)^2} d\omega \lesssim \int_{|\phi| \geq \frac{1}{\mathrm{MN}_2}} \frac{|\sin\phi|}{(\sin\phi)^2} d\phi \lesssim \int_{|\phi| \geq \frac{1}{\mathrm{MN}_2}}^{\frac{\pi}{2}} \frac{1}{|\phi|} d\phi \lesssim \ln(\mathrm{MN}_2).$$

Together with (3.89), this bound yields

$$I_k(v^*) I_j(u^*) \int_{A^c} \|E_k(t, x, v, \omega, u^*, v^*)\|_{L_x^\infty} d\omega \leq \frac{I_k(v^*) I_j(u^*) \ln(\mathrm{MN}_2)}{\mathrm{MN}_2 \sin\alpha_{j,k}}. \tag{3.90}$$

Therefore, combining estimates (3.81), (3.88) and (3.90), we arrive at

$$\int_{\mathbb{S}^2} \|S_{j,k}(t, x, \omega, u^*, v^*)\|_{L_x^\infty} d\omega \lesssim \frac{\ln(\mathrm{MN}_2)}{\mathrm{MN}_2 \sin\alpha_{j,k}} I_k(v^*) I_j(u^*). \tag{3.91}$$

Then, going back to the estimate (3.80) on D^+, we have

$$\begin{aligned}
&\|D^+\|_{L_v^1 L_x^\infty} \\
&\lesssim \frac{M^{2-2s}}{N_2^4} \sum_{k \neq j} \int du^* \int dv^* \int_{\mathbb{S}^2} d\omega \frac{1}{|u^* - v^*|} \|S_{j,k}(t, x, \omega, u^*, v^*)\|_{L_x^\infty} \\
&\lesssim \frac{M^{2-2s}}{N_2^4} \sum_{k \neq j} \frac{\ln(\mathrm{MN}_2)}{\mathrm{MN}_2 \sin\alpha_{j,k}} \int du^* \int dv^* \frac{1}{|u^* - v^*|} I_k(v^*) I_j(u^*).
\end{aligned} \tag{3.92}$$

In this case, since we have that $\sin\alpha_{j,k} \gtrsim \frac{1}{\mathrm{MN}_2}$, we can use estimate (3.66), which is established in the analysis of $Q^-(f_b, f_b)$, to get

$$\int du^* \int dv^* \frac{1}{|u^* - v^*|} I_k(v^*) I_j(u^*) \lesssim \frac{\ln(\mathrm{MN}_2)}{M^2} \int I_j(v^*) dv^* = \frac{\ln(\mathrm{MN}_2)}{M^2} M^{-2} N_2. \tag{3.93}$$

Consequently, combining estimates (3.92) and (3.93), we arrive at

$$\begin{aligned}
N_2^{-1} \|D^+\|_{L_v^1 L_x^\infty} &\lesssim N_2^{-1} \frac{M^{2-2s}}{N_2^4} \sum_{k \neq j}^{J} \frac{\ln(\mathrm{MN}_2)}{\mathrm{MN}_2 \sin\alpha_{j,k}} \frac{\ln(\mathrm{MN}_2)}{M^4} N_2 \\
&\lesssim N_2^{-1} \frac{M^{2-2s}}{N_2^4} \frac{\ln(\mathrm{MN}_2)}{\mathrm{MN}_2} (\mathrm{MN}_2)^4 \frac{\ln(\mathrm{MN}_2)}{M^4} N_2 \\
&= N_2^{-1} M^{1-2s} \left[\ln(\mathrm{MN}_2)\right]^2,
\end{aligned} \tag{3.94}$$

where in the second-to-last inequality we have used that

$$\sum_{k\neq j}^{J}\frac{1}{\sin\alpha_{j,k}}=\sum_{j}^{J}\sum_{i=1}^{MN_2}\sum_{\sin\alpha_{j,k}\sim\frac{i}{\mathrm{MN}_2}}\frac{1}{\sin\alpha_{j,k}}$$
$$\lesssim\sum_{j}^{J}\sum_{i=1}^{\mathrm{MN}_2}\mathrm{MN}_2\sin\alpha_{j,k}\frac{1}{\sin\alpha_{j,k}}\lesssim(\mathrm{MN}_2)^4.$$

This completes the estimate of the $L_v^1L_x^\infty$ norm for D^+.

Case II: $\alpha_{j,k}\sim 0$. (That is, $|j-k|\lesssim 1$.)

In this case, the summands in the double sum $\sum_k^J\sum_j^J$ are reduced to $(\mathrm{MN}_2)^2$, so we only need to use the trivial bound that

$$\int_{\mathbb{S}^2}\|S_{j,k}(t,x,\omega,u^*,v^*)\|_{L_x^\infty}d\omega\lesssim I_k(v^*)I_j(u^*).$$

Then, with the estimate (3.80) on D^+, we use Hardy–Sobolev–Littlewood inequality (A.2) to get

$$\begin{aligned}
&\|D^+\|_{L_v^1L_x^\infty}\\
&\lesssim\frac{M^{2-2s}}{N_2^4}\sum_{k}^{J}\sum_{j}^{J}\int du^*\int dv^*\int_{\mathbb{S}^2}d\omega\frac{1}{|u^*-v^*|}\left\|S_{j,k}(t,x,\omega,u^*,v^*)\right\|_{L_x^\infty}\\
&\lesssim\frac{M^{2-2s}}{N_2^4}\sum_{|j-k|\lesssim 1}^{J}\int du^*\int dv^*\frac{1}{|u^*-v^*|}I_k(v^*)I_j(u^*)\\
&\lesssim\frac{M^{2-2s}}{N_2^4}\sum_{|j-k|\lesssim 1}^{J}\|I_j\|_{L^{\frac{6}{5}}}\|I_k\|_{L^{\frac{6}{5}}}\\
&\lesssim\frac{M^{2-2s}}{N_2^4}(MN_2)^2(M^{-2}N_2)^{\frac{5}{3}}\\
&\lesssim M^{\frac{2}{3}-2s}N_2^{-\frac{1}{3}}.
\end{aligned}\tag{3.95}$$

Combining estimates (3.94) and (3.95) in the two cases, we finally reach

$$N_2^{-1}\|D^+\|_{L_v^1L_x^\infty}\lesssim M^{1-2s}N_2^{-1}\left[\ln(MN_2)\right]^2.\tag{3.96}$$

The $N_2^{\frac{1}{5}}\|\bullet\|_{L_v^{\frac{5}{3}}L_x^\infty}$ estimate.

By the interpolation inequality, we have

$$N_2^{\frac{1}{5}}\|D^+\|_{L_v^{\frac{5}{3}}L_x^\infty}\le\left(N_2^{-1}\|D^+\|_{L_v^1L_x^\infty}\right)^{\frac{1}{5}}\left(N_2^{\frac{1}{2}}\|D^+\|_{L_v^2L_x^\infty}\right)^{\frac{4}{5}}.\tag{3.97}$$

For the $L_v^2L_x^\infty$ norm, by the bilinear estimate (A.6) for Q^+ in Lemma A.4, we have

$$N_2^{\frac{1}{2}}\|D^+\|_{L_v^2L_x^\infty}\lesssim N_2^{\frac{1}{2}}|T_*|\|Q^+(f_{\mathrm{b}},f_{\mathrm{b}})\|_{L_v^2L_x^\infty}$$

$$
\begin{aligned}
&\lesssim N_2^{\frac{1}{2}}|T_*|\|f_{\rm b}\|_{L_v^{\frac{3}{2}}L_x^\infty}\|f_{\rm b}\|_{L_v^2L_x^\infty}\\
&\lesssim |T^*|M^{1-s}M^{1-s}\\
&\lesssim M^{1-s}(\ln\ln\ln M), \qquad (3.98)
\end{aligned}
$$

where we have used the bounds on f_b established in Lemma 3.6 that

$$
\begin{aligned}
&\|f_{\rm b}\|_{L_v^{\frac{3}{2}}L_x^\infty} \le N_2^{-1}\|f_{\rm b}\|_{L_v^1L_x^\infty} + N_2^{\frac{1}{5}}\|f_{\rm b}\|_{L_v^{\frac{5}{3}}L_x^\infty} \le \|f_{\rm b}\|_Z \lesssim M^{1-s},\\
&N_2^{\frac{1}{2}}\|f_{\rm b}\|_{L_v^2L_x^\infty} \lesssim M^{1-s}.
\end{aligned}
$$

Thus, combining estimates (3.96), (3.97) and (3.98), we reach

$$
N_2^{\frac{1}{5}}\|D^+\|_{L_v^{\frac{5}{3}}L_x^\infty} \lesssim N_2^{-\frac{1}{5}}M^{1-\frac{6}{5}s}(\ln\ln\ln M)\ln(MN_2), \tag{3.99}
$$

which is sufficient for our goal. □

3.4. Z-norm bounds on the correction term. Recall the equation (3.3) for the correction term $f_{\rm c}$ that

$$
\begin{cases}
\partial_t f_{\rm c} + v\cdot\nabla_x f_{\rm c} = G,\\
G = \pm Q^{\pm}(f_{\rm c}, f_{\rm a}) \pm Q^{\pm}(f_{\rm a}, f_{\rm c}) \pm Q^{\pm}(f_{\rm c}, f_{\rm c}) - F_{\rm err}.
\end{cases} \tag{3.100}
$$

For $T_* = -M^{s-\frac{d-1}{2}}(\ln\ln\ln M) \le t \le 0$, we are looking for the correction term $f_{\rm c}(t)$ with

$$
\|f_{\rm c}(t)\|_{L_v^{2,r_0}H_x^{s_0}} \lesssim M^{-1/2}. \tag{3.101}
$$

To achieve it, we apply a perturbation argument and work on the stronger Z-norm (3.36). By interpolation inequality, for $d = 2, 3$, we indeed have

$$
\begin{aligned}
\|f\|_{L_v^{2,r_0}H_x^{s_0}} \le \|f\|_{L_v^{2,r_0}H_x^{\frac{d-1}{2}}} &\le \|f\|_{L_v^{2,r_0}L_x^2}^{\frac{3-d}{2}}\|\langle\nabla_x\rangle f\|_{L_v^{2,r_0}L_x^2}^{\frac{d-1}{2}}\\
&\le M^{\frac{d-1}{2}}\|f\|_{L_v^{2,r_0}L_x^2} + M^{\frac{d-3}{2}}\|\langle\nabla_x\rangle f\|_{L_v^{2,r_0}L_x^2} \le \|f\|_Z.
\end{aligned}
$$

Certainly, there are multiple choices of Z-norms. As we are fully in the perturbation regime, we expect the correction term $f_{\rm c}$ to be much smoother and hence we choose the $L_v^{2,r_0}H_x^1$ norm. On the other hand, to beat the difficulties caused by singularities of soft potentials, the $L_v^1L_x^\infty$ and $L_v^{\frac{5}{3}}L_x^\infty$ norms[9] are needed as shown in the following estimate (3.102).

In the section, we first prove a closed estimate for the loss and gain terms in Lemma 3.13 and then use it to conclude the existence of small correction term $f_{\rm c}(t)$ in Proposition 3.14.

[9] The index $\frac{5}{3}$ is just one of the multiple choices. We choose it, as it would not yield much more difficulties in the estimates on the approximation solution and error terms.

Lemma 3.13. (Bilinear Z-norm estimates for loss/gain operator $Q^{\pm}$) *For f_1, f_2, we have*

$$\|Q^{\pm}(f_1, f_2)\|_Z \lesssim \|f_1\|_Z \|f_2\|_Z.$$

Proof. We only need to prove that

$$\|Q^{\pm}(f_1, f_2)\|_Z \lesssim \|f_1\|_Z \left(\|f_2\|_{L_v^1 L_x^\infty}^{1+\frac{5\gamma}{2d}} \|f_2\|_{L_v^{\frac{5}{3}} L_x^\infty}^{\frac{-5\gamma}{2d}} + M^{-1} \|\nabla_x f_2\|_{L_v^1 L_x^\infty}^{1+\frac{5\gamma}{2d}} \|\nabla_x f_2\|_{L_v^{\frac{5}{3}} L_x^\infty}^{\frac{-5\gamma}{2d}} \right). \tag{3.102}$$

since we have that

$$\begin{aligned}
\|f\|_{L_v^1 L_x^\infty}^{1+\frac{5\gamma}{2d}} \|f\|_{L_v^{\frac{5}{3}} L_x^\infty}^{\frac{-5\gamma}{2d}} \leq & N_2^{-1} \|f\|_{L_v^1 L_x^\infty} + N_2^{\frac{2d}{5}+\gamma} \|f\|_{L_v^{\frac{5}{3}} L_x^\infty} \leq \|f\|_Z, \\
M^{-1} \|\nabla_x f\|_{L_v^1 L_x^\infty}^{1+\frac{5\gamma}{2d}} \|\nabla_x f\|_{L_v^{\frac{5}{3}} L_x^\infty}^{\frac{-5\gamma}{2d}} \leq & M^{-1} N_2^{-1} \|\nabla_x f\|_{L_v^1 L_x^\infty} \\
& + M^{-1} N_2^{\frac{2d}{5}+\gamma} \|\nabla_x f\|_{L_v^{\frac{5}{3}} L_x^\infty} \leq \|f\|_Z.
\end{aligned}$$

The $M^{\frac{d-3}{2}} \|\nabla_x \bullet \|_{L_v^{2,r_0} L_x^2}$ and $M^{\frac{d-1}{2}} \| \bullet \|_{L_v^{2,r_0} L_x^2}$ estimates for $Q^{\pm}(f_1, f_2)$.

It suffices to deal with $M^{\frac{d-3}{2}} \|\nabla_x \bullet \|_{L_v^{2,r_0} L_x^2}$ norm as the $M^{\frac{d-1}{2}} \| \bullet \|_{L_v^{2,r_0} L_x^2}$ norm can be estimated in a similar way. For the estimate on Q^-, we use Leibniz rule and Hölder inequality to get

$$\begin{aligned}
& M^{\frac{d-3}{2}} \|\nabla_x Q^-(f_1, f_2)\|_{L_v^{2,r_0} L_x^2} \\
& \leq M^{\frac{d-3}{2}} \|Q^-(\nabla_x f_1, f_2)\|_{L_v^{2,r_0} L_x^2} + M^{\frac{d-3}{2}} \|Q^-(f_1, \nabla_x f_2)\|_{L_v^{2,r_0} L_x^2} \\
& \lesssim M^{\frac{d-3}{2}} \left\| (\nabla_x f_1)(x, v) \int \frac{f_2(x, u)}{|u-v|^{-\gamma}} du \right\|_{L_v^{2,r_0} L_x^2} \\
& \quad + M^{\frac{d-3}{2}} \left\| f_1(x, v) \int \frac{\nabla_x f_2(x, u)}{|u-v|^{-\gamma}} du \right\|_{L_v^{2,r_0} L_x^2} \\
& \lesssim M^{\frac{d-3}{2}} \|\nabla_x f_1\|_{L_v^{2,r_0} L_x^2} \left\| \int \frac{f_2(x, u)}{|u-v|^{-\gamma}} du \right\|_{L_{v,x}^\infty} \\
& \quad + M^{\frac{d-3}{2}} \|f_1\|_{L_v^{2,r_0} L_x^2} \left\| \int \frac{\nabla_x f_2(x, u)}{|u-v|^{-\gamma}} du \right\|_{L_{v,x}^\infty}.
\end{aligned}$$

Then by L^∞ estimate (A.4) in Lemma A.3, we obtain

$$\begin{aligned}
& M^{\frac{d-3}{2}} \|\nabla_x Q^-(f_1, f_2)\|_{L_v^{2,r_0} L_x^2} \\
& \lesssim M^{\frac{d-3}{2}} \|\nabla_x f_1\|_{L_v^{2,r_0} L_x^2} \|f_2\|_{L_v^1 L_x^\infty}^{1+\frac{5\gamma}{2d}} \|f_2\|_{L_v^{\frac{5}{3}} L_x^\infty}^{\frac{-5\gamma}{2d}} \\
& \quad + M^{\frac{d-1}{2}} \|f_1\|_{L_v^{2,r_0} L_x^2} M^{-1} \|\nabla_x f_2\|_{L_v^1 L_x^\infty}^{1+\frac{5\gamma}{2d}} \|\nabla_x f_2\|_{L_v^{\frac{5}{3}} L_x^\infty}^{\frac{-5\gamma}{2d}} \\
& \lesssim \|f_1\|_Z \|f_2\|_Z.
\end{aligned}$$

For the estimate on Q^+, from the conservation of energy that $|v|^2+|u|^2 = |v^*|^2+|u^*|^2$, we use Leibniz rule to get

$$|\langle v\rangle^{r_0}\nabla_x Q^+(f_1, f_2)| \lesssim Q^+(\langle v\rangle^{r_0}|\nabla_x f_1|, |f_2|) + Q^+(\langle v\rangle^{r_0}|f_1|, |\nabla_x f_2|) + Q^+(|\nabla_x f_1|, \langle v\rangle^{r_0}|f_2|) + Q^+(|f_1|, \langle v\rangle^{r_0}|\nabla_x f_2|).$$

Then by bilinear estimate (A.6) on Q^+, we have

$$\begin{aligned} &M^{\frac{d-3}{2}}\|\nabla_x Q^+(f_1, f_2)\|_{L_v^{2,r_0}L_x^2} \\ &\lesssim M^{\frac{d-3}{2}}\|\langle v\rangle^{r_0}\nabla_x f_1\|_{L_v^2L_x^2}\|f_2\|_{L_v^{\frac{d}{d+\gamma}}L_x^\infty} + M^{\frac{d-1}{2}}\|\langle v\rangle^{r_0} f_1\|_{L_v^2L_x^2}M^{-1}\|\nabla_x f_2\|_{L_v^{\frac{d}{d+\gamma}}L_x^\infty} \\ &\quad + M^{-1}\|\nabla_x f_1\|_{L_v^{\frac{d}{d+\gamma}}L_x^\infty} M^{\frac{d-1}{2}}\|\langle v\rangle^{r_0} f_2\|_{L_v^2L_x^2} + \|f_1\|_{L_v^{\frac{d}{d+\gamma}}L_x^\infty} M^{\frac{d-3}{2}}\|\langle v\rangle^{r_0}\nabla_x f_2\|_{L_v^2L_x^2}. \end{aligned}$$

By the interpolation inequality that

$$\|f\|_{L_v^{\frac{d}{d+\gamma}}L_x^\infty} \le \|f\|_{L_v^1L_x^\infty}^{1+\frac{5\gamma}{2d}}\|f\|_{L_v^{\frac{5}{3}}L_x^\infty}^{\frac{-5\gamma}{2d}},$$

we arrive at

$$M^{\frac{d-3}{2}}\|\nabla_x Q^+(f_1, f_2)\|_{L_v^{2,r_0}L_x^2} \lesssim \|f_1\|_Z\|f_2\|_Z.$$

The $N_2^\gamma\|\bullet\|_{L_v^1L_x^\infty}$ and $N_2^{\frac{2d}{5}+\gamma}\|\bullet\|_{L_v^{\frac{5}{3}}L_x^\infty}$ estimates for $Q^\pm(f_1, f_2)$.

For the estimate on Q^-, we use Hölder inequality and the L^∞ estimate (A.4) to get

$$\begin{aligned} N_2^\gamma\|Q^-(f_1, f_2)\|_{L_v^1L_x^\infty} &\lesssim N_2^\gamma\|f_1\|_{L_v^1L_x^\infty}\left\|\int \frac{f_2(x,u)}{|u-v|^{-\gamma}}du\right\|_{L_v^\infty L_x^\infty} \\ &\lesssim N_2^\gamma\|f_1\|_{L_v^1L_x^\infty}\|f_2\|_{L_v^1L_x^\infty}^{1+\frac{5\gamma}{2d}}\|f_2\|_{L_v^{\frac{5}{3}}L_x^\infty}^{\frac{-5\gamma}{2d}} \\ &\lesssim \|f_1\|_Z\|f_2\|_Z. \end{aligned} \tag{3.103}$$

In the same way, we also have

$$\begin{aligned} N_2^{\frac{2d}{5}+\gamma}\|Q^-(f_1, f_2)\|_{L_v^{\frac{5}{3}}L_x^\infty} &\lesssim N_2^{\frac{2d}{5}+\gamma}\|f_1\|_{L_v^{\frac{5}{3}}L_x^\infty}\|f_2\|_{L_v^1L_x^\infty}^{1+\frac{5\gamma}{2d}}\|f_2\|_{L_v^{\frac{5}{3}}L_x^\infty}^{\frac{-5\gamma}{2d}} \\ &\lesssim \|f_1\|_Z\|f_2\|_Z. \end{aligned}$$

For the estimate on Q^+, by the bilinear estimate (A.6) for Q^+ in Lemma A.4, we have

$$N_2^\gamma\|Q^+(f_1, f_2)\|_{L_v^1L_x^\infty} \lesssim N_2^\gamma\|f_1\|_{L_v^1L_x^\infty}\|f_2\|_{L_v^1L_x^\infty}^{1+\frac{5\gamma}{2d}}\|f_2\|_{L_v^{\frac{5}{3}}L_x^\infty}^{\frac{-5\gamma}{2d}} \lesssim \|f_1\|_Z\|f_2\|_Z. \tag{3.104}$$

Similarly, by the bilinear estimate (A.6) in Lemma A.4, we obtain

$$N_2^{\frac{2d}{5}+\gamma}\|Q^+(f_1, f_2)\|_{L_v^{\frac{5}{3}}L_x^\infty} \lesssim N_2^{\frac{2d}{5}+\gamma}\|f_1\|_{L_v^{\frac{5}{3}}L_x^\infty}\|f_2\|_{L_v^1L_x^\infty}^{1+\frac{5\gamma}{2d}}\|f_2\|_{L_v^{\frac{5}{3}}L_x^\infty}^{\frac{-5\gamma}{2d}} \lesssim \|f_1\|_Z\|f_2\|_Z.$$

The $M^{-1}N_2^{\gamma}\|\nabla_x \bullet\|_{L_v^1 L_x^\infty}$ and $M^{-1}N_2^{\frac{2d}{5}+\gamma}\|\nabla_x \bullet\|_{L_v^{\frac{5}{3}} L_x^\infty}$ estimates for $Q^{\pm}(f_1, f_2)$. For the estimate on Q^-, in a similar way to (3.103), we use the Leibniz rule to get

$$\begin{aligned}
&M^{-1}N_2^{\gamma}\|\nabla_x Q^-(f_1, f_2)\|_{L_v^1 L_x^\infty}\\
&\quad\le M^{-1}N_2^{\gamma}\left(\|Q^-(\nabla_x f_1, f_2)\|_{L_v^1 L_x^\infty} + \|Q^-(f_1, \nabla_x f_2)\|_{L_v^1 L_x^\infty}\right)\\
&\quad\lesssim M^{-1}N_2^{\gamma}\|\nabla_x f_1\|_{L_v^1 L_x^\infty}\|f_2\|_{L_v^1 L_x^\infty}^{1+\frac{5\gamma}{2d}}\|f_2\|_{L_v^{\frac{5}{3}} L_x^\infty}^{\frac{-5\gamma}{2d}}\\
&\qquad + N_2^{\gamma}\|f_1\|_{L_v^1 L_x^\infty}M^{-1}\|\nabla_x f_2\|_{L_v^1 L_x^\infty}^{1+\frac{5\gamma}{2d}}\|\nabla_x f_2\|_{L_v^{\frac{5}{3}} L_x^\infty}^{\frac{-5\gamma}{2d}}\\
&\quad\lesssim \|f_1\|_Z\|f_2\|_Z.
\end{aligned}$$

The same also holds for the $M^{-1}N_2^{\frac{2d}{5}+\gamma}\|\nabla_x Q^-(f_1, f_2)\|_{L_v^{\frac{5}{3}} L_x^\infty}$ norm.

For the estimate on Q^+, in a similar way to (3.104), we also have

$$\begin{aligned}
&M^{-1}N_2^{\gamma}\|\nabla_x Q^+(f_1, f_2)\|_{L_v^1 L_x^\infty}\\
&\quad\le M^{-1}N_2^{\gamma}\left(\|Q^+(\nabla_x f_1, f_2)\|_{L_v^1 L_x^\infty} + \|Q^+(f_1, \nabla_x f_2)\|_{L_v^1 L_x^\infty}\right)\\
&\quad\lesssim M^{-1}N_2^{\gamma}\|\nabla_x f_1\|_{L_v^1 L_x^\infty}\|f_2\|_{L_v^1 L_x^\infty}^{1+\frac{5\gamma}{2d}}\|f_2\|_{L_v^{\frac{5}{3}} L_x^\infty}^{\frac{-5\gamma}{2d}}\\
&\qquad + N_2^{\gamma}\|f_1\|_{L_v^1 L_x^\infty}M^{-1}\|\nabla_x f_2\|_{L_v^1 L_x^\infty}^{1+\frac{5\gamma}{2d}}\|\nabla_x f_2\|_{L_v^{\frac{5}{3}} L_x^\infty}^{\frac{-5\gamma}{2d}}\\
&\quad\lesssim \|f_1\|_Z\|f_2\|_Z.
\end{aligned}$$

The estimate for the $M^{-1}N_2^{\frac{2d}{5}+\gamma}\|\nabla_x Q^+(f_1, f_2)\|_{L_v^{\frac{5}{3}} L_x^\infty}$ norm follows the same way by using bilinear estimate (A.6) in Lemma A.4. □

Now, we take a perturbation argument to generate the correction term $f_c(t)$ using the Z-norm bounds on f_a in Proposition 3.8 and the Z-norm bounds on F_{err} in Proposition 3.9.

Proposition 3.14. *Suppose that f_c solves (3.100) with $f_c(0) = 0$. Then for all t such that*

$$T_* = -M^{s-\frac{d-1}{2}}(\ln\ln\ln M) \le t \le 0,$$

we have the bound

$$\|f_c(t)\|_Z \lesssim M^{-1/2}. \tag{3.105}$$

Proof. Let the time interval $T_* \le t \le 0$ be partitioned as

$$T_* = T_n < T_{n-1} < T_{n-2} < \cdots < T_2 < T_1 < T_0 = 0$$

where

$$T_j = \frac{-jM^{s-\frac{d-1}{2}}}{\sqrt{\ln M}}, \quad n = \sqrt{\ln M}(\ln\ln\ln M).$$

Thus, the length of each time interval $I_j = [T_{j+1}, T_j]$ is

$$|I_j| = \frac{M^{s-\frac{d-1}{2}}}{\sqrt{\ln M}}.$$

For $t \in I_j = [T_{j+1}, T_j]$, we rewrite the Eq. (3.100) in Duhamel form

$$f_c(T_j + t) = e^{-(t-T_j)v\cdot\nabla_x} f_c(T_j) + \int_{T_j}^{t} e^{-(t-t_0)v\cdot\nabla_x} G(t_0)dt_0$$

with $f_c(T_0) = 0$. Applying the Z-norm,

$$\begin{aligned}
\|f_c\|_{L^\infty_{I_j}Z} \leq& \|f_c(T_j)\|_Z + \left\| \int_{T_j}^{t} e^{-(t-t_0)v\cdot\nabla_x} G(t_0)dt_0 \right\|_{L^\infty_{I_j}Z} \\
\leq& \|f_c(T_j)\|_Z + |I_j| \|Q^{\pm}(f_c, f_a)\|_{L^\infty_{I_j}Z} \\
&+ |I_j| \|Q^{\pm}(f_a, f_c)\|_{L^\infty_{I_j}Z} + |I_j| \|Q^{\pm}(f_c, f_c)\|_{L^\infty_{I_j}Z} \\
&+ \left\| \int_{T_j}^{t} e^{-(t-t_0)v\cdot\nabla_x} F_{err}(t_0)dt_0 \right\|_{L^\infty_{I_j}Z}.
\end{aligned}$$

For these terms on the second line, we apply the bilinear estimate in Lemma 3.13, and then the estimate (3.47) on $\|f_a\|_{L^\infty_{I_j}Z}$ from Lemma 3.8. For the F_{err} term on the last line, we use the estimate (3.49) in Proposition 3.9. Then we have

$$\|f_c\|_{L^\infty_{I_j}Z} \leq \|f_c(T_j)\|_Z + \frac{C(\ln\ln M)^2}{\sqrt{\ln M}} \|f_c\|_{L^\infty_{I_j}Z} + \frac{CM^{s-\frac{d-1}{2}}}{\sqrt{\ln M}} \|f_c\|^2_{L^\infty_{I_j}Z} + CM^{-1},$$

where C is some absolute constant. Absorbing the $\|f_c\|_{L^\infty_{I_j}Z}$ term on the right gives

$$\|f_c\|_{L^\infty_{I_j}Z} \leq 2\|f_c(T_j)\|_Z + 2CM^{-1}.$$

Applying this successively for $j = 0, 1, \ldots,$ we obtain

$$\|f_c\|_{L^\infty_{I_j}Z} \leq (2^{j+1} - 1)2CM^{-1}.$$

With $j = n = \sqrt{\ln M}(\ln\ln\ln M)$, we arrive at

$$\|f_c(T_*)\|_Z \leq \frac{Ce^{\sqrt{\ln M}\ln\ln M}}{M} = \frac{Ce^{\sqrt{\ln M}\ln\ln M}}{e^{\ln M}} \leq M^{-1/2} \ll 1.$$

□

3.5. Proof of illposedness. We get into the proof the ill-posedness.

Proof of Ill-posedness in Theorem 1.2. Let

$$f_{\mathrm{ex}}(t) = f_{\mathrm{r}}(t) + f_{\mathrm{b}}(t) + f_{\mathrm{c}}(t),$$

with $f_{\mathrm{c}}(t)$ given in Proposition 3.14. By the upper and lower bounds in Lemma 3.3 that

$$\|f_{\mathrm{r}}(0)\|_{L_v^{2,r_0}H_x^{s_0}} \lesssim \frac{1}{\ln\ln M}, \quad \|f_{\mathrm{r}}(T_*)\|_{L_v^{2,r_0}H_x^{s_0}} \gtrsim 1,$$

we can take $t_0 \in [T_*, 0]$ such that $\|f_{\mathrm{r}}(t_0)\|_{L_v^{2,r_0}H_x^{s_0}} = 1$. Note that

$$\|f_{\mathrm{r}}\|_Z \lesssim M^{\frac{d-1}{2}-s}(\ln\ln M)^2, \quad \|v\cdot\nabla_x f_{\mathrm{r}}\|_Z \ll M^{-1}, \quad \|Q^{\pm}(f_{\mathrm{r}}, f_{\mathrm{r}})\|_Z \ll M^{-1},$$

which are established in Lemma 3.7 and Sect. 3.3. Therefore, by the same perturbation argument in Lemma 3.14, we generate an exact solution $g_{\mathrm{ex}}(t)$ to Boltzmann equation

$$g_{\mathrm{ex}}(t) = f_{\mathrm{r}}(t_0) + g_{\mathrm{c}}(t),$$

with $g_{\mathrm{c}}(0) = 0$. This gives that

$$\begin{cases} \|g_{\mathrm{ex}}(0)\|_{L_v^{2,r_0}H_x^{s_0}} = \|f_{\mathrm{r}}(t_0)\|_{L_v^{2,r_0}H_x^{s_0}} = 1, \\ \|g_{\mathrm{c}}(t)\|_{L^\infty([T_*,0];Z)} \lesssim M^{-\frac{1}{2}}. \end{cases}$$

Now, we have two solutions with the decompositions

$$\begin{cases} f_{\mathrm{ex}}(t) = f_{\mathrm{r}}(t) + f_{\mathrm{b}}(t) + f_{\mathrm{c}}(t), \\ g_{\mathrm{ex}}(t) = f_{\mathrm{r}}(t_0) + g_c(t), \end{cases}$$

which gives

$$f_{\mathrm{ex}}(t) - g_{\mathrm{ex}}(t) = (f_{\mathrm{r}}(t) - f_{\mathrm{r}}(t_0)) + f_{\mathrm{b}}(t) + f_{\mathrm{c}}(t) - g_{\mathrm{c}}(t).$$

For $t \in [T_*, 0]$, by Lemma 3.1 and Proposition 3.14, we have

$$\|f_{\mathrm{b}}(t)\|_{L_v^{2,r_0}H_x^{s_0}} \lesssim M^{s_0-s} = \frac{1}{\ln\ln M},$$
$$\|f_{\mathrm{c}}(t)\|_{L_v^{2,r_0}H_x^{s_0}} \le \|f_{\mathrm{c}}(t)\|_Z \lesssim M^{-\frac{1}{2}},$$
$$\|g_{\mathrm{c}}(t)\|_{L_v^{2,r_0}H_x^{s_0}} \le \|g_{\mathrm{c}}(t)\|_Z \lesssim M^{-\frac{1}{2}}.$$

Thus, we obtain

$$\|f_{\mathrm{ex}}(t_0) - g_{\mathrm{ex}}(t_0)\|_{L_v^{2,r_0}H_x^{s_0}} \lesssim \frac{1}{\ln\ln M},$$

and

$$\|f_{\mathrm{ex}}(0) - g_{\mathrm{ex}}(0)\|_{L_v^{2,r_0}H_x^{s_0}} \sim \|f_{\mathrm{r}}(0) - f_{\mathrm{r}}(t_0)\|_{L_v^{2,r_0}H_x^{s_0}} \sim \|f_{\mathrm{r}}(t_0)\|_{L_v^{2,r_0}H_x^{s_0}} = 1,$$

where we have used that $\|f_{\mathrm{r}}(0)\|_{L_v^{2,r_0}H_x^{s_0}} \lesssim \frac{1}{\ln\ln M}$. Hence, we complete the proof. □

Remark 3.15. We actually have found an exact solution $f_{\text{ex}}(t)$ which satisfies the norm deflation property. This is the key to conclude the failure of uniform continuity of the data-to-solution map.

In the end, we prove Corollary 1.3.

Proof of Corollary 1.3. Recall the kernel

$$B(u-v,\omega)=\left(1_{\{|u-v|\le 1\}}|u-v|+1_{\{|u-v|\ge 1\}}|u-v|^{-1}\right)\mathbf{b}\left(\frac{u-v}{|u-v|}\cdot\omega\right),\tag{3.106}$$

and notice the pointwise upper bound estimate

$$\left(1_{\{|u-v|\le 1\}}|u-v|+1_{\{|u-v|\ge 1\}}|u-v|^{-1}\right)\mathbf{b}\left(\frac{u-v}{|u-v|}\cdot\omega\right)\le\frac{1}{|u-v|}\mathbf{b}\left(\frac{u-v}{|u-v|}\cdot\omega\right).\tag{3.107}$$

Therefore, for the kernel $B(u-v,\omega)$ in (3.106), all the same upper bound estimates on f_{b}, f_{r}, f_{a}, F_{err}, and f_{c} follow from the pointwise upper bound estimate (3.107). The only one lower bound on f_{r} we need is given in Remark 3.4. Then by repeating the proof of ill-posedness for the endpoint case $(d,\gamma,r_0)=(3,-1,0)$ in Theorem 1.2, we complete the proof of Corollary 1.3. □

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Declarations

Conflict of interest The authors declare that they have no Conflict of interest.

Appendix A. Sobolev-Type and Time-Independent Bilinear Estimates

Lemma A.1 (Fractional Leibniz rule, [40]). *Suppose $1<r<\infty$, $s\ge 0$ and $\frac{1}{r}=\frac{1}{p_i}+\frac{1}{q_i}$ with $i=1,2$, $1<q_1\le\infty$, $1<p_2\le\infty$. Then*

$$\|\langle\nabla_x\rangle^s(fg)\|_{L^r}\le C\|\langle\nabla_x\rangle^s f\|_{L^{p_1}}\|g\|_{L^{q_1}}+\|f\|_{L^{p_2}}\|\langle\nabla_x\rangle^s g\|_{L^{q_2}}\tag{A.1}$$

where the constant C depends on all of the parameters.

Next, we present the standard Hardy-Littlewood-Sobolev inequality, which is widely used in our various estimates for the soft potential case.

Lemma A.2. *Let $p>1$, $r>1$ and $-d<\gamma\le 0$ with*

$$\frac{1}{p}+\frac{1}{r}=2+\frac{\gamma}{d}.$$

Let $f\in L^p(\mathbb{R}^d)$ and $h\in L^r(\mathbb{R}^d)$, then there exists a constant $C(d,\gamma,p)$, independent of f and h, such that

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(x)|x-y|^{\gamma}h(y)\mathrm{d}x\mathrm{d}y\le C(d,\gamma,p,r)\|f\|_p\|h\|_r. \tag{A.2}$$

In particular, for $p>1$, $q>1$ with

$$1+\frac{1}{q}+\frac{\gamma}{d}=\frac{1}{p},$$

we also have

$$\|f*|\cdot|^{\gamma}\|_{L^q}\le C(d,\gamma,p,q)\|f\|_{L^p}. \tag{A.3}$$

Lemma A.3 (Endpoint case). *Let $d\ge 2$, $-d<\gamma\le 0$, and $1\le p<\frac{d}{d+\gamma}<q\le\infty$. Then for $f\in L^p\left(\mathbb{R}^d\right)\cap L^q\left(\mathbb{R}^d\right)$, it holds that*

$$\int |x|^{\gamma}|f(x)|dx\lesssim \|f\|_{L^p}^{\frac{\frac{q-1}{q}+\frac{\gamma}{d}}{\frac{1}{p}-\frac{1}{q}}}\|f\|_{L^q}^{\frac{-\frac{\gamma}{d}-\frac{p-1}{p}}{\frac{1}{p}-\frac{1}{q}}}. \tag{A.4}$$

In particular, when $\gamma=-1$, $p=1$, and $q>\frac{d}{d-1}$, we have

$$\left\|\int\frac{f(y)}{|x-y|}dy\right\|_{L^\infty_x}\lesssim\|f\|_{L^1}^{1-\frac{1}{d\left(1-\frac{1}{q}\right)}}\|f\|_{L^q}^{\frac{1}{d\left(1-\frac{1}{q}\right)}}. \tag{A.5}$$

Proof. The endpoint case is also known. For completeness, we include a proof. We split the integral into two parts and use Hölder inequality to get

$$\begin{aligned}\int_{\mathbb{R}^d}|x|^{\gamma}|f(x)|dx&\le\int_{|x|\le\eta}|x|^{\gamma}|f(x)|dx+\int_{|x|>\eta}|x|^{\gamma}|f(x)|dx\\&\lesssim\|f\|_{L^q}\eta^{\frac{d}{q'}+\gamma}+\|f\|_{L^p}\eta^{\frac{d}{p'}+\gamma},\end{aligned}$$

where $p'=\frac{p}{p-1}$ and $q'=\frac{q}{q-1}$. Optimizing the choice of η gives the desired estimate that

$$\int_{\mathbb{R}^d}|x|^{\gamma}|f(x)|dx\lesssim\|f\|_{L^p}^{\frac{\frac{q-1}{q}+\frac{\gamma}{d}}{\frac{1}{p}-\frac{1}{q}}}\|f\|_{L^q}^{\frac{-\frac{\gamma}{d}-\frac{p-1}{p}}{\frac{1}{p}-\frac{1}{q}}}.$$

□

The following parts focus on time-independent bilinear estimates for gain/loss terms.

Lemma A.4 ([5, Theorem 2, Corollary 9]). *Let* $1 < p, q, r < \infty$ *and* $-d < \gamma \le 0$ *with*

$$\frac{1}{p} + \frac{1}{q} = 1 + \frac{\gamma}{d} + \frac{1}{r}.$$

Assume the collision kernel

$$B(u-v,\omega) = |u-v|^{\gamma}\, \boldsymbol{b}\left(\frac{u-v}{|u-v|}\cdot\omega\right),$$

with $\boldsymbol{b}(\frac{u-v}{|u-v|}\cdot\omega)$ *satisfying Grad's angular cutoff assumption. Then, it holds that*

$$\left\|Q^{+}(f,g)\right\|_{L^r(\mathbb{R}^d)} \le C\|f\|_{L^p(\mathbb{R}^d)}\|g\|_{L^q(\mathbb{R}^d)}, \tag{A.6}$$

$$\left\|Q^{-}(f,g)\right\|_{L^r(\mathbb{R}^d)} \le C\|f\|_{L^p(\mathbb{R}^d)}\|g\|_{L^q(\mathbb{R}^d)}, \quad p > r. \tag{A.7}$$

Lemma A.5 (L^1 endpoint estimate for Q^+). *For* $\gamma = -1$, *we have*

$$\|Q^{+}(f,g)\|_{L^1} \le \|f\|_{L^1}\|g\|_{L^1}^{1-\frac{1}{d\left(1-\frac{1}{p}\right)}}\|g\|_{L^p}^{\frac{1}{d\left(1-\frac{1}{p}\right)}}, \tag{A.8}$$

$$\|Q^{+}(f,g)\|_{L^1} \le \|f\|_{L^1}^{1-\frac{1}{d\left(1-\frac{1}{p}\right)}}\|f\|_{L^p}^{\frac{1}{d\left(1-\frac{1}{p}\right)}}\|g\|_{L^1}. \tag{A.9}$$

Proof. By the change of variable, we have

$$\begin{aligned}
\|Q^{+}(f,g)\|_{L^1} &\lesssim \int_{\mathbb{S}^{d-1}}\int_{\mathbb{R}^{2d}} \frac{|f(u^*)g(v^*)|}{|u^*-v^*|}\,du\,dv\,d\omega \\
&= \int_{\mathbb{S}^{d-1}}\int_{\mathbb{R}^{2d}} \frac{|f(u^*)g(v^*)|}{|u^*-v^*|}\,du^*dv^*d\omega \\
&\lesssim \|f\|_{L^1}\left\|\int \frac{|g(v^*)|}{|u^*-v^*|}\,dv^*\right\|_{L^\infty}.
\end{aligned}$$

Using the L^∞ estimate (A.5), we get

$$\|Q^{+}(f,g)\|_{L^1} \lesssim \|f\|_{L^1}\|g\|_{L^1}^{1-\frac{1}{d\left(1-\frac{1}{p}\right)}}\|g\|_{L^p}^{\frac{1}{d\left(1-\frac{1}{p}\right)}}.$$

In the same way, we also obtain estimate (A.9). □

Appendix B. Strichartz Estimates

Recall the abstract Strichartz estimates.

Theorem B.1 ([48, Theorem 1.2]). *Suppose that for each time* t *we have an operator* $U(t)$ *such that*

$$\|U(t)f\|_{L^2_x} \lesssim \|f\|_{L^2_x},$$
$$\|U(t)(U(s)^*)f\|_{L^\infty_x} \lesssim |t-s|^{-\sigma}\|f\|_{L^1_x}.$$

Then it holds that

$$\|U(t)f\|_{L^q_t L^p_x} \lesssim \|f\|_{L^2_x}, \tag{B.1}$$

for all sharp σ-admissible exponent pair that

$$\frac{2}{q} + \frac{2\sigma}{p} = \sigma, \quad q \geq 2, \ \sigma > 1. \tag{B.2}$$

The symmetric hyperbolic Schrödinger equation is

$$\begin{cases} i\partial_t \phi + \nabla_\xi \cdot \nabla_x \phi = 0, \\ \phi(0) = \phi_0. \end{cases} \tag{B.3}$$

Note that the linear propagator $U(t) = e^{it\nabla_\xi \cdot \nabla_x}$ satisfies the energy and dispersive estimates

$$\begin{aligned} \|e^{it\nabla_\xi \cdot \nabla_x} \phi_0\|_{L^2_{x\xi}} &\lesssim \|\phi_0\|_{L^2_{x\xi}}, \\ \|e^{it\nabla_\xi \cdot \nabla_x} \phi_0\|_{L^\infty_{x\xi}} &\lesssim t^{-d} \|\phi_0\|_{L^1_{x\xi}}. \end{aligned} \tag{B.4}$$

Then by Theorem B.1, this gives a Strichartz estimate that

$$\|e^{it\nabla_\xi \cdot \nabla_x} \phi_0\|_{L^q_t L^p_{x\xi}} \lesssim \|\phi_0\|_{L^2_{x\xi}}, \quad \frac{2}{q} + \frac{2d}{p} = d, \quad q \geq 2, \ d \geq 2. \tag{B.5}$$

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